The number of irreducible representations of a finite group

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The number of irreducible representations of G

Recall that G denotes a finite group.

Recall that the space $\mathbb{C}[G]$ of *all* $\mathbb{C}$*-valued functions on* G is the vector space underlying the *regular representation* of G.

We introduced the *convolution multiplication* $\star$ on $\mathbb{C}[G]$ by the rule

$$(f_1 \star f_2)(x) = \sum_{yz=x} f_1(y) f_2(z)$$

for $f_1, f_2 \in \mathbb{C}[G]$.

This product makes $\mathbb{C}[G]$ into a (in-general non-commutative) ring. We mostly will avoid invoking general results about rings, and so we define the *center* $\mathbb{C}[G]$ to be the subspace

$$Z = \{f \in \mathbb{C}[G] \mid f \star h = h \star f \quad \forall h \in \mathbb{C}[G]\}.$$

Proposition: The subspace Z coincides with the subspace of $\mathbb{C}[G]$ consisting of those functions which are *constant* on the conjugacy classes of G.

In particular, dim $Z = \#\{\text{conjugacy classes of } G\}$.

Proof: Since $\mathbb{C}[G]$ has a vector space basis consisting of the dirac functions δ_g for $g \in G$, one immediately sees that $f \in Z$ if and only if

$$f \star \delta_g = \delta_g \star f$$

for every $g \in G$.

Note that $\delta_g \star \delta_{g^{-1}} = \delta_1$ is a *multiplicative identity for the operation $\star$, so that for any g, $\delta_{g^{-1}} = (\delta_g)^{-1}$ is a multiplicative inverse.

Thus

$$f \star \delta_g = \delta_g \star f \iff f = \delta_g \star f \star \delta_{g^{-1}}.$$

Now, fix $f \in \mathbb{C}[G]$ and $g \in G$, and let's compute the value of $\delta_g \star f \star \delta_{g^{-1}}$ at an element $h \in G$. We have

$$(\delta_g \star f \star \delta_{g^{-1}})(h) = \sum_{xyz=h} \delta_g(x) f(y) \delta_{g^{-1}}(z) = f(g^{-1}hg)$$

We now conclude that $f \in Z$ if and only if

$$f(h) = f(g^{-1}hg) \quad \forall g, h \in G$$

i.e. if and only if f is constant on the conjugacy classes of G.

Since the characteristic functions ψ_C of the conjugacy classes C of G form a basis for the space of *class functions*, it follows that dim Z is the number of conjugacy classes C of G; this completes the proof of the Proposition.

We write $L_1, L_2, \cdots, L_r$ for a complete set of irreducible representations of G on $\mathbb{C}$-vector spaces, no two of which are isomorphic.

Lemma: Let z be an element of the center $Z \subseteq \mathbb{C}[G]$. For each i there is a scalar $\lambda_i \in \mathbb{C}$ such that for every $v \in L_i$ we have

$$z \star v = \lambda_i v.$$

Proof of Lemma: Note that for each i the mapping "convolution with z" – i.e. the mapping

$$\phi : L_i \to L_i \quad \text{given by } \phi(v) = z \star v$$

– is a homomorphism of G-representations.

Indeed, note for $g \in G$ that – since $z \in Z$ – we have

$$\phi(gv) = \phi(\delta_g \star v) = z \star \delta_g \star v = \delta_g \star z \star v = \delta_g \star \phi(v) = g\phi(v).$$

Now, Schur's Lemma tells us – since L_i is *irreducible* – that the endomorphisms of L_i as a G-representation identify with the scalar operators $\mathbb{C} = \mathbb{C} \cdot \mathrm{id}_{W_i}$.

Thus, there is $\lambda_i \in \mathbb{C}$ such that

$$\phi = \lambda_i \, \mathrm{id}_{W_i};$$

in other words, $z \star w = \phi(w) = \lambda_i w$ for $w \in W_i$, as required.

Theorem: The number r of irreducible representations is equal to $\dim Z$. In particular, r is equal to the number of conjugacy classes in G.

Proof: Write $W_i = \mathbb{C}[G]_{(L_i)}$ for the L_i-isotypic component of the regular representation $\mathbb{C}[G]$.

Thus for each i, W_i is a direct sum of copies of the irreducible representation L_i, and the quotient representation $\mathbb{C}[G]/W_i$ contains no irreducible invariant subspace isomorphic to L_i.

You proved for homework that

$$\mathbb{C}[G] = W_1 \oplus W_2 \oplus \cdots \oplus W_r.$$

In view of this decomposition of $\mathbb{C}[G]$, we may write

$$\delta_1 = f_1 + f_2 + \cdots + f_r$$

for uniquely determined elements $f_i \in W_i$.

Let $z \in Z$. According to the Lemma, there are scalars $\lambda_i \in \mathbb{C}$ for which $z \star v_i = \lambda_i v_i$ for $v_i \in L_i$.

Since W_i is L_i-isotypic, it follows at once that

$$z \star w_i = \lambda_i w_i$$

for each $w_i \in W_i$. In particular,

$$z \star f_i = \lambda_i f_i \quad \text{for } i = 1, 2, \cdots, r.$$

Now we notice that

$$\begin{aligned} z = z \star \delta_1 &= z \star (f_1 + f_2 + \cdots + f_r) \\ &= z \star f_1 + z \star f_2 + \cdots + z \star f_r \\ &= \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_r f_r \end{aligned}$$

This proves that Z is contained in the *span* of the vectors $f_1, f_2, \cdots, f_r$; i.e.

$$Z \subseteq \sum_{i=1}^{r} \mathbb{C} f_i.$$

We conclude that

$$\dim Z \leq \dim \sum_{i=1}^{r} \mathbb{C} f_i \leq r.$$

But on the other hand, we have proved that the *characters* $\chi_i = \chi_{L_i}$ of the irreducible representations form an orthonormal – hence linearly independent – set of *class functions* on G.

According to the preceding Proposition, $\chi_i \in Z$ for each i. This proves that

$$r = \dim \sum_{i=1}^{r} \mathbb{C}\chi_i \le \dim Z.$$

We may now conclude that $\dim Z = r$ as required.

Remarks: With notations as in the proof of the Theorem, note that

- we have an equality $Z = \sum_{i=r}^{r} \mathbb{C}f_i$ of subspaces of $\mathbb{C}[G]$.
- since $\dim Z = r$, conclude that $f_1, f_2, \cdots, f_r$ are *linearly independent*
- Moreover, $f_i \in Z$ for each i.

Bibliography

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