

# Characters of irreducible representations

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## Convolution

We write  $\mathbb{C}[G]$  for the space of functions on  $G$ , viewed as a permutation representation of  $G$  (and we suppress the notation for the homomorphism  $G \rightarrow \mathrm{GL}(\mathbb{C}[G])$ ).

For functions  $f_1, f_2 \in \mathbb{C}[G]$ , we define their *convolution* by the formula

$$(f_1 \star f_2)(x) = \sum_{yz=x} f_1(y)f_2(z).$$

If  $V$  is a  $G$ -representation and  $f \in \mathbb{C}[G]$ , we define

$$f \star v = \sum_{g \in G} f(g)gv$$

for  $v \in V$ .

### Remark:

1. For the basis elements  $\delta_g \in \mathbb{C}[G]$  (i.e. the *Dirac functions*), we have

$$\delta_g \star \delta_h = \delta_{gh}.$$

2. The action of  $G$  on  $\mathbb{C}[G]$  can be described by

$$gf = \delta_g \star f$$

for  $g \in G$  and  $f \in \mathbb{C}[G]$ .

3. Viewing  $\mathbb{C}[G]$  as a  $G$ -representation, the two notions of  $\star$  just introduced actually coincide:

$$f_1 \star f_2 = \sum_{g \in G} f_1(g)\delta_g \star f_2.$$

4. The product  $\star$  makes  $\mathbb{C}[G]$  into a *ring* (in fact, a  $\mathbb{C}$ -algebra) and  $V$  into a  $\mathbb{C}[G]$ -module. Mostly we won't use this fact - at least explicitly - in these notes.

5. Let  $W \subseteq \mathbb{C}[G]$  be an invariant subspace. For any  $f \in \mathbb{C}[G]$ , we have

$$f \star f' \in W \quad \forall f' \in W.$$

6. The element  $\delta_1$  acts as the identity for the  $\star$  operation. Namely, for  $f \in \mathbb{C}[G]$

$$f \star \delta_1 = \delta_1 \star f.$$

This follows easily from the fact that  $\delta_1 \star \delta_g = \delta_g \star \delta_1 = \delta_g$  for all  $g \in G$ .

## Isotypic decomposition

Let  $V$  be a  $G$ -representation and let  $L$  be an irreducible  $G$ -representation.

Consider the set  $\mathcal{S}$  of all invariant subspaces  $S \subseteq V$  for which  $S \simeq L$  as  $G$ -representation.

Set

$$W = \sum_{S \in \mathcal{S}} S;$$

then  $W$  is an invariant subspace of  $V$ .

**Proposition:**  $W$  is isotypic in the sense that any irreducible invariant subspace of  $W$  is isomorphic (as  $G$ -representation) to  $L$ .

Moreover,  $[V/W : L] = 0$ .

*You will prove this in homework.*

We write  $V_{(L)}$  for the invariant subspace  $W$ .

You will also prove:

**Proposition:** If  $L_1, L_2, \dots, L_r$  is a complete set of non-isomorphic irreducible representations of  $G$ , then

$$V = V_{(L_1)} \oplus V_{(L_2)} \oplus \dots \oplus V_{(L_r)}.$$

## Results about the characters of the irreducible representations

### Investigation of certain idempotent elements in $\mathbb{C}[G]$ .

Let  $L$  be an irreducible representation of  $G$  and let  $W_1 = \mathbb{C}[G]_{(L)}$ .

Use complete reducibility to write

$$\mathbb{C}[G] = W_1 \oplus W_2$$

for some invariant subspace  $W_2 \subset \mathbb{C}[G]$ .

Note that  $[W_2 : L] = 0$  by construction.

We now write

$$\delta_1 = e_1 + e_2 \quad \text{for } e_1 \in W_1 \text{ and } e_2 \in W_2.$$

**Proposition:** For  $w_1 \in W_1$  and  $w_2 \in W_2$  we have

$$\begin{aligned} e_1 \star w_1 &= w_1, & e_1 \star w_2 &= 0, \\ e_2 \star w_1 &= 0, & \text{and } e_2 \star w_2 &= w_2. \end{aligned}$$

**Proof:** Fix  $w_2 \in W_2$ . We define a mapping  $\phi : W_1 \rightarrow W_2$  by the rule  $\phi(w_1) = w_1 \star w_2$ .

We note that  $\phi$  is a homomorphism of  $G$ -representations. Indeed, recall the action of  $g \in G$  on  $\mathbb{C}[G]$  is the same as that of  $\delta_g \star$ . Now

$$\phi(\delta_g \star w_1) = (\delta_g \star w_1) \star w_2 = \delta_g \star (w_1 \star w_2) = \delta_g \star \phi(w_1).$$

Since  $W_1$  is  $L$ -isotypic and since  $[W_2 : L] = 0$ , the mapping  $\phi$  must be 0.

Now conclude that

$$0 = w_1 \star w_2 \quad \forall w_1 \in W_1, w_2 \in W_2.$$

A similar argument shows that

$$0 = w_2 \star w_1 \quad \forall w_1 \in W_1, w_2 \in W_2.$$

(For  $w_1 \in W_1$ , one define a homomorphism  $\psi : W_2 \rightarrow W_1$  by the rule  $\psi(w_2) = w_2 \star w_1$ . As before, one argues that  $\psi = 0 \dots$ )

Now notice for  $w_1 \in W_1$  that

$$w_1 = \delta_1 \star w_1 = (e_1 + e_2) \star w_1 = e_1 \star w_1 + e_2 \star w_1 = e_1 \star w_1$$

since  $e_2 \in W_2 \implies e_2 \star w_1 = 0$  by the preceding results. This proves that  $e_1 \star w_1 = w_1$  for all  $w_1 \in W_1$

Similarly, for  $w_2 \in W_2$  we have

$$w_2 = \delta_1 \star w_2 = (e_1 + e_2) \star w_2 = e_1 \star w_2 + e_2 \star w_2 = e_2 \star w_2$$

since  $w_2 \in W_2 \implies e_1 \star w_2 = 0$  by the preceding results.

This completes the proof.

As an immediate consequence, we get:

**Corollary:**

- $e_1 \star e_1 = e_1$
- $e_2 \star e_2 = e_2$
- $e_1 \star e_2 = e_2 \star e_1 = 0$ .

We can actually find a *formula* expressing  $e_1$  in the basis  $\{\delta_g\}$  for  $\mathbb{C}[G]$ :

**Proposition:** Let  $W_1 = \mathbb{C}[G]_{(L)}$  and suppose that  $\mathbb{C}[G] = W_1 \oplus W_2$  for an invariant subspace  $W_2$  as before. Write  $\delta_1 = e_1 + e_2$  with  $e_i \in W_i$ . and let  $\chi$  be the *character* of  $W_1$ . We have

$$e_1 = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) \delta_g.$$

**Proof:** Fix  $x \in G$  and define

$$\Phi : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$$

by the rule

$$\Phi(f) = \delta_{x^{-1}} \star e_1 \star f.$$

We are going to compute the *trace* of  $\Phi$  in two different ways.

First, Note that since  $e_1 \star w_1 = w_1$  for each  $w_1 \in W_1$ , we see that  $\Phi|_{W_1}$  is given

$$w \mapsto \delta_{x^{-1}} \star w$$

. Thus  $\text{tr}(\Phi|_{W_1})$  is given by  $\chi(x^{-1})$ .

Since  $e_1 \star W_2 = 0$ , we conclude that  $\Phi|_{W_2} = 0$  so that

$$\text{tr}(\Phi) = \text{tr}(\Phi|_{W_1}) \oplus \text{tr}(\Phi|_{W_2}) = \text{tr}(\Phi|_{W_1}) = \chi(x^{-1}),$$

On the other hand, let us express  $e_1$  in the basis  $\{\delta_g\}$  of  $\mathbb{C}[G]$ :

$$e_1 = \sum_{g \in G} \lambda_g \delta_g \quad (\lambda_g \in \mathbb{C}).$$

Let us examine the mapping

$$\theta_{x^{-1}g} : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$$

given by

$$w \mapsto \delta_{x^{-1}g} \star w.$$

Recall that  $\mathbb{C}[G]$  is the permutation representation corresponding to the action of  $G$  on itself by left multiplication. We have seen that the trace of the action of an element of  $G$  is the number of fixed points for that action. We conclude that the trace of  $\theta_{x^{-1}g}$  is  $|G|$  if  $x = g$  and otherwise is 0.

Now, the mapping  $\Phi$  is given by

$$\Phi(w) = \delta_{x^{-1}} \star \left( \sum_{g \in G} \lambda_g \delta_g \right) \star w = \sum_{g \in G} \lambda_g \delta_{x^{-1}g} \star w = \sum_{g \in G} \lambda_g \theta_{x^{-1}g}(w).$$

Since the trace is a linear operator, we conclude that

$$\text{tr}(\Phi) = \sum_{g \in G} \lambda_g \text{tr}(\theta_{x^{-1}g}) = \lambda_x |G|.$$

Now comparing our two computations of  $\text{tr}(\Phi)$  we get the formula

$$\lambda_x |G| = \chi(x^{-1})$$

i.e.  $\lambda_x = \chi(x^{-1})/|G|$ .

But then

$$e_1 = \sum_{g \in G} \lambda_g \delta_g = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) \delta_g$$

as required. This completes the proof.

**Remark:** Since  $G$  is a finite group, the eigenvalues of the operation of  $g \in G$  on any  $G$ -representation are roots of unity  $\zeta$ . Notice that  $\bar{\zeta} = \zeta^{-1}$  for any root of unity. In particular, if  $\chi$  is the character of a representation of  $G$ , we have  $\chi(g^{-1}) = \overline{\chi(g)}$ .

**Corollary:** Let  $\chi$  be the character of  $W_1 = \mathbb{C}[G]_{(L)}$ . Then

$$\langle \chi, \chi \rangle = \chi(1).$$

**Proof:** Note that the Proposition allows us to calculate:

$$e_1 \star e_1 = \frac{1}{|G|} \sum_{x, g \in G} \chi(g^{-1}) \chi(x^{-1}) \delta_{gx}.$$

The coefficient of  $\delta_1$  in this expression is precisely

$$\frac{1}{|G|^2} \sum_{g \in G} \chi(g^{-1}) \chi(g) = \frac{1}{|G|^2} \sum_{g \in G} \chi(g) \overline{\chi(g)} = \frac{1}{|G|} \langle \chi, \chi \rangle.$$

On the other hand,  $e_1 = e_1 \star e_1$  and the coefficient of  $\delta_1$  in the expression for  $e_1$  is  $\chi(1^{-1}) = \chi(1)$ .

Thus we see that  $\chi(1) = \langle \chi, \chi \rangle$  as required.

We have so far elided the *multiplicities*  $[\mathbb{C}[G] : L]$  for irreducible representations  $L$ . We are going to state the result here (and maybe prove it later).

**Theorem:** For an irreducible representation  $L$  of  $G$ , the multiplicity  $[\mathbb{C}[G] : L]$  is given by

$$[\mathbb{C}[G] : L] = \dim_{\mathbb{C}} L.$$

**Remark:** If  $\chi, \psi$  are the characters of representations of  $G$ , then  $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle$ .

Indeed,

$$\begin{aligned} \langle \chi, \psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)} = \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{x \in G} \chi(x^{-1}) \psi(x) \\ &= \langle \psi, \chi \rangle. \end{aligned}$$

Now we are able to prove that main Theorem which shows that the characters of the irreducible representations form an *orthonormal set*.

**Theorem:** Let  $U, V$  be irreducible representations of  $G$  with characters  $\chi, \psi$  respectively. Then

$$\langle \chi, \chi \rangle = 1 \quad \text{and} \quad \langle \chi, \psi \rangle = 0.$$

**Proof:** Let  $W = \mathbb{C}[G]_{(U)}$  be the  $U$ -isotypic part of  $\mathbb{C}[G]$  and let  $\chi_W$  be the character of  $W$ .

The preceding Theorem tells us that  $\chi_W = m\chi$  where  $m = \dim U$ .

Now, the preceding Corollary tells us that

$$\langle \chi_W, \chi_W \rangle = \chi_W(1).$$

Note that

$$\langle \chi_W, \chi_W \rangle = \langle m\chi, m\chi \rangle = m^2 \langle \chi, \chi \rangle.$$

On the other hand,

$$\chi_W(1) = m\chi(1) = m^2.$$

Thus we find  $m^2 \langle \chi, \chi \rangle = m^2$ ; since  $m \neq 0$ , conclude  $\langle \chi, \chi \rangle = 1$  as required.

Now let  $Y = \mathbb{C}[G]_{(U)} + \mathbb{C}[G]_{(V)}$  be the sum of the isotypic components.

Note that  $\chi_Y = m\chi + n\psi$  where  $m = \dim U$  and  $n = \dim V$ . Now we have

$$\chi_Y(1) = \langle \chi_Y, \chi_Y \rangle.$$

On the one hand,

$$\chi_Y(1) = m\chi(1) + n\psi(1) = m^2 + n^2.$$

And on the other hand, the preceding corollary shows that

$$\begin{aligned} \langle \chi_Y, \chi_Y \rangle &= \langle m\chi + n\psi, m\chi + n\psi \rangle \\ &= m^2 \langle \chi, \chi \rangle + n^2 \langle \psi, \psi \rangle + 2mn \langle \chi, \psi \rangle \\ &= m^2 + n^2 + 2mn \langle \chi, \psi \rangle. \end{aligned}$$

Thus we find

$$m^2 + n^2 = m^2 + n^2 + 2mn \langle \chi, \psi \rangle$$

so that indeed  $\langle \chi, \psi \rangle = 0$ . This completes the proof.

**Remark:** In order to complete the proof that the irreducible characters form an orthonormal *basis* for the space of class functions on  $G$ , we still need to prove that the number of distinct irreducible representations is equal to the number of conjugacy classes in  $G$ .

## Bibliography