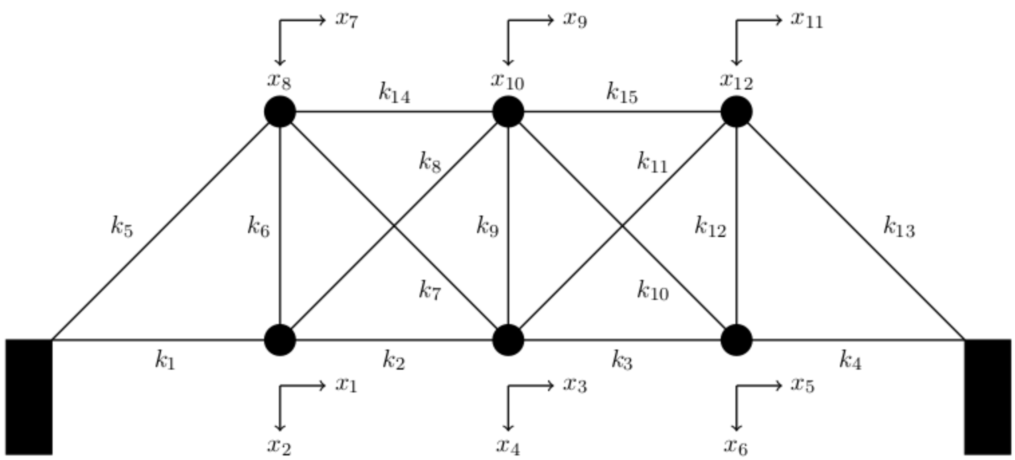
weekk14–springs-bridges

April 22, 2024

1 Least squares and bridges

Suppose we are building a bridge, which we'll view as a spring-system as before:



For simplicity we are going to assume that all angles in the diagram are $\pi/2$ or $\pi/4$.

Using the same material for each "beam" of the bridge, the spring constants are inversely proportional to their length.

There are 15 springs/beams in total, and we take them as follows:

$$k_i = \begin{cases} 9.7500 \cdot 10^6 \text{ N/m} & \text{for } i = 1,2,3,4,6,9,12,14,15 \\ 6.8943 \cdot 10^6 \text{ N/m} & \text{for } i = 5,7,8,10,11,13 \end{cases}$$

In order to test the bridge for structural integrity, we load the three lower masses (where traffic would be) individually with forces of $2 \cdot 10^6$ N, and measure the resulting displacements.

Remark The average car in the US weighs about 4,000 lbs ($\approx$ 1,814 kg), which corresponds to $\approx 1.778 \cdot 10^4$ N. The average semi-truck weighs about 80,000 lbs ($\approx$ 36,287 kg), which corresponds to $\approx 3.556 \cdot 10^5$ N.

[196]:
```
# The matrix B with Bx = e is given as follows:
def sbv(i,n):
    return np.array([1 if i == j else 0 for j in range(n)])

def sbvarray(x,l):
    return sbv(l.index(x),len(l))
```

```python
a = list(range(1,13))

k = np.sqrt(2)/2

B_bridge = np.array([ sbvarray(1,a),                                  # e1 = x1
               sbvarray(3,a) - sbvarray(1,a),                         # e2 = x3 - x1
               sbvarray(5,a) - sbvarray(3,a),                         # e3 = x5 - x3
               sbvarray(5,a),                                         # e4 = x5
               k*(sbvarray(7,a) - sbvarray(8,a)),                     # e5 = k*x7 - k*x8
               sbvarray(2,a) - sbvarray(8,a),                         # e6 = x2 - x8
               k*(sbvarray(3,a) + sbvarray(4,a)
                  - sbvarray(7,a) - sbvarray(8,a)),                   # e7 = k*x3 + k*x4 - k*x7 - k*x8
               k*(-sbvarray(1,a) + sbvarray(2,a)
                  + sbvarray(9,a) - sbvarray(10,a)),                  # e8 = -k*x1 + k*x2 + k*x9 - k*x10
               sbvarray(4,a) - sbvarray(10,a),                        # e9 = x4 - x10
               k*(sbvarray(5,a) + sbvarray(6,a)
                  - sbvarray(9,a) - sbvarray(10,a)),                  # e10 = k*x5 + k*x6 - k*x9 - k*x10
               k*(-sbvarray(3,a) + sbvarray(4,a)
                  + sbvarray(11,a) - sbvarray(12,a)),                 # e11 = -k*e3 + k*e4 + k*e11 - k*e12
               sbvarray(6,a) - sbvarray(12,a),                        # e12 = x6 - x12
                  k*(-sbvarray(11,a) - sbvarray(12,a)),               # e13 = -k*x11 + -k*x12
               -sbvarray(7,a) + sbvarray(9,a),                        # e14 = -x7 + x9
               -sbvarray(9,a) + sbvarray(11,a)                        # e15 = -x9 + x11
             ])
B_bridge
```

[196]:
```
array([[ 1.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ],
       [-1.        ,  0.        ,  1.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ],
       [ 0.        ,  0.        , -1.        ,  0.        ,  1.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
```

```
 0.        ,  0.        ],
[ 0.        ,  0.        ,  0.        ,  0.        ,  1.        ,
  0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
  0.        ,  0.        ],
[ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
  0.        ,  0.70710678, -0.70710678,  0.        ,  0.        ,
  0.        ,  0.        ],
[ 0.        ,  1.        ,  0.        ,  0.        ,  0.        ,
  0.        ,  0.        , -1.        ,  0.        ,  0.        ,
  0.        ,  0.        ],
[ 0.        ,  0.        ,  0.70710678,  0.70710678,  0.        ,
  0.        , -0.70710678, -0.70710678,  0.        ,  0.        ,
  0.        ,  0.        ],
[-0.70710678,  0.70710678,  0.        ,  0.        ,  0.        ,
  0.        ,  0.        ,  0.        ,  0.70710678, -0.70710678,
  0.        ,  0.        ],
[ 0.        ,  0.        ,  0.        ,  1.        ,  0.        ,
  0.        ,  0.        ,  0.        ,  0.        , -1.        ,
  0.        ,  0.        ],
[ 0.        ,  0.        ,  0.        ,  0.        ,  0.70710678,
  0.70710678,  0.        ,  0.        , -0.70710678, -0.70710678,
  0.        ,  0.        ],
[ 0.        ,  0.        , -0.70710678,  0.70710678,  0.        ,
  0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
  0.70710678, -0.70710678],
[ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
  1.        ,  0.        ,  0.        ,  0.        ,  0.        ,
  0.        , -1.        ],
[ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
  0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
 -0.70710678, -0.70710678],
[ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
  0.        , -1.        ,  0.        ,  1.        ,  0.        ,
  0.        ,  0.        ],
[ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
  0.        ,  0.        ,  0.        , -1.        ,  0.        ,
  1.        ,  0.        ]])
```

Let's suppose that `f = [ 0, 2, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0]`. (We keep the numbering on the forces is the same as the numbering on the displacements. Thus, these forces are *downward* on all three masses in the bottom row.)

[ ]:
```
f = 10**6*np.array([0, 2, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0])

K2 = makeK([ 9.7* 10**6 if i in [1,2,3,4,6,9,12,14,15] else 6.8943 * 10**6 for␣
 ↪i in range (1,16)])
```

```
A2 = B_bridge.transpose() @ K2 @ B_bridge
la.solve(A2,ff)
```

1.1 inverse problem

The above shows that, given knowledge of the "spring constants" k_i, and the applied forces f_i, we can estimate the displacements x_i. This is the "*forward problem*".

The inverse problem is this: given measurements of the displacements x_i, find the spring constants k_i.

There are many applications of such *inverse problems*.

For our bridge problem, note that consider the system

$$B^T K B\mathbf{x} = \mathbf{f}$$

We consider a general case where there are m springs and $n/2$ masses, so that there are n "displacement components" (in 2 dimensions).

Notice that

$$\begin{aligned}
KB\mathbf{x} &= \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_m \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= \operatorname{diag}(B\mathbf{x}) \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}
\end{aligned}$$

where for a vector $\mathbf{w} \in \mathbb{R}^m$, we obtain an $m \times m$ matrix

$$\operatorname{diag}(\mathbf{w}) = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_m \end{bmatrix}$$

Thus we have

$$B^T K B\mathbf{x} = B^T \operatorname{diag}(B\mathbf{x})\mathbf{k} = \mathbf{f}$$

which gives some hope for finding the vector $\mathbf{k}$ given knowledge of $\mathbf{x}$ and $\mathbf{f}$.

1.2 the difficulty

Note that in the case of our bridge example, the number of "displacement components" we are tracking is $n = 12$, while the number of springs is $m = 15$.

Note that B^T is an $n \times m$ matrix, that $B\mathbf{x}$ is in $\mathbb{R}^m$, so that $B^T \operatorname{diag}(B\mathbf{x})$ is an $n \times m$ matrix.

In our example, the linear equation $B^T \operatorname{diag}(B\mathbf{x})\mathbf{k} = \mathbf{f}$ amounts to a system of 12 linear equations in 15 unknowns.

In general, if $n < m$, the system is *underdetermined.* As a consequence, the linear equation does not uniquely determine the spring constants k_i – i.e. the entries in the vector $\mathbf{k}$.

One way to fix this:

take measurements $\mathbf{x}$ for *various different force loads* $\mathbf{f}$.

More precisely, consider different force loads $\mathbf{f}_1, \cdots, \mathbf{f}_p$. For each of these force loads, determine the displacement vectors $\mathbf{x}_1, \cdots, \mathbf{x}_p$.

We now obtain a $pn \times m$ matrix system

$$\begin{bmatrix} B^T \operatorname{diag}(B\mathbf{x}_1) \\ B^T \operatorname{diag}(B\mathbf{x}_2) \\ \vdots \\ B^T \operatorname{diag}(B\mathbf{x}_p) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_p \end{bmatrix}.$$

If we choose p sufficiently large and if we make sure that the $np \times m$ coefficient matrix has rank at least m, we expect to find a solution.

[142]:
```
def diag(w):
    n = len(w)
    return np.array([ w[i]*sbv(i,n) for i in range(n) ])

diag([1,2,3,4])
```

[142]:
```
array([[1, 0, 0, 0],
       [0, 2, 0, 0],
       [0, 0, 3, 0],
       [0, 0, 0, 4]])
```

Let's consider some data. We have collected displacement measurements for three different force loads. The displacement measurements are outside of engineering tolerances!

We want to know which spring (=bridge component) is defective.

[206]:
```
# measurements

# for these forces
f1 = 10**6*np.array([0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0])
f2 = 10**6*np.array([0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0])
f3 = 10**6*np.array([0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0])

# measured the following displacements
x1 = np.array([ 0.04465819,  0.69735727,  0.08117014,  0.49972439,  0.0584346 ,
        0.29520329,  0.09315958,  0.52830165, -0.04706315,  0.50115528,
       -0.11445688,  0.25950424])
```

```
x2 =np.array([-0.03453953,  0.49972439,  0.01867222,  0.76169019,  0.03453953,
        0.49283618,  0.16411865,  0.45421336,  0.0034441 ,  0.66577233,
       -0.15233463,  0.44242935])

x3 = np.array([-0.04465819,  0.29520329, -0.08117014,  0.49283618, -0.0584346 ,
        0.69735727,  0.11302599,  0.25807335,  0.04706315,  0.49140529,
       -0.09172868,  0.52687076])
```

[208]:
```
coeffMatrix = np.concatenate([ B_bridge.transpose() @ diag( B_bridge @ x ) for␣
 ↪x in [x1,x2,x3] ] )

kvalues,_,_,_ = la.lstsq(coeffMatrix , np.concatenate([f1,f2,f3]),rcond=None)


[ f"k{i+1} = {kvalues[i]}" for i in range(len(kvalues))]
```

[208]:
```
['k1 = 9699998.607722549',
 'k2 = 2000000.4090927793',
 'k3 = 9700000.122052612',
 'k4 = 9699998.86756195',
 'k5 = 6894300.281687661',
 'k6 = 9700000.126240244',
 'k7 = 6894300.249842933',
 'k8 = 6894299.831309714',
 'k9 = 9699998.96686751',
 'k10 = 6894300.258596171',
 'k11 = 6894299.903274672',
 'k12 = 9700000.298712648',
 'k13 = 6894300.008658496',
 'k14 = 9700000.174220743',
 'k15 = 9700000.624519458']
```

We see that `k2` is different than expected and needs to be replaced!