Problem Set 3

Tufts University
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Math 065, Fall 2025

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Question 1: If n is a positive integer, prove using induction that

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 \;=\; \frac{n^2(n+1)^2}{4}.$$

Base case: $1^3 = 1 = \frac{1^2(1+1)^2}{4}$. Thus, the equation holds for $n = 1$.
Suppose the equation holds for $n = m$.
Then $\sum\limits_{i=1}^{m+1} i^3 \;=\; \sum\limits_{i=1}^{m} i^3 + (m+1)^3 \;=\; \frac{m^2(m+1)^2}{4} + (m+1)^3 \;=\; (m+1)^2\left(\frac{m^2+4m+4}{4}\right) \;=\; (m+1)^2\frac{(m+2)^2}{4} \;=\; \frac{(m+1)^2((m+1)+1)^2}{4}$, and so the equatino holds for $m+1$. Thus the result is shown by induction.

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Question 2: Use induction to prove for $n \in \mathbb{N}$ that

$$(-1)^n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

We have 2 cases, so we should show 2 base cases: For $n = 1$, $(-1)^1 = -1$, and for $n = 2$, $(-1)^2 = 1$.
Now suppose the statement holds for some $n > 2$. Then

$$(-1)^{n+1} = (-1)^n(-1)^1 = \begin{cases} 1(-1) = -1 & \text{if } n \text{ is even,} \\ -1(-1) = 1 & \text{if } n \text{ is odd.} \end{cases}$$

Since $n+1$ is odd if and only if n is even, we have

$$(-1)^{n+1} = \begin{cases} 1 & \text{if } n+1 \text{ is even,} \\ -1 & \text{if } n+1 \text{ is odd.} \end{cases}$$

$\blacksquare$

Question 3: Let $n \in \mathbb{N}$. We may use division with remainder to find $q, r \in \mathbb{N}$ such that

$$n = 4q + r \quad \text{and} \quad 0 \le r < 4.$$

Thus r is the remainder upon division of n by 4.
Let $i \in \mathbb{C}$ be the imaginary number with $i^2 = -1$. Prove that

$$i^n = \begin{cases} 1 & \text{if } r = 0, \\ i & \text{if } r = 1, \\ -1 & \text{if } r = 2, \\ -i & \text{if } r = 3. \end{cases}$$

Hint: What is i^4? More generally, what is i^{4q} for $q \in \mathbb{Z}$?

$i = \sqrt{-1}$. Thus, $i^2 = (\sqrt{-1})^2 = -1$, and so $i^4 = (-1)^2 = 1$. Thus,

$$i^n = i^{4q+r} = i^{4q} i^r = (i^4)^q i^r = 1^q i^r = i^r$$

Thus,

$$i^n = i^r = \begin{cases} 1 & \text{if } r = 0 \\ i & \text{if } r = 1 \\ -1 & \text{if } r = 2 \\ -i & \text{if } r = 3 \end{cases}$$

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Question 4: Complete the following:

(a) **For $p \in \mathbb{N} \setminus \{0, 1\}$, define what it means to say that p is prime.**
We say $p \in \mathbb{N} \setminus \{0, 1\}$ is prime if and only if

$$\forall n \in \mathbb{N}, n|p \iff n = 1 \text{ or } n = p.$$

(b) **Use strong induction to prove that every natural number greater than 1 is a product of primes.**
Base case: 2 is prime (if $n > 2$, $n \neg | 2$. If $n \leq 2$, then $n = 0$, $n \neg | 2$, or $n = 1$ or 2.)
Suppose for all $n < N \in \mathbb{N}$, n is a product of primes. Now consider N: Suppose there exists some $a \in \mathbb{N}$ with $1 < a < N$ such that $a|N$. Then there exists some $b \in \mathbb{N}$ with $1 < b < N$ and $N = ab$. Since a and b are products of primes, and N is a product of a and b, N must be a product of primes.
Suppoose no such a exists. Then the only natural numbers that divide N are 1 and N, and so N must be prime. Thus N is a product of primes.

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Question 5: Let $n \in \mathbb{N}$, $n \geq 1$. Find and prove a non-recurrent formula for

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

***Hint:* Compute S_n for $n = 1, 2, 3$ and guess a formula. Then use induction to prove it. (*Non-recurrent* means your formula for S_n should not involve earlier S_m with $m < n$.)**

$S_N = \sum\limits_{i=1}^{N} \frac{1}{i(i+1)} = \sum\limits_{i=1}^{N-1} \frac{1}{i(i+1)} + \frac{1}{N(N+1)} = S_{N-1} + \frac{1}{N(N+1)}$.
Thus, $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, $S_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$ ... Maybe $S_N = \frac{n}{n+1}$.
Base case is proved above. Suppose this guess holds for some n. Then

$$S_{n+1} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)(n+1)}{(n+1)(n+2)} = \frac{n+1}{n+2} = \frac{n+1}{(n+1)+1},$$

and so the result is shown.

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Question 6: Consider the condition that a sequence a_n satisfies

$$(\clubsuit) \quad a_n = 6a_{n-1} - 9a_{n-2} \quad \text{for each } n \geq 2.$$

(a) **Assume b is a real number different from zero. Find all values $b \neq 0$ such that the sequence $a_n = b^n$ satisfies (♣).**

Suppose

$$b^n = 5b^{n-1} - 9b^{n-2}.$$

Since $b \neq 0$, we can divide by b. Then

$$b^2 = 6b - 9 \quad \Rightarrow \quad b^2 - 6b + 9 = 0.$$

This quadratic has only one solution: $b = 3$.

(b) **If the sequences a_n and b_n both satisfy (♣), explain why the sequence $c_n = a_n + b_n$ also satisfies (♣).**

Suppose $a_n = 6a_{n-1} - 9a_{n-2}$ and $b_n = 6b_{n-1} - 9b_{n-2}$. Then

$c_n = a_n + b_n = a_n = 6a_{n-1} - 9a_{n-2} + b_n = 6b_{n-1} - 9b_{n-2} = 6(a_{n-1} + b_{n-1}) - 9(a_{n-2} - b_{n-2}) = 6c_{n-1} - 9c_{n-2}$

(c) **Show that the sequence $b_n = n3^n$ satisfies (♣).**

$$\begin{aligned} n3^n &= 2n3^n - n3^n \\ &= 2n3^n - 2 \cdot 3^n + 2 \cdot 3^n - n3^n \\ &= 2(n-1)3^n - (n-2)3^n \\ &= 6(n-1)3^{n-1} - 9(n-2)3^{n-2} \end{aligned}$$

(d) **Using your sequences from (a) and (c), use the result from (b) to find a sequence c_n satisfying (♣) that also satisfies $c_0 = -2$ and $c_1 = 6$.**

Let $a_n = 3^n$ and $b_n = n3^n$. From part (b), we have $c_n = xa_n + yb_n$ satisfies (♣). Thus, we must find x and y such that $c_0 = -2$ and $c_1 = 6$.

$c_0 = xa_0 + yb_0 = x + 0 = -2$. Thus, $x = -2$.

$c_1 = -2a_1 + yb_1 = -2 * 3 + y * 3 = 6$. Thus, $y = 4$.

Therefore, $c_n = 4n \cdot 3^n - 2 \cdot 3^n = 2 \cdot 3^n(2n - 1)$ satisfies (♣) and $c_0 = -2$ and $c_1 = 6$.