Review for Exam 1

1. solutions

Problem 1: Let P, Q, and R be logical propositions.

a. Show that $\neg(P \vee Q) \Rightarrow R$ and $P \vee (Q \vee R)$ are logically equivalent.
b. Show that $\neg(P \Leftrightarrow Q)$ and $(\neg P) \Leftrightarrow Q$ are logically equivalent.

Hint: In each case, show that they have the same truth table.

Solution for 1a:
We just show that the propositions coincide for all possible truth values of P, Q, and R.

a. For any propositions A and B, the proposition $A \Rightarrow B$ is logically equivalent to $\neg A \vee B$.

   So
   $$\neg(P \vee Q) \Rightarrow R \text{ is equivalent to } \neg\neg(P \vee Q) \vee R$$
   which in turn is equivalent to $(P \vee Q) \vee R$ by cancelling the double negation. Finally, the result follows since the logical operation $\vee$ is associative.

   Or just check the appropriate columns in the table below.

p	q	r	$\neg(p \vee q) \Rightarrow r$	$p \vee (q \vee r)$
`true`	`true`	`true`	`true`	`true`
`true`	`true`	`false`	`true`	`true`
`true`	`false`	`true`	`true`	`true`
`true`	`false`	`false`	`true`	`true`
`false`	`true`	`true`	`true`	`true`
`false`	`true`	`false`	`true`	`true`
`false`	`false`	`true`	`true`	`true`
`false`	`false`	`false`	`false`	`false`

Solution for 1b:

p	q	$p \Leftrightarrow q$	$\neg(p \Leftrightarrow q)$	$(\neg q) \Leftrightarrow q$
`true`	`true`	`true`	`false`	`false`
`true`	`false`	`false`	`true`	`true`
`false`	`true`	`false`	`true`	`true`
`false`	`false`	`true`	`false`	`false`

Problem 2: Consider the claim

$$(\blacklozenge) \quad \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ such that } 3x + 2y = 2x + 4.$$

a. Either prove that $(\blacklozenge)$ is true, or give a counterexample demonstrating that $(\blacklozenge)$ is not true.

b. Write the negation of the claim above so that the negation occurs as the symbol $\neq$.

Solution for 2b:
The negation is

$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, 3x + 2y \neq 2x + 4.$

Solution for 2a:
Statement $(\blacklozenge)$ is false. To prove this, we must prove the negation of the proposition i.e.

$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, 3x + 2y \neq 2x + 4.$$

We take $x = 1$. Now we must argue for each y that $3 + 2y \neq 6$, i.e. that $2y \neq 3$.

Let $y \in \mathbb{Z}$. Then $2y$ is even while 3 is odd, so indeed $2y \neq 3$. Since y was arbitrary, we have proved the negation of $(\blacklozenge)$.

Problem 3:

Let $p(x)$ be the proposition: "I will go to the concert on day x." and let $q(x)$ be the proposition: "I have an exam on day x."

Using p and q and logical connectives, write the propositions that follow. Write the negation of each of these statements both in mathematical symbols and in English.

a. "I will not go to the concert today if I have an exam tomorrow."
b. "If I do not have an exam tomorrow, I will go to the concert today."
c. "If I do not go to the concert today, I will not have an exam tomorrow."
d. "I will have the exam some day"
e. "I will never go to the concert"

Solution for 3:
write tod for "today" and tom for "tomorrow".

a. $q(\text{tom}) \Rightarrow \neg p(\text{tod}))$
b. $\neg q(\text{tom}) \Rightarrow p(\text{tod})$
c. $\neg p(\text{tod}) \Rightarrow \neg q(\text{tom})$
d. $\exists x, q(x)$
e. $\forall x, \neg p(x)$

Negations:

Recall that $A \Rightarrow B$ is equivalent to $\neg A \vee B$. Thus $\neg(A \Rightarrow B)$ is equivalent to $A \wedge \neg B$.

a. $q(\text{tom}) \wedge p(\text{tod})$.

   In words: "I have an exam tomorrow, and I will go to the concert today."

b. $\neg q(\text{tom}) \wedge \neg p(\text{tod})$

   In words: "I don't have an exam tomorrow, and I will not go to the concert today."

c. $\neg p(\text{tod}) \wedge q(\text{tom})$

   In words: "I won't go to the concert today, and I don't have an exam tomorrow."

d. $\forall x, \neg q(x)$

   In words: "I will never have the exam."

e. $\exists x, p(x)$

   In words: "Some day I will go to the concert."

Problem 4:

Consider the following statements for the set $\mathbb{N} = \{0, 1, 2, ...\}$ of natural numbers. If the statement is true, give a proof, if false give a counterexample.

Then write the negation of the statements and again give a proof or a counterexample.

a. $\exists x \in \mathbb{N}, -x > -3$
b. $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, \exists z \in \mathbb{N}, x = 2y + 4z$
c. $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, y \geq x$
d. $\forall y \in \mathbb{N}, \exists x \in \mathbb{N}, x < y$
e. $\forall x \in \mathbb{N}, -2x < -x$

Solution for 4:

a. statement is true; take $x = 0$ so that $0 = -0 > -3$.

   negation is: $\forall x \in \mathbb{N}, -x \leq -3$; it is false.

b. negation is: $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, \forall z \in \mathbb{N}, x \neq 2y + 4z$.

   The negation is true: take $x = 1$. For any $y, z \in \mathbb{N}$, $2y + 4z$ is even and x is odd, so indeed $x \neq 2y + 4z$.

   So the given statement is false.

c. statement is true; take $x = 0$. We just note that $\forall y \in \mathbb{N}, y \geq 0$.

   negation is: $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y < x$; it is false.

d. negation is: $\exists y \in \mathbb{N}, \forall x \in \mathbb{N}, x \geq y$.

   The negation is true: take $y = 0$ and note that for every $x \in \mathbb{N}$, $x \geq 0 = y$.

   So the given statement is false.

e. negation is: $\exists x \in \mathbb{N}, -2x \geq -x$.

   The negation is true: take $x = 0$ and note that $-2 \cdot 0 = 0 \geq 0 = -0$.

   So the given statement is false.

Problem 5: Let $A = \{x \in \mathbb{R} |\ x^2 \leq 8x\}, B = \{x \in \mathbb{R} |\ x^2 \leq 1\}$

a. Find $A \cap B$.
b. Find $A \cup B$.

Solution for 5:

Notice that $A = \{x \in \mathbb{R} \mid x^2 - 8x \leq 0\}$. Since the graph of $p(x) = x^2 - 8x = x(x - 8)$ is a parabola that opens upwards, and since the roots of p are $0, 8$, we see that $A = [0, 8]$.

Similar reasoning shows that $B = [-1, 1]$.

Thus:

a. $A \cap B = [0, 1]$.
b. $A \cup B = [-1, 8]$

Problem 6:

a. For $n \in \mathbb{N}, n \geq 1$, define

$$S_n = 1^2 + 2^2 + \ldots + n^2 = \sum_{i=1}^{n} i^2.$$

Use mathematical induction to show that

$$S_n = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \geq 1.$$

b. For $n \in \mathbb{N}, n \geq 1$ define

$$T_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = \sum_{i=1}^{n} \frac{1}{2^i}.$$

Use mathematical induction to show that

$$T_n = 1 - \frac{1}{2^n} \text{ for all } n \geq 1.$$

Solution for 6a:
To give the proof by induction, we first address the base case $n = 1$.

Then $S_1 = \sum_{i=1}^{1} i^2 = 1$ and on the other hand

$$\frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1.$$

Thus S_1 is given by the given formula for $n = 1$.

We now prove the induction step. Thus, we assume that $m \in \mathbb{N}_{\geq 1}$ and that

$$S_m = \frac{m(m+1)(2m+1)}{6}.$$

We must prove that

$$(\clubsuit) \quad S_{m+1} = \frac{(m+1)(m+1+1)(2(m+1)+1)}{6} = \frac{(m+1)(m+2)(2m+3)}{6}.$$

Well, by definition we have

$$S_{m+1} = \sum_{i=1}^{m+1} i^2 = \sum_{i=1}^{m} i^2 + (m+1)^2 = S_m + (m+1)^2.$$

Using the induction hypothesis, we find that

$$S_{m+1} = \frac{m(m+1)(2m+1)}{6} + (m+1)^2 = \frac{m(m+1)(2m+1) + 6(m+1)^2}{6}.$$

Performing algebra, we see

$$\begin{aligned} S_{m+1} &= \frac{(m+1)(m(2m+1) + 6(m+1))}{6} = \frac{(m+1)(2m^2 + 7m + 6)}{6} \\ &= \frac{(m+1)(2m+3)(m+2)}{6} \end{aligned}$$

and $(\clubsuit)$ is confirmed.

Now the result follows by induction.

Solution for 6b:
We first address the base case $n = 1$; we must prove that $T_1 = 1/2$.

By definition $T_1 = \sum_{i=1}^{1} \frac{1}{2^i} = 1/2$, as required.

We now prove the induction step. Thus, let $m \in \mathbb{N}_{\geq 1}$ and suppose that $T_m = 1 - \dfrac{1}{2^m}$.

We must argue that (♥) $\quad T_{m+1} = 1 - \dfrac{1}{2^{m+1}}$.

By definition we have

$$T_{m+1} = \sum_{i=1}^{m+1} \frac{1}{2^i} = \sum_{i=1}^{m} \frac{1}{2^i} + \frac{1}{2^{m+1}} = T_m + \frac{1}{2^{m+1}}.$$

Using the induction hypothesis, we find that

$$T_{m+1} = T_m + \frac{1}{2^{m+1}} = 1 - \frac{1}{2^m} + \frac{1}{2^{m+1}}.$$

Now,

$$\begin{aligned} T_{m+1} &= 1 - \frac{1}{2^m} + \frac{1}{2^{m+1}} \\ &= 1 - \frac{1}{2^m}\left(1 - \frac{1}{2}\right) \\ &= 1 - \frac{1}{2^m} \cdot \frac{1}{2} \\ &= 1 - \frac{1}{2^{m+1}}. \end{aligned}$$

This completes the proof of (♥), and the formula now follows by induction.

Problem 7:

Denote by $\mathbb{R}$ the set of real numbers. Define the function

$$F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \text{ by the rule } F(a, b) = (a + b, a - b).$$

a. Define what it means for a function $g : A \to B$ to be one to one.
b. Define what it means for a function $g : A \to B$ to be onto.
c. Define what it means for a function $g : A \to B$ to be a bijection.
d. For the function F defined above, prove or disprove that F is one to one.
e. For the function F defined above, prove or disprove that F is onto.
f. For the function F defined above, prove or disprove that F is a bijection.

Solution for 7:

(d). The function F is one-to-one. Suppose that (a_1, b_1) and (a_2, b_2) are in $\mathbb{R} \times \mathbb{R}$ and that

$$(*) \quad F(a_1, b_1) = F(a_2, b_2).$$

We must argue that $(a_1, b_1) = (a_2, b_2)$.

Condition $(*)$ implies that

$$(a_1 + b_1, a_1 - b_1) = (a_2 + b_2, a_2 - b_2).$$

This leads to the two equations

$$(\clubsuit) \quad a_1 + b_1 = a_2 + b_2 \text{ and } a_1 - b_1 = a_2 - b_2.$$

Adding these equations, we obtain $2a_1 = 2a_2$; after cancelling the 2 we deduce $a_1 = a_2$.

Since $a_1 = a_2$, the first equation of $(\clubsuit)$ shows that $a_1 + b_1 = a_1 + b_2$ so that $b_1 = b_2$ as well.

Thus indeed $(a_1, b_1) = (a_2, b_2)$; this completes the proof that F is one-to-one.

(e). The function F is onto. Suppose that $(x, y) \in \mathbb{R} \times \mathbb{R}$ is an arbitrary point. To prove that F is onto, we must find $(a, b) \in \mathbb{R} \times \mathbb{R}$ for which $F(a, b) = (x, y)$.

We require

$$(a + b, a - b) = (x, y)$$

which amounts to the equations

$$(*) \quad a + b = x \text{ and } a - b = y.$$

Adding the equations $(*)$, we obtain $2a = x + y$. And subtracting the equations $(*)$ we obtains $2b = x - y$.

We set

$$(a, b) = \left(\frac{x + y}{2}, \frac{x - y}{2}\right)$$

and observe that

$$F(a, b) = F\left(\frac{x + y}{2}, \frac{x - y}{2}\right) = (x, y).$$

This completes the proof that F is onto.

(f). We have proved that F is one-to-one and onto; thus F is a bijection.

Problem 8:

Let A, B, C be three sets, and consider the functions

$$f : A \to B, \quad g : B \to C, \quad h = g \circ f : A \to C.$$

a. Give a careful definition for the statement: "f is one-to-one".

b. Prove that if f, g are one-to one, then h is one-to one.

c. Show by example that one can find sets A, B, C and functions $f : A \to B$, $g : B \to C$ such that $h = g \circ f$ is one-to one but g is not one-to one.

Solution for 8:

(b). Suppose that f and g are one-to-one, and let $h = g \circ f$. We prove that h is one-to-one.

To do so, let $a_1, a_2 \in A$ and suppose that $h(a_1) = h(a_2)$. We must argue that $a_1 = a_2$.

Now,

$$h(a_1) = h(a_2) \Rightarrow (\clubsuit) \quad g(f(a_1)) = g(f(a_2)).$$

Since g is one-to-one, equation $(\clubsuit)$ implies that $(\blacklozenge) \quad f(a_1) = f(a_2)$.

Since f is one-to-one, equation $(\blacklozenge)$ implies that $a_1 = a_2$. This completes the proof that $h = -g \circ f$ is one-to-one.

(c). We give two examples (either would be a fine solution!) Of course, there are many more valid solutions as well.

example 1: Let $f : [0, \infty) \to \mathbb{R}$ be the "inclusion map" $f(x) = x$, and let $g : \mathbb{R} \to [0, \infty)$ be the function $g(x) = x^2$.

Then $h = g \circ f : [0, \infty) \to [0, \infty)$ is the mapping $h(x) = x^2$ which has inverse function given by $h^{-1}(y) = \sqrt{y}$. Since h is invertible, it is bijective; in particular, h is one-to-one.

On the other hand, g is not one-to-one. For example,

$$g(-1) = (-1)^2 = 1^2 = g(1) \text{ while } -1 \neq 1.$$

example 2: For a perhaps simpler example: Let $A = \{a\}$ be a singleton set, let $B = \{a, b\}$ be a set with two elements, and let $C = A = \{a\}$.

Define $f : A \to B$ by the rule $f(a) = a$, and define $g : B \to C$ by the rules

$$g(a) = a \text{ and } g(b) = a.$$

Then $h = g \circ f : A = \{a\} \to C = \{a\}$ is the identity map. In particular, h is invertible hence one-to-one.

On the other hand, g is not one-to-one since $g(a) = a = g(b)$ while $a \neq b$ in B.

Problem 9: Assume that A and B are sets with $A \neq \emptyset$ and that $f : A \to B$ a function. Show that f is one to one if and only if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$.

Solution for 9:
$(\Rightarrow)$: Assume that $f : A \to B$ is one-to-one. We must show that $\exists g : B \to A$ with $g \circ f = \mathrm{id}_A$.

Before beginning the proof, note that the hypothesis $A \neq \emptyset$ guarantees that there is at least one element of A; we write $a_0 \in A$ for some fixed choice of an element.

Our first task is to define the function g. For $b \in B$, there are two possibilities: either $b \in f(A)$, or $b \notin f(A)$.

If $b \in f(A)$, then by definition of the image $f(A)$, $\exists a \in A$, $f(a) = b$. We now *choose* some $a \in A$ with $f(a) = b$ and *define* $g(b) = a$.

If $b \notin f(A)$, then we set $g(b) = a_0$.

We have now defined a function $g : B \to A$. It remains to show that $g \circ f = \mathrm{id}_A$.

Let $a_1 \in A$. We must show that $(g \circ f)(a_1) = a_1$; i.e. that $g(f(a_1)) = a_1$.

Now, write $g(f(a_1)) = a$ and note that by definition of the function g, the value a is some choice of an element $a \in A$ for which $f(a) = f(a_1)$.

Since f is one-to-one, conclude that $a = a_1$. This shows that $g(f(a_1)) = a_1$; thus $g \circ f = \mathrm{id}_A$ as required.

$(\Leftarrow)$: Assume that $\exists g : B \to A$ such that $g \circ f = \mathrm{id}_A$. We must prove that f is one-to-one.

To that end, let $a_1, a_2 \in A$ and suppose that $f(a_1) = f(a_2)$. To prove that f is one-to-one, we must show that $a_1 = a_2$.

Applying the function g to the equal elements $f(a_1)$ and $f(a_2)$ gives the equation

$$g(f(a_1)) = g(f(a_2)) \quad \text{i.e.} \quad (g \circ f)(a_1) = (g \circ f)(a_2).$$

Since $g \circ f = \mathrm{id}_A$, this implies

$$\mathrm{id}_A(a_1) = \mathrm{id}_A(a_2) \quad \text{i.e.} \quad a_1 = a_2.$$

This completes the proof that f is one-to-one, as required.