Assignment 2

Question 1: Let G be a group, let $S_1, S_2 \subseteq G$ be subsets and let $H_i = \langle S_i \rangle$ for $i = 1, 2$. Suppose that $\forall x \in S_1$ and $\forall y \in S_2$ we have $xyx^{-1} \in H_2$. Prove that H_1 normalizes H_2; i.e. prove that

$$\forall x \in H_1, \forall y \in H_2, xyx^{-1} \in H_2$$

Question 2: Let $n \in \mathbb{N}$, $n > 0$ and consider the group $S = S(\mathbb{Z}/n\mathbb{Z})$ of permutations of the set $\mathbb{Z}/n\mathbb{Z}$.

a. For $x \in \mathbb{Z}/n\mathbb{Z}$, recall that the additive order $o(x)$ is a divisor of n.

   Describe the *cycle structure* of the element $\sigma \in S$ defined by the rule $\sigma(z) = z + x$. Show that the order of σ is $o(x)$.
b. Suppose that $n = p$ is a prime number, and let $k \in \mathbb{Z}$ with $\gcd(k, p) = 1$. Thus the class $\overline{k}$ of k in $\mathbb{Z}/p\mathbb{Z}$ lies in the group $(\mathbb{Z}/p\mathbb{Z})^\times$ of *units*. The multiplicative order $o\left(\overline{k}\right)$ of $\overline{k}$ is a divisor of $p - 1$.

   Describe the *cycle structure* of the element $\tau \in S$ defined by the rule $\tau(z) = \overline{k} \cdot z$. Show that the order of τ is $o\left(\overline{k}\right)$.

Question 3: Let G be the group of invertible 2×2 matrices with entries in $F = \mathbb{Z}/p\mathbb{Z}$ for a prime number p; the group operation is given by matrix multiplication.

a. Show that $|G| = (p^2 - 1)(p^2 - p)$.
b. Show that $T = \left\{ \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix} \mid t, s \in F^\times \right\}$ is a subgroup of G. Here $F^\times$ denotes the multiplicative group of invertible elements of $F = \mathbb{Z}/p\mathbb{Z}$. Also show that

   $$T \text{ is isomorphic to } F^\times \times F^\times$$

   .
c. Show that $U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in F \right\}$ is a subgroup of G isomorphic to the addtive group F.
d. Show that T normalizes U. Find the order of the group $B = TU$.
e. A line in F^2 is by definition a linear subspace of dimension 1. For any non-zero vector v, the set $Fv = \operatorname{Span}(v)$ is a line. Note that G acts in a natural way on the set of lines in F^2.

   If we write $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the standard basis of F^2, show that B is the stabilizer of the line Fe.
f. Show that G acts transitively on the set of lines in F^2.
g. Conclude that the set of lines in F^2 is in bijection with the set G/B. How many lines are there in F^2?

Question 4: Let G be a group and let Ω be a G-set. If $x, y \in \Omega$ and $x = gy$ for some $g \in G$, prove that the stabilizers $G_x = \mathrm{Stab}_G(x)$ and $G_y = \mathrm{Stab}_G(y)$ are *conjugate*. More precisely, show that

$$G_x = g \cdot G_y \cdot g^{-1}$$

Question 5: Let G be a group. G acts on itself by conjugation: for $g, x \in G$, the action of g on x is given by $\mathrm{Inn}_g\, x = gxg^{-1}$.

a. Prove that the assignment $g \mapsto \mathrm{Inn}_g$ determines a group homomorphism $G \to \mathrm{Aut}(G)$ where $\mathrm{Aut}(G)$ is the group of automorphisms of G.
b. Let $Z = \{g \in G \mid \forall x \in G, gx = xg\}$ be the *center* of G. Prove that $Z = \ker \mathrm{Inn}$.

For the action of G on itself by conjugation, the stabilizer $\mathrm{Stab}_G(x)$ of $x \in G$ is usually written $C_G(x)$ and is called the *centralizer* of x in G. Note that

$$C_G(x) = \{y \in G \mid yxy^{-1} = x\} = \{y \in G \mid yx = xy.\}$$

Question 6: Let $I = I_n$ be a finite set with n elements, and let $S = S_n = S(I_n)$ be the group of permutations of I. Recall that $|S| = n!$.

a. Prove that there are $(n-1)!$ n-cycles in S. **Hint**: If the elements of I are writen $I = \{a_1, a_2, ..., a_n\}$, then the n-cycles $(a_1, a_2, ..., a_n)$ and $(a_2, a_3, ..., a_n, a_1)$ are *equal*.
b. Prove that if σ is an n-cycle in S, then $C_S(\sigma) = \langle \sigma \rangle$