Characters of irreducible representations

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Convolution

We write $\mathbb{C}[G]$ for the space of functions on G, viewed as a permutation representation of G (and we suppress the notation for the homomorphism $G \to \text{GL}(\mathbb{C}[G])$).

For functions $f_1, f_2 \in \mathbb{C}[G]$, we define their *convolution* by the formula

$$(f_1 \star f_2)(x) = \sum_{yz=x} f_1(y) f_2(z).$$

If V is a G-representation and $f \in \mathbb{C}[G]$, we define

$$f \star v = \sum_{g \in G} f(g) g v$$

for $v \in V$.

Remark:

1. For the basis elements $\delta_g \in \mathbb{C}[G]$ (i.e. the *Dirac functions*), we have

   $$\delta_g \star \delta_h = \delta_{gh}.$$

2. The action of G on $\mathbb{C}[G]$ can be described by

   $$gf = \delta_g \star f$$

   for $g \in G$ and $f \in \mathbb{C}[G]$.

3. Viewing $\mathbb{C}[G]$ as a G-representation, the two notions of $\star$ just introduced actually coincide:

   $$f_1 \star f_2 = \sum_{g \in G} f_1(g) \delta_g \star f_2.$$

4. The product $\star$ makes $\mathbb{C}[G]$ into a *ring* (in fact, a $\mathbb{C}$-algebra) and V into a $\mathbb{C}[G]$-module. Mostly we won't use this fact - at least explicitly - in these notes.

5. Let $W \subseteq \mathbb{C}[G]$ be an invariant subspace. For any $f \in \mathbb{C}[G]$, we have

   $$f \star f' \in W \quad \forall f' \in W.$$

6. The element δ_1 acts as the identity for the $\star$ operation. Namely, for $f \in \mathbb{C}$

   $$f \star \delta_1 = \delta_1 \star f.$$

   This follows easily from the fact that $\delta_1 \star \delta_g = \delta_g \star \delta_1 = \delta_g$ for all $g \in G$.

Isotypic decomposition

Let V be a G-representation and let L be an irreducible G-representation.

Consider the set $\mathcal{S}$ of all invariant subspaces $S \subseteq V$ for which $S \simeq L$ as G-representation.

Set
$$W = \sum_{S \in \mathcal{S}} S;$$
then W is an invariant subspace of V.

Proposition: W is *isotypic* in the sense that any irreducible invariant subspace of W is isomorphic (as G-representation) to L.

Moreover, $[V/W : L] = 0$.

You will prove this in homework.

We write $V_{(L)}$ for the invariant subspace W.

You will also prove:

Proposition: If $L_1, L_2, \cdots, L_r$ is a complete set of non-isomorphic irreducible representations of G, then
$$V = V_{(L_1)} \oplus V_{(L_2)} \oplus \cdots \oplus V_{(L_r)}.$$

Results about the characters of the irreducible representations

Investigation of certain idempotent elements in $\mathbb{C}[G]$.

Let L be an irreducible representation of G and let $W_1 = \mathbb{C}[G]_{(L)}$.

Use completely reduciblity to write
$$\mathbb{C}[G] = W_1 \oplus W_2$$
for some invariant subspace $W_2 \subset \mathbb{C}[G]$.

Note that $[W_2 : L] = 0$ by construction.

We now write
$$\delta_1 = e_1 + e_2 \quad \text{for } e_1 \in W_1 \text{ and } e_2 \in W_2.$$

Proposition: For $w_1 \in W_1$ and $w_2 \in W_2$ we have
$$e_1 \star w_1 = w_1, \quad e_1 \star w_2 = 0,$$
$$e_2 \star w_1 = 0, \quad \text{and} \quad e_2 \star w_2 = w_2.$$

Proof: Fix $w_2 \in W_2$. We define a mapping $\phi : W_1 \to W_2$ by the rule $\phi(w_1) = w_1 \star w_2$.

We note that ϕ is a homomorphism of G-representations. Indeed, recall the action of $g \in G$ on $\mathbb{C}[G]$ is the same as that of $\delta_g \star$. Now
$$\phi(\delta_g \star w_1) = (\delta_g \star w_1) \star w_2 = \delta_g \star (w_1 \star w_2) = \delta_g \star \phi(w_2).$$

Since W_1 is L-isotypic and since $[W_2 : L] = 0$, the mapping ϕ must be 0.

Now conclude that
$$0 = w_1 \star w_2 \quad \forall w_1 \in W_1, w_2 \in W_2.$$

A similar argument shows that
$$0 = w_2 \star w_1 \quad \forall w_1 \in W_1, w_2 \in W_2.$$

(For $w_1 \in W_1$, one define a homomorphism $\psi : W_2 \to W_1$ by the rule $\psi(w_2) = w_2 \star w_1$ As before, one argues that $\psi = 0 \ldots$)

Now notice for $w_1 \in W_1$ that

$$w_1 = \delta_1 \star w_1 = (e_1 + e_2) \star w_1 = e_1 \star w_1 + e_2 \star w_1 = e_1 \star w_1$$

since $e_2 \in W_2 \implies e_2 \star w_1 = 0$ by the preceding results. This proves that $e_1 \star w_1 = w_1$ for all $w_1 \in W_1$

Similarly, for $w_2 \in W_2$ we have

$$w_2 = \delta_1 \star w_2 = (e_1 + e_2) \star w_2 = e_1 \star w_2 + e_2 \star w_2 = e_2 \star w_2$$

since $w_2 \in W_2 \implies e_1 \star w_2 = 0$ by the preceding results.

This completes the proof.

As an immediate consequence, we get:

Corollary:

- $e_1 \star e_1 = e_1$
- $e_2 \star e_2 = e_2$
- $e_1 \star e_2 = e_2 \star e_1 = 0.$

We can actually find a *formula* expressing e_1 in the basis $\{\delta_g\}$ for $\mathbb{C}[G]$:

Proposition: Let $W_1 = \mathbb{C}[G]_{(L)}$ and suppose that $\mathbb{C}[G] = W_1 \oplus W_2$ for an invariant subspace W_2 as before. Write $\delta_1 = e_1 + e_2$ with $e_i \in W_i$. and let χ be the *character* of W_1. We have

$$e_1 = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) \delta_g.$$

Proof: Fix $x \in G$ and define

$$\Phi : \mathbb{C}[G] \to \mathbb{C}[G]$$

by the rule

$$\Phi(f) = \delta_{x^{-1}} \star e_1 \star f.$$

We are going to compute the *trace* of Φ in two different ways.

First, Note that since $e_1 \star w_1 = w_1$ for each $w_1 \in W_1$, we see that $\Phi_{|W_1}$ is given

$$w \mapsto \delta_{x^{-1}} \star w$$

. Thus tr$(\Phi_{|W_1})$ is given by $\chi(x^{-1})$.

Since $e_1 \star W_2 = 0$, we conclude that $\Phi_{|W_2}) = 0$ so that

$$\text{tr}(\Phi) = \text{tr}(\Phi_{|W_1}) \oplus \text{tr}(\Phi_{|W_2}) = \text{tr}(\Phi_{|W_1}) = \chi(x^{-1}),$$

On the other hand, let us express e_1 in the basis $\{\delta_g\}$ of $\mathbb{C}[G]$:

$$e_1 = \sum_{g \in G} \lambda_g \delta_g \quad (\lambda_g \in \mathbb{C}).$$

Let us examine the mapping

$$\theta_{x^-1g} : \mathbb{C}[G] \to \mathbb{C}[G]$$

given by

$$w \mapsto \delta_{x^-1g} \star w.$$

Recall that $\mathbb{C}[G]$ is the permutation representation corresponding to the action of G on itself by left multiplication. We have seen that the trace of the action of an element of G is the number of fixed points for that action. We conclude that the trace of $\theta_{x^{-1}g}$ is $|G|$ if $x = g$ and otherwise is 0.

Now, the mapping Φ is given by

$$\Phi(w) = \delta_{x^{-1}} \star \left(\sum_{g \in G} \lambda_g \delta_g \right) \star w = \sum_{g \in G} \lambda_g \delta_{x^{-1}g} \star w = \sum_{g \in G} \lambda_g \theta_{x^{-1}g}(w).$$

Since the trace is a linear operator, we conclude that

$$\mathrm{tr}(\Phi) = \sum_{g \in G} \lambda_g \, \mathrm{tr}(\theta_{x^{-1}g}) = \lambda_x |G|.$$

Now comparing our two computations of $\mathrm{tr}(\Phi)$ we get the formula

$$\lambda_x |G| = \chi(x^{-1})$$

i.e. $\lambda_x = \chi(x^{-1})/|G|$.

But then

$$e_1 = \sum_{g \in G} \lambda_g \delta_g = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) \delta_g$$

as required. This completes the proof.

Remark: Since G is a finite group, the eigenvalues of the operation of $g \in G$ on any G-representation are roots of unity ζ. Notice that $\overline{\zeta} = \zeta^{-1}$ for any root of unity. In particular, if χ is the character of a representation of G, we have $\chi(g^{-1}) = \overline{\chi(g)}$.

Corollary: Let χ be the character of $W_1 = \mathbb{C}[G]_{(L)}$. Then

$$\langle \chi, \chi \rangle = \chi(1).$$

Proof: Note that the Proposition allows us to calculate:

$$e_1 \star e_1 = \frac{1}{|G|} \sum_{x, g \in G} \chi(g^{-1}) \chi(x^{-1}) \delta_{gx}.$$

The coefficient of δ_1 in this expression is precisely

$$\frac{1}{|G|^2} \sum_{g \in G} \chi(g^{-1}) \chi(g) = \frac{1}{|G|^2} \sum_{g \in G} \chi(g) \overline{\chi(g)} = \frac{1}{|G|} \langle \chi, \chi \rangle.$$

On the other hand, $e_1 = e_1 \star e_1$ and the coefficient of δ_1 in the expression for e_1 is $\chi(1^{-1}) = \chi(1)$.

Thus we see that $\chi(1) = \langle \chi, \chi \rangle$ as required.

We have so far elided the *multiplicities* $[\mathbb{C}[G] : L]$ for irreducible representations L. We are going to state the result here (and maybe prove it later).

Theorem: For an irreducible representation L of G, the multiplicity $[\mathbb{C}[G] : L]$ is given by

$$[\mathbb{C}[G] : L] = \dim_{\mathbb{C}} L.$$

Remark: If χ, ψ are the characters of representations of G, then $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle$.

Indeed,

$$\begin{aligned} \langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)} &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{x \in G} \chi(x^{-1}) \psi(x) \\ &= \langle \psi, \chi \rangle. \end{aligned}$$

Now we are able to prove that main Theorem which shows that the characters of the irreducible representations form an *orthonormal set*.

Theorem: Let U, V be irreducible representations of G with characters χ, ψ respectively. Then

$$\langle \chi, \chi \rangle = 1 \quad \text{and} \quad \langle \chi, \psi \rangle = 0.$$

Proof: Let $W = \mathbb{C}[G]_{(U)}$ be the U-isotypic part of $\mathbb{C}[G]$ and let χ_W be the character of W.

The preceding Theorem tells us that $\chi_W = m\chi$ where $m = \dim U$.

Now, the preceding Corollary tells us that

$$\langle \chi_W, \chi_W \rangle = \chi_W(1).$$

Note that

$$\langle \chi_W, \chi_W \rangle = \langle m\chi, m\chi \rangle = m^2 \langle \chi, \chi \rangle.$$

On the other hand,

$$\chi_W(1) = m\chi(1) = m^2.$$

Thus we find $m^2\langle \chi, \chi \rangle = m^2$; since $m \neq 0$, conclude $\langle \chi, \chi \rangle = 1$ as required.

Now let $Y = \mathbb{C}[G]_{(U)} + \mathbb{C}[G]_{(V)}$ be the sum of the isotypic components.

Note that $\chi_Y = m\chi + n\psi$ where $m = \dim U$ and $n = \dim V$. Now we have

$$\chi_Y(1) = \langle \chi_Y, \chi_Y \rangle.$$

On the one hand,

$$\chi_Y(1) = m\chi(1) + n\psi(1) = m^2 + n^2.$$

And on the other hand, the preceding corollary shows that

$$\begin{aligned} \langle \chi_Y, \chi_Y \rangle &= \langle m\chi + n\psi, m\chi + n\psi \rangle \\ &= m^2\langle \chi, \chi \rangle + n^2 \langle \psi, \psi \rangle + 2mn\langle \chi, \psi \rangle \\ &= m^2 + n^2 + 2mn\langle \chi, \psi \rangle. \end{aligned}$$

Thus we find

$$m^2 + n^2 = m^2 + n^2 + 2mn\langle \chi, \psi \rangle$$

so that indeed $\langle \chi, \psi \rangle = 0$. This completes the proof.

Remark: In order to complete the proof that the irreducible characters form an orthonormal *basis* for the space of class functions on G, we still need to prove that the number of distinct irreducible representations is equal to the number of conjugacy classes in G.

Bibliography