Problem Set 2

Question 1:

Write the following sentence using the quantifier notation (that is, the symbols $\exists$, $\forall$). *Notice:* We do not claim these statements are true, so please do not try to prove them

a. Every natural number greater than one is prime
b. There is a natural number greater than one that is neither prime nor composite.
c. For every integer x there exists an integer y such that $xy = 1$.
d. There is an integer x such that for every integer y, $xy = 1$.
e. For every integer x and every integer y, $x + y = y + x$.
f. There is an integer x and an integer y such that x/y is an integer.

Question 2: Write the negation of each of the statements in Question 1, in such a way that the negation appears as late in the sentence as possible.

For each statement, decide whether the negation is true, and confirm your assessment with an argument.

Question 3:

a. Prove that an integer is odd if and only if it is equal to the sum of two consecutive integers.

   Note: Recall that "odd" means "not even", but we gave a characterization of odd numbers in the notes; you are free to use this characterization.
b. Prove or disprove the assertion that every even integer a is the sum of two consecutive even integers.

   Note: Here 12 and 14 are "consecutive even integers."

Question 4: Let $a, b, c \in \mathbb{Z}$. Show that if a does not divide bc, then a does not divide b.

Hint: Prove the *contrapositive.*

Question 5: Let P and Q denote logical propositions. Prove that P is logically equivalent to $(P \wedge Q) \vee (P \wedge \neg Q)$.

Question 6: Let $a, b, c \in \mathbb{Z}$ be *odd* integers. Prove that the equation

$$ax^2 + bx + c = 0$$

has no rational solution, i.e. there is no rational number p/q (with $p, q \in \mathbb{Z}$) which can be substituted for x to yield a true equation.

Hint: Follow the ideas in the proof of the fact that $\sqrt{2}$ is not rational.