ProblemSet 2 – Representations and characters

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In these exercises, G always denotes a finite group and all vector spaces are assumed to be finite dimensional over the field $F = \mathbb{C}$.

In these exercises, you may use results stated but not yet proved in class about characters of representations of G.

1. In this problem, we identify the character χ_Ω of the *permutation representation* $(\rho, F[\Omega])$ of a group G.

   a. Let V be a vector space and let $\Phi : V \to V$ a linear mapping If $\mathcal{B}$ is a basis for V, recall that the *trace* of Φ is defined by

   $$\operatorname{tr}(\Phi) = \operatorname{tr}([\Phi]_{\mathcal{B}}).$$

   apologies – this is just explanatory; it isn't actually a question

   b. Recall that the *dual* of V is the vector space $V^\vee = \operatorname{Hom}_F(V, F)$ of *linear functionals* on V.

   If $b_1, \dots, b_n$ is a basis for V, let ${b_j}^\vee : V \to F$ be defined by ${b_j}^\vee(b_i) = \delta_{i,j}$. Show that ${b_1}^\vee, \dots, {b_n}^\vee$ is a basis for $V^\vee$; it is known as the *dual basis* to $b_1, \dots, b_n$.

   We must show that the vectors $\{{b_i}^\vee\}$ are *linearly independent* and *span* $V^\vee$.

   First, linear independence:

   Suppose that $\alpha_1, \cdots, \alpha_n \in F$ and that

   $$0 = \sum_{i=1}^n \alpha_i {b_i}^\vee$$

   (note that this equality "takes place in the vector space $V^\vee$").

   We must argue that all coefficients α_j are zero. Well, fix j and consider the vector b_j. We apply the functional $\sum_{i=1}^n \alpha_i b_i$ to b_j:

   $$\left(\sum_{i=1}^n \alpha_i {b_i}^\vee\right)(b_j) = \sum_{i=1}^n \alpha_i {b_i}^\vee)(b_j) = \alpha_j.$$

   Since the *functional* $\sum_{i=1}^n \alpha_i {b_i}^\vee$ is equal to 0, we know that $\left(\sum_{i=1}^n \alpha_i {b_i}^\vee\right)(v) = 0$ for every $v \in V$. In particular, this holds when $v = b_j$ and we now conclude that $\alpha_j = 0$.

   This proves linear independence.

   Finally, we prove the vectors span $V^\vee$.

   Let $\phi \in V^\vee$, and for $1 \le i \le n$ write $\alpha_i = \phi(b_i)$. We claim that

   $$(\clubsuit) \quad \phi = \sum_{i=1}^n \alpha_i {b_i}^\vee.$$

   TO prove this equality of functions ("functionals") we must argue that

   $$\phi(v) = \left(\sum_{i=1}^n \alpha_i {b_i}^\vee\right)(v)$$

   for every $v \in V$.

And it suffices to prove the latter equality for vectors v taken from the basis $\{b_i\}$.

But now by construction, for each j we have

$$\phi(b_j) = \alpha_j = \left(\sum_{i=1}^{n} \alpha_i {b_i}^\vee \right) (\alpha_j)$$

This proves (♣) so that the ${b_i}^\vee$ indeed span $V^\vee$.

c. Prove that the trace of the linear mapping $\Phi : V \to V$ is given by the expression

$$\text{tr}(\Phi) = \sum_{i=1}^{n} {b_i}^\vee(\Phi(b_i)).$$

Recall that $\text{tr}(\Phi)$ is defined to be the trace of the matrix $[\Phi]_{\mathcal{B}}$ where $\mathcal{B}$ is a *basis* of V. It is a fact that this definition is *independent* of the choice of basis.

Also recall that the trace of the $n \times n$ matrix $M = (M_{i,j})$ is given by

$$\text{tr}(M) = \sum_{i=1}^{n} M_{i,i}.$$

We consider the basis $\mathcal{B}$ of V, and the dual basis $\mathcal{B}^\vee$ of $V^\vee$, as above.

Recall that the *matrix* $M = [M_{j,i}] = [\Phi]_{\mathcal{B}}$ *of* Φ *in the basis* $\mathcal{B}$ is defined by the condition

$$\Phi(b_i) = \sum_{j=1}^{n} M_{j,i} b_j$$

(for $1 \le i \le n$).

Thus,

$${b_i}^\vee(\Phi(b_i)) = {b_i}^\vee \left(\sum_{j=1}^{n} M_{j,i} b_j \right) = M_{i,i}$$

.

Summing over i we find that

$$\sum_{i=1}^{n} {b_i}^\vee(\Phi(b_i)) = \sum_{i=1}^{n} M_{i,i} = \text{tr}(M) = \text{tr}([\Phi]_{\mathcal{B}}) = \text{tr}(\Phi),$$

as required.

d. Suppose that the finite group G acts on the finite set Ω, and consider the corresponding permutation representation $(\rho, F[\Omega])$ of G. Recall that $F[\Omega]$ is the vector space of all F-values functions on Ω, and that for $f \in F[\Omega]$ and $g \in G$, we have

$$\rho(g)f(\omega) = f(g^{-1}\omega).$$

In particular, we saw in the lecture that

$$\rho(g)\delta_\omega) = \delta_{g\omega},$$

where δ_ω denotes the *Dirac function* at $\omega \in \Omega$.

Show that

$$\text{tr}(\rho(g)) = \#\{\omega \in \Omega \mid g\omega = \omega\};$$

i.e. the trace of $\rho(g)$ is the number of fixed points of the action of g on Ω.

Recall that the vector space $F[\Omega]$ has a basis consisting of the vectors δ_ω for $\omega \in \Omega$.

We write $\delta_\omega^\vee \in F[\Omega]^\vee$ for vectors of the *dual basis*. The linear functional

$$\delta_\omega^\vee : F[\Omega] \to F$$

is defined by

$$\delta_\omega^\vee(\delta_\tau) = \begin{cases} 1 & \tau = \omega \\ 0 & \tau \neq \omega \end{cases}$$

Fix $g \in G$. According to our earlier work, we know that

$$\mathrm{tr}(\rho(g)) = \sum_{\omega \in \Omega} \delta_\omega^\vee(\rho(g)\delta_\omega) = \sum_{\omega \in \Omega} \delta_\omega^\vee(\delta_{g\omega}).$$

Now, $\delta_\omega^\vee(\delta_g \omega)$ is 1 when $\omega = g\omega$ and is 0 otherwise. This shows that $\mathrm{tr}(\rho(g))$ is given by the number of $\omega \in \Omega$ for which $g\omega = \omega$, as required.

2. Let V be a representation of G, suppose that W_1, W_2 are invariant subspaces, and that V is the internal direct sum

$$V = W_1 \oplus W_2.$$

Show that the character χ_V of V satisfies

$$\chi_V = \chi_{W_1} + \chi_{W_2}$$

i.e. for $g \in G$ that

$$\chi_V(g) = \chi_{W_1}(g) + \chi_{W_2}(g).$$

Let $\mathcal{B} = \{b_1, \cdots, b_n\}$ be a basis of W_1 and let $\mathcal{C} = \{c_1, \cdots, c_m\}$ be a basis of W_2.

Since $V = W_1 \oplus W_2$, we know that $\mathcal{B} \cup \mathcal{C} = \{b_1, \cdots, b_n, c_1, \cdots, c_m\}$ is a basis for

We consider the dual basis $b_1^\vee, b_2^\vee, \cdots, b_n^\vee, c_1^\vee, \cdots, c_m^\vee$ of the dual vector space $V^\vee$.

(Be careful! $W_1^\vee$ is not a subspace of $V^\vee$! Instead, it is a *quotient* of $V^\vee \ldots$)

Observe that the functional $b_i^\vee \in V^\vee$ is determined by the rules

$$b_i^\vee(b_j) = \delta_{i,j} \quad \text{and} \quad b_i^\vee(c_j) = 0.$$

Similarly, the functional $c_j^\vee \in V^\vee$ is determined by the rules

$$c_j^\vee(b_i) = 0 \quad \text{and} \quad c_j^\vee(c_i) = \delta_{j,i}.$$

Observe that we can restrict $b_i^\vee$ to W_1, and these restrictions $\{b_i^\vee|_{W_1}\}$ give the basis of $W_1^\vee$ dual to the basis $\{b_i\}$ of W_1.

Similarly the restrictions $\{c_j^\vee|_{W_2}\}$ give the basis of $W_2^\vee$ dual to the basis $\{c_j\}$ of W_2.

Now using the results of the previous problem applied to the mapping $g : W_1 \to W_1$, $g : W_2 \to W_2$ and $g : V \to V$, we see that

$$\chi_{W_1}(g) = \mathrm{tr}(g : W_1 \to W_1) = \sum_{i=1}^n b_i^\vee(g \cdot b_i)$$

$$\chi_{W_2}(g) = \mathrm{tr}(g : W_2 \to W_2) = \sum_{j=1}^m c_j^\vee(g \cdot c_j)$$

$$\chi_V(g) = \mathrm{tr}(g : V \to V) = \sum_{i=1}^n b_i^\vee(g \cdot b_i) + \sum_{j=1}^m c_j^\vee(g \cdot c_j)$$

Thus indeed $\chi_V(g) = \chi_{W_1}(g) + \chi_{W_2}(g)$ for each g, as required.

3. Let $G = A_4$ be the alternating group of order $\dfrac{4!}{2} = 12$.

   We are going to find the *character table* of this group.

   a. Confirm that the following list gives a representative for each of the conjugacy classes of G:

   $$1, (12)(34), (123), (124)$$

   (Note that (123) and (124) are conjugate in S_4, but *not* in A_4).

   What are the *sizes* of the corresponding conjugacy classes?

   Note that the centralizer $C_{A_4}((12)(34))$ contains the group $\langle (12), (34) \rangle$, which has 4 elements. On the other hand, $(12)(34)$ is not central in A_4 (e.g. (23) doesn't commute with $(12)(34)$). Since $[A_4 : \langle (12)(34) \rangle] = 3$ (a prime number), conclude that $C_{A_4}((12)(34)) = \langle (12), (13) \rangle$. We conclude that $(12)(34)$ has exactly $12/4 = 3$ conjugates in A_4.

   Next note that the centralizer $C_{A_4}((123))$ contains the subgroup $\langle (123) \rangle$ of order 3. On the other hand, suppose that $\sigma \in C_{A_4}((123))$. Then $\sigma(123)\sigma^{-1} = (123)$. But we know $\sigma(123)\sigma^{-1} = (\sigma(1)\sigma(2)\sigma(3))$, and now the condition
   $$(123) = (\sigma(1)\sigma(2)\sigma(3))$$
   implies that $\sigma \in \langle (123) \rangle$. Thus $C_{A_4}((123)) = \langle (123) \rangle$ has order 3, and the conjugacy class of (123) has $12/3 = 4$ elements.

   Similarly, the centralizer of (124) has order 3, and its conjugacy class has 4 elements.

   Finally, we should argue that (123) and (124) are not in fact conjugate in A_4. Of course, they are conjugate in S_4 by the transposition (34). Arguing as above, the centralizer of (123) in S_4 is still just equal to $\langle (123) \rangle$. So any element σ of S_4 for which $\sigma(123)\sigma^{-1} = (124)$ has the form $(123)^i(12)$ for some i, and none of those elements is in A_4.

   We have

class rep g	$C_{A_4}(g)$	size of conjugacy class of g
1	12	1
$(12)(34)$	4	3
(123)	3	4
(124)	3	4

   Since
   $$1 + 3 + 4 + 4 = 12$$
   we have found all of the conjugacy classes in A_4.

   b. Let $K = \langle (12)(34), (14)(23) \rangle$. Show that K is a normal subgroup of index 3, so that $G/K \simeq \mathbb{Z}/3\mathbb{Z}$.

   One checks directly that
   $$K = \{1, (12)(34), (14)(23), (13)(24)\}$$
   so that K has order 4 and index 3 as asserted.

   Notice that - as a set - K is the union of $\{1\}$ and the 3-element conjugacy class of $(12)(34)$. This makes clear that $\sigma\tau\sigma^{-1} \in K$ for all $\sigma \in A_4$ and $\tau \in K$, so that K is a *normal subgroup.*

   Since $|G/K| = 3$, of course $G/K \simeq \mathbb{Z}/3\mathbb{Z}$ ("groups of prime order are cyclic").

   Let ζ_3 be a primitive 3rd root of unity in $F^\times$ and for $i = 0, 1, 2$ let $\rho_i : G \to F^\times$ be the unique homomorphism with the following properties:

   i. $\rho_i\left((123)\right) = \zeta^i$
   ii. $K \subseteq \ker \rho_i$.

Explain why $\rho_0 = \mathbf{1}, \rho_1, \rho_2$ determine distinct irreducible (1-dimensional) representations of G.

In fact, let (ρ_1, V_1) and (ρ_2, V_2) be representations of G for which V_1 and V_2 are 1 dimensional. In this case, $\mathrm{GL}(V_i) = \mathrm{GL}_1(F) = F^\times$ is a commutative group.

Since V_1 and V_2 have dimension 1, any isomorphism $\Phi : V_1 \to V_2$ is just given by multiplication with a scalar $\alpha \in F^\times$. So if the representations are isomorphic, we have for each $g \in G$ and $v \in V_1$:

$$\rho_2(g)\Phi(v) = \Phi(\rho_1(g)v) \implies \alpha\rho_2(g)v = \alpha\rho_1(g)v$$

Since $\alpha \neq 0$ and since this holds for all $v \in V_1$, we conclude that $\rho_1(g) = \rho_2(g)$ for each $g \in G$.

In other words, two 1 dimensional representations are isomorphic iff they are *equal* (as functions $G \to F^\times$).

Now, the three homomorphisms ρ_i $(i = 0, 1, 2)$ are clearly distinct, because each maps the element (123) to a different element of $F^\times$. Thus they constitute distinct irreducible 1 dimensional representations of G.

c. Let $\Omega = \{1, 2, 3, 4\}$ on which G acts by the embedding $A_4 \subset S_4$.

Compute the character χ_Ω of the representation $F[\Omega]$. (This means: compute and list the values of χ_Ω at the conjugacy class representatives given in a.)

(*Use the result of problem 1 above*).

According to problem 1, the trace of the action of an element $\sigma \in A_4$ on the permutation representation $F[\Omega]$ is equal to the number of fixed points of σ on Ω.

Let's write χ_Ω for the character of the representation $F[\Omega]$.

Thus, the trace is given by

σ	χ_Ω
1	4
$(12)(34)$	0
(123)	1
(124)	1

d. The span of the vector $\delta_1 + \delta_2 + \delta_3 + \delta_4 \in F[\Omega]$ is an invariant subspace isomorphic to the irreducible representation ρ_0 (the so-called *trivial representation*).

Thus $F[\Omega] = \rho_0 \oplus W$ for a 3-dimensional invariant subspace. Explain why problem 2 shows that the character of W is given by $\chi_W = \chi_\Omega - \mathbf{1}$.

Problem 2 shows that

$$\chi_\Omega = \mathbf{1} + \chi_W.$$

This is an identity of F*-valued functions on* G, and it immediately implies that $\chi_W = \chi_\Omega - \mathbf{1}$ as required.

Now prove that $\langle \chi_W, \chi_W \rangle = 1$ and conclude that W is an irreducible representation.

Write $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ for class representatives $1, (12)(34), (123), (124)$. And write c_i for the order of the *centralizer* of σ_i.

Notice that the *values* of $\chi_W = \chi_\Omega - \mathbf{1}$ are given in the following table:

σ_i	c_i	$\chi_\Omega(\sigma_i)$	$\chi_W(\sigma_i)$
$1 = \sigma_1$	12	4	3
$(12)(34) = \sigma_2$	4	0	-1
$(123) = \sigma_3$	3	1	0
$(124) = \sigma_4$	3	1	0

We calculate

$$\langle \chi_W, \chi_W \rangle = \sum_{i=1}^{4} \frac{1}{c_i} \chi_W(\sigma_i)\overline{\chi_W(\sigma_i)} = \frac{1}{12} 3 \cdot 3 + \frac{1}{4}(-1) \cdot (-1) + \frac{1}{3} 0 \cdot 0 + \frac{1}{3} 0 \cdot 0$$
$$= \frac{9}{12} + \frac{1}{4} = \frac{9+3}{12} = 1$$

It follows from the results described in lecture that a representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$, so we conclude that W is an irreducible representation.

e. Explain why

$$\mathbf{1}, \rho_1, \rho_2, W$$

is a complete set of non-isomorphic irreducible representations of G.

We know that G has 4 conjugacy classes, so up to isomorphism there are exactly 4 irreducible representations of G.

We've already pointed out that $1, \rho_1, \rho_2$ are non-isomorphic irreducible representations each of dimension 1. Now, we've seen that W is an irreducible representation; since W is 2 dimensional, it is not isomorphic to any of the representations $1, \rho_1, \rho_2$.

Thus we have found 4 non-isomorphic irreducible representations, and we can conclude that any irreducible representation is isomorphic to once of these 4.

f. Display the *character table* of $G = A_4$.

	1	$(12)(34)$	(123)	(124)
	12	4	3	3
$\mathbf{1}$	1	1	1	1
ρ_1	1	1	ζ	ζ^2
ρ_2	1	1	ζ^2	ζ
χ_W	3	-1	0	0

Bibliography