week05-02–branch-and-bound

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1 Math087 - Mathematical Modeling

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1.1 Course material (Week 5): Integer programming via Branch & Bound

1.2 Integer programming: summary of some issue(s)

As an example, consider the linear program:

`maximize` $f(x_1, x_2) = x_1 + 5x_2$; i.e. $\mathbf{c} \cdot \mathbf{x}$ where $\mathbf{c} = \begin{bmatrix} 1 & 5 \end{bmatrix}$.

such that $A\mathbf{x} = \begin{bmatrix} 1 & 10 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 20 \\ 2 \end{bmatrix}$ and $\mathbf{x} \geq \mathbf{0}$.

Let's find the optimal solution $\mathbf{x} \in \mathbb{R}^2$, and the optimal *integral* solution $\mathbf{x}$ with $x_1, x_2 \in \mathbb{Z}$.

We'll start by solving the *relaxed* problem, where the integrality condition is ignored:

[3]:
```
from scipy.optimize import linprog
import numpy as np

A = np.array([[1,10],[1,0]])
b = np.array([20,2])
c = np.array([1,5])

result=linprog((-1)*c,A_ub = A, b_ub = b)
print(f"result = {result.x}\nmaxvalue = {(-1)*result.fun:.2f}")
```

```
result = [2.  1.8]
maxvalue = 11.00
```

This calculution shows that an optimal solution with no integer constraint is $\mathbf{x} = \begin{bmatrix} 2 \\ 1.8 \end{bmatrix}$ and that the optimal value is roughly 11.

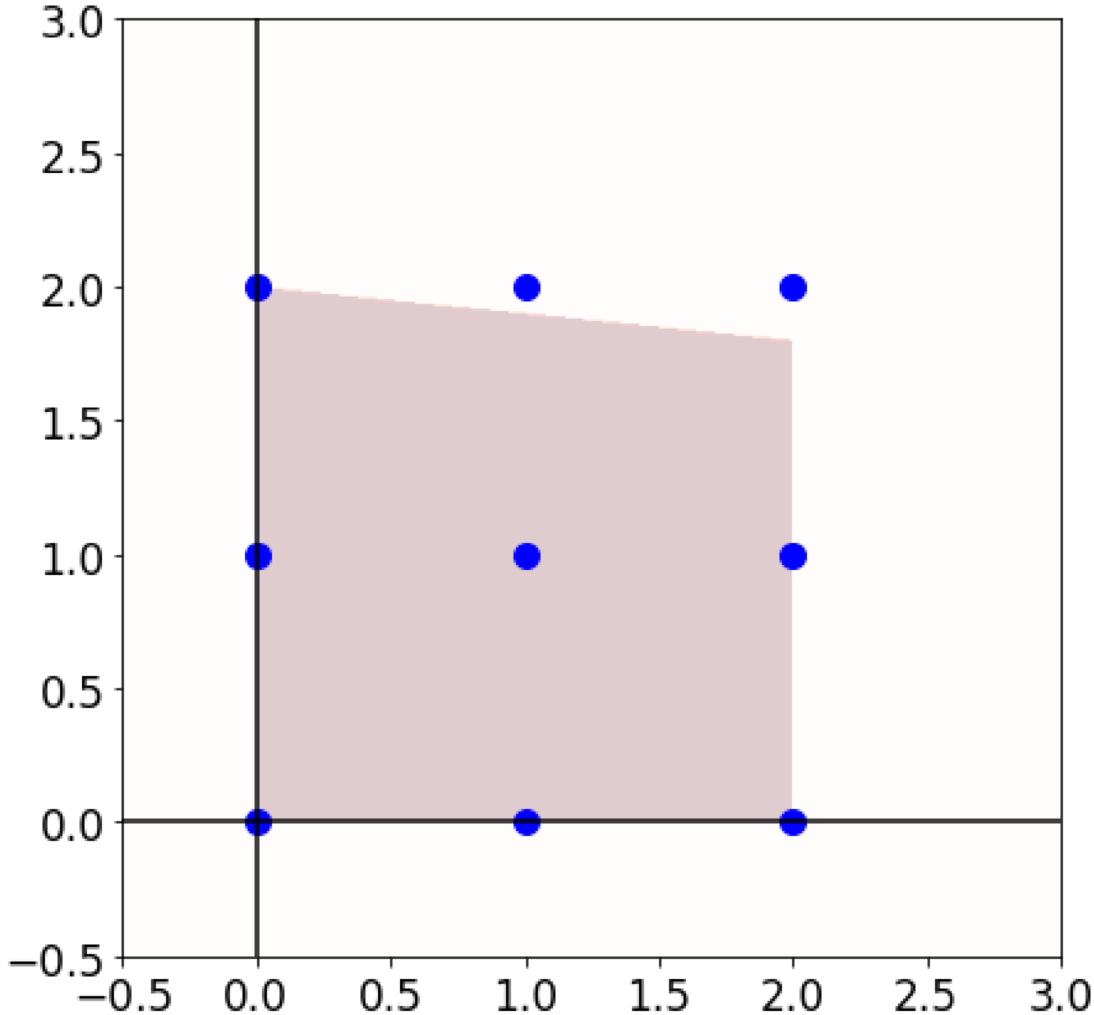
Let's make an image of the feasible set:

[5]:
```
%matplotlib notebook
%matplotlib inline
```

```
import matplotlib.pyplot as plt
import itertools

plt.rcParams.update({'font.size': 17})

# plot the feasible region
d = np.linspace(-.5,3,500)
X,Y = np.meshgrid(d,d)

def vector_le(b,c):
    return np.logical_and.reduce(b<=c)

@np.vectorize
def feasible(x,y):
    p=np.array([x,y])
    if vector_le(A@p,b) and vector_le(np.zeros(2),p):
        return 1.0
    else:
        return 0.0

Z=feasible(X,Y)
```

[3]:

```
fig,ax = plt.subplots(figsize=(7,7))
ax.axhline(y=0, color = "black")
ax.axvline(x=0, color = "black")

# draw the region defind by x >= 0 and Ax <= b.
ax.imshow(Z,
          extent=(X.min(),X.max(),Y.min(),Y.max()),
          origin="lower",
          cmap="Reds",
          alpha = 0.2)

def dot(x,y):
    return ax.scatter(x,y,s=100,color="blue")

# draw the integer points
for i,j in itertools.product(range(3),range(3)):
    dot(i,j)
```



You might imagine that the optimal *integer* solution is just obtained by rounding. Note the following:

$(2, 2)$ is infeasible.

$(2, 1)$ is feasible and $f(2, 1) = 2 + 5 \cdot 1 = 7$

$(1, 2)$ is infeasible

$(1, 1)$ is feasible and $f(1, 1) = 1 + 5 \cdot 1 = 6$

But as it turns out, the optimal integer solution is the point $(0, 2)$ for which $f(0, 2) = 0 + 5 \cdot 2 = 10$.

Of course, this optimal integral solution is nowhere near the optimal non-integral solution. So in general, rounding is inadequate!

How to proceed? Well, in this case there are not very many integral feasible points, so to optimize, we can just check the value of f at all such points!

Consider a linear program in standard form for $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq \mathbf{0}$, with inequality constraint $A\mathbf{x} \leq \mathbf{b}$ which seeks to `maximize` its objective function f.

Here is a systematic way that we might proceed:

Find an integer $M \geq 0$ with the property that

$$\mathbf{x} > M \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \implies \mathbf{x} \text{ is infeasible.}$$

There are $(M+1)^n$ points $\mathbf{x}$ with integer coordinates for which $\mathbf{0} \leq \mathbf{x} \leq M \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

For each of these points $\mathbf{x}$, we do the following: - if $\mathbf{x}$ is infeasible, discard - otherwise, record the pair $(\mathbf{x}, f(\mathbf{x}))$.

When we are finished, we just scan the list of recorded pairs and select that with the largest objective function value; this selection solves the problem.

The strategy just described is systematic, easy to describe, and works OK when $(M+1)^n$ isn't so large. But e.g. if $M = 3$ and $n = 20$, then already

$$(M+1)^n \approx 1.1 \times 10^{12},$$

which gives us a huge number of points to check!!!

2 A more efficient approach: "Branch & Bound"

We are going to describe an algorithm that implements a branch-and-bound strategy to approach the problem described above.

Let's fix some notation; after we formulate some generalities, we'll specialize our treatment to some examples.

2.1 Notation

We consider an `integer linear program`:

(♣) `maximize` $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$

subject to:

- $\mathbf{x} \in \mathbb{Z}^n, \mathbf{x} \geq \mathbf{0}$
- $A\mathbf{x} \leq \mathbf{b}$ for some $A \in \mathbb{R}^{r \times n}$ and $\mathbf{b} \in \mathbb{R}^r$.

Recall that $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$ is the set of *integers*, and $\mathbb{Z}^n$ is the just the set of vectors $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$ where $a_1, a_2, \ldots, a_n \in \mathbb{Z}$.

We are going to suppose that we have some vector

$$\mathbf{M} = \begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}^T \in \mathbb{Z}^n, \quad \mathbf{M} \geq \mathbf{0}$$

with the property that $\mathbf{x} > \mathbf{M} \implies \mathbf{x}$ is infeasible (i.e. $\mathbf{x} > \mathbf{M} \implies A\mathbf{x} > \mathbf{b}$).

In practice, it'll often be the case that $m_1 = m_2 = \cdots = m_n$ but that isn't a requirement for us.

Let's write

$$S = \{\mathbf{x} \in \mathbb{Z}^n \mid \mathbf{0} \leq \mathbf{x} \leq \mathbf{M}\}.$$

Note that the number of elements $|S|$ in the set S is given by the product

$$S = \prod_{i=1}^{n}(m_i + 1) = (m_1 + 1) \times (m_2 + 1) \times \cdots \times (m_n + 1).$$

And according to our assumption, S contains every feasible point $\mathbf{x}$ whose coordinates are integers. So a brute force approach to finding an optimal integral point $\mathbf{x}$ could be achieved just by testing each element of S.

Our goal is to systematically eliminate many of the points in S.

2.2 Algorithm overview

Keep the preceding notations. We sometimes refer to the entries x_i of $\mathbf{x}$ as "variables".

Let's focus on one entry of $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \in \mathbb{Z}^n$, say the j-th entry for some $1 \leq j \leq n$ (we'll say more below about how we should select j). i.e. we focus on the variable x_j.

Now, x_j may take the values $0, 1, 2, \cdots, m_j$, so we consider the following subsets of S:

$$\begin{array}{ccc} S_0 & = & \{\mathbf{x} \in S \mid x_j = 0\} \\ S_1 & = & \{\mathbf{x} \in S \mid x_j = 1\} \\ \vdots & \vdots & \vdots \\ S_{m_j} & = & \{\mathbf{x} \in S \mid x_j = m_j\} \end{array}$$

Thus we have *partitioned* S as a *disjoint union* of certain subsets:

$$S = S_0 \cup S_1 \cup \cdots \cup S_{m_j}$$

For $0 \leq \ell \leq m_j$, write f_ℓ for the maximum value of the objective function on points in S_ℓ:

$$f_\ell = \max\left(f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \mid \mathbf{x} \in S_\ell\right).$$

If we know for some ℓ_0 that the quantity f_{ℓ_0} exceeds f_ℓ for every $\ell \neq \ell_0$, then of course an optimal integral to (♣) is contained in the subset S_{ℓ_0}.

So we can then *prune* all the S_ℓ with $\ell \neq \ell_0$ and continue our search for optimal solutions to ($\clubsuit$) only considering points in S_{ℓ_0}.

We may now *repeat* the above procedure by focusing on a *new entry of* $\mathbf{x}$ (different from the j-th entry), checking only points in S_{ℓ_0}.

Iterating after selection of the subset S_{ℓ_0} is known as *branching*.

The main question now is this: how can we compare the values $f_\ell, 0 \leq \ell \leq m_j$ with one another, in order to carry out our pruning? This is known as *bounding*; so the question is: "how do we bound"?

The answer is to use "relaxed" versions of the integer linear program ($\clubsuit$), obtained by omitting variables (imposing additional equality constraints) and/or eliminating the "integral" requirement.

E.g. using ordinary linear programming, we may find the optimal *value* v_ℓ of the objective function for the linear program obtained from ($\clubsuit$) by considering $\mathbf{x} \in \mathbb{R}^n$ rather than $\in \mathbb{Z}^n$ and by imposing the additional equality constraint $x_j = \ell$.

Then of course $f_\ell \leq v_\ell$ – i.e. v_ℓ is an *upper bound* for f_ℓ.

On the other hand, for any point $\tilde{\mathbf{x}} \in S_\ell$, the must have

$$f(\tilde{x}) = \mathbf{c} \cdot \tilde{\mathbf{x}} \leq f_\ell$$

(since f_ℓ is the maximum of such values!).

These observations give us access to upper and lower bounds for the f_ℓ; we can now *bound* - i.e. prune - S_ℓ if we can demonstrate that a lower bound for f_{ℓ_0} exceeds an upper bound for f_ℓ for $\ell \neq \ell_0$.

2.3 Example

Let's see how this works in practice!

As a guiding heuristic, when we have a (non-integral) optimal point for a linear program, we choose to branch on the variable whose value (for this optimal point) is non-integral but closest to a integer.

Consider again the integer linear program

$( ) $ `maximize` $f(x_1, x_2) = x_1 + 5x_2$; i.e. $\mathbf{c} \cdot \mathbf{x}$ where $\mathbf{c} = \begin{bmatrix} 1 & 5 \end{bmatrix}$.

such that $A\mathbf{x} = \begin{bmatrix} 1 & 10 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 20 \\ 2 \end{bmatrix}$ and $\mathbf{x} \geq \mathbf{0}$ for $\mathbf{x} \in \mathbb{Z}^2$.

We notice that $(*) \quad \mathbf{x} > \begin{bmatrix} 2 & 2 \end{bmatrix}^T \implies \mathbf{x}$ is not feasible.

To begin, we first solve the linear program obtained from ($\diamondsuit$) by consider $\mathbf{x} \in \mathbb{R}^2$. The optimal value is $v = 11$ and an optimal solution is $\mathbf{x} = (2, 1.8)$.

Thus for this optimal solution, x_1 is already an integer, so we branch on x_2.

According to $(*)$, x_2 can take the values $0, 1, 2$; we consider these cases in turn:

- $x_2 = 0$

We solve the linear program obtained from ($\diamondsuit$) be considering $\mathbf{x} \in \mathbb{R}^2$ and by imposing the additional equality constraint $x_2 = 0$; the optimal value is $v_0 = 2$ and an optimal solution is $\mathbf{x} = (2, 0)$.

- $x_2 = 1$

  We solve the linear program obtained from ($\diamondsuit$) be considering $\mathbf{x} \in \mathbb{R}^2$ and by imposing the additional equality constraint $x_2 = 1$; the optimal value is $v_1 = 7$ and an optimal solution is $\mathbf{x} = (2, 1)$.

- $x_2 = 2$

  We solve the linear program obtained from ($\diamondsuit$) be considering $\mathbf{x} \in \mathbb{R}^2$ and by imposing the additional equality constraint $x_2 = 2$; the optimal value is $v_2 = 10$ and an optimal solution is $\mathbf{x} = (0, 2)$.

We also see that $f_0 \le 2, f_1 \le 7$ and (since $(0, 2)$ is an integral solution) $10 \le f_2$; thus f_2 exceeds f_0 and f_1. Thus we `prune` S_0 and S_1.

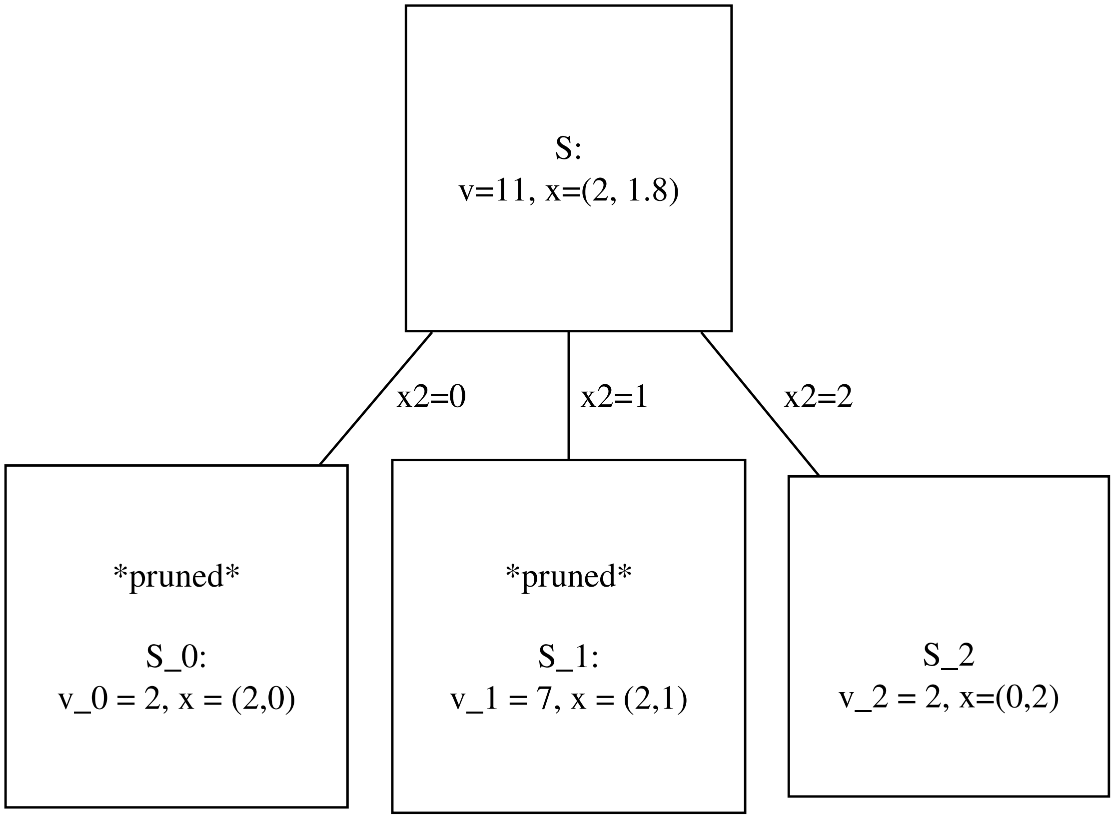
Moreover, since the optimal solution v_2 is actually integral, we find $f_2 = v_2$.

Thus $f_2 = 10$ is the optimal value for ($\diamondsuit$) and an optimal integral solution is $\mathbf{x} = (x_1, x_2) = (0, 2)$.

One often presents this algorithm via a tree diagram, like the following:

[6]:
```python
from graphviz import Graph

## https://www.graphviz.org/
## https://graphviz.readthedocs.io/en/stable/index.html

dot = Graph('bb1')

dot.node('S','S:\nv=11, x=(2, 1.8)',shape="square")

dot.node('S0','*pruned*\n\nS_0:\nv_0 = 2, x = (2,0)',shape="square")
dot.node('S1','*pruned*\n\nS_1:\nv_1 = 7, x = (2,1) ',shape="square")
dot.node('S2','\n\nS_2\nv_2 = 2, x=(0,2)',shape="square")


dot.edge('S','S0','x2=0')
dot.edge('S','S1','x2=1')
dot.edge('S','S2','x2=2')

dot
```

[6]:



Now let's consider a more elaborate example.

3 Example

`$( )` `$` `maximize` $f(\mathbf{x}) = \begin{bmatrix}10 & 7 & 4 & 3 & 1 & 0\end{bmatrix} \cdot \mathbf{x}$

subject to: $\mathbf{x} = \begin{bmatrix}x_1 & x_2 & x_3 & x_4 & x_5 & x_6\end{bmatrix}^T \in \mathbb{R}^5$, $\mathbf{x} \geq \mathbf{0}$,

$x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}$

and $A\mathbf{x} \leq \mathbf{b}$

where $A = \begin{bmatrix} 2 & 6 & 1 & 0 & 0 & 1 \\ 1 & 0 & 2 & -3 & 1 & -1 \\ 2 & -3 & 4 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ 3 \\ 3 \end{bmatrix}$.

Notice that we *aren't* imposing any integral condition on x_6, but we require that $x_i \in \mathbb{Z}$ for $1 \leq i \leq 5$, and even more: these coordinates may only take the value 0 or 1.

The procedure described above can (with perhaps some minor adaptations) be applied to this problem, as we now describe. Note that – unlike the previous example – we now must iterate our procedure.

(A)

We begin by finding an optimal solution to the linear program obtained from (♡) by replacing the condition

$x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}$ with the condition $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T \leq \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$

(note that the variables are already required to be non-negative by other conditions in (♡)).

The optimal value of the objective function is $v_A = 20.21$ and an optimal solution is $\mathbf{x}_A = (1, 0.72, 0.49, 1, 0.21, 0.19)$

Of the non-integer coordinates of $\mathbf{x}^*$, the one closest to an integer is $x_5 = .21$.

Branch on (A) with x_5:

(B): branching from **(A)** with $x_5 = 0$

Find an optimal solution to the linear program of **(A)** with the additional equality constraint $x_5 = 0$. The optimal value is $v_B = 19.44$ and an optimal solution is $\mathbf{x}_B = (1.00, 0.78, 0.33, 0.89, 0.00, 0.00)$.

(C): branching from **(A)** with $x_5 = 1$

Find an optimal solution to the linear program of **(A)** with the additional equality constraint $x_5 = 1$. The optimal value is $v_C = 19.97$ and an optimal solution is $\mathbf{x}_C = (1.00, 0.70, 0.27, 1.00, 1.00, 0.55)$.

We must now branch off of both **(B)** and **(C)**. We'll begin with **(C)**, and branch on x_3:

(D): branching from **(C)** with $x_3 = 0$.

Find an optimal solution to the linear program of **(C)** with equality constraints $x_5 = 1$, $x_3 = 0$. The optimal value is $v_D = 19.83$ and an optimal solution is $\mathbf{x}_D = (1.00, 0.83, 0.00, 1.00, 1.00, 0.00)$.

(E): branching from **(C*)** with $x_3 = 1$.

We try to find an optimal solution to the linear program of **(C)** with equality constraints $x_5 = 1$, $x_3 = 1$. But there are no feasible points.

So we may **Prune (E)**.

We now branch on **(D)** with x_2.

(F): branching from **(D)** with $x_2 = 0$.

Find an optimal solution to the linear program of **(D)** with equality constraints $x_5 = 1$, $x_3 = 0$, $x_2 = 0$. The optimal value is $v_F = 11.0$ and an optimal solution is $\mathbf{x}_F = (1.00, 0.00, 0.00, 0.00, 1.00, 3.92)$.

Observe that $\mathbf{x}_F$ is an integral solution (we have no integrality requirement for x_6) so we can stop examining this branch of the tree.

(G): branching from **(D)** with $x_2 = 1$.

Find an optimal solution to the linear program of **(D)** with equality constraints $x_5 = 1$, $x_3 = 0$, $x_2 = 1$. The optimal value is $v_G = 16$ and an optimal solution is $\mathbf{x}_G = (0.50, 1.00, 0.00, 1.00, 1.00, 0.00)$

We are now done branching "below **(C)**", and so we return and **branch on (B)** with x_4:

(H): branching from **(B)** with $x_4 = 0$.

Find an optimal solution to the linear program of **(B)** with equality constraints $x_5 = 0$, $x_4 = 0$. The optimal value is $v_H = 13.73$ and an optimal solution is $\mathbf{x}_H = (1.00, 0.27, 0.45, 0.00, 0.00, 2.91)$

(I): branching from **(B)** with $x_4 = 1$.

Find an optimal solution to the linear program of **(B)** with equality constraint $x_5 = 0$, $x_4 = 1$. The optimal value is $v_I = 19.4$ and an optimal solution is $\mathbf{x}_I = (1.00, 0.80, 0.20, 1.00, 0.00, 0.00)$

Now **branch on (I)** with x_2:

(J): branching from **(I)** with $x_2 = 0$.

Find an optimal solution to the linear program of **(I)** with equality constraints $x_5 = 0$, $x_4 = 1$, $x_2 = 0$. The optimal value is $v_J = 13$ and an optimal solution is $\mathbf{x}_J = (1.00, 0.00, 0.00, 1.00, 0.00, 2.26)$

This is an integer point, and 13 is larger than the objective function value for the point from **(F)**, so we **Prune (F)**.

(K): branching from **(I)** with $x_2 = 1$.

Find an optimal solution to the linear program of **(I)** with equality constraints $x_5 = 0$, $x_4 = 1$, $x_2 = 1$. The optimal value is $v_K = 15$ and an optimal solution is $\mathbf{x}_K = (0.50, 1.00, 0.00, 1.00, 0.00, 0.00)$

Now **branch from (G)** with x_1:

(L): branching from **(G)** with $x_1 = 0$

Find an optimal solution to the linear program of **(G)** with equality constraints $x_5 = 1$, $x_3 = 0$, $x_2 = 1$, $x_1 = 0$. The optimal value is $v_L = 11$ and an optimal solution is $\mathbf{x}_L = (0.00, 1.00, 0.00, 1.00, 1.00, 0.53)$

This is an integer solution **prune (L)** since the integer solution from **(J)** has an larger objective function value than 11.

(M): branching from **(G)** with $x_1 = 1$

Try to find an optimal solution to the linear program of **(G)** with equality constraints $x_5 = 1$, $x_3 = 0$, $x_2 = 1$, $x_1 = 1$. This linear program is infeasible.

Finally, **branch from (K)** with x_1

(N): branching from **(K)** with $x_1 = 0$

Find an optimal solution to the linear program of **(K)** with equality constraints $x_5 = 0$, $x_4 = 1$, $x_2 = 1$, $x_1 = 0$. The optimal value is $v_N = 14$ and an optimal solution is $\mathbf{x}_N = (0.00, 1.00, 1.00, 1.00, 0.00, 0.00)$

This is an integer solution.

(O): branching from **(K)** with $x_1 = 1$

Try to find an optimal solution to the linear program of **(K)** with equality constraints $x_5 = 0$, $x_4 = 1$, $x_2 = 1$, $x_1 = 1$. This linear program is infeasible.

Finally,we **Prune (H) and (J)** since the objective function for the point in **(N)** exceeds an upper bound for the objective functions in **(H)** and in **(J)**.

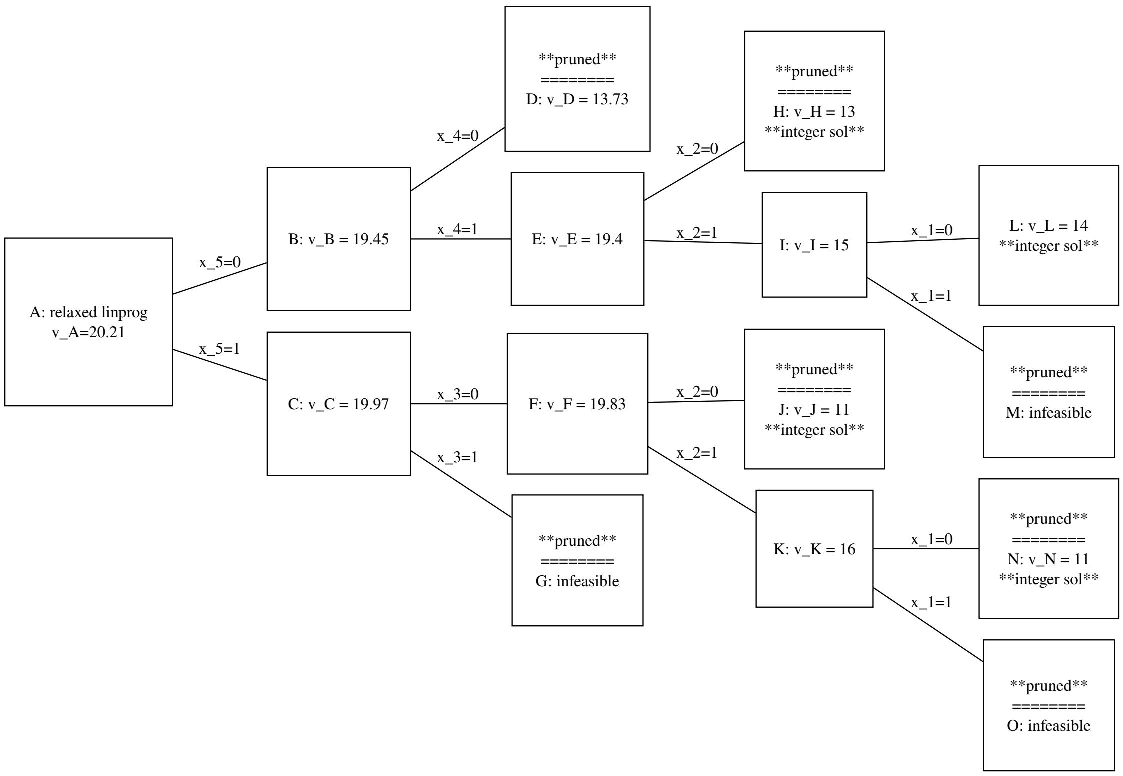
We are now **finished**; the integer point $\mathbf{x}_N = (0.00, 1.00, 1.00, 1.00, 0.00, 0.00)$ with $v_N = 14$ is an optimal point for $(\heartsuit)$.

We have included below `python` code used to find the optimal solutions/values given in **(A)**–**(O)**. Before giving that code, we also construct (via `graphviz`) the *tree* corresponding to the branch-and-bound just carried out; see below.

[7]:
```python
from graphviz import Graph

## https://www.graphviz.org/
## https://graphviz.readthedocs.io/en/stable/index.html

dot = Graph('bb1')
dot.attr(rankdir='LR')
dot.node('A','A: relaxed linprog\nv_A=20.21',shape="square")

dot.node('B','B: v_B = 19.45',shape="square")
dot.node('C','C: v_C = 19.97',shape="square")
dot.node('D','**pruned**\n========\nD: v_D = 13.73',shape="square")
dot.node('E','E: v_E = 19.4',shape="square")
dot.node('F','F: v_F = 19.83',shape="square")
dot.node('G','**pruned**\n========\nG: infeasible',shape="square")
dot.node('H','**pruned**\n========\nH: v_H = 13\n**integer␣
 ↪sol**',shape="square")
dot.node('I','I: v_I = 15',shape="square")
dot.node('J','**pruned**\n========\nJ: v_J = 11\n**integer␣
 ↪sol**',shape="square")
dot.node('K','K: v_K = 16',shape="square")
dot.node('L','L: v_L = 14\n**integer sol**',shape="square")
dot.node('M','**pruned**\n========\nM: infeasible',shape="square")
dot.node('N','**pruned**\n========\nN: v_N = 11\n**integer␣
 ↪sol**',shape="square")
dot.node('O','**pruned**\n========\nO: infeasible',shape="square")


dot.edge('A','B','x_5=0')
dot.edge('A','C','x_5=1')

dot.edge('B','D','x_4=0')
dot.edge('B','E','x_4=1')

dot.edge('C','F','x_3=0')
dot.edge('C','G','x_3=1')

dot.edge('E','H','x_2=0')
dot.edge('E','I','x_2=1')

dot.edge('F','J','x_2=0')
dot.edge('F','K','x_2=1')
```

```
dot.edge('I','L','x_1=0')
dot.edge('I','M','x_1=1')


dot.edge('K','N','x_1=0')
dot.edge('K','O','x_1=1')



dot
```

[7]:



[6]:

```
import numpy as np
from scipy.optimize import linprog

float_formatter = "{:.2f}".format
np.set_printoptions(formatter={'float_kind':float_formatter})

def sbv(index,size):
    return np.array([1.0 if i == index-1 else 0.0 for i in range(size)])

def eq_coords(idx,size):
```

```
    return np.array([sbv(i,size) for i in idx])

A = np.array([[2,6,1,0,0,1],[1,0,2,-3,1,-1],[2,-3,4,1,1,0],[1,1,1,1,-1,0]])
b = np.array([7,-1,3,3])
c = np.array([10,7,4,3,1,0])

bb = 5*[(0,1)] + [(0,None)]

resA = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb)
resB = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=np.array([sbv(5,6)]),b_eq=0)
resC = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=np.array([sbv(5,6)]),b_eq=1)

resD = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,3],6),b_eq=np.array([1,0]))
resE = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,3],6),b_eq=np.array([1,1]))

resF = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,3,2],6),b_eq=np.array([1,0,0]))
resG = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,3,2],6),b_eq=np.array([1,0,1]))

resH = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,4],6),b_eq=np.array([0,0]))
resI = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,4],6),b_eq=np.array([0,1]))

resJ = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,4,2],6),b_eq=np.array([0,1,0]))
resK = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,4,2],6),b_eq=np.array([0,1,1]))



resL = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,3,2,1],6),b_eq=np.array([1,0,1,0]))
resM = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,3,2,1],6),b_eq=np.array([1,0,1,1]))



resN = linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,4,2,1],6),b_eq=np.array([0,1,1,0]))
```

```
resO =␣
 ↪linprog((-1)*c,A_ub=A,b_ub=b,bounds=bb,A_eq=eq_coords([5,4,2,1],6),b_eq=np.
 ↪array([0,1,1,1]))


def report(res):
    return f"OK? {str(res.success):6s}  opt value: {(-1)*res.fun:.2f}  opt␣
 ↪point: {res.x}"

print("A - " + report(resA))
print("B - " + report(resB))
print("C - " + report(resC))
print("D - " + report(resD))
print("E - " + report(resE))
print("F - " + report(resF))
print("G - " + report(resG))
print("H - " + report(resH))
print("I - " + report(resI))
print("J - " + report(resJ))
print("K - " + report(resK))
print("L - " + report(resL))
print("M - " + report(resM))
print("N - " + report(resN))
print("O - " + report(resO))
```

```
A - OK? True    opt value: 20.21  opt point: [1.00 0.72 0.49 1.00 0.21 0.19]
B - OK? True    opt value: 19.44  opt point: [1.00 0.78 0.33 0.89 0.00 0.00]
C - OK? True    opt value: 19.97  opt point: [1.00 0.70 0.27 1.00 1.00 0.55]
D - OK? True    opt value: 19.83  opt point: [1.00 0.83 0.00 1.00 1.00 0.00]
E - OK? False   opt value: 13.62  opt point: [0.01 0.84 1.00 0.88 1.00 1.29]
F - OK? True    opt value: 11.00  opt point: [1.00 0.00 0.00 0.00 1.00 3.92]
G - OK? True    opt value: 16.00  opt point: [0.50 1.00 0.00 1.00 1.00 0.00]
H - OK? True    opt value: 13.73  opt point: [1.00 0.27 0.45 0.00 0.00 2.91]
I - OK? True    opt value: 19.40  opt point: [1.00 0.80 0.20 1.00 0.00 0.00]
J - OK? True    opt value: 13.00  opt point: [1.00 0.00 0.00 1.00 0.00 2.26]
K - OK? True    opt value: 15.00  opt point: [0.50 1.00 0.00 1.00 0.00 0.00]
L - OK? True    opt value: 11.00  opt point: [0.00 1.00 0.00 1.00 1.00 0.53]
M - OK? False   opt value: 22.23  opt point: [1.00 1.00 0.00 1.41 1.00 3.04]
N - OK? True    opt value: 14.00  opt point: [0.00 1.00 1.00 1.00 0.00 0.00]
O - OK? False   opt value: 20.83  opt point: [1.00 1.00 0.21 1.00 0.00 0.68]
```

4 Postcript

It turns out that solving integer programming problems is *hard*. In fact, in computer science integer programming problems are in a class of problems called **NP Hard** problems – see the discussion here..

The algorithm we describe above is a type of `branch and bound algorithm`, which is a common approach. While our description gives pretty good evidence that this approach is effective, we haven't said anything e.g. about the `run time` of our algorithm, etc.

```
[ ]:
```