Review for Exam 2

1. Remarks

The exam covers material discussed in Weeks 5 through 9 of our course.

Topics include:

- cardinality of sets (finite and infinite)
- relations and equivalence relations
- limits of sequences
- Cauchy sequences

This review sheet is provided as additional practice and preparation for the first midterm that takes place Friday Nov 7, 10.30-11:20.

This review sheet will not be collected or graded. Model solutions will be provided later, and we'll discuss these problems in the in-class review on Nov 5. You should also review the homework and examples in the notes.

2. terminology and definitions

- what is a finite set?
- what is a countable set?
- what is meant by the disjoint union of two sets?
- what is a relation? an equivalence relation? a partition of a set?
- how is the relation $\equiv \pmod{n}$ defined on $\mathbb{Z}$?
- if a_n is a sequence or real or rational numbers, what does it mean to say that $\lim_{n \to \infty} a_n = L$?

  what does it mean to say that a_n is a Cauchy sequence?

3. Sample problems

Problem 1:

a. Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is injective but not surjective.

b. Give an example of a function $g : \mathbb{Z} \to \mathbb{Z}$ which is surjective but not injective.

Problem 2: We have seen that for finite sets A and B, we have

$$(\heartsuit) \quad |A \cup B| = |A| + |B| - |A \cap B|.$$

Now, let A, B, C be finite sets.

a. Suppose that

$$|A \cup B \cup C| = |A| + |B| + |C|.$$

Prove that $A \cap B = \emptyset$, $A \cap C = \emptyset$ and $B \cap C = \emptyset$.

b. If $A \cap B \cap C = \emptyset$, is it true that $|A \cup B \cup C| = |A| + |B| + |C|$? Justify your response with a proof or a counter-example.

Problem 3: Let S be a finite set having $n \in \mathbb{N}$ elements, let $d \in \mathbb{N}$.

A sequence of elements of S of length d is a list $a_0, a_1, \ldots, a_{d-1}$ where $a_j \in S$ for each j.

Let $T_{\le d}$ be the set of all sequences of elements of S of length $\le d$.

Give an expression for the cardinality $|T_d|$ in terms of d and n.

Problem 4: Let $f : X \to Y$ be a function. Define a relation $\sim$ on X by the rule: $x \sim x'$ if and only if $f(x) = f(x')$.

a. Show that $\sim$ is an equivalence relation.
b. For $x \in X$, show that the equivalence class $[x]$ is equal to $f^{-1}(f(x))$.

Problem 5: Let $n \in \mathbb{N}$ and write $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. For $(a, b), (a', b') \in \mathbb{Z}^2$ we consider the relation

$$(a, b) \equiv (a', b') \pmod{n} \text{ provided that } a \equiv a' \pmod{n} \text{ and } b \equiv b' \pmod{n}.$$

a. Show that this relation is an equivalence relation on $\mathbb{Z}^2$.

b. Show that there are n^2 equivalence classes in $\mathbb{Z}^2$ for this relation.

Problem 6: Let A and B be sets.

a. If A is finite and B is countably infinite, prove that the union $A \cup B$ is a countably infinite set.
b. Prove that there is a bijection between $A \sqcup A \sqcup A$ and $A \times \{0, 1, 2\}$.
c. Let $f : A \to B$ be an injective function, and suppose that A is infinite and not countable. Prove that B is infinite and not countable.

Problem 7: We showed the following in class (and you may use these statements here):

Let A, B be sets.

i. If $f : A \to B$ is an injective function and if B is countable, then A is either finite or countably infinite.
ii. If $g : B \to A$ is a surjective function and if A is countable, then B is either finite or countably infinite.

a. Prove that if A is a countably infinte set and if $\sim$ is an equivalence relation on A, then the set of equivalence classes is either finite or countably infinite.
b. Show by example for some countable set(s) A with equivalence relation(s) $\sim$ that the set B of equivalence classes can be finite, and that B can be infinite.

Problem 8: Find all solutions x to the following equations in $\mathbb{Z}_{14}$:

a. $[3]_{14} \cdot x + [2]_{14} = [6]_{14}$

   (Note that $[3]_{14} \cdot [5]_{14} = [15]_{14} = [1]_{14}$.)
b. $[2]_{14} \cdot x = [2]_{14}$.

Problem 9: Let $n \in \mathbb{N}$ and consider the mapping $f : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f([x]_n) = [x]_n \cdot [x]_n$.

a. Show that f is well-defined.

   (We obtained a result in class that implies that f is well-defined, but show the details here).
b. Is f injective? Is f surjective? Prove or disprove each statement.

Problem 10: Assume that a_n is a sequence with $\lim_{n\to\infty} a_n = 1$. Prove that $(-1)^n a_n$ is not a Cauchy sequence.

Problem 11: If a_n is a Cauchy sequence, prove that the sequence $b_n = a_n \cdot a_{2n}$ is a Cauchy sequence.

(You can use results from class/the notes - just give a brief description of the result you are using).