ProblemSet 1 – Linear algebra and representations

George McNinch

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F denotes an *algebraically closed field of characteristic 0*. If you like, you can suppose that $F = \mathbf{C}$ is the field of complex numbers.

1. Let V be a finite dimensional vector space over the field F. Suppose that $\phi, \psi : V \to V$ are linear maps. Let $\lambda \in F$ be an eigenvalue of ϕ and write W for the λ-eigenspace of ϕ; i.e.

   $$W = \{v \in V \mid \phi(v) = \lambda v\}.$$

   If $\phi\psi = \psi\phi$ show that W is *invariant* under ψ – i.e. show that $\psi(W) \subseteq W$.

2. Let $n \in \mathbf{N}$ be a non-zero natural number, and let V be an n dimensional F-vector space with a given basis $e_1, e_2, \cdots, e_n$.

   Consider the linear transformation $T : V \to V$ given by the rule

   $$Te_i = e_{i+1 \pmod n}.$$

   In other words

   $$Te_i = \begin{cases} e_{i+1} & i < n \\ e_1 & i = n \end{cases}.$$

   a. Show that T is *invertible* and that $T^n = \mathrm{id}_V$.

   b. Consider the vector $v_0 = \displaystyle\sum_{i=1}^{n} e_i$. Show that v_0 is a 1-eigenvector for T.

   Let $\zeta \in F$ be a primitive n-th root of unity. (e.g. if you assume $F = \mathbf{C}$, you may as well take $\zeta = e^{2\pi i/n}$).

   c. Let $v_1 = \displaystyle\sum_{i=1}^{n} \zeta^i e_i$. Show that v_1 ~~is a ζ-eigenvector~~ is a ζ^{-1}-eigenvector for T.

   d. More generally, let $0 \le j < n$ and let

   $$v_j = \sum_{i=1}^{n} \zeta^{ij} e_i.$$

   Show that v_j ~~is a ζ^j-eigenvector~~ is a ζ^{-j}-eigenvector for T.

   e. Conclude that $v_0, v_1, \cdots, v_{n-1}$ is a basis of V consisting of *eigenvectors* for T, so that T is *diagonalizable*.

   Hint: You need to use the **fact** that eigenvectors for distinct eigenvalues are *linearly independent*.

   What is the *matrix* of T in this basis?

3. Let $G = \mathbb{Z}/3\mathbb{Z}$ be the additive group of order 3, and let ζ be a primitive 3rd root of unity in F.

   To define a representation $\rho : G \to \mathrm{GL}_n(F)$, it is enough to find a matrix $M \in \mathrm{GL}_n(F)$ with $M^3 = 1$; in turn, M determines a representation ρ by the rule $\rho(i + 3\mathbb{Z}) = M^i$.

   Consider the *representation* $\rho_1 : G \to \mathrm{GL}_3(F)$ given by the matrix

   $$\rho_1(1 + 3\mathbb{Z}) = M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}$$

and consider the representation $\rho_2 : G \to \mathrm{GL}_3(F)$ given by the matrix

$$\rho_2(1 + 3\mathbb{Z}) = M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that the *representations* ρ_1 and ρ_2 are *equivalent* (alternative terminology: are *isomorphic*). In other words, find a linear bijection $\Phi : F^3 \to F^3$ with the property that

$$\Phi(\rho_2(g)v) = \rho_1(g)\Phi(v)$$

for every $g \in G$ and $v \in F^3$.

Hint: First find a basis of F^3 consisting of eigenvectors for the matrix M_2.

4. Let V be a n dimensional F-vector space for $n \in \mathbb{N}$.

   Let $\mathrm{GL}(V)$ denote the group

   $$\mathrm{GL}(V) = \{\text{all invertible } F\text{-linear transformations } \phi : V \to V\}$$

   where the group operation is *composition* of linear transformations.

   Recall that $\mathrm{GL}_n(F)$ denotes the group of all invertible $n \times n$ matrices.

   If $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$ is a choice of basis, show that the assignment $\phi \mapsto [\phi]_{\mathcal{B}}$ determines an isomorphism

   $$\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}_n(F).$$

   Here $[\phi]_{\mathcal{B}} = [M_{ij}]$ denotes the *matrix* of ϕ in the basis $\mathcal{B}$ defined by equations

   $$\phi(b_i) = \sum_{k=1}^{n} M_{ki} b_k.$$

Bibliography