PS 06 – Markov Processes

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1. Suppose you're standing on a street with buildings labelled by the integers (specifically, you're in front of the building labelled 0, and suppose that the indices are increasing to the right). Suppose that every minute you flip a coin. If the coin is heads you walk right and if the coin is tails you walk left.

    a. Explain why your position (i.e. the building you're in front of) as a function of time can be modeled as a Markov process.

SOLUTION:

Your position at time t is (randomly chosen to be) one step to the left or to the right of your previous position, and hence is only dependent on your position at time $t - 1$. It doesn't matter (for instance) what sequence of steps took you to get to your position at time $t - 1$, and hence this is a Markov chain.

Somewhat more precisely, the nodes of the *transition diagram* are in one-to-one correspondence with the integers $\{0, \pm 1, \pm 2, \cdots\}$, and each node `[n]` has two outgoing edges. There is an edge `[n] --> [n+1]` labeled with probability 1/2, and an edge `[n-1] <-- [n]` also labeled with probability 1/2. Thus, the probabilities on the outgoing edges always sum to 1.

And the state of the system evolves each minute by applying the rules of this *state machine*; this is the definition of a *Markov process* (or *Markov chain*).

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    b. Can the distance from where you started as a function of time be modeled by a Markov process?

SOLUTION:

Yes. In this case, consider the *transition diagram* whose nodes are *non-negative* integers. For any node `[n]` with `n>0`, there are *two* outgoing edges: `[n] --> [n+1]` and `[n-1] <-- [n]`, both of which are labeled with probability 1/2.

If you are standing in front of building `m` - so that the distance from building `0` is `|m|` - , the effect of a `heads` toss and of a `tails` toss depend on the *sign* of `m`. Namely, if `m>0` and you toss `heads`, you move one building to the right and hence increase your distance to the origin by one unit, and if you toss `tails`, you move one building to the left and hence decrease your distance to the origin by one unit.

On the other hand, if `m<0` and you toss `heads`, your one-building to the right move *increases* your distance to the origin by one unit, and if you toss `tails`, your distance to the origin *decreases* by one unit.

Finally, if your distance is `0`, there is only one outgoing edge `[0] --> [1]` and it has a probability of 1, since a toss of either `heads` or `tails` results in a move placing you exactly 1 unit from the origin.

In summary, `distance to the origin` is governed by the transition diagram we just described, hence is a Markov process.

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c. Now suppose that every minute you flip two coins. If both are heads, you move right, if both are tails you move left and otherwise you stay put. Is your distance from where you started modeled by a Markov process in this scenario? How do you expect this to compare to the process described in part b?

SOLUTION:

This is again a Markov process, with a different transition diagram which we now describe.

Again the nodes of the diagram are non-negative integers `[n]`.

The outgoing edges from `[0]` are a loop `[0] --> [0]` together with an edge `[0] --> [1]`. In this case each edge occurs with probability 1/2. (A result of `heads,head` or `tails,tails` causes a move `[0] --> [1]`, while a mixed toss results in the move `[0] --> [0]`.

For `|n| > 0`, there are 3 outgoing edges from `[n]`:

- `[n] --> [n]` is labeled with probability 1/2 and results from `heads,tails` or `tails,heads`
- `[n] --> [n+1]` is labeled with probability 1/4; if `n>0` this move results from `heads,head` while if `n<0` this move results from `tails,tails`.
- `[n] --> [n-1]` is also labeled with probability 1/4; if `n>0` this move results from `tails,tails` while if `n<0` this move results from `heads,heads`.

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d. For both experiments, compute the probability that you are standing on an odd number for minute $0, 1, 2, 3, 4$.

SOLUTION:

I'll do this calculation with the following code:

```
from pprint import pprint

# we represent the probabilities for a certain state
# of our system using a dictionary
# The initial state is { 0:1 } - this means with probability 1 you are at distance 0
# The state { 0: p0, 1: p1, 2: p2, ... } indicates that with probability pi you are ate distance
#
def prob(state,pos):
    # return the probability recorded in the state dictionary for the indicated position
    if pos in state.keys():
        return state[pos]
    else:
        return 0

def step(f,state):
    # update the state using the transition function f
    # f should be a function of two arguments: f(old_pos,new_pos)
    # should return the probability of transitioning from old_pos to new_pos
    # We use f to update the probabilities, and we return the new state
    results = len(state.keys())
    return { r: f(r-1,r)*prob(state,r-1) + f(r,r)*prob(state,r) + f(r+1,r)*prob(state,r+1)
             for r in range(results+1) }

def iterate(num,f,init):
    # given a starting state `init`, return the state after `num` iterations,
    # using the function `f` as input to `step`.
    if num<=0:
        return init
```

```
    else:
        return iterate(num-1,f,step(f,init))

def prob_odd(state):
    # for a given state, return the probability that the distance from
    # building 0 is *odd*
    return sum([ state[r] for r in state.keys() if r % 2 == 1 ])
```

In the case described in (b), using only one coin, we define a function computing the probabilities for state transitions in the single-coin case. We find the following:

```
# we'll use this function as the update function f when
# calling the `step` function defined above.
#
def one_coin(old_pos,new_pos):
    # return the probability of
    # transition from old_pos to new_pos
    match (old_pos,new_pos):
        case 0,1:  return 1
        case 0,_:  return 0
        case m,n:
            if abs(m-n) == 1:
                return .5
            else:
                return 0


S = [ (m,iterate(m,one_coin,{0:1})) for m in [0,1,2,3,4] ]
T = [ (m,prob_odd(state)) for (m,state) in S ]
pprint(S)
print(T)
=>
[(0, {0: 1}),
 (1, {0: 0.0, 1: 1.0}),
 (2, {0: 0.5, 1: 0.0, 2: 0.5}),
 (3, {0: 0.0, 1: 0.75, 2: 0.0, 3: 0.25}),
 (4, {0: 0.375, 1: 0.0, 2: 0.5, 3: 0.0, 4: 0.125})]
[(0, 0), (1, 1.0), (2, 0.0), (3, 1.0), (4, 0.0)]
```

Here we see that after 0,2,4 steps we are *never* in front of an odd-numbered building, and after 1,3 steps we are *always* in front of an odd-numbered building.

In the case described in (c), we find instead the following results:

```
# we'll use this function as the update function f when
# calling the `step` function defined above.
#
def two_coin(old_pos,new_pos):
    # return the probability of
    # transition from old_pos to new_pos
    match (old_pos,new_pos):
        case 0,1: return .5
        case 0,0: return .5
        case m,n:
            if m==n:
                return .5
            if abs(m-n) == 1:
                return .25
```

```
            else:
                return 0

S = [ (m,iterate(m,two_coin,{0:1})) for m in [0,1,2,3,4] ]
T = [ (m,prob_odd(state)) for (m,state) in S ]
pprint(S)
print(T)
=>
[(0, {0: 1}),
 (1, {0: 0.5, 1: 0.5}),
 (2, {0: 0.375, 1: 0.5, 2: 0.125}),
 (3, {0: 0.3125, 1: 0.46875, 2: 0.1875, 3: 0.03125}),
 (4, {0: 0.2734375, 1: 0.4375, 2: 0.21875, 3: 0.0625, 4: 0.0078125})]
[(0, 0), (1, 0.5), (2, 0.5), (3, 0.5), (4, 0.5)]
```

So after 0 steps, we are (of course) never in front of an odd numbered building, but after 1,2,3 or 4 steps we are in front of an odd numbered building with probability 1/2.

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e. (Optional food for thought) Suppose your friend is playing the same game, but started at position -100. Do you think it is more likely that you two will eventually meet or that you two will never meet? Does this answer change when your friend starts at -1? How about -10000000?

SOLUTION:

It is a fact that two random walkers will eventually meet regardless of where they start.

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2. **Rain or shine** On Planet X, the weather is strangely predictable: The weather is always either sunny, rainy, foggy or snowy. If it rains today, its sunny tomorrow. If it is sunny today, its rainy tomorrow. If its foggy today, its not sunny tomorrow. Finally, the weather is never the same two days in a row. Apart from these rules, the weather is completely random, in that if e.g. its foggy today it is equally likely to be either rainy or snowy tomorrow. You live on Planet X and are trying to figure out what to wear this week, so you'd like to develop a model for the weather.

   a. Explain why the weather can be modeled as a Markov process. Write out the transition matrix, and draw the corresponding finite state machine.

SOLUTION:

The system is governed by a transition diagram for which the probabilities on the outgoing edges from each node sum to 1. Thus, the system is a *Markov process*.

We describe the *transition diagram*. It has 4 nodes: `sunny`, `rainy`, `foggy` and `snowy`. We are going to draw the diagram using `graphviz` so we describe the probabilities as a python dictionary:

```
weather = [ 'sunny', 'rainy', 'foggy', 'snowy' ]

transitions = {
    ('rainy', 'sunny'): 1,
    ('sunny', 'rainy'): 1,
    **{ ('foggy',w): 1/2 for w in weather if w != 'sunny' and w != 'foggy' },
    **{ ('snowy',w): 1/3 for w in weather if w != 'snowy' }
    }
transitions
=>
{('rainy', 'sunny'): 1,
```

```
 ('sunny', 'rainy'): 1,
 ('foggy', 'rainy'): 0.5,
 ('foggy', 'snowy'): 0.5,
 ('snowy', 'sunny'): 0.3333333333333333,
 ('snowy', 'rainy'): 0.3333333333333333,
 ('snowy', 'foggy'): 0.3333333333333333}
```

Now we create the labeled digram:

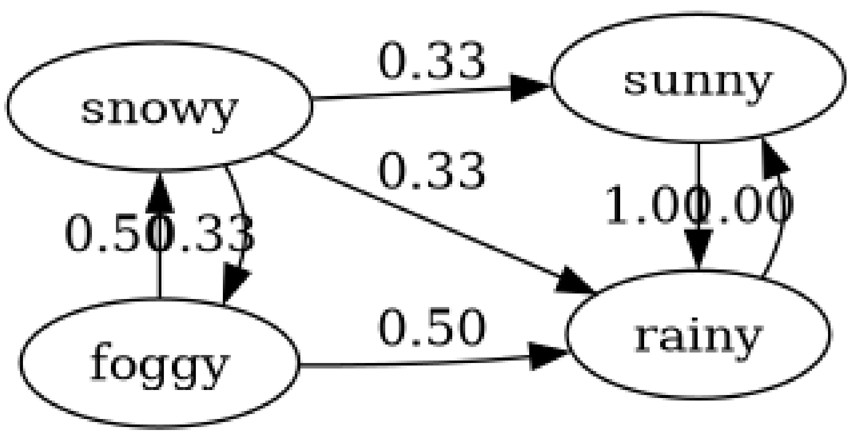
```
from graphviz import Digraph

dot = Digraph()
dot.attr(rankdir='LR')

with dot.subgraph() as c:
    c.attr(rank='same')
    c.node('rainy')
    c.node('sunny')
with dot.subgraph() as c:
    c.attr(rank='same')
    c.node('foggy')
    c.node('snowy')

from itertools import product

for (v1,v2) in product(weather,weather):
    if (v1,v2) in transitions.keys():
        dot.edge(v1,v2,f"{transitions[(v1,v2)]:.02f}")
dot.render('weather.png')
```



And we create the transition matrix:

```
import numpy as np
import numpy.linalg as npl

def transition_prob(v,w):
    # get the probability for the transition v-->w
    if (v,w) in transitions.keys():
        return transitions[(v,w)]
    else:
        return 0

p = np.array([[ transition_prob(v,w) for v in weather] for w in weather ])
```

```
p
=>
array([[0.        , 1.        , 0.        , 0.33333333],
       [1.        , 0.        , 0.5       , 0.33333333],
       [0.        , 0.        , 0.        , 0.33333333],
       [0.        , 0.        , 0.5       , 0.        ]])
```

b. Check whether the conditions for the Perron-Frobenius theorem is satisfied for this problem (aperiodic and strongly connected). Explain your reasoning.

SOLUTION:

The hypothesis of the Frobenius-Perron theorem do not hold. The transition diagram is not *strongly connected*. For example, there is no path from the node `sunny` to the node `foggy`.

c. Do you expect power iteration to be effective for computing the greatest eigenvector of your transition matrix?

SOLUTION:

Because the conclusion of the Frobenius-Perron theorem is not known to hold, it is possible that `p` has more than one eigenvalue with absolute value 1. In that case, power iteration will not help to compute the greatest eigenvector.

d. Find the eigenvalue decomposition for the transition matrix, and the associated eigenvectors. Explain why these values confirm your answer to part 2.

SOLUTION:

(the problem should have read: "Explain why these values confirm your answer to (c)").

Let's look at the eigenvalues of `p`:

```
vals,vecs = npl.eig(p)
vals
=>
array([ 1.        ,  0.40824829, -1.        , -0.40824829])
```

We note that `p` has an eigenvalue 1, but also an eigenvalue -1.

If `v` is an eigenvector with eigenvalue -1, then

```
npl.matrix_power(p,n) @ v = ± v
```

depending on the parity of `n`.

So the long-term behavior of *powers of* `p` fails to stabilize, so we do not expect power iteration to be an effective way of computing the greatest eigenvector.

e. Suppose that the "weather rules" change so that if its sunny today, it is equally likely to be snowy or rainy tomorrow. Write out the new transition matrix, associated finite state machine, and determine whether the conditions for the Perron-Frobenius are satisfied. Compute the eigenvalue decomposition and compare to the previous set of eigenvalues.

Solution:

We update the transition probabilities to reflect the new weather rules:

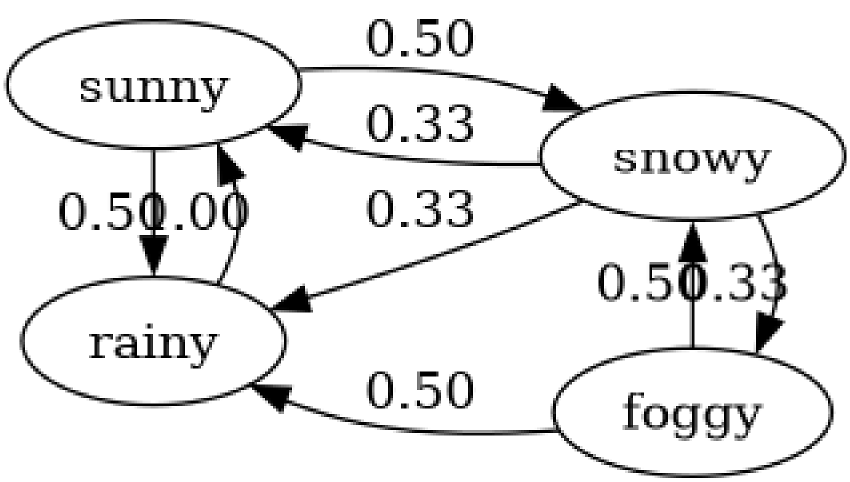
```
new_transitions = {
    ('rainy', 'sunny'): 1,
    ('sunny', 'rainy'): 1/2,
    ('sunny', 'snowy'): 1/2,
    **{ ('foggy',w): 1/2 for w in weather if w != 'sunny' and w != 'foggy' },
    **{ ('snowy',w): 1/3 for w in weather if w != 'snowy' }
    }
```

We get a new diagram as follows:

```
new_dot = Digraph(format='png')
dot.attr(rankdir='LR')

with dot.subgraph() as c:
    c.attr(rank='same')
    c.node('rainy')
    c.node('sunny')
with dot.subgraph() as c:
    c.attr(rank='same')
    c.node('foggy')
    c.node('snowy')

for (v1,v2) in product(weather,weather):
    if (v1,v2) in new_transitions.keys():
        dot.edge(v1,v2,f"{new_transitions[(v1,v2)]:.02f}")
dot.render('new_weather')
```



And we get a new transition matrix q.

```
q = np.array([[ transition_prob(v,w,new_transitions) for v in weather] for w in weather ])

q
=>
array([[0.        , 1.        , 0.        , 0.33333333],
       [0.5       , 0.        , 0.5       , 0.33333333],
       [0.        , 0.        , 0.        , 0.33333333],
       [0.5       , 0.        , 0.5       , 0.        ]])
```

We observe that in this case, the transition diagram is *strong connected*. Moreover, it is also acyclic e.g. because there are cycles of length two (`snowy --> foggy --> snowy` for example) as well as cycles of length three (`sunny --> snowy --> rainy --> sunny`). Since `gcd(2,3) = 1` there only natural number dividing all cycle lengths is 1.

Thus the Frobenius Perron Theorem holds. It promises that `q` has eigenvalue 1 with multiplicity 1. If `v` is an eigenvector with eigenvalue 1, normalized so that `v` is a probability vector, then we know that

```
npl.matrix_power(p,n)
```

converges as $n \to \infty$ to the matrix `B` with 4 columns equal to the vector `v`.

We can observe this phenomenon by compute "big" powers of `q`:

```
npl.matrix_power(q,100)
=>
array([[0.38461538, 0.38461538, 0.38461538, 0.38461538],
       [0.30769231, 0.30769231, 0.30769231, 0.30769231],
       [0.07692308, 0.07692308, 0.07692308, 0.07692308],
       [0.23076923, 0.23076923, 0.23076923, 0.23076923]])
```

We check the eigenvalues:

```
vals,vecs = npl.eig(q)
vals
=>
array([ 1.        , -0.78867513, -0.21132487,  0.        ])
```

In this case, as promised by Frobenius-Perron, there is exactly one eigenvalue with absolute value 1. All other eigenvalues λ have $|\lambda| < 1$.

Note that after normalizing, the eigenvector `vecs[:,0]` is close to the columns of the matrix q^{100} we computed above:

```
# get the eigenvector computed by numpy for eigenvalue 1
# remember that it is the first *column* of the matrix ev,
# not the first row...!!
ev = vecs[:,0]

# normalize to make a probability vector
c = np.array([1,1,1,1]) @ ev  # c is the sum of the entries of ev
p_ev = (1/c)*ev               # the entries of p_ev sum to 1

p_ev
=>
array([0.38461538, 0.30769231, 0.07692308, 0.23076923])
```