

Problem Set week 12

Problem 1: Let $\Gamma = (V, E)$ be an undirected graph. Show that there is an even number of vertices of odd degree. **Hint:** Remember that $\sum_{v \in V} \deg v = 2|E|$.

Problem 2:

- Prove that in any simple graph with $|V| \geq 2$, there are at least two vertices of the same degree.
- Does the result in a. remain valid for graphs which aren't necessarily simple?

Problem 3: Let \mathbb{B} be the graph with two vertices A and B and a unique edge $[A, B]$.

- Let Γ be a bipartite graph with $V = V_1 \sqcup V_2$ and $E \subset \{[v_1, v_2] \mid (v_1, v_2) \in V_1 \times V_2\} \subset \mathcal{P}(V)$.
Show that there is a morphism of graphs $\varphi : \Gamma \rightarrow \mathbb{B}$ such that $\varphi_V(V_1) = \{A\}$ and $\varphi_V(V_2) = \{B\}$.
- Let Γ be any graph and suppose that there is a morphism $\varphi : \Gamma \rightarrow \mathbb{B}$. Prove that Γ is a bipartite graph.

Problem 4: Let $\Gamma = (V, E)$ be an undirected graph.

- Define a relation \sim on V by $a \sim b$ if and only if there is a path in Γ from a to b . Prove that \sim is an equivalence relation.
- For a vertex $a \in V$, let $[a] \subseteq V$ be the equivalence class of a for the equivalence relation from part a. Let $E_a = \{[x, y] \in E \mid x, y \in [a]\}$. Prove that $([a], E_a)$ is a subgraph of Γ .
- For a natural number n , let \mathbb{T}_n be the graph with n vertices $\{v_1, v_2, \dots, v_n\}$ and with edges $\{[v_1, v_1], [v_2, v_2], \dots, [v_n, v_n]\}$. In other words, \mathbb{T}_n has a loop at each vertex and no other edges. Suppose that there is a morphism $\varphi : \Gamma \rightarrow \mathbb{T}_n$.
Prove that if $a, b \in V$ and $\varphi_V(a) \neq \varphi_V(b)$ then $a \not\sim b$.
- Conclude that if there is a morphism $\varphi : \Gamma \rightarrow \mathbb{T}_n$ such that the mapping φ_V on vertices is surjective, then there are at least n equivalence classes in V for the relation \sim .

Problem 5: Let $n \in \mathbb{N}$ and let K_n be the complete (undirected) graph on n vertices.

- a. Let Γ_0 be the subgraph of K_n obtained by removing a single vertex and removing all edges involving that vertex. prove that Γ_0 is isomorphic to K_{n-1} .
- b. Let e_1, e_2 be edges in K_n , and for $i = 1, 2$ let Γ_i be the graph obtained from K_n by deleting the edge e_i . (The vertices of Γ_i are the n vertices of K_n).

Prove that Γ_1 is isomorphic to Γ_2 .

- c. Let $e_1 \neq e_2$ and $f_1 \neq f_2$ be edges in K_n , let Γ_e be the graph obtained from K_n by deleting the edges e_1 and e_2 and let Γ_f be the graph obtained from K_n by deleting the edges f_1 and f_2 . (Again, the vertices of Γ_e and Γ_f are the n vertices of K_n).

Show that in general Γ_e is not isomorphic to Γ_f .

Problem 6 has been deleted.