ProblemSet 1 – Linear algebra and representations **Solutions**

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F denotes an algebraically closed field of characteristic 0. If you like, you can suppose that $F = \mathbb{C}$ is the field of complex numbers.

1. Let V be a finite dimensional vector space over the field F. Suppose that $\phi, \psi : V \to V$ are linear maps. Let $\lambda \in F$ be an eigenvalue of ϕ and write W for the λ -eigenspace of ϕ ; i.e.

$$W = \{ v \in V \mid \phi(v) = \lambda v \}.$$

If $\phi\psi = \psi\phi$ show that W is *invariant* under ψ – i.e. show that $\psi(W) \subset W$.

Solution:

Let $w \in W$. We must show that $x = \psi(w) \in W$. To do this, we must establish that $x = \psi(w)$ is a λ -eigenvector for ϕ .

We have

$$\begin{split} \phi(x) &= \phi(\psi(x)) \\ &= \psi(\phi(w)) \\ &= \psi(\lambda w) \\ &= \lambda \psi(w) \\ &= \lambda x \end{split} \qquad \begin{array}{l} \operatorname{since} \, \phi \circ \psi = \psi \circ \phi \\ \operatorname{since} \, w \text{ is a λ-eigenvector} \\ \operatorname{since} \, \psi \text{ is linear} \end{split}$$

This completes the proof.

2. Let $n \in \mathbb{N}$ be a non-zero natural number, and let V be an n dimensional F-vector space with a given basis e_1, e_2, \cdots, e_n . Consider the linear transformation $T: V \to V$ given by the rule

$$Te_i = e_{i+1 \pmod n}.$$

In other words

$$Te_i = \left\{ \begin{matrix} e_{i+1} & i < n \\ e_1 & i = n \end{matrix} \right..$$

a. Show that T is *invertible* and that $T^n = id_V$.

To check that $T^n = \mathrm{id}_V$, we check that $T^n(e_i) = e_i$ for $1 \le i \le n$.

From the definition, it follows by induction on the natural number m that

$$T^m(e_i) = e_{i+m \pmod n}.$$

Thus $T^n(e_i) = e_{i+n \pmod n} = e_i$. Since this holds for every i, conclude $T^n = \mathrm{id}_V$.

Now T is invertible since its inverse is given by T^{n-1} .

b. Consider the vector $v_0 = \sum_{i=1}^n e_i$. Show that v_0 is a 1-eigenvector for T.

We compute

$$\begin{split} T(v_0) &= T\left(\sum_{i=1}^n e_i\right) = \sum_{i=1}^n T(e_i) \\ &= \sum_{i=1}^n e_{i+1 \pmod n} \\ &= \sum_{j=2}^{n+1} e_{j \pmod n} \\ &= \sum_{j=1}^n e_{j \pmod n} = v_0 \end{split} \tag{let } j = i+1)$$

Thus $T(v_0) = v_0$ so indeed v_0 is a 1-eigenvector.

Let $\zeta \in F$ be a primitive n-th root of unity. (e.g. if you assume $F = \mathbb{C}$, you may as well take $\zeta = e^{2\pi i/n}$).

c. Let
$$v_1 = \sum_{i=1}^n \zeta^i e_i$$
. Show that v_1 is a ζ^{-1} -eigenvector for T .

We compute

$$\begin{split} T(v_1) &= T\left(\sum_{i=1}^n \zeta^i e_i\right) \\ &= \sum_{i=1}^n \zeta^i T(e_i) \\ &= \sum_{i=1}^n \zeta^i e_{i+1 \pmod n} \\ &= \sum_{j=2}^{n+1} \zeta^{j-1} e_{j \pmod n} \\ &= \sum_{j=2}^{n+1} \zeta^{j-1} e_{j \pmod n} \\ &= \zeta^{-1} \sum_{j=2}^{n+1} \zeta^j e_{j \pmod n} \\ &= \zeta^{-1} \sum_{j=1}^n \zeta^j e_{j \pmod n} \\ &= \zeta^{-1} v_1 \end{split} \tag{Since } \zeta^j = \zeta^j \pmod n \ \forall j)$$

Thus $T(v_1)=\zeta^{-1}v_1$ so indeed v_0 is a $\zeta^{-1}\text{-eigenvector.}$

d. More generally, let $0 \le j < n$ and let

$$v_j = \sum_{i=1}^n \zeta^{ij} e_i.$$

Show that v_i is a ζ^{-j} -eigenvector for T.

The calcuation in the solution to part (c) is valid for any n-th root of unity unity ζ . Applying this calculation for ζ^j shows that v_i is a ζ^{-j} -eigenvector for T as required.

e. Conclude that $v_0, v_1, \cdots, v_{n-1}$ is a basis of V consisting of eigenvectors for T, so that T is diagonalizable.

Hint: You need to use the fact that eigenvectors for distinct eigenvalues are linearly independent.

What is the matrix of T in this basis?

Since eigenvectors for distinct eigenvalues are linearly independent, conclude that the vectors $\mathcal{B} = \{v_0, v_1, \cdots, v_{n-1}\}$ are linearly independent. Since there n vectors in \mathcal{B} and since dim V = n, conclude that \mathcal{B} is a *basis* for V.

The matrix of T in the basis \mathcal{B} is given by

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \zeta^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta^{-n+1} \end{bmatrix}$$

(This form explains why an $n \times n$ matrix M is diagonalizable iff F^n has a basis of eigenvectors for M).

3. Let $G = \mathbb{Z}/3\mathbb{Z}$ be the additive group of order 3, and let ζ be a primitive 3rd root of unity in F.

To define a representation $\rho: G \to \mathrm{GL}_n(F)$, it is enough to find a matrix $M \in \mathrm{GL}_n(F)$ with $M^3 = 1$; in turn, M determines a representation ρ by the rule $\rho(i+3\mathbb{Z}) = M^i$.

Consider the representation $\rho_1:G\to \mathrm{GL}_3(F)$ given by the matrix

$$\rho_1(1+3\mathbb{Z}) = M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}$$

and consider the representation $\rho_2:G\to \mathrm{GL}_3(F)$ given by the matrix

$$\rho_2(1+3\mathbb{Z}) = M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that the representations ρ_1 and ρ_2 are equivalent (alternative terminology: are isomorphic). In other words, find a linear bijection $\Phi: F^3 \to F^3$ with the property that

$$\Phi(\rho_2(g)v) = \rho_1(g)\Phi(v)$$

for every $q \in G$ and $v \in F^3$.

Hint: First find a basis of F^3 consisting of eigenvectors for the matrix M_2 .

The matrix M_1 is diagonal, which is to say that the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors for M_1 with respective eigenvalues $1, \zeta, \zeta^2$.

By the work in problem 2, we see that

$$v_1 = e_1 + e_2 + e_3, \quad v_2 = e_1 + \zeta e_2 + \zeta^2 e_3, \quad v_3 = e_1 + \zeta^2 e_2 + \zeta e_3$$

are eigenvectors for M_2 with respective eigenvalues $1, \zeta^2, \zeta$.

Now let $\Phi: F^3 \to F^3$ be the linear transformation for which

$$\Phi(e_1)=v_1,\quad \Phi(e_2)=v_3,\quad \Phi(e_3)=v_2$$

We claim that Φ defines an isomorphism of G-representations

$$(\rho_1,F^3) \xrightarrow{\sim} (\rho_2,F^3).$$

We must check that $\Phi(\rho_1(g)v) = \rho_2(g)\Phi(v)$ for all $g \in G$ and all $v \in F^3$.

Since G is cyclic it suffices to check that

$$(\clubsuit) \quad \Phi(M_1v) = M_2\Phi(v) \quad \forall v \in F^3$$

.

(Indeed, (\clubsuit) amounts to "checking on a generator". If (\clubsuit) holds then for every natural number i a straightforward induction argument shows for every $v \in F^3$ that

$$\begin{split} \Phi(\rho_1(i+3\mathbb{Z})v) &= \Phi(\rho_1(1+3\mathbb{Z})^i v) \\ &= \Phi(M_1^{\ i}v) \\ &= M_2^{\ i} \Phi(v) \\ &= \rho_2(1+3\mathbb{Z})^i \Phi(v) \\ &= \rho_2(i+3\mathbb{Z}) \Phi(v) \end{split}$$

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In turn, it suffices to verify the (\clubsuit) holds for the basis vectors e_1, e_2, e_3 for $V = F^3$.

Since e_1 and v_1 are 1-eigenvectors for M_1 resp. M_2 , we have

$$\Phi(M_1 e_1) = \Phi(e_1) = v_1 = M_2 v_1.$$

Since e_2 and v_3 are ζ -eigenvectors for M_1 resp. M_2 , we have

$$\Phi(M_1e_2)=\Phi(\zeta e_2)=\zeta\Phi(e_2)=\zeta v_3=M_2v_3.$$

Since e_3 and v_2 are ζ^2 -eigenvectors for M_1 resp. M_2 , we have

$$\Phi(M_1e_3) = \Phi(\zeta^2e_3) = \zeta^2\Phi(e_3) = \zeta^2v_2 = M_2v_2.$$

Thus (\clubsuit) holds and the proof is complete.

Alternatively, note that the matrix of Φ in the standard basis is given by

$$[\Phi] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{bmatrix}$$

Now, to prove that $\Phi\circ \rho_1(g)=\rho_2(g)\circ \Phi$, it suffices to check that $M_2[\Phi]=[\Phi]M_1$ i.e. that

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}$$

IN fact, both products yield the matrix

$$\begin{bmatrix} 1 & \zeta & \zeta^2 \\ 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \end{bmatrix}$$

4. Let V be a n dimensional F-vector space for $n \in \mathbb{N}$.

Let GL(V) denote the group

$$GL(V) = \{ \text{all invertible } F\text{-linear transformations } \phi: V \to V \}$$

where the group operation is *composition* of linear transformations.

Recall that $GL_n(F)$ denotes the group of all invertible $n \times n$ matrices.

If $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ is a choice of basis, show that the assignment $\phi \mapsto [\phi]_{\mathcal{B}}$ determines an isomorphism

$$\operatorname{GL}(V) \xrightarrow{\sim} \operatorname{GL}_n(F).$$

Here $[\phi]_{\mathcal{B}} = [M_{ij}]$ denotes the *matrix* of ϕ in the basis \mathcal{B} defined by equations

$$\phi(b_i) = \sum_{k=1}^n M_{ki} b_k.$$

Lets write Φ for the mapping

$$\Phi: \mathrm{GL}(V) \to \mathrm{GL}_n(F)$$

defined above.

An important property – proved in $\it Linear Algebra$ – is that for $\phi, \psi: V \to V$ we have

$$(\heartsuit) \quad [\phi \circ \psi]_{\mathcal{B}} = [\phi]_{\mathcal{B}} \cdot [\psi]_{\mathcal{B}}.$$

In words: "once you choose a basis, composition of linear transformations corresponds to multiplication of the corresponding matrices".

Now, since the matrix of the endomorphism $\phi:V\to V$ is equal to the identity matrix \mathbf{I}_n if and only if $\phi=\mathrm{id}_V,$ (\heartsuit) shows at once that a linear transformation $\phi:V\to V$ is invertible if and only if $[\phi]_{\mathcal{B}}$ is an invertible matrix.

This confirms that Φ is indeed a group homomorphism.

To show that Φ is an *isomorphism*, we exhibit its inverse. Namely, we defined a group homomorphism

$$\Psi: \mathrm{GL}_n(F) \to \mathrm{GL}(V)$$

and check that Ψ is the inverse to Φ .

TO define Ψ , we introduce the linear isomorphism $\beta: F^n \to V$ defined by the rule

$$\beta \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \sum_{i=1}^n a_i b_i.$$

For an invertible matrix M, we define

$$\Psi(M): V \to V$$

by the rule

$$\Psi(M)(v) = \beta M \cdot \beta^{-1} v$$

If $M_1, M_2 \in \mathrm{GL}_n(F)$ then for every $v \in V$ we have

$$\begin{split} \Psi(M_1M_2)v &= \beta M_1M_2 \cdot \beta^{-1}v \\ &= \beta M_1\beta^{-1}\beta M_2 \cdot \beta^{-1}v \\ &= \Psi(M_1)\Psi(M_2)v \end{split}$$

This confirms that Ψ is a group homomorphism.

It remains to observe that for $M \in \mathrm{GL}_n(F)$ we have

$$\Phi \circ \Psi(M) = M$$
,

which amounts to the fact that M is the matrix of $\Psi(M)$, and we must observe for $g \in GL(V)$ hat

$$\Psi \circ \Phi(q) = q$$

which amounts to the observation that the transformation $g:V\to V$ is determined by its effect on the basis vectors b_i and hence by the matrix $\Phi(g)$.

Bibliography