Problem Set 5

Let R be a commutative ring (with identity…).

Question 1:

a. Let M be an R-module, and let I be an index set, and for $i \in I$, let $M_i \subseteq M$ for $i \in I$ be an R-submodule. Write $\sum_{i \in I} M_i$ for the sum of submodules M_i. Thus $\sum_{i \in I} M_i$ coincides with the *submodule generated by the* M_i.

   For a finitely supported function $(*)$ $f : I \to \bigcup_{i \in I} M_i$ for which $f(j) \in M_j$ for each $j \in I$, note that $\Sigma(f) = \sum_{i \in I} f(i)$ is a well-defined element of M.

   Prove that $\sum_{i \in I} M_i = \{\Sigma(f) \mid f \text{ a finitely supported function satisfying } (*)\}$

b. Assume that $M = \sum_{i \in I} M_i$ and that for a finitely supported function f satsifying $(*)$, we have $\Sigma(f) = 0 \Rightarrow f(i) = 0$ for each $i \in I$. Prove that $M \simeq \bigoplus_{i \in I} M_i$.

   One says in this case that M is the *internal direct sum* of the submodules M_i.

c. Let $X_1, X_2 \subseteq M$ be R-submodules. Suppose that $M = X_1 + X_2$ and that $X_1 \cap X_2 = 0$. Prove that $M \simeq X_1 \oplus X_2$.

   One says in this case that M is the *internal direct sum* of X_1 and X_2.

Question 2: Let I be an index set and let M_i be and R-module for each $i \in I$. Let M be the set of all functions $f : I \to \bigcup_{i \in I} M_i$ with such that $f(j) \in M_j$ for each $j \in I$. For $j \in I$ let

$$\pi_j : M \to M_i \text{ be the mapping } \pi_j(f) = f(j).$$

Then M is an R-module in a natural way, and π_j is an R-module homomorphism for each j.

a. Explain why (M, π_j) forms a *product* of the M_i in the category $\mathrm{mod}(R)$.

   We usually write $M = \prod_{i \in I} M_i$ for this R-module.

b. Suppose that I is a finite set, let $\left(\prod_{i \in I} M_i, \pi_i\right)$ be a product of the M_i, and let $\left(\bigoplus_{i \in I} M_i, \iota_i\right)$ be a coproduct of the M_i. Show that there is an isomorphism

$$\Phi : \bigoplus_{i \in I} M_i \to \prod_{i \in I} M_i$$

   of R-modules such that for $i, j \in I$ we have

$$\pi_j \circ \Phi \circ \iota_i = \begin{cases} \text{id if } i = j \\ 0 \ \text{ otherwise.} \end{cases}$$

Question 3: Let M, N be R-modules. Show that there is a short exact sequence

$$0 \to M \xrightarrow{\iota_M} M \oplus N \xrightarrow{\pi_N} N \to 0$$

where $\iota_M : M \to M \oplus N$ and $\iota_N : N \to M \oplus M$ are the mappings defining the direct sum (coproduct) $M \oplus N$, and $\pi_M : M \oplus N \simeq M \times N \to M$ and $\pi_N : M \oplus N \simeq M \times N \to N$ are the mappings defining the product $M \times N$.

Question 4: Let M, N be free R-modules. Thus there is some set B and function $\beta : B \to M$ such that M is a free R-module $\beta : B \to M$, and similarly for N.

Prove that $M \oplus N$ is a free R-module.

Question 5: Let M be an R-module, let $I \subseteq R$ be an ideal. Assume that $ax = 0$ for each $a \in I$ and each $x \in M$. Show that M has the structure of an R/I-module.

Question 6: Let I be an ideal of R and let M be an R-module.

Let IM be the R-submodule of M generated by the set

$$\{ax \mid a \in I, x \in M\}.$$

a. Prove that the R-module M/IM has the structure of an R/I-module. (Use Question 5.)
b. If M is a free R-module on $\beta : B \to M$, prove that M/IM is a free R/I-module on

$$\beta' = \pi \circ \beta : B \to M/I$$

where $\pi : M \to M/I$ is the quotient morphism.