Math 065 - Bridge to Higher Math (Fall 2025)

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1 Sets

We introduces the language of set theory which is the basic notation used for reading and writing Mathematics.

We need to start somewhere, so although this is not a proper definition, we call a collection of objects a *set*.

For example, the collection of all students in our class is a set.

There are some sets of numbers that are used over and over in Math and we reserve some particular letters to design them. For instance, we write

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

for the set of *natural numbers.* [1]

Similarly, we use the symbols $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ for the sets of integers, rational numbers, real numbers and complex numbers respectively.

The set of integers ("*zahlen*" in German) is given by

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}.$$

The rational numbers consist in fractions $\frac{a}{b}$ for integers a, b and $b \neq 0$; these fractions satisfy the condition $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$. The set of all rational numbers is then given by

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid \ a, b \in \mathbb{Z}, b \neq 0 \right\}$$

The set of real numbers $\mathbb{R}$ is the one you used in Calculus.

You may have seen complex number when trying to solve quadratic equations

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

it is endowed with a natural addition and multiplication. Addition of complex numbers is carried out component-wise, using the addition you know in the real numbers. Multiplication is defined using the associative, commutative and distributive properties and the rule that $i^2 = -1$.

Definition 1.0.1. We indicate that an object a is a member of a set A using the symbol $a \in A$. For example $2 \in \mathbb{N}$ and $-2 \in \mathbb{Z}$, while $\frac{1}{2} \notin \mathbb{Z}$.

Remark 1.0.2. (a) By convention, we will normally denote sets with capital letters, and elements with lower-case letters. Thus $a, b \in A$.

(b) For sets with a finite number of elements, we can specify the set by just listing the elements. For example, if A denotes the set consisting of the first three letters of the English alphabet, then we can simply write

$$A = \{a, b, c\}.$$

(c) Sometimes we specify a set using *set builder notation*. This means that we indicate some *predicate* required for membership in the set. Thus

$$B = \{a \in A \mid \Phi a\}$$

[1] I like to assume that the natural numbers contain 0, although mathematicians are evenly divided on that issue. It is better when you use this symbol that you clarify what you mean.

is the set of all elements a in A for which the predicate Φ is true. This should be read: B is the set of all a in A *such that* Φa holds.

For example, the even integers are defined by

$$E = \{x \in \mathbb{Z} \mid x \text{ is even}\}$$

Thus E is the set of all x in $\mathbb{Z}$ *such that* is even.

Definition 1.0.3. When every element of a set A is also an element of another set B, we say that A is a *subset* of B and write $A \subseteq B$.

The condition $A \subseteq B$ allows the possibility that A and B are equal; if we do not want to allow for this option, we can write $A \subset B$ or more explicitly $A \subsetneq B$.

Example 1.0.4. (a) The number 2 is a natural number. We say that 2 is an element of the set of natural numbers and we can write $2 \in \mathbb{N}$.

(b) When we write $\{2\}$ we mean a set whose only element is the number 2. It is not correct to say that $\{2\} \in \mathbb{N}$. Instead we write $\{2\} \subseteq \mathbb{N}$.

(c) We have natural inclusions

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

In fact, all these sets are different, so we could also write

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

(d) The empty set is the set that has no elements. It is denoted with the symbol $\emptyset$. So, $\emptyset = \{\ \}$.

Two sets are equal if they contain precisely the same elements. This means that they are contained in each other. Conversely, if we have two sets A, B with $A \subseteq B, B \subseteq A$, then $A = B$.

Example 1.0.5. Consider:

(a) Define

$$A = \{x \in \mathbb{R} \,|\, x^3 - 3x^2 + 2x = 0\}, B = \{0, 1, 2\}$$

Let us show that $A = B$.

First we show that $B \subseteq A$. As B is given as a finite collection of real numbers, we can just check that all elements in B satisfy the condition that is required for a real number to be in A.

That is, we need to see that

$$0^3 - 3 \times 0^2 + 2 \times 0 = 0, 1^3 - 3 \times 1^2 + 2 \times 1 = 0, 2^3 - 3 \times 2^2 + 2 \times 2 = 0$$

These equalities are all satisfied, so $B \subseteq A$.

We now need to prove the converse inclusion. Essentially, this means solving the equation $x^3 - 3x^2 + 2x = 0$ and proving that the only possible solutions are $0, 1, 2$.

Factoring, we find

$$x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x - 1)(x - 2)$$

For a product to be 0, one of the factors needs to be zero. This leads us to $x = 0, x = 1, x = 2$.

(b) Define

$$C = \{x \in \mathbb{R} \,|\, x^3 - 3x^2 + 2x > 0\}, D = \{x \in \mathbb{R} \,|\, 0 < x < 1 \text{ or } 2 < x < \infty\}$$

Let us show that $C = D$.

We saw already that we can factor $x^3 - 3x^2 + 2x = x(x-1)(x-2)$. A product of three numbers is positive if all of them are positive or two of them are negative and one positive. Now, $x - 1 > 0$ is equivalent to $x > 1$, $x - 2 > 0$ is equivalent to $x > 2$ and of course $x > 0$ is equivalent to $x > 0$. So, all three factors are positive for $x > 2$ while two of the factors are negative and the third positive for $0 < x < 1$. Using interval notation as in Calculus, the two sets are $(2, \infty)$, $(0, 1)$ respectively. The set C is then composed of these two pieces. (we will introduce notation for this in a second) and this is precisely the way we defined D.

We introduce some basic operations on sets. Let A and B be sets.

Definition 1.0.6. The *union* $A \cup B$ of the sets A B is the set whose elements are in either A or B (or both). In symbols,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Definition 1.0.7. The *intersection* $A \cap B$ of the sets A B is the set whose elements are in bot A and B. In symbols,

$$AB = \{x \mid x \in A \text{ and } x \in B\}.$$

A and B are said to be *disjoint* if their intersection is the empty set – i.e $A \cap B = \emptyset$.

Definition 1.0.8. The *difference* $A - B$ of the sets A and B is the set of elements that are in the first set and not in the second set:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Definition 1.0.9. If the set A is a subset of a set U, the *complement* of A (in U) is defined to be the set

$$\bar{A} = \{x \mid x \notin A\} = U - A.$$

Unions and intersections can be taken for several (more than two) sets, even for infinite collections.

Set operations are sometimes represented and visualized using *Venn Diagrams*; each set is represented as a shape, and the results of the set operations are represented by certain regions. For example:

Example 1.0.4

Example 1.0.10. (a) In Example 1.0.5 (b), we showed that $C = D$. By its definition, the set D is equal to the union $(0, 1) \cup (2, \infty)$. Thus, this set is the union of two intervals in the real line.

(b) The intersection $(0, 1) \cap (2, \infty) = \emptyset$. So , the expression $C = (0, 1) \cup (2, \infty)$ shows that C is the *disjoint union* of two open intervals in the real line.

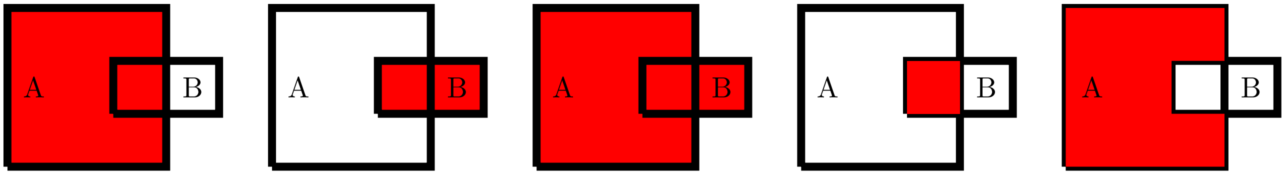

Figure 1: From left to right $A, B, A \cup B, A \cap B, A - B$

(c) For every $n \in \mathbb{N} - \{0\}$ define the semiopen interval of the real line A_n by $A_n = \left(-\frac{1}{n}, n\right]$. The parentheses (on the left indicates that $-\frac{1}{n}$ is not in the set while the bracket] means that n is. Then

$$\bigcap_{n \in \mathbb{N}-\{0\}} A_n = [0, 1]$$

First $[0, 1]$ is contained in each A_n and therefore in its intersection. Also any real number greater than 1 is not contained in A_1 and therefore not contained in the intersection of all A_n. The sequence $-\frac{1}{k}$ has limit zero.

This means that, for any strictly negative number x, we can find some k such that x is smaller than $-\frac{1}{k}$ and therefore $x \notin A_k$ and a fortiori, not in the intersection of all of the A_n.

Similarly, we can compute the union:

$$\bigcup_{n \in \mathbb{N}-\{0\}} A_n = (-1, \infty) :$$

No number smaller than or equal to -1 is in any A_k, therefore, it cannot be in its union. The numbers between -1 and 0 are in A_1 and therefore in the union.

Any positive number is smaller than some natural number m and therefore it is in A_m.

Definition 1.0.11. The *cartesian product* $A \times B$ of the sets A and B is the set whose elements are pairs of elements the first one in A the second one in B:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

Example 1.0.12. (a) You are already familiar with at least one cartesian product. The set of real numbers is represented geometrically as a real line. The cartesian product of two real lines is the set of pairs of real numbers. This is a representation of the points in the plane with each point determined by its two coordinates. That is, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the set of points in the plane.

(b) If $A = \{x \in \mathbb{R} \,|\, x^3 - 3x^2 + 2x = 0\}, B = \{x \in \mathbb{R} \,|\, x^2 - 4 = 0\}$. We have seen that we can write $A = \{0, 1, 2\}$. Similarly, $B = \{2, -2\}$. Therefore,

$$A \times B = \{(0, 2), (1, 2), (2, 2), (0, -2), (1, -2), (2, -2)\}$$

Definition 1.0.13. The *power set* $\mathscr{P}(A)$ of a set A is the set consisting of all subsets of A:

$$\mathscr{P}(A) = \{B \mid B \subseteq A\}$$

Example 1.0.14. (a) Take $A = \emptyset$. Then, $\mathscr{P}(\emptyset) = \{\emptyset\}$. This is a set whose only element is the empty set. In particular, $\mathscr{P}(\emptyset) \neq \emptyset$, as it contains one element.

(b) Take $A = \{1\}$. Then,

$$\mathscr{P}(A) = \{\emptyset, \{1\}\}$$

(c) Take $A = \{1, 2\}$. Then,

$$\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

(d) We see from the previous examples that every time that we add a new element to a set, we double the number of elements in $\mathscr{P}(A)$. We should expect that if A has n elements, then $\mathscr{P}(A)$ has 2^n elements. We can see this as follows: when we construct a subset of A, for each of its elements, we need to decide whether it belongs or not to the subset. This gives two options for each element. These options can be combined in any arbitrary way, so there are in total 2^n possibilities.

Bibliography