# Schur's Lemma and irreducible representations

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Let $G$ be a finite group and $F$ an algebraically closed field of characteristic 0.

## Some notational convention(s)

We will sometimes just write $V$ for a representation $(\rho, V)$ of a group $G$, leaving implicit the mapping

$$G \to \mathrm{GL}(V).$$

One exception is when $(\rho, V)$ is a representation for which $\dim V = 1$. In this case, we denote this representation by the notation $\rho$. For example, for any group $G$ one has the *trivial representation* $\mathbb{1}$: the vector space is simply the 1-dimensional space $F$, and the mapping

$$\mathbf{1} : G \to \mathrm{GL}(V) = \mathrm{GL}_1(F) = F^\times$$

is given by $g \mapsto 1$ for each $g \in G$.

Thus for example $\mathbf{1} \oplus \mathbf{1}$ is a two-dimensional $G$-representation.

## Irreducible Representations

A representation $(\rho, V)$ of $G$ is *irreducible* (sometimes one says *simple*) provided that $V \neq 0$ and for any invariant subspace $W$ of $V$, either $W = 0$ or $W = V$.

**Theorem:** Let $(\rho, V)$ be a finite dimensional representation of $G$. Then $V$ is isomorphic to a direct sum of irreducible representations:

$$V = L_1 \oplus L_2 \oplus \cdots \oplus L_r.$$

**Proof:** First of all, we *claim* that $V$ has an invariant subspace which is irreducible as a representation for $G$.

Indeed, we proceed by induction on the dimension $\dim V$. If $\dim V = 1$, then $V$ is irreducible since the only *linear* subspaces of $V$ are 0 and $V$.

Now suppose that $\dim V > 1$. If $V$ is irreducible, we are done. Otherwise, $V$ has a non-0 invariant subspace $W$ with $\dim W < \dim V$. By the inductive hypotheses, $W$ has an invariant subspace which is irreducible as $G$-representation. This completes the proof of the *claim*.

We now prove the Theorem again by induction on $\dim V$. If $\dim V = 1$, again $V$ is already irreducible and the proof is complete. [1]

Now, suppose that $\dim V > 1$. Choose an irreducible invariant subspace $L_1 \subseteq V$ and use complete reducibility to write $V = L_1 \oplus V'$ for an invariant subspace $V'$.

If $V' = 0$, then $V = L_1$ is a direct sum of irreducible representations. Otherwise, $\dim V' = \dim V - \dim L_1 < \dim V$ and so by the induction hypothesis $V'$ is the direct sum $V' = L_2 \oplus \cdots \oplus L_r$ for certain irreducible representations $L_j$.

Now notice that

$$V = L_1 \oplus V' = L_1 \oplus L_2 \oplus \cdots \oplus L_r$$

as required.

[1] When $V = 0$ the result is still true, since $V$ is the "direct sum" of an empty collection of irreducible representations.

**Example:** Suppose that $G$ is the cyclic group $\mathbb{Z}/m\mathbb{Z}$.

If $\zeta$ is an $m$-th root of unity in $F^\times$, then

$$\rho_\zeta : G \to \mathrm{GL}_1(F) = F^\times$$

defined by $\rho_\zeta(i + m\mathbb{Z}) = \zeta^i$ determines an irreducible representation of $G$.

Every irreducible representation of $G$ is isomorphic to $\rho_\zeta$ for some root $\zeta$ of $T^m - 1 \in F[T]$.

For $G$-representations $V$ and $W$, write

$$\mathrm{Hom}_G(V, W)$$

for the space of all $G$-homomorphisms $\Phi : V \to W$. If $V = W$, write

$$\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V)$$

for the space of $G$-endomorphisms.

Notice that $\mathrm{End}_G(V)$ is a *ring* (in fact, an $F$-algebra) under composition of endomorphisms.

**Theorem:** Let $L, L'$ be irreducible representations for $G$.

a. We have $\mathrm{End}_G(L) = F$.

b. $\dim_F \mathrm{Hom}_G(L, L') = \begin{cases} 1 & \text{if} \quad L \simeq L' \\ 0 & \quad\ \text{else} \end{cases}$

**Proof:** (a). This is essentially the content of *Schur's Lemma*. We claim first that $\mathrm{End}_G(L)$ is a *division algebra*.

For this, it suffices to argue that any non-zero element $\phi$ of $\mathrm{End}_G(L)$ has an inverse.

Since $L$ is irreducible and since the kernel of $\phi$ is non-zero, $\ker \phi = 0$. Since $V$ is finite dimensional, it follows that $\phi$ is bijective and therefore invertible.

To see that $\mathrm{End}_G(L) = F$, it remains to observe that when $F$ is *algebraically closed*, any finite dimensional division algebra $D$ with $F \subset Z(D)$ satisfies $D = F$.

Now (b) follows at once from (a).

# Permutation representations and homomorphisms

Let $G$ act *transitively* on the set $\Omega$, fix $\omega \in \Omega$ and let $H = \mathrm{Stab}_G(\omega)$ be the *stabilizer of* $x$. Since the action of $G$ is *transitive*, $\Omega = G.\omega$ is the $G$-orbit of $\omega$, and $\Omega$ may be identified with $G/H$.

**Proposition:** Let $V$ be a $G$-representation and let $x \in V$ be a non-zero vector such that $hx = x$ for all $h \in H$.

a. There is a unique homomorphism of $G$-representations

$$\Phi : F[\Omega] \to V$$

with the property that $\Phi(\delta_\omega) = x$.

b. If the $G$-representation $V$ is *irreducible*, then $V$ is isomorphic to a direct summand of $F[\Omega]$.

**Proof:** (a). Every element $\tau$ of $\Omega$ may be written in the form $\tau = g\omega$ for some $g \in G$. Define $\Phi$ by the rule

$$\Phi(\delta_{g\omega}) = gx$$

for all $g \in G$.

Notice that $\delta_{g\omega} = \delta_{g'\omega} \iff g^{-1}g' \implies gx = g'x$, so $\Phi$ is a well-defined linear mapping.

Let's check that $\Phi$ is a $G$-homomorphism. Let $\gamma \in G$. We must argue that $\Phi(\gamma v) = \gamma\Phi(v)$, and it suffices to prove this identity when $v = \delta_{g\omega}$ is a basis vector in $F[\Omega]$.

Now,

$$\Phi(\gamma\delta_{g\omega}) = \Phi(\delta_{\gamma g\omega}) = \gamma g. = \gamma(g.x) = \gamma\Phi(\delta_{g\omega});$$

this shows that $\Phi$ is indeed a $G$-homomorphism.

Finally, suppose that $\Psi : F[\Omega] \to V$ is any $G$-homomorphism with $\Psi(\delta_\omega) = x$. Then for $g \in G$,

$$gx = g\Psi(\delta_\omega) = \Psi(g\delta_\omega) = \Psi(\delta_{g\omega}).$$

which shows that $\Psi$ is given by precisely the same formula as $\Phi$; this proves the uniqueness.

(b). The homomorphism constructed in (a) is nonzero since $x$ is contained in its image. Since $V$ is irreducible, this homomorphism is *surjective*. Let $K \subset F[\Omega]$ be the *kernel* of this homomorphism. By complete reducibity, there is a subrepresentation $W$ of $F[\Omega]$ such that $F[\Omega] = K \oplus W$.

On the one hand, $F[\Omega]/K = (W \oplus K)/K \simeq W$, and on the other hand, the homomorphism $F[\Omega] \to V$ induces an isomorphism $F[\Omega]/K \simeq V$. Thus $W \simeq V$, so indeed the irreducible representation $W$ is a direct summand of $F[\Omega]$.

**Remark:** Let $\Phi : F[\Omega] \to V$ be the mapping of the proposition, so $x \in V$ is fixed by $V$. If $f \in F[\Omega]$, then

$$\Phi(f) = \sum_{g \in G} f(g)gx.$$

# The Regular Representation

Note that the group $G$ acts on the set $\Omega = G$ by left multiplication. The resulting *permutation representation* $F[\Omega] = F[G]$ is called the *regular representation*.

Note that the action of $G$ on itself is *transitive*, and the stabilizer $H$ of an element (say, $1 \in G$) is the *trivial subgroup*.

**Theorem:** Every irreducible representation is isomorphic to a subrepresentation of the regular representation $F[G]$.

**Proof:** The Theorem follows at once from the Proposition in the previous section.

**Corollary:** Up to isomorphism, $G$ has only finitely many irreducible representations.

**Proof:** Write the regular representation as a direct sum

$$F[\Omega] = L_1 \oplus L_2 \oplus \cdots \oplus L_r$$

of irreducible representations $L_i$.

For each $i$, let $\pi_i : F[\Omega] \to L_i$ be the *projection* onto $L_i$ for this direct sum decomposition, and notice that

$$\mathrm{id}_V = \sum_{i=1}^{r} \pi_i.$$

If $L \subset F[\Omega]$ is an irreducible invariant subspace, it follows that for some $i$, $\pi_i(L) \neq 0$. Since $L$ and $L_i$ are irreducible, $\pi_i$ induces an isomorphism $L_i \xrightarrow{\sim} L$.

# Characters and class functions

**We are now going to assume** $F = \mathbb{C}$

Let $V$ be a representation of $G$ and consider the $\mathbb{C}$-valued function

$$\chi = \chi_V : G \to \mathbb{C}$$

defined by the rule

$$\chi(g) = \mathrm{tr}(g : V \to V)$$

where $\mathrm{tr}(g)$ denotes the *trace* of the linear endomorphism of $V$ determined by $g$.

If $\rho : G \to \mathrm{GL}(V)$ denotes the homomorphism determining the representation, $\chi_V(g) = \mathrm{tr}(\rho(g))$.

**Proposition:** The character $\chi$ of a representation of $G$ is constant on the *conjugacy classes* of the group $G$.

Recall that a conjugacy class $C \subseteq G$ is an equivalence class for the relation

$$g \sim h \iff g = xhx^{-1} \quad \text{for some } x \in G.$$

Thus, a conjugacy class has the form

$$C = \{xyx^{-1} \mid x \in G\}$$

for some $y \in G$.

**Proof of Proposition:** If $g \sim h$ we must show that $\chi(g) = \chi(h)$. But we have $g = xhx^{-1}$ so that $\rho(g) = \rho(x)\rho(h)\rho(x)^{-1}$.

Now the result follows since for any $m \times m$ matrices $M, P$ with $P$ invertible we have

$$\text{tr}(PMP^{-1}) = \text{tr}(M).$$

Let us write $\text{Cl}(G)$ for the space of $\mathbb{C}$-valued *class functions* on $G$.

Thus for any representation $V$ of $G$, we have $\chi = \chi_V \in \text{Cl}(G)$.

We introduce a *hermitian inner product* $\langle \cdot, \cdot \rangle$ on $\text{Cl}(G)$ by the rule

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{x \in G} \phi(x)\overline{\psi(x)}.$$

Thus

$$\langle \cdot, \cdot \rangle : \text{Cl}(G) \times \text{Cl}(G) \to \mathbb{C}$$

is linear in the first variable and conjugate linear in the second variable.

**Proposition:**

a. $\dim \text{Cl}(G)$ is equal to the number of conjugacy classes in $G$.
b. The hermitian inner product $\langle \cdot, \cdot \rangle$ is *positive definite* on $\text{Cl}(G)$.

**Sketch:** For a conjugacy class $C$, let $\theta_C$ denote the *characteristic function* of $C$; thus $\theta_C \in \text{Cl}(G)$ and it is clear that the functions $\{\theta_C\}$ form a basis for $\text{Cl}(G)$. This proves (a).

For (b), consider two conjugacy classes $C, C'$ and compute:

$$\langle \theta_C, \theta_{C'} \rangle = \frac{1}{|G|} \sum_{x \in G} \theta_C(g)\overline{\theta_{C'}(g)} = \delta_{C,C'} \frac{|C|}{|G|}$$

where $\delta_{C,C'}$ denotes the "Kronecker delta". Since $\dfrac{|C|}{|G|}$ is a positive real number, this suffices to confirm that the inner product is positive definite.

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# Bibliography