

Problem Set 3

Question 1:

Let G be a group and let $H, K \subseteq G$ be subgroups of G . Suppose that H and K are both normal in G , and that $H \cap K = \{1\}$. Recall that HK is a subgroup of G . Prove that the natural map $H \times K \rightarrow HK$ given by $(h, k) \mapsto h \cdot k$ is a group isomorphism.

Recall that the group structure on the cartesian product is given for $(h, k), (h', k') \in H \times K$ by:

$$(h, k) \cdot (h', k') = (hh', kk'), \quad (h, k)^{-1} = (h^{-1}, k^{-1}) \quad \text{and} \quad 1_{H \times K} = (1_H, 1_K);$$

this is the **direct product** of G and H .

Question 2:

Let $\varphi : G \rightarrow H$ be a surjective group homomorphism, and suppose that $N \subseteq H$ is a normal subgroup of H . Prove that $\varphi^{-1}(N) = \{g \in G \mid \varphi(g) \in N\}$ is a normal subgroup of G .

Question 3:

Suppose that G and G' are groups, let H, K be subgroups of G , and let H', K' be subgroups of G' .

Assume that

- H normalizes K and H' normalizes K' .
- $G = \langle K, H \rangle = KH$ and $G' = \langle K', H' \rangle = K'H'$.
- $K \cap H = \{1\}$ and $K' \cap H' = \{1\}$.
- there are group isomorphisms $\varphi : H \xrightarrow{\sim} H'$ and $\psi : K \xrightarrow{\sim} K'$. Since H normalizes K , for $h \in H$ we know that

the restriction of Inn_h to K determines an automorphism of K ; similarly, for $h' \in H'$, $\text{Inn}_{h'}$ determines an automorphism of K' .

We finally suppose

- for $h \in H$ and $k \in K$ we have $\psi(\text{Inn}_h k) = \text{Inn}_{\varphi(h)} \psi(k)$.

Then there is a group isomorphism $\Phi : G \rightarrow G'$ given for $(k, h) \in KH = G$ by the rule

$$\Phi(k, h) = \psi(k)\varphi(h) \in K'H' = G'$$

.

Question 4:

For a prime number p , write $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and let

$$H_p = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_p \right\}$$

so that H_p is a subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$ of order p^3 . (You should at least think through why this is so, though you needn't submit the details).

- a. Prove that H_2 is isomorphic to $D_8 = D_{2,4}$, the dihedral group with 8 elements.

Hint: find $\sigma, \tau \in H_2$ with $o(\sigma) = 4$, $o(\tau) = 2$ which have the property that $\tau\sigma\tau = \sigma^{-1}$. Then $H_2 = \langle \sigma \rangle \cdot \langle \tau \rangle$. Now use the solution to [Question 3](#).

- a. Show that H_p is a p -Sylow subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$.

Question 5:

Let G be a finite group, let p be a prime number, and let $P \in \mathrm{Syl}_p(G)$. Let

$$H = N_G(P) = \{g \in G \mid \mathrm{Inn}_g P = P\}$$

be the normalizer of P in G . Prove that $N_G(H) = H$. (In words: the normalizer of a Sylow p -subgroup is self-normalizing).

Question 6: Suppose that F is a field.

- a. Show that the ideals of F are $\{0\}$ and F .
b. Deduce that if R is any commutative ring (with $0_R \neq 1_R$), then any homomorphism

$$\varphi : F \rightarrow R$$

is injective.

Question 7:

Let $D \in \mathbb{Z}$ and suppose that D is square-free - i.e. for any prime number p , $p^2 \nmid D$.

If $D \equiv 1 \pmod{4}$ let

$$\omega = \frac{1 + \sqrt{D}}{2}$$

and show that $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$ forms a subring of \mathbb{C} .