Real Analysis II

1. Week 1 [2026-01-14]

1.1. Limits

See §13.1 in [Fitzpatrick].

Let $A \subseteq \mathbb{R}^n$ for some $n \in \mathbb{N}$.

Definition 1.1.1: A point $\vec{x_0} \in A$ is a limit point of A if there is a sequence $\{\vec{x}_m\}$ in A such that $\lim_{n\to\infty} \vec{x}_m = \vec{x_0}$.

Definition 1.1.2: Let $f : A \to \mathbb{R}$ be a function and $\vec{x}_0 \in \mathbb{R}^n$ a limit point of A and $\ell \in \mathbb{R}$. Then $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = \ell$ provided that for every sequence $\{\vec{x}_m\}$ in A

$$\lim_{n\to\infty} \vec{x}_m = \vec{x}_0 \quad \Rightarrow \quad \lim_{n\to\infty} f(\vec{x}_m) = \ell.$$

If $\vec{x}_0 \in A$, f is continuous at $\vec{x}_0 \in A$ provided that $\lim\limits_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$.

See [Fitzpatrick] Theorem 13.3 for result showing that limits are compatible with sums, products, and quotients of functions.

Alternatively, one can characterize continuity limits without using sequences. For example:

Theorem 1.1.3: Let $f : A \to \mathbb{R}$, $\vec{x}_0$ a limit point of A and $\ell \in \mathbb{R}$. Then $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = \ell$ if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in A \setminus \{\vec{x}_0\} \text{ and } |\vec{x} - \vec{x}_0| < \delta \Rightarrow |f(\vec{x}) - \ell| < \varepsilon.$$

See [Fitzpatrick] Theorem 13.7 for a proof.

1.2. Derivatives for functions $\mathbb{R} \to \mathbb{R}$.

See §4 in [Fitzpatrick].

Definition 1.2.1: A neighborhood of a point $c \in \mathbb{R}$ is an open interval (a, b) with $c \in (a, b)$.

Let us write I for a neighborhooda of the point $x_0 \in \mathbb{R}$.

Definition 1.2.2: If $f : I \to \mathbb{R}$ is a function, then f is differentiable at c if

$$f'(c) = \lim_{t\to 0} \frac{f(c) - f(c+t)}{t}$$

exists, in which case $f'(c)$ is said to be the derivative of f at c.

Proposition 1.2.3: Let I be neighborhood of x_0 and $f : I \to \mathbb{R}$ a function. If f is differentiable at x_0, then f is continuous at x_0.

For the proof, see [Fitzpatrick] Prop. 4.5

Remark 1.2.4: Let $I \subseteq \mathbb{R}$ be an interval, and consider a strictly monotonic, continuous function $f : I \to \mathbb{R}$. Then

- f is one-to-one.
- The Intermediate Value Theorem ([Fitzpatrick] Theorem 3.11) implies that $J = f(I)$ is an interval.
- $f^{-1} : J \to \mathbb{R}$ is continuous.

For the proofs of these assertions, see [Fitzpatrick] §3.6, especially Theorem 3.29.

Theorem 1.2.5: Let $f : I \to \mathbb{R}$ be strictly monotonic and continuous.

If f is differentiable at x_0 then $f^{-1} : J \to \mathbb{R}$ is differentiable at $y_0 = f(x_0$ and

$$(f^{-1})'(y_0) = 1/f'(x_0).$$

See [Fitzpatrick] Theorem 4.11 for a proof.

Theorem 1.2.6 (The Chain Rule): Suppose that $f : I \to \mathbb{R}$ is differentiable at x_0. Let J be an open interval for which $f(I) \subseteq J$ and suppose that $g : J \to \mathbb{R}$ is differentiable at $f(x_0)$.

Then $g \circ f : I \to \mathbb{R}$ is differentiable at x_0, and

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

See [Fitzpatrick] Theorem 4.14 for a proof.

Proposition 1.2.7: Let $f : I \to \mathbb{R}$ be differentiable at x_0. If x_0 is either a maximizer or a minimizer of the function f, then $f'(x_0) = 0$.

See [Fitzpatrick] Lemma 4.16 for a proof.

Theorem 1.2.8 (The Mean Value Theorem): Let $[a, b]$ be a closed bounded interval in $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function such that the restriction $f : (a, b) \to \mathbb{R}$ is differentiable. Then $\exists x_0 \in (a, b)$ such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

See [Fitzpatrick] Theorems 4.17, 4.18.

1.3. Partial derivatives for functions $\mathbb{R}^n \to \mathbb{R}$.

Definition 1.3.1: Let $\vec{x} \in \mathbb{R}^n$ and $r > 0$. The open ball about $\vec{x}$ of radius r is by definition

$$B_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^n \mid |\vec{x} - \vec{y}| < r\}.$$

A subset $\mathcal{O} \subseteq \mathbb{R}^n$ is said to be an open subset provided that $\forall \vec{x} \in \mathcal{O}, \exists r > 0$ such that $B_r(\vec{x}) \subseteq \mathcal{O}$.

For these definitions, see [Fitzpatrick] §10.3.

We write $\vec{e}_i$ for the standard basis vectors of $\mathbb{R}^n$.

For example, if $n = 3$, then

$$\vec{e}_1 = \vec{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \vec{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

If $n = 5$, then for example

$$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Definition 1.3.2: Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let $f : \mathcal{O} \to \mathbb{R}$. For a natural number $1 \leq i \leq n$ and for $\vec{x} \in \mathcal{O}$, we say that f has the ith partial derivative provided that the limit

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \lim_{t \to 0} \frac{f(\vec{x} + t\vec{e}_i) - f(\vec{x})}{t}$$

exists. If the limit exists, we call $\frac{\partial f}{\partial x_i}(\vec{x})$ the i-th partial derivative of f at $\vec{x}$.

To simplify the notation, we sometimes write

$$f_{x_i}(\vec{x}) \text{ rather than } \frac{\partial f}{\partial x_i}(\vec{x}).$$

If f has ith partial derivative for each $\vec{x} \in \mathcal{O}$, the partial derivative determines a function

$$f_{x_i} = \frac{\partial f}{\partial x_i} : \mathcal{O} \to \mathbb{R}$$

.

See [Fitzpatrick] §13.2.

We sometimes modify this notation in the following manner. For example, if $n = 3$, we often write vectors in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and then we write e.g. $\frac{\partial f}{\partial y}$ or just f_y rather than $\frac{\partial f}{\partial x_2} = f_{x_2}$.