ProblemSet 1 – Linear algebra and representations **Solutions**

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F denotes an *algebraically closed field of characteristic 0*. If you like, you can suppose that $F = \mathbf{C}$ is the field of complex numbers.

1. Let V be a finite dimensional vector space over the field F. Suppose that $\phi, \psi : V \to V$ are linear maps. Let $\lambda \in F$ be an eigenvalue of ϕ and write W for the λ-eigenspace of ϕ; i.e.

$$W = \{v \in V \mid \phi(v) = \lambda v\}.$$

If $\phi\psi = \psi\phi$ show that W is *invariant* under ψ – i.e. show that $\psi(W) \subseteq W$.

Solution:
Let $w \in W$. We must show that $x = \psi(w) \in W$. To do this, we must establish that $x = \psi(w)$ is a λ-eigenvector for ϕ.

We have

$$\begin{aligned} \phi(x) &= \phi(\psi(x)) & \\ &= \psi(\phi(w)) & \text{since } \phi \circ \psi = \psi \circ \phi \\ &= \psi(\lambda w) & \text{since } w \text{ is a } \lambda\text{-eigenvector} \\ &= \lambda\psi(w) & \text{since } \psi \text{ is linear} \\ &= \lambda x & \end{aligned}$$

This completes the proof.

2. Let $n \in \mathbf{N}$ be a non-zero natural number, and let V be an n dimensional F-vector space with a given basis $e_1, e_2, \cdots, e_n$.

Consider the linear transformation $T : V \to V$ given by the rule

$$Te_i = e_{i+1 \pmod n}.$$

In other words

$$Te_i = \begin{cases} e_{i+1} & i < n \\ e_1 & i = n \end{cases}.$$

a. Show that T is *invertible* and that $T^n = \mathrm{id}_V$.

To check that $T^n = \mathrm{id}_V$, we check that $T^n(e_i) = e_i$ for $1 \le i \le n$.

From the definition, it follows by induction on the natural number m that

$$T^m(e_i) = e_{i+m \pmod n}.$$

Thus $T^n(e_i) = e_{i+n \pmod n} = e_i$. Since this holds for every i, conclude $T^n = \mathrm{id}_V$.

Now T is invertible since its inverse is given by T^{n-1}.

b. Consider the vector $v_0 = \sum_{i=1}^{n} e_i$. Show that v_0 is a 1-eigenvector for T.

We compute

$$\begin{aligned}
T(v_0) = T\left(\sum_{i=1}^{n} e_i\right) &= \sum_{i=1}^{n} T(e_i) \\
&= \sum_{i=1}^{n} e_{i+1 \pmod n} \\
&= \sum_{j=2}^{n+1} e_{j \pmod n} && \text{(let } j = i+1\text{)} \\
&= \sum_{j=1}^{n} e_{j \pmod n} = v_0
\end{aligned}$$

Thus $T(v_0) = v_0$ so indeed v_0 is a 1-eigenvector.

Let $\zeta \in F$ be a primitive n-th root of unity. (e.g. if you assume $F = \mathbf{C}$, you may as well take $\zeta = e^{2\pi i/n}$).

c. Let $v_1 = \sum_{i=1}^{n} \zeta^i e_i$. Show that v_1 is a ζ^{-1}-eigenvector for T.

We compute

$$\begin{aligned}
T(v_1) &= T\left(\sum_{i=1}^{n} \zeta^i e_i\right) \\
&= \sum_{i=1}^{n} \zeta^i T(e_i) \\
&= \sum_{i=1}^{n} \zeta^i e_{i+1 \pmod n} \\
&= \sum_{j=2}^{n+1} \zeta^{j-1} e_{j \pmod n} && \text{(let } j = i+1\text{)} \\
&= \zeta^{-1} \sum_{j=2}^{n+1} \zeta^j e_{j \pmod n} \\
&= \zeta^{-1} \sum_{j=1}^{n} \zeta^j e_{j \pmod n} && \text{(since } \zeta^j = \zeta^{j \pmod n}\ \forall j\text{)} \\
&= \zeta^{-1} v_1
\end{aligned}$$

Thus $T(v_1) = \zeta^{-1} v_1$ so indeed v_0 is a ζ^{-1}-eigenvector.

d. More generally, let $0 \le j < n$ and let

$$v_j = \sum_{i=1}^{n} \zeta^{ij} e_i.$$

Show that v_j is a ζ^{-j}-eigenvector for T.

The calcuation in the solution to part (c) is valid for *any* n-th root of unity unity ζ. Applying this calculation for ζ^j shows that v_j is a ζ^{-j}-eigenvector for T as required.

e. Conclude that $v_0, v_1, \cdots, v_{n-1}$ is a basis of V consisting of *eigenvectors* for T, so that T is *diagonalizable*.

Hint: You need to use the **fact** that eigenvectors for distinct eigenvalues are *linearly independent*.

What is the *matrix* of T in this basis?

Since eigenvectors for distinct eigenvalues are linearly independent, conclude that the vectors $\mathcal{B} = \{v_0, v_1, \cdots, v_{n-1}\}$ are linearly independent. Since there n vectors in $\mathcal{B}$ and since $\dim V = n$, conclude that $\mathcal{B}$ is a *basis* for V.

The matrix of T in the basis $\mathcal{B}$ is given by

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \zeta^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta^{-n+1} \end{bmatrix}$$

(This form explains why an $n \times n$ matrix M is *diagonalizable* iff F^n has a basis of eigenvectors for M).

3. Let $G = \mathbb{Z}/3\mathbb{Z}$ be the additive group of order 3, and let ζ be a primitive 3rd root of unity in F.

   To define a representation $\rho : G \to \mathrm{GL}_n(F)$, it is enough to find a matrix $M \in \mathrm{GL}_n(F)$ with $M^3 = 1$; in turn, M determines a representation ρ by the rule $\rho(i + 3\mathbb{Z}) = M^i$.

   Consider the *representation* $\rho_1 : G \to \mathrm{GL}_3(F)$ given by the matrix

$$\rho_1(1 + 3\mathbb{Z}) = M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}$$

   and consider the representation $\rho_2 : G \to \mathrm{GL}_3(F)$ given by the matrix

$$\rho_2(1 + 3\mathbb{Z}) = M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   Show that the *representations* ρ_1 and ρ_2 are *equivalent* (alternative terminology: are *isomorphic*). In other words, find a linear bijection $\Phi : F^3 \to F^3$ with the property that

$$\Phi(\rho_2(g)v) = \rho_1(g)\Phi(v)$$

   for every $g \in G$ and $v \in F^3$.

   Hint: First find a basis of F^3 consisting of eigenvectors for the matrix M_2.

   The matrix M_1 is *diagonal*, which is to say that the *standard basis vectors* $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are *eigenvectors* for M_1 with respective eigenvalues $1, \zeta, \zeta^2$.

   By the work in problem 2, we see that

$$v_1 = e_1 + e_2 + e_3, \quad v_2 = e_1 + \zeta e_2 + \zeta^2 e_3, \quad v_3 = e_1 + \zeta^2 e_2 + \zeta e_3$$

   are eigenvectors for M_2 with respective eigenvalues $1, \zeta^2, \zeta$.

   Now let $\Phi : F^3 \to F^3$ be the linear transformation for which

$$\Phi(e_1) = v_1, \quad \Phi(e_2) = v_3, \quad \Phi(e_3) = v_2$$

   .

   We claim that Φ defines an isomorphism of G-representations

$$(\rho_1, F^3) \xrightarrow{\sim} (\rho_2, F^3).$$

   We must check that $\Phi(\rho_1(g)v) = \rho_2(g)\Phi(v)$ for all $g \in G$ and all $v \in F^3$.

Since G is *cyclic* it suffices to check that

$$(\clubsuit) \quad \Phi(M_1 v) = M_2 \Phi(v) \quad \forall v \in F^3$$

.

(*Indeed*, $(\clubsuit)$ amounts to "checking on a generator". If $(\clubsuit)$ holds then for every natural number i a straightforward induction argument shows for every $v \in F^3$ that

$$\begin{aligned}
\Phi(\rho_1(i+3\mathbb{Z})v) &= \Phi(\rho_1(1+3\mathbb{Z})^i v) \\
&= \Phi({M_1}^i v) \\
&= {M_2}^i \Phi(v) \\
&= \rho_2(1+3\mathbb{Z})^i \Phi(v) \\
&= \rho_2(i+3\mathbb{Z})\Phi(v)
\end{aligned}$$

)

In turn, it suffices to verify the $(\clubsuit)$ holds for the basis vectors e_1, e_2, e_3 for $V = F^3$.

Since e_1 and v_1 are 1-eigenvectors for M_1 resp. M_2, we have

$$\Phi(M_1 e_1) = \Phi(e_1) = v_1 = M_2 v_1.$$

Since e_2 and v_3 are ζ-eigenvectors for M_1 resp. M_2, we have

$$\Phi(M_1 e_2) = \Phi(\zeta e_2) = \zeta \Phi(e_2) = \zeta v_3 = M_2 v_3.$$

Since e_3 and v_2 are ζ^2-eigenvectors for M_1 resp. M_2, we have

$$\Phi(M_1 e_3) = \Phi(\zeta^2 e_3) = \zeta^2 \Phi(e_3) = \zeta^2 v_2 = M_2 v_2.$$

Thus $(\clubsuit)$ holds and the proof is complete.

Alternatively, note that the matrix of Φ in the standard basis is given by

$$[\Phi] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{bmatrix}$$

Now, to prove that $\Phi \circ \rho_1(g) = \rho_2(g) \circ \Phi$, it suffices to check that $M_2[\Phi] = [\Phi]M_1$ i.e. that

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}$$

IN fact, both products yield the matrix

$$\begin{bmatrix} 1 & \zeta & \zeta^2 \\ 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \end{bmatrix}$$

4. Let V be a n dimensional F-vector space for $n \in \mathbb{N}$.

   Let $\mathrm{GL}(V)$ denote the group

   $$\mathrm{GL}(V) = \{\text{all invertible } F\text{-linear transformations } \phi : V \to V\}$$

   where the group operation is *composition* of linear transformations.

Recall that $\mathrm{GL}_n(F)$ denotes the group of all invertible $n \times n$ matrices.

If $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$ is a choice of basis, show that the assignment $\phi \mapsto [\phi]_{\mathcal{B}}$ determines an isomorphism

$$\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_n(F).$$

Here $[\phi]_{\mathcal{B}} = [M_{ij}]$ denotes the *matrix* of ϕ in the basis $\mathcal{B}$ defined by equations

$$\phi(b_i) = \sum_{k=1}^{n} M_{ki} b_k.$$

Lets write Φ for the mapping

$$\Phi : \mathrm{GL}(V) \to \mathrm{GL}_n(F)$$

defined above.

An important property – proved in *Linear Algebra* – is that for $\phi, \psi : V \to V$ we have

$$(\heartsuit) \quad [\phi \circ \psi]_{\mathcal{B}} = [\phi]_{\mathcal{B}} \cdot [\psi]_{\mathcal{B}}.$$

In words: "once you choose a basis, composition of linear transformations corresponds to multiplication of the corresponding matrices".

Now, since the matrix of the endomorphism $\phi : V \to V$ is equal to the identity matrix $\mathbf{I}_n$ if and only if $\phi = \mathrm{id}_V$, $(\heartsuit)$ shows at once that a linear transformation $\phi : V \to V$ is invertible if and only if $[\phi]_{\mathcal{B}}$ is an invertible matrix.

This confirms that Φ is indeed a group homomorphism.

To show that Φ is an *isomorphism*, we exhibit its inverse. Namely, we defined a group homomorphism

$$\Psi : \mathrm{GL}_n(F) \to \mathrm{GL}(V)$$

and check that Ψ is the inverse to Φ.

TO define Ψ, we introduce the linear isomorphism $\beta : F^n \to V$ defined by the rule

$$\beta \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i.$$

For an invertible matrix M, we define

$$\Psi(M) : V \to V$$

by the rule

$$\Psi(M)(v) = \beta M \cdot \beta^{-1} v$$

If $M_1, M_2 \in \mathrm{GL}_n(F)$ then for every $v \in V$ we have

$$\begin{aligned} \Psi(M_1 M_2)v &= \beta M_1 M_2 \cdot \beta^{-1} v \\ &= \beta M_1 \beta^{-1} \beta M_2 \cdot \beta^{-1} v \\ &= \Psi(M_1)\Psi(M_2)v \end{aligned}$$

This confirms that Ψ is a *group homomorphism*.

It remains to observe that for $M \in \mathrm{GL}_n(F)$ we have

$$\Phi \circ \Psi(M) = M,$$

which amounts to the fact that M is the matrix of $\Psi(M)$, and we must observe for $g \in \mathrm{GL}(V)$ hat

$$\Psi \circ \Phi(g) = g$$

which amounts to the observation that the transformation $g : V \to V$ is determined by its effect on the basis vectors b_i and hence by the matrix $\Phi(g)$.

Bibliography