Characters of irreducible representations

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Convolution

We write $\mathbb{C}[G]$ for the space of functions on G, viewed as a permutation representation of G (and we suppress the notation for the homomorphism $G \to \text{GL}(\mathbb{C}[G])$).

For functions $f_1, f_2 \in \mathbb{C}[G]$, we define their *convolution* by the formula

$$(f_1 \star f_2)(x) = \sum_{yz=x} f_1(y) f_2(z).$$

If V is a G-representation and $f \in \mathbb{C}[G]$, we define

$$f \star v = \sum_{g \in G} f(g) g v$$

for $v \in V$.

Remark:

1. For the basis elements $\delta_g \in \mathbb{C}[G]$ (i.e. the *Dirac functions*), we have

$$\delta_g \star \delta_h = \delta_{gh}.$$

2. The action of G on $\mathbb{C}[G]$ can be described by

$$gf = \delta_g \star f$$

   for $g \in G$ and $f \in \mathbb{C}[G]$.

3. Viewing $\mathbb{C}[G]$ as a G-representation, the two notions of $\star$ just introduced actually coincide:

$$f_1 \star f_2 = \sum_{g \in G} f_1(g) \delta_g \star f_2.$$

4. The product $\star$ makes $\mathbb{C}[G]$ into a *ring* (in fact, a $\mathbb{C}$-algebra) and V into a $\mathbb{C}[G]$-module. Mostly we won't use this fact - at least explicitly - in these notes.

5. Let $W \subseteq \mathbb{C}[G]$ be an invariant subspace. For any $f \in \mathbb{C}[G]$, we have

$$f \star f' \in W \quad \forall f' \in W.$$

6. The element δ_1 acts as the identity for the $\star$ operation. Namely, for $f \in \mathbb{C}$

$$f \star \delta_1 = \delta_1 \star f.$$

   This follows easily from the fact that $\delta_1 \star \delta_g = \delta_g \star \delta_1 = \delta_g$ for all $g \in G$.

Isotypic decomposition

Let V be a G-representation and let L be an irreducible G-representation.

Consider the set $\mathcal{S}$ of all invariant subspaces $S \subseteq V$ for which $S \simeq L$ as G-representation.

Set

$$W = \sum_{S \in \mathcal{S}} S;$$

then W is an invariant subspace of V.

Proposition: W is *isotypic* in the sense that any irreducible invariant subspace of W is isomorphic (as G-representation) to L.

Moreover, $[V/W : L] = 0$.

You will prove this in homework.

We write $V_{(L)}$ for the invariant subspace W.

You will also prove:

Proposition: If $L_1, L_2, \cdots, L_r$ is a complete set of non-isomorphic irreducible representations of G, then

$$V = V_{(L_1)} \oplus V_{(L_2)} \oplus \cdots \oplus V_{(L_r)}.$$

Results about the characters of the irreducible representations

Investigation of certain idempotent elements in $\mathbb{C}[G]$.

Let L be an irreducible representation of G and let $W_1 = \mathbb{C}[G]_{(L)}$.

Use completely reduciblity to write

$$\mathbb{C}[G] = W_1 \oplus W_2$$

for some invariant subspace $W_2 \subset \mathbb{C}[G]$.

We now write

$$\delta_1 = e_1 + e_2 \quad \text{for } e_1 \in W_1 \text{ and } e_2 \in W_2.$$

Proposition: For $w_1 \in W_1$ and $w_2 \in W_2$ we have

$$\begin{aligned} e_1 \star w_1 = w_1 \quad e_1 \star w_2 = 0 \\ e_2 \star w_1 = 0 \quad e_2 \star w_2 = w_2. \end{aligned}$$

Bibliography