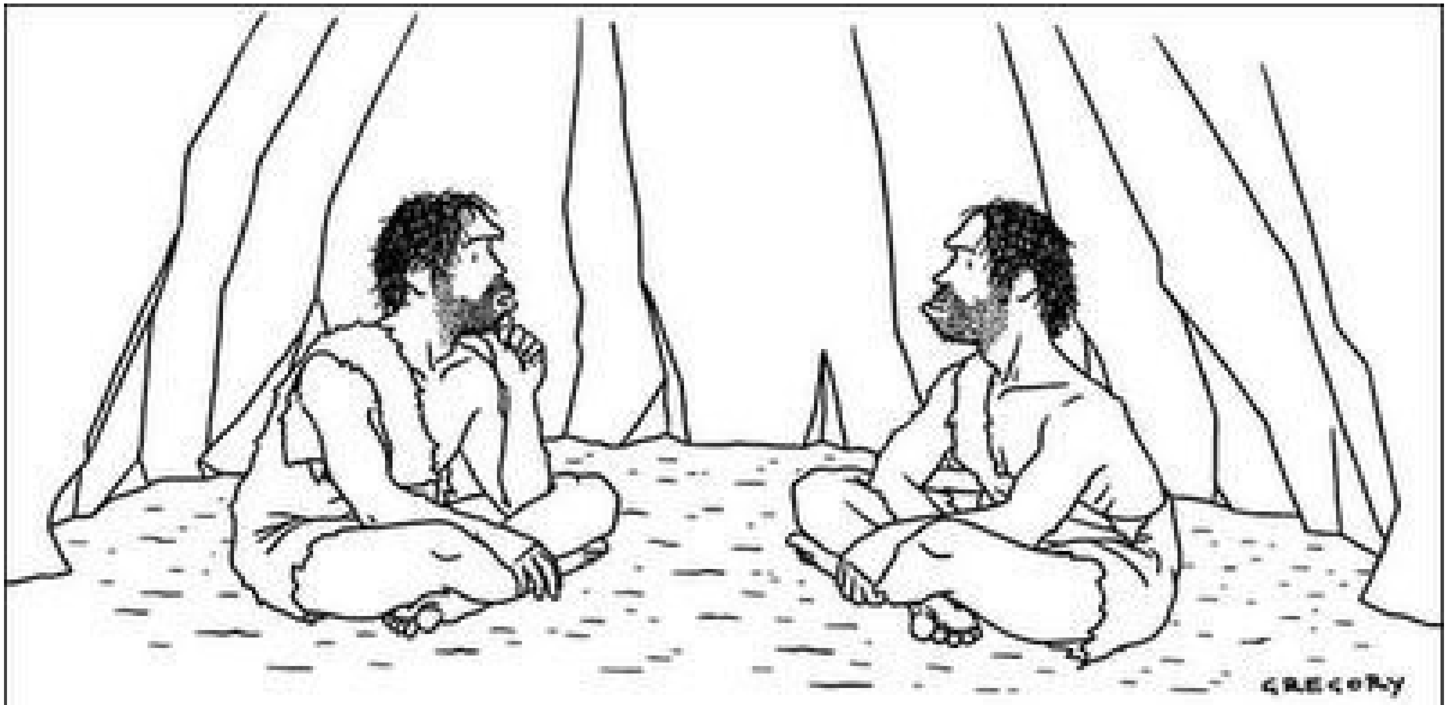
Lecture 3:
Omitted Variables Bias and Multivariate Regression



"Something's just not right—our air is clean, our water is pure, we all get plenty of exercise, everything we eat is organic and free-range, and yet nobody lives past thirty."

Olivier Deschenes, UCSB, ESM 296, Winter 2018

Outline of Lecture 3

- Omitted Variables Bias and Multiple Regression

- Sampling Distribution of OLS Estimator in Multiple Regression

- Homoskedasticity vs. Heteroskedasticity

- Hypothesis Tests (covered in supplementary notes & homework)

Omitted Variables Bias

- Consider the simple model with two regressors:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$$

- Suppose that the variable X_{2i} is <u>omitted</u> from the regression (either because of model specification error, or maybe because you don't have data on the variable X_{2i})

- Then the regression model becomes:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \mathrm{v}_i, \quad \mathrm{v}_i = \beta_2 X_{2i} + u_i$$

- Q: LSA #1 is now $E[v_i|X_{1i}]=0$. Is it satisfied here?

Key Result:

- Let $\mathrm{Corr}(X_{1i},X_{2i}) = \rho_{12} \neq 0$

(Note: LSA #1 not satisfied, i.e., $\mathrm{Corr}(X_{1i},v_i) \neq 0$ even if $\mathrm{Corr}(X_{1i},u_i)=0$)

- Then, we can prove that the OLS estimator has the following probability limit:

$$\boxed{\hat{\beta}_1 \xrightarrow{p} \beta_1 + \beta_2 \rho_{12} \frac{\sigma_{X2}}{\sigma_{X1}}}$$

- This says that as the sample size increases, $\hat{\beta}_1$ does <u>not</u> get close to the true β_1 with high probability

Implications:

- If the regressor X_{1i} is correlated with a variable that:

  (i) is also a predictor of the dependent variable Y_i,

  (ii) has been omitted from the model, then the OLS estimator will suffer from <u>omitted variable bias</u> (i.e. OLS is not consistent)

- In the house value example, omitted variable bias will arise if NOx concentrations are correlated with other predictors of house values (for example: house size, noise levels, other pollutants, etc) <u>and</u> if these factors are not controlled for in the regression

Conclusion on Omitted Variables Bias

- Omitted variable bias is a problem whether the sample size is small or large. Even in the limit experiment when $n \to \infty$, the OLS estimator remains inconsistent

- Whether this "bias" is large or small depends on:

  (i) the magnitude of the correlation between X_{1i} and the omitted variable (X_{2i} in the example). The larger $|\rho_{12}|$, the larger is the bias

  (ii) the magnitude of the regression coefficient on the omitted variable (β_2 in the example)

- The direction of the bias depends on the sign of ρ_{12} and β_2. If $\rho_{12}>0$ and $\beta_2>0$, then the OLS estimator <u>overstates</u> β_1

Solutions to Problem of Omitted Variables Bias:

- **1. Add more variables to the regression model**
- Effectively, this improves the credibility of the assumption $E[u_i|\boldsymbol{X}_i]=0$ (LSA#1)
- Why: the more variables you include, the more potential relevant predictors of Y_i you include
- However: there is a bias/variance tradeoff in finite samples (including more regressors reduces the risk of OV bias but it also reduces the precision of OLS estimator)
  - Moreover, some important factors may be unobservable so it impossible to directly include controls for them
- **2. Later:** Matching, Instrumental variables regression, Panel data models, and also controlled random experiments

The Population Multiple Regression Model

- Consider the case of <u>two</u> regressors:

  $$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i \qquad i = 1,\ldots,n$$

- β_0 = unknown population intercept
- β_1 = effect of a change in X_1 on Y, holding X_2 constant
- β_2 = effect of a change in X_2 on Y, holding X_1 constant
- u_i = the regression error (omitted factors)

Interpretation of Coefficients in Multiple Regression

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i, \qquad i = 1,\ldots,n$$

- Consider changing X_1 by ΔX_1 while holding X_2 constant:

- Population regression function ***before*** the change in X_1:
- $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$

- Population regression function, ***after*** the change in X_1 :
- $Y + \Delta Y = \beta_0 + \beta_1(X_1 + \Delta X_1) + \beta_2 X_2$

- Difference: $\Delta Y = \beta_1 \Delta X_1$

- **Implications:**

$$\beta_1 = \frac{\Delta Y}{\Delta X_1}, \text{holding } X_2 \text{ constant}$$

- Similarly,

$$\beta_2 = \frac{\Delta Y}{\Delta X_2}, \text{holding } X_1 \text{ constant}$$

- β_0 = predicted value of Y when $X_1 = X_2 = 0$
  - Rarely a useful parameter

OLS Estimator in Multivariate Regression

- Recall the formula from <u>bivariate</u> regression
  $Y_i = \beta_0 + \beta_1 X_{1i} + u_i$:

$$\hat{\beta}_1 = \frac{S_{X_1 Y}}{S_{X_1}^2}$$

- Equivalent formula in <u>multivariate</u> setting
  $Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$:

$$\hat{\beta}_1 = \frac{S_{\tilde{X}_1 Y}}{S_{\tilde{X}_1}^2}$$

- Where $\tilde{X}_{1i}$ is the fitted <u>residual</u> from a regression of X_{1i} on a constant term and <u>all</u> the other regressors (here only X_{2i})

Multiple Regression in STATA

```
regress price nox rooms, robust;

Linear regression                                      Number of obs =     206
                                                       F(  2,   203) =   78.47
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.5923
                                                       Root MSE      =  6019.3

------------------------------------------------------------------------------
             |               Robust
       price |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         nox |  -1062.208   357.8614    -2.97   0.003    -1767.811   -356.6063
       rooms |   9836.748   924.3718    10.64   0.000     8014.146    11659.35
       _cons |  -33216.07   6655.565    -4.99   0.000    -46338.97   -20093.17
------------------------------------------------------------------------------
```

$$\widehat{\text{Price}} = -33216 - 1062 \times NOX + 9837 \times ROOMS$$

Interpretation:

- Recall the bivariate regression of *Price* on *NOx (Lec 2)*:

$$\hat{\text{Price}} = 38068 - 2776 \times NOX$$

- Now include number of rooms (*Rooms*) as well:

$$\hat{\text{Price}} = -33216 - 1062 \times NOX + 9837 \times ROOMS$$

- What happens to the coefficient on *NOx*?
- Why? (*Note*: corr(*NOx*, *Rooms*) = -0.29)

- $\Rightarrow$ In the model that omits *Rooms*, the regression attributes some of the effect of *Rooms* to *NOx*

Application of the Partial OLS Estimator Formula

- Recall $\hat{\beta}_1 = \dfrac{S_{\tilde{X}_1 Y}}{S^2_{\tilde{X}_1}}$

- Step 1: Regress *NOx* on *Rooms*, get fitted residuals:

```
. regress nox rooms, robust;

Linear regression                                      Number of obs =     206
                                                       F(  1,   204) =   18.19
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.0837
                                                       Root MSE      =  1.0977

------------------------------------------------------------------------------
             |               Robust
         nox |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       rooms |  -.4806996   .1127166    -4.26   0.000    -.7029385   -.2584607
       _cons |   8.549037    .715984    11.94   0.000     7.137359    9.960715
------------------------------------------------------------------------------
```

```
. predict nox_resid, residual;
```

- Recall $\hat{\beta}_1 = \dfrac{S_{\tilde{X}_1 Y}}{S^2_{\tilde{X}_1}}$

- Step 2: Compute $\hat{\beta}_1$

```
. summm price nox_resid;

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
       price |       206    22723.11    9381.108       5000      50001
   nox_resid |       206   -9.20e-10    1.095036  -1.802523   3.107652

. correlate price nox_resid, cov;
(obs=206)

             |    price nox_re~d
-------------+------------------
       price |  8.8e+07
   nox_resid |  -1273.7   1.1991
```

$$\boxed{\hat{\beta}_1 = \frac{-1273.7}{1.095^2} = -1062}$$

Key Concept 6.4

The Least Squares Assumptions in the Multiple Regression Model

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \ldots + \beta_k X_{ki} + u_i, i = 1, \ldots, n, \text{ where}$$

1. u_i has conditional mean zero given $X_{1i}, X_{2i}, \ldots, X_{ki}$; that is,

$$E(u_i | X_{1i}, X_{2i}, \ldots, X_{ki}) = 0.$$

2. $(X_{1i}, X_{2i}, \ldots, X_{ki}, Y_i), i = 1, \ldots, n$ are independently and identically distributed (i.i.d.) draws from their joint distribution.
3. Large outliers are unlikely: $X_{1i}, \ldots, X_{ki}$ and Y_i have nonzero finite fourth moments.
4. There is no perfect multicollinearity.

LSA#1 is key: An implication is that each regressor X is uncorrelated with the regression error u_i

<u>LSA#1 implies that there is no omitted variable in the model</u>

Discussion of the LSA's for Multivariate Model

- LSA1: $E[u_i|X_{1i}, X_{2i}, ..., X_{Ki}] = 0$
- $\Rightarrow$Key assumption: implies that the OLS estimator is consistent (i.e. no omitted variables bias)
- $\Rightarrow$Remember that it is **not testable** without more information
- LSA2 and LSA3: technical assumptions, always maintained in this class
- LSA4: The regressors are perfectly multi-collinear if one of the regressors is a perfect linear function of another
  - We rule this out

Discussion of Perfect Multi-Collinearity

- LSA4: The regressors are perfectly multi-collinear if one of the regressors is a perfect linear function of some of the others

- Example: X_{1i}=(=1 if observation i is male)
  X_{2i}=(=1 if observation i is female)
  So: $X_{1i} + X_{2i} = 1$, perfectly collinear with intercept

- LSA4 is "testable". If two (or more) regressors are perfectly collinear, Stata will throw one out of the regression model

- It simply means that you cannot separately identify the effect of the multi-collinear regressors on Y

Example**:** Suppose you accidentally include *NOX* twice in the regression:

```
regress price nox nox, robust
note: nox omitted because of collinearity

Linear regression                                      Number of obs =     206
                                                       F(  1,   204) =   44.86
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.1146
                                                       Root MSE      =    8849

------------------------------------------------------------------------------
             |               Robust
       price |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         nox |  -2775.674   414.4046    -6.70   0.000    -3592.739   -1958.608
         nox |  (omitted)
       _cons |   38068.27   2222.545    17.13   0.000     33686.17    42450.38
------------------------------------------------------------------------------
```

“Imperfect” Multi-Collinearity

- Two variables that are highly correlated with each other, although not perfectly (i.e. correlation coefficient close to 1 or -1)

- The more multi-collinear X_1 and X_2 are, the more “unstable” the OLS estimates of β_1 and β_2 become, and also the larger their standard errors become

- Detectable by examining data and regression results. See additional notes online

Key Concept 6.5

Large Sample Distribution of $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_k$

If the least squares assumptions (Key Concept 6.4) hold, then in large samples the OLS estimators $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_k$ are jointly normally distributed and each $\hat{\beta}_j$ is distributed $N(\beta_j, \sigma^2_{\hat{\beta}_j}), j = 0, \ldots, k$.

Same results as in the bivariate regression model

OLS estimator is distributed with a Normal distribution (when n is large) due to the Central Limit Theorem (CLT)

<u>Implication 1</u>. Can use Normal distribution for hypothesis tests

<u>Implication 2</u>. Formula for covariance matrix of OLS estimator depends on assumption of <u>homoskedasticity</u> or <u>heteroskedasticity</u> of the regression errors

***** Here we always proceed with the assumption of heteroskedasticity**

Heteroskedasticity and Homoskedasticity

- **What do these two terms mean?**

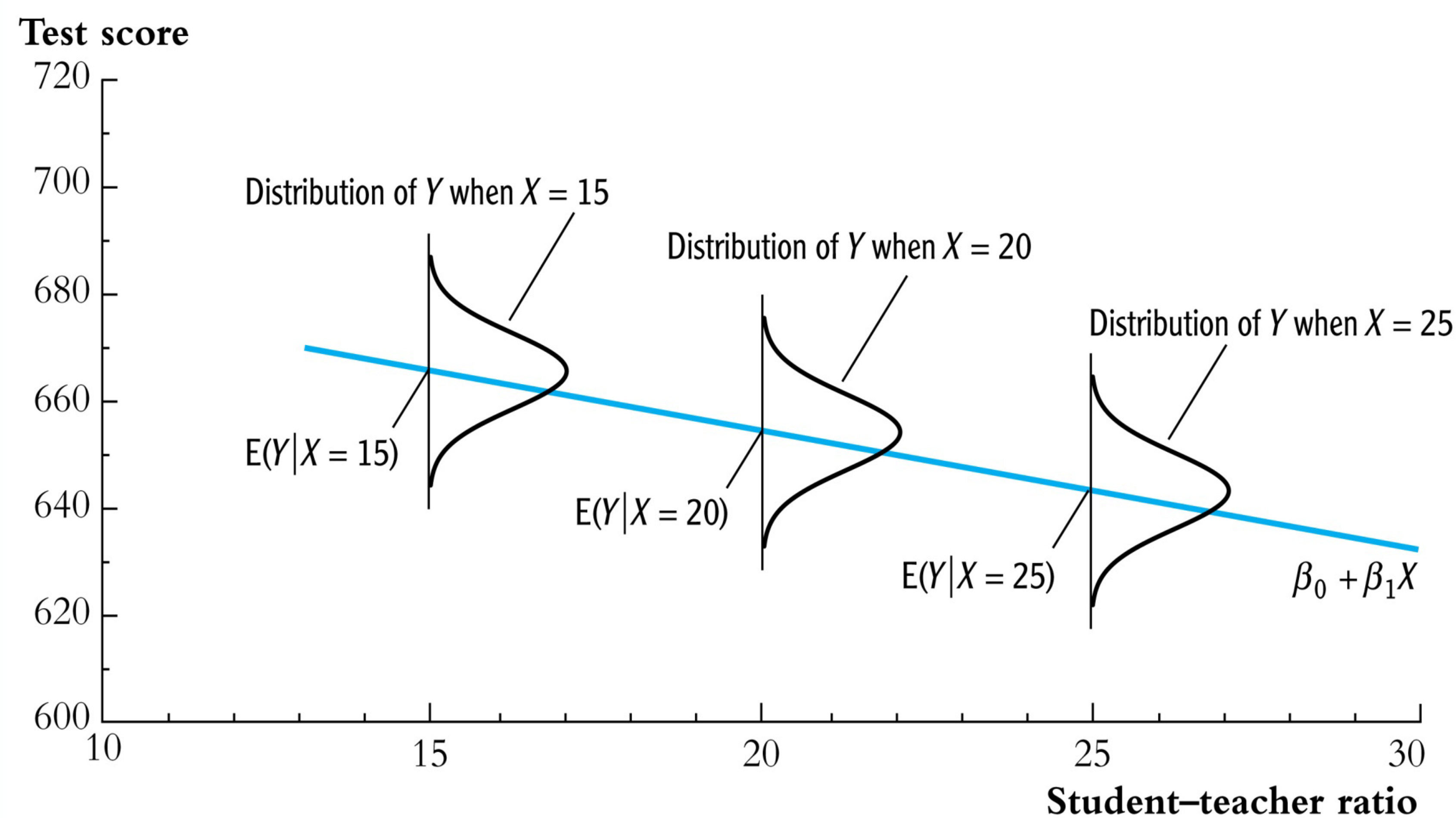
- If Var(u_i|X_i=x) is **constant** – that is, if the variance of the conditional distribution of u_i given X_i does not depend on X_i – then u_i is said to be ***homoskedastic***

- Otherwise, u_i is ***heteroskedastic***

- Since it involves the unobserved regression error term, it is in general difficult to directly assess heteroskedasticity by looking at the data, especially in multivariate models

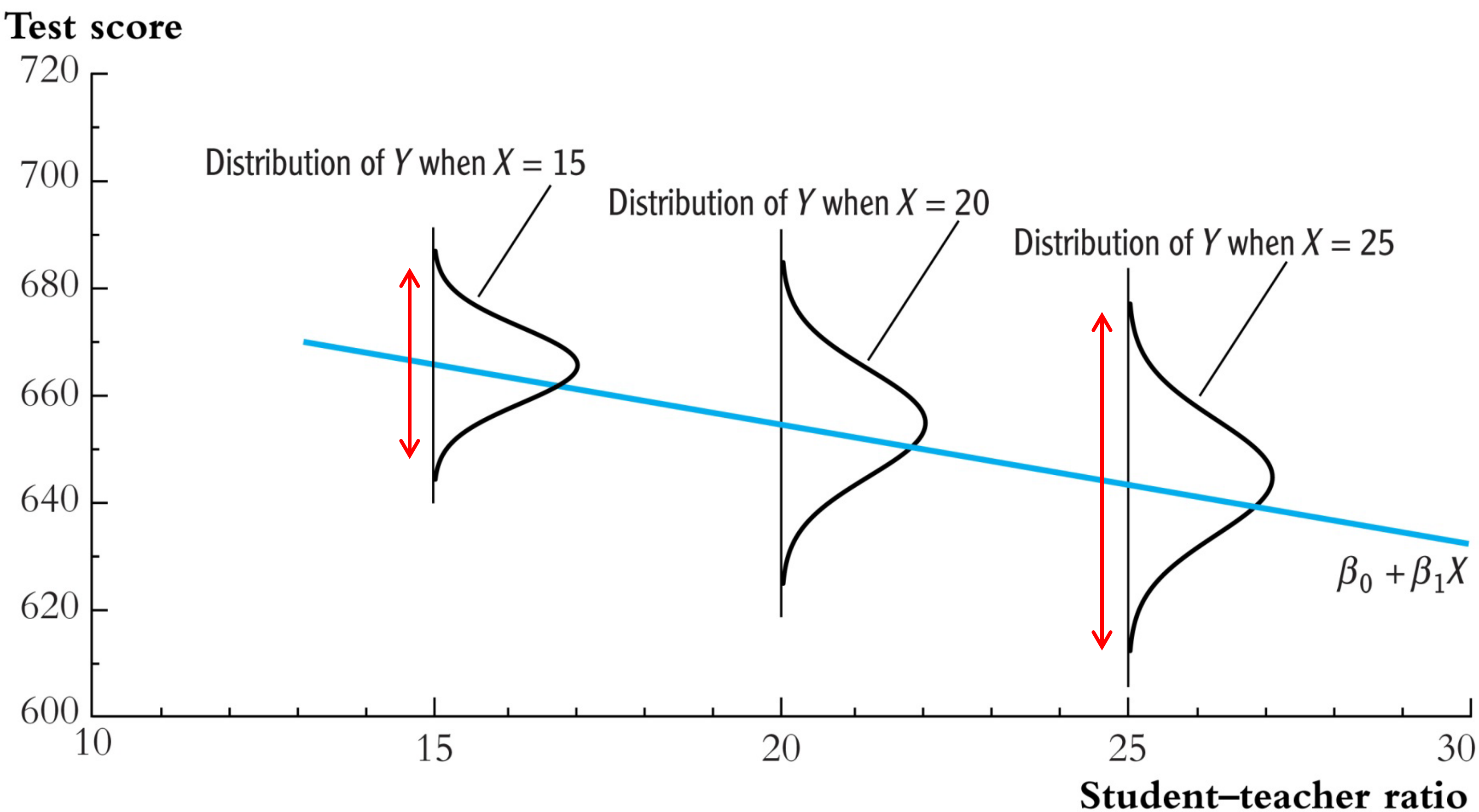
- Approach: assume heteroskedastic errors and adjust our methods of inference to account for it in a general way

Homoskedasticity in a picture:



⇒The variance of *u* (difference between Y and the blue population regression line) <u>does not</u> depend on *x* (i.e. spread of distribution of u|X does not change with X)

Heteroskedasticity in a picture



The variance of *u does* depends on *x*: *u* is heteroskedastic

In fact here, Var(u|X) increase as X increase

Implications of Homoskedasticity:

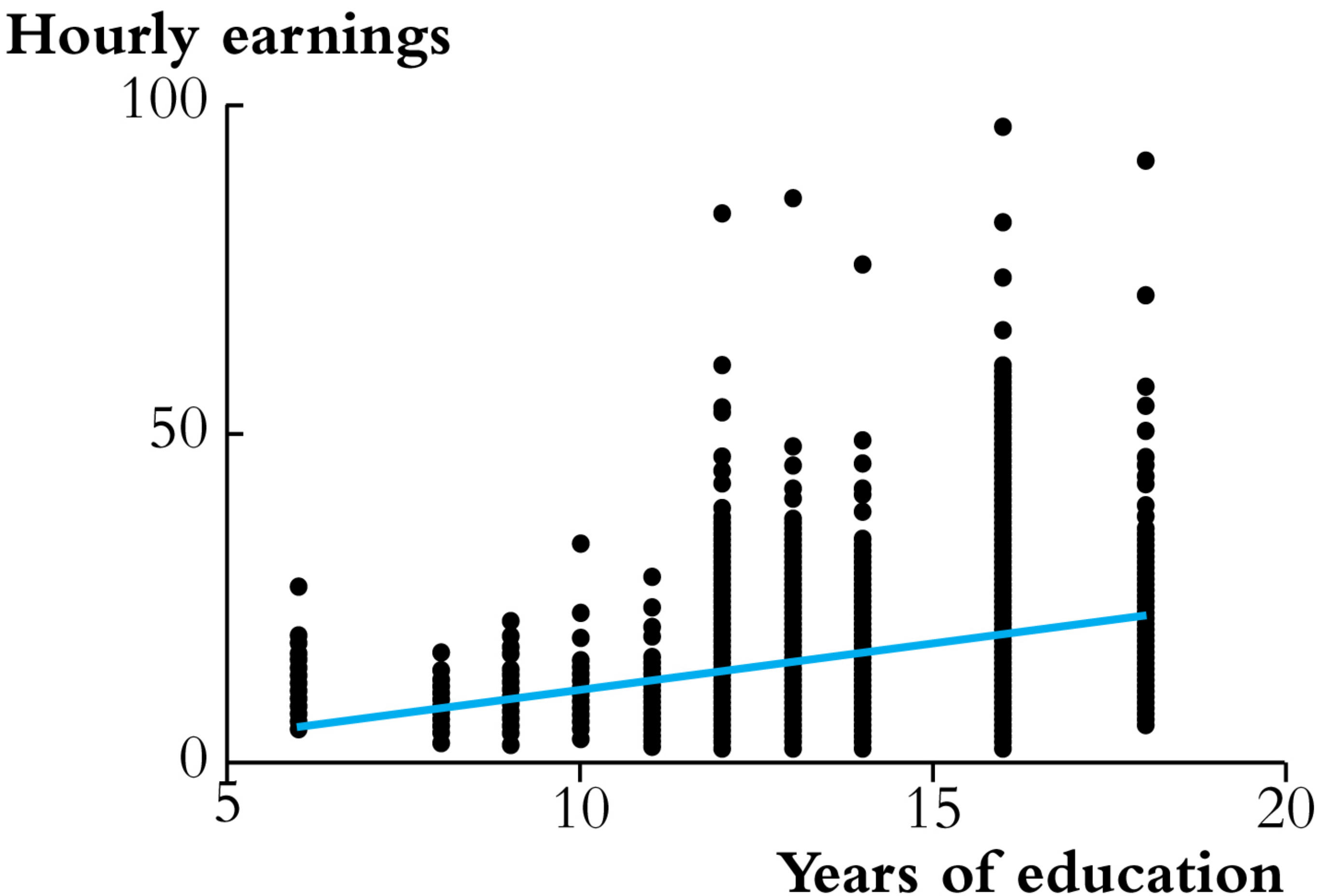
- Homoskedasticity of the error term $Var(u_i|X_i) = \sigma^2$ implies that the ***<u>conditional variance of Y given X is also constant</u>***:

- Consider simple bivariate model $\boldsymbol{Y_i = \beta_0 + \beta_1 X_i + u_i}$

$$
\begin{aligned}
&Var(Y \mid X) = Var(\beta_0 \mid X) + Var(\beta_1 X \mid X) + Var(u \mid X) \\
&= \beta_0 Var(1 \mid X) + \beta_1 Var(X \mid X) + \sigma^2 \\
&= 0 + 0 + \sigma^2
\end{aligned}
$$

- **[Note that all covariance terms are equal to 0 (by LSA#1)]**

Looking at data scatter plot to assess homoskedasticity



Is Var(Y|X) constant?

Scatter plot and regression line for hourly wages vs. years of education (data source: Current Population Survey)

Sampling Variance of OLS Estimator <u>Without</u> Homoskedasticity in Bivariate Model

- Recall the earlier result

- When the sample size n grows large, under assumptions LSA#1, LSA#2, and LSA#3, *and* **<u>without</u>** assuming homoskedasticity you can prove that:

$$\hat{\beta}_1 \overset{A}{\approx} N\left(\beta_1 , \frac{Var[(X_i - \mu_X)u_i]}{nVar(X_i)^2} \right)$$

The standard errors reported by STATA under the "regress y x, <u>robust</u>" command is an estimate of the square root of the sampling variance of the OLS estimator

Sampling Variance of OLS Estimator in Multivariate Regression

- The same logic applies here, but the formulas for the variance of the sampling distribution is more complicated (come to office hours if you want to know…)

- ** The OLS estimator has an approximately normal sampling distribution:

$$\hat{\beta}_j \overset{A}{\approx} N\left(\beta_j, \sigma^2_{\hat{\beta}_j}\right)$$

- You should assume (at least in ESM 296) that $\sigma^2_{\hat{\beta}_j}$ is derived under heteroskedasticity

Conclusion on Heteroskedasticity:

- 1. Whether the errors are homoskedastic or heteroskedastic does not change how we estimate the slope coefficients in all of our regression models

- 2. The sampling covariance matrix (i.e. Var($\hat{\beta}$)) derived under the assumption of heteroskedasticity simplifies (when *n* is large) to the theoretically-correct covariance matrix in the special case of homoskedasticity

  - **So, we always use heteroskedasticity-robust standard errors and inference**

```
. regress price rooms;

(…)

------------------------------------------------------------------------------
       price |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       rooms |   10347.35   620.3613    16.68   0.000     9124.209    11570.49
       _cons |  -42296.93   3921.414   -10.79   0.000    -50028.63   -34565.23
------------------------------------------------------------------------------

. regress price rooms, robust;

Linear regression                                      Number of obs =     206
                                                       F(  1,   204) =  148.50
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.5769
                                                       Root MSE      =  6116.7

------------------------------------------------------------------------------
             |               Robust
       price |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       rooms |   10347.35   849.1011    12.19   0.000     8673.211    12021.49
       _cons |  -42296.93   5426.635    -7.79   0.000    -52996.41   -31597.44
------------------------------------------------------------------------------
```