Exam 1 Solutions

Problem 1. (6 points)

Let A and B be sets and let $f : A \to B$ be a function. State what it means to say that f is an *invertible function.*

Solution for 1:
The function f is invertible if there is a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

Problem 2. (6 points)

Consider the regions

$$A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\} \text{ and } B = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$$

Indicate whether each of the following statements is True or False by circling the correct choice. (Here you do not need to write an argument justifying your choice).

(a). **True** / **False** : $\forall t \geq -1, (t, 2) \in A \cap B$

(b). **True** / **False** : $\exists t \in \mathbb{R}, (t, -t) \in A \cup B$

Solution for 2:

(a). False. (Explanation not needed, but if $t = -1$ then $(-1, 2) \notin A \cap B$)).

(b). True. (Explanation not needed, but take $t = 0$. Then $(0, -0) = (0, 0) \in A \cup B$.)

Problem 3. (8 points)

Consider the proposition:

$$(\heartsuit) \quad \forall x \in \mathbb{N}, \exists y \in \mathbb{N} \text{ such that } xy \text{ is even.}$$

(a). Write the negation of ($\heartsuit$) so that the adjective "odd" is used instead of "even".

(b). Decide which is true: the statement ($\heartsuit$) or its negation. Justify your decision.

Solution for 3:

(a). The negation is $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, xy$ is odd.

(b). The statement ($\heartsuit$) is true. Indeed, to prove this let $x \in \mathbb{N}$ be arbitrary, and take $y = 2$. Then $xy = 2x$ is even, as required.

Problem 4. (10 points)

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the function given by

$$F(a, b) = (2a + b, b + 1).$$

Prove that the function F is onto (i.e. surjective).

Solution for 4:
Let $(x, y) \in \mathbb{R} \times \mathbb{R}$ be arbitrary. To prove that F is onto, we must find an element

$$(a, b) \in \mathbb{R} \times \mathbb{R} \text{ for which } F(a, b) = (x, y).$$

Thus, we require that

$$x = 2a + b \text{ and } y = b + 1.$$

We solve this system of equations for a and b.

The second equation gives $b = y - 1$. Substituting this equation into the first equation yields

$$x = 2a + y - 1, \text{so that } a = \frac{x - y + 1}{2}.$$

This shows that

$$F\left(\frac{x - y + 1}{2}, y - 1\right) = (x, y)$$

and completes the proof that F is onto.

Problem 5. (10 points)

For $n \in \mathbb{N}$ define

$$S_n = \sum_{i=0}^{n} \frac{1}{3^i}.$$

Use mathematical induction to prove that

$$S_n = \frac{1}{2} \cdot \left(\frac{3^{n+1} - 1}{3^n}\right).$$

Solution for 5:
Proceeding by induction, we first check the base-case – i.e. the case $n = 0$.

When $n = 0$, $S_0 = \sum_{i=0}^{0} \frac{1}{3^i} = \frac{1}{3^0} = 1$. On the other hand, when $n = 0$,

$$\frac{1}{2} \cdot \frac{3^{n+1} - 1}{3^n} = \frac{1}{2} \cdot \frac{3^{0+1} - 1}{3^0} = \frac{1}{2}(2) = 1.$$

This confirms the assertion in the base case.

For the induction step, let $k \in \mathbb{N}$ and suppose that

$$S_k = \frac{1}{2} \cdot \left(\frac{3^{k+1} - 1}{3^k}\right).$$

We must prove that

$$S_{k+1} = \frac{1}{2} \cdot \left(\frac{3^{k+2} - 1}{3^{k+1}}\right).$$

By definition, we know that

$$S_{k+1} = \sum_{i=0}^{k+1} \frac{1}{3^i} = S_k + \frac{1}{3^{k+1}}.$$

Thus by the induction hypothesis we have

$$S_{k+1} = \frac{1}{2} \cdot \left(\frac{3^{k+1} - 1}{3^k}\right) + \frac{1}{3^{k+1}}.$$

Getting a common denominatory, we find

$$S_{k+1} = \frac{\left(3^{k+2} - 3\right) + 2}{2 \cdot 3^{k+1}} = \frac{1}{2} \cdot \left(\frac{3^{k+2} - 1}{3^{k+1}}\right);$$

this completes the proof of the induction step.

The indicated formula now follows by induction.