Review for Final Exam

1. problems

Problem 1: Suppose that there are only 4 kinds of donuts left in the shop: jelly, coconut, plain and chocolate, and the shop has at least a dozen of each of these 4 kinds.

- In how many ways can you choose a dozen donuts so that at least 4 are chocolate AND at most 3 are coconut?
- In how many ways can you choose 12 donuts so that at least 4 are chocolate, at most 3 are coconut AND at most one is jelly?

Solution for 1 part 1:
Compare with problem 1 on the problem set from Week 11 (see posted solution).

We need to choose 12 donuts among 4 different flavors. It is like getting a box with room for 12 donuts and three separators to be placed between the flavors. We need to choose the spots for the separators.

The total number of possible configurations is given by $\binom{12+4-1}{4-1} = \binom{15}{3}$.

If at least 4 are chocolate, we are only choosing the remaining 8. The number of ways of choosing a dozen donuts so that at least 4 are chocolate is

$$\binom{8+3}{3} = \binom{11}{3}.$$

If on top of the 4 chocolate, we want 4 at least coconut, we would be choosing only 4 of the donuts. There are

$$\binom{4+3}{3} = \binom{7}{3}$$

ways of doing this.

We want to exclude the last possibility. Hence there are

$$\binom{11}{3} - \binom{7}{3}$$

ways of picking 12 donuts with at least 4 chocolate and at most 3 coconut.

Solution for 1 part 2:

We first count in how many ways we can pick 12 donuts so that at least 4 are chocolate and at least 2 are jelly. This amounts to choosing 6 donuts, so the number of options is

$$\binom{6+3}{3} = \binom{9}{3}.$$

We now count in how many ways we can pick 12 donuts so that at least 4 are chocolate, at least 4 are coconut and at least 2 are jelly. This amounts to choosing 2 donuts, so the number of options is

$$\binom{2+3}{3} = \binom{5}{3}.$$

Let us call U the choices of 12 donuts with at least 4 chocolate. We computed in part a that $|U| = \binom{11}{3}$.

Let us call A the choices of 12 donuts with at least 4 chocolate and at least 4 coconut. We computed in part a that $|A| = \binom{7}{3}$.

Let us call B the choices of 12 donuts with at least 4 chocolate and at least 2 jelly. We computed above that $|B| = \binom{9}{3}$.

Then, $A \cap B$ are the choices of 12 donuts with at least 4 chocolate and both at least 4 coconut AND at least 2 jelly. We computed $|A \cap B| = \binom{5}{3}$.

Note now that $A \cup B$ are the choices of 12 donuts with at least 4 chocolate and either at least 4 coconut or at least 2 jelly. Then

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= \binom{7}{3} + \binom{9}{3} - \binom{5}{3} \end{aligned}$$

The choices we are after are the complement of $A \cup B$ in U. The cardinality of this set is

$$|U| - |A \cup B| = \binom{11}{3} - \binom{7}{3} - \binom{9}{3} + \binom{5}{3}.$$

Problem 2:

a. Use the formula

$$\binom{n}{2} = n \cdot (n-1)/2$$

to prove algebraically that

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}.$$

b. Give a counting argument to confirm the equation in (a).

Solution for 2:
For a, note that

$$\begin{aligned}\binom{n}{2} &= n(n-1)/2 = \frac{n^2-n}{2} \\ &= \frac{(n-k+k)^2-n}{2} = \frac{(n-k)^2+2(n-k)k+k^2-(n-k)-k}{2} \\ &= \frac{(n-k)^2-(n-k)+k^2-k}{2} \\ &= \binom{n-k}{2} + (n-k)\cdot k + \binom{k}{2}.\end{aligned}$$

For b, recall that $\binom{m}{2}$ represents the number of ways of choosing 2 elements from a set of size m.

Now, given a set U of size n, we can view the set as the disjoint union $U = A \cup B$ of a set A of size k and a set B of size $n-k$.

Any choice of a 2 element subset of U is one of the following:

- a subset of A
- a subset of B
- a pair $\{a, b\}$ where $a \in A$ and $b \in B$.

There are $\binom{n-k}{2}$ 2-element subsets of A, $\binom{k}{2}$ 2-element subsets of B, and $(n-k)\cdot k$ two-element subset of the form $\{a, b\}$ where $a \in A$ and $b \in B$.

Since these possibilities are disjoint, it follows that there are

$$\binom{n-k}{2} + (n-k)\cdot k + \binom{k}{2}$$

two-elements subsets of U. Since the number of two-elements subset of U is also given by $\binom{n}{2}$, the identity follows.

Problem 3:

a. Give the definition of for: the sequence a_n has limit L.
b. Give the definition of Cauchy sequence.
c. Prove that if a_n is not a Cauchy sequence, then a_n does not have a limit.

Solution for 3:
For a and b, see definitions in the course notes.

For c, suppose that a_n is not Cauchy. Negating the definition of Cauchy sequence, we can choose $\varepsilon_0 > 0$ such that

$$\forall M \in \mathbb{N}, \quad \exists n_0, m_0 \geq M \text{ such that } |a_n - a_m| \geq \varepsilon_0.$$

Now suppose by way of contradiction that a_n converges to the real number L. In the definition of convergence, take $\varepsilon = \varepsilon_0/2$.

There is $M \in \mathbb{N}$ such that $\forall n \in \mathbb{N}, n \geq M \Rightarrow |a_n - L| < \varepsilon_0/2$.

Since a_n is not Cauchy, there are natural numbers $n_m, m_0 \geq M$ such that

$$|a_{n_0} - a_{m_0}| \geq \varepsilon_0.$$

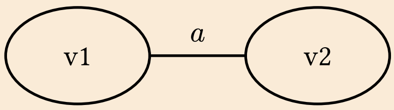
Now, using the triangle inequality we see that

$$|a_{n_0} - a_{m_0}| = |a_{n_0} - L - a_{m_0} + L| \leq |a_{n_0} - L| + |a_{m_0} - L| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0$$

which shows that $|a_{n_0} - a_{m_0}| < \varepsilon_0$. This contradiction shows that a_n does not converge.

Problem 4: For a natural number $n \in \mathbb{N}$, the complete graph K_n on n vertices has $\binom{n}{2}$ edges. If Γ is any (undirected) graph with n vertices and $\binom{n}{2}$ edges, must Γ be isomorphic to K_n?

Solution for 4:
No. Consider $n = 2$. Then the complete graph is given by



Now consider the graph Γ with

Then K_2 and Γ both have two vertices and one edge, but these graphs are not isomorphic. Indeed, Γ has a vertex of degree 0, while K_2 has no vertices of degree 0.

Problem 5: Let K_n be the complete graph with n vertices, and let Γ be the graph obtained from K_n by deleting one edge. If $n \geq 3$, prove that Γ is a connected graph.

Solution for 5:
Let V be the set of vertices and E the set of edges of K_n, and let $e \in E$. Let $a, b \in V$ be the endpoints of the edge e; i.e. $\iota(e) = [a, b]$.

Note that V is also the set of vertices of Γ, while $E \setminus \{e\}$ is the set of edges of Γ. To see that Γ is connected, we must show that there is a path in Γ between any two vertices in V.

Let $c, d \in V$ be distinct vertices. If $\{c, d\} \neq \{a, b\}$ then the edge f with $\iota(f) = [c, d]$ is an edge of Γ: i.e. $f \in E \setminus \{e\}$.

Thus c, f, d is a path in Γ connecting c to d.

So it only remains to handle the case $\{c, d\} = \{a, b\}$; i.e. we must find a path in Γ connecting a and b.

Well, since $|V| \geq 3$, we can choose a third vertex $x \in V \setminus \{a, b\}$. Then the edges f_1, f_2 with $\iota(f_1) = [a, x]$ and $\iota(f_2) = [b, x]$ are edges in Γ; i.e. $f_1, f_2 \in E \setminus \{e\}$.

Now,

$$a, f_1, c, f_2, b$$

is a path in Γ from a to b. Now $a \sim b$. This confirms that Γ is connected.

Problem 6:

a. Consider the assignment

$$f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$

given by $(a/b, c/d) \mapsto (a+b)/(c+d)$. Prove or disprove that this is a well defined function.

b. Consider the assignment

$$f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$

given by $(a/b, c/d) \mapsto (ad+bc)/bd$. Prove or disprove that this is a well defined function.

Solution for 6a:
The function $f(a/b, c/d) = (a+b)/(c+d)$ is **not well-defined**.

Indeed, note that $1/1 = 2/2$. Since $(1/1, 0/1) = (2/2, 0/1)$ in $\mathbb{Q} \times \mathbb{Q}$, if f were well-defined, it would follow that $f(1/1, 0/1) = f(2/2, 0/1)$.

But let us compare: using the given definition we see that

$$f(1/1, 0/1) = (1+1)/(1+0) = 2.$$

while

$$f(2/2, 0/1) = (2+2)/(0+1) = 4.$$

Since these values differ, f is not well-defined.

Solution for 6b:
The function

$$f(a/b, c/d) = (ad + bc)/bd$$

is well-defined. (Note that this is just the formula for addition of rational numbers, so it had better be well-defined!!)

Let's check this assertion. So, suppose that $a/b = a'/b'$ and $c/d = c'/d'$ in $\mathbb{Q}$. This means that

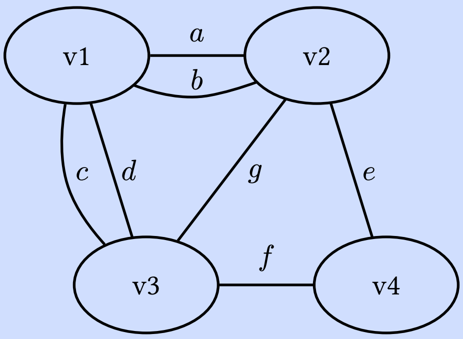
$$(\clubsuit) \quad ab' = a'b \text{ and } cd' = c'd.$$

To prove that f is well-defined, we must argue that $f(a/b, c/d) = f(a'/b', c'/d')$. Well, using $(\clubsuit)$ we see that

$$\begin{aligned} f(a/b, c/d) &= (ad + bc)/bd \\ &= b'd'(ad + bc)/bdb'd' \\ &= (b'd'ad + b'd'bc)/bdb'd' \\ &= (bd'a'd + b'dbc')/bdb'd' \\ &= (d'a' + b'c')/b'd' \\ &= f(a'/b', c'/d'). \end{aligned}$$

This confirms that the given function is well-defined.

Problem 7: Find an Eulerian path for the folowing graph, or prove that there can be no such path.

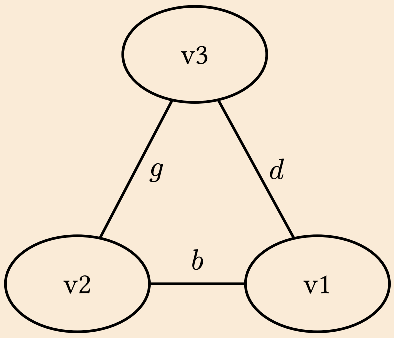


Solution for 7:
Note that each vertex has even degree, so a result in the notes implies that there is a Euler circuit for the graph. We use the algorithm suggested by the proof of that result to find one.

First, we find a circuit in the graph. For example,

$$v_1, a, v_2, e, v_4, f, v_3, c, v_1.$$

Removing the edges involved in that circuit, we are left with the subgraph



which has the "obvious" Euler circuit $v_1, b, v_2, g, v_3, d, v_1$.

Concatenating these circuits, we find an Euler circuit

$$v_1, a, v_2, e, v_4, f, v_3, c, v_1,, b, v_2, g, v_3, d, v_1$$

in the original graph.