

## Problem Set week 11

**Problem 1:** Let  $A$  be a ring and let (♥)  $0 \rightarrow X \xrightarrow{j} Y \xrightarrow{\psi} Z \rightarrow 0$  be a short exact sequence of  $A$ -modules.

Prove that the following are equivalent:

- There is an isomorphism  $\varphi : X \oplus Z \xrightarrow{\sim} Y$  such that  $\psi \circ \varphi = \pi_2$  and  $\varphi^{-1} \circ j = \iota_1$  where  $\pi_2 : X \oplus Z \rightarrow Z$  is  $(x, z) \mapsto z$ , and  $\iota_1 : X \rightarrow X \oplus Z$  is  $x \mapsto (x, 0)$ .

- There is a **section** to  $\psi$ ; i.e., there is an  $A$ -module homomorphism  $\sigma : Z \rightarrow Y$  such that

$$\psi \circ \sigma = \text{id}_Z.$$

- There is a **retract** of  $j$ ; i.e. there is an  $A$ -module homomorphism  $\rho : Y \rightarrow X$  such that

$$\rho \circ j = \text{id}_X.$$

The short exact sequence (♥) is said to be *split exact* if it satisfies these equivalent conditions.

**Problem 2:** Let

$$0 \rightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \rightarrow 0$$

be a short exact sequence of  $A$ -modules.

Prove that for an  $A$ -module  $W$ , the sequence

$$(♣) \quad 0 \rightarrow \text{Hom}_A(W, X) \xrightarrow{f \mapsto f \circ \iota} \text{Hom}_A(W, Y) \xrightarrow{g \mapsto g \circ \pi} \text{Hom}_A(W, Z)$$

is *exact*.

NB. A sequence of  $A$ -modules and  $A$ -module homomorphisms is *exact* if for each interior term

$$\dots S \xrightarrow{f} T \xrightarrow{g} Y \dots$$

of the sequence,  $\ker g = \text{im } f$ . In the sequence (♣),  $\text{Hom}_A(P, Y)$  is not an interior term.

**Remark:** The exactness of (♣) for each short exact sequence is often expressed by saying that the functor  $\text{Hom}_A(W, -)$  is *left-exact*.

So your task is to check exactness of (♣) at the interior terms  $\text{Hom}_A(W, X)$  and  $\text{Hom}_A(W, Y)$ .

**Problem 3:** An  $A$ -module  $P$  is said to **projective** if for every surjective homomorphism of  $A$ -modules  $\pi : M \rightarrow N$  and every  $A$ -module homomorphism  $f : P \rightarrow N$ , there is an  $A$ -module homomorphism  $\alpha : P \rightarrow M$  such that  $\pi \circ \alpha f$ .

- a. Prove that any free  $A$ -module is projective.
- b. Prove that a module  $P$  is projective if and only if  $P$  is a direct summand of a free module; i.e. if there is a free  $A$ -module  $F$  and an isomorphism of  $A$ -modules  $F \simeq P \oplus Q$  for some  $A$ -module  $Q$ .

**Problem 4:** Prove that the following are equivalent:

- a.  $P$  is projective
- b. for every short exact sequence of  $A$ -modules  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  the sequence of  $A$ -modules

$$0 \rightarrow \text{Hom}_A(P, X) \rightarrow \text{Hom}_A(P, Y) \rightarrow \text{Hom}_A(P, Z) \rightarrow 0$$

is exact.

**Remark:** Thus  $P$  is projective if and only if the left-exact functor  $\text{Hom}_A(P, -)$  is exact (in the sense that  $\text{Hom}_A(P, -)$  carries short exact sequences to short exact sequences.).

**Problem 5:** Let  $A$  be a commutative ring, let  $B = A \times A$ , and let  $\pi : B \rightarrow A$  be the projection ring homomorphism given by  $\pi(x, y) = x$ .

Using  $\pi$ , we can view  $A$  as a  $B$ -module. Prove that  $A$  is a projective  $B$ -module that is not a free  $B$ -module.

**Problem 6:** Let  $K$  be a field and let  $a, b \in K \setminus \{0, 1\}$  with  $a \neq b$ . Let  $q = T(T - a)(T - b) \in K[T]$  and consider the polynomial  $f = S^2 - q \in K[S, T]$ .

You proved in the previous homework set that:

- $A = K[S, T]/\langle f \rangle$  is an integral domain. Write  $s, t$  for the images of  $S, T$  in the quotient ring  $A$ ; thus  $s^2 = q(t) = t(t - a)(t - b)$  in  $A$ .
  - The map  $K[T] \rightarrow k[t]$  for which  $T \mapsto t$  is an isomorphism.
  - $A$  is a free  $K[t]$ -module on a basis  $\{1, s\}$ .
- a. Define  $N : A \rightarrow K[t]$  by the rule  $N(\alpha) = N(f + gs) = f^2 - g^2 \cdot q(t)$  for  $\alpha = f + gs$  with  $f, g \in K[t]$ .

Then  $N$  is multiplicative (i.e. it is a monoid homomorphism) and  $\alpha \in A$  is a unit in  $A$  if and only if  $N(\alpha) \in K[T]^\times = K^\times$ .

- b. Write  $\mathfrak{m} = \langle s, t \rangle$  for the ideal of  $A$  generated by  $s$  and  $t$ . Show that the ideal  $\mathfrak{m}$  is not principal.

**Hint:** Suppose to the contrary that  $\alpha = f + g \cdot s \in A$  is a generator for  $\mathfrak{m}$ , for  $f, g \in k[t]$ . Then  $\alpha \mid t \Rightarrow N(\alpha) \mid t^2$ . Now use a degree argument to show that  $g = 0$ .

- c. Show that  $\mathfrak{m}$  is a maximal ideal of  $A$ .

**Remark 1:** Essentially the same arguments show that the ideals

$$\langle s, t - a \rangle \text{ and } \langle s, t - b \rangle$$

are maximal and not principal.

**Remark 2:** The field of fractions  $F = \text{Frac}(A) = K(s, t)$  of  $A$  is the *field of rational functions* on the *elliptic curve* over  $K$  defined by the cubic equation  $S^2 = T(T - a)(T - b)$ .

If we had consider instead the quadratic equation  $S^2 = T(T - a)$ , the analogous ideal  $\mathfrak{m} = \langle s, t \rangle$  is principal (!).

**Problem 7:** Keep the notations of the previous problem. We are going to show that  $\mathfrak{m}$  is a projective  $A$ -module.

- a. First, explain why the condition that  $\mathfrak{m}$  is not a principal ideal implies that  $\mathfrak{m}$  is not a free  $A$ -module.
- b. Explain why  $1 = u \cdot t + v \cdot (t - a)(t - b)$  for some  $u, v \in k[t]$ .
- c. Show for every  $z \in \mathfrak{m}$  that  $\frac{(t-a)(t-b)z}{s} \in A$ ; note that *a priori*  $\frac{(t-a)(t-b)z}{s}$  is an element of the field of fractions of  $A$ .
- d. Consider the surjective  $A$ -module homomorphism

$$\pi : A^2 \rightarrow \mathfrak{m} \text{ given by } \pi \begin{pmatrix} x \\ y \end{pmatrix} = xs + yt.$$

Show that  $\sigma : \mathfrak{m} \rightarrow A^2$  given by the rule

$$\sigma(z) = \begin{pmatrix} v(t-a)(t-b)z/s \\ uz \end{pmatrix}$$

is a *section* to  $\pi$  (where  $u, v$  are as in (b) above).

- e. Conclude using problem (1) that  $\mathfrak{m}$  is isomorphic to a direct summand of the free module  $A^2$ .