

## Assignment 2

**Question 1: Deleted** (see comments on the course web page.)

Let  $G$  be a group, let  $S_1, S_2 \subseteq G$  be subsets and let  $H_i = \langle S_i \rangle$  for  $i = 1, 2$ . Suppose that  $\forall x \in S_1$  and  $\forall y \in S_2$  we have  $xyx^{-1} \in H_2$ . Prove that  $H_1$  normalizes  $H_2$ ; i.e. prove that  $\forall x \in H_1, \forall y \in H_2, xyx^{-1} \in H_2$

**Question 2:** Let  $n \in \mathbb{N}$ ,  $n > 0$  and consider the group  $S = S(\mathbb{Z}/n\mathbb{Z})$  of permutations of the set  $\mathbb{Z}/n\mathbb{Z}$ .

- For  $x \in \mathbb{Z}/n\mathbb{Z}$ , recall that the additive order  $o(x)$  is a divisor of  $n$ .

Describe the *cycle structure* of the element  $\sigma \in S$  defined by the rule  $\sigma(z) = z + x$ . Show that the order of  $\sigma$  is  $o(x)$ .

- Suppose that  $n = p$  is a prime number, and let  $k \in \mathbb{Z}$  with  $\gcd(k, p) = 1$ . Thus the class  $\bar{k}$  of  $k$  in  $\mathbb{Z}/p\mathbb{Z}$  lies in the group  $(\mathbb{Z}/p\mathbb{Z})^\times$  of *units*. The multiplicative order  $o(\bar{k})$  of  $\bar{k}$  is a divisor of  $p - 1$ .

Describe the *cycle structure* of the element  $\tau \in S$  defined by the rule  $\tau(z) = \bar{k} \cdot z$ . Show that the order of  $\tau$  is  $o(\bar{k})$ .

**Question 3:** Let  $G$  be the group of invertible  $2 \times 2$  matrices with entries in  $F = \mathbb{Z}/p\mathbb{Z}$  for a prime number  $p$ ; the group operation is given by matrix multiplication.

- Show that  $|G| = (p^2 - 1)(p^2 - p)$ .
- Show that  $T = \left\{ \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix} \mid t, s \in F^\times \right\}$  is a subgroup of  $G$ . Here  $F^\times$  denotes the multiplicative group of invertible elements of  $F = \mathbb{Z}/p\mathbb{Z}$ . Also show that

$$T \text{ is isomorphic to } F^\times \times F^\times$$

- Show that  $U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in F \right\}$  is a subgroup of  $G$  isomorphic to the additive group  $F$ .
- Show that  $T$  normalizes  $U$ . Find the order of the group  $B = TU$ .

- A line in  $F^2$  is by definition a linear subspace of dimension 1. For any non-zero vector  $v$ , the set  $Fv = \text{Span}(v)$  is a line. Note that  $G$  acts in a natural way on the set of lines in  $F^2$ .

If we write  $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  for the standard basis of  $F^2$ , show that  $B$  is the stabilizer of the line  $Fe$ .

- Show that  $G$  acts transitively on the set of lines in  $F^2$ .
- Conclude that the set of lines in  $F^2$  is in bijection with the set  $G/B$ . How many lines are there in  $F^2$ ?

**Question 4:** Let  $G$  be a group and let  $\Omega$  be a  $G$ -set. If  $x, y \in \Omega$  and  $x = gy$  for some  $g \in G$ , prove that the stabilizers  $G_x = \text{Stab}_G(x)$  and  $G_y = \text{Stab}_G(y)$  are *conjugate*. More precisely, show that

$$G_x = g \cdot G_y \cdot g^{-1}$$

**Question 5:** Let  $G$  be a group.  $G$  acts on itself by conjugation: for  $g, x \in G$ , the action of  $g$  on  $x$  is given by  $\text{Inn}_g x = gxg^{-1}$ .

- Prove that the assignment  $g \mapsto \text{Inn}_g$  determines a group homomorphism  $G \rightarrow \text{Aut}(G)$  where  $\text{Aut}(G)$  is the group of automorphisms of  $G$ .
- Let  $Z = \{g \in G \mid \forall x \in G, gx = xg\}$  be the *center* of  $G$ . Prove that  $Z = \ker \text{Inn}$ .

For the action of  $G$  on itself by conjugation, the stabilizer  $\text{Stab}_G(x)$  of  $x \in G$  is usually written  $C_G(x)$  and is called the *centralizer* of  $x$  in  $G$ . Note that

$$C_G(x) = \{y \in G \mid yxy^{-1} = x\} = \{y \in G \mid yx = xy\}$$

**Question 6:** Let  $I = I_n$  be a finite set with  $n$  elements, and let  $S = S_n = S(I_n)$  be the group of permutations of  $I$ . Recall that  $|S| = n!$ .

- Prove that there are  $(n - 1)! n$ -cycles in  $S$ . **Hint:** If the elements of  $I$  are written  $I = \{a_1, a_2, \dots, a_n\}$ , then the  $n$ -cycles  $(a_1, a_2, \dots, a_n)$  and  $(a_2, a_3, \dots, a_n, a_1)$  are *equal*.
- Prove that if  $\sigma$  is an  $n$ -cycle in  $S$ , then  $C_S(\sigma) = \langle \sigma \rangle$