ProblemSet 2 – Representations and characters

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In these exercises, G always denotes a finite group and all vector spaces are assumed to be finite dimensional over the field $F = \mathbb{C}$.

In these exercises, you may use results stated but not yet proved in class about characters of representations of G.

1. In this problem, we identify the character χ_Ω of the *permutation representation* $(\rho, F[\Omega])$ of a group G.

   a. Let V be a vector space and let $\Phi : V \to V$ a linear mapping If $\mathcal{B}$ is a basis for V, recall that the *trace* of Φ is defined by
   $$\mathrm{tr}(\Phi) = \mathrm{tr}([\Phi]_{\mathcal{B}}).$$

   b. Recall that the *dual* of V is the vector space $V^\vee = \mathrm{Hom}_F(V, F)$ of *linear functionals* on V.

      If $b_1, \dots, b_n$ is a basis for V, let ${b_j}^\vee : V \to F$ be defined by ${b_j}^\vee(b_i) = \delta_{i,j}$. Show that ${b_1}^\vee, \dots, {b_n}^\vee$ is a basis for $V^\vee$; it is known as the *dual basis* to $b_1, \dots, b_n$.

   c. Prove that the trace of the linear mapping $\Phi : V \to V$ is given by the expression
   $$\mathrm{tr}(\Phi) = \sum_{i=1}^{n} {b_i}^\vee(\Phi(b_i)).$$

   d. Suppose that the finite group G acts on the finite set Ω, and consider the corresponding permutation representation $(\rho, F[\Omega])$ of G. Recall that $F[\Omega]$ is the vector space of all F-values functions on Ω, and that for $f \in F[\Omega]$ and $g \in G$, we have
   $$\rho(g)f(\omega) = f(g^{-1}\omega).$$
   In particular, we saw in the lecture that
   $$\rho(g)\delta_\omega) = \delta_{g\omega},$$
   where δ_ω denotes the *Dirac function* at $\omega \in \Omega$.

      Show that
   $$\mathrm{tr}(\rho(g)) = \#\{\omega \in \Omega \mid g\omega = \omega\};$$
   i.e. the trace of $\rho(g)$ is the number of fixed points of the action of g on Ω.

2. Let V be a representation of G, suppose that W_1, W_2 are invariant subspaces, and that V is the internal direct sum
$$V = W_1 \oplus W_2.$$

   Show that the character χ_V of V satisfies
$$\chi_V = \chi_{W_1} + \chi_{W_2}$$
   i.e. for $g \in G$ that
$$\chi_V(g) = \chi_{W_1}(g) + \chi_{W_2}(g).$$

3. Let $G = A_4$ be the alternating group of order $\dfrac{4!}{2} = 12$.

   We are going to find the *character table* of this group.

a. Confirm that the following list gives a representative for each of the conjugacy classes of G:

$$1, (12)(34), (123), (124)$$

(Note that (123) and (124) are conjugate in S_4, but *not* in A_4).

What are the *sizes* of the corresponding conjugacy classes?

b. Let $K = \langle (12)(34), (14)(23) \rangle$. Show that K is a normal subgroup of index 3, so that $G/K \simeq \mathbb{Z}/3\mathbb{Z}$.

Let ζ_3 be a primitive 3rd root of unity in $F^\times$ and for $i = 0, 1, 2$ let $\rho_i : G \to F^\times$ be the unique homomorphism with the following properties:

  i. $\rho_i((123)) = \zeta^i$
  ii. $K \subseteq \ker \rho_i$.

Explain why $\rho_0 = \mathbf{1}, \rho_1, \rho_2$ determine distinct irreducible (1-dimensional) representations of G.

c. Let $\Omega = \{1, 2, 3, 4\}$ on which G acts by the embedding $A_4 \subset S_4$.

Compute the character χ_Ω of the representation $F[\Omega]$. (This means: compute and list the values of χ_Ω at the conjugacy class representatives given in a.)

(*Use the result of problem 1 above*).

d. The span of the vector $\delta_1 + \delta_2 + \delta_3 + \delta_4 \in F[\Omega]$ is an invariant subspace isomorphic to the irreducible representation ρ_0 (the so-called *trivial representation*).

Thus $F[\Omega] = \rho_0 \oplus W$ for a 3-dimensional invariant subspace. Explain why problem 2 shows that the character of W is given by $\chi_W = \chi_\Omega - \mathbf{1}$.

Now prove that $\langle \chi_W, \chi_W \rangle = 1$ and conclude that W is an irreducible representation.

e. Explain why

$$\mathbf{1}, \rho_1, \rho_2, W$$

is a complete set of non-isomorphic irreducible representations of G.

f. Display the *character table* of $G = A_4$.

Bibliography