ProblemSet 3 – Solutions

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In these exercises, G always denotes a finite group. Unless indicated otherwise, all vector spaces are assumed to be finite dimensional over the field $F = \mathbb{C}$. The representation space V of a representation of G is always assumed to be finite dimensional over $\mathbb{C}$.

1. Let $\phi : G \to F^\times$ be a group homomorphism; since $F^\times = \mathrm{GL}_1(F)$, we can think of ϕ as a 1-dimensional representation (ϕ, F) of G.

   If V is any representation of G, we can form a *new* representation $\phi \otimes V$. The underlying vector space for this representation is just V, and the "new" action of an element $g \in G$ on a vector v is given by the rule

   $$g \star v = \phi(g)gv.$$

   a. Prove that if V is irreducible, then $\phi \otimes V$ is also irreducible.

      We prove the following statement: $(*)$ if $W \subset V$ is a subspace, then W is invariant for the *original* action of G if and only if it is invariant for the $\star$ action of G.

      First note that $(*)$ immediately implies the assertion of (a).

      To test invariance, let $w \in W$ and let $g \in G$. Since W is a linear subspace and since $\phi(g)$ is a non-zero scalar, it is immediate that

      $$gw \in W \iff g \star w = \phi(g)gw \in W$$

      Since this holds for all w and all g, $(*)$ follows.

   b. Prove that if χ denotes the *character* of V, then the character of $\phi \otimes V$ is given by $\phi \cdot \chi$; in other words, the trace of the action of $g \in G$ on $\phi \otimes V$ is given by

      $$\chi_{\phi\otimes V}(g) = \mathrm{tr}(v \mapsto g \star v) = \phi(g)\chi(g).$$

      We just need to compute the trace of the linear mapping $V \to V$ given by $v \mapsto g \star v$.

      If the action of g on V is given by the linear mapping $\rho(g)$, then

      $$\chi_V(g) = \mathrm{tr}(\rho(g)).$$

      Now, the $\star$-action of g is given by the linear mapping $v \mapsto g \star v = \phi(g)\rho(g)v$.

      So $\chi_{\phi\otimes V}(g) = \mathrm{tr}(\phi(g)\rho(g))$. For any scalar $s \in k$, trace of the linear mapping $s\rho(g)$ is given by

      $$\mathrm{tr}(s\rho(g)) = s\,\mathrm{tr}(\rho(g)) = s\chi_V(g)$$

      ("linearity of the trace").

      Thus

      $$\chi_{\phi\otimes V}(g) = \mathrm{tr}(\phi(g)\rho(g)) = \phi(g)\chi_V(g).$$

   c. Recall that in class we saw that S_3 has an irreducible representation V_2 of dimension 2 whose character ψ_2 is given by

g	1	(12)	(123)
ψ_2	2	0	-1

Observe that $\operatorname{sgn}\psi = \psi$ and conclude that $V_2 \simeq \operatorname{sgn} \otimes V_2$, where $\operatorname{sgn} : S_n \to \{\pm 1\} \subset \mathbb{C}^\times$ is the *sign homomorphism.*

On the other hand, S_4 has an irreducible representation V_3 of dimension 3 whose character ψ_3 is given by

g	1	(12)	(123)	(1234)	$(12)(34)$
ψ_3	3	1	0	-1	-1

(I'm not asking you to confirm that ψ_3 is irreducible, though it would be straightforward to check that $\langle \psi_3, \psi_3 \rangle = 1$).

Prove that $V_3 \not\simeq \operatorname{sgn} \otimes V_3$ as S_4-representations.

(In particular, S_4 has *at least two* irreducible representations of dimension 3.)

We first consider the representation V_2 of S_3. Write χ_2 of the character of this irreducible representation. The character of $\operatorname{sgn}\chi_2$ is then given by the product $\operatorname{sgn}\chi_2$.

g	1	(12)	(123)
ψ_2	2	0	-1
sgn	1	-1	1
$\operatorname{sgn}\psi_2$	2	0	-1

Inspecting the table we see that $\psi_2 = \operatorname{sgn}\psi_2$. This shows that V_2 is isomorphic to $\operatorname{sgn} \otimes V_2$ as representations for S_3.

2. Let V be a representation of G.

For an irreducible representation L, consider the set

$$\mathcal{S} = \{ S \subseteq V \mid S \simeq L \}$$

of all invariant subspaces that are isomorphic to L as G-representations.

Put

$$V_{(L)} = \sum_{S \in \mathcal{S}} S.$$

a. Prove that $V_{(L)}$ is an invariant subspace, and show that $V_{(L)}$ is isomorphic to a direct sum

$$V_{(L)} \simeq L \oplus \cdots \oplus L$$

as G-representations.

First of all, we note more generally that if I is an index set and if $W_i \subset V$ is a G-invariant subspace for each $i \in I$, then $\sum_{i \in I} W_i$ is again an invariant subspace. (The proof is straightforward from the definitions). This confirms that $V_{(L)}$ is an invariant subspace.

To prove the remaining assertion, we proceed as follows.

Let us say that a G-representation W is L-isotypic if every irreducible invariant subspace of W is isomorphic to L.

It is immediate that $V_{(L)}$ is L-isotypic. We are going to prove:

If W is an L-*isotypic* G-representation, then W is isomorphic to a direct sum

$$W \simeq L \oplus \cdots \oplus L.$$

Proceed by induction on $\dim W$. If $\dim W = 0$ then $W = \{0\}$ and the result is immediate (W is the direct sum of zero copies of L).

Now observe that if $\dim W > 0$ then W contains an invariant subspace isomorphic to L, so that $\dim W \geq \dim L$.

Now if $\dim W = \dim L$, then $W \simeq L$ and the result holds in this case.

Finally, suppose that $\dim W > \dim L$ and let $S \subset W$ be an invariant subspace with $S \simeq L$.

By complete reducibility we may find an invariant subspace $U \subset W$ such that W is the internal direct sum

$$W = S \oplus U.$$

Since $\dim W = \dim S + \dim U$, we have $\dim U < \dim W$. Moreover, U is also L-isotypic. So by induction on dimension, we know that

$$U \simeq L \oplus \cdots \oplus L,$$

(say, a direct sum of d copies of L).

But then

$$W = S \oplus U \simeq L \oplus (L \oplus \cdots \oplus L) = L \oplus L \oplus \cdots \oplus L$$

is isomorphic to a direct sum of $d+1$ copies of L.

b. Prove that the *quotient representation* $V/V_{(L)}$ has no invariant subspaces isomorphic to L as G-representations.

Write $\pi : V \to V/V_{(L)}$ for the quotient map $v \mapsto v + V_{(L)}$; thus π is a surjective homomorphism of G-representations.

Suppose by way of contradiction that $S \subset V/V_{(L)}$ is an invariant subspace isomorphic to L, and let $S' \subset V$ be the inverse image under π of S:

$$S' = \pi^{-1}(S).$$

Then S' is an invariant subspace of V containing $V_{(L)}$, and the restriction of π to S' defines a surjective mapping

$$\pi_{|S'} : S' \to S \simeq L.$$

If K denotes the kernel of $\pi_{|S'}$, then complete reducibility implies that there is an invariant subspace M of V such that S' is the internal direct sum

$$(*) \quad S' = K \oplus M.$$

In particular, the invariant subspace M is isomorphic to L as G-representations. But then by definition we have $M \subset V_{(L)}$ contradicting the condition $M \cap K = \{0\}$ which must hold by $(*)$. This contradiction proves the result.

c. If $L_1, L_2, \cdots, L_m$ is a complete set of non-isomorphic irreducible representations for G, prove that V is the internal direct sum

$$V = \bigoplus_{i=1}^{m} V_{(L_i)}.$$

We first note that V is equal to the sum

$$\sum_{i=1}^{m} V_{(L_i)};$$

indeed, if $W = \sum_{i=1}^{m} V_{(L_i)}$, then by complete reducibility $V = W \oplus W'$ for an invariant subspace W'. But if $W' \neq 0$ then W' contains an irreducible invariant subspace, so that $W' \cap V_{(L_i)} \neq 0$ for some i and hence $W' \cap W \neq 0$; this is impossible since the internal sum $V = W \oplus W'$ is direct. This argument shows that $W' = 0$ and hence that $V = W$.

Finally, we show that the sum

$$\sum_{i=1}^{m} V_{(L_i)}$$

is *direct*, i.e. that for each j we have

$$(\clubsuit) \quad V_{(L_j)} \cap \left(\sum_{i \neq j} V_{(L_i)} \right) = 0.$$

Wrote I for the intersection appearing in ($\clubsuit$); thus, I is an invariant subspace of V. If I is non-zero, it has an irreducible invariant subspace S. Since $I \subset V_{(L_j)}$ and since $V_{(L_j)}$ is L_j-isotypic, we conclude that

$$S \simeq L_j.$$

But then $S \cap V_{(L_i)} = 0$ for every $i \neq j$ so that

$$S \cap \left(\sum_{i \neq j} V_{(L_i)} \right) = 0.$$

Since $I \subset \left(\sum_{i \neq j} V_{(L_i)} \right)$, we conclude that $I = 0$.

This completes the proof that V is the direct sum of the $V_{(L_i)}$, as required.

3. Let χ be the character of a representation V of G. For $g \in G$ prove that $\overline{\chi(g)} = \chi(g^{-1})$.

   Is it true for any arbitrary class function $f : G \to \mathbb{C}$ that $\overline{f(g)} = f(g^{-1})$ for every g? (Give a proof or a counterexample…)

   Let $\rho(g) : V \to V$ denote the linear automorphism of V determined by the action of $g \in G$. Then $\chi(g) = \mathrm{tr}(\rho(g))$.

   Now, since $\rho(g)$ has *finite order*, say n, its minimal polynomial divides $T^n - 1 \in \mathbb{C}[T]$, and hence every eigenvalue of $\rho(g)$ is an n-th root of unity.

   For any n-th root of unity ζ, note that $\overline{\zeta} = \zeta^{-1}$.

   Write $\alpha_1, \cdots, \alpha_d$ for the eigenvalues of $\rho(g)$, with multiplicity (so that $d = \dim V$). Notice that $\rho(g^{-1})$ has eigenvalues $\alpha_1^{-1}, \cdots, \alpha_d^{-1}$.

   Thus

$$\chi(g) = \sum_{i=1}^{d} \alpha_i \quad \text{and} \quad \chi(g^{-1}) = \sum_{i=1}^{d} \alpha_i^{-1}.$$

   Now, we see that

$$\overline{\chi(g)} = \sum_{i=1}^{d} \overline{\alpha_i} = \sum_{i=1}^{d} \alpha_i^{-1} = \chi(g^{-1})$$

   as required.

   It is *not* true that $\overline{f(g)} = f(g^{-1})$ for every class function f and every $g \in G$. Indeed, let $f = \alpha\delta_1$ be a multiple of the characterisitic function δ_1 of the trivial conjugacy class $\{1\}$.

   Then $\overline{f(1)} = \overline{\alpha}$ while $f(1^{-1}) = f(1) = \alpha$, so that if $\alpha \notin \mathbb{R}$, we have $\overline{f(1)} \neq f(1^{-1})$.

4. For a prime number p, let $k = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Let V be an n-dimensional vector space over $\mathbb{F}_p$ for some natural number n, and let

$$\langle \cdot, \cdot \rangle : V \times V \to k$$

   be a non-degenerate bilinear form on V.

   (*A common example would be to take* $V = \mathbb{F}_{p^n}$ *the field of order* p^n, *and* $\langle \alpha, \beta \rangle = \mathrm{tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(\alpha\beta)$ *the trace pairing*).

   Let us fix a non-trivial group homomorphism $\psi : k \to \mathbb{C}^\times$ (recall that $k = \mathbb{Z}/p\mathbb{Z}$ is an additive group, while $\mathbb{C}^\times$ is multiplicative). Thus

$$\psi(\alpha + \beta) = \psi(\alpha)\psi(\beta) \quad \text{for all} \quad \alpha, \beta \in k.$$

   If you want an explicit choice, set $\psi(j + p\mathbb{Z}) = \exp(j \cdot 2\pi i/p) = \exp(2\pi i/p)^j$.

   For a vector $v \in V$, consider the mapping $\Psi_v : V \to \mathbb{C}^\times$ given by the rule

$$\Psi_v(w) = \psi(\langle w, v \rangle).$$

a. Show that Ψ_v is a group homomorphism $V \to \mathbb{C}^\times$.

For $w, w' \in V$ notice that

$$\begin{aligned}
\Psi_v(w+w') &= \psi(\langle w+w', v\rangle) & \\
&= \psi(\langle w, v\rangle + \langle w', v\rangle) & \text{since the form is bilinear} \\
&= \psi(\langle w, v\rangle) \cdot \psi(\langle w', v\rangle) & \text{since } \psi \text{ is a group homom} \\
&= \Psi_v(w) \cdot \Psi_v(w') & \text{by definition.}
\end{aligned}$$

This confirms that Ψ_v is a group homomorphism.

b. Show that the assignment $v \mapsto \Psi_v$ is injective (one-to-one).

(This assignment is a function $V \to \mathrm{Hom}(V, \mathbb{C}^\times)$. In fact, it is a group homomorphism. Do you see why? How do you make $\mathrm{Hom}(V, \mathbb{C}^\times)$ into a group?)

One checks that $\mathrm{Hom}(V, \mathbb{C}^\times)$ is a multiplicitive group (this is the *dual group* $\widehat{V}$ of V, mentioned in the lectures); the product of $\phi, \psi \in \widehat{V}$ is given by the rule $g \mapsto \phi(g) \cdot \psi(g)$.

We note that the assignment $v \mapsto \Psi_v$ is a group homomorphism. For $v, v' \in V$ we must argue that $\Psi_{v+v'} = \Psi_v \Psi_{v'}$.

For $w \in W$ we have

$$\begin{aligned}
\Psi_{v+v'}(w) =& \psi(\langle v+v', w\rangle) & \\
=& \psi(\langle v, w\rangle + \langle v', w\rangle) & \text{since the form is bilinear} \\
=& \psi(\langle v, w\rangle) \cdot \psi(\langle v', w\rangle) & \text{since } \psi \text{ is a group homom} \\
=& \Psi_v(w) \cdot \Psi_{v'}(w) & \text{by defn}
\end{aligned}$$

Now to show that $v \mapsto \Psi_v$ is injective, it is enough to argue that the kernel of this mapping is $\{0\}$.

So, suppose that Ψ_v is the identity element of $\widehat{V}$. In other words, suppose that $\Psi_v(w) = 1$ for every $w \in V$. This shows that $\psi(\langle v, w\rangle) = 1$ for every $w \in V$. Since ψ is a non-trivial homomorphism $\mathbb{F}_p \to \mathbb{C}^\times$, we know that $\ker \psi = \{0\}$ (remember that k has prime order…) and we conclude that $\langle v, w\rangle = 0$ for every $w \in W$.

(Note that $\langle v, w\rangle = 0$ is an equality in $k = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$).

Since the form $\langle \cdot, \cdot \rangle$ is non-degenerate, so we may now conclude that $v = 0$.

This proves that the kernel of the mapping $v \mapsto \Psi_v$ is $\{0\}$, hence the mapping is injective.

c. Show that any group homomorphism $\Psi : V \to \mathbb{C}^\times$ has the form $\Psi = \Psi_v$ for some $v \in V$.

Conclude that there are exactly $|V| = q^n$ group homomorphisms $V \to \mathbb{C}^\times$.

We observed in class that for any finite abelian group A, there is an isomorphism $A \simeq \widehat{A}$.

In particular, $|A| = |\widehat{A}|$.

Applying this in the case $A = V$, we conclude that

$$|V| = |\widehat{V}| = |\,\mathrm{Hom}(V, \mathbb{C}^\times)|.$$

Now, we have define an *injective* mapping

$$v \mapsto \Psi_v : V \to \widehat{(V)}.$$

Since the domain and co-domain of this mapping are finite of the same order, the mapping $v \mapsto \Psi_v$ is also *surjective*.

Thus the *pigeonhole principal* shows that every homomorphism $\Psi : V \to \mathbb{C}^\times$ has the form $\Psi = \Psi_v$ for some $v \in V$, as required.

Bibliography