Problem Set 6

Let R be a commutative ring (with identity). We write 0 for the trivial R-module $\{0\}$.

Question 1: Consider a diagram $\mathcal{A}$ of the form:

$$\ldots \to A_{i-1} \overset{d_{i-1}}{\to} A_i \overset{d_i}{\to} A_{i+1} \overset{d_{i+1}}{\to} A_{i+2} \to \ldots$$

where for $i \in \mathbb{Z}$, A_i is an R-module and $d_i : A_i \to A_{i+1}$ an R-module homomorphism. Then $\mathcal{A}$ is said to be a *complex* provided that $d^2 = 0$; i.e. that for each $i \in \mathbb{Z}$ we have $d_i \circ d_{i-1} = 0$. This implies that $\operatorname{im} d_{i-1} \subseteq \ker d_i$.

And the complex $\mathcal{A}$ is said to be *exact* provided that $\forall i \in \mathbb{Z}, \operatorname{im} d_{i-1} = \ker d_i$.

Let $\mathcal{A}$ be a complex:

a. For $i \in \mathbb{Z}$ write $H^i(\mathcal{A})$ for the R-module $\ker d_i / \operatorname{im} d_{i-1}$. Show that $\mathcal{A}$ is exact if and only if $H^i(\mathcal{A}) = 0$ for each $i \in \mathbb{Z}$.

b. For R-modules X, Y, Z, we view a diagram $(\clubsuit)$ $\ 0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0$ as a complex provided that $g \circ f = 0$ by taking

$$A_i = 0 \quad (i \leq 0), A_1 = X, A_2 = Y, A_3 = Z, \text{and } A_j = 0 (4 \leq j)$$

as well as $d_1 = f$, $d_2 = g$ and $d_j = 0$ for $j \neq 1, 2$.

We say that the complex $(\clubsuit)$ is a *short exact sequence* provided that $(\clubsuit)$ is an exact complex.

Prove that $(\clubsuit)$ is a short exact sequence if and only if (i) f is injective, (ii) $\ker(g) = \operatorname{im}(f)$, and (iii) g is surjective.

c. Let $\varphi : M \to N$ be an R-module homomorphism. Show that

$$0 \to \ker \varphi \overset{\iota}{\to} M \overset{\overline{\pi}}{\to} \operatorname{im} \varphi \to 0$$

is a short exact sequence, where $\iota : \ker \varphi \to M$ and $\pi : M \to \operatorname{im} \varphi$ are the inclusion mapping and the quotient mapping, respectively.

Question 2: Let M, N be R-modules. Show that there is a short exact sequence

$$0 \to M \overset{\iota_M}{\to} M \oplus N \overset{\pi_N}{\to} N \to 0$$

where $\iota_M : M \to M \oplus N$ and $\iota_N : N \to M \oplus M$ are the mappings for the direct sum (coproduct) $M \oplus N$, and $\pi_M : M \oplus N \simeq M \times N \to M$ and $\pi_N : M \oplus N \simeq M \times N \to N$ are the mappings for the product $M \times N$.

Question 3: For ideals $I, J \subseteq R$, the product of I and J is the ideal generated by

$$\{xy \mid x \in I, y \in J\}.$$

a. Prove that $IJ \subseteq I \cap J$.
b. If $P \subset R$ is a *prime ideal* and if $IJ \subseteq P$, prove that either $I \subseteq P$ or $J \subseteq P$.

Question 4: An element $a \in R$ is said to be nilpotent if $\exists N \in \mathbb{N}, a^N = 0$.

For an ideal I of R and $n \in \mathbb{N}$ we define the ideal I^n inductively as follows:

- $I^0 = R$, and
- for $n > 0$, $I^n = I \cdot I^{n-1}$

An ideal I is nilpotent if $\exists N \in \mathbb{N}, I^N = 0$.

a. If $a \in R$ is nilpotent, prove that $1 - ab \in R^\times$ for every $b \in R$, where $R^\times$ is the set of units of R.
b. Let $I = \langle a_1, a_2, ..., a_m \rangle$ for $a_i \in R$ be a finitely generated ideal. Prove that if a_i is nilpotent for all i, then I is a nilpotent ideal.

Question 5: Let G be a group, and let $R[G]$ be the monoid algebra of G. Thus $R[G]$ is a free R-module with a basis $\{e(g) \mid g \in G\}$ and the multiplication satisfies $e(g)e(h) = e(gh)$ for $g, h \in G$.

a. Prove that

$$I = \left\{ \sum_{g \in G} a_g e(g) \in R[G] \,\middle|\, \sum_{g \in G} a_g = 0 \right\}$$

is an 2-sided ideal of $R[G]$ and that the R-algebra $R[G]/I$ is isomorphic to R. I is called the augmentation ideal of $R[G]$.

b. Let p be a prime number, let $G = \langle \sigma \rangle \simeq \mathbb{Z}/p\mathbb{Z}$ be the cyclic group of order p (written multiplicatively), and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field of p elements. Show that the augmentation ideal I of $\mathbb{F}_p[G]$ has an $\mathbb{F}_p$-basis consisting of $\{e(1) - e(\sigma^i) \mid i = 1, ..., p-1\}$, and show that I is a nilpotent ideal.

Question 6: Let $R[T]$ be the polynomial ring in a single variable over R. Recall for $f \in R[T]$ that $\langle f \rangle = f \cdot R[T]$ denotes the principal ideal generated by f.

a. Let

$$f = T^n + a_{n-1}T^{n-1} + \dots + a_1 T + a_0 \text{ for } n \in \mathbb{N} \text{ and } a_i \in R.$$

Prove that the quotient ring $R[T]/f \cdot R[T]$ is a free R-module with a basis

$$\left\{ \overline{T^i} = T^i + f \cdot R[T] \mid 0 \leq i \leq n-1 \right\}.$$

b. Prove that

$$\mathbb{Z}[T]/\langle 2T \rangle$$

is not a free $\mathbb{Z}$-module. Describe this ring as a $\mathbb{Z}$-module (i.e. as an abelian group).

Question 7: A ring R is said to be a local ring if it has a unique maximal ideal.

a. Prove that if R is local with unique maximal ideal M, then every element of $R \setminus M$ is a unit in R.

b. Conversely, prove that if the set of non-units in R forms an ideal M, then R is local with unique maximal ideal M.

c. Prove for a prime $p \in \mathbb{Z}$ that $R \subset \mathbb{Q}$ defined by

$$R = \{a/b \mid a, b \in \mathbb{Z}, b \not\equiv 0 (\operatorname{mod} p)\}$$

is a local ring with unique maximal ideal $pR = \langle p \rangle$.