Problem Set week 11

Problem 1: Suppose that there are four donuts available at a certain bakery: Boston cream, pumpkin spice, glazed, and chocolate.

Boxes containing 6 donuts are sold.

An industrious group of bakers has packed boxes will all possible assortments and created a stack of boxes for each possible assortment. Thus there is stack for "6 Boston creams", another stack for "4 Boston cream, 1 glazed, 1 chocolate", etc.

a. Imagine choosing a box of donuts from a random stack.

   Do you expect the events "the box contains at least 2 chocolate donuts" and "the box contains at most 1 pumpkin spice donut" are independent?

   Don't make any calculations yet, just consider the problem (or discuss with colleagues in the class!) We are going to calculate in the remainder of the problem!

b. Use the method of stars-and-bars to determine the number of stacks of donut boxes. (Think of donut style – Boston cream, glazed, etc. – as the bins.)

c. Use the method of stars-and-bars to determine the number of stacks for which each box contains at least two chocolate donuts.

d. What is the probability that a box from a randomly chosen stack contains at least two chocolate donuts?

e. What is the probability that a box from a randomly chosen stack contains at most one pumpkin spice donut?

f. What is the probability that a box from a randomly chosen stack contains at least two chocolate donuts and at most one pumpkin spice donut?

g. If a box from a randomly chosen stack has at least two chocolate donuts, what is the probability that it has at most 1 pumpkin spice donut. (This is the *conditional probability*).

h. After these calculations, answer the question: are the events "at least 2 are chocolate" and "at most 1 is pumpkin spice" independent?

Solution for 1b:
You are counting the number of ways of arranging 6 donuts in to 4 bins. Using the method of "stars and bars", that number is

$$\binom{6+4-1}{4-1} = \binom{9}{3} = \frac{9!}{3! \cdot (9-3)!} = 84.$$

Solution for 1cd:
c. To count the configurations with at least two chocolate donuts, you must the number of ways of arranging 4 donuts in 4 bins. Using the method of "stars and bars", that number is

$$\binom{4+4-1}{4-1} = \binom{7}{3} = 35.$$

d. The probability of choosing a box with at least two chocolate donuts is thus equal to the ratio:

$$\binom{7}{3} / \binom{9}{3} = \frac{35}{84} = \frac{5}{12}$$

Solution for 1e:
The probability of choosing at most one pumpkin spice donut is 1 minus the probability of choosing a box with at least 5 pumpkin spice donuts.

The number of configurations with at least 2 pumpkin spice donuts is the same as that of "at least 2 chocolate", i.e.

$$\binom{7}{3} = 35$$

and so the probability of choosing a box with at least 5 pumpkin spice donuts is

$$\binom{7}{3} / \binom{9}{3} = \frac{35}{84} = \frac{5}{12}.$$

Thus, the probability of choosing at most one is given by

$$1 - \left(\binom{7}{3} / \binom{9}{3} \right) = 1 - \frac{5}{12} = \frac{7}{12}.$$

Solution for 1f:
To count the configurations with least 2 chocolate donuts and at least 2 pumpkin spice donuts, you must count the ways of arranging 2 donuts in 4 bins. Using stars-and-bars, this is

$$\binom{2+4-1}{4-1} = \binom{5}{3} = 10.$$

Thus the probability of "at least 2 chocolates and at least 2 pumpkin spice" is

$$\binom{5}{3} / \binom{9}{3} = \frac{10}{84} = \frac{5}{42}.$$

The probability that at least two are chocolate and at most 1 is pumpkin spice is then given by the difference:

$P(\text{at least two chocolate}) - P(\text{at least two chocolate and at least two pumpkin spice})$

i.e.

$$\frac{5}{12} - \frac{5}{42} = \frac{35-10}{84} = \frac{25}{84}.$$

Solution for 1gh:
g. We can compute the conditional probability as the quotient of the probability of the intersection ("at least two choc AND at most one pumpkin spice") by the probability of "at least two choc").

This is equal to:

$$\left(\frac{25}{84}\right) / \left(\frac{5}{12}\right) = \frac{25 \cdot 12}{84 \cdot 5} = \frac{5 \cdot 12}{84} = \frac{5}{7}.$$

h. The events are not independent. Indeed, the probability of "at most 1 pumpkin spice" is $7/12$, but the conditional probability "at most 1 pumpkin spice, if two chocolates are chosen" is $5/7$. Since the probabilities differ, the events are dependent.

Problem 2: Two events in a finite probability space are said to be *independent* if $P(A|B) = P(A)$. Prove or disprove:

a. If A and B are independent, then A and $B^c = S \setminus B$ are independent.
b. If $P(A \mid B) = P(B \mid A)$, then A and B are independent.
c. If A and B are independent, then $P(A \mid B) = P(B \mid A)$.

Solution for 2a:
This is true. Suppose A and B are independent, then

$$P(A) = P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Further, note that $P(B) = 1 - P(B^c)$. Combining these two expressions yields

$$P(A \mid B)(1 - P(B^c)) = P(A \cap B) \Rightarrow P(A|B) - P(A|B)P(B^c) = P(A \cap B).$$

But, $P(A \mid B) = P(A)$ by independence and so the last expression is really

$$P(A) - P(A)P(B^c) = P(A \cap B).$$

Rearranging, we get

$$P(A) - P(A \cap B) = P(A)P(B^c).$$

Observe that $P(A) - P(A \cap B)$ is asking for the probability that we get A but not B, this is $P(A \cap B^c)$. So we have

$$P(A \cap B^c) = P(A)P(B^c).$$

Finally then we can rearrange one last time to show independence,

$$P(A) = \frac{P(A \cap B^c)}{P(B^c)} = P(A \mid B^c).$$

Solution for 2b:
This is false. Let S be the set of outcomes of tossing an ordinary 6-sided die. Let A be the event, “the outcome is at most 2”. and let B be the event “the outcome is at least five”.

Then, $A \cap B = \emptyset$. So, $P(A) = 1/3, P(B) = 1/3, P(A \cap B) = 0$. Therefore

$$P(A \mid B) = \frac{0}{1/3} = 0 \neq P(A), \quad P(B \mid A) = \frac{0}{1/3} = 0 \neq P(B)$$

So, the two events are not independent but $P(A \mid B) = P(B \mid A)$.

Solution for 2c:
This is false. Let S be the set of outcomes of tossing a 6-sided die. Let A be the event, "the outcome is at most 2". Let B be the event "the outcome is even". Then, $A \cap B = \{2\}$.

So, $P(A) = 1/3, P(B) = 1/2, P(A \cap B) = 1/6$.

Therefore

$$P(A \mid B) = (1/6)(1/2) = 1/3 = P(A); \quad \text{and} \quad P(B \mid A) = (1/6)(1/3) = 1/2 = P(B).$$

So, the two events are independent but $P(A \mid B) \neq P(B \mid A)$.

Problem 3: In the contract bridge card game, a player is dealt a 13-card hand from a 52-card deck. There are 13 kinds of cards: Ace, King, Queen, Jack, 10, 9, … 2, and 4 suits: clubs (♣), diamons (♦), hearts (♥), spades (♥). Each suit contains exactly one card of each kind.

a. What is the probability of being dealt a hand with no Jack, Queen, or King?
b. What is the probability of being dealt a hand with all 4 kings and exactly 3 queens.

Solution for 3a:
Setting aside the Jacks, Queens and Kings, the 13 cards in a hand should be chosen among the remaining 52-12=40 cards in the deck. Hence, the probability is given by

$$\binom{52-12}{13} \Big/ \binom{52}{13} = \binom{40}{13} \Big/ \binom{52}{13}.$$

Solution for 3b:
There is only one way of choosing all four kings while there are $\binom{4}{3}$ ways of choosing 3 queens and there are $\binom{52-8}{6} = \binom{44}{6}$ ways of choosing the remaining six cards of the hand.. Hence the probability is given by the expression

$$\frac{1 \cdot \binom{4}{3} \cdot \binom{44}{6}}{\binom{52}{13}}.$$

Problem 4: You hold a bag of ten coins. Nine of them are fair, but one is loaded - it shows heads with probability $9/10$. You draw out a coin at random and begin flipping it. The first five tosses are $HHHTH$.

Let's write A for the event: "the chosen coin is fair" and B for the event "the outcomes of five consecutive coin tosses are $HHHTH$."

a. What is the probability $P(A)$? (This probability is only concerned with selecting a coin from the bag; it is unrelated to the outcome of the coin tosses.)
b. What is the probability $P(B|A)$?

   (Notice for this computation that you don't have to view this as a conditional probability; this is just the probability of getting "$HHHTH$" from a fair coin!)
c. Find $P(A|B)$. What is the probability that you are flipping a fair coin?

Solution for 4:

a. As there are 10 coins in the bag nine of which are fair, $P(A)$ is equal to $\frac{9}{10}$.

b. The probability $P(B|A)$ is the chance of obtaining HHHTH from a fair coin, which is equal to $(1/2)^5$.

c. From the definition of the conditional probabilities $P(A|B)$ and $P(B|A)$ we see that

$$P(A|B) = \frac{P(B \cap A)}{P(B)} = \frac{P(B|A) \cdot P(A)}{P(B)}.$$

The probability $P(B|A)$ is the chance of obtaining HHHTH from a fair coin, which is equal to $(1/2)^5$.

Similarly, writing A^c for the complementary event ("the chosen coin is not fair"), we get $P(B|A^c) = (9/10)^4$. (Here 4 is the number of "H"s in "HHHTH").

On the other hand, B can be written as the disjoint union

$$B = (B \cap A) \cup (B \cap A^c)$$

so that

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A^c) \\ &= P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c) \\ &= (1/2)^5 \cdot (9/10) + (9/10)^4 \cdot (1/10) \end{aligned}$$

Thus $P(\text{coin is fair}|\text{flips are "HHHTH"}) = P(A|B)$ is given by

$$\frac{(1/2)^5 \cdot (9/10)}{(1/2)^5 \cdot (9/10) + (9/10)^4 \cdot (1/10)} = \frac{3125}{3854} \approx 0.8108$$

In particular, there is roughly an 80% chance that the coin is fair, given this sequence of tosses.