

## Problem Set 5

Let  $R$  be a commutative ring (with identity...).

### Question 1:

- a. Let  $M$  be an  $R$ -module, and let  $I$  be an index set, and for  $i \in I$ , let  $M_i \subseteq M$  for  $i \in I$  be an  $R$ -submodule. Write  $\sum_{i \in I} M_i$  for the sum of submodules  $M_i$ . Thus  $\sum_{i \in I} M_i$  coincides with the *submodule generated by the  $M_i$* .

For a finitely supported function  $(*) f : I \rightarrow \bigcup_{i \in I} M_i$  for which  $f(j) \in M_j$  for each  $j \in I$ , note that  $\Sigma(f) = \sum_{i \in I} f(i)$  is a well-defined element of  $M$ .

Prove that  $\sum_{i \in I} M_i = \{\Sigma(f) \mid f \text{ a finitely supported function satisfying } (*)\}$

- b. Assume that  $M = \sum_{i \in I} M_i$  and that for a finitely supported function  $f$  satisfying  $(*)$ , we have  $\Sigma(f) = 0 \Rightarrow f(i) = 0$  for each  $i \in I$ . Prove that  $M \simeq \bigoplus_{i \in I} M_i$ .

One says in this case that  $M$  is the *internal direct sum* of the submodules  $M_i$ .

- c. Let  $X_1, X_2 \subseteq M$  be  $R$ -submodules. Suppose that  $M = X_1 + X_2$  and that  $X_1 \cap X_2 = 0$ . Prove that  $M \simeq X_1 \oplus X_2$ .

One says in this case that  $M$  is the *internal direct sum* of  $X_1$  and  $X_2$ .

**Question 2:** Let  $I$  be an index set and let  $M_i$  be an  $R$ -module for each  $i \in I$ . Let  $M$  be the set of all functions  $f : I \rightarrow \bigcup_{i \in I} M_i$  with such that  $f(j) \in M_j$  for each  $j \in I$ . For  $j \in I$  let

$$\pi_j : M \rightarrow M_j \text{ be the mapping } \pi_j(f) = f(j).$$

Then  $M$  is an  $R$ -module in a natural way, and  $\pi_j$  is an  $R$ -module homomorphism for each  $j$ .

- a. Explain why  $(M, \pi_j)$  forms a *product* of the  $M_i$  in the category  $\text{mod}(R)$ .

We usually write  $M = \prod_{i \in I} M_i$  for this  $R$ -module.

- b. Suppose that  $I$  is a finite set, let  $(\prod_{i \in I} M_i, \pi_i)$  be a product of the  $M_i$ , and let  $(\bigoplus_{i \in I} M_i, \iota_i)$  be a coproduct of the  $M_i$ . Show that there is an isomorphism

$$\Phi : \bigoplus_{i \in I} M_i \rightarrow \prod_{i \in I} M_i$$

of  $R$ -modules such that for  $i, j \in I$  we have

$$\pi_j \circ \Phi \circ \iota_i = \begin{cases} \text{id} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

**Question 3:** (question 3 removed – will appear next week!)

**Question 4:** Let  $M, N$  be free  $R$ -modules. Thus there is some set  $B$  and function  $\beta : B \rightarrow M$  such that  $M$  is a free  $R$ -module  $\beta : B \rightarrow M$ , and similarly for  $N$ .

Prove that  $M \oplus N$  is a free  $R$ -module.

**Question 5:** Let  $M$  be an  $R$ -module, let  $I \subseteq R$  be an ideal. Assume that  $ax = 0$  for each  $a \in I$  and each  $x \in M$ . Show that  $M$  has the structure of an  $R/I$ -module.

**Question 6:** Let  $I$  be an ideal of  $R$  and let  $M$  be an  $R$ -module.

Let  $IM$  be the  $R$ -submodule of  $M$  generated by the set

$$\{ax \mid a \in I, x \in M\}.$$

- Prove that the  $R$ -module  $M/IM$  has the structure of an  $R/I$ -module. (Use [Question 5](#).)
- If  $M$  is a free  $R$ -module on  $\beta : B \rightarrow M$ , prove that  $M/IM$  is a free  $R/I$ -module on

$$\beta' = \pi \circ \beta : B \rightarrow M/IM$$

where  $\pi : M \rightarrow M/IM$  is the quotient morphism, where  $IM$  is the  $R$ -submodule  $IM = \{ax \mid a \in I, x \in M\}$  of  $M$ .