

The notion of a code and some examples

George McNinch

2024-02-14

The idea

We want to *transmit* information over a potentially noisy channel. So we want to *encode* our information in some way that permits us to later *decode* it even in the presence of transmission errors.

We want to exploit *algebra* to create our codes. We will use as our *alphabet* a finite field $K = \mathbb{F}_q$

Some recollections about finite fields

We pause to recall some information about finite fields. We plan to return to this topic.

First of all, any finite field K contains a copy of the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for some prime number $p > 0$. Moreover, K can then be viewed as a finite-dimensional vector space over \mathbb{F}_p ; as such, if $d = \dim_{\mathbb{F}_p} K$ we see that $\#K = |K| = p^d$.

Thus every finite field has order a power of some prime number p .

On the other hand, for any prime power $q = p^d$ there is a finite field \mathbb{F}_q which is *unique up to isomorphism*.

For example, $\mathbb{F}_9 = \mathbb{F}_3[i] = \mathbb{F}_3 + \mathbb{F}_3i$ where $i^2 = -1 = 2 \in \mathbb{F}_3$. In fact,

$$\mathbb{F}_9 \simeq \mathbb{F}_3[T]/\langle T^2 + 1 \rangle$$

(quotient of the *polynomial ring* $\mathbb{F}_3[T]$ by the principal ideal generated by the minimal polynomial $T^2 + 1$ of the element $i \in \mathbb{F}_9$. This isomorphism identifies i with the coset $T + \langle T^2 + 1 \rangle$).

Codewords as vectors

We are going to study what are known as *linear codes* C . A linear code C is a linear subspace $C \subseteq \mathbb{F}_q^n$ for some natural number n .

Thus a *code word* is a vector $\mathbf{v} \in C$, and we can write $\mathbf{v} = (v_1, v_2, \dots, v_n)$ for elements v_i in our *alphabet* $k = \mathbb{F}_q$.

We write k for the *dimension* of C ; i.e. $k = \dim_{\mathbb{F}_q} C$. We say that C is an $[n, k]$ -code, or more precisely an $[n, k]_q$ -code.

Specifying a code via a generator matrix.

Let C be an $[n, k]_q$ -code, and choose a *basis* b_1, \dots, b_k for C as \mathbb{F}_q -vector space.

Since $C \subset \mathbb{F}_q^n = \mathbb{F}_q^{1 \times n}$, we view elements of C as $1 \times n$ *row vectors*.

Now form the matrix $k \times n$ matrix G whose *rows* are the $1 \times n$ vectors $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$:

$$G = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_k \end{bmatrix}.$$

G is known as a *generator matrix* for the code C .

Notice that we recover C from G as the *image* of the linear transformation $\mathbb{F}_q^k \rightarrow \mathbb{F}_q^n$ determined by *right-multiplication with G* :

$$C = \{\mathbf{x}G \mid \mathbf{x} \in \mathbb{F}_q^{1 \times n}\} = \text{image}(\mathbf{x} \mapsto \mathbf{x}G).$$

Standard form

Let us write $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ for the *standard basis* for \mathbb{F}_q^n .

Lemma: Let C be an $[n, k]_q$ -code, and let G be a generator matrix for C .

- After possibly replacing the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ with the basis $\mathbf{e}_{\sigma(1)}, \dots, \mathbf{e}_{\sigma(n)}$ for some $\sigma \in S_n$, we may suppose that the first k -columns of the $k \times n$ matrix are linearly independent.
- If the conclusion a. holds, then C has a generator matrix of the form

$$G' = [\mathbf{I}_k \mid A]$$

for some $k \times (n - k)$ matrix A , where \mathbf{I}_k denotes the $k \times k$ identity matrix.

Proof of the Lemma:

- By construction, G has k linearly independent rows (its rows are a *basis* for C). Since G is $k \times n$ it follows that the rank of G is equal to k .

Since the *row rank* of a matrix is equal to the *column rank* of the matrix, it follows that G has k linearly independent columns. After possibly re-ordering the order of these columns, we may arrange that G has the required form.

- Suppose that the first k -columns of G are linearly independent and consider the projection mapping

$$\pi : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^k \quad \text{given by the rule} \quad \pi(x_1, \dots, x_n) = (x_1, \dots, x_k).$$

Write $1, 2, \dots, b_k \in \mathbb{F}_q^{1 \times n}$ for the *rows* of G .

Since the first k -columns of G are linearly independent, the rank of the $k \times k$ matrix whose rows are the $1 \times k$ -vectors $\pi(1), \pi(2), \dots, \pi(k)$ is equal to k .

This shows that the dimension of $\pi(C)$ is $\geq k$, and hence that $\pi(C) = \mathbb{F}_q^k$. In other words, the restriction $\pi|_C : C \rightarrow \mathbb{F}_q^k$ of π to C is *surjective*.

In view of the surjectivity, for $i = 1, 2, \dots, k$ we may choose a vector $i' \in \pi^{-1}(\mathbf{e}_i) \cap C$.

Let G' whose rows are the vectors i' :

$$G' = \begin{bmatrix} 1' \\ 2' \\ \vdots \\ k' \end{bmatrix}.$$

Since

$$\begin{bmatrix} \pi(1') \\ \pi(2') \\ \vdots \\ \pi(k') \end{bmatrix} = \mathbf{I}_k,$$

G' has the required properties.

We say that a $(k \times n)$ generator matrix G for the $[n, k]_q$ -code C is in *standard form* if it has the form

$$G = [\mathbf{I}_k \mid A]$$

for some $k \times (n - k)$ matrix A ; thus the Lemma asserts that (after possibly changing the ordering of the coordinates on \mathbb{F}_q^n) every code C has a generator matrix in standard form.

Check matrices

The preceding discussion described the subspace C by giving *generators* for the vector space. In contrast, we may also specify a subspace using *linear equations*.

So: let C be an $[n, k]_q$ -code.

Consider the quotient linear mapping $x \mapsto x + C$:

$$\mathbb{F}_q^n \rightarrow \mathbb{F}_q^n / C \simeq \mathbb{F}_q^{n-k}$$

There is an $(n - k) \times n$ matrix H which represents this linear mapping; then

$$C = \text{Null}(H) = \{x \in \mathbb{F}_q^{1 \times n} \mid Hx^T = 0\};$$

i.e. we see that C is the null space for some $(n - k) \times n$ matrix H .

We say that such a matrix is a *check matrix* for C . For a vector $x \in \mathbb{F}_q^{1 \times n}$, we can check membership in C using H : we have $x \in C \iff Hx^T = 0$.

Proposition: Suppose that C is an $[n, k]_q$ -code and that

$$G = [\mathbf{I}_k \mid A]$$

is a generator matrix for C in standard form. Then the $(n - k) \times n$ matrix

$$H = [-A^T \mid \mathbf{I}_{n-k}]$$

is a check matrix for C .

Proof: We observe that

$$H \cdot G^T = [-A^T \mid \mathbf{I}_{n-k}] \left[\begin{array}{c} \mathbf{I}_k \\ A^T \end{array} \right] = -A^T \cdot \mathbf{I}_k + \mathbf{I}_{n-k} \cdot A^T = -A^T + A^T = \mathbf{0}.$$

Since the rows of G are a basis for C , this shows that $Hx^T = 0$ for every $x \in C$, i.e. $C \subset \text{Null}(H)$.

Now, H clearly has rank $(n - k)$, so $\dim C = \dim \text{Null}(H)$ and hence $C = \text{Null}(H)$. This completes the proof.

Weights and distance

For a vector $v = (v_1, v_2, \dots, v_n) \in \mathbb{F}_q^n = \mathbb{F}_q^{1 \times n}$ we define $\text{weight}(v)$ to be the number of non-zero entries; i.e.

$$\text{weight}(v) = \#\{i \mid v_i \neq 0\}.$$

For two vectors $v, w \in \mathbb{F}_q^n$ the *distance* between v and w is defined to be

$$\text{dist}(v, w) = \text{weight}(v - w).$$

Thus $\text{dist}(v, w)$ represents the number of coordinates in which v and w differ.

For a subspace $C \subset \mathbb{F}_q^n$ - i.e. a *code* - we define the *minimal distance* of C to be

$$d = \min\{\text{dist}(v, w) \mid v, w \in C, v \neq w\}.$$

The following lemma is immediate:

Lemma: $d = \min\{\text{weight}(v) \mid v \in C, v \neq 0\}$.

If d is the *minimal distance* of the $[n, k]_q$ code C , we say that C is an $[n, k, d]_q$ -code.

Example

We investigate an example using SageMath.

Let's create the field k having 3 elements, and the standard vector space $V=k^9$

```
K = GF(27);
V = VectorSpace(k,9)
print(k)
print(V)
```

=>

Finite Field in z_3 of size 3^3

Vector space of dimension 9 over Finite Field in z_3 of size 3^3

Now let's create a certain 3 dimensional subspace C of V – a $[9, 3]_3$ code – essentially by giving its *generator matrix*.

```
C = V.subspace([V([1,0,0,1,1,0,1,1,2]),
                 V([0,1,0,1,0,1,1,2,1]),
                 V([0,0,1,0,1,1,2,1,1])])
```

C

=>

Vector space of degree 9 and dimension 3 over Finite Field in z_3 of size 3^3

Basis matrix:

```
[1 0 0 1 1 0 1 1 2]
[0 1 0 1 0 1 1 2 1]
[0 0 1 0 1 1 2 1 1]
```

In order to manipulate the generator matrix *as a matrix*, we create the MatrixSpace of the right dimensions, and *coerce* the basis of C into a matrix:

```
MM = MatrixSpace(K,3,9)
G = MM.matrix(C.basis())
```

G

=>

```
[1 0 0 1 1 0 1 1 2]
[0 1 0 1 0 1 1 2 1]
[0 0 1 0 1 1 2 1 1]
```

We investigate the *weights* of non-zero vectors in C :

```
# count the non-zero entries in a vector
```

```
def weight(v):
    r = [x for x in v if x != 0]
    return len(r)
```

```
# we now find the minimum of the weight of v for non-zero vectors v in C
```

```
min([ weight(v) for v in C if v != 0 ])
```

=> 6

This shows that C is a $[9, 3, 6]_3$ -code.

Notice that the generator matrix G is in standard form. Let's extract from G the matrix A which is the 3×6 matrix for which $G = [I \mid A]$

```
A = MatrixSpace(K,3,6).matrix([b[3:9] for b in G])
```

A

=>

```
[1 1 0 1 1 2]
[1 0 1 1 2 1]
[0 1 1 2 1 1]
```

We can now form the 6×9 *check matrix* $H = [-A.T \mid I]$ as above.

```
# form the 6x6 identity matrix
i6=MatrixSpace(K,6,6).one()

# we construct H via *block_matrix*
H=block_matrix([[ -A.transpose(), i6]],
               subdivide=False)

H
=>
[2 2 0 1 0 0 0 0 0]
[2 0 2 0 1 0 0 0 0]
[0 2 2 0 0 1 0 0 0]
[2 2 1 0 0 0 1 0 0]
[2 1 2 0 0 0 0 1 0]
[1 2 2 0 0 0 0 0 1]
```

We can confirm that $H * G.T == 0$ and that H has rank 6:

```
H * G.T == 0
=>
True

rank(H)
=>
6
```

And indeed, if we use SAGE to check the `right_kernel` of the matrix H , we get exactly the subspace C .

```
H.right_kernel() == C
=>
True
```

So H is indeed a check-matrix for the code C .

Bibliography

Bibliography