

## Problem Set week 13

**Problem 1:** Let  $A$  be a local ring with unique maximal ideal  $\mathfrak{m}$ , and write  $k = A/\mathfrak{m}$  for the residue field of  $A$ .

Suppose that  $M$  is a free  $A$ -module of finite rank  $n$ . Let  $\alpha_1, \dots, \alpha_n \in M/\mathfrak{m}M$  be a  $k = A/\mathfrak{m}$ -basis for  $M/\mathfrak{m}M$ , and let  $x_1, \dots, x_n \in M$  be elements for which  $\alpha_i = x_i + \mathfrak{m}M$  for  $i = 1, \dots, n$ . Prove that  $x_1, \dots, x_n$  forms an  $A$ -basis for  $M$ .

**Problem 2:** Let  $A$  be a commutative ring and  $P \subset A$  a prime ideal. Denote by  $A_P$  the *localization* of  $A$  at  $P$ ; thus  $A_P$  is a local ring with unique maximal ideal  $\mathfrak{m} = P^e = P \cdot A_P$ .

- Prove that  $A_P/P \cdot A_P$  is isomorphic to the field of fractions of the integral domain  $A/P$ .
- If  $P$  is a maximal ideal of  $A$  conclude that the fields  $A/P$  and  $A_P/P \cdot A_P$  are isomorphic.

**Problem 3:** Let  $A$  be a PID,  $n \in \mathbb{N}$ , let  $F$  be a free  $A$ -module of finite rank and let  $\Phi \in \text{End}_A(F)$  be an  $A$ -linear endomorphism.

For a prime  $p \in A$ , write  $A_{pA}$  for the localization of  $A$  at  $pA$ . Note that  $\Phi$  determines an  $A_{pA}$ -homomorphism

$$\text{id} \otimes \Phi : A_{pA} \otimes_A F \rightarrow A_{pA} \otimes_A F.$$

- Fix an  $A$ -basis  $\mathcal{B}$  for  $F$ . For  $v \in M$ , we write

$$v = \sum_{b \in \mathcal{B}} \alpha_b b \text{ for a function } \alpha : \mathcal{B} \rightarrow A$$

and we write  $[v]_{\mathcal{B}} = \alpha \in A^{\mathcal{B}}$ , where  $A^{\mathcal{B}}$  is the module of all functions  $\mathcal{B} \rightarrow A$  (recall that  $|\mathcal{B}| = n < \infty!$ )

Thus  $v \mapsto [v]_{\mathcal{B}} : F \rightarrow A^{\mathcal{B}}$  is an isomorphism of  $A$ -modules where  $A^{\mathcal{B}}$  is the module of

Show that there is a matrix  $M \in \text{Mat}_{\mathcal{B} \times \mathcal{B}}(A)$  for which

$$[\Phi v]_{\mathcal{B}} = M \cdot [v]_{\mathcal{B}} \text{ for } v \in F.$$

- Let  $d = \det(\Phi)$ . Prove that  $\text{id} \otimes \Phi$  is an isomorphism – i.e. is an *automorphism* of  $A_{pA} \otimes_A F$  – if and only if  $\gcd(d, p) = 1$ .

In particular if  $d \neq 0$ , then  $\text{id} \otimes \Phi$  is an automorphism for all but finitely many primes  $p \in A$ .

**Problem 4:** Let  $A$  be a commutative ring and let  $M, N, P$  be  $A$ -modules. Prove that there is an isomorphism

$$M \otimes_A (N \oplus P) \simeq (M \otimes_A N) \oplus (M \otimes_A P).$$

**Problem 5:** Let  $A$  be a PID and let  $p, q \in A$ . Write  $d = \gcd(p, q) \in A$  for a greatest common divisor.

Prove that:

$$(A/pA) \otimes_A (A/q) \simeq A/dA.$$

**Problem 6:** Let  $F$  be a field and let  $V, W$  be finite dimensional vector spaces over  $F$ .

Recall that the dual space  $V^*$  is the vector space  $\text{Hom}_F(V, F)$ .

- Show that  $\dim_F V = \dim_F V^*$ . **Hint:** exhibit a basis for  $V^*$ .
- Show that there is an isomorphism

$$V^* \otimes_A W \xrightarrow{\sim} \text{Hom}_F(V, W).$$

**Hint:** Use the mapping property of  $\otimes$  to define the indicated map. Basis considerations show that this map is surjective. Now compare the dimension of the domain and co-domain.

Let  $A$  be a commutative ring, let  $M, M', N, N'$  be  $A$ -modules and let  $\varphi : M \rightarrow N$  and  $\varphi' : M' \rightarrow N'$  be  $A$ -module homomorphisms. There is a unique homomorphism of  $A$ -modules

$$\varphi \otimes \varphi' : M \otimes_A N \rightarrow M' \otimes_A N'$$

such that

$$(\varphi \otimes \varphi')(m \otimes n) = \varphi(m) \otimes \varphi'(n).$$

**Problem 7:** Let  $F$  be a field and let  $\varphi : V \rightarrow W$  be a homomorphism of  $F$ -vector spaces (a “linear transformation”) and let  $X$  be an  $F$ -vector space.

If  $\varphi$  is injective, prove that  $\text{id}_X \otimes \varphi : X \otimes_F V \rightarrow X \otimes_F W$  is injective.

**Remark:** This shows that the functor  $X \otimes_F -$  is *exact* for a field  $F$ ; indeed, combine the preceding observation with the result proved in class that the functor  $Y \otimes_A -$  is always right exact. In general, an  $A$ -module  $Y$  is said to be **flat** if  $Y \otimes_A -$  is exact.

**Problem 8:** Let  $A$  be a commutative ring and let  $M$  be an  $A$ -module. If  $F$  is a free  $A$ -module on  $\beta : \mathcal{B} \rightarrow F$ , prove that  $F \otimes_A M$  is isomorphic to  $\bigoplus_{b \in \mathcal{B}} M$ , a direct sum of copies of  $M$  indexed by  $\mathcal{B}$ .