Problem Set 3

Question 1: If n is a positive integer, prove using induction that

$$1^3 + 2^3 + 3^3 + \ldots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4}$$

Question 2: Use induction to prove for $n \in \mathbb{N}$ that

$$(-1)^n = \begin{cases} 1 \text{ if } n \text{ is even} \\ -1 \text{ if } n \text{ is odd.} \end{cases}$$

Question 3: Let $n \in \mathbb{N}$. We may use *division with remainder* to find $q, r \in \mathbb{N}$ for which

$$n = 4q + r \text{ and } 0 \leq r < 4.$$

Thus r is the *remainder* upon division of n by 4.

Let $i \in \mathbb{C}$ be the “imaginary” complex number with $i^2 = -1$ and prove that

$$i^n = \begin{cases} 1 \text{ if } r = 0 \\ i \text{ if } r = 1 \\ -1 \text{ if } r = 2 \\ -i \text{ if } r = 3. \end{cases}$$

Hint: What is i^4? More generally, what is i^{4q} for $q \in \mathbb{Z}$?

Question 4:

a. For $p \in \mathbb{N} - \{0, 1\}$, define what it means to say that p is prime.
b. Use **strong induction** to prove that every natural number greater than 1 is a product of primes.

Question 5: Let $n \in \mathbb{N}, n \geq 1$. Find and prove a non-recurrent formula for

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

Hint: Write $S_n = \sum_{i=1}^{n} 1/(i(i+1))$ and compute S_n for $n = 1, 2, 3$ and see if you can guess a formula for S_n. Then use induction to prove that your formula is correct.

"Non-recurrent" means that in describing S_n, your formula shouldn't mention terms of the form S_m for $m < n$. For example, one might try to solve this problem by responding that

$$S_n = S_{n-1} + \frac{n}{n+1}$$

but that formula is recurrent.

Question 6: Consider the condition that a sequence a_n satisfies

$$(\clubsuit) \quad a_n = 6a_{n-1} - 9a_{n-2} \text{ for each } n \geq 2$$

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a. Assume that b is a real number different from zero. Find all values $b \neq 0$ such that the sequence $a_n = b^n$ satisfies $(\clubsuit)$.
b. If the sequence a_n and b_n both satisfy the condition in $(\clubsuit)$, explain why the sequence

$$c_n = a_n + b_n$$

also satisfies $(\clubsuit)$.
c. Show that the sequence $b_n = n3^n$ satisfies the $(\clubsuit)$.
d. Using your sequences from a. and c., use the result from b. to find a sequence c_n satisfying $(\clubsuit)$ that also satisfies

$$c_0 = -2 \text{ and } c_1 = 6.$$