# Problem Set 4

**Question 1**: Let $f : A \to B$ be a function. Show that $f$ is onto if and only if there exists a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$.

**Question 2**: Consider the mapping

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$$

given by the rule $f(a, b) = (a + b, a - b)$.

a. Prove or disprove that $f$ is one to one.
b. Prove or disprove that $f$ is onto

**Question 3**: Let $f : A \to B$ be a function. We do not suppose that $f$ is a bijection, so $f^{-1}$ is not defined as a function.

If $X \subseteq A$, we defined the subset $f(X) \subseteq B$ as follows:

$$f(X) = \{y \in B \mid \exists x \in X, y = f(x)\}.$$

If $Y \subseteq B$, we defined the subset $f^{-1}(Y) \subseteq A$ as follows:

$$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}.$$

a. If $Y \subseteq B$, show that $f(f^{-1}(Y)) \subseteq Y$.
b. Show that if $\forall Y \subseteq B, \quad f(f^{-1}(Y)) = Y$, then $f$ is onto.
c. Show that if $f$ is onto, then $\forall Y \subseteq B, f(f^{-1}(Y)) = Y$.
d. Give an example demonstrating that $f(f^{-1}(Y)) = Y$ can fail when $f$ is not onto.

**Question 4**:

a. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 4x^3 + 13$. Is $f$ a bijection? Carefully justify your answer.
b. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 4x^3 + 13$. Is $f$ a bijection? Carefully justify your answer.

**Question 5**: Let $f : S \to T$ be a function, and let $A$ and $B$ be subsets of $S$.

a. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.
b. Give an example to show that in general equality does not hold; i.e. give an example for which $f(A \cap B) \subsetneq f(A) \cap f(B)$.

**Question 6**: Let $A, B, C$ be sets and let $f : A \to B$ and $g : B \to C$ be functions.

a. Suppose that $g \circ f$ is surjective. Prove that $g$ is surjective.
b. Suppose that $g \circ f$ is injective. Prove that $f$ is injective.

**Question 7**: (compare with Proposition 4.6.1 in the notes)

Let $A, B$ be sets. Suppose that $f : A \to B$ is a function.

a. Suppose that $f$ is injective. Prove that for any set $C$ and any pair of functions
$$g_1 : C \to A \text{ and } g_2 : C \to A$$
such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.
b. Conversely, suppose that for every set $C$ and for every pair of functions
$$g_1 : C \to A \text{ and } g_2 : C \to A,$$
the equality $f \circ g_1 = f \circ g_2$ implies that $g_1 = g_2$. Prove that $f$ is injective.