Exam 2 solutions

Problem 1.

Give careful answers to the following:

(a). What does it mean to say that a set A is *countably infinite*?

(b). What does it mean to say that the sequence a_n is *Cauchy*?

Solution for 1:

(a). A is countably infinite if there is a *bijection* $\varphi : A \to \mathbb{N}$, where $\mathbb{N}$ is the set of *natural numbers*.

(b). The sequence a_n of real (or rational) numbers is *Cauchy* if for every real number $\varepsilon > 0$, there is a natural number $m \in \mathbb{N}$ with the property that $\forall n_1, n_2 \geq m$, $|a_{n_1} - a_{n_2}| < \varepsilon$.

Problem 2.

Indicate whether each of the following statements is True or False by circling the correct choice. (Here you do not need to write an argument justifying your choice).

(a). **True** / **False** : Define a relation $\sim$ on $\mathbb{R}$ by the condition: $a \sim b$ if and only if $|a| = |b + 1|$. Then $\sim$ is an equivalence relation.

(b). **True** / **False** : Let $n \in \mathbb{N}$ with $5 \leq n$ and let X be the set

$$X = \left\{(a_0, a_1, a_2, a_3, a_4) \mid a_i \in I_n, i \neq j \Rightarrow a_i \neq a_j\right\}$$

of all sequences of *distinct* elements of $I_n = \{0, 1, ..., n - 1\}$ of length 5. Then

$$|X| = \frac{n!}{(n-5)!}.$$

Solution for 2a:
False. This relation is not reflexive – e..g $0 \nsim 0$ since $|0| \neq |0 + 1|$.

Solution for 2b:
True. To form a sequence $(a_0, a_1, ..., a_4)$ of 5 distinct elements of I_n, there are n choices for a_0. Once a_0 has been chosen, there are $n - 1$ choices for a_1. Similarly, there are then $n - 2$ choices for a_2. In this manner we see that the number of such sequences is

$$n(n-1)(n-2)...(n-4) = n!/(n-5)!.$$

Problem 3.

Let A, B be sets. Recall that we proved the following:

(♣): If $g : A \to B$ is a surjective function and if A is countable, then B is either finite or countably infinite.

(a). For any non-empty set Y, $\mathbb{R} \times Y$ is infinite and not countable.

Hint: Consider the mapping $\pi : \mathbb{R} \times Y \to \mathbb{R}$ given by $\pi(x, y) = x$. Is the function π surjective?

(b). Let $\mathbb{C}$ be the set of complex numbers; thus $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$. Is $\mathbb{C}$ countably infinite? Explain.

Solution for 3a:
The function $\pi : \mathbb{R} \times Y \to \mathbb{R}$ is surjective. Indeed, since Y is non-empty there is an element $y_0 \in Y$. For any $a \in \mathbb{R}$, we see that $\pi(a, y_0) = a$; this confirms that π is surjective.

Now, the contra-positive of the conclusion of (♣) says for a surjective function $g : A \to B$ that if B is infinite and not countable, then A is infinite and not countable.

We have seen that $\mathbb{R}$ is infinite and not countable. Applying the observation of the previous paragraph to the function $\pi : \mathbb{R} \times Y \to \mathbb{R}$, since $\mathbb{R}$ is infinite and not countable we may conclude that $\mathbb{R} \times Y$ is infinite and not countable.

Solution for 3b:
$\mathbb{C}$ is infinite and not countable.

To prove this, first observe that there is a bijection

$$\mathbb{R} \times \mathbb{R} \to \mathbb{C} \text{ given by the rule } (a, b) \mapsto a + bi.$$

Thus to prove that $\mathbb{C}$ is infinite and not countable, it is enough to prove that $\mathbb{R} \times \mathbb{R}$ is infinite and not countable. And in turn, $\mathbb{R} \times \mathbb{R}$ is infinite and not countable by the result of **3a**.

Problem 4.

For any finite sets X and Y we know that

$$(*) \quad |X \cup Y| = |X| + |Y| - |X \cap Y|.$$

Using $(*)$, show that if A, B, C are finite sets for which $A \cap B = A \cap C = B \cap C = \emptyset$, then

$$|A \cup B \cup C| = |A| + |B| + |C|.$$

Solution for 4:

Using $(*)$ we first see that

$$(\clubsuit) \quad |A \cup B \cup C| = |A| + |B \cup C| - |A \cap (B \cup C)|.$$

Now, another application of $(*)$ shows that

$$(\blacklozenge) \quad |B \cup C| = |B| + |C| - |B \cap C|.$$

Substitution of $(\blacklozenge)$ into $(\clubsuit)$ gives

$$|A \cup B \cup C| = |A| + B| + |C| - |B \cap C| - |A \cap (B \cup C)|,$$

so the result will follow once we confirm that $B \cap C = \emptyset$ and $A \cap (B \cup C) = \emptyset$, since in that case $|B \cap C| = 0$ and $|A \cap (B \cup C)| = 0$.

But $B \cap C = \emptyset$ by hypothesis.

And again by hypothesis we see that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = \emptyset \cup \emptyset = \emptyset.$$

This completes the argument.

Problem 5.

Consider the sequence $a_n = \dfrac{2n+1}{3n-2}$. Use the *definition* of the limit to prove that

$$\lim_{n\to\infty} a_n = \frac{2}{3}.$$

Solution for 5:
To prove the statement using the definition, let $\varepsilon > 0$. We must find $m \in \mathbb{N}$ with the property that

$$n \geq m \Rightarrow |a_n - 2/3| < \varepsilon.$$

Notice that

$$\begin{aligned} \frac{2n+1}{3n-2} - \frac{2}{3} &= \frac{6n+3-2(3n-2)}{(3n-2)\cdot 3} \\ &= \frac{6n+3-6n+4}{(3n-2)\cdot 3} \\ &= \frac{7}{9n-6}. \end{aligned}$$

So we need to arrange that

$$\frac{7}{9n-6} < \varepsilon, \quad \text{or equivalently that} \quad \frac{9n-6}{7} > \frac{1}{\varepsilon}.$$

This latter inequality is equivalent to

$$n > \frac{7}{9\varepsilon} + \frac{6}{9}, \quad \text{i.e.} \quad n > \frac{7+6\varepsilon}{9\varepsilon}.$$

Now use the *archimedean property* of $\mathbb{R}$ to choose $m \in \mathbb{N}$ such that $m > \dfrac{7+6\varepsilon}{9\varepsilon}$.

Then for any $n \geq m$ we see by our choice of m that

$$|a_n - 2/3| = \left|\frac{2n+1}{3n-2} - \frac{2}{3}\right| = \left|\frac{7}{9n-6}\right| \leq \left|\frac{7}{9m-6}\right| < \varepsilon.$$

This confirms that a_n converges to $2/3$, as required.