Problem Set week 11

Problem 1: Let A be a ring and let $(\heartsuit)\quad 0 \to X \xrightarrow{j} Y \xrightarrow{\psi} Z \to 0$ be a short exact sequence of A-modules.

Prove that the following are equivalent:

a. There is an isomorphism $\varphi : X \oplus Z \xrightarrow{\sim} Y$ such that $\psi \circ \varphi = \pi_2$ and $\varphi^{-1} \circ j = \iota_1$ where

$$\pi_2 : X \oplus Z \to Z \text{ is } (x, z) \mapsto z, \text{and } \iota_1 : X \to X \oplus Z \text{ is } x \mapsto (x, 0).$$

b. There is a section to ψ; i.e., there is an A-module homomorphism $\sigma : Z \to Y$ such that

$$\psi \circ \sigma = \mathrm{id}_Z .$$

c. There is a retract of j; i.e. there is an A-module homomorphism $\rho : Y \to X$ such that

$$\rho \circ j = \mathrm{id}_X .$$

The short exact sequence $(\heartsuit)$ is said to be *split exact* if it satisfies these equivalent conditions.

Problem 2: Let

$$0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0$$

be a short exact sequence of A-modules.

Prove that for an A-module W, the sequence

$$(\clubsuit)\quad 0 \to \mathrm{Hom}_A(W, X) \xrightarrow{f \mapsto \iota \circ f} \mathrm{Hom}_A(W, Y) \xrightarrow{g \mapsto \pi \circ g} \mathrm{Hom}_A(W, Z)$$

is *exact*.

NB. A sequence of A-modules and A-module homomorphisms is *exact* if for each interior term

$$\ldots S \xrightarrow{f} T \xrightarrow{g} Y \ldots$$

of the sequence, $\ker g = \mathrm{im}\, f$. In the sequence $(\clubsuit)$, $\mathrm{Hom}_A(P, Y)$ is not an interior term.

Remark: The exactness of $(\clubsuit)$ for each short exact sequence is often expressed by saying that the functor $\mathrm{Hom}_A(W, -)$ is *left-exact*.

So your task is to check exactness of $(\clubsuit)$ at the interior terms $\mathrm{Hom}_A(W, X)$ and $\mathrm{Hom}_A(W, Y)$.

Problem 3: An A-module P is said to projective if for every surjective homomorphism of A -modules $\pi : M \to N$ and every A-module homomorphism $f : P \to N$, there is an A-module homomorphism $\alpha : P \to M$ such that $\pi \circ \alpha f$.

a. Prove that any free A-module is projective.
b. Prove that a module P is projective if and only if P is a direct summand of a free module; i.e. if there is a free A-module F and an isomorphism of A-modules $F \simeq P \oplus Q$ for some A-module Q.

Problem 4: Prove that the following are equivalent:

a. P is projective
b. for every short exact sequence of A-modules $0 \to X \to Y \to Z \to 0$ the sequence of A-modules

$$0 \to \operatorname{Hom}_A(P, X) \to \operatorname{Hom}_A(P, Y) \to \operatorname{Hom}_A(P, Z) \to 0$$

is exact.

Remark: Thus P is projective if and only if the left-exact functor $\operatorname{Hom}_A(P, -)$ is exact (in the sense that $\operatorname{Hom}_A(P, -)$ carries short exact sequences to short exact sequences.).

Problem 5: Let A be a commutative ring, let $B = A \times A$, and let $\pi : B \to A$ be the projection ring homomorphism given by $\pi(x, y) = x$.

Using π, we can view A as a B-module. Prove that A is a projective B-module that is not a free B-module.

Problem 6: Let K be a field and let $a, b \in K \setminus \{0, 1\}$ with $a \neq b$. Let $q = T(T-a)(T-b) \in K[T]$ and consider the polynomial $f = S^2 - q \in K[S, T]$.

You proved in the previous homework set that:

- $A = K[S,T]/\langle f \rangle$ is an integral domain. Write s, t for the images of S, T in the quotient ring A; thus $s^2 = q(t) = t(t-a)(t-b)$ in A.
- The map $K[T] \to k[t]$ for which $T \mapsto t$ is an isomorphism.
- A is a free $K[t]$-module on a basis $\{1, s\}$.

a. Define $N : A \to K[t]$ by the rule $N(\alpha) = N(f + gs) = f^2 - g^2 \cdot q(t)$ for $\alpha = f + gs$ with $f, g \in K[t]$.

   Then N is multiplicitive (i.e. it is a monoid homomorphism) and $\alpha \in A$ is a unit in A if and only if $N(\alpha) \in K[T]^\times = K^\times$.

b. Write $\mathfrak{m} = \langle s, t \rangle$ for the ideal of A generated by s and t. Show that the ideal $\mathfrak{m}$ is not principal.

   Hint: Suppose to the contrary that $\alpha = f + g \cdot s \in A$ is a generator for $\mathfrak{m}$, for $f, g \in k[t]$. Then $\alpha \mid t => N(\alpha) \mid t^2$. Now use a degree argument to show that $g = 0$.

c. Show that $\mathfrak{m}$ is a maximal ideal of A.

Remark 1: Essentially the same arguments show that the ideals

$$\langle s, t - a \rangle \text{ and } \langle s, t - b \rangle$$

are maximal and not principal.

Remark 2: The field of fractions $F = \mathrm{Frac}(A) = K(s, t)$ of A is the *field of rational functions* on the *elliptic curve* over K defined by the cubic equation $S^2 = T(T-a)(T-b)$.

If we had consider instead the quadratic equation $S^2 = T(T-a)$, the analogous ideal $\mathfrak{m} = \langle s, t \rangle$ is princpal (!).

Problem 7: Keep the notations of the previous problem. We are going to show that $\mathfrak{m}$ is a projective A-module.

a. First, explain why the condition that $\mathfrak{m}$ is not a principal ideal implies that $\mathfrak{m}$ is not a free A-module.
b. Explain why $1 = u \cdot t + v \cdot (t-a)(t-b)$ for some $u, v \in k[t]$.
c. Show for every $z \in \mathfrak{m}$ that $\dfrac{(t-a)(t-b)z}{s} \in A$; note that *a priori* $\dfrac{(t-a)(t-b)z}{s}$ is an element of the field of fractions of A.
d. Consider the surjective A-module homomorphism
$$\pi : A^2 \to \mathfrak{m} \text{ given by } \pi\begin{pmatrix} x \\ y \end{pmatrix} = xs + yt.$$
Show that $\sigma : \mathfrak{m} \to A^2$ given by the rule
$$\sigma(z) = \begin{pmatrix} v(t-a)(t-b)z/s \\ uz \end{pmatrix}$$
is a *section* to π (where u, v are as in (b) above).
e. Conclude using problem (1) that $\mathfrak{m}$ is isomorphic to a direct summand of the free module A^2.