Introducing the character table of a finite group

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Let G a finite group; we consider finite dimensional representations of G on $\mathbb{C}$-vector spaces.

Previews!

Let $L_1, \cdots, L_m$ be a complete set of non-isomorphic irreducible representations for the finite group G, and let χ_i be the *character* of L_i.

Recall that *any* G-representation V can be written as a direct sum of irreducible subrepresentations:

$$G = \bigoplus_{j=1}^{N} S_j$$

where each S_j is an irreducible representation. For each $1 \le i \le m$, we say that the *multiplicity of* L_i *in* V – written $[V : L_i]$ – is the natural number given by

$$[V : L_i] = \#\{j \mid S_j \simeq L_i\}$$

Thus we have

$$V \simeq \bigoplus_{i=1}^{m} L_i^{\oplus[V:L_i]}$$

where the notation $W^{\oplus d}$ means

$$W^{\oplus d} = W \oplus W \oplus \cdots \oplus W \quad (d \text{ copies}).$$

Next week, we are going to sketch a proof of the following facts

Theorem:

a. The number m of irreducible representations of G is equal to the number of *conjugacy classes* in G.
b. $\chi_1, \cdots, \chi_m$ are an *orthonormal basis* for the space $\mathrm{Cl}(G)$ of $\mathbb{C}$-value class functions on G.
c. For any G-representation V, let χ be the character of V. Then the multiplicity $[V : L_i]$ is given by

   $$[V : L_i] = \langle \chi, \chi_i \rangle.$$

   In particular, write $k_i = [V : L_i]$; then

   $$\chi = k_1\chi_1 + k_2\chi_2 + \cdots + k_m\chi_m.$$

d. The multiplicity with which L_i appears in the *regular representation* $F[G]$ is given by

   $$[F[G] : L_i] = \dim_F L_i.$$

   In particular, if $d_i = \dim_F L_i$, then

   $$|G| = \sum_{i=1}^{m} d_i^2.$$

The Hermitian inner product on class functions, again

Enumerate the conjugacy classes of G, say $C_1, \cdots, C_m$ and choose a representative $g_i \in C_i$ for each i.

Write $c_i = |C_G(g_i)|$ for the number of elements in the centralizer of g_i, and notice that

$$|\operatorname{Cl}(g_i)| = |G|/|C_G(g_i)| = |G|/c_i.$$

Recall that for two class functions $f_1, f_2 \in \operatorname{Cl}(G)$ we have defined

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

Lemma: We have

$$\langle f_1, f_2 \rangle = \frac{1}{c_i} \sum_{i=1}^{m} f_1(g)\overline{f_2(g)}.$$

The character table.

The matrix is known as the *character table of the group* G. Consider the $m \times m$ matrix whose rows are indexed by the irreducible characters $\chi_1, \cdots, \chi_m$ and whose columns are indexed by the conjugacy class representatives $g_1, \ldots, g_m$, and whose entry in the i-th row and j-th column is given by $\chi_i(g_j)$.

$$\begin{array}{c|cccc} & g_1 & g_2 & \cdots & g_m \\ & c_1 & c_2 & \cdots & c_m \\ \hline \chi_1 & \chi_1(g_1) & \chi_1(g_2) & \cdots & \chi_1(g_m) \\ \chi_2 & \chi_2(g_1) & \chi_2(g_2) & \cdots & \chi_2(g_m) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi_m & \chi_m(g_1) & \chi_m(g_2) & \cdots & \chi_m(g_m) \end{array}$$

Remark:

a. The fact that the χ_i form an orthonormal basis for the space $\operatorname{Cl}(G)$ is equivalent to the statement that the above matrix is *unitary*, in the sense that for $1 \le i, j \le m$ we have

$$\sum_{k=1}^{m} c_k \chi_i(g_k)\overline{\chi_j(g_k)} = \delta_{i,j}.$$

b. Since the transpose of unitary matrix is also unitary, we find for $1 \le i, j \le m$ that

$$\sum_{k=1}^{m} c_i \chi_k(g_i)\overline{\chi_k(g_j)} = \delta_{i,j}.$$

We are now going to compute the character table for a few finite groups

Cyclic groups

Let $G = \mathbb{Z}/N\mathbb{Z}$ for a natural number N. If ζ is a primitive N-th root of unity, we have for $0 \le i < N$ a homomorphism

$$\rho_i : G \to F^\times$$

determined by the equation

$$\rho_i(1 + N\mathbb{Z}) = \zeta^i.$$

Since G is abelian, its conjugacy classes are *singletons*; thus the number of irreducible representations is $N = |G|$.

Each ρ_i determines a 1-dimensional irreducible representation, and $i \neq j \implies \rho_i \not\simeq \rho_j$.

Orthonormality of characters implies that

$$\langle \rho_i, \rho_j \rangle = \delta_{i,j}$$

This equality can actually be deduce in a more elementary way. For example, write $\mathbf{1} = \rho_0$.

Note that

$$\langle \rho_1, \mathbf{1} \rangle = \frac{1}{N} \sum_{i=0}^{N} \zeta^i = 0$$

since ζ is a root of $\dfrac{T^N - 1}{T - 1} = T^N + T^{N-1} + \cdots + T + 1$.

Consider a function $f \in F[\mathbb{Z}/N\mathbb{Z}]$. Using orthonormality of characters we deduce

$$f = \sum_{i=0}^{N-1} \langle f, \rho_i \rangle \rho_i.$$

The assigment $i \mapsto \langle f, \rho_i \rangle$ is usual known as the *discrete Fourier transform* of f.

The group $D_6 = S_3$.

Let $G = S_3$ be the symmetric group on 3 letters, so that $|G| = 6$.

Note that the subgroup $H = \langle (123) \rangle \subset G$ has index 2 and is thus a *normal* subgroup. (In fact, H is the *alternating group* A_3).

Notice that the centralizer of (123) coincides with H, so that (123) has exactly $[G : H] = 2$ conjugates (namely, (123) and (132)). On the other hand, the centralizer of (12) is $\langle (12) \rangle$ so that (12) has 3 conjugates; namely (12), (13) and (23).

Thus 1, (12), and (123) is a full set of representatives for the conjugacy classes of G.

In particular, we expect to find 3 irreducible representations of G.

There are exactly two homomorphisms $G \to F^\times$ which contain H in their kernel; one is the trivial mapping **1**, and the other is the sign homomorphism $\mathrm{sgn} : G \to F^\times$; it is the unique mapping for which $H \subset \ker \mathrm{sgn}$ and $\mathrm{sgn}((12)) = -1$.

Thus **1** and sgn are 1-dimensional irreducible representations of G.

It remains to find one more irreducible representation.

Let $\Omega = \{1, 2, 3\}$ and consider the standard action of $G = S_3$ on Ω. Write $\chi = \chi_\Omega$ for the *character* of this representation.

You will see for homework that $\chi(\sigma)$ is equal to the number of fixed points of σ on Ω.

We have seen that the trivial representation **1** appears in the representation $F[\Omega]$; thus

$$F[\Omega] = W \oplus \mathbf{1}.$$

You will argue in homework that the character of W is given by $\psi = \chi - \mathbf{1}$.

Thus χ and $\chi - \mathbf{1}$ are given by:

	$\chi(\sigma)$	$\psi(\sigma)$
1	3	2
(12)	1	0
(123)	0	-1

Now, we compute

$$\langle \psi, \psi \rangle = 1/6 \cdot 2 \cdot 2 + 1/2 \cdot 0 \cdot 0 + 1/3 \cdot -1 \cdot -1 = 2/3 + 1/3 = 1$$

This shows that ψ is an irreducible representation. Thus the character table of G is given by

	1	(12)	(123)
	6	2	3
1	1	1	1
sgn	1	-1	1
ψ	2	0	1

Bibliography