

Introducing the character table of a finite group

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Let G a finite group; we consider finite dimensional representations of G on \mathbb{C} -vector spaces.

Previews!

Let L_1, \dots, L_m be a complete set of non-isomorphic irreducible representations for the finite group G , and let χ_i be the *character* of L_i .

Recall that *any* G -representation V can be written as a direct sum of irreducible subrepresentations:

$$V = \bigoplus_{j=1}^N S_j$$

where each S_j is an irreducible representation. For each $1 \leq i \leq m$, we say that the *multiplicity of L_i in V* – written $[V : L_i]$ – is the natural number given by

$$[V : L_i] = \#\{j \mid S_j \simeq L_i\}$$

Thus we have

$$V \simeq \bigoplus_{i=1}^m L_i^{\oplus [V : L_i]}$$

where the notation $W^{\oplus d}$ means

$$W^{\oplus d} = W \oplus W \oplus \dots \oplus W \quad (d \text{ copies}).$$

Next week, we are going to sketch a proof of the following facts

Theorem:

- The number m of irreducible representations of G is equal to the number of *conjugacy classes* in G .
- χ_1, \dots, χ_m are an *orthonormal basis* for the space $\text{Cl}(G)$ of \mathbb{C} -value class functions on G .
- For any G -representation V , let χ be the character of V . Then the multiplicity $[V : L_i]$ is given by

$$[V : L_i] = \langle \chi, \chi_i \rangle.$$

In particular, write $k_i = [V : L_i]$; then

$$\chi = k_1 \chi_1 + k_2 \chi_2 + \dots + k_m \chi_m.$$

- The multiplicity with which L_i appears in the *regular representation* $F[G]$ is given by

$$[F[G] : L_i] = \dim_F L_i.$$

In particular, if $d_i = \dim_F L_i$, then

$$|G| = \sum_{i=1}^m d_i^2.$$

The Hermitian inner product on class functions, again

Enumerate the conjugacy classes of G , say C_1, \dots, C_m and choose a representative $g_i \in C_i$ for each i .

Write $c_i = |C_G(g_i)|$ for the number of elements in the centralizer of g_i , and notice that

$$|\text{Cl}(g_i)| = |G|/|C_G(g_i)| = |G|/c_i.$$

Recall that for two class functions $f_1, f_2 \in \text{Cl}(G)$ we have defined

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Lemma: We have

$$\langle f_1, f_2 \rangle = \frac{1}{c_i} \sum_{i=1}^m f_1(g_i) \overline{f_2(g_i)}.$$

The character table.

The matrix is known as the *character table of the group G* . Consider the $m \times m$ matrix whose rows are indexed by the irreducible characters χ_1, \dots, χ_m and whose columns are indexed by the conjugacy class representatives g_1, \dots, g_m , and whose entry in the i -th row and j -th column is given by $\chi_i(g_j)$.

	g_1	g_2	\dots	g_m
	c_1	c_2	\dots	c_m
χ_1	$\chi_1(g_1)$	$\chi_1(g_2)$	\dots	$\chi_1(g_m)$
χ_2	$\chi_2(g_1)$	$\chi_2(g_2)$	\dots	$\chi_2(g_m)$
\vdots	\vdots	\vdots	\ddots	\vdots
χ_m	$\chi_m(g_1)$	$\chi_m(g_2)$	\dots	$\chi_m(g_m)$

Remark:

- The fact that the χ_i form an orthonormal basis for the space $\text{Cl}(G)$ is equivalent to the statement that the above matrix is *unitary*, in the sense that for $1 \leq i, j \leq m$ we have

$$\sum_{k=1}^m c_k \chi_i(g_k) \overline{\chi_j(g_k)} = \delta_{i,j}.$$

- Since the transpose of unitary matrix is also unitary, we find for $1 \leq i, j \leq m$ that

$$\sum_{k=1}^m c_i \chi_k(g_i) \overline{\chi_k(g_j)} = \delta_{i,j}.$$

We are now going to compute the character table for a few finite groups

Cyclic groups

Let $G = \mathbb{Z}/N\mathbb{Z}$ for a natural number N . If ζ is a primitive N -th root of unity, we have for $0 \leq i < N$ a homomorphism

$$\rho_i : G \rightarrow F^\times$$

determined by the equation

$$\rho_i(1 + N\mathbb{Z}) = \zeta^i.$$

Since G is abelian, its conjugacy classes are *singletons*; thus the number of irreducible representations is $N = |G|$.

Each ρ_i determines a 1-dimensional irreducible representation, and $i \neq j \implies \rho_i \not\cong \rho_j$.

Orthonormality of characters implies that

$$\langle \rho_i, \rho_j \rangle = \delta_{i,j}$$

This equality can actually be deduce in a more elementary way. For example, write $\mathbf{1} = \rho_0$.

Note that

$$\langle \rho_1, \mathbf{1} \rangle = \frac{1}{N} \sum_{i=0}^N \zeta^i = 0$$

since ζ is a root of $\frac{T^N - 1}{T - 1} = T^N + T^{N-1} + \dots + T + 1$.

Consider a function $f \in F[\mathbb{Z}/N\mathbb{Z}]$. Using orthonormality of characters we deduce

$$f = \sum_{i=0}^{N-1} \langle f, \rho_i \rangle \rho_i.$$

The assignment $i \mapsto \langle f, \rho_i \rangle$ is usual known as the *discrete Fourier transform* of f .

The group $D_6 = S_3$.

Let $G = S_3$ be the symmetric group on 3 letters, so that $|G| = 6$.

Note that the subgroup $H = \langle (123) \rangle \subset G$ has index 2 and is thus a *normal* subgroup. (In fact, H is the *alternating group* A_3).

Notice that the centralizer of (123) coincides with H , so that (123) has exactly $[G : H] = 2$ conjugates (namely, (123) and (132)). On the other hand, the centralizer of (12) is $\langle (12) \rangle$ so that (12) has 3 conjugates; namely (12) , (13) and (23) .

Thus 1 , (12) , and (123) is a full set of representatives for the conjugacy classes of G .

In particular, we expect to find 3 irreducible representations of G .

There are exactly two homomorphisms $G \rightarrow F^\times$ which contain H in their kernel; one is the trivial mapping $\mathbf{1}$, and the other is the sign homomorphism $\text{sgn} : G \rightarrow F^\times$; it is the unique mapping for which $H \subset \ker \text{sgn}$ and $\text{sgn}((12)) = -1$.

Thus $\mathbf{1}$ and sgn are 1-dimensional irreducible representations of G .

It remains to find one more irreducible representation.

Let $\Omega = \{1, 2, 3\}$ and consider the standard action of $G = S_3$ on Ω . Write $\chi = \chi_\Omega$ for the *character* of this representation.

You will see for homework that $\chi(\sigma)$ is equal to the number of fixed points of σ on Ω .

We have seen that the trivial representation $\mathbf{1}$ appears in the representation $F[\Omega]$; thus

$$F[\Omega] = W \oplus \mathbf{1}.$$

You will argue in homework that the character of W is given by $\psi = \chi - \mathbf{1}$.

Thus χ and $\chi - \mathbf{1}$ are given by:

	$\chi(\sigma)$	$\psi(\sigma)$
1	3	2
(12)	1	0
(123)	0	-1

Now, we compute

$$\langle \psi, \psi \rangle = 1/6 \cdot 2 \cdot 2 + 1/2 \cdot 0 \cdot 0 + 1/3 \cdot (-1) \cdot (-1) = 2/3 + 1/3 = 1$$

This shows that ψ is an irreducible representation. Thus the character table of G is given by

	1	(12)	(123)
	6	2	3
1	1	1	1
sgn	1	-1	1
ψ	2	0	1

Bibliography