Representations and the symmetric group

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Rank preferences

I want to describe in brief the ideas investigated in (Diaconis 1989), which amounts as an application of representation theory on the symmetric group to statistics.

First, an example

Suppose we were to ask people to rank their preferred ice cream flavors from the following ordered list:

- [`pistachio`, `chocolate`, `strawberry`, `vanilla`, `neopolitan`]

Numbering the flavors [`0`, `1`, `2`, `3`, `4`] in order, we can represent an individual's preference using a permutation $\sigma \in S_5$.

For example the preference list

- [`neopolitan`, `strawberry`, `chocolate`, `vanilla`, `pistachio`]

corresponds to the product of transposition $(04)(12)$.

So our survey data amounts to a list of elements $\sigma_1, \sigma_2, \ldots, \sigma_N \in S_5$.

Ranking in more generality

In general, n will denote the number of items to be ranked, and we assume given a *list* of ranking data:

$$\Sigma : \sigma_1, \sigma_2, \cdots, \sigma_m$$

The main statistic of interest is the *frequency function*

$$f = f_\Sigma : S_n \to \mathbb{C}$$

given for $\sigma \in S_n$ by the rule $f(\sigma) = \#\{\sigma_i \mid \sigma = \sigma_i\}$.

Idea: View f as a vector in the regular representation $\mathbb{C}[S_n]$. We want to understand how f decomposes in some natural descriptions of $\mathbb{C}[S_n]$.

Irreducible representations of the symmetric group

Recall that two elements $\sigma, \tau \in S_n$ are *conjugate* $\iff$ they have the same *cycle structure*. We state this in the following form:

Lemma: Conjugacy classes in S_n are in bijection with *partitions* $\lambda \vdash n$.

For example, there are 5 partitions of 4; they are $\lambda = (1,1,1,1)$, $\lambda = (2,2)$, $\lambda = (2,1,1)$, $\lambda = (3,1)$, and $\lambda = (4)$.

As a consequence, we know:

Proposition: Isomorphism classes of irreducible representations of S_n on complex vector spaces are in bijection with partitions $\lambda \vdash n$.

Here is a quick overview of some facts about the representations of S_n, without proofs:

In fact, for each partition $\lambda \vdash n$, there is a construction of an irreducible representation V_λ for S_n. To begin the construction, associate with λ the subgroup

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_r}$$

let $\Omega_\lambda = S_n/S_\lambda$ be the set of left cosets of S_λ in S_n.

Thus Ω_λ is a set on which the symmetric group S_n acts, and we may consider the *permutation representation* $\mathbb{C}[\Omega_\lambda]$, and for brevity we write $M^\lambda = \mathbb{C}[\Omega_\lambda]$.

One defines the *dominance ordering* on partitions of n by the rule

$$\lambda \trianglelefteq \mu$$

if and only if for each ℓ

$$\sum_{i=1}^{\ell} \lambda_i \leq \sum_{i=1}^{\ell} \mu_i.$$

There is a labeling of irreducible representations of S_n by partition; let us write S^λ for the irreducible representation corresponding to λ. It is known as a *Specht representation*.

Theorem: $[M^\lambda : S^\lambda] = 1$ and $[M^\lambda : S^\mu] > 0 \implies \lambda \trianglelefteq \mu$.

As stated, the Theorem appears to depend on our knowledge of the irreducible representations, but in fact it gives us a way to define them.

For a fixed λ, consider all homomorphisms of S_n-representations

$$M^\mu \to M^\lambda \quad \text{for } \lambda \trianglelefteq \mu;$$

lets write $R^\lambda \subset M^\lambda$ for the *sum of the image of all of these homomorphisms.*

The Theorem implies that $M^\lambda/R^\lambda = S^\lambda$ is irreducible.

Here are some special cases:

- if $\lambda = (n)$ then $M^\lambda = S^\lambda = \mathbf{1}$ is the trivial representation.
- if $\lambda = (1, 1, \cdots, 1)$ them $S^\lambda = \text{sgn}$ is the (1-dimensional) *sign representation*.
- if $\lambda = (n-1, 1)$ then M^λ is the permutation representation $\mathbb{C}[\{1, 2, \cdots, n\}]$, and

  $$M^\lambda = S^\lambda \oplus \mathbf{1}$$

  so that dim $S^\lambda = n - 1$.

  Observe that indeed $(n-1, 1) \trianglelefteq (n)$.
- when $n = 3$, we saw previously that S_3 has 3 irreducible representations. They can be described as
  - $S^{(3)} = \mathbf{1}$ the trivial representation
  - $S^{(1,1,1)} = \text{sgn}$ the sign representation
  - $S^{(2,1)}$, an irreducible representation of dimension 2

Description of the regular representation of S_n.

Theorem: We have

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n} R^\lambda$$

where for each $\lambda \vdash n$, the representation R^λ is S^λ-isotypic; i.e. every irreducible representation contained in R^λ is isomorphic to S^λ.

Moreover,

$$R^\lambda = S^\lambda \oplus \cdots \oplus S^\lambda$$

and the number of summands is dim S^λ.

In particular, given $f \in \mathbb{C}[S_n]$, we may decompose f according to this decomposition of $\mathbb{C}[S_n]$; in other words, we may write

$$f = \sum_{\lambda \vdash n} f_\lambda$$

where $f_\lambda \in R^\lambda$.

We want to consider this description when f is the *frequency function* $f = f_\Sigma$ for some data

$$\Sigma : \sigma_1, \sigma_2, \cdots, \sigma_m$$

with $\sigma_i \in S_n$.

Lemma: For $f \in \mathbb{C}[S_n]$, the component $f_{(n)}$ corresponding to the trivial representation $\mathbf{1} = S^{(n)}$ is the *average value* of f:

$$f_{(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma).$$

First-order effects

Bibliography

Bibliography

Diaconis, Persi. 1989. "A Generalization of Spectral Analysis with Application to Ranked Data." *The Annals of Statistics* 17 (3): 949–79. https://www.jstor.org/stable/2241705.