Problem Set week 9

Let A be a commutative ring (with identity).

Problem 1: Suppose that A is an integral domain. Prove that the polynomial ring $A[T]$ is an integral domain.

Problem 2: Let F be a field and let $f \in F[T]$ be a polynomial of degree ≥ 1. An element $\alpha \in F$ is a root of f if $f(\alpha) = 0$. Recall that $f(\alpha) = \varphi(f)$ is the result of applying the ring homomorphism $\varphi : F[T] \to F$ determined by the properties: $\varphi_{|F} = \mathrm{id}$ and $\varphi(T) = \alpha$.

a. Prove that if $\deg f = 2$ or $\deg f = 3$ then f is irreducible if and only if f has no root in F.
b. Prove that in general, there are reducible polynomials with no root in f provided that $\deg f \geq 4$.

Problem 3: Let $d \in \mathbb{Z}$, let $f = T^2 - d \in \mathbb{Z}[T]$ and consider the ring $\mathbb{Z}\left[\sqrt{d}\right] \subset \mathbb{C}$. Assume that $\forall a \in \mathbb{Z}, d \neq a^2$. In that case, one knows that $\sqrt{d} \notin \mathbb{Q}$.

a. Explain why $f = T^2 - d$ is irreducible in $\mathbb{Q}[T]$ and prove that

$$\mathbb{Q}[T]/\langle T^2 - d\rangle \simeq \mathbb{Q}\left[\sqrt{d}\right].$$

(We often write $\mathbb{Q}\left(\sqrt{d}\right) = \mathbb{Q}\left[\sqrt{d}\right]$ for this field – explanation later!)

b. Expain why $f = T^2 - d$ is irreducible in $\mathbb{Z}[T]$ and prove that

$$\mathbb{Z}[T]/\langle T^2 - d\rangle \simeq \mathbb{Z}\left[\sqrt{d}\right]$$

is a free $\mathbb{Z}$-module of rank 2.

c. Prove that $\mathbb{Q}\left(\sqrt{d}\right)$ may be identified with the field of fractions of $\mathbb{Z}\left[\sqrt{d}\right]$.

Problem 4: Let $A = \mathbb{Z}[i]$. Then A is a Euclidean Domain with norm given by

$$N(a + bi) = a^2 + b^2 \text{ for } a, b \in \mathbb{Z}$$

– you are free to use this; for a reference see [Dummit-Foote, § 8.1 example 3 p. 272].

a. Show that $N(\alpha) = \pm 1 \Leftrightarrow \alpha \in A^\times$.
b. Show for any integer n that $A/A \cdot n$ is a ring with n^2 elements.
c. Show that 7 is irreducible in A. **Hint:** is it possible to write $7 = a^2 + b^2$ for $a, b \in \mathbb{Z}$? Conclude that $A/A \cdot 7$ is a field with 49 elements.

Problem 5: Again let $A = \mathbb{Z}[i]$ and keep the notation N for the norm on A.

a. Show that if $N(\alpha)$ is a prime integer, then α is irreducible and hence prime.
b. Show that $2 + i$ and $2 - i$ are both prime in A.
c. Prove that $A/A \cdot (2 + i)$ and $A/A \cdot (2 - i)$ are both fields, each with 5 elements.
d. Prove that

$$A/A \cdot 5 \simeq A/A \cdot (2 + i) \times A/A \cdot (2 - i).$$

   (Use the Chinese Remainder Theorem.)

Problem 6: Let K be an uncountable field (e.g. $K = \mathbb{R}$ or $K = \mathbb{C}$). Prove that $K(T)$ contains a K-linearly independent subset which is uncountable.

Hint: Consider the set

$$\left\{ \frac{1}{T - \alpha} \mid \alpha \in K \right\}.$$

Problem 7: Let K be a field, and let $p_1, p_2, \ldots, p_r \in K[T]$ be pairwise non-associate irreducible polynomials for some $r \in \mathbb{N}$. Let S be a new polynomial variable and consider the polynomial

$$f = S^2 - p_1 \ldots p_r \in K[S, T].$$

a. Prove that if $r > 0$ then f is irreducible in $K(T)[S]$.

   Hint: Argue that f has no root in $K(T)$ just as in the classical proof that $\sqrt{2}$ is irrational.
b. Explain how to deduce that f is irreducible – and hence prime – in $K[T, S]$.
c. Let $A = K[T, S]/\langle f \rangle$ and write s for the image of S in A. Prove that the integral domain A is a free $K[T]$-module on a basis $\{1, s\}$.
d. Prove that $K(T)[S]/\langle f \rangle$ may be identified with the field of fractions of $K[S, T]/\langle f \rangle$.