ProblemSet 3 – representation theory

George McNinch

2024-02-16

Work any 3 of the following 4 problems.

In these exercises, G always denotes a finite group. Unless indicated otherwise, all vector spaces are assumed to be finite dimensional over the field $F = \mathbb{C}$. The representation space V of a representation of G is always assumed to be finite dimensional over $\mathbb{C}$.

1. Let $\phi : G \to F^\times$ be a group homomorphism; since $F^\times = \mathrm{GL}_1(F)$, we can think of ϕ as a 1-dimensional representation (ϕ, F) of G.

   If V is any representation of G, we can form a *new* representation $\phi \otimes V$. The underlying vector space for this representation is just V, and the "new" action of an element $g \in G$ on a vector v is given by the rule

   $$g \star v = \phi(g) g v.$$

   a. Prove that if V is irreducible, then $\phi \otimes V$ is also irreducible.

   b. Prove that if χ denotes the *character* of V, then the character of $\phi \otimes V$ is given by $\phi \cdot \chi$; in other words, the trace of the action of $g \in G$ on $\phi \otimes V$ is given by

   $$\chi_{\phi \otimes V}(g) = \mathrm{tr}(v \mapsto g \star v) = \phi(g)\chi(g).$$

   c. Recall that in class we saw that S_3 has an irreducible representation V_2 of dimension 2 whose character ψ_2 is given by

g	1	(12)	(123)
ψ_2	2	0	-1

   Observe that $\mathrm{sgn}\,\psi = \psi$ and conclude that $V_2 \simeq \mathrm{sgn} \otimes V_2$, where $\mathrm{sgn} : S_n \to \{\pm 1\} \subset \mathbb{C}^\times$ is the *sign homomorphism.*

   On the other hand, S_4 has an irreducible representation V_3 of dimension 3 whose character ψ_3 is given by

g	1	(12)	(123)	(1234)	(12)(34)
ψ_3	3	1	0	-1	-1

   (I'm not asking you to confirm that ψ_3 is irreducible, though it would be straightforward to check that $\langle \psi_3, \psi_3 \rangle = 1$).

   Prove that $V_3 \not\simeq \mathrm{sgn} \otimes V_3$ as S_4-representations.

   (In particular, S_4 has *at least two* irreducible representations of dimension 3.)

2. Let V be a representation of G.

   For an irreducible representation L, consider the set

   $$\mathcal{S} = \{ S \subseteq V \mid S \simeq L \}$$

   of all invariant subspaces that are isomorphic to L as G-representations.

   Put

   $$V_{(L)} = \sum_{S \in \mathcal{S}} S.$$

a. Prove that $V_{(L)}$ is an invariant subspace, and show that $V_{(L)}$ is isomorphic to a direct sum

$$V_{(L)} \simeq L \oplus \cdots \oplus L$$

as G-representations.

b. Prove that the *quotient representation* $V/V_{(L)}$ has no invariant subspaces isomorphic to L as G-representations.

c. If $L_1, L_2, \cdots, L_m$ is a complete set of non-isomorphic irreducible representations for G, prove that V is the internal direct sum

$$V = \bigoplus_{i=1}^{m} V_{(L_i)}.$$

3. Let χ be the character of a representation V of G. For $g \in G$ prove that $\overline{\chi(g)} = \chi(g^{-1})$.

   Is it true for any arbitrary class function $f : G \to \mathbb{C}$ that $\overline{f(g)} = f(g^{-1})$ for every g? (Give a proof or a counterexample…)

4. For a prime number p, let $k = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Let V be an n-dimensional vector space over $\mathbb{F}_p$ for some natural number n, and let

$$\langle \cdot, \cdot \rangle : V \times V \to k$$

be a non-degenerate bilinear form on V.

(*A common example would be to take $V = \mathbb{F}_{p^n}$ the field of order p^n, and $\langle \alpha, \beta \rangle = \mathrm{tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(\alpha\beta)$ the trace pairing*).

Let us fix a non-trivial group homomorphism $\psi : k \to \mathbb{C}^\times$ (recall that $k = \mathbb{Z}/p\mathbb{Z}$ is an additive group, while $\mathbb{C}^\times$ is multiplicative). Thus

$$\psi(\alpha + \beta) = \psi(\alpha)\psi(\beta) \quad \text{for all} \quad \alpha, \beta \in k.$$

If you want an explicit choice, set $\psi(j + p\mathbb{Z}) = \exp(j \cdot 2\pi i/p) = \exp(2\pi i/p)^j$.

For a vector $v \in V$, consider the mapping $\Psi_v : V \to \mathbb{C}^\times$ given by the rule

$$\Psi_v(w) = \psi(\langle w, v \rangle).$$

a. Show that Ψ_v is a group homomorphism $V \to \mathbb{C}^\times$.

b. Show that the assignment $v \mapsto \Psi_v$ is injective (one-to-one).

   (This assignment is a function $V \to \mathrm{Hom}(V, \mathbb{C}^\times)$. In fact, it is a group homomorphism. Do you see why? How do you make $\mathrm{Hom}(V, \mathbb{C}^\times)$ into a group?)

c. Show that any group homomorphism $\Psi : V \to \mathbb{C}^\times$ has the form $\Psi = \Psi_v$ for some $v \in V$.

   Conclude that there are exactly $|V| = q^n$ group homomorphisms $V \to \mathbb{C}^\times$.

Bibliography