Problem Set 6

Tufts University
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Math 065
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1. Let R be a commutative ring (with identity). We write 0 for the trivial R-module $\{0\}$.

   Consider a diagram $\mathcal{A}$ of the form:

   $$\cdots \to A_{i-1} \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} A_{i+2} \to \cdots$$

   where for $i \in \mathbb{Z}$, A_i is an R-module and $d_i : A_i \to A_{i+1}$ is an R-module homomorphism. Then $\mathcal{A}$ is said to be a *complex* provided that $d^2 = 0$; i.e., that for each $i \in \mathbb{Z}$ we have $d_i \circ d_{i-1} = 0$. This implies that $\operatorname{im} d_{i-1} \subseteq \ker d_i$.

   And the complex $\mathcal{A}$ is said to be *exact* provided that for all $i \in \mathbb{Z}$, $\operatorname{im} d_{i-1} = \ker d_i$.

   Let $\mathcal{A}$ be a complex:

   (a) For $i \in \mathbb{Z}$ write $H^i(\mathcal{A})$ for the R-module $\ker d_i / \operatorname{im} d_{i-1}$. Show that $\mathcal{A}$ is exact if and only if $H^i(\mathcal{A}) = 0$ for each $i \in \mathbb{Z}$.

   (b) For R-modules X, Y, Z, we view a diagram

   $$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

   as a complex provided that $g \circ f = 0$ by taking $A_i = 0$ for $i \le 0$, $A_1 = X$, $A_2 = Y$, $A_3 = Z$, and $A_j = 0$ for $j \ge 4$ as well as $d_1 = f$, $d_2 = g$, and $d_j = 0$ for $j \ne 1, 2$.

   We say that the complex $0 \to X \to Y \to Z \to 0$ is a *short exact sequence* provided that it is an exact complex.

   Prove that $0 \to X \to Y \to Z \to 0$ is a short exact sequence if and only if (i) f is injective, (ii) $\ker(g) = \operatorname{im}(f)$, and (iii) g is surjective.

   (c) Let $\phi : M \to N$ be an R-module homomorphism. Show that

   $$0 \to \ker \phi \xrightarrow{\iota} M \xrightarrow{\overline{\pi}} \operatorname{im} \phi \to 0$$

   is a short exact sequence, where $\iota : \ker \phi \to M$ and $\pi : M \to \operatorname{im} \phi$ are the inclusion mapping and the quotient mapping, respectively.

2. Let M, N be R-modules. Show that there is a short exact sequence

$$0 \to M \xrightarrow{\iota_M} M \oplus N \xrightarrow{\pi_N} N \to 0$$

where $\iota_M : M \to M \oplus N$ and $\iota_N : N \to M \oplus N$ are the inclusion maps, and $\pi_M : M \oplus N \cong M \times N \to M$ and $\pi_N : M \oplus N \cong M \times N \to N$ are the projections.

3. For ideals $I, J \subseteq R$, the *product* of I and J is the ideal generated by $\{xy \mid x \in I, y \in J\}$.

   (a) Prove that $IJ \subseteq I \cap J$.

   (b) If $P \subseteq R$ is a *prime ideal* and if $IJ \subseteq P$, prove that either $I \subseteq P$ or $J \subseteq P$.

4. An element $a \in R$ is said to be *nilpotent* if $\exists N \in \mathbb{N}, a^N = 0$.

   For an ideal I of R and $n \in \mathbb{N}$ we define the ideal I^n inductively as follows:

   - $I^0 = R$, and
   - for $n > 0, I^n = I \cdot I^{(n-1)}$.

   An ideal I is *nilpotent* if $\exists N \in \mathbb{N}, I^N = 0$.

   (a) If $a \in R$ is nilpotent, prove that $1 - ab$ is a unit in R^* for every $b \in R$, where R^* is the set of units of R.

   (b) Let $I = \langle a_1, a_2, \ldots, a_m \rangle$ for $a_i \in R$ be a finitely generated ideal. Prove that if a_i is nilpotent for all i, then I is a nilpotent ideal.

5. Let G be a group, and let $R[G]$ be the *monoid algebra* of G. Thus $R[G]$ is a free R-module with a basis $\{e(g) \mid g \in G\}$ and the multiplication satisfies $e(g)e(h) = e(gh)$ for $g, h \in G$.

   (a) Prove that

   $$I = \left\{ \sum_{g \in G} a_g e(g) \in R[G] \;\middle|\; \sum_{g \in G} a_g = 0 \right\}$$

   is a two-sided ideal of $R[G]$ and that the R-algebra $R[G]/I$ is isomorphic to R. I is called the *augmentation ideal* of $R[G]$.

   (b) Let p be a prime number, let $G = \langle \sigma \rangle \cong \mathbb{Z}/p\mathbb{Z}$ be the cyclic group of order p (written multiplicatively), and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field of p elements. Show that the augmentation ideal I of $\mathbb{F}_p[G]$ has a basis consisting of $\{e(1) - e(\sigma^i) \mid i = 1, \ldots, p - 1\}$, show that $(e(1) - e(\sigma^i))^p = 0$ for each i and deduce that I is a nilpotent ideal.

6. Let $R[T]$ be the polynomial ring in a single variable over R. Recall that for $f \in R[T]$, $\langle f \rangle = f \cdot R[T]$ denotes the principal ideal generated by f.

(a) Let $f = T^n + a_{n-1}T^{n-1} + \ldots + a_1 T + a_0$ for $n \in \mathbb{N}$ and $a_i \in R$. Prove that the quotient ring $R[T]/\langle f \rangle$ is a free R-module with basis $\{\overline{T^i} = T^i + \langle f \rangle \mid 0 \leq i \leq n-1\}$.

(b) Prove that $\mathbb{Z}[T]/\langle 2T \rangle$ is not a free $\mathbb{Z}$-module. Describe this ring as a $\mathbb{Z}$-module (i.e., as an abelian group).

7. A ring R is said to be a *local ring* if it has a unique maximal ideal.

   (a) Prove that if R is local with unique maximal ideal M, then every element of $R \setminus M$ is a unit in R.

   (b) Conversely, prove that if the set of non-units in R forms an ideal M, then R is local with unique maximal ideal M.

   (c) Prove for a prime $p \in \mathbb{Z}$ that

   $$R \subseteq \mathbb{Q}$$

   defined by

   $$R = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \not\equiv 0 \pmod{p} \right\}$$

   is a local ring with unique maximal ideal $pR = \langle p \rangle$.