Invariant subspaces & complete reducibility

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Invariant subspaces

Let (ρ, V) be a *representation* of the group G on the F-vector space V.

If W is a *subspace* of F [1], one says that W is a *sub-representation*, or that W is an *invariant subspace*, provided that

$$\rho(g)W \subseteq W \quad \forall g \in G.$$

If W is a sub-representation, then W "*is*" itself a representation of G in a natural way, since ρ determines a group homomorphism

$$g \mapsto \rho(g)_{|W} : G \to \mathrm{GL}(W).$$

Proposition: If (ρ, V) and (ψ, W) are G-representations and if $\Phi : V \to W$ is a homomorphism of G-representations then $\ker \Phi$ is a subrepresentation of V and $\Phi(V)$ is a subrepresentation of W.

Recollections on vector subspaces and direct sums

Let W_1 and W_2 be F-vector subspaces of the vector space V. We can form the *direct sum* $W_1 \oplus W_2$.

And the defining property of the direct sum tells us that we get a linear mapping

$$\phi : W_1 \oplus W_2 \to V$$

by the rule $\phi(w_1, w_2) = w_1 + w_2$.

Suppose the following hold:

- $W_1 + W_2 = V$ – i.e. ϕ is surjective, and
- $\ker \phi = 0$ – i.e. $W_1 \cap W_2 = \{0\}$.

Under these conditions, ϕ determines an isomorphism $W_1 \oplus W_2 \simeq V$, and one says that V is the *internal direct sum* of the subspaces W_1 and W_2.

Remark: More generally, if $W_1, W_2, \cdots, W_n$ are subspaces of V, suppose that

- $V = \sum_{i=1}^n W_i$, and
- for each i we have $W_i \cap \left(\sum_{j \neq i} W_j\right) = 0$.

Then V is the *internal direct sum* $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

Example: Let $\phi : V \to V$ be a linear mapping with $\dim V < \infty$, and suppose that ϕ is *diagonalizable* i.e. that V has a basis consisting of *eigenvectors* for ϕ.

Let $\lambda_1, \cdots, \lambda_k \in F$ be the *distinct eigenvalues* of ϕ, and let

$$V_i = \{x \in V \mid \phi(x) = \lambda_i x\}$$

be the λ_i-eigenspace.

[1] The term "subspace" means "vector subspace". One might also say "F-subspace" to emphasize the scalars.

Then

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

i.e. V is the internal direct sum of the eigenspaces for ϕ.

Proposition: Let W be a subspace of V where $\dim V < \infty$. Then there is a subspace W' of V for which V is the internal direct sum of W and W'.

Remark The analogue of the property described by the Proposition fails for abelian groups in general. Consider $A = \mathbb{Z}/4\mathbb{Z}$. For the subgroup $B = 2\mathbb{Z}/4\mathbb{Z}$ (of order 2), there is no subgroup B' for which $A = B \oplus B'$.

Sketch of proof of Proposition: Choose a basis $\beta_1, \dots, \beta_\ell$ for the F-vector space V/W. Now *choose* vectors $b_1, \cdots, b_\ell \in V$ so that

$$\beta_i = b_i + W \in V/W.$$

Let W' be the *span* of $b_1, \cdots, b_\ell$; i.e

$$W' = \sum_{i=1}^{\ell} F b_i.$$

We are going to show that V is the internal direct sum of W and W'.

For $v \in V$, viewing $v + W$ as an element of V/W we may write

$$v + W = a_1\beta_1 + \cdots + a_\ell\beta_\ell$$

for scalars $a_i \in F$.

Let $w' = \sum_{i=1}^{\ell} a_i b_i \in W'$. It is clear that $w = v - w' \in W$. Since $v = w + w'$ we have showed that $W + W' = V$.

Finally the linear independence of the β_i shows the only element of W' contained in W is 0; thus $W \cap W' = \{0\}$ so that $V = W \oplus W'$.

With notation as in the statement of the Proposition, one says that the subspace W' is a *complement* to the subspace W.

Complements and projections

Given a subspace $W \subset V$ and a complement W', so that $V = W \oplus W'$, we get a *projection operator*

$$\pi : V = W \oplus W' \to V = W \oplus W' \quad \text{via} \quad \pi(x, y) = (x, 0)$$

The mapping π satisfies the following properties:

P1. $\pi^2 = \pi$, and

P2. $\pi(V) = W$.

We say that a linear mapping $\pi : V \to V$ is a *projection onto* W provided that conditions P1 and P2 hold.

Lemma: Suppose that $\pi : V \to V$ is a linear mapping. Then π is projection onto W if and only if $\pi(W) = W$ and the restriction of π to W is the identity mapping id_W.

Proof of Lemma: For a linear map $\pi : V \to V$ for which $\pi(V) = W$, we must show that $\pi^2 = \pi$ if and only if $\pi_{|W} = \mathrm{id}_W$.

($\Rightarrow$): Suppose that $w \in W$. Since $\pi(V) = W$, we may write $w = \pi(v)$ for some $v \in V$. Then $\pi^2 = \pi$ shows that $\pi^2(v) = \pi(v) \implies \pi(\pi(v)) = \pi(v)$ so that $\pi(w) = w$.

($\Leftarrow$): Suppose that $v \in V$. We have $\pi(v) \in W$, and since $\pi_{|W}$ is the identity, we find $\pi^2(\pi(v)) = \pi(v)$. Since this holds for every v, we have $\pi^2 = \pi$ as required.

Proposition: There is a bijection between the following:

- *complements* W' to W in V
- projections $\pi : V \to V$ onto W

Proof: We've already described how to build a projection π from a complement W'.

Given a projection π, take $W' = \ker \pi$. We must argue that W' is a complement to W in V.

Suppose $x \in W \cap W'$. Since $x \in W$, the Lemma shows that $x = \pi(x)$. But on the other hand since $x \in W' = \ker \pi$ we find that $x = \pi(x) = 0$. This proves that $W \cap W' = \{0\}$.

Finally we must show that $V = W + W'$. Let $v \in V$. Then $w = \pi(v) \in W$ by P2. Now,

$$v = \pi(v) + (v - \pi(v)) = w + (v - \pi(v))$$

and it just remains to see that $v - \pi(v) \in W'$. But by P1,

$$\pi(v - \pi(v)) = \pi(v) - \pi^2(v) = \pi(v) - \pi(v) = 0.$$

Complete reducibility of G representations.

Let G be a finite group and (ρ, V) a representation of G.

Definition: We say that (ρ, V) is *completely reducible* if for every subrepresentation $W \subseteq V$, there is a subrepresentation $W' \subseteq V$ such that V is the internal direct sum of W and W' as vector spaces.

Theorem: Let F be a field of char. 0 and let G be a finite group. Then every representation of G on a finite dimensional F-vector space is completely reducible.

Proof: Let (ρ, V) be a (finite dimensional) representation of G and let $W \subset V$ be a *subrepresentation*.

We *choose* a vector space complement, which by the Proposition above amounts to the choice of a projection operator $\pi : V \to V$ onto the subspace W.

We form a new linear mapping

$$\tilde{\pi} : V \to V$$

by the rule

$$\tilde{\pi} = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g)^{-1}.$$

We are going to prove:

(i) $\tilde{\pi}$ is a homomorphism of G-representations, and

(ii) $\tilde{\pi}$ is a *projection*.

Together (i) and (ii) imply the Theorem. Indeed, if (ii) holds, one knows that $W' = \ker \tilde{\pi}$ is a *complement to* W. Since $\tilde{\pi}$ is a homomorphism of G-representations, one knows that its kernel W' is a subrepresentation.

To prove (i), let $h \in G$ and $v \in V$ and observe

$$\begin{aligned}
\tilde{\pi}(\rho(h)v) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g)^{-1}(\rho(h)v) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1}h)(v) \\
&= \frac{1}{|G|} \sum_{x \in G} \rho(hx) \circ \pi \circ \pi(x^{-1})(v) \\
&= \frac{1}{|G|} \rho(h) \sum_{x \in G} \rho(x) \circ \pi \circ \rho(x)^{-1}(v) \\
&= \rho(h)\tilde{\pi}(v)
\end{aligned}$$

Thus $\tilde{\pi}$ is indeed a homomorphism of G-representations.

To prove (ii), we observe that for each $g \in G$, the mapping $\rho(g) \circ \pi \circ \rho(g)^{-1}$ is also a projection onto W. Indeed, since W is a subrepresentation, $\rho(g)W = W$, so that $\rho(g) \circ \pi \circ \rho(g)^{-1}(V) \subseteq W$. On the other hand, since $\pi_{|W}$ is the identity mapping, $\rho(g) \circ \pi \circ \rho(g)^{-1}(w) = w$ for any $w \in W$ so the Lemma above shows that $\rho(g) \circ \pi \circ \rho(g) - 1$ is a projection onto W.

Now it is clear that $\sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1})$ maps V to W. Since each mapping $\rho(g) \circ \pi \circ \rho(g^{-1})$ is the identity on W, it follows that

$$\tilde{\pi} = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1})$$

is the identity mapping on W, so $\tilde{\pi}$ is a projection by the Lemma above.

This completes the proof of the Theorem.

Bibliography