ProblemSet 1 – Optimization

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1. An optimization question in auto manufacturing

An automobile manufacturer makes a profit of \$1,500 per unit on the sale of a certain car model. It is estimated that for every \$100 of rebate, the number of units of this model sold in a given month will increase by 15%.

a. What amount of rebate will maximize the manufacturers profit for the month? Model the question as a single-variable optimization problem.

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2. Computing yields with multi-variate optimization

A chemist is synthesizing a compound. In the last step, she must dissolve her reagents in a solution with a particular pH level H, for $1.2 \leq H \leq 2.7$, and heated to a temperature T (in degrees Celsius), for $66 \leq T \leq 98$. Her goal is to maximize her percent yield as a percentage of the initial mass of the reagents.

The equation determining the percentage $F(H, T)$ is

$$F(H,T) = -0.038 \cdot T^2 - 0.223 \cdot T \cdot H - 10.982 \cdot H^2 + 7.112 \cdot T + 60.912 \cdot H - 3$$

1. Find the optimal temperature and pH level in the allowed range.

To find optimal values let us consider partial derivatives:

$$\partial F \quad 0.223T - 21.964H + 60.912 = 0$$

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3. Blood typing

Human blood is generally classified in the "ABO" system, with four blood types: `A`, `B`, `O`, and `AB`. These four types reflect six gene pairs (genotypes), with blood type `A` corresponding to gene pairs `AA` and `AO`, blood type `B` corresponding to gene pairs `BB` and `BO`, blood type `O` corresponding to gene pair `OO`, and blood type `AB` corresponding to gene pair `AB`. Let p be the proportion of gene `A` in the population, let q be the proportion of gene `B` in the population, and let r be the proportion of gene `O` in the population. Observe that $p + q + r = 1$.

a. The Hardy-Weinberg principle implies that:

Assuming the validity of $(\clubsuit)$, what is the probability that an individual has genotype `AA`? `BB`? `OO`? What is the probability

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4. Newton's method and root finding

a. microprocessors

   One of the uses of Newton's method is in implementing division on microprocessors, where only addition and multiplication are available as primitive operations. To compute $x = a/b$, first the root of $f(x) = 1/x - b$ is found using Newton's method, then the fraction is computed with one last multiplication by a.

   Find the Newton iteration needed to solve $f(x) = 0$ and explain why it is well-suited to this purpose. (**Note**: We are trying to approximate division, so we shouldn't actually use division functions implemented in `python`...)

Given an initial guess x_0 for the reciprocal of b, Newton's method refines the guess by the rule

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

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