Problem Set week 13

Problem 1: Let A be a local ring with unique maximal ideal $\mathfrak{m}$, and write $k = A/\mathfrak{m}$ for the residue field of A.

Suppose that M is a free A-module of finite rank n. Let $\alpha_1, ..., \alpha_n \in M/\mathfrak{m}M$ be a $k = A/\mathfrak{m}$-basis for $M/\mathfrak{m}M$, and let $x_1, ..., x_n \in M$ be elements for which $\alpha_i = x_i + \mathfrak{m}M$ for $i = 1, ...n$. Prove that $x_1, ..., x_n$ forms an A-basis for M.

Problem 2: Let A be a commutative ring and $P \subset A$ a prime ideal. Denote by A_P the *localization* of A at P; thus A_P is a local ring with unique maximal ideal $\mathfrak{m} = P^e = P \cdot A_P$.

a. Prove that $A_P/P \cdot A_P$ is isomorphic to the field of fractions of the integral domain A/P.
b. If P is a maximal ideal of A conclude that the fields A/P and $A_P/P \cdot A_P$ are isomorphic.

Problem 3: Let A be a PID, $n \in \mathbb{N}$, let F be a free A-module of finite rank and let $\Phi \in \operatorname{End}_A(F)$ be an A-linear endomorphism.

For a prime $p \in A$, write A_{pA} for the localization of A at pA. Note that Φ determines an A_{pA}-homomorphism

$$\mathrm{id} \otimes \Phi : A_{pA} \otimes_A F \to A_{pA} \otimes_A F.$$

a. Fix an A-basis $\mathcal{B}$ for F. For $v \in M$, we write

$$v = \sum_{b \in \mathcal{B}} \alpha_b b \text{ for a function } \alpha : \mathcal{B} \to A$$

and we write $[v]_{\mathcal{B}} = \alpha \in A^{\mathcal{B}}$, where $A^{\mathcal{B}}$ is the module of all functions $\mathcal{B} \to A$ (recall that $|\mathcal{B}| = n < \infty$!)

Thus $v \mapsto [v]_{\mathcal{B}} : F \to A^{\mathcal{B}}$ is an isomorphism of A-modules where $A^{\mathcal{B}}$ is the module of

Show that there is a matrix $M \in \operatorname{Mat}_{\mathcal{B} \times \mathcal{B}}(A)$ for which

$$[\Phi v]_{\mathcal{B}} = M \cdot [v]_{\mathcal{B}} \text{ for } v \in F.$$

b. Let $d = \det(\Phi)$. Prove that $\mathrm{id} \otimes \Phi$ is an isomorphism – i.e. is an *automorphism* of $A_{pA} \otimes_A F$ – if and only if $\gcd(d, p) = 1$.

In particular if $d \neq 0$, then $\mathrm{id} \otimes \Phi$ is an automorphism for all but finitely many primes $p \in A$.

Problem 4: Let A be a commutative ring and let M, N, P be A-modules. Prove that there is an isomorphism

$$M \otimes_A (N \oplus P) \simeq (M \otimes_A N) \oplus (N \otimes_A P).$$

Problem 5: Let A be a PID and let $p, q \in A$. Write $d = \gcd(p, q) \in A$ for a greatest common divisor.

Prove that:

$$(A/pA) \otimes_A (A/q) \simeq A/dA.$$

Problem 6: Let F be a field and let V, W be a finite dimensional vector spaces over F.

Recall that the dual space V^* is the vector space $\mathrm{Hom}_F(V, F)$.

a. Show that $\dim_F V = \dim_F V^*$. **Hint:** exhibit a basis for V^*.
b. Show that there is an isomorphism

$$V^* \otimes_A W \xrightarrow{\sim} \mathrm{Hom}_F(V, W).$$

Hint: Use the mapping property of $\otimes$ to define the indicated map. Basis considerations show that this map is surjective. Now compare the dimension of the domain and co-domain.

Let A be a commutative ring. let M, M', N, N' be A-modules and let $\varphi : M \to N$ and $\varphi' : M \to N'$ be A-module homomorphisms. There is a unique homomorphism of A-modules

$$\varphi \otimes \varphi' : M \otimes_A N \to M' \otimes_A N'$$

such that

$$(\varphi \otimes \varphi')(m \otimes n) = \varphi(m) \otimes \varphi'(n).$$

Problem 7: Let F be a field and let $\varphi : V \to W$ be a homomorphism of F-vector spaces (a "linear transformation") and let X be an F-vector space.

If φ is injective, prove that $\mathrm{id}_X \otimes \varphi : X \otimes_F V \to X \otimes_F W$ is injective.

Remark: This shows that the functor $X \otimes_F -$ is *exact* for a field F; indeed, combine the preceding observation with the result proved in class that the functor $Y \otimes_A -$ is always right exact. In general, an A-module Y is said to be flat if $Y \otimes_A -$ is exact.

Problem 8: Let A be a commutative ring and let M be an A-module. If F is a free A-module on $\beta : \mathcal{B} \to F$, prove that $F \otimes_A M$ is isomorphic to $\bigoplus_{b \in \mathcal{B}} M$, a direct sum of copies of M index by $\mathcal{B}$.