Representations and the symmetric group

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Irreducible representations of finite abelian groups

Let A be a finite abelian group, written *additively*.

We set $\widehat{A} = \operatorname{Hom}(A, \mathbb{C}^\times)$ for the set of all group homomorphisms $A \to \mathbb{C}^\times$. We can make $\widehat{A}$ into a group by declaring for $\phi, \psi \in \widehat{A}$ that

$$\phi \cdot \psi : A \to \mathbb{C}^\times$$

is the mapping $a \mapsto \phi(a)\psi(a)$.

Proposition: $\widehat{A}$ is an abelian group, and $\widehat{A} \simeq A$.

Sketch: When $A = \mathbb{Z}/n\mathbb{Z}$ is *cyclic* let $\zeta = \exp(2\pi i/n)$ be primitive nth root of unity.

For $j \in \mathbb{Z}/n\mathbb{Z}$, define $\phi_j \in \widehat{\mathbb{Z}/n\mathbb{Z}}$ by

$$\phi_j(i + n\mathbb{Z}_)) = \zeta^{ij}.$$

One checks that $j \mapsto \phi_j$ defines an isomorphism of groups

$$\mathbb{Z}/n\mathbb{Z} \to \widehat{\mathbb{Z}/n\mathbb{Z}}.$$

Now the Proposition follows by using that

$$\widehat{A \times B} \simeq \widehat{A} \times \widehat{B}$$

for abelian group A and B.

Observe that elements of $\widehat{A}$ determine 1-dimensional representations, which are necessarily irreducible. Since a 1-dimensional representation coincides with its trace, our results on the characters of irreducible representations imply that the functions $\mathbb{C}$-valued functions $\widehat{A}$ form an orthonormal basis for $\mathbb{C}[A]$.

Let us write $\widehat{A} = \{\phi_1, \phi_2, \cdots, \phi_n\}$ where $n = |A|$.

Given a function $f : A \to \mathbb{C}$, i.e. an element $f \in \mathbb{C}[A]$, we may write

$$f = \sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i.$$

The Fourier transform of f is the function

$$\hat{f} : \widehat{A} \to \mathbb{C}$$

given by

$$\hat{f}(\phi) = \langle f, \phi \rangle.$$

Remark: When $A = \mathbb{Z}/n\mathbb{Z}$, one often views the values $f(i)$ for $i + n\mathbb{Z}$ as *samples* of some periodic function of (say) a real variable t, viewed as *time*.

In that case, the domain of $\hat{f}$ is viewed as *frequency*.

Remark: In any event, for abelian A we have two natural bases for $\mathbb{C}[A]$: the functions δ_a for $a \in A$, and the functions $\widehat{A}$.

How to study $\mathbb{C}[G]$ for non-abelian G?

Idea: the *matrix coefficients* of linear representations define functions on G. If $\rho : G \to \mathrm{GL}(V)$ is a linear representation, let $b_1, \ldots, b_n$ be a basis of V and let $b_1^\vee, \cdots, b_n^\vee$ be the dual basis.

For $1 \le i, j \le n$ we get a function

$$\rho_{i,j} : G \to \mathbb{C}$$

by the rule

$$\rho_{i,j}(g) = b_i^\vee(\rho(g) b_j).$$

Claim: Let L be an irreducible representation and let $\mathbb{C}[G]_{(L)}$ be the isotypic component of the regular representation. Then the functions $\rho_{i,j}$ provide a *basis* for $\mathbb{C}[G]_{(L)}$ where ρ defines the irreducible representation *dual* to L.

We omit the proof, and refer the reader e.g.(James and Liebeck 2001) for the notion of the *dual* of a representation.

But recall that we know that $[\mathbb{C}[G]_{(L)} : L] = \dim L$ so that $\dim \mathbb{C}[G]_{(L)} = (\dim L)^2 = \dim \mathrm{End}(L)$.

Irreducible representations of the symmetric group

Recall that two elements $\sigma, \tau \in S_n$ are *conjugate* $\iff$ they have the same *cycle structure*. We state this in the following form:

Lemma: Conjugacy classes in S_n are in bijection with *partitions* $\lambda \vdash n$.

For example, there are 5 partitions of 4; they are $\lambda = (1,1,1,1)$, $\lambda = (2,2)$, $\lambda = (2,1,1)$, $\lambda = (3,1)$, and $\lambda = (4)$.

As a consequence, we know:

Proposition: Isomorphism classes of irreducible representations of S_n on complex vector spaces are in bijection with partitions $\lambda \vdash n$.

Here is a quick overview of some facts about the representations of S_n, without proofs:

In fact, for each partition $\lambda \vdash n$, there is a construction of an irreducible representation V_λ for S_n. To begin the construction, associate with λ the subgroup

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_r}$$

let $\Omega_\lambda = S_n / S_\lambda$ be the set of left cosets of S_λ in S_n.

Thus Ω_λ is a set on which the symmetric group S_n acts, and we may consider the *permutation representation* $\mathbb{C}[\Omega_\lambda]$, and for brevity we write $M^\lambda = \mathbb{C}[\Omega_\lambda]$.

One defines the *dominance ordering* on partitions of n by the rule

$$\lambda \trianglelefteq \mu$$

if and only if for each ℓ

$$\sum_{i=1}^{\ell} \lambda_i \le \sum_{i=1}^{\ell} \mu_i.$$

There is a labeling of irreducible representations of S_n by partition; let us write S^λ for the irreducible representation corresponding to λ. It is known as a *Specht representation*.

Theorem: $[M^\lambda : S^\lambda] = 1$ and $[M^\lambda : S^\mu] > 0 \implies \lambda \trianglelefteq \mu$.

As stated, the Theorem appears to depend on our knowledge of the irreducible representations, but in fact it gives us a way to define them.

For a fixed λ, consider all homomorphisms of S_n-representations

$$M^\mu \to M^\lambda \quad \text{for } \lambda \trianglelefteq \mu;$$

lets write $R^\lambda \subset M^\lambda$ for the *sum of the image of all of these homomorphisms.*

The Theorem implies that $M^\lambda / R^\lambda = S^\lambda$ is irreducible.

Here are some special cases:

- if $\lambda = (n)$ then $M^\lambda = S^\lambda = \mathbf{1}$ is the trivial representation.
- if $\lambda = (1, 1, \cdots, 1)$ them $S^\lambda =$ sgn is the (1-dimensional) *sign representation*.
- if $\lambda = (n-1, 1)$ then M^λ is the permutation representation $\mathbb{C}[\{1, 2, \cdots, n\}]$, and

  $$M^\lambda = S^\lambda \oplus \mathbf{1}$$

  so that dim $S^\lambda = n - 1$.

  Observe that indeed $(n-1, 1) \trianglelefteq (n)$.
- when $n = 3$, we saw previously that S_3 has 3 irreducible representations. They can be described as
  - $S^{(3)} = \mathbf{1}$ the trivial representation
  - $S^{(1,1,1)} =$ sgn the sign representation
  - $S^{(2,1)}$, an irreducible representation of dimension 2

Rank preferences

I want to describe in brief the ideas investigated in (Diaconis 1989), which amounts as an application of representation theory on the symmetric group to statistics.

First, an example

Suppose we were to ask people to rank their preferred ice cream flavors from the following ordered list:

- [`pistachio`, `chocolate`, `strawberry`, `vanilla`, `neopolitan`]

Numbering the flavors [`0`, `1`, `2`, `3`, `4`] in order, we can represent an individual's preference using a permutation $\sigma \in S_5$.

For example the preference list

- [`neopolitan`, `strawberry`, `chocolate`, `vanilla`, `pistachio`]

corresponds to the product of transposition $(04)(12)$.

So our survey data amounts to a list of elements $\sigma_1, \sigma_2, \ldots, \sigma_N \in S_5$.

Ranking in more generality

In general, n will denote the number of items to be ranked, and we assume given a *list* of ranking data:

$$\Sigma : \sigma_1, \sigma_2, \cdots, \sigma_m$$

The main statistic of interest is the *frequency function*

$$f = f_\Sigma : S_n \to \mathbb{C}$$

given for $\sigma \in S_n$ by the rule $f(\sigma) = \#\{\sigma_i \mid \sigma = \sigma_i\}$.

Idea: View f as a vector in the regular representation $\mathbb{C}[S_n]$. We want to understand how f decomposes in some natural descriptions of $\mathbb{C}[S_n]$.

Bibliography

Bibliography

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