Problem Set 6

Question 1: Let A and B be finite sets

a. Prove that if there is a one to one function $f : A \to B$ then B has at least as many elements as A (i.e. show that $|A| \leq |B|$).
b. Prove that if there is an onto function $f : A \to B$ then B has at most as many elements as A (i.e. show that $|B| \leq |A|$).
c. Let A and B be finite sets both with n elements. Prove that a function $f : A \to B$ is injective if and only if it is surjective.
d. Prove that the equivalence in (c) is false if A is infinite. In particular, give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is injective, but not surjective, and a function $g : \mathbb{N} \to \mathbb{N}$ which is surjective, but not injective.

Solution for 1ab:
Write $|A| = n$ and $|B| = m$

a. We must show that $n \leq m$. We know that

$$B = f(A) \cup (B \setminus f(A))$$

and that this union is *disjoint* – i.e. $f(A) \cap (B \setminus f(A)) = \emptyset$. Thus Prop. 6.2.1 shows that

$$|B| = |f(A)| + |B \setminus f(A)|.$$

Now, view f as a function $f : A \to f(A)$. Then f is injective by hypothesis, and f is surjective by the definition of $f(A)$. Thus $f : A \to f(A)$ is a bijection so that $n = |A| = |f(A)|$.

Now we see that $m = |B| = |f(A)| + |B \setminus f(A)| = n + |B \setminus f(A)|$ so indeed $m \geq n$.

b. Define a mapping $h : B \to A$ as follows: for $b \in B$, we know that $f : A \to B$ is onto, so there is at least one element $a \in A$ with $f(a) = b$. Choose one such element a and set $h(b) = ja$.

Now note that by construction $f \circ h = \mathrm{id}_B$; this implies that h is one-to-one.

Since h is one-to-one, the result in (a) shows that $m \leq n$.

Solution for 1cd:

c. First suppose that $f : A \to B$ is injective. Then $f : A \to f(A)$ is a bijection so that $|A| = |f(A)| = n$. Since $|B| = n$, Prop 5.2.9 in the notes then shows that $f(A) = B$ so that f is onto.

   On the other hand, if $f : A \to B$ is surjective, we find as in the solution to (b) a mapping $h : B \to A$ with the property that $f \circ h = \mathrm{id}_B$. Then h is injective and we just argued that h is therefore a surjective (and hence a bijection).

   Thus h has an inverse function $h^{-1} : A \to B$. We claim that in fact $f = h^{-1}$. Indeed,

$$\begin{aligned} f \circ h = \mathrm{id}_b &\Rightarrow f^{-1} \circ f \circ h = f^{-1} \circ \mathrm{id}_b \\ &\Rightarrow h = f^{-1}. \end{aligned}$$

   This shows that f is invertible (with inverse h) and hence f is onto, as required.

d. The mapping $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 1$ is injective but not surjective.

   The mapping $g : \mathbb{N} \to \mathbb{N}$ given by

$$g(x) = \begin{cases} 0 \text{ if } x = 0 \\ x - 1 \text{ if } x > 0 \end{cases}$$

   is surjective but not injective.

Question 2: For a finite set A, recall that the number of elements in the set is denoted by $|A|$.

Assume that A, B, C are finite sets.

a. Prove that if $|A \cup B| = |A| + |B|$ then $A \cap B = \emptyset$.
b. Prove or disprove:

   $|A \cup B \cup C| = |A| + |B| + |C|$ iff $A \cap B = \emptyset, B \cap C = \emptyset,$ and $C \cap A = \emptyset$

Solution for 2:
We know that

$$(\clubsuit) \quad |A \cup B| = |A| + |B| - |A \cap B|.$$

a. Using ($\clubsuit$), our hypothesis implies that $|A \cap B| = 0$. But the only set with 0 elements is the empty set, thus $A \cap B = \emptyset$.

b. The statement is true.

($\Rightarrow$) : Suppose that $|A \cup B \cup C| = |A| + |B| + |C|$. Now ($\clubsuit$) shows that

$$|A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$$

and that

$$|B \cup C| = |B| + |C| - |B \cap C|.$$

We find that

$$\begin{aligned} |A| + |B| + |C| = |A \cup (B \cup C)| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|. \end{aligned}$$

Thus

$$0 = |B \cap C| + |A \cap (B \cup C)|.$$

Since both terms are non-negative, find that $|B \cap C| \Rightarrow B \cap C = \emptyset$ and

$$0 = |A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)|$$

which shows that $A \cap B = \emptyset$ and $A \cap C = \emptyset$.

($\Leftarrow$) : Assume $A \cap B = B \cap C = C \cap A = \emptyset$.

Using ($\clubsuit$) twice we have

$$\begin{aligned} (*) \quad |A \cup B \cup C| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|. \end{aligned}$$

Since $B \cap C = \emptyset$, we know that $|B \cap C| = 0$. And since $A \cap B = \emptyset$ and $A \cap C = \emptyset$ we know that $|(A \cap B) \cup (A \cap C)| = |\emptyset| = 0$.

This ($*$) shows that $|A \cup B \cup C| = |A| + |B| + |C|$ as required.

Question 3: Let S be a finite set of characters, and let T be the set of all finite (but arbitrarily long) sequences of characters in S. Thus, if $S = \{A, G, C, T\}$, then T is the set of all possible DNA sequences. If S is the set of keys on your computer keyboard, then T is the set of all possible sentences in the english language.

Prove that T is a countable set.

Solution for 3:
If $|S| = n$, then the set T_d of sequences of length d of characters of S is finite of order $|T_d| = n^d$.

Now notice that the set of sequences T is the union

$$T = \bigcup_{d \in \mathbb{N}} T_d.$$

Thus T is a countable union of finite sets.

Now use the following fact:

- (♠) If $X = \cup_{d \in \mathbb{N}} X_d$ where each X_d is finite of order $n_d > 0$, then X is countably infinite.

  To prove (♠), for each d list the elements of X_d:

  $$x(d)_1, x(d)_2, ..., x(d)_{n_d}.$$

  Now define a function $f : \mathbb{N} \to X$ as follows: For $m \in \mathbb{N}$, find the (unique) value of $d \in \mathbb{N}$ for which

  $$\sum_{i=0}^{d} n_i \leq m < \sum_{i=0}^{d+1} n_i.$$

  Let $k = \sum_{i=0}^{d} n_i$ and let $f(m) = x(d)_{m-k}$. One checks that f is a bijection.

Question 4: For $n \in \mathbb{N}, n \geq 1$, the expression $n!$ denotes the number of possible lists of length n made of n objects in which there are no repeats. By convention $0! = 1$ while we do not define the factorial of a negative integer.

For $n \in \mathbb{N}$ recall that I_n denotes the set of number

$$I_n = \{0, 1, ..., n-1\}.$$

For $n, m \in \mathbb{N}$, the number $\binom{n}{m}$ denotes the number of subsets A of I_n with $|A| = m$. Notice that $\binom{n}{m} = 0$ if $m > n$.

a. For $n > 0$ prove that $n! = (n-1)! \cdot n$.

b. For $n > 0$ prove that

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}.$$

Hint: Note that for subset $A \subset I_n$ of size m, either $n - 1 \in A$ or $n - 1 \notin A$.

If $n - 1 \in A$, then $A \setminus \{n-1\}$ is a subset of I_{n-1} of size $m - 1$.

If $n - 1 \notin A$ then A is a subset of I_{n-1} of size m.

c. Prove that if $0 \leq m \leq n$, then $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.

Hint: Proceed by induction on n. After treating the base case, the induction hypothesis should be:

$$\forall m, 0 \leq m \leq k \Rightarrow \binom{k}{m} = \frac{k!}{m!(k-m)!}.$$

And you must prove:

$$\forall m, 0 \leq m \leq k+1 \Rightarrow \binom{k+1}{m} = \frac{(k+1)!}{m!(k+1-m)!}.$$

To carry out this proof, use the result of part b.

Solution for 4ab:

a. Let $n > 0$. To count the number of lists of length n, we notice that there are n choices for the first entry of the list, and that the remainder of the list is a list of length $n-1$.

   Thus the number of such lists is the product

   $$n \times \# \text{ of lists of length } n-1$$

   Now the formula $n! = (n-1)! \cdot n$ is immediate.

b. To compute $\binom{n}{m}$ we must count the subsets $A \subset I_n$ of size m. For each subset A, there are two possibilies: $n-1 \in A$ or $n-1 \notin A$.

   If $n-1 \in A$ then $A = A' \cup \{n-1\}$ for a subset $A' \subseteq I_{n-1}$ of size $m-1$. By induction on n, there are $\binom{n-1}{m-1}$ such subsets A.

   If $n-1 \notin A$ then $A \subseteq I_{n-1}$. By induction on n, there are $\binom{n-1}{m}$ such subsets A.

   Since these possiblies account for all subsets A, we see that indeed

   $$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}.$$

Solution for 4c:

c. Proceed by induction on $n > 0$. The base case is $n = 1$. Just notice:

   $$\binom{1}{0} = 1 = \frac{1!}{0!(1-0)!} \text{ and } \binom{1}{1} = 1 = \frac{1!}{1!(1-1)!}.$$

   Now suppose that $M \in \mathbb{N}$ and that for any $0 \leq k \leq M$ we have

   $$\binom{M}{k} = \frac{M!}{k!(M-k)!}.$$

   Using (b) and the induction hypothesis we see that

   $$\begin{aligned}\binom{M+1}{k} &= \binom{M}{k-1} + \binom{M}{k} \\ &= \frac{M!}{(k-1)!(M-k+1)!} + \frac{M!}{k!(M-k)!} = (\clubsuit).\end{aligned}$$

   Getting a common denominator of $k!(M+1-k)!$, we see using (a) that

   $$(\clubsuit) = \frac{M! \cdot k + M! \cdot (M+1-k)}{k!(M+1-k)!} = \frac{M!(M+1)}{k!(M+1-k)!} = \frac{(M+1)!}{k!(M+1-k)!}$$

   as required.

Question 5: Assume that $A_1, A_2, \ldots, A_k$ are finite sets with $|A_i| = n_i \in \mathbb{N}$ for $1 \leq i \leq k$. Then the cartesian product $A_1 \times A_2 \times \ldots \times A_k$ is finite of cardinality $n_1 n_2 \ldots n_k$.

Hint: Prove the assertion by induction on k. Notice for $k > 1$ that

$$A_1 \times A_2 \times \ldots \times A_k = (A_1 \times A_2 \times \ldots \times A_{k-1}) \times A_k.$$

Solution for 5:
We will use the fact that if A and B are finite sets, we have

$$(\heartsuit) \quad |A \times B| = |A| \times |B|.$$

Proceed by induction on $k \geq 1$. When $k = 1$, $A_1 \times \ldots \times A_k = A_1$ has order n_1 as required.

Now suppose that $m \in \mathbb{N}$ and that it is known that $A_1 \times \ldots \times A_m$ has order $n_1 n_2 \ldots n_m$.

Then using $(\heartsuit)$ we see that

$$\begin{aligned} |A_1 \times A_2 \times \ldots \times A_m \times A_{m+1}| &= |(A_1 \times A_2 \times \ldots \times A_m) \times A_{m+1}| \\ &= |A_1 \times A_2 \times \ldots \times A_m| \cdot |A_{m+1}| \quad (\diamondsuit). \end{aligned}$$

Now by the inductive hypothesis we see that

$$(\diamondsuit) = (n_1 n_2 \ldots n_m) \cdot n_{m+1} = n_1 n_2 \ldots n_{m+1}$$

as required.

Question 6: Let A and B be sets. We are going to define the *disjoint union* of A and B.

To this end, let $I = \{l, r\}$ be a set with two elements l and r (“left” and “right”).

Consider the cartesian product $(A \cup B) \times I$. We define the disjoint union to be

$$A \sqcup B = \{(a, l) \mid a \in A\} \cup \{(b, r) \mid b \in B\}.$$

We write

$$\iota_l : A \to A \sqcup B \text{ and } \iota_r : B \to A \sqcup B$$

for the functions defined by

$$\iota_l(a) = (a, l) \text{ and } \iota_r(b) = (b, r).$$

Notice that by definition, $A \sqcup B = \iota_l(A) \cup \iota_r(B)$.

a. Explain why ι_l and ι_r are injective functions.
b. Explain why

$$\iota_l(A) \cap \iota_r(B) = \emptyset.$$

c. Recall that we proved in class: if $|A| = n$ and $|B| = m$ then $A \sqcup B$ is finite of cardinality $n + m$.

   If A is a finite set, prove that $\mathbb{Z} \sqcup A$ is a countably infinite set.

Solution for 6:

a. For $a, a' \in A$, we have $(a, l) = (a', l)$ if and only if $a = a'$. This shows that ι_l is injective. A similar argument shows that ι_r is injective.
b. Elements (x, y) and (x', y') in the cartesian product $(A \cup B) \times I$ are equal if and only if $x = x'$ and $y = y'$

   Any element of $\iota_l(A)$ has the form (a, l) and any element of $\iota_r(B)$ has the form (b, r). Since $(a, l) \neq (b, r)$ for any $a \in A$ and $b \in B$, conclude that $\iota_l(A) \cap \iota_r(B) = \emptyset$.
c. Note that $\mathbb{Z}$ is in bijection with $\mathbb{Z}$. So it suffices to prove that if A is finite, then $A \sqcup \mathbb{N}$ is countably infinite.

   For this, suppose that $|A| = n$ and write $A = \{a_0, ..., a_{(n-1\}}$

$$f : A \sqcup \mathbb{N} \to \mathbb{N} \text{ by } f(x) = \begin{cases} i \text{ if } x = \iota_l(a_i) \\ n + k \text{ if } x = \iota_r(k) \text{ for } k \in \mathbb{N} \end{cases}.$$

   One checks that f is one-to-one and onto.