

ProblemSet 2 – Representations and characters

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In these exercises, G always denotes a finite group and all vector spaces are assumed to be finite dimensional over the field $F = \mathbb{C}$.

In these exercises, you may use results stated but not yet proved in class about characters of representations of G .

1. In this problem, we identify the character χ_Ω of the *permutation representation* $(\rho, F[\Omega])$ of a group G .

- a. Let V be a vector space and let $\Phi : V \rightarrow V$ a linear mapping. If \mathcal{B} is a basis for V , recall that the *trace* of Φ is defined by

$$\text{tr}(\Phi) = \text{tr}([\Phi]_{\mathcal{B}}).$$

apologies – this is just explanatory; it isn't actually a question

- b. Recall that the *dual* of V is the vector space $V^\vee = \text{Hom}_F(V, F)$ of *linear functionals* on V .

If b_1, \dots, b_n is a basis for V , let $b_j^\vee : V \rightarrow F$ be defined by $b_j^\vee(b_i) = \delta_{i,j}$. Show that $b_1^\vee, \dots, b_n^\vee$ is a basis for V^\vee ; it is known as the *dual basis* to b_1, \dots, b_n .

We must show that the vectors $\{b_i^\vee\}$ are *linearly independent* and *span* V^\vee .

First, linear independence:

Suppose that $\alpha_1, \dots, \alpha_n \in F$ and that

$$0 = \sum_{i=1}^n \alpha_i b_i^\vee$$

(note that this equality “takes place in the vector space V^\vee ”).

We must argue that all coefficients α_j are zero. Well, fix j and consider the vector b_j . We apply the functional $\sum_{i=1}^n \alpha_i b_i^\vee$ to b_j :

$$\left(\sum_{i=1}^n \alpha_i b_i^\vee \right) (b_j) = \sum_{i=1}^n \alpha_i b_i^\vee(b_j) = \alpha_j.$$

Since the *functional* $\sum_{i=1}^n \alpha_i b_i^\vee$ is equal to 0, we know that $\left(\sum_{i=1}^n \alpha_i b_i^\vee \right) (v) = 0$ for every $v \in V$. In particular, this holds when $v = b_j$ and we now conclude that $\alpha_j = 0$.

This proves linear independence.

Finally, we prove the vectors span V^\vee .

Let $\phi \in V^\vee$, and for $1 \leq i \leq n$ write $\alpha_i = \phi(b_i)$. We claim that

$$(\clubsuit) \quad \phi = \sum_{i=1}^n \alpha_i b_i^\vee.$$

TO prove this equality of functions (“functionals”) we must argue that

$$\phi(v) = \left(\sum_{i=1}^n \alpha_i b_i^\vee \right) (v)$$

for every $v \in V$.

And it suffices to prove the latter equality for vectors v taken from the basis $\{b_i\}$.

But now by construction, for each j we have

$$\phi(b_j) = \alpha_j = \left(\sum_{i=1}^n \alpha_i b_i^\vee \right) (\alpha_j)$$

This proves (♣) so that the b_i^\vee indeed span V^\vee .

- c. Prove that the trace of the linear mapping $\Phi : V \rightarrow V$ is given by the expression

$$\text{tr}(\Phi) = \sum_{i=1}^n b_i^\vee(\Phi(b_i)).$$

Recall that $\text{tr}(\Phi)$ is defined to be the trace of the matrix $[\Phi]_{\mathcal{B}}$ where \mathcal{B} is a *basis* of V . It is a fact that this definition is *independent* of the choice of basis.

Also recall that the trace of the $n \times n$ matrix $M = (M_{i,j})$ is given by

$$\text{tr}(M) = \sum_{i=1}^n M_{i,i}.$$

We consider the basis \mathcal{B} of V , and the dual basis \mathcal{B}^\vee of V^\vee , as above.

Recall that the *matrix* $M = [M_{j,i}] = [\Phi]_{\mathcal{B}}$ of Φ in the basis \mathcal{B} is defined by the condition

$$\Phi(b_i) = \sum_{j=1}^n M_{j,i} b_j$$

(for $1 \leq i \leq n$).

Thus,

$$b_i^\vee(\Phi(b_i)) = b_i^\vee \left(\sum_{j=1}^n M_{j,i} b_j \right) = M_{i,i}$$

.

Summing over i we find that

$$\sum_{i=1}^n b_i^\vee(\Phi(b_i)) = \sum_{i=1}^n M_{i,i} = \text{tr}(M) = \text{tr}([\Phi]_{\mathcal{B}}) = \text{tr}(\Phi),$$

as required.

- d. Suppose that the finite group G acts on the finite set Ω , and consider the corresponding permutation representation $(\rho, F[\Omega])$ of G . Recall that $F[\Omega]$ is the vector space of all F -values functions on Ω , and that for $f \in F[\Omega]$ and $g \in G$, we have

$$\rho(g)f(\omega) = f(g^{-1}\omega).$$

In particular, we saw in the lecture that

$$\rho(g)\delta_\omega = \delta_{g\omega},$$

where δ_ω denotes the *Dirac function* at $\omega \in \Omega$.

Show that

$$\text{tr}(\rho(g)) = \#\{\omega \in \Omega \mid g\omega = \omega\};$$

i.e. the trace of $\rho(g)$ is the number of fixed points of the action of g on Ω .

Recall that the vector space $F[\Omega]$ has a basis consisting of the vectors δ_ω for $\omega \in \Omega$.

We write $\delta_\omega^\vee \in F[\Omega]^\vee$ for vectors of the *dual basis*. The linear functional

$$\delta_\omega^\vee : F[\Omega] \rightarrow F$$

is defined by

$$\delta_\omega^\vee(\delta_\tau) = \begin{cases} 1 & \tau = \omega \\ 0 & \tau \neq \omega \end{cases}$$

Fix $g \in G$. According to our earlier work, we know that

$$\text{tr}(\rho(g)) = \sum_{\omega \in \Omega} \delta_\omega^\vee(\rho(g)\delta_\omega) = \sum_{\omega \in \Omega} \delta_\omega^\vee(\delta_{g\omega}).$$

Now, $\delta_\omega^\vee(\delta_{g\omega})$ is 1 when $\omega = g\omega$ and is 0 otherwise. This shows that $\text{tr}(\rho(g))$ is given by the number of $\omega \in \Omega$ for which $g\omega = \omega$, as required.

2. Let V be a representation of G , suppose that W_1, W_2 are invariant subspaces, and that V is the internal direct sum

$$V = W_1 \oplus W_2.$$

Show that the character χ_V of V satisfies

$$\chi_V = \chi_{W_1} + \chi_{W_2}$$

i.e. for $g \in G$ that

$$\chi_V(g) = \chi_{W_1}(g) + \chi_{W_2}(g).$$

Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis of W_1 and let $\mathcal{C} = \{c_1, \dots, c_m\}$ be a basis of W_2 .

Since $V = W_1 \oplus W_2$, we know that $\mathcal{B} \cup \mathcal{C} = \{b_1, \dots, b_n, c_1, \dots, c_m\}$ is a basis for

We consider the dual basis $b_1^\vee, b_2^\vee, \dots, b_n^\vee, c_1^\vee, \dots, c_m^\vee$ of the dual vector space V^\vee .

(Be careful! W_1^\vee is not a subspace of V^\vee ! Instead, it is a *quotient* of $V^\vee \dots$)

Observe that the functional $b_i^\vee \in V^\vee$ is determined by the rules

$$b_i^\vee(b_j) = \delta_{i,j} \quad \text{and} \quad b_i^\vee(c_j) = 0.$$

Similarly, the functional $c_j^\vee \in V^\vee$ is determined by the rules

$$c_j^\vee(b_i) = 0 \quad \text{and} \quad c_j^\vee(c_i) = \delta_{j,i}.$$

Observe that we can restrict b_i^\vee to W_1 , and these restrictions $\{b_i^\vee|_{W_1}\}$ give the basis of W_1^\vee dual to the basis $\{b_i\}$ of W_1 .

Similarly the restrictions $\{c_j^\vee|_{W_2}\}$ give the basis of W_2^\vee dual to the basis $\{c_j\}$ of W_2 .

Now using the results of the previous problem applied to the mapping $g : W_1 \rightarrow W_1$, $g : W_2 \rightarrow W_2$ and $g : V \rightarrow V$, we see that

$$\chi_{W_1}(g) = \text{tr}(g : W_1 \rightarrow W_1) = \sum_{i=1}^n b_i^\vee(g \cdot b_i)$$

$$\chi_{W_2}(g) = \text{tr}(g : W_2 \rightarrow W_2) = \sum_{j=1}^m c_j^\vee(g \cdot c_j)$$

$$\chi_V(g) = \text{tr}(g : V \rightarrow V) = \sum_{i=1}^n b_i^\vee(g \cdot b_i) + \sum_{j=1}^m c_j^\vee(g \cdot c_j)$$

Thus indeed $\chi_V(g) = \chi_{W_1}(g) + \chi_{W_2}(g)$ for each g , as required.

3. Let $G = A_4$ be the alternating group of order $\frac{4!}{2} = 12$.

We are going to find the *character table* of this group.

- a. Confirm that the following list gives a representative for each of the conjugacy classes of G :

$$1, (12)(34), (123), (124)$$

(Note that (123) and (124) are conjugate in S_4 , but *not* in A_4).

What are the *sizes* of the corresponding conjugacy classes?

Note that the centralizer $C_{A_4}((12)(34))$ contains the group $\langle (12), (34) \rangle$, which has 4 elements. On the other hand, $(12)(34)$ is not central in A_4 (e.g. (23) doesn't commute with $(12)(34)$). Since $[A_4 : \langle (12)(34) \rangle] = 3$ (a prime number), conclude that $C_{A_4}((12)(34)) = \langle (12), (34) \rangle$. We conclude that $(12)(34)$ has exactly $12/4 = 3$ conjugates in A_4 .

Next note that the centralizer $C_{A_4}((123))$ contains the subgroup $\langle (123) \rangle$ of order 3. On the other hand, suppose that $\sigma \in C_{A_4}((123))$. Then $\sigma(123)\sigma^{-1} = (123)$. But we know $\sigma(123)\sigma^{-1} = (\sigma(1)\sigma(2)\sigma(3))$, and now the condition

$$(123) = (\sigma(1)\sigma(2)\sigma(3))$$

implies that $\sigma \in \langle (123) \rangle$. Thus $C_{A_4}((123)) = \langle (123) \rangle$ has order 3, and the conjugacy class of (123) has $12/3 = 4$ elements.

Similarly, the centralizer of (124) has order 3, and its conjugacy class has 4 elements.

Finally, we should argue that (123) and (124) are not in fact conjugate in A_4 . Of course, they are conjugate in S_4 by the transposition (34) . Arguing as above, the centralizer of (123) in S_4 is still just equal to $\langle (123) \rangle$. So any element σ of S_4 for which $\sigma(123)\sigma^{-1} = (124)$ has the form $(123)^i(12)$ for some i , and none of those elements is in A_4 .

We have

class rep g	$C_{A_4}(g)$	size of conjugacy class of g
1	12	1
$(12)(34)$	4	3
(123)	3	4
(124)	3	4

Since

$$1 + 3 + 4 + 4 = 12$$

we have found all of the conjugacy classes in A_4 .

- b. Let $K = \langle (12)(34), (14)(23) \rangle$. Show that K is a normal subgroup of index 3, so that $G/K \simeq \mathbb{Z}/3\mathbb{Z}$.

One checks directly that

$$K = \{1, (12)(34), (14)(23), (13)(24)\}$$

so that K has order 4 and index 3 as asserted.

Notice that - as a set - K is the union of $\{1\}$ and the 3-element conjugacy class of $(12)(34)$. This makes clear that $\sigma\tau\sigma^{-1} \in K$ for all $\sigma \in A_4$ and $\tau \in K$, so that K is a *normal subgroup*.

Since $|G/K| = 3$, of course $G/K \simeq \mathbb{Z}/3\mathbb{Z}$ ("groups of prime order are cyclic").

Let ζ_3 be a primitive 3rd root of unity in F^\times and for $i = 0, 1, 2$ let $\rho_i : G \rightarrow F^\times$ be the unique homomorphism with the following properties:

- $\rho_i((123)) = \zeta_3^i$
- $K \subseteq \ker \rho_i$.

Explain why $\rho_0 = \mathbf{1}, \rho_1, \rho_2$ determine distinct irreducible (1-dimensional) representations of G .

In fact, let (ρ_1, V_1) and (ρ_2, V_2) be representations of G for which V_1 and V_2 are 1 dimensional. In this case, $\text{GL}(V_i) = \text{GL}_1(F) = F^\times$ is a commutative group.

Since V_1 and V_2 have dimension 1, any isomorphism $\Phi : V_1 \rightarrow V_2$ is just given by multiplication with a scalar $\alpha \in F^\times$. So if the representations are isomorphic, we have for each $g \in G$ and $v \in V_1$:

$$\rho_2(g)\Phi(v) = \Phi(\rho_1(g)v) \implies \alpha\rho_2(g)v = \alpha\rho_1(g)v$$

Since $\alpha \neq 0$ and since this holds for all $v \in V_1$, we conclude that $\rho_1(g) = \rho_2(g)$ for each $g \in G$.

In other words, two 1 dimensional representations are isomorphic iff they are *equal* (as functions $G \rightarrow F^\times$).

Now, the three homomorphisms ρ_i ($i = 0, 1, 2$) are clearly distinct, because each maps the element (123) to a different element of F^\times . Thus they constitute distinct irreducible 1 dimensional representations of G .

- c. Let $\Omega = \{1, 2, 3, 4\}$ on which G acts by the embedding $A_4 \subset S_4$.

Compute the character χ_Ω of the representation $F[\Omega]$. (This means: compute and list the values of χ_Ω at the conjugacy class representatives given in a.)

(Use the result of problem 1 above).

According to problem 1, the trace of the action of an element $\sigma \in A_4$ on the permutation representation $F[\Omega]$ is equal to the number of fixed points of σ on Ω .

Let's write χ_Ω for the character of the representation $F[\Omega]$.

Thus, the trace is given by

σ	χ_Ω
1	4
(12)(34)	0
(123)	1
(124)	1

- d. The span of the vector $\delta_1 + \delta_2 + \delta_3 + \delta_4 \in F[\Omega]$ is an invariant subspace isomorphic to the irreducible representation ρ_0 (the so-called *trivial representation*).

Thus $F[\Omega] = \rho_0 \oplus W$ for a 3-dimensional invariant subspace. Explain why problem 2 shows that the character of W is given by $\chi_W = \chi_\Omega - \mathbf{1}$.

Problem 2 shows that

$$\chi_\Omega = \mathbf{1} + \chi_W.$$

This is an identity of F -valued functions on G , and it immediately implies that $\chi_W = \chi_\Omega - \mathbf{1}$ as required.

Now prove that $\langle \chi_W, \chi_W \rangle = 1$ and conclude that W is an irreducible representation.

Write $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ for class representatives $1, (12)(34), (123), (124)$. And write c_i for the order of the *centralizer* of σ_i .

Notice that the *values* of $\chi_W = \chi_\Omega - \mathbf{1}$ are given in the following table:

σ_i	c_i	$\chi_\Omega(\sigma_i)$	$\chi_W(\sigma_i)$
$1 = \sigma_1$	12	4	3
$(12)(34) = \sigma_2$	4	0	-1
$(123) = \sigma_3$	3	1	0
$(124) = \sigma_4$	3	1	0

We calculate

$$\begin{aligned}\langle \chi_W, \chi_W \rangle &= \sum_{i=1}^4 \frac{1}{c_i} \chi_W(\sigma_i) \overline{\chi_W(\sigma_i)} = \frac{1}{12} 3 \cdot 3 + \frac{1}{4} (-1) \cdot (-1) + \frac{1}{3} 0 \cdot 0 + \frac{1}{3} 0 \cdot 0 \\ &= \frac{9}{12} + \frac{1}{4} = \frac{9+3}{12} = 1\end{aligned}$$

It follows from the results described in lecture that a representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$, so we conclude that W is an irreducible representation.

e. Explain why

$$\mathbf{1}, \rho_1, \rho_2, W$$

is a complete set of non-isomorphic irreducible representations of G .

We know that G has 4 conjugacy classes, so up to isomorphism there are exactly 4 irreducible representations of G .

We've already pointed out that $\mathbf{1}, \rho_1, \rho_2$ are non-isomorphic irreducible representations each of dimension 1. Now, we've seen that W is an irreducible representation; since W is 2 dimensional, it is not isomorphic to any of the representations $\mathbf{1}, \rho_1, \rho_2$.

Thus we have found 4 non-isomorphic irreducible representations, and we can conclude that any irreducible representation is isomorphic to one of these 4.

f. Display the *character table* of $G = A_4$.

	1	(12)(34)	(123)	(124)
	12	4	3	3
1	1	1	1	1
ρ_1	1	1	ζ	ζ^2
ρ_2	1	1	ζ^2	ζ
χ_W	3	-1	0	0

Bibliography