Problem Set 8 (solutions)

Question 1: For integers $a, b \in \mathbb{Z}$, let us say that a is near b if the numbers are no more than 2 units apart. Thus 4 is near 6 and 4 is near 2, but 4 is not near 7.

Since the distance between a and b can be computed using the absolute value function, the relation is given by

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid |x - y| \leq 2\}.$$

Write $a \sim b$ to express that a is near to b.

a. is $\sim$ reflexive?
b. is $\sim$ symmetric?
c. is $\sim$ antisymmetric?
d. is $\sim$ transitive?

Solution for 1:

a. yes it is reflexive: For $x \in \mathbb{Z}$, $|x - x| = 0 \leq 2 \Rightarrow x \sim x$.
b. yes it is symmetric: For $x, y \in \mathbb{Z}$

$$x \sim y \Rightarrow |x - y| \leq 2 \Rightarrow |y - x| \leq 2 \Rightarrow y \sim x.$$

c. No it is not antisymmetric: For example,

$$1 \sim 0 \text{ since } |1 - 0| \leq 2 \text{ and } 0 \sim 1 \text{ since } |0 - 1| \leq 2$$

but $0 \neq 1$.

d. no it is not transitive. For example

$$0 \sim 2 \text{ since } |0 - 2| \leq 2 \text{ and } 2 \sim 4 \text{ since } |2 - 4| \leq 2$$

but $0 \nsim 4$ since $|0 - 4| > 2$.

Question 2: How many distinct equivalence relations are there on the set $A = \{1, 2, 3\}$?

Hint: How many partitions are there of the set $A = \{1, 2, 3\}$?

Solution for 2:
There is one eq. relation on A for which there is exactly 1 equivalence class.

There is one eq. relation on A for which there are exactly 3 equivalence classes.

It remains to count the eq. relations on A for which there are exactly 2 equivalence classes.

For any such eq. relation, one equivalence class will have size 2 and the other will have size 1. Thus the resulting parition is completely determined by the choice of a 2-element subset of $\{1, 2, 3\}$. There are exactly

$$\binom{3}{2} = 3$$

such choices.

Conclusion: there $1 + 1 + 3 = 5$ distinct equivalence relations on $\{1, 2, 3\}$.

Question 3: For $n \in \mathbb{N}$, recall that $\mathbb{Z}_n$ denotes the set of congruence classes modulo n:

$$\mathbb{Z}_n = \{[x]_n \mid x \in \mathbb{Z}\} = \{[0]_n, [1]_n, ..., [n-1]_n\}.$$

For each of the following, decide whether the indicated rule determines a well-defined function. Explain your conclusion.

a. $f_1 : \mathbb{Z} \to \mathbb{Z}_5$ given by $f_1(z) = [2 \cdot z]_5$.
b. $f_2 : \mathbb{Z}_{10} \to \mathbb{Z}_5$ given by $f_2([z]_{10}) = [2 \cdot z]_5$
c. $f_3 : \mathbb{Z}_5 \to \mathbb{Z}_{10}$ given by $f_3([z]_5) = [2 \cdot z]_{10}$.

Solution for 3:

a. The rule is well-defined, since elements of $\mathbb{Z}$ aren't intepreted here as equivalence classes.

b. The rule is well-defined. Suppose that $[z]_{10} = [z']_{10}$. Thus $z \equiv z' \pmod{10}$, so there is $k \in \mathbb{Z}$ with $z - z' = 10 \cdot k$. We just need to check that

$$f_1([z]_{10}) = f_1\left([z']_{10}\right).$$

But

$$f_1([z]_{10}) = [2z]_5 \text{ and } f_1\left([z']_{10}\right) = [2z']_5.$$

So we need to argue that $2z \equiv 2z' \pmod{5}$.

We know that $z - z' = 10 \cdot k \Rightarrow 2z - 2z' = 20 \cdot k = 5 \cdot (4k)$. This shows that

$$5 \mid 2z - 2z'$$

so indeed $2z \equiv 2z' \pmod{5}$.

We conclude that f_1 is well-defined.

c. Again, the rule is well-defined. Suppose that $[z]_5 = [z']_5$. This $z \equiv z' \pmod{5}$ so that $z - z' = 5k$ for $k \in \mathbb{Z}$.

We need to check that

$$f_2([z]_5) = f_2\left([z']_5\right).$$

Since

$$f_2([z]_5) = [2 \cdot z]_{10} \text{ and } f_2\left([z']_5\right) = [2 \cdot z']_{10},$$

we need to argue that $2z \equiv 2z' \pmod{10}$.

For this, notice that

$$2z - 2z' = 2(z - z') = 2 \cdot (5k) = 10k.$$

Thus $10 \mid 2z - 2z'$ so that $2z \equiv 2z' \pmod{10}$.

This confirms that f_2 is well-defined.

Question 4:

a. Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y + [2]_{10} = [8]_{10}$ or show that there aren't any.
b. Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y \cdot [2]_{10} = [8]_{10}$ or show that there aren't any.
c. Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y \cdot [2]_{10} = [3]_{10}$ or show that there aren't any.

Solution for 4:

a. solution is $y = [6]_{10}$.
b. solutions are $y = [4]_{10}$ and $y = [9]_{10}$.
c. there are no solutions. Indeed, if $y \in \mathbb{Z}_{10}$ were a solution, writ=e $y = [z]$ for $z \in \mathbb{Z}$. Then we would know that

$$2z \equiv 3 (\operatorname{mod} 10) \Rightarrow 10 \mid 2z - 3.$$

Thus for some $k \in \mathbb{Z}$ we have $2z - 3 = 10k$. But then $3 = 2z - 10k = 2(z - 5k)$ which shows that $2 \mid 3$. This contradiction shows that there are no solutions.

Question 5: Let $A = (\mathbb{N} - \{0\}) \times (\mathbb{N} - \{0\})$ be the set of pairs of strictly positive natural numbers. Define a relation on A by

$$\forall (a,b),(c,d) \in A, (a,b) \sim (c,d) \text{ if and only if } a+d = b+c.$$

a. Show that $\sim$ is an equivalence relation.

b. Denote by B the set of equivalence classes of elements of A by the relation $\sim$ defined in part (a).

   To cut down on notational clutter, let's agree to write

   $$[a,b] \text{ for the equivalence class } [(a,b)] \text{ of the element } (a,b) \in A.$$

   Define an "addition" on B by the rule

   $$[a,b] \oplus [c,d] = [a+c, b+d].$$

   Show that this rule determines a well-defined addition of the cosets.

   For this, you must show: if

   $$[a,b] = [a',b'] \text{ and } [c,d] = [c',d']$$

   then $[a,b] \oplus [c,d] = [a',b'] \oplus [c',d']$.

c. Show that the assignment

   $$f : B \to \mathbb{Z} \text{ given by } f([a,b]) = a - b$$

   is a well-defined function.

d. List three distinct representatives of the equivalence class $[a,b] \in B$ for which $f([a,b]) = 0$.

e. Show that the function f is a bijection from B to $\mathbb{Z}$.

f. Show that the addition of equivalence classes defined in part (b) corresponds under the function f to the addition of integers.

   In other words, show that

   $$f([a,b]) + f([c,d]) = f([a,b] \oplus [c,d]).$$

g. Could you define a product $\otimes$ on B corresponding by f to the product $\times$ on $\mathbb{Z}$?

Solution for 5abcd:

a. check!
b. Suppose that $[a, b] = [a', b']$ and $[c, d] = [c', d']$. Thus, $a + b' = a' + b$ and $c + d' = c' + d$. The addition formula results in the following:

$$[a, b] \oplus [c, d] = [a + c, b + d] \text{ and } [a', b'] \oplus [c', d'] = [a' + c', b' + d'].$$

To see that $\oplus$ is well-defined, we must argue that

$$(a + c, b + d) \sim (a' + c', b' + d').$$

But this follows since

$$\begin{aligned}(a + c) + (b' + d') &= (a + b') + (c + d') = (a' + b) + (c' + d) \\ &= (a' + c') + (b + d)\end{aligned}$$

c. Suppose that $[a, b] = [c, d]$ Thus $a + d = c + b$ so that

$$(*) \quad a - b = c - d.$$

We must argue that $f([a, b]) = f([c, d])$. But using $(*)$ we see that

$$f([a, b]) = a - b = c - d = f([c, d])$$

which completes the proof.
d. Let $x = [1, 1] = [2, 2] = [3, 3]$ and $f(x) = 0$. Thus $(1, 1), (2, 2)$ and $(3, 3)$ are 3 distinct representatives of this equivalence class x.

Solution for 5ef:

e. To see that f is a bijection, define $g : \mathbb{Z} \to B$ by the rule

$$g(z) = \begin{cases} [z+1, 1] \text{ if } z \geq 0 \\ [1, -z+1] \text{ if } z < 0 \end{cases}.$$

We notice that if $z \geq 0$ then

$$(f \circ g)(z) = f([z+1, 1]) = z + 1 - 1 = z$$

while if $z < 0$ then

$$(f \circ g)(z) = f([1, -z+1]) = 1 - (-z+1) = z.$$

Thus $f \circ g = \mathrm{id}_{\mathbb{Z}}$.

On the other hand, for $[a, b] \in B$, we have

$$(g \circ f)([a, b]) = g(a - b).$$

If $a - b \geq 0$ then $g(a - b) = [a - b + 1, 1] = [a, b]$ since $(a - b + 1, 1) \sim (a, b)$.

Similarly, if $a - b < 0$ then $g(a - b) = [1, -(a - b) + 1] = [a, b]$ since

$$(1, -(a - b) + 1) \sim (a, b).$$

This shows that $(f \circ g) = \mathrm{id}_B$.

Thus f is invertible (with inverse g) and hence a bijection.

f. Let $[a, b]$ and $[c, d]$ in B. We must check that

$$(*) \quad f([a, b] \oplus [c, d]) = f([a, b]) + f([c, d]).$$

On the one hand,

$$f([a, b] \oplus [c, d]) = f([a + c, b + d]) = (a + c) - (b + d))$$

and on the other hand

$$f([a, b]) + f([c, d]) = (a - b) + (c - d).$$

Since $(a + c) - (b + d) = (a - b) + (c - d)$, condition $(*)$ indeed holds.