Bridge to Higher Mathematics

1. Lecture 1

1.1. Logical propositions and quantifiers

When writing about mathematics, we will often use the language of predicate logic or first-order logic.

First of all, a proposition is a statement that can be classified as either true or false.

We can combine proposition to form new ones:

Definition 1.1.1: Let P and Q be propositions:

- the proposition $P \land Q$ (read: "P and Q") is true if both P and Q are true.
- the proposition $P \lor Q$ (read: "P or Q") is true if either P is true or Q is true (or both, of course).
- the proposition $\neg P$ (read: "not P") is true if P is false.
- the proposition $P \Rightarrow Q$ (read: "P implies Q") is equivalent to $Q \lor \neg P$.

For example:

- $2 > 0$ is a proposition (and it is true).
- $3 = 0$ is a proposition (and it is false).
- The proposition $(2 > 0) \land (3 = 0)$ is false, while $(2 > 0) \lor (3 = 0)$ is true.

Definition 1.1.2: A logical predicate is family of propositions depending on a variable.

More precisely, if a is a variable, we can consider a proposition Φa for each possible value of a; we say that Φa is a predicate.

For example:

- the statement $a^2 - 1 < 0$ is a predicate Φa depending on the variable a. For real numbers a, the corresponding proposition Φa is true for a in the interval $(-1, 1)$ and false otherwise.

Definition 1.1.3: If a is a *variable* and Φ is a logical *predicate* which depends on a, we write

- $\forall a, \Phi a$ for the proposition that Φa holds for all possible value of the variable a.
- $\exists a, \Phi a$ for the proposition that there exists *some* value of the variable a for which the proposition Φa is true.
- $\exists! a, \Phi a$ for the proposition that there exists a *unique* value of the variable a for which the proposition Φa is true.

Here are some examples:

- The proposition $\forall a \in \mathbb{R}, a^2 \geq 0$ means that for all real numbers a, the condition $a^2 \geq 0$ holds. This proposition is true!

- The proposition $\exists a \in \mathbb{R}, a^2 - 4 = 0$ means that there exists real numbers a for which the condition $a^2 - 4 = 0$. In fact, $a = 2$ and $a = -2$ both work, so the proposition is true.
- The proposition $\exists ! a \in \mathbb{R}, a^3 = -8$ means that there exists a unique real number a satisfying the indicated equation. In fact $a = -2$ is the only solution, so the proposition is true.

1.2. Sets

We now introduce the language of set theory which is the basic notation used for reading and writing Mathematics.

We need to start somewhere, so although this is not a proper definition, we call a collection of objects a set.

For example:

- the collection of all students in our class is a set
- the collection of all negative real numbers is a set

  the collection of all pairs (n, m) of natural numbers is a set

1.2.1. Notations for standard mathematical sets

There are some sets of numbers that are used over and over in Math and we reserve some particular letters to design them.

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers.
- $\mathbb{Z} = \{\ldots - 2, -1, 0, 1, 2, \ldots\}$ is the set of integers ("*zahlen*" in German).
- $\mathbb{Q}$, the set of rational numbers is

  $$\mathbb{Q} = \{n/m \mid \text{for integers } n, m \text{ with } m \neq 0\},$$

  where fractions satisfy the condition $n/m = a/b$ if and only if $nb = ma$.

  ($\mathbb{Q}$ as in "q" for "quotient").
- $\mathbb{R}$, the set of real numbers that you know from calculus.
- $\mathbb{C}$, the set of complex numbers, where a complex number z has the form $z = a + bi$ for real numbers a and b.

1.2.2. Basic properties of sets.

We indicate that an object a is a member of a set A using the symbol $a \in A$. For example $2 \in \mathbb{N}$ and $-2 \in \mathbb{Z}$, while $1/2 \in \mathbb{Q}$ but $1/2 \notin \mathbb{Z}$.

By convention, we will normally denote sets with capital letters, and elements with lower-case letters. Thus $a, b \in A$.

For sets with a finite number of elements, we can specify the set by just listing the elements. For example, if A denotes the set consisting of the first three letters of the English alphabet, then we can simply write $A = \{a, b, c\}$.

Sometimes we specify a set using set builder notation. This means that we indicate some predicate required for membership in the set. Thus

$$B = \{a \in A \mid \Phi a\}$$

is the set of all elements a in A for which the predicate Φ is true.

This should be read: "B is the set of all a in A *such that* Φa holds."

For example, the set E of even integers can be expressed in set builder notation using the predicate "is even"; thus we can write

$$E = \{x \in \mathbb{Z} \mid x \text{ is even}\}$$

read this as: "E is the set of all x in $\mathbb{Z}$ *such that* x is even".

Definition 1.2.2.1: If A and B are sets, then $A = B$ provided that $x \in A \Leftrightarrow x \in B$

Definition 1.2.2.2: When every element of a set A is also an element of another set B, we say that A is a subset of B and write $A \subseteq B$.

A more precise statement is as follows: if $\forall a \in A$ we have $a \in B$, then $A \subseteq B$.

Note that $A \subseteq B$ allows the possibility that A and B are equal. The condition $A \subset B$ – or more precisely $A \subsetneq B$ – is defined by $A \subseteq B$ and $\exists b \in B, b \notin A$.

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Example 1.2.2.1:

(a). The number 2 is a natural number. We say that 2 is an element of – or is a member of - the set of natural numbers, and we indicate this by writing $2 \in \mathbb{N}$.

(b). The notation $\{2\}$ means a set whose only element is the number 2. It is not correct to write that $\{2\} \in \mathbb{N}$. Instead, we write $\{2\} \subseteq \mathbb{N}$.

(c). There are natural inclusions

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

In fact, all these sets are different, so we could also write

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$$

(d). The empty set is the set that has no elements. It is denoted with the symbol $\emptyset$. So, $\emptyset = \{\}$.

(e). Any set A has the empty set as a subset: $\emptyset \subseteq A$.

(f). And any set A has itself as a subset: $A \subseteq A$.

1.2.3. Equality of sets

Two sets A and B are equal provided they have precisely the same elements. Thus we have

$$A = B \Leftrightarrow (A \subseteq B) \text{ and } (B \subseteq A)$$

Example 1.2.3.1:

(a). Define

$$A = \{x \in \mathbb{R} \mid x^3 - 3x^2 + 2x = 0\}, \text{and } B = \{0, 1, 2\}.$$

We argue that $A = B$.

First we show that $B \subseteq A$. As B is given as a finite collection of real numbers, we can just check that all elements in B satisfy the condition that is required for a real number to be in A.

That is, we need to see that each $0, 1, 2$ satisfy the condition for membership in A. Thus we must confirm that

$$0^3 - 3 \times 0^2 + 2 \times 0 = 0, \qquad 1^3 - 3 \times 1^2 + 2 \times 1 = 0, \qquad 2^3 - 3 \times 2^2 + 2 \times 2 = 0.$$

These equalities are all satisfied, so indeed $B \subseteq A$.

We now need to prove the converse inclusion, i.e. that $A \subseteq B$. Essentially, this means finding the solutions to the equation $x^3 - 3x^2 + 2x = 0$ and proving that these solutions are among $0, 1, 2$.

Factoring the polynomial, we find

$$x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x - 1)(x - 2)$$

For a product to be 0, one of the factors needs to be zero. This leads us to $x = 0, x = 1$, or $x = 2$ as required.

(b). Define

$$C = \{x \in \mathbb{R} \mid x^3 - 3x^2 + 2x > 0\}, \quad D = \{x \in \mathbb{R} \mid 0 < x < 1 \text{ or } 2 < x < \infty\}$$

Let us show that $C = D$.

We saw already seen the factorization $x^3 - 3x^2 + 2x = x(x - 1)(x - 2)$; thus

$$C = \{x \in \mathbb{R} \mid x(x - 1)(x - 2) > 0\}.$$

A product of three numbers is positive if all of them are positive or two of them are negative and one positive. Now,

- $x - 1 > 0$ is equivalent to $x > 1$,
- $x - 2 > 0$ is equivalent to $x > 2$, and of course
- $x > 0$ is equivalent to $x > 0$.

So a real number x is a member of the set C if all three of these three inqualities hold, or if exactly one of these inequalities hold. So, all three factors are positive for $x > 2$ while two of the factors are negative and the third positive for $0 < x < 1$. Using interval notation as in Calculus, the two sets are $(2, \infty)$ and $(0, 1)$ respectively. The set C is then composed of these two pieces (we will introduce notation for this soon) and this is precisely the way D was defined.

1.2.4. Operations on sets

In this section, we introduce some basic operations. Let A and B be sets.

Definition 1.2.4.1: The union $A \cup B$ is the set whose elements are in either A or B (or both). In symbols,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Definition 1.2.4.2: The intersection $A \cap B$ is the set whose elements are in both A and B. In symbols,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Definition 1.2.4.3: A and B are said to be disjoint if their intersection is the empty set – i.e if $A \cap B = \emptyset$.

If A and B are disjoint, the union $A \cup B$ is sometimes called a disjoint union, since an element $x \in A \cup B$ satisfies either $x \in A$ or $x \in B$ but not both.

Definition 1.2.4.4: The difference $A - B$ of the sets A and B is the set of elements that are in the first set and not in the second set:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Definition 1.2.4.5: If the set A is a subset of a set U, the complement of A (in U) is defined to be the set

$$\bar{A} = \{x \in U \mid x \notin A\} = U - A.$$

Remark: Unions and intersections can be taken for several (more than two) sets, even for infinite collections.

Set operations are sometimes represented and visualized using *Venn Diagrams*; each set is represented as a shape, and the results of the set operations are represented by certain regions, as in Figure 1.

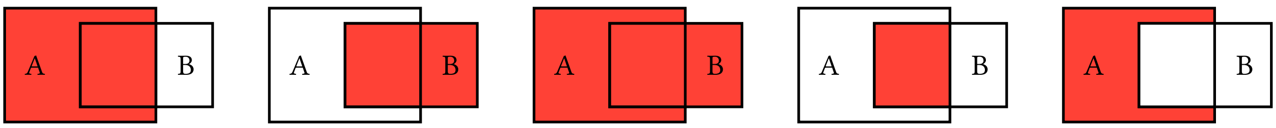

Figure 1: From left to right: A, B, $A \cup B$, $A \cap B$, $A - B$

Example 1.2.4.1:

(a). In Example 1.2.3.1, we showed that $C = D$. By its definition, the set D is equal to the union $(0, 1) \cup (2, \infty)$. Thus, this set is the union of two intervals in the real line.

(b). The intersection of $(0, 1)$ and $(2, \infty)$ is empty; i.e. $(0, 1) \cap (2, \infty) = \emptyset$. So , the expression $C = (0, 1) \cup (2, \infty)$ shows that C is the disjoint union of two open intervals in the real line.

(c). For every $n \in \mathbb{N} - \{0\}$ define the semiopen interval of the real line A_n by $A_n = \left(-\frac{1}{n}, n\right]$. The parentheses (on the left indicates that $-\frac{1}{n}$ is not in the set while the bracket] means that n is. Then

$$\bigcap_{n \in \mathbb{N} - \{0\}} A_n = [0, 1]$$

First of all $[0, 1]$ is contained in each A_n and therefore in its intersection. Also any real number greater than 1 is not contained in A_1 and therefore not contained in the intersection of all A_n.

Now, the sequence $\left\{-\frac{1}{k}\right\}$ has limit zero. Thus for any strictly negative number x, we can find some k such that x is smaller than $-\frac{1}{k}$ and therefore $x \notin A_k$ and *a fortiori* x is not in the intersection of all of the A_n.

Similarly, we can compute the union:

$$\bigcup_{n \in \mathbb{N} - \{0\}} A_n = (-1, \infty).$$

No number smaller than or equal to −1 is in any A_k, therefore, it cannot be in its union. The numbers between -1 and 0 are in A_1 and therefore in the union.

Any positive number is smaller than some natural number m and therefore it is in A_m.

Definition 1.2.4.6: The cartesian product $A \times B$ of the sets A and B is the set whose elements are ordered pairs of elements with the first one in A the second one in B:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

Example 1.2.4.2:

(a). You are already familiar with at least one cartesian product. The set of real numbers is represented geometrically as a real line. The cartesian product of two real lines is the set of pairs of real numbers. This is a representation of the points in the plane with each point determined by its two coordinates. That is, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the set of points in the plane.

(b). If $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_m\}$ for natural numbers $n, m \in \mathbb{N}$, then $A \times B = \{(a_s, b_t) : 1 \leq s \leq n, 1 \leq t \leq m\}$.

(c). Let $A = \{x \in \mathbb{R} \mid x^3 - 3x^2 + 2x = 0\}, B = \{x \in \mathbb{R} \mid x^2 - 4 = 0\}$. We have seen in Example 1.2.3.1 that A may written $A = \{0, 1, 2\}$. Similarly, $B = \{2, -2\}$. Therefore,

$$A \times B = \{(0, 2), (1, 2), (2, 2), (0, -2), (1, -2), (2, -2)\}.$$

Definition 1.2.4.7: The power set $\mathcal{P}(A)$ of a set A is the set consisting of all subsets of A:

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}$$

Example 1.2.4.3:

(a). Take $A = \emptyset$. Then, $\mathcal{P}(\emptyset) = \{\emptyset\}$. This is a set whose only element is the empty set. In particular,

$$\mathcal{P}(\emptyset) \neq \emptyset$$

, as it contains one element.

(b). If $A = \{a\}$ is a singleton set – i.e. a set with exactly one element – then $\mathcal{P}(A) = \{\emptyset, \{a\}\}$ contains exactly 2 elements.

(c). If $A = \{a, b\}$ is a set with exactly two elements, then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ contains exactly $4 = 2^2$ elements.

(d). We see from the previous examples that every time that we add a new element to a set, we double the number of elements in $\mathcal{A}$. We should expect that if A has exactly n elements, then $\mathcal{P}(A)$ has exactly 2^n elements.

We can see this as follows: to construct a subset of A, we must decide for each element of A whether or not the element belongs to the subset. This gives two options for each element. These options can be combined in any arbitrary way, so there are in total 2^n possibilities.