Problem Set 9

Question 1:

a. Let b_n be the sequence given by $b_0 = 0, b_1 = .1, b_2 = .01, b_3 = .001$, and in general b_n has a 1 in decimal spot n and $0'$s elsewhere.

   Otherwise said: for $n \geq 1$ $b_n = \frac{1}{10^n}$. Show that
$$\lim_{n\to\infty} b_n = 0.$$

b. Let d_n be the sequence given by $d_0 = 0, d_1 = .9, b_2 = .99, d_3 = .999, \ldots$ and in general d_n has a 9 in each decimal spot up to and including spot n and $0'$s after that.

   Otherwise said: for $n \geq 1$, $d_n = \frac{10^n - 1}{10^n}$. Show that
$$\lim_{n\to\infty} d_n = 1.$$

Question 2: Observe that
$$10^6 = 142857 \times 7 + 1.$$
Explain why this implies that the decimal expansion of $1/7$ is given by
$$\frac{1}{7} = 0.142857\overline{142857}\ldots$$

Question 3: Using the definition of limit, show that if $a_n = \frac{6n+1}{3n-1}$, then
$$\lim_{n\to\infty} a_n = 2.$$

Question 4:

a. By negating the definition, express the condition "a_n is not a Cauchy sequence" so that the last inequality in your sentence has a symbol "greater than" ($\geq$).

b. Show that $a_n = \frac{(-1)^n(6n+1)}{3n-1}$ is not a Cauchy sequence.

Question 5: Assume that a_n is a Cauchy sequence of real numbers. Prove that the sequence $b_n = a_{2n}$ is a Cauchy sequence.

Question 6: Let $d \in \mathbb{N}_{>0}$. Show that the sequence $a_n = 1/n^d$ is a Cauchy sequence.

Question 7: Let a_n and a'_n be sequences.

a. If $a_n + a'_n$ is a Cauchy sequence, is it true that a_n is Cauchy and a'_n is Cauchy? Give a proof or a counter-example.
b. If $a_n \cdot a'_n$ is a Cauchy sequence, is it true that a_n is Cauchy and a'_n is Cauchy? Give a proof or a counter-example.

Question 8: Assume that a_n is a sequence for which

$$\lim_{n \to \infty} a_n = 0.$$

If b_n is a sequence such that $\exists m \in \mathbb{N}$ for which

$$n \geq m \Rightarrow -a_n \leq b_n \leq a_n,$$

prove that

$$\lim_{n \to \infty} b_n = 0.$$