

Midterm Project 2 – PageRank

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2024-03-15

Instructions

There are 6 questions contained in the presentation below. You should submit to gradescope a report as a PDF file containing responses to those questions, written in the style of a *lab report*: you should write in such a manner that a reader who didn't know the assignment in advance could read and follow your response.

In some cases, you are asked to write python code. You should include the text of the code embedded in your submission.

(And: You don't have to include any discussion on the **for-fun** section at the end...)

PageRank

Markov chains formed the original basis for the [PageRank algorithm](#) used by [Google](#) to rank web-pages by “importance”.

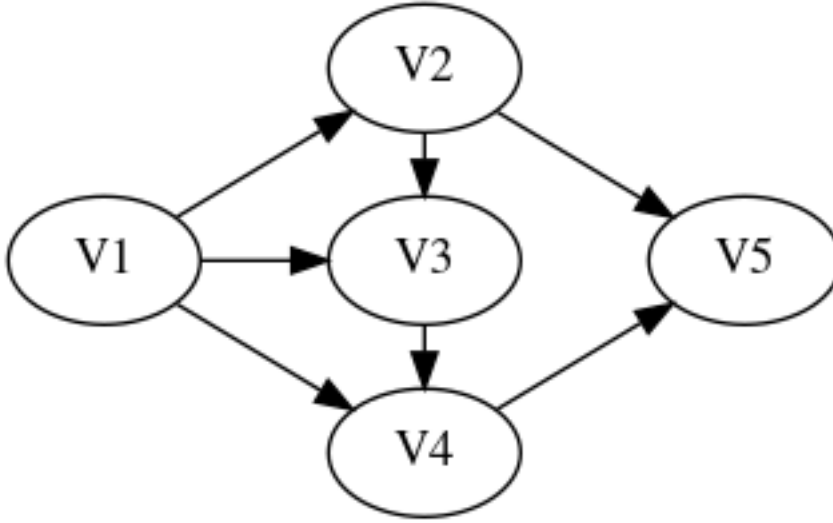
Consider the following basic model of surfing the web. A user begins surfing at a particular website. From the set of outgoing links from this site, a choice is made, with each link given equal probability. This transition brings the user to a new page, where the process is repeated. Let's model this system as a Markov Chain: the nodes in our transition diagram correspond to a certain collection of websites, and there is a directed edge from node A to node B if site A contains a hyperlink to the site B. Consider a fixed node A: to assign the probabilities on an edge $A \rightarrow B$ we just need to count the number $N(A)$ of outgoing links from site A; then all edges $A \rightarrow B$ will be given probability $1/N(A)$.

Number the nodes a_0, \dots, a_{n-1} and write \mathbf{e}_i for the standard basis vectors. $0 \leq i \leq n-1$. We wish to define the adjacency matrix A . It should have the property that $A\mathbf{e}_i$ is the sum of all the standard basis vectors \mathbf{e}_j for which there is an edge $a_i \rightarrow a_j$. A little thought shows that the s, t entry satisfies

$$A_{s,t} = \begin{cases} 1 & \text{if there is an edge } a_t \rightarrow a_s \\ 0 & \text{otherwise} \end{cases}$$

Using A , one can create the “naive” transition matrix T whose entries are the probabilities discussed above: as in the course notebooks, one views the directed graph as the transition diagram of a probabilistic finite-state machine, and T is the associated transition matrix.

Suppose the sites we consider were described by the following directed graph:



Treating this naively, we obtain the following adjacency matrix and “naive” transition matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & 0 \end{bmatrix}$$

Notice that V5 is a *sink* - there are no outgoing edges (the site V5 has no links to any of the sites under consideration). This is problematic – the transition diagram we obtain isn’t strongly connected and the corresponding transition matrix isn’t *stochastic* (the entries in the last column don’t sum to 1).

Now, the model is supposed to represent a *random web-surfer*, so to recover a stochastic matrix, we model the transition from a *sink* node – i.e. a node with no “out-connections” – by having the surfer randomly choose a site (i.e. a node) from all sites with equal probability.

In the example, this corresponds to replacing the final column of T by the column

$$\frac{1}{5} [1 \ 1 \ 1 \ 1 \ 1]^T$$

The matrix T_1 obtained in this manner is *stochastic*, though it may well not satisfy the conclusion of the Perron-Frobenius Theorem.

We now introduce a *damping factor*:

The PageRank theory holds that an imaginary surfer who is randomly clicking on links will eventually stop clicking.

The probability, at any step, that the person will stop is a damping factor p .

We model this by declaring that with probability p the imaginary surfer will “stop clicking”. The transition that occurs in this case lands the surfer on a new node chosen at random from *all* available nodes (with equal probability).

For the example, for the above diagram, we obtain the matrix

$$C = (1 - p)T_1 + \frac{p}{5} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which we call the PageRank transition matrix for the damping probability p .

- Given an adjacency matrix A , complete the following description of the i, j entry of the corresponding PageRank transition matrix C for the probability p .

- If the j -th column of the matrix A is equal to $\mathbf{0}$, then for each $i=0, \dots, n-1$:

$$C_{i,j} = \underline{\hspace{2cm}}$$

- Suppose that the sum of the entries of the j -th column of the matrix A is equal to $s > 0$, then for each $i=0, \dots, n-1$:

$$\text{If } A_{i,j} = 0, \text{ then } C_{i,j} = \underline{\hspace{2cm}}$$

$$\text{If } A_{i,j} = 1, \text{ then } C_{i,j} = \underline{\hspace{2cm}}$$

- Use your description in 1. to write a python function to create the PageRank transition matrix from an adjacency matrix A .

Here is a suggested starting point:

```
import numpy as np

def make_transition(A,p):
    (n,m) = A.shape
    if n==m:
        C = np.zeros((n,n))    ## create a matrix of
        #                      ## zeros of the correct size
        #
        #                      ## insert code to modify
        #                      ## the entries C[i,j]
        #
        return C
```

Recall that you can obtain the *sum* of the entries of the i th column of the matrix A using the following:

```
s=np.ones(n) @ A[:,i]
```

And you can *test* whether the i th column of A is equal to the zero-vector using $(A[:,i] == 0).all()$; this expression will be True if all entries in the i th column are equal to 0.

You can use this expression in a conditional statement:

```
if (A[:,i] == 0).all() then
    ## do something...
```

I recommend that you *build up* the matrix C by assigning values to the entries one column at a time, by using a *loop* as follows:

```
for j in range(n):
    ## insert code to assign values for the j-th column...
```

You can assign the value of C in the i th row and j th column using an expression like

```
C[i,j] = ...
```

Test your code on the example above, as follows:

```
>>> float_formatter = "{:.5f}".format
>>> np.set_printoptions(formatter={'float_kind':float_formatter})
>>> A = np.array([[ 0, 0, 0, 0, 0 ],
                  [ 1, 0, 0, 0, 0 ],
                  [ 1, 1, 0, 0, 0 ],
                  [ 1, 0, 1, 0, 0 ],
```

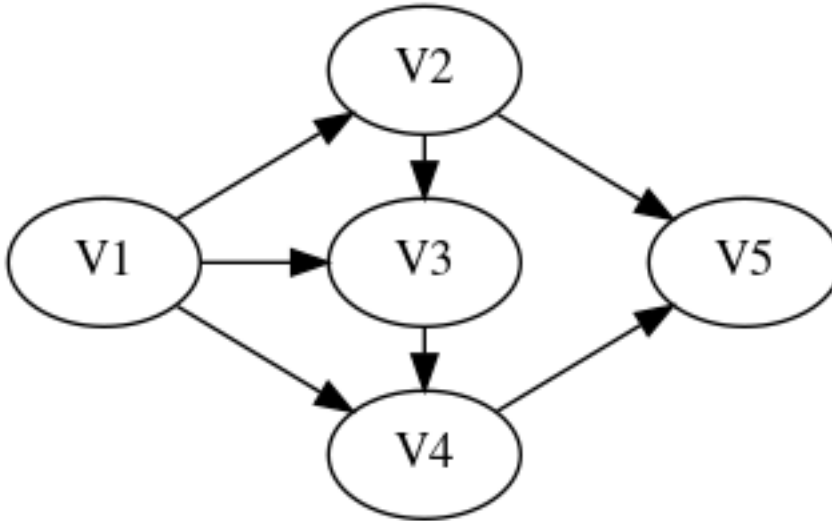
```

[ 0, 1, 0, 1, 0]])

>>> ## you should see the following output
>>> make_transition(A,p=0.8)
array([[0.16000, 0.16000, 0.16000, 0.16000, 0.20000],
       [0.22667, 0.16000, 0.16000, 0.16000, 0.20000],
       [0.22667, 0.26000, 0.16000, 0.16000, 0.20000],
       [0.22667, 0.16000, 0.36000, 0.16000, 0.20000],
       [0.16000, 0.26000, 0.16000, 0.36000, 0.20000]])

```

3. Find a 1-eigenvector for the PageRank transition matrix C when $p=.8$ and when $p=.4$ for the directed graph



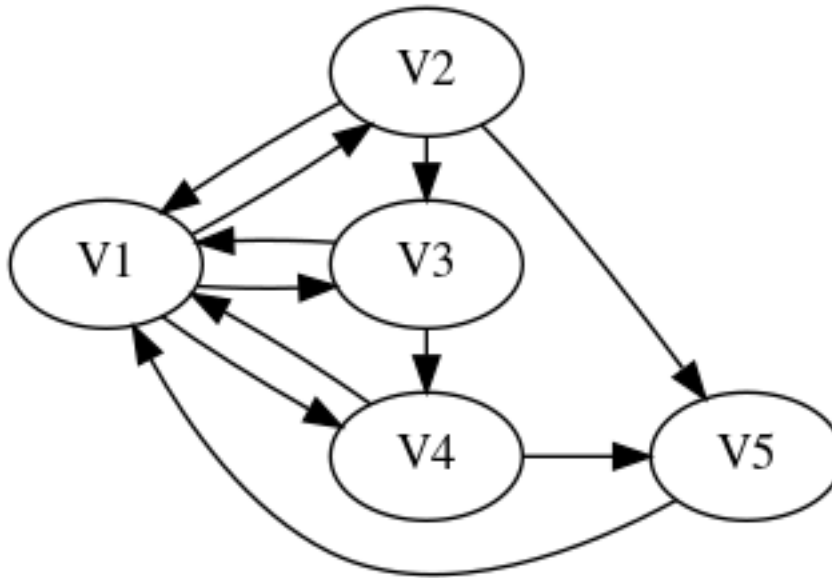
Be sure to normalize your eigenvector to obtain *probability vectors*.

Explain what the entries in this vector predict about the probability of a random surfer landing on one of the five web-sites corresponding to the nodes in the diagram.

Compare the information you get from the eigenvector with that obtained by studying powers of C calculated using `np.linalg.matrix_power(C,m)`.

Remark: PageRank uses the long-term probability as a proxy for the importance of the page – the probability with which a random surfer lands on a page in the long-run determines its ranking.

4. The following diagram describes the same nodes as in problem 2., but includes some additional edges (i.e. links).



Assess the impact of the additional edges on the page rankings (when $p=.8$ and when $p=.4$) – i.e. compare the ranking obtained for this diagram with that obtained in problem 2. Before making your comparison, you’ll need to first compute the adjacency matrix for this new configuration, and then use your code to find the corresponding PageRank transition matrix.

5. For *any* directed graph, explain why the corresponding PageRank transition matrix for $p>0$ is a stochastic matrix corresponding to a strongly connected aperiodic transition diagram. In particular, explain why the conclusion of the *Perron-Frobenius Theorem* holds for this matrix.
6. The following code can be used to extract formatted data (json) from a file. The parameter to the function is a file name; this json file should contain data like

```
[
  {
    "from": "Blue Whale",
    "to": "Snail"
  },
  {
    "from": "Blue Whale",
    "to": "Alligator"
  },
  ...
]
```

(Here we view these animal names as “sites”; a from/to pair indicates a link.)

and the return value is a pair (sites,A) where sites is a list of strings containing the name of each “site” that appears in one of the from/to pairs, and A is the adjacency matrix determined by the indicated links.

```
import json
import numpy as np
float_formatter = "{:.5f}".format
np.set_printoptions(formatter={'float_kind':float_formatter})

def bv(it,items):
    return np.array([1.0 if i == items.index(it)
                     else 0.0 for i in range(len(items))])

## >>> bv("c",["a","b","c","d"])
## array([0.00000, 0.00000, 1.00000, 0.00000])
```

```
def adj_from_json(json_file):
    with open(json_file) as f:
        adj_data = json.load(f)

    dict = {}

    for i in adj_data:
        lfrom = i['from']
        lto = i['to']
        if lfrom in dict.keys():
            dict[lfrom].add(lto)
        else:
            dict[lfrom] = set()
            dict[lfrom].add(lto)
        if not(lto in dict.keys()):
            dict[lto] = set()

    sites = list(dict.keys())

    A = np.array([sum([bv(l_to,sites) for l_to in dict[l_from]],
                      np.zeros(len(sites)))
                  for l_from in sites])
    return (sites, A)
```

Now, here is a file containing data from which you should build an adjacency matrix using the function `adj_from_json`:

[data.json](#)

To use this file, follow these instructions:

Local computer instructions

If you use python on your local computer, save this file to your drive and run python from a shell in the same directory. You can then call

```
(ll,A) = adj_from_json("data.json")
```

Colab instructions

If you use colab, proceed as follows. Save the file `data.json` to your local computer, and then upload that file to your google drive.

To enable colab to see your google drive, execute the following commands in a cell:

```
from google.colab import drive
drive.mount('/gdrive')
```

Now change to the correct directory in your google drive by executing the following in a cell in colab (you do need the % symbol):

```
%cd /gdrive/My\ Drive/
```

Note for example that you can use the following command to see the contents of the currently selected directory of your google drive:

```
%ls
```

Now you can run

```
(ll,A) = adj_from_json("data.json"):
in a colab cell.
```

Once you have produced the adjacency matrix A from `data.json`, use the `make_transition` function you wrote in the previous exercises to build the PageRank transition matrix C from the adjacency matrix A , for $p=0.8$

Now find the page-rankings of these pages, using both the eigenvector method and the power-iteration method. For each method, report the top ten “sites” by name (when $p=0.8$). When using the power-iteration method, report the number of iterations used.

Note that you can get the site name corresponding to index i as follows. If you used `(ll,A) = adj_from_json("data.json")` to build the adjacency matrix then the list `ll` contains the names, and in particular `ll[i]` is the name for index i .

What happens if you reduce the value of p ? How much must you vary p to see any change in the “top-ten” list?

7. The damping probability $0 < p < 1$ forces the PageRank transition matrix to satisfy the conclusion of the Perron-Frobenius Theorem. As we discussed above, from the point-of-view of ranking web pages this quantity represents the probability that a random web-surfer gets bored and makes a random new choice of sites. In particular, this provides a reasonable *self-contained* explanation for the use of this PageRank transition matrix.

On the other hand, one might hope to use the PageRank approach to rank other “linked material”. For example, one might try to rank published academic papers by citation.

More precisely: given a collection of academic papers, consider a directed graph whose nodes are the papers and for which there is a directed edge from **paper A** to **paper B** if **paper A** contains a citation to **paper B**.

Suppose that each paper in the collection has at least one citation to another paper in the collection; under this assumption, the “naive” transition matrix T is stochastic. But this transition matrix may fail to satisfy the conclusion of the Perron-Frobenius Theorem. One can “fix” this problem as before by introducing a “damping probability” $0 < p \leq 1$ – i.e. by replacing T with the matrix

$$(1-p)T + \frac{p}{N}\mathbf{1}^{N \times N}$$

where N is the number of papers in the collection and $\mathbf{1}^{N \times N}$ denotes the $N \times N$ matrix all of whose entries are equal to 1.

Do you think this is a reasonable proposal? What would be the explanation for the damping probability from the point-of-view of the model?

For Fun:

Here is some code ¹ for performing a web-crawl:

[web-crawl code](#)

It contains a function with the following signature

```
def crawl(start_url,
          save_file,
          n,
          verbose=True,
          follow_relative_links=True,
          max_links_per_page=float('inf'),
          blacklist=[],
          whitelist=[]):
```

¹[2024-03-08] I needed to install some packages via conda to get it running just now:

```
mamba install lxml beautifulsoup4
```

I believe it works in colab.

which when invoked will “scrape/crawl” the web, beginning at `start_url`. It finds all links in this first site, and then proceeds to recursively follow these links. The parameter `n` is the number of pages to include in the adjacency data to be reported (roughly speaking, the recursive search can stop once `n` pages have been found).

For example, you could try something like

```
crawl("https://www.comap.com/contests/mcm-icm", "comap.txt", 500)
```

When finished, this function will output a json file with name `save_file`. This file will contain `from/to` pairs for links found in the search.

You can then use the above function `adj_from_json` to build an adjacency matrix from the links in the file built by `crawl`.

In turn, you can build the PageRank transition matrix from this adjacency matrix, and use either the eigenvalue method or the iteration method to rank the pages that you have considered.

There are a number of comments in the code for `crawl` that are worth reading if you want to play with this code...
