Problem Set week 13

Problem 1: Let $\Gamma = (V, E)$ be an undirected graph. Show that there is an even number of vertices of odd degree. **Hint:** Remember that $\sum_{v \in V} \deg v = 2|E|$.

Problem 2:

a. Prove that in any simple graph with $|V| \geq 2$, there are at least two vertices of the same degree.

b. Does the result in a. remain valid for graphs which aren't necessarily simple?

Problem 3: Let $\mathbb{B}$ be the graph with two vertices A and B and a unique edge $[A, B]$.

a. Let Γ be a bipartite graph with $V = V_1 \sqcup V_2$ and $E \subset \{[v_1, v_2] \mid (v_1, v_2) \in V_1 \times V_2\} \subset \mathcal{P}(V)$.

   Show that there is a morphism of graphs $\varphi : \Gamma \to \mathbb{B}$ such that $\varphi_V(V_1) = \{A\}$ and $\varphi_V(V_2) = \{B\}$.

b. Let Γ be any graph and suppose that there is a morphism $\varphi : \Gamma \to \mathbb{B}$. Prove that Γ is a bipartite graph.

Problem 4: Let $\Gamma = (V, E)$ be an undirected graph.

a. Define a relation $\sim$ on V by $a \sim b$ if and only if there is a path in Γ from a to b. Prove that $\sim$ is an equivalence relation.

b. For a vertex $a \in V$, let $[a] \subseteq V$ be the equivalence class of a for the equivalence relation from part a. Let $E_a = \{[x, y] \in E \mid x, y \in [a]\}$. Prove that $([a], E_a)$ is a subgraph of Γ.

c. For a natural number n, let $\mathbb{T}_n$ be the graph with n vertices $\{v_1, v_2, ..., v_n\}$ and with edges $\{[v_1, v_1], [v_2, v_2], ..., [v_n, v_n]\}$. In other words, $\mathbb{T}_n$ has a loop at each vertex and no other edges. Suppose that there is a morphism $\varphi : \Gamma \to \mathbb{T}_n$.

   Prove that if $a, b \in V$ and $\varphi_V(a) \neq \varphi_V(b)$ then $a \nsim b$.

d. Conclude that if there is a morphism $\varphi : \Gamma \to \mathbb{T}_n$ such that the mapping φ_V on vertices is surjective, then there are at least n equivalence classes in V for the relation $\sim$.

Problem 5: Let $n \in \mathbb{N}$ and let K_n be the complete (undirected) graph on n vertices.

a. Let Γ_0 be the subgraph of K_n obtained by removing a single vertex and removing all edges involving that vertex. prove that Γ_0 is isomorphic to K_{n-1}.

b. Let e_1, e_2 be edges in K_n, and for $i = 1, 2$ let Γ_i be the graph obtained from K_n by deleting the edge e_i. (The vertices of Γ_i are the n vertices of K_n).

   Prove that Γ_1 is isomorphic to Γ_2.

c. Let $e_1 \neq e_2$ and $f_1 \neq f_2$ be edges in K_n, let Γ_e be the graph obtained from K_n by deleting the edges e_1 and e_2 and let Γ_f be the graph obtained from K_n by deleting the edges f_1 and f_2. (Again, the vertices of Γ_e and Γ_f are the n vertices of K_n).

   Show that in general Γ_e is not isomorphic to Γ_f.

Problem 6: Let $\Gamma = (V, E)$ be a directed graph. Recall that for $a, b \in V$, an edge $[a, b]$ is directed, so that $a \neq b \Rightarrow [a, b] \neq [b, a]$.

A source vertex is a vertex $a \in V$ such that any edge $e \in E$ incident to a has the form $[a, x]$ for some $x \in V$.

Similarly, a sink vertex is a vertex $b \in V$ such that any edge $e \in E$ incident b has the form $[y, b]$ for some $y \in V$.

Suppose that there are m source vertices and n sink vertices. Prove that there is a morphism of graphs $\varphi : \Gamma \to K_{m,n}$ such that the mapping φ_V on vertices is surjective, where $K_{m,n}$ is the complete bi-partite graph of type (m, n).

Remark: One can view a bipartite graph as a directed graph in a natural way.