Problem Set week 12

Problem 1: Let $\Gamma = (V, E)$ be an undirected graph. Show that there is an even number of vertices of odd degree. **Hint:** Remember that $\sum_{v \in V} \deg v = 2|E|$.

Solution for 1:
Every edge adds to the degree of two vertices. Hence, the sum of the degrees of all vertices is twice the number of edges, which is an even number.

Suppose that there are j vertices $v_1, \ldots, v_j$ are the vertices of odd degree. Write $v_{j+1}, \ldots, v_d$ for the vertices of even degree.

Thus

$$\deg v_1 = 2m_1 + 1, \ldots, \deg v_j = 2m_j + 1$$

and

$$\deg v_{j+1} = 2m_{j+1}, \ldots, \deg v_d = 2m_d.$$

Then the sum of all the degrees is

$$\begin{aligned} 2|E| &= \deg v_1 + \ldots + \deg_j + d_{v(j+1)} + \ldots + \deg v_d \\ &= 2m_1 + 1 + \ldots + 2m_j + 1 + 2m_{j+1} + \ldots + 2m_d \\ &= 2(m_1 + \ldots + m_j + m_{j+1} + \ldots + m_d) + j \end{aligned}$$

Therefore,

$$j = 2(|E| - (m_1 + \ldots + m_j + m_{j+1} + \ldots + m_d))$$

is even. That is, there is an even number of vertices of odd degree.

Problem 2:

a. Prove that in any simple graph with $|V| \geq 2$, there are at least two vertices of the same degree.
b. Does the result in a. remain valid for graphs which aren't necessarily simple?

Solution for 2:

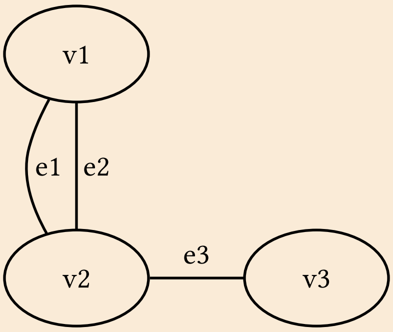
a. A simple graphs does not have loops or multiple-edges. Therefore, the degree of any vertex is a number between 0 and $n - 1$ where n is the number of vertices of the graph. There cannot be vertices with degree 0 and vertices with degree $n - 1$ in the same graph because if there is a vertex with degree 0, then no edges go to this vertex and therefore there are at most $n - 2$ edges coming out of any of the remaining vertices.

   We thus have n vertices and at most $n - 1$ possible degrees (either $\{0, 1, \ldots n - 2\}$ or $\{1, \ldots, n - 2, n - 1\}$). By the pigeonhole principle, two of the vertices must have the same degree.

b. The result is not necessarily true for non simple graphs as the example below shows: take a graph with vertices v_1, v_2, v_3, edges e_1, e_2, e_3 and incidence map

$$\iota(e_1) = \{v_1, v_2\} = \iota(e_2), \iota(e_3) = \{v_2, v_3\}.$$

   Then $\deg v_1 = 2, \deg v_2 = 3, \deg v_3 = 1$ are all different.



Problem 3: Let $\mathbb{B}$ be the graph with two vertices A and B and a unique edge $[A, B]$.

a. Let Γ be a bipartite graph with $V = V_1 \sqcup V_2$ and $E \subset \{[v_1, v_2] \mid (v_1, v_2) \in V_1 \times V_2\} \subset \mathcal{P}(V)$.

   Show that there is a morphism of graphs $\varphi : \Gamma \to \mathbb{B}$ such that $\varphi_V(V_1) = \{A\}$ and $\varphi_V(V_2) = \{B\}$.

b. Let Γ be any graph and suppose that there is a morphism $\varphi : \Gamma \to \mathbb{B}$. Prove that Γ is a bipartite graph.

Solution for 3:

a. If Γ is a bipartite graph as indicated, we define a morphism $\varphi : \Gamma \to \mathbb{B}$ as follows.

First, we define the map φ_V on vertices by the rule

$$\varphi_V(x) = \begin{cases} A \text{ if } x \in V_1 \\ B \text{ if } x \in V_2 \end{cases}$$

and we define the map φ_E on edges by the rule

$$\varphi_E(e) = [A, B] \quad \text{for } e \in E.$$

To confirm that φ is a morphism, we must observe that φ preserves the incidence relation. But by hypothesis, every edge $e \in E$ has $\iota(e) = [a, b]$ for $(a, b) \in V_1 \times V_2$. Since $\varphi_V(a) = A$ and $\varphi_V(b) = B$, $\varphi_V(a)$ and $\varphi_V(b)$ are indeed the endpoints of the edge $[A, B]$ of $\mathbb{B}$, as required.

b. Let $\varphi : \Gamma \to \mathbb{B}$ be a morphism of graphs where Γ has vertices V and edges E. Consider

$$V_1 = \varphi_V^{-1}(\{A\}) \subseteq V \text{ and } V_2 = \varphi_V^{-1}(\{B\}) \subseteq V.$$

We claim that Γ is bipartite, for the decomposition $V = V_1 \cup V_2$. The only thing to argue is that if $e \in E$ is an edge with $\iota(e) = [x, y]$ then either

$$(x, y) \in V_1 \times V_2 \quad \text{or} \quad (x, y) \in V_1 \times V_2.$$

Since φ is a morphism and since $\varphi(e) = [A, B]$, it follows that either $\varphi(x) = A$ and $\varphi(y) = B$ - in which case

$$(x, y) \in A \times B$$

or $\varphi(x) = B$ and $\varphi(y) = A$ in which case

$$(y, x) \in A \times B.$$

Thus Γ is indeed bipartite.

Problem 4: Let $\Gamma = (V, E)$ be an undirected graph.

a. Define a relation $\sim$ on V by $a \sim b$ if and only if there is a path in Γ from a to b. Prove that $\sim$ is an equivalence relation.

b. For a vertex $a \in V$, let $[a] \subseteq V$ be the equivalence class of a for the equivalence relation from part a. Let $E_a = \{[x, y] \in E \mid x, y \in [a]\}$. Prove that $([a], E_a)$ is a subgraph of Γ.

c. For a natural number n, let $\mathbb{T}_n$ be the graph with n vertices $\{v_1, v_2, ..., v_n\}$ and with edges $\{[v_1, v_1], [v_2, v_2], ..., [v_n, v_n]\}$. In other words, $\mathbb{T}_n$ has a loop at each vertex and no other edges. Suppose that there is a morphism $\varphi : \Gamma \to \mathbb{T}_n$.

   Prove that if $a, b \in V$ and $\varphi_V(a) \neq \varphi_V(b)$ then $a \nsim b$.

d. Conclude that if there is a morphism $\varphi : \Gamma \to \mathbb{T}_n$ such that the mapping φ_V on vertices is surjective, then there are at least n equivalence classes in V for the relation $\sim$.

Solution for 4:

a. We more-or-less did this problem in a later lecture; see proof of Prop 13.1.3 and 13.1.4 in the lecture notes.

b. To see that E_a is a subgraph, we just need to observe that whenever $v, w \in [a]$ if there is an edge $e \in E$ of Γ with $\iota(e) = [a, b]$, then $e \in E_a$. But this holds by definition of E_a.

c. Suppose that $a, b \in V$ and that $a \sim b$. Consider a path

$$v_0 = a, e_1, v_1, ..., v_{n-1}, e_n, v_n = b$$

   in Γ.

   Since φ is a morphism, we see that

$$\varphi_V(v_0) = \varphi_V(a), \varphi_E(e_1), \varphi_V(v_1), ..., \varphi_V(v_{n-1}), \varphi_E(e_n), \varphi_V(v_n) = \varphi_V(b)$$

   a path in $\mathbb{T}$ between $\varphi_V(a)$ and $\varphi_V(b)$. In other words $\varphi_V(a) \sim \varphi_V(b)$ in the graph $\mathbb{T}$.

   But it is straightforward to see that the equivalence classes for $\sim$ in $\mathbb{T}$ are precise the singleton sets.

   Thus $\varphi_V(a) \sim \varphi_V(b) \Rightarrow \varphi_V(a) = \varphi_V(b)$. We have proved $a \sim b \Rightarrow \varphi_V(a) = \varphi_v(b)$, which is equivalent to

$$\varphi_V(a) \neq \varphi_V(b) \Rightarrow a \nsim b.$$

d. This is an immediate consequence of c.

Problem 5: Let $n \in \mathbb{N}$ and let K_n be the complete (undirected) graph on n vertices.

a. Let Γ_0 be the subgraph of K_n obtained by removing a single vertex and removing all edges involving that vertex. prove that Γ_0 is isomorphic to K_{n-1}.

b. Let e_1, e_2 be edges in K_n, and for $i = 1, 2$ let Γ_i be the graph obtained from K_n by deleting the edge e_i. (The vertices of Γ_i are the n vertices of K_n).

   Prove that Γ_1 is isomorphic to Γ_2.

c. Let $e_1 \neq e_2$ and $f_1 \neq f_2$ be edges in K_n, let Γ_e be the graph obtained from K_n by deleting the edges e_1 and e_2 and let Γ_f be the graph obtained from K_n by deleting the edges f_1 and f_2. (Again, the vertices of Γ_e and Γ_f are the n vertices of K_n).

   Show that in general Γ_e is not isomorphic to Γ_f.

Solution for 5ab:

a. Let v_0 be the vertex of K_n which is deleted in the formation of Γ_0.

   The graph Γ_0 is simple and has $n - 1$ vertices. To see that it is isomorphic to K_n we just must argue that for every pair v, w of vertices of Γ_0, there is an edge between them.

   Viewing v, w as vertices of K_n, there is an edge e of K_n with $\iota_{K_n}(e) = [v, w]$. Since $v_0 \notin \{v, w\}$, e is an edge of Γ_0. Thus indeed Γ_0 has an edge between every pair of its vertices.

b. Let $\iota_{K_n}(e_1) = [a_1, b_1]$ and $\iota_{K_n}(e_2) = [a_2, b_2]$. Write V_n for the vertices of K_n; thus $|V_n| = n$.

   Now, $V_n \setminus \{a_1, b_1\}$ and $V_n \setminus \{a_2, b_2\}$ are sets each with cardinality $n - 2$. We choose a bijection

   $$\gamma_0 : V_n \setminus \{a_1, b_1\} \to V_n \setminus \{a_2, b_2\}$$

   and we define

   $$\gamma : V_n \to V_n \text{ by } \gamma(x) = \begin{cases} a_2 \text{ if } x = a_1 \\ b_2 \text{ if } x = b_1 \\ \gamma_0(x) \text{ otherwise} \end{cases}.$$

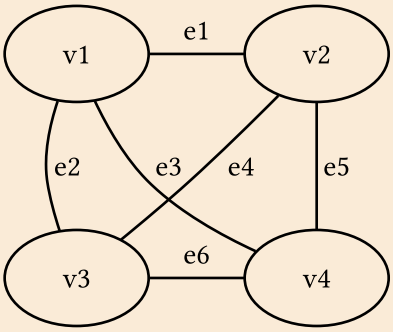
   Observe that f is a bijection by construction.

   We claim that there is a isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ for which $\varphi_V = \gamma$.

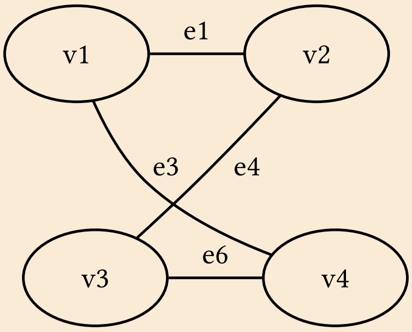
   Indeed, for an edge e of Γ_1 with $\iota(e) = [v, w]$, we define $\varphi_E(e)$ to be the unique edge f of Γ_2 with $\iota(f) = [\gamma(v), \gamma(w)]$.

   By construction, φ_V and φ_E determine a morphism. We've seen that φ_V is bijective, and it is straightforward to confirm that φ_E is bijective. Thus φ is the required isomorphism.

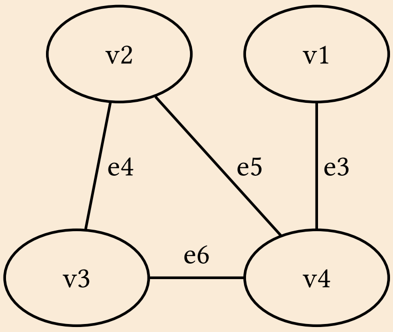
Solution for 5c:
Let $n = 4$; let's label K_4 as follows:



Let Γ_1 be obtained by removing e_2 and e_5



and let Γ_2 be obtained by removing e_1 and e_2.



Then Γ_2 has a vertex of degree 1 –namely v_1. But Γ_1 has no vertex of degree 1. So Γ_1 and Γ_2 are not isomorhpic.

Problem 6 has been deleted.