

# ProblemSet 3 – Solutions

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In these exercises,  $G$  always denotes a finite group. Unless indicated otherwise, all vector spaces are assumed to be finite dimensional over the field  $F = \mathbb{C}$ . The representation space  $V$  of a representation of  $G$  is always assumed to be finite dimensional over  $\mathbb{C}$ .

1. Let  $\phi : G \rightarrow F^\times$  be a group homomorphism; since  $F^\times = \text{GL}_1(F)$ , we can think of  $\phi$  as a 1-dimensional representation  $(\phi, F)$  of  $G$ .

If  $V$  is any representation of  $G$ , we can form a *new* representation  $\phi \otimes V$ . The underlying vector space for this representation is just  $V$ , and the “new” action of an element  $g \in G$  on a vector  $v$  is given by the rule

$$g \star v = \phi(g)gv.$$

- a. Prove that if  $V$  is irreducible, then  $\phi \otimes V$  is also irreducible.

We prove the following statement:  $(*)$  if  $W \subset V$  is a subspace, then  $W$  is invariant for the *original* action of  $G$  if and only if it is invariant for the  $\star$  action of  $G$ .

First note that  $(*)$  immediately implies the assertion of (a).

To test invariance, let  $w \in W$  and let  $g \in G$ . Since  $W$  is a linear subspace and since  $\phi(g)$  is a non-zero scalar, it is immediate that

$$gw \in W \iff g \star w = \phi(g)gw \in W$$

Since this holds for all  $w$  and all  $g$ ,  $(*)$  follows.

- b. Prove that if  $\chi$  denotes the *character* of  $V$ , then the character of  $\phi \otimes V$  is given by  $\phi \cdot \chi$ ; in other words, the trace of the action of  $g \in G$  on  $\phi \otimes V$  is given by

$$\chi_{\phi \otimes V}(g) = \text{tr}(v \mapsto g \star v) = \phi(g)\chi(g).$$

We just need to compute the trace of the linear mapping  $V \rightarrow V$  given by  $v \mapsto g \star v$ .

If the action of  $g$  on  $V$  is given by the linear mapping  $\rho(g)$ , then

$$\chi_V(g) = \text{tr}(\rho(g)).$$

Now, the  $\star$ -action of  $g$  is given by the linear mapping  $v \mapsto g \star v = \phi(g)\rho(g)v$ .

So  $\chi_{\phi \otimes V}(g) = \text{tr}(\phi(g)\rho(g))$ . For any scalar  $s \in k$ , trace of the linear mapping  $s\rho(g)$  is given by

$$\text{tr}(s\rho(g)) = s \text{tr}(\rho(g)) = s\chi_V(g)$$

(“linearity of the trace”).

Thus

$$\chi_{\phi \otimes V}(g) = \text{tr}(\phi(g)\rho(g)) = \phi(g)\chi_V(g).$$

- c. Recall that in class we saw that  $S_3$  has an irreducible representation  $V_2$  of dimension 2 whose character  $\psi_2$  is given by

$g$	1	(12)	(123)
$\psi_2$	2	0	-1

Observe that  $\text{sgn } \psi = \psi$  and conclude that  $V_2 \simeq \text{sgn} \otimes V_2$ , where  $\text{sgn} : S_n \rightarrow \{\pm 1\} \subset \mathbb{C}^\times$  is the *sign homomorphism*.

On the other hand,  $S_4$  has an irreducible representation  $V_3$  of dimension 3 whose character  $\psi_3$  is given by

$g$	1	(12)	(123)	(1234)	(12)(34)
$\psi_3$	3	1	0	-1	-1

(I'm not asking you to confirm that  $\psi_3$  is irreducible, though it would be straightforward to check that  $\langle \psi_3, \psi_3 \rangle = 1$ ).

Prove that  $V_3 \not\simeq \text{sgn} \otimes V_3$  as  $S_4$ -representations.

(In particular,  $S_4$  has *at least two* irreducible representations of dimension 3.)

We first consider the representation  $V_2$  of  $S_3$ . Write  $\chi_2$  of the character of this irreducible representation. The character of  $\text{sgn } \chi_2$  is then given by the product  $\text{sgn } \chi_2$ .

$g$	1	(12)	(123)
$\psi_2$	2	0	-1
$\text{sgn}$	1	-1	1
$\text{sgn } \psi_2$	2	0	-1

Inspecting the table we see that  $\psi_2 = \text{sgn } \psi_2$ . This shows that  $V_2$  is isomorphic to  $\text{sgn} \otimes V_2$  as representations for  $S_3$ .

2. Let  $V$  be a representation of  $G$ .

For an irreducible representation  $L$ , consider the set

$$\mathcal{S} = \{S \subseteq V \mid S \simeq L\}$$

of all invariant subspaces that are isomorphic to  $L$  as  $G$ -representations.

Put

$$V_{(L)} = \sum_{S \in \mathcal{S}} S.$$

a. Prove that  $V_{(L)}$  is an invariant subspace, and show that  $V_{(L)}$  is isomorphic to a direct sum

$$V_{(L)} \simeq L \oplus \cdots \oplus L$$

as  $G$ -representations.

First of all, we note more generally that if  $I$  is an index set and if  $W_i \subset V$  is a  $G$ -invariant subspace for each  $i \in I$ , then  $\sum_{i \in I} W_i$  is again an invariant subspace. (The proof is straightforward from the definitions). This confirms that  $V_{(L)}$  is an invariant subspace.

To prove the remaining assertion, we proceed as follows.

Let us say that a  $G$ -representation  $W$  is  $L$ -isotypic if every irreducible invariant subspace of  $W$  is isomorphic to  $L$ .

It is immediate that  $V_{(L)}$  is  $L$ -isotypic. We are going to prove:

If  $W$  is an  $L$ -isotypic  $G$ -representation, then  $W$  is isomorphic to a direct sum

$$W \simeq L \oplus \cdots \oplus L.$$

Proceed by induction on  $\dim W$ . If  $\dim W = 0$  then  $W = \{0\}$  and the result is immediate ( $W$  is the direct sum of zero copies of  $L$ ).

Now observe that if  $\dim W > 0$  then  $W$  contains an invariant subspace isomorphic to  $L$ , so that  $\dim W \geq \dim L$ .

Now if  $\dim W = \dim L$ , then  $W \simeq L$  and the result holds in this case.

Finally, suppose that  $\dim W > \dim L$  and let  $S \subset W$  be an invariant subspace with  $S \simeq L$ .

By complete reducibility we may find an invariant subspace  $U \subset W$  such that  $W$  is the internal direct sum

$$W = S \oplus U.$$

Since  $\dim W = \dim S + \dim U$ , we have  $\dim U < \dim W$ . Moreover,  $U$  is also  $L$ -isotypic. So by induction on dimension, we know that

$$U \simeq L \oplus \cdots \oplus L,$$

(say, a direct sum of  $d$  copies of  $L$ ).

But then

$$W = S \oplus U \simeq L \oplus (L \oplus \cdots \oplus L) = L \oplus L \oplus \cdots \oplus L$$

is isomorphic to a direct sum of  $d + 1$  copies of  $L$ .

- b. Prove that the *quotient representation*  $V/V_{(L)}$  has no invariant subspaces isomorphic to  $L$  as  $G$ -representations.

Write  $\pi : V \rightarrow V/V_{(L)}$  for the quotient map  $v \mapsto v + V_{(L)}$ ; thus  $\pi$  is a surjective homomorphism of  $G$ -representations.

Suppose by way of contradiction that  $S \subset V/V_{(L)}$  is an invariant subspace isomorphic to  $L$ , and let  $S' \subset V$  be the inverse image under  $\pi$  of  $S$ :

$$S' = \pi^{-1}(S).$$

Then  $S'$  is an invariant subspace of  $V$  containing  $V_{(L)}$ , and the restriction of  $\pi$  to  $S'$  defines a surjective mapping

$$\pi|_{S'} : S' \rightarrow S \simeq L.$$

If  $K$  denotes the kernel of  $\pi|_{S'}$ , then complete reducibility implies that there is an invariant subspace  $M$  of  $V$  such that  $S'$  is the internal direct sum

$$(*) \quad S' = K \oplus M.$$

In particular, the invariant subspace  $M$  is isomorphic to  $L$  as  $G$ -representations. But then by definition we have  $M \subset V_{(L)}$  contradicting the condition  $M \cap K = \{0\}$  which must hold by  $(*)$ . This contradiction proves the result.

- c. If  $L_1, L_2, \dots, L_m$  is a complete set of non-isomorphic irreducible representations for  $G$ , prove that  $V$  is the internal direct sum

$$V = \bigoplus_{i=1}^m V_{(L_i)}.$$

We first note that  $V$  is equal to the sum

$$\sum_{i=1}^m V_{(L_i)};$$

indeed, if  $W = \sum_{i=1}^m V_{(L_i)}$ , then by complete reducibility  $V = W \oplus W'$  for an invariant subspace  $W'$ . But if  $W' \neq 0$  then  $W'$  contains an irreducible invariant subspace, so that  $W' \cap V_{(L_i)} \neq 0$  for some  $i$  and hence  $W' \cap W \neq 0$ ; this is impossible since the internal sum  $V = W \oplus W'$  is direct. This argument shows that  $W' = 0$  and hence that  $V = W$ .

Finally, we show that the sum

$$\sum_{i=1}^m V_{(L_i)}$$

is *direct*, i.e. that for each  $j$  we have

$$(\clubsuit) \quad V_{(L_j)} \cap \left( \sum_{i \neq j} V_{(L_i)} \right) = 0.$$

Wrote  $I$  for the intersection appearing in  $(\clubsuit)$ ; thus,  $I$  is an invariant subspace of  $V$ . If  $I$  is non-zero, it has an irreducible invariant subspace  $S$ . Since  $I \subset V_{(L_j)}$  and since  $V_{(L_j)}$  is  $L_j$ -isotypic, we conclude that

$$S \simeq L_j.$$

But then  $S \cap V_{(L_i)} = 0$  for every  $i \neq j$  so that

$$S \cap \left( \sum_{i \neq j} V_{(L_i)} \right) = 0.$$

Since  $I \subset \left( \sum_{i \neq j} V_{(L_i)} \right)$ , we conclude that  $I = 0$ .

This completes the proof that  $V$  is the direct sum of the  $V_{(L_i)}$ , as required.

3. Let  $\chi$  be the character of a representation  $V$  of  $G$ . For  $g \in G$  prove that  $\overline{\chi(g)} = \chi(g^{-1})$ .

Is it true for any arbitrary class function  $f : G \rightarrow \mathbb{C}$  that  $\overline{f(g)} = f(g^{-1})$  for every  $g$ ? (Give a proof or a counterexample...)

Let  $\rho(g) : V \rightarrow V$  denote the linear automorphism of  $V$  determined by the action of  $g \in G$ . Then  $\chi(g) = \text{tr}(\rho(g))$ .

Now, since  $\rho(g)$  has *finite order*, say  $n$ , its minimal polynomial divides  $T^n - 1 \in \mathbb{C}[T]$ , and hence every eigenvalue of  $\rho(g)$  is an  $n$ -th root of unity.

For any  $n$ -th root of unity  $\zeta$ , note that  $\bar{\zeta} = \zeta^{-1}$ .

Write  $\alpha_1, \dots, \alpha_d$  for the eigenvalues of  $\rho(g)$ , with multiplicity (so that  $d = \dim V$ ). Notice that  $\rho(g^{-1})$  has eigenvalues  $\alpha_1^{-1}, \dots, \alpha_d^{-1}$ .

Thus

$$\chi(g) = \sum_{i=1}^d \alpha_i \quad \text{and} \quad \chi(g^{-1}) = \sum_{i=1}^d \alpha_i^{-1}.$$

Now, we see that

$$\overline{\chi(g)} = \sum_{i=1}^d \bar{\alpha}_i = \sum_{i=1}^d \alpha_i^{-1} = \chi(g^{-1})$$

as required.

It is *not* true that  $\overline{f(g)} = f(g^{-1})$  for every class function  $f$  and every  $g \in G$ . Indeed, let  $f = \alpha \delta_1$  be a multiple of the characteristic function  $\delta_1$  of the trivial conjugacy class  $\{1\}$ .

Then  $\overline{f(1)} = \bar{\alpha}$  while  $f(1^{-1}) = f(1) = \alpha$ , so that if  $\alpha \notin \mathbb{R}$ , we have  $\overline{f(1)} \neq f(1^{-1})$ .

4. For a prime number  $p$ , let  $k = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$  be the field with  $p$  elements. Let  $V$  be an  $n$ -dimensional vector space over  $\mathbb{F}_p$  for some natural number  $n$ , and let

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow k$$

be a non-degenerate bilinear form on  $V$ .

(A common example would be to take  $V = \mathbb{F}_{p^n}$  the field of order  $p^n$ , and  $\langle \alpha, \beta \rangle = \text{tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(\alpha\beta)$  the trace pairing).

Let us fix a non-trivial group homomorphism  $\psi : k \rightarrow \mathbb{C}^\times$  (recall that  $k = \mathbb{Z}/p\mathbb{Z}$  is an additive group, while  $\mathbb{C}^\times$  is multiplicative). Thus

$$\psi(\alpha + \beta) = \psi(\alpha)\psi(\beta) \quad \text{for all } \alpha, \beta \in k.$$

If you want an explicit choice, set  $\psi(j + p\mathbb{Z}) = \exp(j \cdot 2\pi i/p) = \exp(2\pi i/p)^j$ .

For a vector  $v \in V$ , consider the mapping  $\Psi_v : V \rightarrow \mathbb{C}^\times$  given by the rule

$$\Psi_v(w) = \psi(\langle w, v \rangle).$$

- a. Show that  $\Psi_v$  is a group homomorphism  $V \rightarrow \mathbb{C}^\times$ .

For  $w, w' \in V$  notice that

$$\begin{aligned}\Psi_v(w + w') &= \psi(\langle w + w', v \rangle) \\ &= \psi(\langle w, v \rangle + \langle w', v \rangle) && \text{since the form is bilinear} \\ &= \psi(\langle w, v \rangle) \cdot \psi(\langle w', v \rangle) && \text{since } \psi \text{ is a group homomorphism} \\ &= \Psi_v(w) \cdot \Psi_v(w') && \text{by definition.}\end{aligned}$$

This confirms that  $\Psi_v$  is a group homomorphism.

- b. Show that the assignment  $v \mapsto \Psi_v$  is injective (one-to-one).

(This assignment is a function  $V \rightarrow \text{Hom}(V, \mathbb{C}^\times)$ . In fact, it is a group homomorphism. Do you see why? How do you make  $\text{Hom}(V, \mathbb{C}^\times)$  into a group?)

One checks that  $\text{Hom}(V, \mathbb{C}^\times)$  is a multiplicative group (this is the *dual group*  $\widehat{V}$  of  $V$ , mentioned in the lectures); the product of  $\phi, \psi \in \widehat{V}$  is given by the rule  $g \mapsto \phi(g) \cdot \psi(g)$ .

We note that the assignment  $v \mapsto \Psi_v$  is a group homomorphism. For  $v, v' \in V$  we must argue that  $\Psi_{v+v'} = \Psi_v \Psi_{v'}$ .

For  $w \in W$  we have

$$\begin{aligned}\Psi_{v+v'}(w) &= \psi(\langle v + v', w \rangle) \\ &= \psi(\langle v, w \rangle + \langle v', w \rangle) && \text{since the form is bilinear} \\ &= \psi(\langle v, w \rangle) \cdot \psi(\langle v', w \rangle) && \text{since } \psi \text{ is a group homomorphism} \\ &= \Psi_v(w) \cdot \Psi_{v'}(w) && \text{by definition}\end{aligned}$$

Now to show that  $v \mapsto \Psi_v$  is injective, it is enough to argue that the kernel of this mapping is  $\{0\}$ .

So, suppose that  $\Psi_v$  is the identity element of  $\widehat{V}$ . In other words, suppose that  $\Psi_v(w) = 1$  for every  $w \in V$ . This shows that  $\psi(\langle v, w \rangle) = 1$  for every  $w \in V$ . Since  $\psi$  is a non-trivial homomorphism  $\mathbb{F}_p \rightarrow \mathbb{C}^\times$ , we know that  $\ker \psi = \{0\}$  (remember that  $k$  has prime order...) and we conclude that  $\langle v, w \rangle = 0$  for every  $w \in W$ .

(Note that  $\langle v, w \rangle = 0$  is an equality in  $k = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ ).

Since the form  $\langle \cdot, \cdot \rangle$  is non-degenerate, so we may now conclude that  $v = 0$ .

This proves that the kernel of the mapping  $v \mapsto \Psi_v$  is  $\{0\}$ , hence the mapping is injective.

- c. Show that any group homomorphism  $\Psi : V \rightarrow \mathbb{C}^\times$  has the form  $\Psi = \Psi_v$  for some  $v \in V$ .

Conclude that there are exactly  $|V| = q^n$  group homomorphisms  $V \rightarrow \mathbb{C}^\times$ .

We observed in class that for any finite abelian group  $A$ , there is an isomorphism  $A \simeq \widehat{\widehat{A}}$ .

In particular,  $|A| = |\widehat{A}|$ .

Applying this in the case  $A = V$ , we conclude that

$$|V| = |\widehat{V}| = |\text{Hom}(V, \mathbb{C}^\times)|.$$

Now, we have defined an *injective* mapping

$$v \mapsto \Psi_v : V \rightarrow \widehat{V}.$$

Since the domain and co-domain of this mapping are finite of the same order, the mapping  $v \mapsto \Psi_v$  is also *surjective*.

Thus the *pigeonhole principle* shows that every homomorphism  $\Psi : V \rightarrow \mathbb{C}^\times$  has the form  $\Psi = \Psi_v$  for some  $v \in V$ , as required.

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## Bibliography