

# (Representation Theory) Notes on Representations

George McNinch

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Let  $G$  be a finite group, let  $F$  be a field (algebraically closed, char. 0).

A representation of  $G$  is a vector space  $V$  together with a group homomorphism

$$\rho : G \rightarrow \mathrm{GL}(V).$$

If we choose a basis  $\mathcal{B}$  of  $V$ , we obtain from  $\rho$  an assignment  $g \mapsto [\rho(g)]_{\mathcal{B}}$  which is a group homomorphism  $G \rightarrow \mathrm{GL}_n(F)$ .

## First example

Consider the finite cyclic group  $G = \mathbb{Z}/n\mathbb{Z}$  for a natural number  $n > 0$ .

Suppose that  $M$  is an  $m \times m$  matrix for which  $M^n = \mathbf{I}$ . Using  $M$ , we get a representation

$$\rho_M : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathrm{GL}_m(F)$$

by the rule

$$\rho_M(i + n\mathbb{Z}) = M^i.$$

One might consider the question: If  $M$  and  $M'$  are two  $m \times m$  matrices for which  $M^n = (M')^n = \mathbf{I}$ , when are the representations  $\rho_M$  and  $\rho_{M'}$  “the same”?

Consider the following  $3 \times 3$  matrices, each of which satisfies  $M^3 = \mathbf{I}$ :

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 \\ 3/2 & 0 & 0 \\ 0 & -1/3 & 0 \end{pmatrix}.$$

where  $\zeta$  is a root of  $\frac{T^3 - 1}{T - 1} = T^2 + T + 1$ .<sup>1</sup>

In fact, all three of these matrices have eigenvalues  $1, \zeta, \zeta^2$  each with multiplicity 1.

## Example: two commuting matrices

Suppose that  $M, N$  are  $m \times m$  matrices for which  $M^n = \mathbf{I}, N^k = \mathbf{I}$  and  $MN = NM$ .

We get a representation

$$\rho : \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z} \rightarrow \mathrm{GL}_m(F)$$

by the rule

$$\rho(i + n\mathbb{Z}, j + k\mathbb{Z}) = M^i N^j.$$

**Remark** Let  $\mu$  be an eigenvalue of the matrix  $M$ , and let  $V_\mu \subset F^m$  be the corresponding *eigenspace*.

Since  $M$  and  $N$  commute, a homework exercise shows that  $V_\mu$  is *invariant* under the action of  $N$ .

Then

$$N|_{V_\mu} : V_\mu \rightarrow V_\mu$$

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<sup>1</sup>If  $F = \mathbb{C}$  then we can take  $\zeta = e^{2\pi i/3}$ .

is a linear transformation satisfying  $(N|_{V_\mu})^k = \text{id}$ .

In particular, we may choose a basis of  $V_\mu$  of eigenvectors for  $N|_{V_\mu}$ .

In this way, we see that  $V = F^m$  has a basis consisting of vectors which are simultaneously eigenvectors for  $N$  and for  $M$ .

## Construction of representations from group actions

Suppose that the (finite) group  $G$  acts on the (finite) set  $\Omega$ . We are going to use this action to define a representation of  $G$ .

We consider the vector space  $F[\Omega]$  of all  $F$ -valued functions on  $\Omega$ .

For  $g \in G$ , we consider the linear transformation

$$\rho(g) : F[\Omega] \rightarrow F[\Omega]$$

defined for  $f \in F[\Omega]$  by the rule

$$\rho(g)(f)(\omega) = f(g^{-1}\omega).$$

**Proposition:** (a).  $\rho(g)$  is an  $F$ -linear map.

(b).  $\rho(gh) = \rho(g)\rho(h)$  for  $g, h \in G$ .

(c).  $\rho(g) : F[\Omega] \rightarrow F[\Omega]$  is invertible.

**Proof of (b):** Let  $f \in F[\Omega]$ . For  $\tau \in \Omega$  we have

$$\rho(gh)f(\tau) = f((gh)^{-1}\tau) = f(h^{-1}g^{-1}\tau) = \rho(h)f(g^{-1}\tau) = \rho(g)\rho(h)f(\tau).$$

Since this holds for all  $\tau$ , conclude that

$$\rho(gh)f = \rho(g)\rho(h)f.$$

Since this holds for all  $f$ , conclude that  $\rho(gh) = \rho(g)\rho(h)$  as required.

The Proposition shows that

$$\rho : G \rightarrow \text{GL}(F[\Omega])$$

is a representation of  $G$ .

Here is an alternate perspective on  $F[\Omega]$  and the action of  $G$ .

For  $\omega \in \Omega$ , consider the *Dirac* function  $\delta_\omega \in F[\Omega]$  defined by

$$\delta_\omega(\tau) = \begin{cases} 1 & \omega = \tau \\ 0 & \text{else} \end{cases}$$

**Proposition:** The collection  $\delta_\omega$  forms a *basis* for the  $F$ -vector space  $F[\Omega]$ .

**Sketch:** For  $f \in F[\Omega]$ , we have

$$f = \sum_{\omega \in \Omega} f(\omega)\delta_\omega.$$

This shows that the  $\delta_\omega$  *span*  $F[\Omega]$ .

Now, suppose  $0 = \sum_{\omega \in \Omega} a_\omega \delta_\omega$  for  $a_\omega \in F$  for each  $\omega \in \Omega$ .

To prove linear independence, we must argue that  $a_\omega = 0$  for every  $\omega$ .

Set  $f = \sum_{\omega \in \Omega} a_\omega \delta_\omega$ . For  $\tau \in \Omega$ , the function  $f$  satisfies

$$f(\tau) = \sum_{\omega \in \Omega} a_\omega \delta_\omega(\tau) = a_\tau.$$

Since  $f = 0$  - i.e. “ $f$  is the function which is takes value zero at every argument” - we conclude that  $a_\tau = 0$  for each  $\tau$ . This proves linear independence.

Let's consider the action of  $G$  on these basis vectors.

**Proposition:** For  $g \in G$  and  $\omega \in \Omega$ , we have

$$\rho(g)\delta_\omega = \delta_{g\omega}.$$

**Proof:** For  $\tau \in \Omega$  we have

$$\rho(g)\delta_\omega(\tau) = \delta_\omega(g^{-1}\tau) = \begin{cases} 1 & \omega = g^{-1}\tau \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & g\omega = \tau \\ 0 & \text{else} \end{cases}$$

This shows that  $\rho(g)\delta_\omega(\tau) = \delta_{g\omega}(\tau)$ ; since this holds for all  $\tau$  we conclude  $\rho(g)\delta_\omega = \delta_{g\omega}$  as required.

**Remark:** Viewing  $F[\Omega]$  as the vector space with basis  $\delta_\omega$  gives us a perhaps simpler proof that  $\rho(gh) = \rho(g)\rho(h)$ .

Indeed, for  $\omega \in \Omega$  we have

$$\rho(gh)\delta_\omega = \delta_{gh\omega} = \rho(g)\delta_{h\omega} = \rho(g)(\rho(h)\delta_\omega) = \rho(g)\rho(h)\delta_\omega.$$

## Homomorphisms

Let  $(\rho, V)$  and  $(\psi, W)$  be representations of  $G$ . a linear mapping  $\Phi : V \rightarrow W$  is a *homomorphism of  $G$ -representations* provided that for every  $v \in V$  and for every  $g \in G$  we have

$$\Phi(\rho(g)v) = \psi(g)\Phi(v).$$

The  $G$ -representations  $(\rho, V)$  and  $(\psi, W)$  are isomorphic if there is a homomorphism of  $G$ -representations which is *invertible*.

### Example

For any  $G$  one has the so-called *trivial representation*  $\mathbb{1}$ . This representation has vector space  $V = F$  (it is 1 dimensional!) and the mapping

$$G \rightarrow \text{GL}(V) = \text{GL}_1(F) = F^\times$$

is just the trivial homomorphism  $g \mapsto 1$ .

One often writes  $\mathbb{1}$  for the *underlying vector space*  $F$ .

Let  $G$  act on the (finite) set  $\Omega$ . Then there is a homomorphism of  $G$ -representations

$$\mathbb{1} \rightarrow F[\Omega].$$

Viewing  $\mathbb{1} = F$ , the scalar  $c$  is mapped to the *constant function with value  $c$* .

Put another way, the scalar  $c$  is mapped to the vector

$$c \sum_{\omega \in \Omega} \delta_\omega.$$

## Bibliography