

## PS 8 – Binomial & Poisson distribution

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1. For a whole number  $N \geq 1$ , recall the following identity:

$$(X + Y)^N = \sum_{j=0}^N \binom{N}{j} X^j Y^{N-j}$$

where  $X, Y$  are *variables*.

- a. Explain why the identity  $(X + Y)^N = (Y + X)^N$  implies that

$$\binom{N}{j} = \binom{N}{N-j}$$

for each  $0 \leq j \leq N$ .

- b. Explain why the identity

$$\begin{aligned}(X + Y)^N &= (X + Y)(X + Y)^{N-1} \\ &= X(X + Y)^{N-1} + Y(X + Y)^{N-1}\end{aligned}$$

implies that

$$\binom{N}{j} = \binom{N-1}{j} + \binom{N-1}{j-1}$$

for each  $0 \leq j \leq N-1$ .

**Hint:** In each case, observe what the indicated identity says about the coefficient of  $X^j Y^{N-j}$  in the given expression(s).

2. Using the identities in 1., one can argue inductively that

$$\binom{N}{j} = \frac{N!}{j! \cdot (N-j)!}$$

(in fact, this formula has been used in the notebook!)

Use this identity to compute the following limits:

$$\lim_{N \rightarrow \infty} \frac{1}{N^3} \binom{2N}{3} \quad \text{and} \quad \lim_{N \rightarrow \infty} \frac{1}{e^N} \binom{N}{N-4}.$$

3. Suppose you have estimated the probability that your pet songbird will sing during a one-hour time period is 0.35.

For the following, you should use the *binomial distribution*.

- a. Indicate an expression for the probability that the bird will sing during a ten-minute time period.

- b. Indicate an expression for the probability that the bird will not sing during a twenty-minute time period.
4. Suppose that the probability that an automobile accident occurs during a 24 hour period in a certain stretch of freeway is given by the number  $p$ ,  $0 < p < 1$ .

Assume that the random variable  $X$  describing the *number of automobile accidents* is given by the Poisson distribution.

Thus the probability that there are  $k$  accidents is given by

$$P(X = k) = e^{-p} \cdot \frac{p^k}{k!}$$

Give an expression for the probability that there no more than 3 accidents in a 24-hour period.

5. *Jane's Fish Tank Emporium (JFTE)* yet again

Recall that in a [notebook](#) last week, we discussed the operation of *JFTE* by considering the question: what is the optimal ordering strategy for fish tanks: ordering *on-demand*?, or putting in place a *standing order*?

Our simulation used a python class *JFTE*; the *constructor* of the class *JFTE* (i.e. its member function `__init__`) creates the `customers` instance variable. For this, the version in the notebook invokes the function

```
def customer(prob=1./7):
    return rng.choice([1,0],p=[prob,1-prob])
```

In [this-week's-notebook], we created a function `arrival` which takes two arguments: `p` and `num_max`; this function returns the integer `k` with probability determined by the Poisson distribution and base probability `p`, where

`0 <= k <= num_max`.

Edit the *JFTE* class so that its constructor uses the Poisson distribution to simulate arrival.

Recall that the constructor is the function `__init__`; it has the form:

```
def __init__(self,N,prob=1./7):
    # ...
    self.reset()
```

You need to replace “# ...” with code to create the instance variable `self.customers` and assign it to be a list of integers containing `N` values returned by the function `arrival(prob,5)`.

You may now apply these strategy functions to an instance of the *JFTE* class constructed using the Poisson customer-arrival function.

Run the simulation 10 times with both strategy functions, as was done in the notebook. Discuss similarities/differences between the results obtained in the notebook.