Problem Set 9

Question 1:

a. Let b_n be the sequence given by $b_0 = 0, b_1 = .1, b_2 = .01, b_3 = .001$, and in general b_n has a 1 in decimal spot n and $0'$s elsewhere.

Otherwise said: for $n \geq 1$ $b_n = \frac{1}{10^n}$. Show that

$$\lim_{n \to \infty} b_n = 0.$$

b. Let d_n be the sequence given by $d_0 = 0, d_1 = .9, b_2 = .99, d_3 = .999, \ldots$ and in general d_n has a 9 in each decimal spot up to and including spot n and $0'$s after that.

Otherwise said: for $n \geq 1$, $d_n = \frac{10^n - 1}{10^n}$. Show that

$$\lim_{n \to \infty} d_n = 1.$$

Solution for 1:

a. Let $\varepsilon > 0$. Now use the archimedean property of real (or rational) numbers to find a natural number m such that $m > -\log_{10}(\varepsilon) = \log_{10}(\varepsilon^{-1})$. Since $\log_{10}$ is an increasing function, we see that

$$10^m > \varepsilon^{-1} \text{ so that } 10^{-m} < \varepsilon.$$

Then for $n \geq m$, we have

$$|b_n - 0| = |b_n| = 10^{-n} \leq 10^{-m} < \varepsilon.$$

This confirms that $\lim_{n \to \infty} b_n = 0$.

b. Let $\varepsilon > 0$. Then

$$d_n = \frac{10^n - 1}{10^n} = 1 - \frac{1}{10^n}.$$

The constant sequence 1 converges to 1. By (a), the sequence $\frac{1}{10^n}$ converges to 0.

Using results from the notes we now see that

$$\lim_{n \to \infty} d_n = \lim_{n \to \infty} \left(1 - \frac{1}{10^n}\right) = \lim_{n \to \infty} 1 - \lim_{n \to \infty} \frac{1}{10^n} = 1 - 0 = 1.$$

Question 2: Observe that

$$10^6 = 142857 \times 7 + 1.$$

Explain why this implies that the decimal expansion of $1/7$ is given by

$$\frac{1}{7} = 0.142857\overline{142857}...$$

Solution for 2:
The equation shows that

$$\frac{10^6}{7} = 142857 + \frac{1}{7}.$$

Multiplying by 10^{-6} we find that

$$\frac{1}{7} = 0.142857 + 10^{-6} \cdot \left(\frac{1}{7}\right).$$

This shows that the first 6 decimal digits are 142857, and since the decimal expansion of $10^{-6} \cdot \left(\frac{1}{7}\right)$ begins

$$10^{-6} \cdot \left(\frac{1}{7}\right) = 0.000000142857...$$

this equation shows that the decimal digits indeed repeat as indicated.

Question 3: Using the definition of limit, show that if $a_n = \frac{6n+1}{3n-1}$, then

$$\lim_{n \to \infty} a_n = 2.$$

Solution for 3:
Let $\varepsilon > 0$. Notice that

$$|a_n - 2| = \left|\frac{6n+1}{3n-1} - 2\right| = \left|\frac{(6n+1) - 2(3n-1)}{3n-1}\right| = \frac{3}{3n-1} < \frac{3}{3n-3} < \frac{1}{n-1}.$$

Now use the archimedean property of $\mathbb{R}$ to choose a natural number m so that $m > \varepsilon^{-1} + 1$.

Then

$$m - 1 > \frac{1}{\varepsilon} \text{ so that } \frac{1}{m-1} < \varepsilon.$$

Now for $n \geq m$ we see that

$$|a_n - 2| < \frac{1}{n-1} \leq \frac{1}{m-1} < \varepsilon.$$

Thus $\lim_{n\to\infty} a_n = 2$.

Question 4:

a. By negating the definition, express the condition "a_n is not a Cauchy sequence" so that the last inequality in your sentence has a symbol "greater than" ($\geq$).
b. Show that $a_n = \dfrac{(-1)^n(6n+1)}{3n-1}$ is not a Cauchy sequence.

Solution for 4:

a. a_n is not Cauchy provided that $\exists \varepsilon > 0$ such that $\forall m \in \mathbb{N}$, $\exists n_1, n_2 \geq m$ for which

$$|a_{n_1} - a_{n_2}| \geq \varepsilon.$$

b. Lets write $b_n = \dfrac{6n+1}{3n-1}$ and $a_n = (-1)^n b_n = (-1)^n \dfrac{6n+1}{3n-1}$.

In the preceding problem, we showed that a_n converges to 2.

We can therefore find $m_0 \in \mathbb{N}$ for which $n \geq m_0 \Rightarrow |a_n - 2| < \frac{1}{2}$.

Notice that

$$-\frac{1}{2} < a_n - 2 < \frac{1}{2} \text{ so that } \frac{3}{2} < a_n < \frac{5}{2}.$$

In particular $|a_n| \geq 3/2$.

To prove that b_n is not Cauchy, let $\varepsilon = 3/2$, and let $m \in \mathbb{N}$ be arbitrary. Choose an even number $n_1 \geq \max(m, m_0)$ and set $n_2 = n_1 + 1$ (so n_2 is odd.)

Notice that

$$\left|b_{n_1} - b_{n_2}\right| = \left|(-1)^{n_1} a_{n_1} - (-1)^{n_2} a_{n_2}\right| = \left|a_{n_1} + a_{n_2}\right| \geq \left|a_{n_1}\right|.$$

Since $n_1 \geq m_0$ we see that

$$\left|b_{n_1} - b_{n_2}\right| \geq \left|a_{n_1}\right| \geq \frac{3}{2}.$$

Thus the formulation in (a) shows that b_n is not Cauchy.

Question 5: Assume that a_n is a Cauchy sequence of real numbers. Prove that the sequence $b_n = a_{2n}$ is a Cauchy sequence.

Solution for 5:
Let $\varepsilon > 0$. Since a_n is Cauchy, there is $m \in \mathbb{N}$ such that

$$(*) \quad n_1, n_2 \geq m \Rightarrow |a_{n_1} - a_{n_2}| < \varepsilon.$$

Now, let $n_1, n_2 \geq m$ To show that $b_n = a_{2n}$ is Cauchy, we claim that

$$\left|b_{n_1} - b_{n_2}\right| < \varepsilon.$$

Notice that $2n_1 \geq n_1 \geq m$ and $2n_2 \geq n_2 \geq m$. Thus we see that

$$\left|b_{n_1} - b_{n_2}\right| = \left|a_{2n_1} - a_{2n_2}\right| < \varepsilon$$

by an application of $(*)$. This completes the proof that b_n is Cauchy.

Question 6: Let $d \in \mathbb{N}_{>0}$. Show that the sequence $a_n = 1/n^d$ is a Cauchy sequence.

Solution for 6:
In fact, we will show that $\lim_{n\to\infty} 1/n^d = 0$. For any $d \in \mathbb{N}_{>0}$, one knows that $\dfrac{1}{n^d} \leq \dfrac{1}{n}$.

Now, let $\varepsilon > 0$ and choose m so that $m > \varepsilon^{-1}$. Thus $\dfrac{1}{m} < \varepsilon$.

Now, for $n \geq m$ we see that

$$|a_n - 0| = |a_n| = \frac{1}{n^d} \leq \frac{1}{n} \leq \frac{1}{m} < \varepsilon.$$

This proves that $\lim_{n\to\infty} 1/n^d = 0$.

Since we know that convergent sequences are Cauchy, it follows that $1/n^d$ is Cauchy, as required.

Question 7: Let a_n and a_n' be sequences.

a. If $a_n + a_n'$ is a Cauchy sequence, is it true that a_n is Cauchy and a_n' is Cauchy? Give a proof or a counter-example.
b. If $a_n \cdot a_n'$ is a Cauchy sequence, is it true that a_n is Cauchy and a_n' is Cauchy? Give a proof or a counter-example.

Solution for 7:

a. Let $a_n = n$ and $a'_n = -n$. Neither sequence is Cauchy since neither sequence is *bounded*. But $a_n + a'_n$ is the constant sequence 0, and constant sequences are Cauchy. This provides a example where $a_n + a'_n$ is Cauchy but neither a_n nor a'_n is Cauchy.

b. Let $a_n = \frac{1}{n}$ for $n \in \mathbb{N}_{>0}$. and $a'_n = n$. Then a'_n is not bounded hence not Cauchy. But the sequence $a_n \cdot a'_n$ is the constant sequence 1 which is Cauchy.

   This provides a example where $a_n \cdot a'_n$ is Cauchy but a'_n is not Cauchy.

Question 8: Assume that a_n is a sequence for which

$$\lim_{n\to\infty} a_n = 0.$$

If b_n is a sequence such that $\exists m \in \mathbb{N}$ for which

$$n \geq m \Rightarrow -a_n \leq b_n \leq a_n,$$

prove that

$$\lim_{n\to\infty} b_n = 0.$$

Solution for 8:
Note: In the statement, I should have stipulated either that all a_n are non-negative, or I should have written the condition on b_n in the form

$$-|a_n| < b_n < |a_n| \qquad \text{i.e. } |b_n| < |a_n|.$$

Let $\varepsilon > 0$. Since a_n converges to 0, we may choose $m \in \mathbb{N}$ such that

$$n \geq m \text{ implies that } |a_n - 0| = |a_n| < \varepsilon.$$

To prove that b_n converges to 0 as well, let $n \geq m$.

Then

$$|b_n| < |a_n| < \varepsilon.$$

This completes the proof that $\lim_{n\to\infty} b_n = 0$.