Review for Exam 1

1. Remarks

The exam covers material discussed in Weeks 1 through 4 of our course.

This review sheet is provided as additional practice and preparation for the first midterm that takes place Friday October 3, 10.30-11:20.

This review sheet will not be collected or graded. Model solutions will be provided later, and we'll discuss these problems in the in-class review on Oct 1. Please note that the exam will not be a copy of this review sheet: you should also review the homework and examples in the notes.

2. Sample problems

Problem 1: Let P and Q be logical propositions.

a. Show that $\neg(P \vee Q) \Rightarrow R$ and $P \vee (Q \vee R)$ are logically equivalent.
b. Show that $\neg(P \Leftrightarrow Q)$ and $(\neg P) \Leftrightarrow Q$ are logically equivalent.

Hint: In each case, show that they have the same truth table.

Problem 2: Consider the claim

$$(\blacklozenge) \quad \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ such that } 3x + 2y = 2x + 4.$$

a. Either prove that $(\blacklozenge)$ is true, or give a counterexample demonstrating that $(\blacklozenge)$ is not true.
b. Write the negation of the claim above so that the negation occurs as the symbol $\neq$.

Problem 3:

Let $p(x)$ be the proposition: "I will go to the concert on day x." and let $q(x)$ be the proposition: "I have an exam on day x."

Using p and q and logical connectives, write the propositions that follow. Write the negation of each of these statements both in mathematical symbols and in English.

a. "I will not go to the concert today if I have an exam tomorrow."
b. "If I do not have an exam tomorrow, I will go to the concert today."
c. "If I do not go to the concert today, I will not have an exam tomorrow."
d. "I will have the exam some day"
e. "I will never go to the concert"

Problem 4:

Consider the following statements for the set $\mathbb{N} = \{0, 1, 2, ...\}$ of natural numbers. If the statement is true, give a proof, if false give a counterexample.

Then write the negation of the statements and again give a proof or a counterexample.

a. $\exists x \in \mathbb{N}, -x > -3$
b. $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, \exists z \in \mathbb{N}, x = 2y + 4z$
c. $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, y \geq x$
d. $\forall y \in \mathbb{N}, \exists x \in \mathbb{N}, x < y$
e. $\forall x \in \mathbb{N}, -2x < -x$

Problem 5: Let $A = \{x \in \mathbb{R} |\ x^2 \leq 8x\}, B = \{x \in \mathbb{R} |\ x^2 \leq 1\}$

a. Find $A \cap B$.
b. Find $A \cup B$.

Problem 6:

a. For $n \in \mathbb{N}, n \geq 1$, define

$$S_n = 1^2 + 2^2 + ... + n^2 = \sum_{i=1}^{n} i^2.$$

Use mathematical induction to show that

$$S_n = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \geq 1.$$

b. For $n \in \mathbb{N}, n \geq 1$ define

$$T_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = \sum_{i=1}^{n} \frac{1}{2^i}.$$

Use mathematical induction to show that

$$T_n = 2^n - \frac{1}{2^n} \text{ for all } n \geq 1.$$

Problem 7:

Denote by $\mathbb{R}$ the set of real numbers. Define the function

$$F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \text{ by the rule } F(a, b) = (a + b, a - b).$$

a. Define what it means for a function $g : A \to B$ to be one to one.
b. Define what it means for a function $g : A \to B$ to be onto.
c. Define what it means for a function $g : A \to B$ to be a bijection.
d. For the function F defined above, prove or disprove that F is one to one.
e. For the function F defined above, prove or disprove that F is onto.
f. For the function F defined above, prove or disprove that F is a bijection.

Problem 8:

Let A, B, C be three sets, and consider the functions

$$f : A \to B, \quad g : B \to C, \quad h = g \circ f : A \to C.$$

a. Give a careful definition for the statement: "f is one-to-one".

b. Prove that if f, g are one-to one, then h is one-to one.

c. Show by example that one can find sets A, B, C and functions $f : A \to B, g : B \to C$ such that $h = g \circ f$ is one-to one but g is not one-to one.

Problem 9: Assume that A and B are sets with $A \neq \emptyset$ and that $f : A \to B$ a function. Show that f is one to one if and only if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$.