

# (Representation Theory) Notes on Groups & Linear Algebra

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2024-01-17

In the first lecture, we discussed some examples of groups and some basics of linear algebra.

## Groups

- the elements of the *cyclic group*  $\mathbb{Z}/n\mathbb{Z}$  are the equivalence classes of integers under the relation “ $\equiv \pmod{n}$ ”  
this group is *additive*
- we observed that the mapping  $\phi : \mathbb{R} \rightarrow \mathbf{S}^1$  given by  $\phi(t) = e^{2\pi i t}$  is a *group homomorphism* since  $\phi(t+s) = \phi(t)\phi(s)$  for all  $t, s \in \mathbb{R}$ .

we observed that  $\ker \phi = \mathbb{Z}$ , and that - by the **First Isomorphism Theorem** -  $\phi$  induces an isomorphism

$$\bar{\phi} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbf{S}^1.$$

- for a non-zero natural number the symmetric group  $S_n$  is the collection of all bijections  $I_n \rightarrow I_n$  where  $I_n = \{1, 2, \dots, n\}$ .  
We may sometimes use *cycle notation* for elements of  $S_n$ .

The subgroup

$$H = \langle (1234), (14)(23) \rangle$$

has order 8 and is sometimes called the *dihedral group*  $D_4$  or  $D_8$  – it has order 8.

- Let  $F$  be a field.

Recall that typically examples are:  $F = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}$  for a prime number  $p$ .

The set

$$\mathrm{GL}_n(F) = \{\text{all invertible } n \times n \text{ matrices with entries in } F\}$$

forms a group under matrix multiplication.

The determinant function yields a group homomorphism

$$\det : \mathrm{GL}_n(F) \rightarrow F^\times$$

(here  $F^\times$  means  $F \setminus \{0\}$ , which is a commutative group under multiplication in the field  $F$ ).

## Linear Algebra

Let  $F$  be a field. An  $F$ -vector space  $V$  is an additive abelian group together with an operation of *scalar multiplication* – this amounts to a function

$$F \times V \rightarrow V$$

– satisfying certain axioms.

If  $V, W$  are  $F$ -vector spaces, a linear mapping  $T : V \rightarrow W$  is a function which satisfies

$$T(\alpha v + w) = \alpha T(v) + T(w).$$

Let's suppose that  $V$  is *finite dimensional* and that  $\phi : V \rightarrow V$ .

We write  $\phi^2 = \phi \circ \phi$  and more generally  $\phi^n = \phi \circ \phi^{n-1}$ .

## trace, det, char poly

The *trace* of a matrix  $M = [M_{ij}]$  is the sum of the diagonal entries:

$$\text{tr}(M) = \sum_{i=1}^n M_{ii}.$$

I'm assuming you recall the definition of the *determinant*  $\det M$ .

The characteristic polynomial  $\text{cp}_M(X) \in F[X]$  of  $M$  is defined to be

$$\text{cp}_M(X) = \det(M - X \cdot \mathbf{I}_n).$$

For a linear transformation  $\phi$  we define

- $\text{tr}(\phi) = \text{tr}([\phi]_{\mathcal{B}})$
- $\det(\phi) = \det([\phi]_{\mathcal{B}})$
- $\text{cp}_{\phi}(X) = \text{cp}_{[\phi]_{\mathcal{B}}}(X)$

**Proposition**  $\text{tr}(\phi)$ ,  $\det(\phi)$ , and  $\text{cp}_{\phi}(X)$  are independent of the choice  $\mathcal{B}$  of basis for  $V$ .

The main point here is that if  $\mathcal{B}$  and  $\mathcal{B}'$  are two basis for  $V$ , there is an invertible matrix ("change of basis matrix")  $P$  for which

$$[\phi]_{\mathcal{B}} = P[\phi]_{\mathcal{B}'}P^{-1}.$$

## Evaluation of polynomials at linear transformations

Suppose that  $f = f(X) \in F[X]$  is a polynomial; thus

$$f = \sum_{i=0}^N a_i X^i$$

for some coefficients  $a_i \in F$ .

We may *evaluate* the polynomial  $f$  at the linear endomorphism  $\phi$ :

$$f(\phi) = \sum_{i=0}^N a_i \phi^i.$$

**Proposition** Let  $\phi : V \rightarrow V$  be a linear transformation, and let

$$I = \{f \in F[X] \mid f(\phi) = 0\}.$$

Then  $I$  is an *ideal* in the polynomial ring  $F[X]$ . In particular, there is a unique monic polynomial  $m_{\phi}(X) \in F[X]$  for which  $I = m_{\phi}(X)F[X]$ .

In particular, if  $f \in F[X]$  and  $f(\phi) = 0$ , then  $m_{\phi} \mid f$ .

**Theorem (Cayley-Hamilton)** Let  $\phi : V \rightarrow V$  be a linear transformation, and let  $\text{cp}(X) = \text{cp}_{\phi}(X) \in F[X]$  be the characteristic polynomial.

Then  $\text{cp}(\phi) = 0$ .

Recall that the *eigenvalues* of  $\phi$  are precisely the roots of the characteristic polynomial. The Cayley-Hamilton Theorem implies that any root of the minimal polynomial is an eigenvalue. In fact, we have the converse as well:

**Proposition:** If  $\lambda \in F$  is an eigenvalue of  $\phi$  – i.e. a root of the characteristic polynomial – then  $\lambda$  is a root of the minimal polynomial.

**Theorem:**  $\phi$  is *diagonalizable* – i.e.  $V$  has a basis of eigenvectors for  $\phi$  – if and only if the minimal polynomial has no multiple roots.

**Remark:** This theorem should be proved in Math215-216 here at Tufts using the *Fundamental Theorem for modules over a PID*. We don't need the full force of this result in our class.

### Example

Suppose that  $\phi : V \rightarrow V$  satisfies  $\phi^N = \text{id}_V$  for some positive natural number  $N$ .

We suppose that  $F$  is *algebraically closed* and of *characteristic zero*.

Notice that the polynomial  $f(X) = X^N - 1 \in F[X]$  has distinct roots.

(If  $F = \mathbb{C}$ , these roots are exactly

$$\{\exp(2\pi ki/n) \mid 0 \leq k < N\}.$$

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Since the minimal polynomial of  $\phi$  divides  $f$ , we see that the minimal polynomial has distinct roots and hence  $\phi$  is diagonalizable by the **Theorem** quoted above.

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### Bibliography