Assignment 1

Question 1: Let F be a field, let I be a set, and let $V = V_I$ the set of finitely supported F-valued functions on I.

a. Explain why V is an F-vector space.
b. For $i \in I$, let δ_i be the function $\delta_i : I \to F$ given by the rule

$$\delta_i(j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ else} \end{cases}$$

Prove that $\mathcal{B} = \{\delta_i \mid i \in I\}$ is a basis for V_I as F-vector space. You need to prove that $\mathcal{B}$ spans V and that $\mathcal{B}$ is linearly independent.

c. If $f : I \to J$ is a bijective function, prove that there is a linear mapping $\Phi_f : V_I \to V_J$ for which $\Phi_f(\delta_i) = \delta_{f(i)}$ for each $i \in I$, and that Φ_f determines an isomorphism of F-vector space $V_I \to V_J$.

Question 2: Let F be a field, let V be an F-vector space, and let $\mathrm{GL}(V)$ be the group of invertible F-linear maps $V \to V$.

a. Let $e \in V$, let $\varphi : V \to F$ be an F-linear map, and suppose that $\varphi(e) = 0$. Consider the mapping

$$\Psi = \Psi_{\varphi,e} : V \to V$$

defined by the rule $\Psi(v) = v + \varphi(v)e$ for $v \in V$. Prove that Ψ is F-linear and invertible, so that $\Psi \in \mathrm{GL}(V)$.

b. If $\dim_F V > 1$ prove that $\mathrm{GL}(V)$ is non-abelian.

Hint/Suggestion: Let $e_1, e_2 \in V$ be linearly independent vectors. Explain why you can choose linear maps $\varphi_1, \varphi_2 : V \to F$ for which $\varphi_i(e_i) = 1$ and $\varphi_i(e_j) = 0$ if $i \neq j$. Now show that $\Psi_1 = \Psi_{\varphi_1,e_1}$ and $\Psi_2 = \Psi_{\varphi_2,e_2}$ don't commute.