

ProblemSet 1 – Linear algebra and representations

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F denotes an *algebraically closed field of characteristic 0*. If you like, you can suppose that $F = \mathbf{C}$ is the field of complex numbers.

1. Let V be a finite dimensional vector space over the field F . Suppose that $\phi, \psi : V \rightarrow V$ are linear maps. Let $\lambda \in F$ be an eigenvalue of ϕ and write W for the λ -eigenspace of ϕ ; i.e.

$$W = \{v \in V \mid \phi(v) = \lambda v\}.$$

If $\phi\psi = \psi\phi$ show that W is *invariant* under ψ – i.e. show that $\psi(W) \subseteq W$.

2. Let $n \in \mathbf{N}$ be a non-zero natural number, and let V be an n dimensional F -vector space with a given basis e_1, e_2, \dots, e_n .

Consider the linear transformation $T : V \rightarrow V$ given by the rule

$$Te_i = e_{i+1 \pmod n}.$$

In other words

$$Te_i = \begin{cases} e_{i+1} & i < n \\ e_1 & i = n \end{cases}.$$

- a. Show that T is *invertible* and that $T^n = \text{id}_V$.

- b. Consider the vector $v_0 = \sum_{i=1}^n e_i$. Show that v_0 is a 1-eigenvector for T .

Let $\zeta \in F$ be a primitive n -th root of unity. (e.g. if you assume $F = \mathbf{C}$, you may as well take $\zeta = e^{2\pi i/n}$).

- c. Let $v_1 = \sum_{i=1}^n \zeta^i e_i$. Show that v_1 is a ζ -eigenvector for ϕ .

- d. More generally, let $0 \leq j < n$ and let

$$v_j = \sum_{i=1}^n \zeta^{ij} e_i.$$

Show that v_j is a ζ^j -eigenvector for ϕ .

- e. Conclude that v_0, v_1, \dots, v_{n-1} is a basis of V consisting of *eigenvectors* for ϕ , so that ϕ is *diagonalizable*.

Hint: You need to use the **fact** that eigenvectors for distinct eigenvalues are *linearly independent*.

What is the *matrix* of ϕ in this basis?

3. Let $G = \mathbb{Z}/3\mathbb{Z}$ be the additive group of order 3, and let ζ be a primitive 3rd root of unity in F .

To define a representation $\rho : G \rightarrow \text{GL}_n(F)$, it is enough to find a matrix $M \in \text{GL}_n(F)$ with $M^3 = 1$; in turn, M determines a representation ρ by the rule $\rho(i + 3\mathbb{Z}) = M^i$.

Consider the *representation* $\rho_1 : G \rightarrow \text{GL}_3(F)$ given by the matrix

$$\rho_1(1 + 3\mathbb{Z}) = M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}$$

and consider the representation $\rho_2 : G \rightarrow \text{GL}_2(F)$ given by the matrix

$$\rho_2(1 + 3\mathbb{Z}) = M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that the *representations* ρ_1 and ρ_2 are *equivalent* (alternative terminology: are *isomorphic*). In other words, find a linear bijection $\Phi : F^2 \rightarrow F^2$ with the property that

$$\Phi(\rho_2(g)v) = \rho_1(g)\Phi(v)$$

for every $g \in G$ and $v \in F^2$.

Hint: First find a basis of F^2 consisting of eigenvectors for the matrix M_2 .

4. Let V be a n dimensional F -vector space for $n \in \mathbb{N}$.

Let $\text{GL}(V)$ denote the group

$$\text{GL}(V) = \{\text{all invertible } F\text{-linear transformations } \phi : V \rightarrow V\}$$

where the group operation is *composition* of linear transformations.

Recall that $\text{GL}_n(F)$ denotes the group of all invertible $n \times n$ matrices.

If $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ is a choice of basis, show that the assignment $\phi \mapsto [\phi]_{\mathcal{B}}$ determines an isomorphism

$$\text{GL}(V) \xrightarrow{\sim} \text{GL}_n(F).$$

Here $[\phi]_{\mathcal{B}} = [M_{ij}]$ denotes the *matrix* of ϕ in the basis \mathcal{B} defined by equations

$$\phi(b_i) = \sum_{k=1}^n M_{ki} b_k.$$

Bibliography