

# week02-01-multivariable-optimization

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## 2 Week 2: Multi-variable Optimization

### 2.1 Optimization of functions of several variables

Consider a function  $f(x, y)$  of two variables. You learned in Calculus 3 (*vector calculus*) how to search for the points  $(x, y)$  at which  $f$  assumes its maximum and minimum value. Let's briefly recapitulate this story.

Recall that for a function of a single variable, critical points are those points for which the tangent line is horizontal. In the single variable case, the criteria depends instead on the *tangent plane*.

Recall that the surface defined by  $z = f(x, y)$  can be *parameterized* by  $\mathbf{r}(x, y) = (x, y, f(x, y))$ . So a **normal vector** to this surface at a point  $P = (x_0, y_0, f(x_0, y_0))$  on the surface is given by the *cross product*

$$\left( \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right)_P.$$

Alternatively, consider  $F(x, y, z) = z - f(x, y)$  so that the surface is defined by  $F = 0$ . Then the gradient  $\nabla F$  defines a normal vector at each point  $P$ , where

$$(\nabla F)_P = \left( \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j} + \frac{\partial F}{\partial z} \mathbf{k} \right)_P.$$

Both points of view can be useful, but here they lead to the same formula. At the point  $P = (x_0, y_0, f(x_0, y_0))$  one has a normal to the surface defined by  $z = f(x, y)$  given by

$$\mathbf{n}|_P = \left( \mathbf{i} + \frac{\partial f}{\partial x} \mathbf{k} \right)_P \times \left( \mathbf{j} + \frac{\partial f}{\partial y} \mathbf{k} \right)_P = \left( \mathbf{k} - \frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} \right)_P$$

Now, the **tangent plane** at  $P$  to the surface  $z = f(x, y)$  is just the plane orthogonal to this normal vector  $\mathbf{n}_P$ . Thus, the tangent plane at  $P$  is horizontal – parallel to the  $x, y$ -plan – just in case this normal vector points in the  $\mathbf{k}$  direction – i.e. provided that

$$(\clubsuit) \quad \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = 0 \quad \text{and} \quad \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} = 0$$

Mimicking the one variable case, we say that the points  $(x_0, y_0)$  for which the tangent plane to the surface  $P = (x_0, y_0, f(x_0, y_0))$  is horizontal are the *critical points*.

So we find the critical points by simultaneously solving the equations ( $\clubsuit$ ).

There is a *second derivative test* which gives information about the “max/min status” of these critical points.

To use this test, consider the matrix of second partial derivatives

$$M(x_0, y_0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \Big|_{(x_0, y_0)}.$$

For a reasonable class of functions, the “mixed partials”  $\frac{\partial^2 f}{\partial x \partial y}$  and  $\frac{\partial^2 f}{\partial y \partial x}$  coincide.

Remember that the determinant of a  $2 \times 2$  matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is  $ad - bc$ .

So, the *determinant* of  $M$  is the expression

$$D = D(x_0, y_0) = \left( \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^2 \right) \Big|_{(x_0, y_0)}$$

**Theorem:** (Second derivative test) Suppose that  $(x_0, y_0)$  is a critical point.

- If  $D > 0$  and  $\frac{\partial^2 f}{\partial x^2} \Big|_{(x_0, y_0)} < 0$ , then  $f(x, y)$  has a relative maximum at  $(x_0, y_0)$ .
- If  $D > 0$  and  $\frac{\partial^2 f}{\partial x^2} \Big|_{(x_0, y_0)} > 0$ , then  $f(x, y)$  has a relative minimum at  $(x_0, y_0)$ .
- If  $D < 0$ , then  $f(x, y)$  has a saddle point at  $(x_0, y_0)$ .
- If  $D = 0$ , the second derivative test is inconclusive.

Let’s examine some *graphs* in order to see examples of critical opints & relative max/mins.

We are going to use the python library `matplotlib` to draw graphs.

```
[1]: import matplotlib.pyplot as plt
import numpy as np

# https://matplotlib.org/mpl_toolkits/mplot3d/tutorial.html
# https://matplotlib.org/3.3.1/gallery/mplot3d/surface3d.html

def draw_graph(f,x,y,x0,y0,elev_azim=[]):
    X,Y = np.meshgrid(x,y)

    fig = plt.figure(figsize=(20,20))
    for idx,(e,a) in enumerate(elev_azim,start=1):

        ax = fig.add_subplot(1,len(elev_azim),idx,projection='3d',elev=e,azim=a)
```

```

ax.plot_wireframe(X,Y,f(X,Y))

ax.plot(x,y0*np.ones(y.shape), zs= f(x,y0), color="red", linewidth=3)
ax.plot(x0*np.ones(x.shape),y, zs= f(x0,y), color="red", linewidth=3)

return fig

```

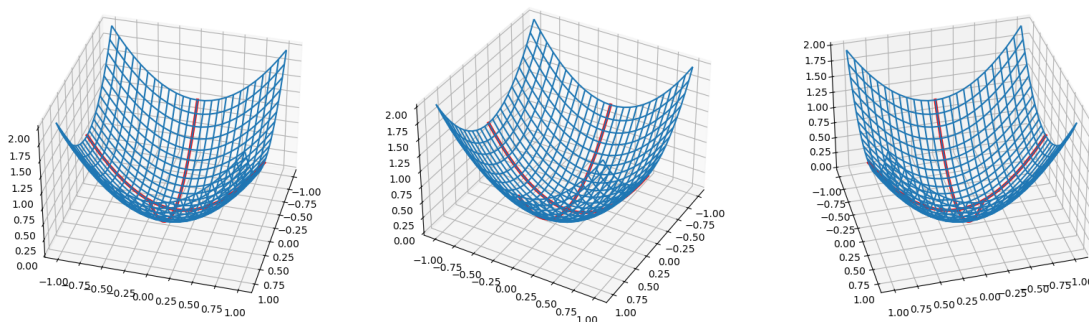
Now we define some functions  $f(x)$  and  $g(x)$  and use the function= `draw_graph` that we just defined to render an image of their graphs.

```

[2]: def f(x,y):
      return x**2 + y**2

af=draw_graph(f,
              x=np.linspace(-1,1,25),
              y=np.linspace(-1,1,25),
              x0=0,
              y0=0,
              elev_azim=[(35,15),(35,30),(35,75)])

```



Note that  $(0,0)$  is a critical point for  $f(x) = x^2 + y^2$ . Moreover,

$$D(x,y) = \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4.$$

Since  $D(0,0) > 0$  and  $\frac{\partial^2 f}{\partial x^2} = 2 > 0$ , the second derivative test shows that the function  $f$  has a **relative min** at  $(0,0)$  (& this is confirmed by the viewing the image).

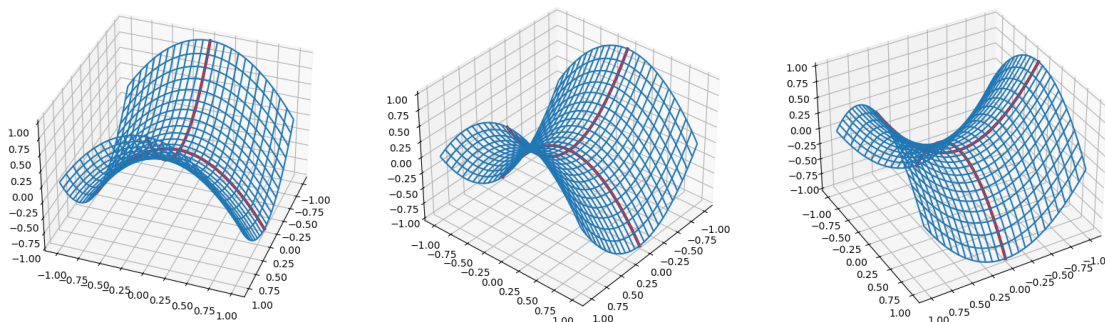
```

[3]: def g(x,y):
      return x**2 - y**2

ag=draw_graph(g,
              x=np.linspace(-1,1,25),
              y=np.linspace(-1,1,25),
              x0=0,

```

```
y0=0,
elev_azim=[(35,20),(35,40),(35,60)]
```



Again,  $(0, 0)$  is the only critical point for  $g(x, y) = x^2 - y^2$ . Moreover,

$$D(x, y) = \det \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = -4.$$

Since  $D(0, 0) < 0$ , the second derivative test shows that the function  $g$  has a **saddle point** at  $(0, 0)$  (& this is confirmed by examining the graph).

## 2.2 Example: Television manufacturer

A company that makes TV sets sells two models: a 19" set and a 21" set.

Their annual production costs are \\$ 195 per TV for the 19" model and \\$ 225 per TV for the 21" model, plus \\$ 400,000 per year in fixed costs.

They expect to sell their production to a single wholesaler who will pay a base price of \\$ 339 per 19" TV and \\$399 per 21" TV. The wholesaler receives a volume discount calculated as 1¢ per 19" TV + 0.3¢ per 21" TV for the 19" models and 1¢ per 21" TV + 0.4¢ per 19" TV for the 21" models.

How many 19" and 21" TVs should be produced to maximize the profits?

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Let's go through our modeling procedure. Let's set up the problem and ask the right questions. What are our variables? -  $s$  = # of 19" TVs produced -  $t$  = # of 21" TVs produced -  $p$  = selling price of each 19" TV -  $q$  = selling price of each 21" TV -  $C$  = cost of production -  $R$  = total revenue of sales -  $P$  = total profit

What do we *know* to start with??

- $p(s, t) = 339 - 0.01s - 0.003t$  dollars
- $q(s, t) = 399 - 0.004s - 0.01t$  dollars
- $R(s, t) = ps + qt = 339s - 0.01s^2 - 0.003st + 399t - 0.004st - 0.01t^2$   
 $= 339s + 399t - 0.01s^2 - 0.01t^2 - 0.007st$  dollars

- $C(s, t) = 400,000 + 195s + 225t$  dollars
- $P(s, t) = R(s, t) - C(s, t)$   
 $= -400,000 + 144s + 174t - 0.01s^2 - 0.01t^2 - 0.007st$  dollars

Of course, our goal is to maximize profit – i.e. to find  $(s_0, t_0)$  for which  $P(s_0, t_0)$  is at a maximum.

According to the discussion above, we should compute the partial derivatives of  $P$  and simultaneously solve the equations

$$0 = \frac{\partial P}{\partial s} = \frac{\partial P}{\partial t}$$

So we need to solve the equations:

$$\frac{\partial P}{\partial s} = 144 - 0.02s - 0.007t = 0$$

$$\frac{\partial P}{\partial t} = 174 - 0.02t - 0.007s = 0$$

This amounts to solving the matrix equation

$$\begin{bmatrix} 0.02 & 0.007 \\ 0.007 & 0.02 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 144 \\ 174 \end{bmatrix}$$

which we can solve using *row reduction* on the corresponding augmented matrix:

$$\left[ \begin{array}{cc|c} 0.02 & 0.007 & 144 \\ 0.007 & 0.02 & 174 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & .35 & 7200 \\ 1 & 2.857 & 24857.14 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & .35 & 7200 \\ 0 & 2.507 & 17657.14 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & .35 & 7200 \\ 0 & 1 & 7043.135 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & 4735.04273504 \\ 0 & 1 & 7042.73504274 \end{array} \right]$$

We find that the function  $P$  has exactly one critical point which occurs at  $(s_0, t_0) = (4735, 7043)$ .

Alternatively, we can use **numpy** to solve the matrix equation, as follows:

```
[4]: A= np.array([[.02,.007],[.007,.02]])
      b=np.array([144,174])
      np.linalg.solve(A,b)
```

```
[4]: array([4735.04273504, 7042.73504274])
```

Now, let's apply the second derivative test to investigate this critical point.

The matrix of second derivatives is

$$\begin{bmatrix} \frac{\partial^2 P}{\partial s^2} & \frac{\partial^2 P}{\partial s \partial t} \\ \frac{\partial^2 P}{\partial s \partial t} & \frac{\partial^2 P}{\partial t^2} \end{bmatrix} = \begin{bmatrix} -0.02 & -0.007 \\ -0.007 & -0.02 \end{bmatrix}$$

which has determinant  $(0.02)^2 - (.007)^2 > 0$ .

Since  $\frac{\partial^2 P}{\partial s^2} = -0.02 < 0$ , the second derivative test shows that  $P$  has a local maximum at  $(s_0, t_0)$ , and we conclude that profit is maximized there.

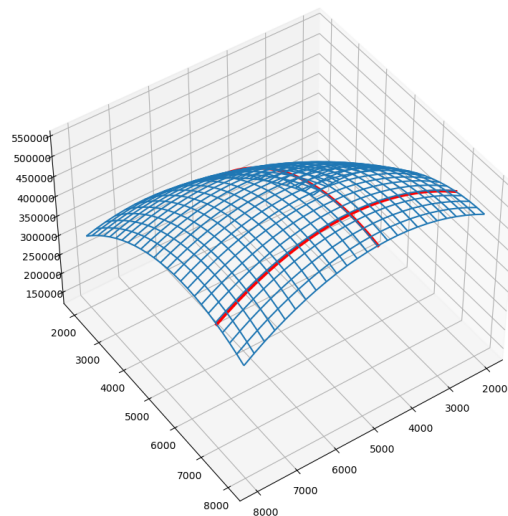
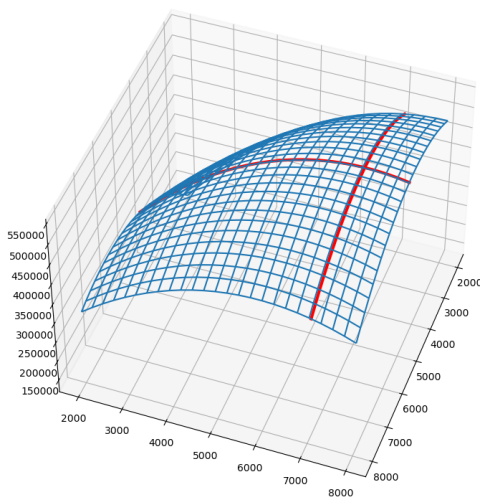
(Technically, we should check for minima “on the boundary” but in this case that would be the point  $(0, 0)$  which clearly doesn’t maximize  $P$ ).

Let’s produce a (or a few...) graph(s) to confirm our work:

```
[5]: s = np.linspace(2000,8000,25)
      t = np.linspace(2000,8000,25)

      def p(s,t):
          return -400000 + 144*s + 174*t - 0.01*s**2 - 0.01*t**2 - .007*s*t

      a=draw_graph(p,
                   x=s,
                   y=t,
                   x0=4735,
                   y0=7043,
                   elev_azim=[(45,20),(45,55)])
```



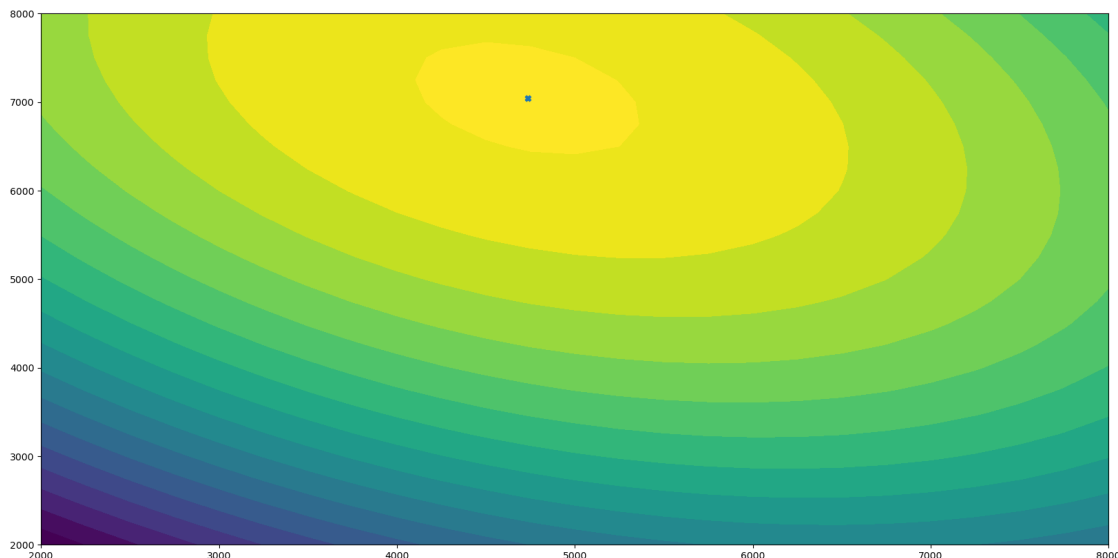
```
[6]: # contour plot

      S,T = np.meshgrid(s,t)

      figc = plt.figure(figsize=(20,10))
      axc = figc.add_subplot()
```

```
axc.contourf(S,T,p(S,T),levels=20
             , extend='both')
axc.scatter(4735,7043,marker="X")
```

[6]: <matplotlib.collections.PathCollection at 0x7f6e3c60dbd0>



## 2.3 Sensitivity Analysis (the television example, continued)

Just as in the single-variable case, we should be able to perform *Sensitivity Analysis* for our optimization problems.

Let's start by picking a parameter we want to change.

Consider the *Price Elasticity* parameter,  $a$  for 19" TVs. Thus,  $a$  is the amount the selling price of the 19" TVs decreases per 19" TV sold (due to the volume discount).

In the original description of the problem, this is described as:

volume discount calculated as 1¢ per 19" TV

Thus, we considered above the case  $a = 0.01$ .

Rewriting our parameters using  $a$ , we see that the selling price of 19" TVs is given by  $p(s, t) = 339 - as - 0.003t$  dollars.

In turn, rewriting the profit equation with  $a$ , we obtain:

$$P(s, t) = 144s + 174t - as^2 - 0.01t^2 - 0.007st - 400000$$

We now look for optimal values  $s = s(a)$  and  $t = t(a)$  depending on  $a$ .

We need to solve the system:

$$\begin{cases} 0 &= \frac{\partial P}{\partial s} = 144 - 2as - 0.007t \\ 0 &= \frac{\partial P}{\partial t} = 174 - .02t - 0.007s \end{cases}$$

Solving this system, find that  $s = s(a) = \frac{144 - 0.007t}{2a}$  so that

$$174 - 0.02t - 0.007 \cdot \frac{144 - 0.007t}{2a} = 0$$

We now find that

$$t = 8,700 - \frac{581,700}{40,000a - 49}$$

and

$$s = \frac{1,662,000}{40,000a - 49}$$

Now we check the sensitivity:

$$S(s, a) = \frac{ds}{da} \cdot \frac{a}{s} \quad \text{and} \quad S(t, a) = \frac{dt}{da} \cdot \frac{a}{t}$$

Thus

$$S(s, a) = \frac{66,480,000,000}{(40,000a - 49)^2} \cdot \frac{40,000a^2 - 49a}{1,662,000}$$

and

$$S(t, a) = \frac{23,268,000,000}{(40,000a - 49)^2} \cdot \frac{40,000a^2 - 49a}{8,700 \cdot (40,000a - 49) - 581,700}$$

The sensitivity near our guess of  $a = 0.01$  is thus

$$S(s, 0.01) \approx -1.1 \quad \text{and} \quad S(t, 0.01) \approx 0.2$$

**Interpretation:** If the price elasticity increases by 10% (i.e. the warehouse receives a bigger bulk discount) the optimal value of  $s$  decreases by 11% and the optimal value of  $t$  increases by 2.7%