Math087 - PS02 due 2026-01-30

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2025-01-30

1. Blood typing

   Human blood is generally classified in the "ABO" system, with four blood types: `A`, `B`, `O`, and `AB`. These types are determined by pairs of alleles at the `ABO` gene locus, with blood type `A` corresponding to genotypes `AA` and `AO`, blood type `B` corresponding to genotypes `BB` and `BO`, blood type `O` corresponding to genotype `OO`, and blood type `AB` corresponding to genotype `AB`.

   Let `p` be the proportion of allele `A` in the population, let `q` be the proportion of allele `B`, and let `r` be the proportion of allele `O`. Note that `p + q + r = 1`.

   The Hardy-Weinberg principle implies that:

   > (♣) The quantities p, q, and r remain constant from generation to generation, as do the frequencies of occurrence of the different genotypes `AA`, `AO`, … .

   a. Assuming the validity of (♣), what is the probability that an individual has genotype `AA`? `BB`? `OO`?

      What is the probability that an individual is heterozygous (has two different alleles)?

      Express your responses using the quantities p, q and r.

   b. Still assuming the validity of (♣), what is the maximum possible proportion of heterozygotes in a population, and what values of `p`, `q`, and `r` achieve this maximum?

      Perform this computation in two different ways:

      - directly maximize a function of only two variables
      - use the method of Lagrange multipliers.

   c. If you could somehow increase the total "pool" from `p + q + r = 1` to `p + q + r = 1.01`, how would the maximum proportion of heterozygotes (in part (b)) change? How does the Lagrange multiplier λ relate to this change?

2. Newton's Method

   In class, we investigated use of Newton's method to find solutions to some equations. In this problem, you should use the framework introduced in class to carry out similar investigations. Namely, you should use `python` code implementing Newton's Method and use this code to investigate the following questions.

   You can check your solutions using `scipy.optimize.newton` as well.

   a. Let $f(x) = x^3 - 6x + 2$. The equation $0 = f(x)$ has three solutions in $\mathbb{R}$.

      Find all three roots by trying different starting guesses. What do you notice about which initial guesses lead to which roots?

b. Let $g(x) = x - \cos(x)$. Find a solution to $0 = g(x)$.

   Remember that you'll need to compute $g'(x)$ to carry out Newton's method.

c. Consider $h(x) = x^{1/3}$. What happens when you carry out Newton's method for $h(x)$ with an initial seed of $x_0 = 0.1$.

   Explain what goes wrong and why Newton's method fails in this case. **Hint**: look at what happens to the derivative near the root."

3. A linear program

   Consider the optimization problem: find the max of $f(x, y) = 3x + y$ and of $g(x, y) = x - 2y$ subject to the following constraints:

$$
\begin{aligned}
y &\leq 5 \\
y &\geq 0 \\
-2x + y &\leq 1 \\
2x + y &\leq 17 \\
3x + y &\geq 6 \\
-3x + y &\leq -18
\end{aligned}
$$

   a. Draw the feasible region. Label the boundary curves and corner points.

   b. Find the maximum value of f subject to the constraints and the point where it occurs.

   c. Verify your answer using the 'linprog' solver found in the 'scipy' library.