Let G a finite group; we consider finite dimensional representations of G on $\mathbb{C}$-vector spaces.

Previews!

Let $L_1, \cdots, L_m$ be the complete list of irreducible representations for the finite group G, and let χ_i be the *character* of L_i.

Next week, we are going to prove the following:

Theorem:

a. The number m of irreducible representations of G is equal to the number of *conjugacy classes* in G.

b. $\chi_1, \cdots, \chi_m$ are an *orthonormal basis* for the space $\mathrm{Cl}(G)$ of $\mathbb{C}$-value class functions on G.

c. For any G-representation V, let χ be the character of V.

Enumerate the conjugacy classes of G, say $C_1, \cdots, C_m$ and choose a representative $g_i \in C_i$ for each i.

Consider the $m \times m$ matrix whose rows are indexed by the irreducible characters $\chi_1, \cdots, \chi_m$ and whose columns are indexed by the conjugacy class representatives $g_1, \ldots, g_m$, and whose entry in the i-th row and j-th column is given by $\chi_i(g_j)$. Write $c_i = |C_i|$ for the number of elements in the ith conjugacy class.

This matrix is known as the *character table of the group G*.

$$\begin{array}{c|cccc}
 & g_1 & g_2 & \cdots & g_m \\
 & c_1 & c_2 & \cdots & c_m \\
\hline
\chi_1 & \chi_1(g_1) & \chi_1(g_2) & \cdots & \chi_1(g_m) \\
\chi_2 & \chi_2(g_1) & \chi_2(g_2) & \cdots & \chi_2(g_m) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\chi_m & \chi_m(g_1) & \chi_m(g_2) & \cdots & \chi_m(g_m)
\end{array}$$

Remark:

a. The fact that the χ_i form an orthonormal basis for the space $\mathrm{Cl}(G)$ is equivalent to the statement that the above matrix is *unitary*, in the sense that for $1 \leq i, j \leq m$ we have

$$\sum_{k=1}^{m} c_k \chi_i(g_k) \overline{\chi_j(g_k)} = \delta_{i,j}.$$

b. Since the transpose of unitary matrix is also unitary, we find for $1 \leq i, j \leq m$ that

$$\sum_{k=1}^{m} c_i \chi_k(g_i) \overline{\chi_k(g_j)} = \delta_{i,j}.$$

Bibliography