Problem Set 6

Question 1: Let A and B be finite sets

a. Prove that if there is a one to one function $f : A \to B$ then B has at least as many elements as A (i.e. show that $|A| \leq |B|$).
b. Prove that if there is an onto function $f : A \to B$ then B has at most as many elements as A (i.e. show that $|B| \leq |A|$).
c. Let A and B be finite sets both with n elements. Prove that a function $f : A \to B$ is injective if and only if it is surjective.
d. Prove that the equivalence in (c) is false if A is infinite. In particular, give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is injective, but not surjective, and a function $g : \mathbb{N} \to \mathbb{N}$ which is surjective, but not injective.

Question 2: For a finite set A, recall that the number of elements in the set is denoted by $|A|$.

Assume that A, B, C are finite sets.

a. Prove that if $|A \cup B| = |A| + |B|$ then $A \cap B = \emptyset$.
b. Prove or disprove:

   $|A \cup B \cup C| = |A| + |B| + |C|$ if and only if $A \cap B = \emptyset, B \cap C = \emptyset,$ and $C \cap A = \emptyset$

Question 3: Let S be a finite set of characters, and let T be the set of all finite (but arbitrarily long) sequences of characters in S. Thus, if $S = \{A, G, C, T\}$, then T is the set of all possible DNA sequences. If S is the set of keys on your computer keyboard, then T is the set of all possible sentences in the english language.

Prove that T is a countable set.

Question 4: For $n \in \mathbb{N}, n \geq 1$, the expression $n!$ denotes the number of possible lists of length n made of n objects in which there are no repeats. By convention $0! = 1$ while we do not define the factorial of a negative integer.

For $n \in \mathbb{N}$ recall that I_n denotes the set of number

$$I_n = \{0, 1, ..., n-1\}.$$

For $n, m \in \mathbb{N}$, the number $\binom{n}{m}$ denotes the number of subsets A of I_n with $|A| = m$. Notice that $\binom{n}{m} = 0$ if $m > n$.

a. For $n > 0$ prove that $n! = (n-1)! \cdot n$.

b. For $n > 0$ prove that

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}.$$

Hint: Note that for subset $A \subset I_n$ of size m, either $n-1 \in A$ or $n-1 \notin A$.

If $n-1 \in A$, then $A \setminus \{n-1\}$ is a subset of I_{n-1} of size $m-1$.

If $n-1 \notin A$ then A is a subset of I_{n-1} of size m.

c. Prove that if $0 \leq m \leq n$, then $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.

Hint: Proceed by induction on n. After treating the base case, the induction hypothesis should be:

$$\forall m, 0 \leq m \leq k \Rightarrow \binom{k}{m} = \frac{k!}{m!(k-m)!}.$$

And you must prove:

$$\forall m, 0 \leq m \leq k+1 \Rightarrow \binom{k+1}{m} = \frac{(k+1)!}{m!(k+1-m)!}.$$

To carry out this proof, use the result of part b.

Question 5: Assume that $A_1, A_2, ..., A_k$ are finite sets with $|A_i| = n_i \in \mathbb{N}$ for $1 \leq i \leq k$. Then the cartesian product $A_1 \times A_2 \times ... \times A_k$ is finite of cardinality $n_1 n_2 ... n_k$.

Hint: Prove the assertion by induction on k. Notice for $k > 1$ that

$$A_1 \times A_2 \times ... \times A_k = (A_1 \times A_2 \times ... \times A_{k-1}) \times A_k.$$

Question 6: Let A and B be sets. We are going to define the *disjoint union* of A and B.

To this end, let $I = \{l, r\}$ be a set with two elements l and r ("left" and "right").

Consider the cartesian product $(A \cup B) \times I$. We define the disjoint union to be

$$A \sqcup B = \{(a, l) \mid a \in A\} \cup \{(b, r) \mid b \in B\}.$$

We write

$$\iota_l : A \to A \sqcup B \text{ and } \iota_r : B \to A \sqcup B$$

for the functions defined by

$$\iota_{l(a)} = (a, l) \text{ and } \iota_{r(b)} = (b, r).$$

Notice that by definition, $A \sqcup B = \iota_l(A) \cup \iota_r(B)$.

a. Explain why ι_l and ι_r are injective functions.
b. Explain why

$$\iota_l(A) \cap \iota_r(B) = \emptyset.$$

c. Recall that we proved in class: if $|A| = n$ and $|B| = m$ then $A \sqcup B$ is finite of cardinality $n + m$.

   If A is a finite set, prove that $\mathbb{Z} \sqcup A$ is a countably infinite set.