

ProblemSet 4 – Finite fields and codes

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1. Let q be a power of a prime $p > 3$ and let $k = \mathbb{F}_q$.

For a homogeneous polynomial $F \in k[X, Y, Z, W]$, let us write

$$V(F) = \{P = (x : y : z : w) \in \mathbb{P}_k^3 \mid F(x, y, z, w) = 0\}$$

for the set of solutions of the equation $F = 0$ in \mathbb{P}_k^3 .

For $a \in k^\times$, consider the polynomial

$$F_a = XY + Z^2 - aW^2 \in k[X, Y, Z, W].$$

- a. If $4 \mid q - 1$ show that

$$|V(F_a)| = |V(X^2 + Y^2 + Z^2 - aW^2)|$$

Hint: First show that $X^2 + Y^2 + Z^2 - aW^2$ is obtained from F_a by a linear change of variables.

- b. If $a = 1$, show that $|V(F_1)| = q^2 + 2q + 1$.

Hint: Making a linear change of variables, first show that $|V(F_1)| = |V(G)|$ where $G = XY + ZW$.

To count the points $(x : y : z : w)$ in $V(G)$, first count the points with $xy = 0$ (and hence also $zw = 0$), and then the points with $xy \neq 0$.

Let $S = \{a^2 \mid a \in k\}$.

- c. Show that $|S| = \frac{q+1}{2}$. Conclude that there are $q - \frac{q+1}{2} = \frac{q-1}{2}$ non-squares in k .

- d. If $a \in S$, show that $|V(F_a)| = |V(F_1)| = q^2 + 2q + 1$.

- e. If $a \in k$, $a \notin S$, show for any $\alpha \in k^\times$ that there are exactly $q + 1$ pairs $(c, d) \in k \times k$ with $c^2 - ad^2 = \alpha$.

Hint: We may identify $\ell = \mathbb{F}_{q^2} = \mathbb{F}_q[\sqrt{a}]$. Under this identification, the norm homomorphism $N = N_{\ell/k} : \ell^\times \rightarrow k^\times$ is given by the formula

$$N(c + d\sqrt{a}) = (c + d\sqrt{a})(c - d\sqrt{a}) = c^2 - ad^2.$$

On the other hand, by Galois Theory, we have $N(x) = x \cdot x^q = x^{1+q}$ for any $x \in \ell$. Thus $N(\ell^\times) = k^\times$ and $|\ker N| = q + 1$.

- f. If $a \in k$, $a \notin S$ show that $|V(F_a)| = q^2 + 1$

Hint: Notice that the equation $Z^2 - aW^2 = 0$ has no solutions $(z : w) \in \mathbb{P}_k^1$, and use (e) to help count.

2. Let $f = T^{11} - 1 \in \mathbb{F}_4[T]$.

- a. Show that $T^{11} - 1$ has a root in \mathbb{F}_{4^5} .

- b. If $\alpha \in \mathbb{F}_{4^5}$ is a primitive element – i.e. an element of order $4^5 - 1$, find an element $a = \alpha^i \in \mathbb{F}_{4^5}$ of order 11, for a suitable i .

- c. Show that the minimal polynomial g of a over \mathbb{F}_4 has degree 5, and that the roots of g are powers of a . Which powers?

- d. Show that $f = g \cdot h \cdot (T - 1)$ for another irreducible polynomial $h \in \mathbb{F}_4[T]$ of degree 5. The roots of h are again powers of a . Which powers?
- e. Show that $\langle f \rangle$ is a $[11, 6, d]_4$ code for which $d \geq 4$.
3. Consider the following variant of a Reed-Solomon code: let $\mathcal{P} \subset \mathbb{F}_q$ be a subset with $n = |\mathcal{P}|$ and write $\mathcal{P} = \{a_1, \dots, a_n\}$. Let $1 \leq k \leq n$ and write $\mathbb{F}_q[T]_{<k}$ for the space of polynomial of degree $\leq k$, and let $C \subset \mathbb{F}_q^n$ be given by
- $$C = \{(p(a_1), \dots, p(a_n)) \mid p \in \mathbb{F}_q[T]_{<k}\}.$$
- a. Prove that C is a $[n, k, n - k]_q$ -code.
- b. If $P = \mathbb{F}_q^\times$, prove that C is a *cyclic code*.
- c. If $q = p$ is *prime* and if $P = \mathbb{F}_p$, prove that C is a *cyclic code*.