

(Representation Theory) Invariant subspaces & complete reducibility

George McNinch

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Invariant subspaces

Let (ρ, V) be a *representation* of the group G on the F -vector space V .

If W is a *subspace* of V ¹, one says that W is a *sub-representation*, or that W is an *invariant subspace*, provided that

$$\rho(g)W \subseteq W \quad \forall g \in G.$$

If W is a sub-representation, then W “is” itself a representation of G in a natural way, since ρ determines a group homomorphism

$$g \mapsto \rho(g)|_W : G \rightarrow \mathrm{GL}(W).$$

Proposition: If (ρ, V) and (ψ, W) are G -representations and if $\Phi : V \rightarrow W$ is a homomorphism of G -representations then $\ker \Phi$ is a subrepresentation of V and $\Phi(V)$ is a subrepresentation of W .

Recollections on vector subspaces and direct sums

Let W_1 and W_2 be F -vector subspaces of the vector space V . We can form the *direct sum* $W_1 \oplus W_2$.

And the defining property of the direct sum tells us that we get a linear mapping

$$\phi : W_1 \oplus W_2 \rightarrow V$$

by the rule $\phi(w_1, w_2) = w_1 + w_2$.

Suppose the following hold:

- $W_1 + W_2 = V$ – i.e. ϕ is surjective, and
- $\ker \phi = 0$ – i.e. $W_1 \cap W_2 = \{0\}$.

Under these conditions, ϕ determines an isomorphism $W_1 \oplus W_2 \simeq V$, and one says that V is the *internal direct sum* of the subspaces W_1 and W_2 .

Remark: More generally, if W_1, W_2, \dots, W_n are subspaces of V , suppose that

- $V = \sum_{i=1}^n W_i$, and
- for each i we have $W_i \cap (\sum_{j \neq i} W_j) = 0$.

Then V is the *internal direct sum* $V = W_1 \oplus W_2 \oplus \dots \oplus W_n$.

Example: Let $\phi : V \rightarrow V$ be a linear mapping with $\dim V < \infty$, and suppose that ϕ is *diagonalizable* i.e. that V has a basis consisting of *eigenvectors* for ϕ .

Let $\lambda_1, \dots, \lambda_k \in F$ be the *distinct eigenvalues* of ϕ , and let

$$V_i = \{x \in V \mid \phi(x) = \lambda_i x\}$$

be the λ_i -eigenspace.

¹The term “subspace” means “vector subspace”. One might also say “ F -subspace” to emphasize the scalars.

Then

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

i.e. V is the internal direct sum of the eigenspaces for ϕ .

Proposition: Let W be a subspace of V where $\dim V < \infty$. Then there is a subspace W' of V for which V is the internal direct sum of W and W' .

Remark The analogue of the property described by the Proposition fails for abelian groups in general. Consider $A = \mathbb{Z}/4\mathbb{Z}$. For the subgroup $B = 2\mathbb{Z}/4\mathbb{Z}$ (of order 2), there is no subgroup B' for which $A = B \oplus B'$.

Sketch of proof of Proposition: Choose a basis $\beta_1, \dots, \beta_\ell$ for the F -vector space V/W . Now choose vectors $b_1, \dots, b_\ell \in V$ so that

$$\beta_i = b_i + W \in V/W.$$

Let W' be the span of b_1, \dots, b_ℓ ; i.e

$$W' = \sum_{i=1}^{\ell} Fb_i.$$

We are going to show that V is the internal direct sum of W and W' .

For $v \in V$, viewing $v + W$ as an element of V/W we may write

$$v + W = a_1\beta_1 + \cdots + a_\ell\beta_\ell$$

for scalars $a_i \in F$.

Let $w' = \sum_{i=1}^{\ell} a_i b_i \in W'$. It is clear that $w = v - w' \in W$. Since $v = w + w'$ we have showed that $W + W' = V$.

Finally the linear independence of the β_i shows the only element of W' contained in W is 0; thus $W \cap W' = \{0\}$ so that $V = W \oplus W'$.

With notation as in the statement of the Proposition, one says that the subspace W' is a *complement* to the subspace W .

Complements and projections

Given a subspace $W \subset V$ and a complement W' , so that $V = W \oplus W'$, we get a *projection operator*

$$\pi : V = W \oplus W' \rightarrow V = W \oplus W' \quad \text{via} \quad \pi(x, y) = (x, 0)$$

The mapping π satisfies the following properties:

P1. $\pi^2 = \pi$, and

P2. $\pi(V) = W$.

We say that a linear mapping $\pi : V \rightarrow V$ is a *projection onto* W provided that conditions P1 and P2 hold.

Lemma: Suppose that $\pi : V \rightarrow V$ is a linear mapping. Then π is projection onto W if and only if $\pi(W) = W$ and the restriction of π to W is the identity mapping id_W .

Proof of Lemma: For a linear map $\pi : V \rightarrow V$ for which $\pi(V) = W$, we must show that $\pi^2 = \pi$ if and only if $\pi|_W = \text{id}_W$.

(\Rightarrow): Suppose that $w \in W$. Since $\pi(V) = W$, we may write $w = \pi(v)$ for some $v \in V$. Then $\pi^2 = \pi$ shows that $\pi^2(v) = \pi(v) \Rightarrow \pi(\pi(v)) = \pi(v)$ so that $\pi(w) = w$.

(\Leftarrow): Suppose that $v \in V$. We have $\pi(v) \in W$, and since $\pi|_W$ is the identity, we find $\pi^2(\pi(v)) = \pi(v)$. Since this holds for every v , we have $\pi^2 = \pi$ as required.

Proposition: There is a bijection between the following:

- complements W' to W in V
- projections $\pi : V \rightarrow V$ onto W

Proof: We've already described how to build a projection π from a complement W' .

Given a projection π , take $W' = \ker \pi$. We must argue that W' is a complement to W in V .

Suppose $x \in W \cap W'$. Since $x \in W$, the Lemma shows that $x = \pi(x)$. But on the other hand since $x \in W' = \ker \pi$ we find that $x = \pi(x) = 0$. This proves that $W \cap W' = \{0\}$.

Finally we must show that $V = W + W'$. Let $v \in V$. Then $w = \pi(v) \in W$ by P2. Now,

$$v = \pi(v) + (v - \pi(v)) = w + (v - \pi(v))$$

and it just remains to see that $v - \pi(v) \in W'$. But by P1,

$$\pi(v - \pi(v)) = \pi(v) - \pi^2(v) = \pi(v) - \pi(v) = 0.$$

Complete reducibility of G representations.

Let G be a finite group and (ρ, V) a representation of G .

Definition: We say that (ρ, V) is *completely reducible* if for every subrepresentation $W \subseteq V$, there is a subrepresentation $W' \subseteq V$ such that V is the internal direct sum of W and W' as vector spaces.

Theorem: Let F be a field of char. 0 and let G be a finite group. Then every representation of G on a finite dimensional F -vector space is completely reducible.

Proof: Let (ρ, V) be a (finite dimensional) representation of G and let $W \subset V$ be a *subrepresentation*.

We *choose* a vector space complement, which by the Proposition above amounts to the choice of a projection operator $\pi : V \rightarrow V$ onto the subspace W .

We form a new linear mapping

$$\tilde{\pi} : V \rightarrow V$$

by the rule

$$\tilde{\pi} = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g)^{-1}.$$

We are going to prove:

- (i) $\tilde{\pi}$ is a homomorphism of G -representations, and
- (ii) $\tilde{\pi}$ is a *projection*.

Together (i) and (ii) imply the Theorem. Indeed, if (ii) holds, one knows that $W' = \ker \tilde{\pi}$ is a *complement* to W . Since $\tilde{\pi}$ is a homomorphism of G -representations, one knows that its kernel W' is a subrepresentation.

To prove (i), let $h \in G$ and $v \in V$ and observe

$$\begin{aligned} \tilde{\pi}(\rho(h)v) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g)^{-1}(\rho(h)v) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1}h)(v) \\ &= \frac{1}{|G|} \sum_{x \in G} \rho(hx) \circ \pi \circ \rho(x^{-1})(v) \\ &= \frac{1}{|G|} \rho(h) \sum_{x \in G} \rho(x) \circ \pi \circ \rho(x)^{-1}(v) \\ &= \rho(h)\tilde{\pi}(v) \end{aligned}$$

Thus $\tilde{\pi}$ is indeed a homomorphism of G -representations.

To prove (ii), we observe that for each $g \in G$, the mapping $\rho(g) \circ \pi \circ \rho(g)^{-1}$ is also a projection onto W . Indeed, since W is a subrepresentation, $\rho(g)W = W$, so that $\rho(g) \circ \pi \circ \rho(g)^{-1}(V) \subseteq W$. On the other hand, since $\pi|_W$ is the identity mapping, $\rho(g) \circ \pi \circ \rho(g)^{-1}(w) = w$ for any $w \in W$ so the Lemma above shows that $\rho(g) \circ \pi \circ \rho(g)^{-1}$ is a projection onto W .

Now it is clear that $\sum_{g \in G} \rho(g) \circ \pi \circ \rho(g)^{-1}$ maps V to W . Since each mapping $\rho(g) \circ \pi \circ \rho(g)^{-1}$ is the identity on W , it follows that

$$\tilde{\pi} = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g)^{-1}$$

is the identity mapping on W , so $\tilde{\pi}$ is a projection by the Lemma above.

This completes the proof of the Theorem.

Bibliography