Block codes

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Bounds for block codes, continued

Let $C \subset A^n$ be a block code, where $q = |A|$, and suppose that d is the *minimal distance* of C.

Recall that we showed that C "corrects up to t errors", where

$$t = \lfloor (d-1)/2 \rfloor.$$

And recall that $A_q(n,d)$ is the maximal size of a code $C \subset A^n$ with minimal distance d.

Let

$$\delta(m) = 1 + \binom{n}{1}(q-1) + \binom{n}{2}(q-1)^2 + \cdots + \binom{n}{m}(q-1)^m = \sum_{j=0}^{m} \binom{n}{j}(q-1)^j.$$

The (closed) ball $B_m(u)$ of radius m in A^n satisfies $|B_m(u)| = \delta(m)$.

Recall that the *Gilbert-Varshamov* result gave in some sense a *lower bound* result for $A_q(n,d)$; it showed that

$$A_q(n,d) \cdot \delta(d) \geq q^n.$$

We now give an *upper bound* for $A_q(n,d)$ known as the *sphere-packing bound*.

Theorem (*Sphere-packing bound*) Let $t = \lfloor (d-1)/2 \rfloor$. Then

$$A_q(n,d) \cdot \delta(t) \leq q^n.$$

Proof Let $C \subset A^n$ be a code of minimal distance d with $|C| = A_q(n,d)$.

Suppose that $u, v \in C$, and suppose that $w \in B_t(u) \cap B_t(v)$. Thus we have

$$\text{dist}(u,v) \leq \text{dist}(u,w) + \text{dist}(w,v) \leq 2t \leq d-1.$$

Since d is the minimal distance of C it follows that $u = v$. This shows that

$$u \neq v \implies B_t(u) \cap B_t(v) = \emptyset.$$

Thus the union $\displaystyle\bigcup_{u \in C} B_t(u)$ is disjoint, so that

$$|\bigcup_{u \in C} B_t(u)| = |C| \cdot \delta(t).$$

Since $\displaystyle\bigcup_{u \in C} B_t(u) \subseteq A^n$, the Theorem follows at once.

Remark A code is said to be *perfect* if it meets the sphere packing bound; i.e. if $|C| \cdot \delta(t) = q^n$.

We'll have some examples of perfect codes later; meanwhile note that to have a perfect code of length n and given t, we need $\delta(t) \mid q^n$. This doesn't happen too often…:

```
def delta(n,q,m):
  return sum([ binomial(n,j) * (q-1)**j for j in range(m+1) ])

def test(n,q,t):
  return Mod(q^n,delta(n,q,t)) == 0

q = 2
t = 2

list(filter(lambda n: test(n,q,t),range(1,200)))
=>
[1, 2, 5, 90]

q = 2
t = 3
list(filter(lambda n: test(n,q,t), range(1,200)))
=>
[1, 2, 3, 7, 23]

q=3
t=2
list(filter(lambda n: test(n,q,t),range(1,200)))
=>
[1, 2, 11]
```

Lemma (*Plotkin Lemma*) Let $C \subset A^n$, $|A| = q$, and suppose the maximal distance of C is d. Then

$$|C| \left(d + \frac{n}{q} - n \right) \leq d.$$

Proof Fix $1 \leq j \leq n$ and for each $a \in A$ write λ_a for the number of times a appears as the jth coordinate of a codeword in C.

i.e.

$$\lambda_a = |\{(u_1, u_2, \cdots, u_n) \in C \mid u_j = a\}|.$$

Of course, we have

$$(\clubsuit) \quad \sum_{a \in A} \lambda_a = |C|.$$

Moreover,

$$(\diamondsuit) \quad \sum_{a \in A} \left(\lambda_a - \frac{|C|}{q} \right)^2 \geq 0$$

since the sum of non-negative terms is non-negative.

Expanding each summand in $(\diamondsuit)$ and using $(\clubsuit)$ we see that

$$\begin{aligned}
0 &\leq \sum_{a \in A} \left(\lambda_a^2 - \frac{2|C|}{q}\lambda_a + \frac{|C|^2}{q^2} \right) \\
&= \left(\sum_{a \in A} \lambda_a^2 \right) - \frac{2|C|}{q} \left(\sum_{a} \lambda_a \right) + \frac{|C|^2}{q} \\
&= \left(\sum_{a \in A} \lambda_a^2 \right) - \frac{2|C|^2}{q} + \frac{|C|^2}{q} \\
&= \left(\sum_{a \in A} \lambda_a^2 \right) - \frac{|C|^2}{q}
\end{aligned}$$

Thus

$$(\heartsuit) \quad \sum_{a \in a} \lambda_a^2 \geq \frac{|C|^2}{q}.$$

Now write $S = \sum_{u,v \in C} \mathrm{dist}(u, v)$, and let S_j be the contribution that the j-th coordinate makes to this sum. More precisely,

$$S_j = \sum_{a \in A} \lambda_a \left(|C| - \lambda_a\right).$$

Of course,

$$S = \sum_j S_j.$$

Using ($\clubsuit$) and ($\heartsuit$) we find

$$\begin{aligned} S_j &= \sum_{a \in A} \lambda_a |C| - \sum_{a \in A} \lambda_a^2 \\ &= |C|^2 - \sum_{a \in A} \lambda_a^2 \\ &\leq |C|^2 - \frac{|C|^2}{q} \end{aligned}$$

Finally, since d is the minimal distance of C we have

$$d|C|(|C| - 1) \leq S$$

and on the other hand we have established

$$S = \sum_j S_j \leq n \left(|C|^2 - \frac{|C|^2}{q} \right)$$

Combining these inequalities (and canceling a factor of $|C|$) we find

$$\begin{aligned} & d\left(|C| - 1\right) \leq n|C|(1 - 1/q) \\ \implies & d|C| - n|C|(1 - 1/q) \leq d \\ \implies & |C| \left(d - n + \frac{n}{q} \right) \leq d; \end{aligned}$$

this completes the proof.

Asymptotics of codes

(*sketch/motivation*)

If we wish to send a large amount of data with short length codes, we have to cut up a string of n “bits” of data into strings of some fixed length n_0.

If the probability of decoding the string of length n_0 is p, then the probability of decoding the string of length n is p^{n/n_0}. For fixed n_0, note that

$$p^{n/n_0} \to 0 \quad \text{as } n \to \infty$$

On the other hand, Shanon’s Theorem promises us that we should be able to send the string of n bits through a channel with some given capacity Λ which encodes almost Λn bits of information and then decode correctly with probability approaching 1.

Now, a proof of Shannon’s theorem e.g. for the *binary symmetric channel* uses the fact that the average number of errors which occur in the transmission of n bits is $(1 - \phi)n$.

Thus, to satisfy Shannon’s Theorem, our code should be able to correct a number of errors that is *linear in* n – i.e. we want to construct codes of length n for which the *minimal distance* grows linearly with n.

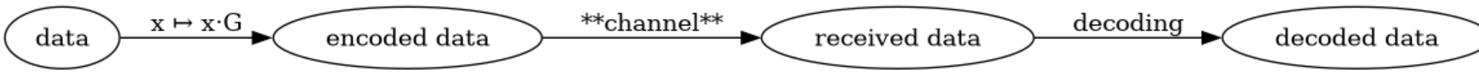
Defn A *sequence of asymptotically good codes* is a sequence $\{C_n\}$ where C_n is a code of length n, dimension $k(n)$ and minimal distance $d(n)$ for which $d(n)/n$ and $k(n)/n$ are bounded away from zero (as $n \to \infty$).

In some sense, the goal of constructing asymptotically good codes hopefully makes clear the utility of the preceding result on *bounds* for codes.

Decoding

Let C be a (linear) $[n,k,d]_q$-code.

The following diagram outlines components of usage of such a code for data transmission:



So, we begin with *data* $\mathbf{x} \in \mathbb{F}_q^k$. We encode it using a generator matrix G for the code C:

$$\mathbf{x} \mapsto \mathbf{x} \cdot G$$

Now, this vector in $\mathbb{F}_q^n$ is somehow *transmitted through the channel*; the received data is a vector $\mathbf{v} \in \mathbb{F}_q^n$, possible suffering transmission errors.

This leaves the decoding step: how do we hope to recover from $\mathbf{v}$ the data vector $\mathbf{x} \in \mathbb{F}_q^k$?

Standard array decoding

Here is a fairly simple-to-describe procedure for decoding.

For each coset $\mathbf{b} + C$ in $\mathbb{F}_q^n$, find an element with *minimal weight*.

Now, to decode the vector $\mathbf{v}$, find the coset containing $\mathbf{v}$, and write $\mathbf{w}$ for the element (chosen previously) of minimal length.

Notice that $\mathbf{v} - \mathbf{w} \in C$ (since both vectors are in C); we decode to this vector.

Example Let's consider a $[5,2]_2$ code with $k = \mathbb{F}_2$.

```
K = GF(2);
V = VectorSpace(K,5)

C= V.subspace([V([1,0,1,1,0]),
               V([0,1,1,0,1])])

W = V.subspace([V([0,0,1,0,0]),
                V([0,0,0,1,0]),
                V([0,0,0,0,1])])

def weight(v):
    r = [x for x in v if x != 0]
    return len(r)


min([ weight(v) for v in C if v != 0])
=>
3
```

This confirms that the minimal weight is `d=3`.

```
# build the coset of C with representative v, and sort the vectors in order of
# increasing weight

def coset(v):
```

```
    c =  [ v + c for c in C ]
    c.sort(key = lambda x: weight(x))
    return list(c)

# build the lookup array
# rows are the cosets of C in V, vectors ordered by increasing weight
#
lookup = [ coset(w) for w in W ]
```

Now we can perform *nearest neighbor decoding*. To decode the vector `w`, we find the row `c` of the `lookup` array containing `w`, and return `w - c[0]`.

```
def decode(w):
  c = [ x for x in lookup if w in x ][0]
  return w - c[0]

#  vectors in C are decoded to themselves, of course. e.g.
[  (c,decode(c)) for c in C ]
=>
[((0, 0, 0, 0, 0), (0, 0, 0, 0, 0)),
 ((1, 0, 1, 1, 0), (1, 0, 1, 1, 0)),
 ((2, 0, 2, 2, 0), (2, 0, 2, 2, 0)),
 ((0, 1, 1, 0, 1), (0, 1, 1, 0, 1)),
 ((1, 1, 2, 1, 1), (1, 1, 2, 1, 1)),
 ((2, 1, 0, 2, 1), (2, 1, 0, 2, 1)),
 ((0, 2, 2, 0, 2), (0, 2, 2, 0, 2)),
 ((1, 2, 0, 1, 2), (1, 2, 0, 1, 2)),
 ((2, 2, 1, 2, 2), (2, 2, 1, 2, 2))]
```

We should be able to correct `(d-1)/2 = (3-1)/2 = 1` error.

```
# consider "error vectors" of weight 1

[ (e,[(c+e, decode(c+e), decode(c+e) ==  c) for c in C]) for e in V.basis()]
=>
[((1, 0, 0, 0, 0),
 [((1, 0, 0, 0, 0), (0, 0, 0, 0, 0), True),
  ((2, 0, 1, 1, 0), (1, 0, 1, 1, 0), True),
  ((0, 0, 2, 2, 0), (2, 0, 2, 2, 0), True),
  ((1, 1, 1, 0, 1), (0, 1, 1, 0, 1), True),
  ((2, 1, 2, 1, 1), (1, 1, 2, 1, 1), True),
  ((0, 1, 0, 2, 1), (2, 1, 0, 2, 1), True),
  ((1, 2, 2, 0, 2), (0, 2, 2, 0, 2), True),
  ((2, 2, 0, 1, 2), (1, 2, 0, 1, 2), True),
  ((0, 2, 1, 2, 2), (2, 2, 1, 2, 2), True)]),
((0, 1, 0, 0, 0),
 [((0, 1, 0, 0, 0), (0, 0, 0, 0, 0), True),
  ((1, 1, 1, 1, 0), (1, 0, 1, 1, 0), True),
  ((2, 1, 2, 2, 0), (2, 0, 2, 2, 0), True),
  ((0, 2, 1, 0, 1), (0, 1, 1, 0, 1), True),
  ((1, 2, 2, 1, 1), (1, 1, 2, 1, 1), True),
  ((2, 2, 0, 2, 1), (2, 1, 0, 2, 1), True),
  ((0, 0, 2, 0, 2), (0, 2, 2, 0, 2), True),
  ((1, 0, 0, 1, 2), (1, 2, 0, 1, 2), True),
  ((2, 0, 1, 2, 2), (2, 2, 1, 2, 2), True)]),
((0, 0, 1, 0, 0),
 [((0, 0, 1, 0, 0), (0, 0, 0, 0, 0), True),
  ((1, 0, 2, 1, 0), (1, 0, 1, 1, 0), True),
```

```
  ((2, 0, 0, 2, 0), (2, 0, 2, 2, 0), True),
  ((0, 1, 2, 0, 1), (0, 1, 1, 0, 1), True),
  ((1, 1, 0, 1, 1), (1, 1, 2, 1, 1), True),
  ((2, 1, 1, 2, 1), (2, 1, 0, 2, 1), True),
  ((0, 2, 0, 0, 2), (0, 2, 2, 0, 2), True),
  ((1, 2, 1, 1, 2), (1, 2, 0, 1, 2), True),
  ((2, 2, 2, 2, 2), (2, 2, 1, 2, 2), True)]),
((0, 0, 0, 1, 0),
 [((0, 0, 0, 1, 0), (0, 0, 0, 0, 0), True),
  ((1, 0, 1, 2, 0), (1, 0, 1, 1, 0), True),
  ((2, 0, 2, 0, 0), (2, 0, 2, 2, 0), True),
  ((0, 1, 1, 1, 1), (0, 1, 1, 0, 1), True),
  ((1, 1, 2, 2, 1), (1, 1, 2, 1, 1), True),
  ((2, 1, 0, 0, 1), (2, 1, 0, 2, 1), True),
  ((0, 2, 2, 1, 2), (0, 2, 2, 0, 2), True),
  ((1, 2, 0, 2, 2), (1, 2, 0, 1, 2), True),
  ((2, 2, 1, 0, 2), (2, 2, 1, 2, 2), True)]),
((0, 0, 0, 0, 1),
 [((0, 0, 0, 0, 1), (0, 0, 0, 0, 0), True),
  ((1, 0, 1, 1, 1), (1, 0, 1, 1, 0), True),
  ((2, 0, 2, 2, 1), (2, 0, 2, 2, 0), True),
  ((0, 1, 1, 0, 2), (0, 1, 1, 0, 1), True),
  ((1, 1, 2, 1, 2), (1, 1, 2, 1, 1), True),
  ((2, 1, 0, 2, 2), (2, 1, 0, 2, 1), True),
  ((0, 2, 2, 0, 0), (0, 2, 2, 0, 2), True),
  ((1, 2, 0, 1, 0), (1, 2, 0, 1, 2), True),
  ((2, 2, 1, 2, 0), (2, 2, 1, 2, 2), True)])]
```

On the other hand, we shouldn't expect to correct 2 errors:

```
# consider an "error vector" with weight 2
f = V([0,1,0,1,0])

[ (c+f, decode(c+f), decode(c+f) == c) for c in C ]
=>
[((0, 1, 0, 0, 1), (0, 1, 1, 0, 1), False),
 ((1, 1, 1, 1, 1), (1, 1, 2, 1, 1), False),
 ((2, 1, 2, 2, 1), (2, 1, 0, 2, 1), False),
 ((0, 2, 1, 0, 2), (0, 2, 2, 0, 2), False),
 ((1, 2, 2, 1, 2), (1, 2, 0, 1, 2), False),
 ((2, 2, 0, 2, 2), (2, 2, 1, 2, 2), False),
 ((0, 0, 2, 0, 0), (0, 0, 0, 0, 0), False),
 ((1, 0, 0, 1, 0), (1, 0, 1, 1, 0), False),
 ((2, 0, 1, 2, 0), (2, 0, 2, 2, 0), False)]
```

Standard array decoding is pretty costly, though. The array we construct amounts to a list of $q^{nk} \times q^k$ vectors each of length n.

Syndrome decoding

Let C as before an $[n,k,d]_q$-code, and suppose that H is a *check matrix* for C.

If the vector **v** is sent, and the error pattern **e** appears, so that $\mathbf{v}+\mathbf{e}$ is received, we observe that

$$H(\mathbf{v}+\mathbf{e})^T = H\mathbf{e}^T.$$

So: we create a *lookup table* whose entries are pairs $(H.\mathbf{e}^T, \mathbf{e})$ for $\mathbf{e} \in V$ with weight $\mathbf{e} \le (d-1)/2$; the first entry is an element of $\mathbb{F}_q^{n-k}$.

To decode a received vector **v**, we compute its syndrome $\mathbf{w} = H\mathbf{v}^T$. If no more than $(d-1)/2$ errors occured, we will find an entry $(\mathbf{w}, \mathbf{e})$ in the table.

Now we decode to $\mathbf{v} - \mathbf{e}$.

Example: We consider

```
K = GF(3);
V = VectorSpace(K,5)

C= V.subspace([V([1,0,1,1,0]),
               V([0,1,1,0,1])])

# generator matrix
G = MatrixSpace(K,2,5).matrix(C.basis())

A = MatrixSpace(K,2,3).matrix([b[2:5] for b in G])

# check matrix
H=block_matrix([[-A.transpose(),MatrixSpace(K,3,3).one()]],
               subdivide=False)

def weight(v):
    r = [x for x in v if x != 0]
    return len(r)

min([ weight(v) for v in C if v != 0])
=>
3
```

We create a *lookup table*: for each vector $v \in \mathbb{F}_q^n$ of `weight <= 1`; the `keys` of the lookup table are the *syndromes* Hv^T.

```
lookup = { tuple(H*v):v for v in V if weight(v) < 2 }
lookup
=>
{(0, 0, 0): (0, 0, 0, 0, 0),
 (2, 2, 0): (1, 0, 0, 0, 0),
 (1, 1, 0): (2, 0, 0, 0, 0),
 (2, 0, 2): (0, 1, 0, 0, 0),
 (1, 0, 1): (0, 2, 0, 0, 0),
 (1, 0, 0): (0, 0, 1, 0, 0),
 (2, 0, 0): (0, 0, 2, 0, 0),
 (0, 1, 0): (0, 0, 0, 1, 0),
 (0, 2, 0): (0, 0, 0, 2, 0),
 (0, 0, 1): (0, 0, 0, 0, 1),
 (0, 0, 2): (0, 0, 0, 0, 2)}
```

Decoding a vector $\mathbf{v}$ is easy: compute the syndrome $H.\mathbf{v}^T$ and use it to locate the *error vector* e in the lookup table. Then return $\mathbf{v} - e$.

```
def decode(v):
  return v-lookup[tuple(H*v)]
```

Once again we can check that vectors in C are decoded as themselves:

```
[ (decode(c), c==decode(c)) for c in C ]
=>
[((0, 0, 0, 0, 0), True),
 ((1, 0, 1, 1, 0), True),
 ((2, 0, 2, 2, 0), True),
 ((0, 1, 1, 0, 1), True),
 ((1, 1, 2, 1, 1), True),
 ((2, 1, 0, 2, 1), True),
```

```
 ((0, 2, 2, 0, 2), True),
 ((1, 2, 0, 1, 2), True),
 ((2, 2, 1, 2, 2), True)]
```

Again, we should be able to correct `(d-1)/2 = (3-1)/2 = 1` error.

```
# consider "error vectors" of weight 1

[ (e,[(c+e, decode(c+e), decode(c+e) ==  c) for c in C]) for e in V.basis()]
=>
[((1, 0, 0, 0, 0),
  [((1, 0, 0, 0, 0), (0, 0, 0, 0, 0), True),
   ((2, 0, 1, 1, 0), (1, 0, 1, 1, 0), True),
   ((0, 0, 2, 2, 0), (2, 0, 2, 2, 0), True),
   ((1, 1, 1, 0, 1), (0, 1, 1, 0, 1), True),
   ((2, 1, 2, 1, 1), (1, 1, 2, 1, 1), True),
   ((0, 1, 0, 2, 1), (2, 1, 0, 2, 1), True),
   ((1, 2, 2, 0, 2), (0, 2, 2, 0, 2), True),
   ((2, 2, 0, 1, 2), (1, 2, 0, 1, 2), True),
   ((0, 2, 1, 2, 2), (2, 2, 1, 2, 2), True)]),
 ((0, 1, 0, 0, 0),
  [((0, 1, 0, 0, 0), (0, 0, 0, 0, 0), True),
   ((1, 1, 1, 1, 0), (1, 0, 1, 1, 0), True),
   ((2, 1, 2, 2, 0), (2, 0, 2, 2, 0), True),
   ((0, 2, 1, 0, 1), (0, 1, 1, 0, 1), True),
   ((1, 2, 2, 1, 1), (1, 1, 2, 1, 1), True),
   ((2, 2, 0, 2, 1), (2, 1, 0, 2, 1), True),
   ((0, 0, 2, 0, 2), (0, 2, 2, 0, 2), True),
   ((1, 0, 0, 1, 2), (1, 2, 0, 1, 2), True),
   ((2, 0, 1, 2, 2), (2, 2, 1, 2, 2), True)]),
 ((0, 0, 1, 0, 0),
  [((0, 0, 1, 0, 0), (0, 0, 0, 0, 0), True),
   ((1, 0, 2, 1, 0), (1, 0, 1, 1, 0), True),
   ((2, 0, 0, 2, 0), (2, 0, 2, 2, 0), True),
   ((0, 1, 2, 0, 1), (0, 1, 1, 0, 1), True),
   ((1, 1, 0, 1, 1), (1, 1, 2, 1, 1), True),
   ((2, 1, 1, 2, 1), (2, 1, 0, 2, 1), True),
   ((0, 2, 0, 0, 2), (0, 2, 2, 0, 2), True),
   ((1, 2, 1, 1, 2), (1, 2, 0, 1, 2), True),
   ((2, 2, 2, 2, 2), (2, 2, 1, 2, 2), True)]),
 ((0, 0, 0, 1, 0),
  [((0, 0, 0, 1, 0), (0, 0, 0, 0, 0), True),
   ((1, 0, 1, 2, 0), (1, 0, 1, 1, 0), True),
   ((2, 0, 2, 0, 0), (2, 0, 2, 2, 0), True),
   ((0, 1, 1, 1, 1), (0, 1, 1, 0, 1), True),
   ((1, 1, 2, 2, 1), (1, 1, 2, 1, 1), True),
   ((2, 1, 0, 0, 1), (2, 1, 0, 2, 1), True),
   ((0, 2, 2, 1, 2), (0, 2, 2, 0, 2), True),
   ((1, 2, 0, 2, 2), (1, 2, 0, 1, 2), True),
   ((2, 2, 1, 0, 2), (2, 2, 1, 2, 2), True)]),
 ((0, 0, 0, 0, 1),
  [((0, 0, 0, 0, 1), (0, 0, 0, 0, 0), True),
   ((1, 0, 1, 1, 1), (1, 0, 1, 1, 0), True),
   ((2, 0, 2, 2, 1), (2, 0, 2, 2, 0), True),
   ((0, 1, 1, 0, 2), (0, 1, 1, 0, 1), True),
   ((1, 1, 2, 1, 2), (1, 1, 2, 1, 1), True),
   ((2, 1, 0, 2, 2), (2, 1, 0, 2, 1), True),
```

```
  ((0, 2, 2, 0, 0), (0, 2, 2, 0, 2), True),
  ((1, 2, 0, 1, 0), (1, 2, 0, 1, 2), True),
  ((2, 2, 1, 2, 0), (2, 2, 1, 2, 2), True)])]
```

And again we don't expect to correct more than a single error with this code.

```
# consider an "error vector" with weight 2
f = V([0,1,1,0,0])

[ (c+f, decode(c+f), decode(c+f) == c) for c in C ]
=>
[((0, 1, 1, 0, 0), (0, 1, 1, 0, 1), False),
 ((1, 1, 2, 1, 0), (1, 1, 2, 1, 1), False),
 ((2, 1, 0, 2, 0), (2, 1, 0, 2, 1), False),
 ((0, 2, 2, 0, 1), (0, 2, 2, 0, 2), False),
 ((1, 2, 0, 1, 1), (1, 2, 0, 1, 2), False),
 ((2, 2, 1, 2, 1), (2, 2, 1, 2, 2), False),
 ((0, 0, 0, 0, 2), (0, 0, 0, 0, 0), False),
 ((1, 0, 1, 1, 2), (1, 0, 1, 1, 0), False),
 ((2, 0, 2, 2, 2), (2, 0, 2, 2, 0), False)]
```

Bibliography