Notes on Representations

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Let G be a finite group, let F be a field (algebraically closed, char. 0).

A representation of G is a vector space V together with a group homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

If we choose a basis $\mathcal{B}$ of V, we obtain from ρ an assignment $g \mapsto [\rho(g)]_{\mathcal{B}}$ which is a group homomorphism $G \to \mathrm{GL}_n(F)$.

First example

Consider the finite cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ for a natural number $n > 0$.

Suppose that M is an $m \times m$ matrix for which $M^n = \mathbf{I}$. Using M, we get a representation

$$\rho_M : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_m(F)$$

by the rule

$$\rho_M(i + n\mathbb{Z}) = M^i.$$

One might consider the question: If M and M' are two $m \times m$ matrices for which $M^n = (M')^n = \mathbf{I}$, when are the representations ρ_M and $\rho_{M'}$ "the same"??

Consider the following 3×3 matrices, each of which satisfies $M^3 = \mathbf{I}$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 \\ 3/2 & 0 & 0 \\ 0 & -1/3 & 0 \end{pmatrix}.$$

where ζ is a root of $\dfrac{T^3 - 1}{T - 1} = T^2 + T + 1.$ [1]

In fact, all three of these matrices have eigenvalues $1, \zeta, \zeta^2$ each with multiplicity 1.

Example: two commuting matrices

Suppose that M, N are $m \times m$ matrices for which $M^n = \mathbf{I}$, $N^k = \mathbf{I}$ and $MN = NM$.

We get a representation

$$\rho : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z} \to \mathrm{GL}_m(F)$$

by the rule

$$\rho(i + n\mathbb{Z}, j + k\mathbb{Z}) = M^i N^j.$$

Remark Let μ be an eigenvalue of the matrix M, and let $V_\mu \subset F^m$ be the corresponding *eigenspace*.

Since M and N commute, a homework exercise shows that V_μ is *invariant* under the action of N.

Then

$$N_{|V_\mu} : V_\mu \to V_\mu$$

[1] If $F = \mathbb{C}$ then we can take $\zeta = e^{2\pi i/3}$.

is a linear transformation satisfying $(N_{|V_\mu})^k = \text{id}$.

In particular, we may choose a basis of V_μ of eigenvectors for $N_{|V_\mu}$

In this way, we see that $V = F^m$ has a basis consisting of vectors which are simultaneously eigenvectors for N and for M.

Construction of representations from group actions

Suppose that the (finite) group G acts on the (finite) set Ω. We are going to use this action to define a representation of G.

We consider the vector space $F[\Omega]$ of all F-valued functions on Ω.

For $g \in G$, we consider the linear transformation

$$\rho(g) : F[\Omega] \to F[\Omega]$$

defined for $f \in F[\Omega]$ by the rule

$$\rho(g)(f)(\omega) = f(g^{-1}\omega).$$

Proposition: (a). $\rho(g)$ is an F-linear map.

(b). $\rho(gh) = \rho(g)\rho(h)$ for $g, h \in G$.

(c). $\rho(g) : F[\Omega] \to F[\Omega]$ is *invertible*.

Proof of (b): Let $f \in F[\Omega]$. For $\tau \in \Omega$ we have

$$\rho(gh)f(\tau) = f((gh)^{-1}\tau) = f(h^{-1}g^{-1}\tau) = \rho(h)f(g^{-1}\tau) = \rho(g)\rho(h)f(\tau).$$

Since this holds for all τ, conclude that

$$\rho(gh)f = \rho(g)\rho(h)f.$$

Since this holds for all f, conclude that $\rho(gh) = \rho(g)\rho(h)$ as required.

The Proposition shows that

$$\rho : G \to \text{GL}(F[\Omega])$$

is a representation of G.

Here is an alternate perspective on $F[\Omega]$ and the action of G.

For $\omega \in \Omega$, consider the *Dirac* function $\delta_\omega \in F[\Omega]$ defined by

$$\delta_\omega(\tau) = \begin{cases} 1 & \omega = \tau \\ 0 & \text{else} \end{cases}$$

Proposition: The collection δ_ω forms a *basis* for the F-vector space $F[\Omega]$.

Sketch: For $f \in F[\Omega]$, we have

$$f = \sum_{\omega \in \Omega} f(\omega)\delta_\omega.$$

This shows that the δ_ω *span* $F[\Omega]$.

Now, suppose $0 = \sum_{\omega \in \Omega} a_\omega \delta_\omega$ for $a_\omega \in F$ for each $\omega \in \Omega$.

To prove linear independence, we must argue that $a_\omega = 0$ for every ω.

Set $f = \sum_{\omega \in \Omega} a_\omega \delta_\omega$. For $\tau \in \Omega$, the function f satisfies

$$f(\tau) = \sum_{\omega \in \Omega} a_\omega \delta_\omega(\tau) = a_\tau.$$

Since $f = 0$ - i.e. "f is the function which is takes value zero at every argument" – we conclude that $a_\tau = 0$ for each τ. This proves linear independence.

Let's consider the action of G on these basis vectors.

Proposition: For $g \in G$ and $\omega \in \Omega$, we have
$$\rho(g)\delta_\omega = \delta_{g\omega}.$$

Proof: For $\tau \in \Omega$ we have
$$\rho(g)\delta_\omega(\tau) = \delta_\omega(g^{-1}\tau) = \begin{cases} 1 & \omega = g^{-1}\tau \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & g\omega = \tau \\ 0 & \text{else} \end{cases}$$

This shows that $\rho(g)\delta_\omega(\tau) = \delta_{g\omega}(\tau)$; since this holds for all τ we conclude $\rho(g)\delta_\omega = \delta_{g\omega}$ as required.

Remark: Viewing $F[\Omega]$ as the vector space with basis δ_ω gives us a perhaps simpler proof that $\rho(gh) = \rho(g)\rho(h)$.

Indeed, for $\omega \in \Omega$ we have
$$\rho(gh)\delta_\omega = \delta_{gh\omega} = \rho(g)\delta_{h\omega} = \rho(g)(\rho(h)\delta_\omega) = \rho(g)\rho(h)\delta_\omega.$$

Homomorphisms

Let (ρ, V) and (ψ, W) be representations of G. a linear mapping $\Phi : V \to W$ is a *homomorphism of G-representations* provided that for every $v \in V$ and for every $g \in G$ we have
$$\Phi(\rho(g)v) = \psi(g)\Phi(v).$$

The G-representations (ρ, V) and (ψ, W) are isomorphic is there is a homomorphism of G-representations which is *invertible*.

Example

For any G one has the so-called *trivial representation* $\mathbb{1}$. This representation has vector space $V = F$ (it is 1 dimensional!) and the mapping
$$G \to \mathrm{GL}(V) = \mathrm{GL}_1(F) = F^\times$$
is just the trivial homomorphism $g \mapsto 1$.

One often writes $\mathbb{1}$ for the *underlying vector space F*.

Let G act on the (finite) set Ω. Then there is a homomorphism of G-representations
$$\mathbb{1} \to F[\Omega].$$

Viewing $\mathbb{1} = F$, the scalar c is mapped to the *constant function with value c.*

Put another way, the scalar c is mapped to the vector
$$c \sum_{\omega \in \Omega} \delta_\omega.$$

Bibliography