Problem Set 4

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Math 065
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Questions

Question 1: Let G be a group and let $H, K \subseteq G$ be subgroups of G. Suppose that H and K are both normal in G, and that $H \cap K = \{1\}$. Recall that HK is a subgroup of G. Prove that the natural map $H \times K \to HK$ given by $(h, k) \mapsto h \cdot k$ is a group isomorphism.

Recall that the group structure on the Cartesian product is given for $(h, k), (h', k') \in H \times K$ by:

$$(h, k) \cdot (h', k') = (hh', kk'), \quad (h, k)^{-1} = (h^{-1}, k^{-1}), \quad \text{and } 1_{H \times K} = (1_H, 1_K);$$

this is the **direct product** of G and H.

Question 2: Let $\phi : G \to H$ be a surjective group homomorphism, and suppose that $N \subseteq H$ is a normal subgroup of H. Prove that

$$\phi^{-1}(N) = \{g \in G \mid \phi(g) \in N\}$$

is a normal subgroup of G.

Update: The hypothesis that ϕ is surjective is not needed.

Question 3: Suppose that G and G' are groups, let H, K be subgroups of G, and let H', K' be subgroups of G'.

Assume that:

- H normalizes K and H' normalizes K'.
- $G = \langle K, H \rangle = KH$ and $G' = \langle K', H' \rangle = K'H'$.
- $K \cap H = \{1\}$ and $K' \cap H' = \{1\}$.
- There are group isomorphisms $\phi : H \to \tilde{H}'$ and $\psi : K \to \tilde{K}'$. Since H normalizes K, for $h \in H$ we know that the restriction of Inn_h to K determines an automorphism of K; similarly, for $h' \in H'$, $\mathrm{Inn}_{h'}$ determines an automorphism of K'.

We finally suppose that for $h \in H$ and $k \in K$ we have

$$\psi(\operatorname{Inn}_h k) = \operatorname{Inn}_{\phi(h)} \psi(k).$$

Then there is a group isomorphism $\Phi : G \to G'$ given for $(k,h) \in KH = G$ by the rule

$$\Phi(k,h) = \psi(k)\phi(h) \in K'H' = G'.$$

Update: I really should have written that $\Phi : G \to G'$ is defined by the rule

$$\Phi(kh) = \psi(k)\phi(h) \ \in K'H' = G' \text{ for } kh \in KH = G.$$

Note that under the hypotheses, G may be identified as a set with the direct product $H \times K$ - that is why I wrote (k,h) for an element of G.

Question 4: For a prime number p, write $FF_p = \mathbb{Z}/p\mathbb{Z}$ and let

$$H_p = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a,b,c \in FF_p \right\}$$

so that H_p is a subgroup of $\mathrm{GL}_3(FF_p)$ of order p^3. (You should at least think through why this is so, though you needn't submit the details).

- Prove that H_2 is isomorphic to $D_8 = D_{2\cdot 4}$, the dihedral group with 8 elements.

  Hint: Find $\sigma, \tau \in H_2$ with $o(\sigma) = 4$, $o(\tau) = 2$ which have the property that $\tau\sigma\tau = \sigma^{-1}$. Then $H_2 = \langle\sigma\rangle \cdot \langle\tau\rangle$. Now use the solution to Question 3.

- Show that H_p is a p-Sylow subgroup of $\mathrm{GL}_3(FF_p)$.

Question 5: Let G be a finite group, let p be a prime number, and let $P \in \mathrm{Syl}_p(G)$. Let $H = N_G(P) = \{g \in G \mid \operatorname{Inn}_g P = P\}$ be the normalizer of P in G. Prove that $N_G(H) = H$. (In words: the normalizer of a Sylow p-subgroup is self-normalizing).

Question 6: Suppose that F is a field.

- Show that the ideals of F are $\{0\}$ and F.
- Deduce that if R is any commutative ring (with $0_R \neq 1_R$), then any homomorphism $\phi : F \to R$ is injective.

Question 7: Let $D \in \mathbb{Z}$ and suppose that D is square-free - i.e., for any prime number p, $p^2 \nmid D$.

If $D \equiv 1 \pmod 4$ let $\omega = \frac{1+\sqrt{D}}{2}$ and show that

$$\mathbb{Z}[\omega] = \{a + b\omega \mid a,b \in \mathbb{Z}\}$$

forms a subring of $\mathbb{C}$.