Problem Set 2 Solutions

Tufts University
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Question 1: Write the following sentence using the quantifier notation (that is, the symbols $\exists$, $\forall$). Notice: We do not claim these statements are true, so please do not try to prove them.

(a) Every natural number greater than one is prime.
$\forall n \in \mathbb{N},\ n > 1$, n is prime.

(b) There is a natural number greater than one that is neither prime nor composite.
$\exists n \in \mathbb{N},\ n > 1$ such that (n not prime) $\wedge$ (n not composite).

(c) For every integer x there exists an integer y such that $xy = 1$.
$\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z}$ such that $xy = 1$.

(d) There is an integer x such that for every integer y, $xy = 1$.
$\exists x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z},\ xy = 1$.

(e) For every integer x and every integer y, $x + y = y + x$.
$\forall x, y \in \mathbb{Z},\ x + y = y + x$.

(f) There is an integer x and an integer y such that x/y is an integer.
$\exists x, y \in \mathbb{Z}$, such that $\frac{x}{y} \in \mathbb{Z}$.

Question 2: Write the negation of each of the statements in Question 1, in such a way that the negation appears as late in the sentence as possible. For each statement, decide whether the negation is true, and confirm your assessment with an argument.

(a) $\exists n \in \mathbb{N}$, $n > 1$ such that n is not prime
True: $4 \in \mathbb{N}$ and $4 > 1$, but $4 = 2 * 2$, which means 4 is not prime.

(b) $\forall n \in \mathbb{N}$ with $n > 1$, n is either prime or composite.
True: If a natural number greater than 1 is not prime, it is composite by definition.

(c) $\exists x \in \mathbb{Z}$ such that $\forall y \in \mathbb{Z}$, $xy \neq 1$.
True: Suppose this statement were false. Then for all $x \in \mathbb{Z}$, there would exist some $y \in \mathbb{Z}$ such that $xy = 1$. However, $x \cdot y = 1$ implies $x = \frac{1}{y}$. If $y \neq 1$, then $\frac{1}{y} = x \notin \mathbb{Z}$. if $y = 1$, then for any $x \neq 1$, we know $xy \neq 1$. Thus, we have a contradiction, and so the statement must be true.

(d) $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $xy \neq 1$.
True: let $y = 5$. Then $xy = 1$ implies $x = \frac{1}{5} \notin \mathbb{Z}$. Since $x \in \mathbb{Z}$, $xy \neq 1$, and so the staement is true.

(e) $\exists x, y \in \mathbb{Z}$ such that $x + y \neq y + x$.
False: We know that the commutative property holds for integers under addition!

(f) $\forall x, y \in \mathbb{Z}$, $\frac{x}{y} \notin \mathbb{Z}$.
False: Let $x = 4$ and $y = 2$. Then $\frac{x}{y} = 2 \in \mathbb{Z}$.

Question 3: (a) **Prove that an integer is odd if and only if it is equal to the sum of two consecutive integers.**

($\Rightarrow$) Let $n \in \mathbb{Z}$ be odd. Then

$$n = 2a + 1 = a + (a + 1)$$

for some $a \in \mathbb{Z}$. Since a and $(a + 1)$ are two consecutive integers, this direction is shown.

($\Leftarrow$)

Suppose $n \in \mathbb{Z}$ is the sum of two consecutive integers. Then

$$n = a + (a + 1) = 2a + 1$$

for $a \in \mathbb{Z}$. This is the definition of an odd number, and so n is odd.

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(b) **Prove or disprove the assertion that every even integer a is the sum of two consecutive even integers.**

The statement is false. Suppose we could write 12 as the sum of two consecutive even integers. Then there must exist $a \in \mathbb{Z}$ such that

$$12 = 2a + (2a + 2) = 4a + 2 = 2(2a + 1).$$

Then this implies $2a + 1 = 6$, which implies $a = \frac{5}{2} \notin \mathbb{Z}$, a contradiction. Thus, 12 cannot be expressed as a sum of two consecutive even integers, and so the statement is disproved.

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Question 4: Let $a, b, c \in \mathbb{Z}$. Show that if a does not divide bc, then a does not divide b. (Hint: Prove the contrapositive.)

Contrapositive: Let $a, b, c \in \mathbb{Z}$. If $a|b$, then $a|bc$.

Suppose $a|b$. Then there exists $m \in \mathbb{Z}$ such that $a \cdot m = b$. Thus, $(a \cdot m)c = bc$, and so $a \cdot (mc) = bc$, which implies $a|bc$. Thus, the contrapositive is proved.

Therefore, if $a \nmid bc$, then $a \nmid b$

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Question 5: Let P and Q denote logical propositions. Prove that P is logically equivalent to $(P \wedge Q) \vee (P \wedge \neg Q)$.

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$(P \wedge Q) \vee (P \wedge \neg Q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

Thus, $P = (P \wedge Q) \vee (P \wedge \neg Q)$

Question 6: Let $a, b, c \in \mathbb{Z}$ be odd integers. Prove that the equation $ax^2 + bx + c = 0$ has no rational solution, i.e. there is no rational number p/q (with $p, q \in \mathbb{Z}$) which can be substituted for x to yield a true equation. (Hint: Follow the ideas in the proof that $\sqrt{2}$ is not rational.)

Proof by contradiction: Suppose there exists some rational number $\frac{p}{q} \in \mathbb{Q}$ that solves the equation $f(x) = ax^2 + bx + c = 0$. (Suppose $f(\frac{p}{q}) = 0$).

Let $\frac{p}{q}$ be in reduced form, i.e., $p \nmid q$.

Then

$$a\left(\frac{p}{q}\right)^2 + b\frac{p}{q} + c = 0,$$

which implies

$$ap^2 + bpq + q^2 = 0.$$

Now consider the parity of p and q. This is where we use the fact that a, b, c are all odd numbers:

Case 1: p and q are odd. Then ap^2, bpq, and q^2 are all odd. Three odd numbers cannot sum to an even number (0 is even), so this case must be false.

Case 2: p is odd and q is even. Then ap^2 is odd, while bpq and q^2 are even. An odd number added to an even number cannot be even, and so this case must also be false. The same argument shows that we cannot have p even and q odd either.

Case 3: p and q are both even. This case gives us three even numbers summing to 0, which works just fine.

Thus, p and q must both be even! However, this implies that $\frac{p}{q}$ is not in reduced form, a contradiction.

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