Problem Set 8

Question 1: For integers $a, b \in \mathbb{Z}$, let us say that a is near b if the numbers are no more than 2 units apart. Thus 4 is near 6 and 4 is near 2, but 4 is not near 7.

Since the distance between a and b can be computed using the absolute value function, the relation is given by

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid |x - y| \leq 2\}.$$

Write $a \sim b$ to express that a is near to b.

a. is $\sim$ reflexive?
b. is $\sim$ symmetric?
c. is $\sim$ antisymmetric?
d. is $\sim$ transitive?

Question 2: How many distinct equivalence relations are there on the set $A = \{1, 2, 3\}$?

Hint: How many partitions are there of the set $A = \{1, 2, 3\}$?

Question 3: For $n \in \mathbb{N}$, recall that $\mathbb{Z}_n$ denotes the set of congruence classes modulo n:

$$\mathbb{Z}_n = \{[x]_n \mid x \in \mathbb{Z}\} = \{[0]_n, [1]_n, ..., [n-1]_n\}.$$

For each of the following, decide whether the indicated rule determines a well-defined function. Explain your conclusion.

a. $f_1 : \mathbb{Z} \to \mathbb{Z}_5$ given by $f_1(z) = [2 \cdot z]_5$.
b. $f_2 : \mathbb{Z}_{10} \to \mathbb{Z}_5$ given by $f_2([z]_{10}) = [2 \cdot z]_5$
c. $f_3 : \mathbb{Z}_5 \to \mathbb{Z}_{10}$ given by $f_3([z]_5) = [2 \cdot z]_{10}$.

Question 4:

a. Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y + [2]_{10} = [8]_{10}$ or show that there aren't any.
b. Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y \cdot [2]_{10} = [8]_{10}$ or show that there aren't any.
c. Find all solutions $y \in \mathbb{Z}_{10}$ of the equation $y \cdot [2]_{10} = [3]_{10}$ or show that there aren't any.

Question 5: Let $A = (\mathbb{N} - \{0\}) \times (\mathbb{N} - \{0\})$ be the set of pairs of strictly positive natural numbers. Define a relation on A by

$$\forall (a, b), (c, d) \in A, (a, b) \sim (c, d) \text{ if and only if } a + d = b + c.$$

a. Show that $\sim$ is an equivalence relation.

b. Denote by B the set of equivalence classes of elements of A by the relation $\sim$ defined in part (a).

   To cut down on notational clutter, let's agree to write

   $$[a, b] \text{ for the equivalence class } [(a, b)] \text{ of the element } (a, b) \in A.$$

   Define an "addition" on B by the rule

   $$[a, b] \oplus [c, d] = [a + c, b + d].$$

   Show that this rule determines a well-defined addition of the cosets.

   For this, you must show: if

   $$[a, b] = [a', b'] \text{ and } [c, d] = [c', d']$$

   then $[a, b] \oplus [c, d] = [a', b'] \oplus [c', d']$.

c. Show that the assignment

   $$f : B \to \mathbb{Z} \text{ given by } f([a, b]) = a - b$$

   is a well-defined function.

d. List three distinct representatives of the equivalence class $[a, b] \in B$ for which $f([a, b]) = 0$.

e. Show that the function f is a bijection from B to $\mathbb{Z}$.

f. Show that the addition of equivalence classes defined in part (b) corresponds under the function f to the addition of integers.

   In other words, show that

   $$f([a, b]) + f([c, d]) = f([a, b] \oplus [c, d]).$$

g. Could you define a product $\otimes$ on B corresponding by f to the product $\times$ on $\mathbb{Z}$?