Implementation: The conjugacy problem in right-angled Artin groups

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Abstract

In 2009, Crisp, Godelle and Wiest constructed a linear-time algorithm to solve the conjugacy problem in right-angled Artin groups. This algorithm has now been implemented in Python, and the code is freely available on GitHub. This document provides a summary of how the code works. As well as determining whether two elements w_1, w_2 are conjugate in a RAAG A_Γ, our code also returns a conjugating element $x \in A_\Gamma$ such that $w_1 = x^{-1}w_2x$, if w_1 and w_2 are conjugate.

1 INTRODUCTION

In their paper '*The conjugacy problem in subgroups of right-angled Artin groups*', Crisp, Godelle and Wiest created a linear-time algorithm to solve the conjugacy problem in right-angled Artin groups (RAAGs) [1]. This algorithm has now been implemented as a Python program, and is freely available on GitHub.

This document provides an overview of the code. We recommend the reader first takes a look at the original paper [1], to understand the key tools and steps of this algorithm.

Remark 1.1. Our Python code requires the `networkx` module. You may also need to install Pillow and nose. Details on how to install these modules can be found here: `https://networkx.org/documentation/stable/install.html`.

1.1 Notation

Throughout we let A_Γ denote a RAAG with defining graph Γ. We let N denote the number of vertices in the generating set, i.e. the size of the standard generating set for A_Γ. We assume generators commute if and only if there does not exist an edge between corresponding vertices in Γ, to match with the convention used in [1].

Examples are provided both in the code as well as this document to assist the user. Throughout this summary, we will often use the RAAG defined in Example 2.4 from [1], which has the following presentation:

$$A_\Gamma = \langle a_1, a_2, a_3, a_4 \mid [a_1, a_4] = 1, [a_2, a_3] = 1, [a_2, a_4] = 1\rangle. \tag{1}$$

This RAAG is defined by the following graph:

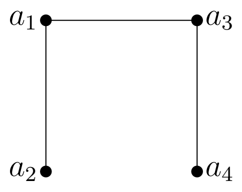


Figure 1: Defining graph Γ

2 SUMMARY OF ALGORITHM

We summarise the key steps of the algorithm from [1]. For further details on piling representations, see Chapter 14 of [2].

<u>ALGORITHM</u>: Conjugacy problem in RAAGs.

Input:

1. RAAG A_Γ: number of vertices of the defining graph, and list of commuting generators.
2. Words v, w representing group elements in A_Γ.

Step 1: Cyclic reduction

Produce the piling representation $\pi(v)$ of the word v, and apply cyclic reduction to $\pi(v)$ to produce a cyclically reduced piling p. Repeat this step for the word w to get a cyclically reduced piling q.

Step 2: Factorisation

Factorise each of the pilings p and q into non-split factors. If the collection of subgraphs do not coincide, **Output** = `False`.
Otherwise, continue to Step 3.

Step 3: Compare non-split factors

If $p = p^{(1)} \dots p^{(k)}$ and $q = q^{(1)} \dots q^{(k)}$ are the factorisations found in Step 2, then for each $i = 1, \dots, k$, do the following:

(i) Transform the non-split cyclically reduced pilings $p^{(i)}$ and $q^{(i)}$ into pyramidal pilings $\tilde{p}^{(i)}$ and $\tilde{q}^{(i)}$.
(ii) Produce the words representing these pilings in cyclic normal form $\sigma^*\left(\tilde{p}^{(i)}\right)$ and $\sigma^*\left(\tilde{q}^{(i)}\right)$.
(iii) Decide whether $\sigma^*\left(\tilde{p}^{(i)}\right)$ and $\sigma^*\left(\tilde{q}^{(i)}\right)$ are equal up to a cyclic permutation. If not, **Output** = `False`.

Output = `True`.

Example 2.1. We provide an example of how to implement this algorithm with our Python code, using the RAAG A_Γ defined in Equation 1. Consider the following two words:

$$w_1 = a_2^{-2} a_4^{-1} a_3 a_2 a_4 a_1 a_2 a_1^{-1} a_2^2 a_4^{-1}, \quad w_2 = a_4 a_3 a_4^{-1} a_2 a_1 a_2 a_1^{-1} a_4^{-1}.$$

These represent conjugate elements in A_Γ, since

$$w_1 = \left(a_4^2 a_2^2\right)^{-1} \cdot w_2 \cdot \left(a_4^2 a_2^2\right).$$

To check this, we input the following code. See Section 3 on how to input words and commutation relations from a RAAG.

```
      w_1 = [-2,-2,-4,3,2,4,1,2,-1,2,2,-4]
      w_2 = [4,3,-4,2,1,2,-1,-4]
CGW_Edges = [(1,4), (2,3), (2,4)]
            is_conjugate(w_1, w_2, 4, CGW_Edges)
```

The output from this is

```
True, [-2, -2, -4, -4]
```

The first argument is a `True` or `False` statement of whether w_1 and w_2 are conjugate. The second argument returns a conjugating element x such that $w_1 = x^{-1}w_2x$.

It is important to note that the order of the input words determines the conjugating element. In particular, when we input w_1 following by w_2, we obtain a conjugator x such that $w_1 = x^{-1}w_2x$. If we reverse the order of w_1 and w_2, the program will return the inverse of this conjugator, since $w_2 = xw_1x^{-1}$.

3 INPUTS

3.1 Words

The approach of using symbols $a, b, c...$ to represent letters in a word is limiting, since there are only finitely many letters in the alphabet. Instead, we use the positive integers to represent the generators, and the negative integers to represent their inverses:

Generator	Representation	Generator	Representation
a	1	a^{-1}	-1
b	2	b^{-1}	-2
c	3	c^{-1}	-3
...	...	...	...

This also matches with our convention of ordering which will be shortlex, i.e.

$$1 < -1 < 2 < -2 < \dots$$

We note this convention is the opposite of the normal form convention in [1].

Words in A_Γ are represented by lists of integers. The following table gives some examples. Here ε denotes the empty word.

Word	Representation
abc	`[1,2,3]`
ε	`[]`
$a^{-2}c^3b^{-1}$	`[-1,-1,3,3,3,-2]`
$a^{100}b^{-1}$	`([1]*100)+[-2]`
$abc...xyz$	`list(range(1,27))`

3.2 Commuting generators

In many of the functions in our code, we need to define which generators commute in A_Γ. This is represented by a list of tuples, each of which represents a pair of commuting generators. Here are some examples:

Group	Commuting generators list
$\mathbb{F}_n$	`[]`
$\mathbb{Z}^2$	`[(1,2)]`
$\mathbb{F}_2 \times \mathbb{Z}$	`[(1,3),(2,3)]`
$\mathbb{Z}^n$	`[(i,j) for i in range(1,n) for j in range(i+1,n+1)]`

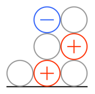
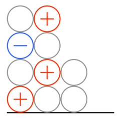
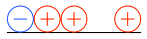
By convention, we do not include edges (n, n), and we order our list by shortlex ordering. For example, if $(3, 4)$ is in our list, we do not need to also include $(4, 3)$. Also, the code does not require inverses of generators to be taken into account. In particular, for every tuple (n, m) in the list, we do not need to also include $(-n, m)$, $(n, -m)$ or $(-n, -m)$.

This list should contain precisely one tuple for each non-edge in the defining graph. If no list is entered, the program takes `[]` as default (i.e. the free group).

Remark 3.1. In the Python code, we use the command `commuting_elements` to denote commuting generators.

4 PILINGS

Pilings are represented by lists of lists. Each list within the main list corresponds to a column of the piling, reading from bottom to top. A '+' bead is represented by `1`, a '−' bead is represented by `-1` and a '0' bead is represented by `0`. Here are some examples:

Piling	Representation
	`[[0],[1,0,-1],[0,1,0]]`
	`[[1,0,-1,0],[0,1,0,1],[0,0],[]]`
	`[[-1],[1],[1],[],[1]]`

We note the empty piling is represented by `[[]*N]` where N denotes the number of generators.

Reading pilings as a list of lists is not as user friendly as a pictorial representation. We refer the reader to Appendix A, which explains how to use the `draw_piling()` function. This allows the user to create a pictorial representation of pilings.

4.1 Programs

One of the most useful steps of the algorithm is converting words, which represent group elements in a RAAG, into piling representations. One key fact from these constructions is that if two words $u, v \in A_\Gamma$ represent the same group element, then the piling representations for u and v will be the same. In particular, every group element in A_Γ is uniquely determined by its piling. We describe this function here.

Function: `piling(w, N, commuting_elements=[])`

Input:

1. Word w representing a group element in A_Γ.
2. N = number of vertices in defining graph.
3. List of commuting generators.

Output: Piling representation of w, as a list of lists.

By construction, the piling will reduce any trivial cancellations in w after shuffling, so we do not require our input word w to be reduced.

Example 4.1. Suppose you wish to compute the piling for the word $ab^2a^{-1}b \in \mathbb{F}_2$. Then you would input `piling([1,2,2,-1,2], 2)`, which outputs the following:

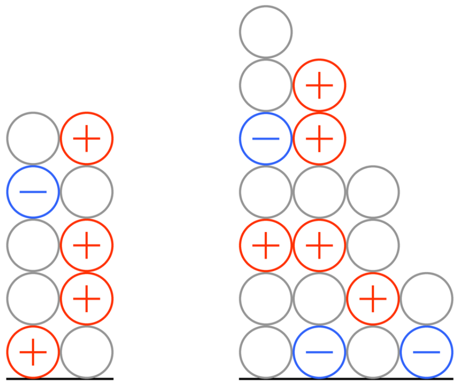
```
[[1, 0, 0, -1, 0], [0, 1, 1, 0, 1]]
```

Similarly we could compute the piling for the word $a_2^{-2}a_4^{-1}a_3a_2a_4a_1a_2a_1^{-1}a_2^2a_4^{-1} \in A_\Gamma$ from Equation 1. This outputs the following piling:

```
[[0, 0, 1, 0, -1, 0, 0], [-1, 0, 1, 0, 1, 1], [0, 1, 0, 0], [-1, 0]]
```

Figure 2 gives a pictorial representation of these pilings. Each list represents a column in the piling.

We now present a function which computes the normal form representative from a piling. For us, this is the shortlex shortest word which represents the piling. We remind the reader that this convention is the opposite ordering for normal forms in [1].



(a) (b)

Figure 2: $ab^2a^{-1}b \in \mathbb{F}_2$ and $a_2^{-2}a_4^{-1}a_3a_2a_4a_1a_2a_1^{-1}a_2^2a_4^{-1} \in A_\Gamma$

Function: `word(p, commuting_elements=[])`

Input:

1. $p =$ piling.
2. List of commuting generators.

Output: normal form word which represents the piling p.

Example 4.2. The following is an example of how to use this function for the following piling in $A_\Gamma = \langle a, b, c \mid ac = ca \rangle$:

```
p=[[1,0],[0,0,-1],[-1,0]]
w=word(p,[(1,3)])
print(w)
```

The output is:

```
[1,-3,-2]
```

5 CYCLIC REDUCTION

After constructing pilings from our input words, the next step of the conjugacy algorithm is to cyclically reduce each piling.

Function: `cyclically_reduce(p, commuting_elements=[])`

Input:

1. $p =$ piling.
2. List of commuting generators.

Output:

1. Cyclically reduced piling.
2. Conjugating element.

Example 5.1. Let $w = abca^{-1} \in \langle a, b, c \mid bc = cb \rangle$. We can compute all possible cyclic reductions on w as follows:

```
p=piling([1,2,3,-1], 3,[(2,3)])
p_cr=cyclically_reduce(p,[(2,3)])
w=word(p_cr[0],[(2,3)])
print(w, p_cr[1])
```

The output from this is:

```
[2,3], [1]
```

This function is the first example where we output a conjugating element. In this example, when we cyclically reduce $w = a \cdot bc \cdot a^{-1}$ to the word bc, we have taken out the conjugating element a.

6 GRAPHS

We remind the reader that for the functions in this section, we need to import the `networkx` module into Python. This makes working with the defining graph much easier, in particular computing induced subgraphs and connected components.

The next step in the conjugacy algorithm is to compute the non-split factors from a word based on the defining graph. First, we need a method to input our defining graph.

Function: `graph_from_edges(N, commuting_edges=[])`

Input:

1. N = number of vertices in defining graph.
2. List of commuting generators.

Output: `networkx` graph, which represents the defining graph.

We recall that in [1], the convention is to add edges for non-commuting vertices. This is precisely what this function does - the output is a graph on N vertices, with edges between vertices (x, y) if and only if (x, y) is not in the list of commuting generators.

Example 6.1. Let's compute the defining graph from Equation 1. We input the following:

```
CGW_Edges = [(1,4), (2,3), (2,4)]
g = graph_from_edges(4, CGW_Edges)
nx.draw(g)
```

This outputs the following graph:

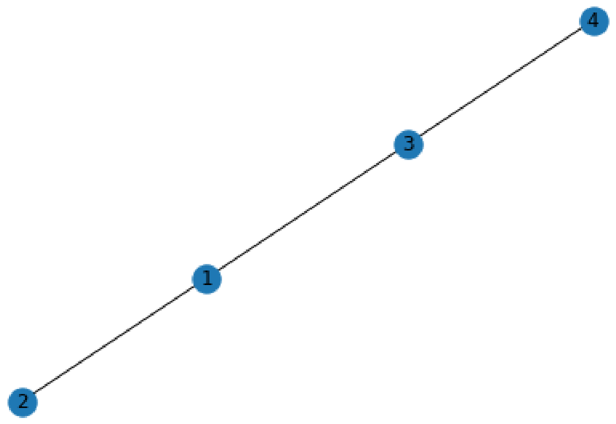


Figure 3: `networkx` graph defined in Example 6.1.

6.1 Factorising

We now want to build up a function which takes our defining graph g and our input word w, and computes the non-split factors related to w.

Function: `factorise(g, p)`

Input:

1. $g =$ `networkx` graph representing the defining graph of A_Γ.
2. $p =$ piling.

Output: List of subgraphs of g, one for each non-split factor of p.

This function first checks which columns in the piling contain a `1` or `-1` bead (this is the `supp_piling()` function). From this subset of vertices, we build the induced subgraph, and then return a list of the connected components of this subgraph. Each of these connected components represents the non-split factors.

At this stage, we have extracted each of the subgraphs which correspond to the factors. We now want to extract pilings which represent each factor.

Function: `graphs_to_nsfactors(l, w, N, commuting_elements)`

Input:

1. $l =$ list of subgraphs representing connected components.
2. $w =$ word representing a group element of A_Γ.
3. $N =$ number of vertices in defining graph.
4. List of commuting generators.

Output: list of the non-split factors from w represented by pilings.

Example 6.2. We use the example from Figure 4 of [1]. We input the following:

```
w = [2,3,-4]
CGW_Edges = [(1,4), (2,3), (2,4)]
g = graph_from_edges(4, CGW_Edges)
graphs = factorise(g, piling(w, 4, CGW_Edges))
print(graphs_to_nsfactors(graphs, w, 4, CGW_Edges))
```

The output is

```
[[[0], [1], [], []], [[0], [], [1, 0], [0, -1]]]
```

where the first piling represents the factor a_2, and the second piling represents the factor $a_3a_4^{-1}$.

7 PYRAMIDAL PILINGS

For the final step in the algorithm, we need a function which converts pilings into pyramidal pilings. After this, we need to check when two pyramidal pilings are equal up to cyclic permutation.

Function: `pyramidal_decomp(p, commuting_elements=[])`

Input:

1. $p =$ non-split piling.
2. List of commuting generators.

Output: factors p_0, p_1 as pilings.

It is important to note that the input piling must be non-split, otherwise we cannot achieve a pyramidal piling and the function will run forever.

Example 7.1. Let's compute the decomposition in Figure 5 of [1]. We input the following:

```
CGW_Edges = [(1,4), (2,3), (2,4)]
p = [[0,1,0,-1,0], [0,1,0,1], [0,1,0,0], [-1,0]]
decomp = pyramidal_decomp(p, CGW_Edges)
print(decomp)
```

The output is then

```
([[0], [], [0, 1], [-1, 0]], [[1, 0, -1, 0], [0, 1, 0, 1], [0, 0], []])
```

Function: `pyramidal(p, N, commuting_elements=[])`

Input:

1. $p =$ non-split cyclically reduced piling.

2. N = number of vertices in defining graph.

3. List of commuting generators.

Output:

1. Pyramidal piling of p.

2. Conjugating element.

The `pyramidal()` function iteratively applies `pyramidal_decomp()` until the factor p_0 is empty. Since the only operation we apply on the piling is cyclic permutations, we can add these steps to our conjugating element.

Example 7.2. Again we take the same example above. Our input is:

```
CGW_Edges = [(1,4), (2,3), (2,4)]
p = [[0,1,0,-1,0], [0,1,0,1], [0,1,0,0], [-1,0]]
pyr = pyramidal(p, 4, CGW_Edges)
```

Figure 4 gives the pictorial representation of this pyramidal piling. The conjugating element output in this example is `[-4,3,-4]`.

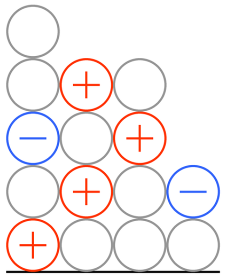


Figure 4: Pyramidal piling.

Function: `is_cyclic_permutation(w,v)`

Input: Two words w, v representing group elements in A_Γ.
Output: If w is a cyclic permutation of v, the function outputs two arguments:

1. `True`

2. Conjugating element.

Otherwise, return `False`.

8 IMPLEMENTATION OF THE CONJUGACY PROBLEM

8.1 Simple Cases

If we want to solve the conjugacy problem in either $\mathbb{F}_n$ or $\mathbb{Z}^n$, the algorithm for implementing the conjugacy problem is more straightforward, and using the `is_conjugate()` function detailed below is wasteful. Certainly in the latter case, the conjugacy problem is trivial; it is equivalent to checking if two words are equal, which is just a case of comparing the number of occurrences of each letter in the words. For free groups, two cyclically reduced words are conjugate if and only if they are cyclic permutations of each other. Hence one can make a shorter program for free groups by simply using the `cyclically_reduce()` and `is_cyclic_permutation()` functions.

For general RAAGs, we now have the necessary functions to implement the conjugacy problem.

Function: `is_conjugate(w1, w2, N, commuting_elements=[])`

Input:

1. Two words w_1, w_2 representing group elements in A_Γ.
2. $N =$ number of vertices in defining graph.
3. List of commuting generators.

Output: If w_1 is conjugate to w_2 in A_Γ then return:

1. `True`
2. Conjugating element x such that $w_1 = x^{-1}w_2x$.

Otherwise, return `False`.

We note that when computing the conjugating element, we find a reduced word x which represents an element which conjugates w_1 to w_2.

A THE 'DRAW PILING' FUNCTION

The representation of pilings as a list of lists of `1`s, `-1`s and `0`s is not very readable in the Python interpreter. For small pilings it is tolerable, but for larger ones the `draw_piling()` function is useful.

`draw_piling()` can take a total of nine arguments, however only the first one, which is the piling to be drawn, is needed. The function returns nothing, but by default it will show a picture of the piling in a new window, as well as save a `PNG` file of it.

The following table lists all the arguments of the `draw_piling()` function and what they do:

Argument	Type	Description
piling	Piling	The (mandatory) piling to draw.
scale	Float	The scale to draw the piling at. Default is 100.0.
plus_colour	Colour	The colour to draw the '+' beads. Default is red.
zero_colour	Colour	The colour to draw the '0' beads. Default is grey.
minus_colour	Colour	The colour to draw the '−' beads. Default is blue.
anti_aliasing	Integer	The super-sampling resolution for anti-aliasing. Only allows positive integers. Default is 4.
filename	String	The filename to save the piling as. Default is "piling.png".
show	Boolean	Whether to show the piling in a window. Default is True.
save	Boolean	Whether to save the piling. Default is True.

B SOLUTION TO THE WORD PROBLEM

With the tools that pilings.py provides, it is not hard to solve the Word Problem in any given RAAG. The following code defines a function that decides if a given word is equal to the identity:

```
def identity(w, N, commuting_elements=[]):
p=piling(w, N, commuting_elements)#generate reduced piling
reduced_w=word(p, commuting_elements)#read off reduced word
return(reduced_w==[])#return whether it equals the identity
```

The following code defines a function that decides if two given words are equal in a given RAAG:

```
def equal(w1, w2, N, commuting_elements=[]):
#generate reduced pilings
p1=piling(w1, N, commuting_elements)
p2=piling(w2, N, commuting_elements)
#read off reduced words
reduced_w1=word(p1, commuting_elements)
reduced_w2=word(p2, commuting_elements)
#return whether they are equal
return(reduced_w1==reduced_w2)
```

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Whilst we have made our best efforts to debug and correctly implement code for this algorithm, there may still be mistakes! If you spot any errors or issues with our code, please let us know.

REFERENCES

[1] John Crisp, Eddy Godelle, and Bert Wiest. The conjugacy problem in subgroups of right-angled artin groups. *Journal of Topology*, 2:442–460, 2009.

[2] Dan Margalit and Matt Clay, editors. *Office Hours with a Geometric Group Theorist.* Princeton University Press, 2017.

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