Graduate Algebra

Definition 2.3.1

Contents

1. Week 1 [2025-09-03]

We'll begin by recalling some basic sorts of algebra that you more-or-less encountered before.

1.1. Notations and recollections

We reserve the following letters:

- $\mathbb{N}$ for the set of *natural numbers* $0, 1, 2, \ldots$
- $\mathbb{Z}$ for the set of *integers*, i.e. for all $\pm n$ for $n \in \mathbb{N}$
- $\mathbb{Q}$ for the set of *rational numbers* m/n for $m, n \in \mathbb{Z}$ with $n \neq 0$
- $\mathbb{R}$ for the set of *real numbers*, and
- $\mathbb{C}$ for the set of *complex numbers* $a + bi$ for $a, b \in \mathbb{R}$.

In this first lecture, I want to recall some of the main objects of study in algebra, including: groups, rings and fields. Ultimately, the goal today is to prove an analogue of Cayley's Theorem - see Theorem 1.6.1 and Theorem 1.7.1 about embedding arbitrary groups in some standard groups.

1.2. Groups

Recall that a group is a set G together with a binary operation $\cdot : G \times G \to G$ satifsying the following:

- associativity: $\forall x, y, z \in G, (xy)z = x(yz)$
- identity: $\exists e \in G, xe = ex = x$.
- inverses: $\forall x \in G, \exists y \in G, xy = yx = 1$.

Remark 1.2.1:

a. We usually write 1 or sometimes 1_G rather than e for the inverse element of G. + we usually write x^{-1} for the inverse of $x \in G$

b. there are *uniqueness* results that I'm eliding here; the identity 1 of G is unique, and the inverse x^{-1} of an element is unique. These statements are *consequences* of the above axioms (they don't require additional assumption.)

c. A group is abelian if $\forall a, b \in G, ab = ba$

d. Sometimes we write groups additively; in that case, 0 is the identity element and and the inverse of $a \in G$ is $-a \in G$. We always insist that additive groups are abelian.

Definition 1.2.2: For groups G and H, a function $\varphi : G \to H$ is a group homomorphism provided that $\forall x, y \in G, \varphi(xy) = \varphi(x)\varphi(y)$.

Definition 1.2.3: Let $\varphi : G \to H$ be a group homomorphism. The kernel of φ is

$$\ker \varphi = \{g \in G \mid \varphi(g) = 1.\}$$

Remark 1.2.4: If $\varphi : G \to H$ is a group homomorphism, $\ker \varphi$ is a subgroup of G – i.e. $\ker \varphi$ is non-empty, and is closed under multiplication and under taking inverses.

Proposition 1.2.5: Let $\varphi : G \to H$ be a group homomorphism. Then φ is an injective (or one-to-one) function if and only if $\ker \varphi = \{1_G\}$.

1.3. Rings

Definition 1.3.1: A ring is an additive abelian group R together with a binary operation of multiplication

$$\cdot : R \times R \to R$$

which satisfies the following:

- multiplication is associative: $\forall a, b, c \in R, (ab)c = a(bc)$.
- there is a multiplicative identity: $\exists 1 \in R, \forall a \in R, 1a = a1 = a$.
- distribution laws: $\forall a, b, c \in R, a(b+c) = ab + ac$ and $(b+c)a = ba + ca$.

The ring R is commutative provided that $\forall a, b \in R, ab = ba$.

Example 1.3.2:

a. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are commutative rings.
b. For a natural number $n > 1$, the ring $\mathrm{Mat}_n(\mathbb{Z})$ of $n \times n$ matrices with coefficients in $\mathbb{Z}$ is a non-commutative ring.

Definition 1.3.3: For a commutative ring R, an element $a \in R$ is a unit provided that $\exists v \in R, uv = vu = 1$.

The set $R^\times$ of units in R is a group under the multiplication of R.

1.4. Fields

Definition 1.4.1: A field is a commutative ring F such that $\forall a \in F, a \neq 0 \Rightarrow a$ is a unit.

Example 1.4.2: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields, but $\mathbb{Z}$ is not a field.

1.5. Linear Algebra

Definition 1.5.1: If F is a field, a vector space over F – or an F-vector space – is an additive abelian group V together with an operation of scalar multiplication

$$F \times V \to V$$

written $(t, v) \mapsto tv$, subject to the following:

- identity: $\forall v \in V, 1v = v$.
- associativity: $\forall a, b \in F$ and $v \in V$, $a(bv) = (ab)v$.
- distributive laws: $\forall a, b \in F$ and $v, w \in V$, $(a+b)v = av + bv$ and $a(v+w) = av + aw$.

Remark 1.5.2: Probably in a linear algebra class you saw results stated for vector spaces over $\mathbb{R}$ or $\mathbb{C}$; however, "most" results in linear algebra remain valid for vector space over F.

Example 1.5.3: Let I be any set, and let V be the set of all functions $f : I \to F$ which have finite support. Recall that the support of f is $\{x \in I \mid f(x) \neq 0\}$.

Then V is a vector space. (The addition and scalar multiplication operations are define "pointwise" – see homework.)

Remark 1.5.4: Recall that a basis of a vector space is subset B of V which is linearly indepent and spans V.

The vector space of finitely supported functions $I \to F$ has a basis $B = \{\delta_i \mid i \in I\}$, where

$$\delta_i : I \to F$$

is the function defined by $\delta_i(j) = 0$ if $i \neq j$ and $\delta_i(i) = 1$.

Definition 1.5.5: If V and W are F-vector spaces, an F-linear map $\varphi : V \to W$ is a homomorphism of additive groups which satisfies the condition

$$\forall t \in F, \forall v \in V, \varphi(tv) = t\varphi(v).$$

Definition 1.5.6: If V is an F-vector space, the general linear group $\mathrm{GL}(V)$ is the set

$$\{\varphi : V \to V \mid \varphi \text{ is } F\text{-linear and invertible.}\}$$

$\mathrm{GL}(V)$ is a group whose operation is given by composition of linear transformations.

Remark 1.5.7: If V is finite dimensional, so that V is isomorphic to F^n as F-vector spaces, linear algebra shows that $\mathrm{GL}(V)$ is isomorphic to the group GL_n of $n \times n$ matrices with non-zero determinant, where $n = \dim_F V$ and where the operation in GL_n is given by matrix multiplication.

1.6. Cayley's Theorem

Let Ω be any set. The set $S(\Omega)$ of all bijective functions $\psi : \Omega \to \Omega$ is a group whose operation is composition of functions.

Theorem 1.6.1 (Cayley's Theorem): Let G be any group. Then G is isomorphic to a subgroup of $S(\Omega)$ for some Ω.

Proof: Let $\Omega = G$. For $g \in G$, define a mapping $\lambda_g : G \to G$ by the rule

$$\lambda_g(h) = gh.$$

We are going to argue that the mapping $g \mapsto \lambda_g$ defines an injective group homomorphism $G \to S(\Omega) = S(G)$.

First of all, we note that $\lambda_1 = \mathrm{id}$. Indeed, to check this identity of functions, let $h \in \Omega = G$. Then

$$\lambda_1(h) = 1h = h = \mathrm{id}(h);$$

this confirms $\lambda_1 = \mathrm{id}$.

Next, we note that for $g_1, g_2 \in G$, we have $(*) \quad \lambda_{g_1} \circ \lambda g_2 = \lambda_{g_1 g_2}$. Again, to confirm this identify of functions, we let $h \in \Omega = G$. Then

$$\big(\lambda_{g_1} \circ \lambda_{g_2}\big)h = \lambda_{g_1}\big(\lambda_{g_2}(h)\big) = \lambda_{g_1}(g_2 h) = g_1(g_2 h) = (g_1 g_2)h = \lambda_{g_1 g_2}(h)$$

as required.

Now, using $(*)$ we see for $g \in G$ that $\lambda_g \circ \lambda_{g^{-1}} = \lambda_1 = \mathrm{id} = \lambda_{g^{-1}} \circ \lambda_g$, which proves that λ_g is bijective; thus indeed $\lambda_g \in S(\Omega) = S(G)$.

Moreover, $(*)$ shows that the mapping $\lambda : G \to S(G)$ given by $g \mapsto \lambda_g$ is a group homomorphism.

It remains to see that λ is injective. If $g \in \ker \lambda$, then $\lambda_g = \mathrm{id}$. Thus $1 = \mathrm{id}(1) = \lambda_g(1) = g1 = g$. Thus $g = 1$ so that $\ker \lambda = \{1\}$ which confirms that λ is injective by Proposition 1.2.5. This completes the proof. ■

1.7. A linear analogue of Cayley's Theorem.

Let F be a field.

Theorem 1.7.1: Let G be any group. Then G is isomorphic to a subgroup of $\mathrm{GL}(V)$ for some F-vector space V.

Proof: The proof is quite similar to the proof of Cayley's Theorem.

Let V be the vector space of all finitely supported functions $f : G \to F$. Recall that V has a basis $B = \{\delta_g \mid g \in G.\}$

We are going to define an injective group homomorphism $G \to \mathrm{GL}(V)$.

For $g \in G$, we may define an F-linear mapping $\lambda_g : V \to V$ by defining the value of λ_g at each vector in B. We set $\lambda_g(\delta_h) = \delta_{gh}$.

Recall that a typical element v of V has the form

$$v = \sum_{i=1}^{n} t_i \delta_{h_i}$$

for scalars $t_i \in F$ and elements $g_i \in G$; since λ_g is F-linear, we have

$$\lambda_g(v) = \sum_{i=1}^{n} t_i \delta_{g h_i}.$$

We now show that $\lambda_1 = \mathrm{id}$. To prove this, since the functions $V \to V$ are linear, it is enough to argue that the functions agree at each element of the basis B of V. Well, for $h \in G$,

$$\lambda_1(\delta_h) = \delta_{1h} = \delta_h = \mathrm{id}(\delta_h)$$

as required.

We next show for $g_1, g_2 \in G$ that $(*) \quad \lambda_{g_1} \circ \lambda_{g_2} = \lambda_{g_1 g_2}$. Again, it suffices to argue that these functions agree at each element δ_h of B. For $h \in G$ we have:

$$\big(\lambda_{g_1} \circ \lambda_{g_2}\big)(\delta_h) = \lambda_{g_1}\big(\lambda_{g_2}\delta_h\big) = \lambda_{g_1}\big(\delta_{g_2 h}\big) = \delta_{g_1(g_2 h)} = \delta_{(g_1 g_2)h} = \lambda_{g_1 g_2}\delta_h$$

as required.

Now, for $g \in G$ we see that by $(*)$ that

$$\mathrm{id} = \lambda_1 = \lambda_g \circ \lambda_{g^{-1}}$$

which proves that λ_g is invertible and hence in $\mathrm{GL}(V)$.

Moreover, $(*)$ shows that the assignment $\lambda : G \to \mathrm{GL}(V)$ given by the rule $g \mapsto \lambda_g$ is a group homomorphism.

It remains to argue that λ is injective. Suppose that $x \in \ker \lambda$, so that $\mathrm{id} = \lambda_x$.

Then $\delta_1 = \mathrm{id}(\delta_1) = \lambda_x(\delta_1) = \delta_{x1} = \delta_x$. This implies that $1 = x$ so that indeed the kernel of λ is trivial and thus λ is injective by Proposition 1.2.5.

■

2. Week 2 [2025-09-08]

This week, we'll discuss quotients, and we'll begin our discussion of group actions.

2.1. The Quotient of a set by an equivalence relation

Let S be a set and let R be a relation on S. Formally, R is an assignment $R : S \times S \to \text{Prop}$ – in other words, for $a, b \in S$, $R(a,b)$ is the proposition that a and b are related; of course $R(a,b)$ may or may not hold.

We often use a symbol $\sim$ or $\underset{R}{\sim}$ to indicate this proposition; thus $R(a,b) \Leftrightarrow a \underset{R}{\sim} b$.

Definition 2.1.1: The relation $\sim$ is an equivalence relation if the following properties hold:

- reflexive: $\forall s \in S, s \sim s$.
- symmetric: $\forall s_1, s_2 \in S, s_1 \sim s_2 \Rightarrow s_2 \sim s_1$
- transitive: $\forall s_1, s_2, s_3 \in S, s_1 \sim s_2$ and $s_2 \sim s_3 \Rightarrow s_1 \sim s_3$

Definition 2.1.2: If $\sim$ is an equivalence relation on the set S, a quotient of S by $\sim$ is a set $\bar{S}$ together with a surjective function $\pi : S \to \bar{S}$ with the following properties:

(Quot 1) $\forall a, b \in S, a \sim b \Rightarrow \pi(a) = \pi(b)$

(Quot 2) Let T be any set and let f be any function $f : S \to T$ such that $\forall a, b \in S, a \sim b \Rightarrow f(a) = f(b)$. Then there is a function $\bar{f} : S \to T$ for which $f = \bar{f} \circ \pi$.

Proposition 2.1.3: Suppose that $\left(\bar{S}_1, \pi_1\right)$ and $\left(\bar{S}_2, \pi_2\right)$ are two quotients of the set S by the equivalence relation $\sim$. Let

$$\bar{\pi_2} : \bar{S}_1 \to \bar{S}_2$$

be the mapping determined by the quotient property for $\left(\bar{S}_1, \pi_1\right)$ using

$$T = \bar{S}_2 \quad \text{and} \quad f = \pi_2 : S \to \bar{S}_2,$$

and let

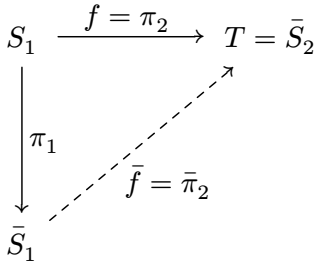
$$\bar{\pi}_1 : \bar{S}_2 \to \bar{S}_1$$

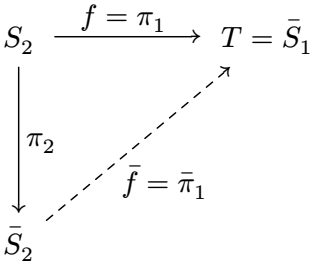
be the mapping determined by the quotient property for $\left(\bar{S}_2, \pi_2\right)$ using

$$T = \bar{S}_1 \quad \text{and} \quad f = \pi_1 : S \to \bar{S}_2.$$

Then the maps $\pi_2' : \bar{S}_1 \to \bar{S}_2$ and $\pi_{1'} : \bar{S}_2 \to \bar{S}_1$ are inverse to one another, and in particular π_1' and π_2' are bijections.

Proof: By the definition of quotients, we have commutative diagrams





In particular, we have $\pi_2 = \bar{\pi}_2 \circ \pi_1$ and $\pi_1 = \bar{\pi}_1 \circ \pi_2$

Substitution now yields

$$\pi_1 = \bar{\pi}_1 \circ \bar{\pi}_2 \circ \pi_1$$

and

$$\pi_2 = \bar{\pi}_2 \circ \bar{\pi}_1 \circ \pi_2$$

Since π_1 and π_2 are surjective, we conclude that $\mathrm{id} = \bar{\pi}_1 \circ \bar{\pi}_2$ and $\mathrm{id} = \bar{\pi}_2 \circ \bar{\pi}_1$ so indeed the indicated functions are inverse to one another. ■

Remark 2.1.4: The point of the Proposition is that a quotient is completely determined by the property indicated in the definition – this property is an example of what is known as a universal property or sometimes as a universal mapping property. The conclusion of the Proposition shows that any two ways of constructing a quotient are equivalent in a strong sense.

One way of constructing the quotient is by considering equivalence classes, as follows:

Definition 2.1.5: For an equivalence relation $\sim$ on a set S, the equivalence class $[s]$ of an element $s \in S$ is the subset of S defined by

$$[s] = \{x \in S \mid x \sim s\}.$$

Proposition 2.1.6: Equivalence classes for the equivalence relation $\sim$ have the following properties for arbitrary $s, s' \in S$:

a. $s \sim s' \Leftrightarrow [s] = [s']$
b. $[s] \neq [s'] \Leftrightarrow [s] \cap [s'] = \emptyset$

Proof: Review! ■

Theorem 2.1.7 (*Existence of quotients*): For any equivalence relation $\sim$ on a set S, there is a quotient $\left(\bar{S}, \pi\right)$.

Proof: We consider the set $\bar{S} = \{[s] \mid s \in S\}$ of equivalence classes and the mapping $\pi : S \to \bar{S}$ given by the rule $\pi(s) = [s]$.

Proposition 2.1.6 confirms condition (a) of Definition 2.1.2.

For condition (b) of Definition 2.1.2 suppose that T is a set and that $f : S \to T$ is a function with the property that $\forall a, b \in S, a \sim b \Rightarrow f(a) = f(b)$. We must exhibit a function $\bar{f} : \bar{S} \to T$ with the property $f = \bar{f} \circ \pi$. If $\bar{f}$ exists, it must satisfy $\bar{f}([a]) = a$ for $a \in S$. On the other hand, in view of Proposition 2.1.6 (a), the rule $[a] \mapsto f(a)$ indeed determines a well-defined function $\bar{f} : \bar{S} \to T$. Moreover, the identity $f = \bar{f} \circ \pi$ evidently holds.

■

Remark 2.1.8: We gave an explicit construction of the quotient using equivalence classes. On the other hand, if one has a quotient $\left(\bar{S}, \pi\right)$, the equivalence class $[x]$ of an element $x \in S$ is equal to $\pi^{-1}(\pi(x))$.

Proposition 2.1.9: If $\sim$ is an equivalence relation on the set S, then S is the disjoint union of the equivalence classes.

Proof: Each element $x \in S$ is contained in the equivalence classes $[x]$, so it only remains to prove that if two equivalence classes have a common element, they are equal. For this, let $x, y \in S$ and suppose that $z \in [x] \cap [y]$. Then $x \sim z$ and $y \sim z$ so that $x \sim y$ by transitivity; thus $[x] = [y]$.

■

2.2. Sub-groups

Let G be a group (when giving definitions, we'll write G multiplicatively).

Definition 2.2.1: A subgroup of G is a non-empty subset $H \subseteq G$ such that H is closed under the operations of multiplication in G and inversion in G. In other words,

$$\forall a, b \in G, ab \in H \text{ and } a^{-1} \in H$$

.

Example 2.2.2: Consider the group $G = \mathbb{Z} \times \mathbb{Z}$ where the operation is componentwise addition. Check the following!

a. $H_1 = \{(a, b) \in G \mid 2a + 3b = 0\}$ is a subgroup.
b. $H_2 = \{n(2, 2) + m(1, 2) \mid n, m \in \mathbb{Z}\}$ is a subgroup.

The collection of subgroups of G has a natural partial order given by *containment.*

Proposition 2.2.3: (Constructing subgroups)

a. If H_i for $i \in I$ is a family of subgroups of G, indexed by some set I, then the intersection $\bigcap_{i \in I} H_i$ is again a subgroup of G.
b. Let $S \subseteq G$ be a subset. There is a unique smallest subgroup $H(S) = \langle S \rangle$ containing S. In other words, for any subgroup H' of G with $S \subseteq H'$, we have $\langle S \rangle \subseteq H'$.

Remark 2.2.4:

a. If $S, T \subseteq G$ are subsets, we often write $\langle S, T \rangle$ for $\langle S \cup T \rangle$. If $S = \{s_1, s_2, ..., s_n\}$ we often write $\langle S \rangle = \langle s_1, s_2, ..., s_n \rangle$.

b. The subgroup in Example 2.2.2(b) is precisely $\langle (2,2),(1,2)\rangle$.
c. For any group G and $a \in G$, $\langle a\rangle := \langle \{a\}\rangle$ is the cyclic subgroup generated by a. If G is multiplicative, then $\langle a\rangle = \{a^n \mid n \in Z\}$ while if G is addtive then $\langle a\rangle = \{na \mid n \in Z\}$.

Proposition 2.2.5: If $\varphi : G \to H$ is a group homomorphism, then $\ker \varphi$ is a subgroup of G.

Proposition 2.2.6: If $X \subseteq G$ is a non-empty subset of G, then X is a subgroup if and only if
$(*)$ $\quad \forall a, b \in X, ab^{-1} \in X$.

Proof: ($\Rightarrow$): Immediate from the definition of a subgroup.

($\Leftarrow$): Assume that $(*)$ holds. We must show that X is a subgroup.

We first argue that X contains the identity element. Since X is non-empty, there is an element $x \in X$. Condition $(*)$ then shows that $xx^{-1} = 1 \in X$ as required.

We now show that X is closed under inversion. Let $x \in X$. Since $1 \in X$, we apply $(*)$ with $a = 1$ and $b = x$ to learn that $1x^{-1} = x^{-1} \in X$, as required.

Finally, we show that X is closed under multiplication. Let $x, y \in X$. We have already seen that $y^{-1} \in X$. Now apply $(*)$ with $a = x$ and $b = y^{-1}$ to learn that

$$ab^{-1} = x(y^{-1})^{-1} = xy \in X$$

as required. ■

Proposition 2.2.7: Let $f : G \to H$ be a group homomorphism and let $S \subseteq G$ be any subset. Then

$$f(\langle S\rangle) \subseteq \langle f(S)\rangle.$$

Proof: Since f is a homomorphism, for any subgroup $K \subseteq G$, the image

$$f(K) = \{f(x) \mid x \in K\}$$

is a subgroup of H. Thus $f(\langle S\rangle)$ is a subgroup containing $\langle f(S)\rangle$ of H where $f(S)$ is the image of the set S via the function f. It now follows from Proposition 2.2.3 that $f(\langle S\rangle)$ is contained in the subgroup $\langle f(S)\rangle$ generated by $f(S)$, as required. ■

2.3. Group actions

Definition 2.3.1: Let G be a group and let Ω be a set. An action of G on Ω is a mapping

$$G \times \Omega \to \Omega \text{ written } (g, x) \mapsto gx$$

such that for each $x \in \Omega$ we have

- $1x = x$
- $\forall g, h \in G, (gh)x = g(hx)$.

For brevity, sometimes we say that Ω is a G-space.

Proposition 2.3.2: An action of a group G on a set Ω determines a homomorphism $f : G \to S(\Omega)$ such that $f(g)(x) = gx$ for $g \in G$ and $x \in \Omega$.

Conversely, given a homomorphism $f : G \to S(\Omega)$, there is an action of G on Ω given by $gx = f(g)(x)$ for each $g \in G$ and $x \in \Omega$.

Definition 2.3.3: Suppose that Ω is a G-space. The G-conjugacy relation on Ω is defined as follows: for $x, y \in \Omega$, $x \underset{G}{\sim} y$ provided that $\exists g \in G, gx = y$.

Proposition 2.3.4: The G-conjugacy relation on Ω is an equivalence relation.

Definition 2.3.5: Let Ω be a G-space, and let $\varphi : \Omega \to \Omega/\sim$ be the quotient mapping for the G-conjugacy relation; see Definition 2.1.2. For $x \in \Omega$, the orbit $\mathcal{O}_x = Gx$ of G through x is the subset of Ω defined by

$$\mathcal{O}_x = \varphi^{-1}(\varphi(x)).$$

Thus the G-orbits are the equivalence classes for the relation $\underset{G}{\sim}$; see Remark 2.1.8.

Equivalently, we have $\mathcal{O}_x = \{gx \mid g \in G.\}$

Proposition 2.3.6: Ω is the disjoint union of the G-orbits in Ω.

Proof: This follows from Proposition 2.1.9. ■

Remark 2.3.7: Each orbit $\mathcal{O}_x$ is itself a G-set.

2.4. Quotients of groups

Let G be a group and let H be a subgroup of G. There is an action of H on the set G by right multiplication: for $h \in H$ and $g \in G$ we can define $h \cdot g = gh^{-1}$.

We are going to consider the quotient of G by the equivalence relation of H-conjugacy; this equivalence relation is defined by

$$g \sim g' \Leftrightarrow \exists h \in H, g = g'h.$$

Definition 2.4.1: The left quotient of G by H is the quotient $(\pi, G/H)$ of G by the equivalence relation of H-conjugacy defined using the action of H on G by right multiplication as described above.

Remark 2.4.2:

a. Of course, one can use an explicit model for the quotient by taking G/H to be the set of equivalence classes in G for the H-conjugacy relation.

b. The equivalence classes for the relation of H-conjugacy defined by the action of right multiplication are precisely the left cosets of H in G. The class of $x \in G$ has the form

$$xH = \{xh \mid h \in H\}$$

.

For $x \in G$,

$$\pi^{-1}(\pi(x)) = xH.$$

c. We can also consider the action of H on G by left multiplication. This action determines an equivalence relation of H-conjugacy, and the quotient of G by this equivalence relation is called the right quotient of G by H and is written $(\pi, H \setminus G)$. In this case, the equivalence classes are the right cosets where the class of $x \in G$ has the form $Hx = \{hx \mid h \in H\}$.

For $x \in G$, we have $\pi^{-1}(\pi(x)) = Hx$.

Proposition 2.4.3: There is an action

$$\alpha : G \times G/H \to G/H$$

of the group G on the set G/H such that

$$\forall g, x \in G, \text{we have } \alpha(g, \pi(x)) = \pi(gx)$$

where $\pi : G \to G/H$ is the quotient map.

Proof: To define the action map α, first fix $g \in G$. We are going to define the mapping

$$\alpha(g, -) : G/H \to G/H.$$

Consider the mapping $\pi_g : G \to G/H$ given by the rule $\pi_g(x) = \pi(gx)$. This mapping has the property that $x \underset{H}{\sim} x' \Rightarrow \pi_g(x) = \pi_{g(x')}$. Indeed,

$$x \underset{H}{\sim} x' \Rightarrow \exists h, x = x'h \Rightarrow \pi_g(x) = \pi(gx) = \pi(gx'h) = \pi(gx') = \pi_g(x')$$

by the defining property of π; see Definition 2.1.2. Again using Definition 2.1.2 we find the desired mapping $\alpha(g, -) : G/H \to G/H$ with the property that

$$(\clubsuit) \quad \alpha(g, -) \circ \pi = \pi_g.$$

We now assemble the mappings $\alpha(g,-)$ to get a mapping $\alpha : G \times G/H \to G/H$ which satisfies $\alpha(g,\pi(x)) = \pi(gx)$ for each $g, x \in G$, and it remains to check that α determines an action as in Definition 2.3.1.

Of course, using (♣), we have $\alpha(1,-) \circ \pi = \pi_1 = \pi = \mathrm{id} \circ \pi$; since π is surjective, it follows that $\alpha(1,-) = \mathrm{id}$. Thus $\alpha(1,z) = z$ for each $z \in G/H$, which shows that α satsifies the first requirement of Definition 2.3.1.

Now suppose that $g_1, g_2 \in G$. To complete the proof, we must veryify the remaining requirement of Definition 2.3.1; thus we must show that

$$(\heartsuit) \quad \alpha(g_1, \alpha(g_2,-)) = \alpha(g_1 g_2, -)$$

On the one hand, using (♣) we find that

$$\alpha(g_1 g_2, -) \circ \pi = \pi_{g_1 g_2};$$

on the other hand, for $z \in G$ we have

$$\begin{aligned}(\alpha(g_1,\alpha(g_2,-)) \circ \pi)(z) &= \alpha(g_1, \alpha(g_2, \pi(z))) \\ &= \alpha\Big(g_1, \pi_{g_2}(z)\Big) \qquad \text{by } (\clubsuit) \\ &= \alpha(g_1, \pi(g_2 z)) \\ &= \pi(g_1(g_2 z)) \qquad \text{by } (\clubsuit)\end{aligned}$$

Since π is surjective, $(\heartsuit)$ follows at once. This completes the proof.

■

2.5. Quotients of groups and orbits.

Definition 2.5.1: Suppose that G acts on Ω_1 and on Ω_2. A morphism of G-sets $\varphi : \Omega_1 \to \Omega_2$ is a function φ with the property that $\forall g \in G$ and $\forall x \in \Omega_1$, we have $\varphi(gx) = g\varphi(x)$.

The morphism of G-sets φ is an isomorphism (of G-sets) if there is a morphism of G-sets $\psi : \Omega_2 \to \Omega_1$ such that $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$.

Suppose that G acts on Ω and let $x \in \Omega$.

Definition 2.5.2: The stabilizer of x in G is the subgroup $\mathrm{Stab}_G(x) = \{g \in G \mid gx = x\}$.

Proposition 2.5.3: Write $H = \mathrm{Stab}_G(x)$ and recall that $\pi : G \to G/H$ is the quotient mapping. There is a unique isomorphism of G-sets $\gamma : G/H \to \mathcal{O}_x$ with the property that

$$\gamma(\pi(1)) = x.$$

Proof: The rule $g \mapsto gx$ determines a surjective mapping $\alpha_x : G \to \mathcal{O}_x$. Recall that the action of H on G by right multiplication determines an equivalence relation $\sim$ on G used to construct the quotient G/H.

For $g_1, g_2 \in G$ we find that

$$g_1 \sim g_2 \Rightarrow \exists h \in H, g_1 = g_2 h \Rightarrow \alpha(g_1) = \alpha(g_2 h) = g_2 h x = g_2 x = \alpha(g_2)$$

since $h \in H = \mathrm{Stab}_G(x) \Rightarrow hx = x$.

Thus Definition 2.1.2 shows that there is a mapping $\gamma : G/H \to \mathcal{O}_x$ such that $\gamma \circ \pi = \alpha_x$. To see that γ is a morhpism of G-sets, it suffices to show that (♣) $\forall g, g'$ we have

$$\gamma(g \cdot \pi(g')) = g \cdot \gamma(\pi(g')).$$

Now by the definition of the G-action on G/H we have $g \cdot \pi(g') = \pi(gg')$; see Proposition 2.4.3. Thus $\gamma(g \cdot \pi(g')) = \gamma(\pi(gg')) = \alpha_x(gg') = gg' \cdot x$. On the other hand, $g \cdot \gamma(\pi(g')) = g \cdot \alpha_{x(g')} = g \cdot g' \cdot x$ which confirms (♣). This shows that γ is indeed a morphism of G-sets.

Since α_x is surjective and $\gamma \circ \pi = \alpha_x$, also γ is surjective. It only remains to see that γ is injective. Suppose that $z, z' \in G/H$ such that $\gamma(z) = \gamma(z')$. Since $\pi : G \to G/H$ is surjective, we may choose $g, g' \in G$ with $z = \pi(g)$ and $z' = \pi(g')$. Now

$$\gamma(z) = \gamma(z') \Rightarrow \gamma(\pi(g)) = \gamma(\pi(g')) \Rightarrow \alpha_{x(g)} = \alpha_{x(g')} \Rightarrow gx = g'x.`$$

We now conclude that $g^{-1}gx = x$ so that $g^{-1}g \in \mathrm{Stab}_G(x) = H$. Since the quotient mapping π is constant on H-orbits, $z = \pi(g) = \pi(gg^{-1}g') = \pi(g') = z'$. This shows that γ is injective and completes the proof. ■

Definition 2.5.4: The action of G on Ω is transitive if there is a single G-orbit on Ω. Equivalently, the action is transitive if the quotient $\Omega/\sim$ is a singleton set.

Example 2.5.5: Let I be a set and let $G = S(I)$ be the group of permutations of I. Fix $x \in I$ and let $H = \mathrm{Stab}_G(x)$. Notice that G acts on I. Moreover, the G-orbit of x is precisely I - in other words, the action of G on I is transitive.

Notice that $H = S(I - \{x\})$.

Now Proposition 2.5.3 gives an isomorphism of G-sets $G/H \to I$; i.e. $S(I)/S(I - \{x\}) \to I$.

2.6. The product of subgroups

Definition 2.6.1: If $H, K \subseteq G$ are two subgroups, then H normalizes K if for each $g \in H$ we have $\mathrm{Inn}_g K \subseteq K$ (in other words, $\forall x \in K$, $gxg^{-1} \in K$).

Definition 2.6.2: Let H, K be subsets of G. The product of H and K is the subset

$$HK := \{xy \mid x \in H, y \in K\}$$

.

Proposition 2.6.3: Suppose that H, K are subgroups of G and that H normalizes K. Then $\langle H, K\rangle = HK$. In particular, HK a subgroup of G.

Proof: Let $X = HK$. Since any subgroup of G which contains both H and K clearly contains X, it only remains to argue that X is a subgroup. For this, we use Proposition 2.2.6. First note that $1 = 1 \cdot 1 \in X$, so X is non-empty. Now, let $a_1, b_2 \in X$. We must argue that $a_1 a_2^{-1} \in X$. By definition, there are elements $x_1, x_2 \in H$ and y_1, y_2 in K with $a_i = x_i y_i$ for $i = 1, 2$. We now compute

$$a_1 a_2^{-1} = x_1 y_1 (x_2 y_2)^{-1} = x_1 y_1 y_2^{-1} x_2^{-1} = (x_1 x_2^{-1}) \cdot (x_2 y_1 y_2^{-1} x_2^{-1}).$$

We notice that $x_1 x_2^{-1} \in H$. Moreover, $y_1 y_2^{-1} \in K$; since H normalizes K it follows that $x_2 y_1 y_2^{-1} x_2^{-1} \in K$.

We have now argued that $a_1 a_2^{-1}$ has the form xy for $x \in H$ and $y \in K$ so that $a_1 a_2^{-1} \in X$. Now Proposition 2.2.6 indeed shows that $X = HK$ is a subgroup. ■

Proposition 2.6.4: Let H, K be subgroups of G and let $\varphi : H \times K \to HK$ be the natural mapping given by $\varphi(h, k) = hk$.

a. For each $\alpha \in HK$, the set $\varphi^{-1}(\alpha)$ is in bijection with $H \cap K$.
b. In particular, if $H \cap K = \{1\}$, then φ is bijective.

Proof: Let $\alpha = hk \in HK$. Note for any $x \in H \cap K$ that $\varphi(hx, x^{-1}k) = \alpha$ so that $(hx, x^{-1}k) \in \varphi^{-1}(\alpha)$. We argue that the mapping

$$\gamma : H \cap K \to \varphi^{-1}(\alpha) \text{ given by } \gamma(x) = (hx, x^{-1}k)$$

is bijective. Well, if $(h_1, k_1) \in \varphi^{-1}(\alpha)$ then $\varphi(h_1, k_1) = \varphi(h, k)$ so that $h_1 k_1 = hk$ and thus $h^{-1} h_1 = k k_1^{-1}$. Now set $x = h^{-1} h_1 = k k_1^{-1} \in H \cap K$ and observe that $(h_1, k_1) = \gamma(x)$. This shows that γ is surjective. To see that γ is injective, suppose that $\gamma(x) = \gamma(x')$ for $x \in H \cap K$. Then

$$(hx, x^{-1}k) = (hx', x'^{-1}k) \Rightarrow hx = hx' \Rightarrow x = x'.$$

So γ is injective and the proof of a. is complete.

Now, the mapping φ is surjective by the definition of HK. To prove b. we suppose that

$$H \cap K = \{1\}.$$

According to a. the fiber $\varphi^{-1}(\alpha)$ is a singleton for each $\alpha \in HK$; this shows that φ is injective and confirms b.

■

Corollary 2.6.5: If G is a finite group and H, K subgroups of G, then

$$|HK| = |H| \cdot |K| \ / \ |H \cap K|.$$

Proof: This is a consequence of Proposition 2.6.4. ■

Let's introduce some examples of groups in order to investigate this a bit more.

Example 2.6.6: For $n \in \mathbb{N}$ with $n \geq 1$, consider the symmetric group $S = S_n$ viewed as $S(\mathbb{Z}/n\mathbb{Z})$ where $\mathbb{Z}/n\mathbb{Z}$ denotes the collection of integers modulo n.

Consider the elements $\sigma, \tau \in S$ defined by the rules $\sigma(i) = i + 1$ and $\tau(i) = -i$ where the addition and negation occurs in $\mathbb{Z}/n\mathbb{Z}$.

Viewed as permutations, σ identifies with an n-cycle and τ identifies with a product of disjoint transpositions:

$$\sigma = (1, 2, ..., n) \text{ and } \tau = (1, n-1)(2, n-2)... = \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (i, n-i).$$

In particular, σ has order n and τ has order 2. Moreover,

$$(\heartsuit) \quad \tau\sigma\tau = \sigma^{-1}$$

Condition $(\heartsuit)$ shows that the subgroup $\langle \tau \rangle$ normalizes the subgroup $\langle \sigma \rangle$. Thus Proposition 2.6.3 shows that

$$\langle \sigma, \tau \rangle = \langle \sigma \rangle \langle \tau \rangle.$$

We call $D = \langle \sigma, \tau \rangle$ the dihedral group of order n. Note that $(\heartsuit)$ shows that $\langle \tau \rangle$ normalizes $\langle \sigma \rangle$ so that $D = \langle \tau \rangle \cdot \langle \sigma \rangle$.

We claim:

- $|D| = 2n$. In fact, D is usually written D_{2n}.

To prove the claim, we apply Corollary 2.6.5; we just need to argue that

$$(\clubsuit) \quad \langle \sigma \rangle \cap \langle \tau \rangle = \{1\}.$$

Since σ has order n and τ has order 2, $(\clubsuit)$ is immediate if n is odd.

Now suppose that $n = 2k$ is even. The unique subgroup of order 2 in $\langle \sigma \rangle$ is generated by σ^k. To prove $(\clubsuit)$ we must argue that $\sigma^k \neq \tau$.

Suppose the contrary. If $\sigma^k = \tau$ then $\sigma(n) = \tau(n) \in \mathbb{Z}/n\mathbb{Z}$. Since $\sigma^k(n) \equiv n + k (\mathtt{mod}\ n)$ while $\tau(n) = -n \equiv n (\mathtt{mod}\ n)$, we conclude that $n + k \equiv n (\mathtt{mod}\ n)$; thus $k \equiv 0 (\mathtt{mod}\ n)$ i.e. $2k \mid k$, which yields a contradiction as $k \geq 1$. This completes the proof $(\clubsuit)$.

2.7. Lagrange's Theorem

Let H be a subgroup of the group G and write G/H for the (left) quotient, as above. Recall that the H-cosets xH are the H-orbits for this action.

Theorem 2.7.1: There is a bijection $\varphi : (G/H) \times H \to G$ for which $\{\varphi(z, h) \mid h \in H\}$ is an H-orbit (i.e. a left H-coset) for each $z \in G/H$.

Proof: Indeed, using the axiom of choice we select for each $z \in G/H$ an element $g_z \in \pi^{-1}(z)$ where $\pi : G \to G/H$ is the quotient map.

Now define $\varphi : (G/H) \times H \to G$ by the rule $\varphi(z,h) = g_z h$.

To see that φ is onto, let $g \in G$. One then knows that $g \sim g_z$ for some $z \in G/H$. Since $\pi^{-1}(z) = g_z H$ it follows that $g = g_z h$ for some $h \in H$, so $g = \varphi(z,h)$.

To see that φ is injective, suppose that $\varphi(z,h) = \varphi(z',h')$. Then $g_z h = g_{z'} h'$ so that

$$(g_{z'})^{-1} g_z \in H \Rightarrow g_z \sim g_{z'} \Rightarrow z = z'.$$

Now $g_z h = g_z h' \Rightarrow h = h'$ which completes the proof that φ is injective. The remaining assertion follows from the definition of φ. ■

Corollary 2.7.2: Suppose that G is a finite group and that H is a subgroup of G. Then

$$|G| = |G/H| \cdot |H|.$$

Proof: Indeed, for finite sets X and Y, we have $|X \times Y| = |X|\ |Y|$. ■

3. Week 3 [2025-09-15]

3.1. Normal subgroups

Subgroups of the form $\ker \varphi$ have a property that ordinary subgroups might lack; in this section we describe this property.

Proposition 3.1.1: Let G be a group.

a. For $g \in G$, the assignment $x \mapsto gxg^{-1}$ determines a group isomorphism

$$\mathrm{Inn}_x : G \to G$$

b. The assignment $x \mapsto \mathrm{Inn}_x$ determines a group homomorphism $G \to \mathrm{Aut}(G)$ where $\mathrm{Aut}(G)$ is the group of *automorphisms* of G.

Proof sketch:

- First check that Inn_x is a group homomorphism.
- Then check that $(\blacklozenge)$ $\quad \mathrm{Inn}_x \circ \mathrm{Inn}_y = \mathrm{Inn}_{xy}$ for all $x, y \in G$.
- Next, check that $\mathrm{Inn}_1 = \mathrm{id}$. Using $(\blacklozenge)$, this shows that $(\mathrm{Inn}_x)^{-1} = \mathrm{Inn}_{x^{-1}}$ so indeed Inn_x is an *automorphism* of G.
- Finally, $(\blacklozenge)$ shows that Inn is a group homomorphism.

■

Definition 3.1.2: A subset $N \subseteq G$ is a normal subgroup of G if N is a subgroup of G and if for any $g \in G$ and for any $x \in N$, we have $gxg^{-1} \in N$.

Using earlier notation, a subgroup N is normal if $\forall g \in G, \mathrm{Inn}_g\, N \subseteq N$.

Remark 3.1.3: If N is a normal subgroup then for every $g \in G$ we have $\mathrm{Inn}_g\, N = N$.

Indeed, our assumption means for every g that $\mathrm{Inn}_g\, N \subseteq N$. Thus $\mathrm{Inn}_g^{-1} \circ \mathrm{Inn}_g\, N \subseteq \mathrm{Inn}_g^{-1} N$ so that $N \subseteq \mathrm{Inn}_g^{-1} N$. Since this holds for every g, we find that $\mathrm{Inn}_g\, N \subseteq N \subseteq \mathrm{Inn}_g\, N$ for every g; this confirms the assertion.

Proposition 3.1.4: Let H be a subgroup of G.

a. Suppose $G = \langle S \rangle$ for some subset $S \subseteq G$. Then H is normal in G if and only if $\mathrm{Inn}_x\, H = H$ for each $x \in S$.
b. If $H = \langle T \rangle$ for some subset $T \subseteq H$, then H is normal in G if and only if $\forall t \in T, \forall x \in G, \mathrm{Inn}_x\, t \in H$.

Proof:

a. $(\Rightarrow)$: This follows from the definition of normal subgroup.

($\Leftarrow$): Write $N = \{g \in G \mid \mathrm{Inn}_g H = H\}$ and check that N is a subgroup of G. It is clear that $H \subseteq N$ and by construction H is a normal subgroup of N. Now our assumption shows that $S \subseteq N$ so that $G = \langle S \rangle \subseteq N \Rightarrow N = G$ and thus H is normal in G.

b. ($\Rightarrow$): Again, this implication follows from the definition of normal subgroup.

($\Leftarrow$): Fix $x \in G$; we must argue that $\mathrm{Inn}_x H \subseteq H$. We know that Inn_x is a group homomorphism; see Proposition 3.1.1. It follows from Proposition 2.2.7

$$\mathrm{Inn}_x(\langle T \rangle) \subseteq \langle \mathrm{Inn}_x(T) \rangle$$

which indeed shows that $\mathrm{Inn}_x H \subseteq H$.

■

Proposition 3.1.5:

Let $N = \ker \varphi$ where $\varphi : G \to H$ is a group homomorphism. Then N is a normal subgroup of G.

Proof: We have already observed that N is a subgroup. Now let $g \in G$ and $x \in N$ so that $\varphi(x) = 1$. Now

$$\varphi\big(\mathrm{Inn}_g(x)\big) = \varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g^{-1}) = \varphi(g)\varphi(g)^{-1} = 1$$

so that $\mathrm{Inn}_g N \subseteq N$ as required.

■

Example 3.1.6:

Consider the group $\mathrm{GL}_2(\mathbb{Q})$. For $x \in \mathbb{Q}$ write

$$\alpha(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Note for $x, y \in \mathbb{Q}$ that $\alpha(x+y) = \alpha(x) + \alpha(y)$; thus $\alpha : \mathbb{Q} \to \mathrm{GL}_2(\mathbb{Q})$ is an injective group homomorphism whose image

$$U_{\mathbb{Q}} = \{\alpha(x) \mid x \in \mathbb{Q}\} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q} \right\}$$

is a subgroup of $\mathrm{GL}_2(\mathbb{Q})$.

Observe that $U_{\mathbb{Z}} = \{\alpha(x) \mid x \in \mathbb{Z}\}$ is a subgroup of $U_{\mathbb{Q}}$.

For $t \in \mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$, write

$$h(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

For $t, s \in \mathbb{Q}^\times$, we have $h(ts) = h(t)h(s)$ so that $h : \mathbb{Q}^\times \to \mathrm{GL}_2(\mathbb{Q})$ is an injective group homomorphism whose image

$$H = \{h(t) \mid t \in \mathbb{Q}^\times) = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{Q}^\times \right\}$$

is a subgroup of $\mathrm{GL}_2(\mathbb{Q})$.

We observe for $t \in \mathbb{Q}^\times$ and $x \in \mathbb{Q}$ that

$$h(t)\alpha(x)h(t)^{-1} = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix} = \alpha(tx).$$

This shows that $\forall h \in H, \mathrm{Inn}_h\, U_{\mathbb{Q}} \subseteq U_{\mathbb{Q}}$ so that H normalizes $U_{\mathbb{Q}}$.

Let $n \in \mathbb{Z}$ with $n > 1$ and consider the subgroup $C = C_n = \langle h(n) \rangle$ of H.

The generator $h(n)$ satisfies $\mathrm{Inn}_{h(n)}\, U_{\mathbb{Z}} \subset U_{\mathbb{Z}}$ since for $x \in \mathbb{Z}$

$$h(n)\alpha(x)h(n)^{-1} = \alpha(nx) \in U_{\mathbb{Z}}.$$

Note however that $\mathrm{Inn}_{h(n)}\, U_{\mathbb{Z}}$ is a proper subset of $U_{\mathbb{Z}}$; indeed, identifying $U_{\mathbb{Z}}$ with $\mathbb{Z}$, the image subgroup $\mathrm{Inn}_{h(n)}\, U_{\mathbb{Z}}$ identifies with $n\mathbb{Z}$ and of course $n\mathbb{Z}$ has index n in $\mathbb{Z}$.

The group $U_{\mathbb{Z}}$ is not normalized by $C = \langle h(n) \rangle$ e.g. since $\mathrm{Inn}^{-1}_{h(n)}\, U_{\mathbb{Z}} = \mathrm{Inn}_{h(n^{-1})}\, U_{\mathbb{Z}} \not\subseteq U_{\mathbb{Z}}$; indeed

$$\mathrm{Inn}_{h(n^{-1})}\, \alpha(1) = \alpha\left(\frac{1}{n}\right) = \begin{pmatrix} 1 & \frac{1}{n} \\ 0 & 1 \end{pmatrix} \notin U_{\mathbb{Z}}.$$

This example shows the following: there exists a group G, subgroups H, K of G, and a subset $S \subseteq G$ for which

$H = \langle S \rangle$ such that $\mathrm{Inn}_x\, K \subseteq K$ for all $x \in S$ even though H does not normalize K.

Of course, if we insist that $\mathrm{Inn}_x\, K = K$ for all $x \in S$ then H will normalize K; see Proposition 3.1.4.

In the proof of Proposition 3.1.4 we gave a definition of the normalizer of K in H, namely

$$N_H(K) = \{h \in H \mid \mathrm{Inn}_h\, K = K\}.$$

This example shows why in that definition one needs to insist that $\mathrm{Inn}_h\, K = K$ for each $h \in H$ rather than simple $\mathrm{Inn}_h\, K \subseteq K$.

3.2. Quotient groups

Theorem 3.2.1: Let N be a subgroup of G, and write $\left(\pi_{G/N}, G/N\right)$ for the quotient. If N is a normal subgroup, then G/N is a group for which

a. the multiplication $\mu : G/N \times G/N \to G/N$ satisfies

$$\forall g, g' \in G, \pi(g)\pi(g') = \pi(gg')$$

b. the identity is given by $1_{G/N} = \pi(1_G)$,

c. inversion satisfies $\forall g \in G, \pi(g)^{-1} = \pi(g^{-1})$.

Moreover, the quotient map $\pi_{G/N} : G \to G/N$ is a group homomorphism.

Proof: We first confirm that there is a mapping $\mu : G/N \times G/N \to G/N$ satisfying the condition in a.

We observe that $G/N \times G/N$ may be viewed as the quotient of the product group $G \times G$ by the subgroup $N \times N$; i.e. as $(G \times G)/(N \times N)$.

Consider the function

$$\varphi : G \times G \to G/N$$

given by

$$\varphi(g, g') = \pi_{G/N}(gg').$$

We claim that φ is constant on the $N \times N$ orbits in $G \times G$. Indeed, suppose that $(g, g') = (g_1, g_1')(h, h')$ for $g, g', g_1, g_1' \in G$ and $h, h' \in H$. Thus $g = g_1 h$ and $g' = g_1' h'$. Then

$$\varphi(g, g') = \pi_{G/N}(gg') = \pi_{G/N}(g_1 h \cdot g_1' h') = \pi_{G/N}(g_1 g_1' g_1'^{-1} h g_1' h') = \pi_{G/N}(g_1 g_1') = \varphi(g_1, g_1')$$

since N a normal subgroup

$\Rightarrow g_1'^{-1} h g_1' \in N \Rightarrow g_1'^{-1} h g_1' h' \in N$.

Thus there is a mapping $\mu : G/N \times G/N \to G/N$ which satisfies $\mu \circ \pi_{G \times G/N \times N} = \varphi$ and μ clearly satisfies a.

Next we confirm that there is an inversion mapping $G/N \to G/N$ that satifies b. For this, one just checks that the mapping $G \to G/N$ given by $g \mapsto \pi_{G/N}(g^{-1})$ is constant on N-orbits. Let $g, g' \in G$ and $h \in N$ and suppose that $g = g'h$. We must argue that

$$\pi_{G/N}(g^{-1}) = \pi_{G/N}(g'^{-1}).$$

We have

$$(g'h)^{-1} = h^{-1} g'^{-1} = g'^{-1} g' h^{-1} g'^{-1}$$

so indeed

$$\pi_{G/N}(g^{-1}) = \pi_{G/N}\big((g'h)^{-1}\big) = \pi_{G/N}(g'^{-1} g' h^{-1} g'^{-1}) = \pi_{G/N}(g'^{-1})$$

since $g' h^{-1} g'^{-1} \in N$ by the normality of N in G.

It remains to cofirm that the group axioms hold.

To confirm associativity in G/N, let $z, z', z'' \in G/N$. We must argue that $(zz')z'' = z(z'z'')$. Since π is surjective we can write $z = \pi(g)$, $z' = \pi(g')$ and $z'' = \pi(g'')$ for $g, g', g'' \in G$. Now we see using a. twice that

$$(zz')z'' = (\pi(g)\pi(g'))\pi(g'')) = \pi(gg')\pi(g'') = \pi((gg')g'').$$

A similar calculation shows that

$$z(z'z'') = \pi(g(g'g''))$$

and now the result follows by associativity in G.

Similar calculations confirm that the $\pi_{G/N}(1)$ acts as an identity and that $\pi_{G/N}(g^{-1})$ is the inverse of $\pi(G/N)(g)$.

Finally, it follows from the definitions that $\pi_{G/N}$ is a group homomorphism. ■

Example 3.2.2:

If G is an abelian group, then Inn_x is the trivial homomorphism for each $x \in G$, and in particular every subgroup of G is normal.

Let's consider an additive abelian group A and B any subgroup. Write $\pi : A \to A/B$ for the quotient mapping.

For $a \in A$, we often view $\pi(a)$ as the coset $a + B = \{a + x \mid x \in B\}$.

We see for $a, a' \in A$ that $\pi(a) = \pi(a') \Leftrightarrow a - a' \in B$.

3.3. First isomorphism theorem

Theorem 3.3.1:

Let $\varphi : G \to H$ be a group homomorphism, and let $K = \ker \varphi$. Assume that φ is surjective. Then there is a unique isomorphism of groups $\overline{\varphi} : G/K \to H$ such that $\varphi = \overline{\varphi} \circ \pi$ where $\pi : G \to G/K$ is the quotient homomorphism.

Proof: We first observe that – provided it exists – $\overline{\varphi}$ is unique. Indeed, for any $z \in G/K$ we may write $z = \pi(g)$ for $g \in G$ and then our assumption guarantees that

$$(*) \quad \overline{\varphi}(z) = \overline{\varphi}(\pi(g)) = \varphi(g).$$

So it just remains to argue that $(*)$ determines a group isomorphism.

We first check that $(*)$ determines a group homomorphism. Indeed, for $z, z' \in G/K$ with $z = \pi(g)$ and $z' = \pi(g')$ for $g, g' \in G$, we have

$$\overline{\varphi}(zz') = \overline{\varphi}(\pi(g)\pi(g')) = \overline{\varphi}(\pi(gg')) = \varphi(gg') = \varphi(g)\varphi(g') = \overline{\varphi}(\pi(g))\overline{\varphi}(\pi(g')) = \overline{\varphi(z)\varphi(z')}.$$

Now we observe that since φ is surjective, and since $\pi : G \to G/K$ is surjective, then $\overline{\varphi}$ is surjective.

Finally, we check that φ is injective. For this, it suffices to show that $\ker \varphi = \{1\}$; see Proposition 1.2.5.

So, let $z \in \ker \varphi \subseteq G/K$ and write $z = \pi(g)$ for $g \in G$. We know that

$$1_H = \overline{\varphi}(z) = \overline{\varphi}(\pi(g)) = \varphi(g)$$

and we conclude that $\varphi(g) = 1 \Rightarrow g \in \ker \varphi$. Since $g \in \ker \varphi$, we know that $\pi(g) = \pi(1)$, in other words, $z = \pi(g)$ is the identity element of the quotient group G/K. This proves that $\ker \overline{\varphi}$ is trivial so that $\overline{\varphi}$ is injective.

■

3.4. Groups acting on Groups

Let G and H be groups and suppose that G acts on the set H.

Definition 3.4.1:

We say that G acts by automorphisms on H if for each $g \in G$, the mapping

$$h \mapsto g \cdot h : H \to H$$

is an automorphism of the group H.

Remark 3.4.2:

To give an action of G on H by automorphisms is the same as to give a group homomorphism $G \to \mathrm{Aut}(H)$.

Proposition 3.4.3:

If G acts on H by automorphisms, the set of fixed points

$$H^G = \{x \in H \mid \forall g \in G, g \cdot x = x\}$$

is a subgroup of H.

Proof: Notice that $1 \in H^G$ since each each group automorphism $\psi : H \to H$ satisfies $\psi(1) = 1$.

Let $x, y \in H^G$. We must argue that $x^{-1}y \in H^G$.

We first argue that $x^{-1} \in H^G$. For this, let $g \in G$. Since the action of g is an automorphism of H and since $g \cdot x = x$, we see that

$$1 = g \cdot 1 = g \cdot xx^{-1} = (g \cdot x)(g \cdot x^{-1}) = x(g \cdot x^{-1}).$$

This shows that $g \cdot x^{-1} = x^{-1}$ so that $x^{-1} \in H^G$.

Now again let $g \in G$. We must argue that $g \cdot x^{-1}y = x^{-1}y$. Since g acts as an automorphism of H we see that

$$g \cdot x^{-1}y = (g \cdot x^{-1})(g \cdot y) = x^{-1}y$$

since $x^{-1}, y \in H^G$.

■

Example 3.4.4:

G acts in itself by inner automorphisms. This action is determined by the group homomorphism $\mathrm{Inn} : G \to \mathrm{Aut}(G)$.

In this case, the subgroup $G^G = G^{\mathrm{Inn}(G)}$ of fixed points is precisely the center $Z = Z(G)$:

$$Z = \left\{x \in G \mid \mathrm{Inn}_g\, x = x\right\} = \{x \in G \mid gx = xg \quad \forall g \in G.\}$$

For $x \in G$, the stabilizer $\mathrm{Stab}_G(x)$ is known as the centralizer:

$$\mathrm{Stab}_G(x) = C_G(x) = \{g \in G \mid \mathrm{Inn}_g\, x = x\} = \{g \in G \mid gx = xg\}.$$

And the orbit of x is known as the conjugacy class of x:

$$\mathcal{O}_x = \mathrm{Cl}(x) = \{\mathrm{Inn}_g\, x \mid g \in G\} = \{gxg^{-1} \mid g \in G\}.$$

Proposition 2.5.3 gives a bijection

$$\mathrm{Cl}(x) \simeq G/C_G(x).$$

Proposition 3.4.5: The center of a group G is a normal subgroup of G.

3.5. p-groups

Definition 3.5.1:

For a prime number p, a finite p-group is a finite group G whose order is a power of p.

Let G be a finite p-group and suppose that G acts on the finite set E, and write E^G for the set of elements of E fixed by the action of G; thus $E^G = \{x \in E \mid \forall g \in G, g \cdot x = x\}$.

Proposition 3.5.2:

With notation as above, we have $|E| \equiv |E^G| \pmod{p}$.

Proof: Indeed, the complement $E \setminus E^G$ is the disjoint union of non-trivial orbits of G, each of which has order divisible by p.

■

Proposition 3.5.3:

Suppose that G acts by automorphisms on a second p-group H. The fixed points H^G form a non-trivial subgroup.

Proof: First of all, the fixed points form a subgroup because the action of an element $g \in G$ is a group automorphism of H. In more detail, since H^G is a non-empty subset of G, it is enough to argue that for every $x, y \in H^G$, we have $x^{-1}y \in H^G$.

We first argue that $x^{-1} \in H^G$. For $g \in G$, we have

$$1 = g \cdot 1 = g \cdot xx^{-1} = (g \cdot x)(g \cdot x^{-1}) = x(g \cdot x^{-1}).$$

Thus $g \cdot x^{-1}$ is an inverse of x so indeed $x^{-1} = g \cdot x^{-1}$. We now show that $x^{-1}y \in H^G$. For this again let $g \in G$ be arbitrary. We have

$$g \cdot x^{-1}y = (g \cdot x^{-1})(g \cdot y) = x^{-1}y$$

which shows that $x^{-1}y \in H^G$.

Now Proposition 3.5.2 shows that p divides the order of the subgroup H^G, so H^G is indeed non-trivial..

■

Theorem 3.5.4: The center of a non-trivial p-group is non-trivial.

Proof: If G is a non-trivial p-group, consider the action of G on itself by conjugation. The subgroup of fixed points is precisely the center of G, and Proposition 3.5.3 implies that this subgroup is non-trivial.

■

Corollary 3.5.5:

Let G be a finite p-group with $|G| = p^n$. There is a series of subgroups

$$\{1\} = G_n \subset G_{n-1} \subset \ldots \subset G_0 = G$$

such that G_i is normal in G for each $0 \leq i < n$ and such that G_i/G_{i+1} is cyclic of order p for $0 \leq i < n-1$.

Proof sketch: We proceed by induction on $|G|$. If $|G| = 1$ so that G is the trivial group, the assertion is immediate.

Now suppose given a non-trivial p-group G and suppose that the result holds for all p-groups of order $< |G|$.

Let Z be the center of G. Then Z is non-trivial by Theorem 3.5.4. Thus G/Z is p group with order $< |G|$.

By induction there is a sequence of subgroups

$$\{1\} = H_m \subset H_{m-1} \subset \ldots \subset H_0 = G/Z.$$

such that H_i is normal in G/Z and H_i/H_{i+1} is cyclic of order p for each $i < m$.

Let $G_i = \pi^{-1}(H_i) \subset G$, where $\pi : G \to G/Z$ is the quotient homomorphism.

One must check the following:

- G_i is a normal subgroup of G,
- $G_i/G_{i+1} \simeq H_i/H_{i+1}$ is cyclic of order p for each i.

Since $G_m = \ker \pi = Z$ we have the sequence in G:

$$\{1\} \subset Z = G_m \subset G_{m-1} \subset \ldots \subset G_1 \subset G_0 = G.$$

Thus to complete the proof of the Theorem, we must demonstrate that Z has a suitable sequence of subgroup.

Thus it remains to prove the Theorem in case G is an *abelian* p-group. This proof is addressed in the homework.

■

4. Week 4 [2025-09-22]

4.1. Sylow subgroups

Let G be a finite group of order $n = p^m q$ with p a prime and with $\gcd(p, q) = 1$.

Theorem 4.1.1 (Sylow's Theorem):

There exists a subgroup of G having order p^m; such a subgroup is known as a *Sylow subgroup*, or a *Sylow p-subgroup*. Moreover:

a. Any two Sylow p-subgroups are conjugate by an element of G.
b. Any p-subgroup of G is contained in a Sylow p-subgroup.
c. If r denotes the number of p-Sylow subgroups, then $r \equiv 1 (\mathtt{mod}\ p)$ and $r \mid q$.
d.

For the proof, we consider the set E of all subsets of G having order p^m. The action of G on itself by translation induces an action of G on E: for $X \in E$, evidently $g \cdot X \in E$ where $g \cdot X = \{g \cdot x \mid x \in X\}$.

One knows that $|E| = \binom{|G|}{p^m} = \binom{p^m q}{p^m}$.

Proposition 4.1.2: $\binom{p^m q}{p^m} \equiv q (\mathtt{mod}\ p)$.

Proof: Let X and Y be indeterminants; we work in the polynomial ring $(\mathbb{Z}/p\mathbb{Z})[X, Y]$. Write $n = p^m q$ and consider

$$(X+Y)^n = ((X+Y)^{p^m})^q = (X^{p^m} + Y^{p^m})^q = \sum_{i=0}^{q} \binom{q}{i} (X^{p^m})^i (Y^{p^m})^{q-i}.$$

On the other hand, we have

$$(X+Y)^n = \sum_{i=0}^{n} \binom{n}{i} X^i Y^{n-i}.$$

and the required result follows by comparing the coefficient of $X^{p^m} Y^{(q-1)p^m}$ in the two expressions. ∎

Definition 4.1.3: Let G be a group and let $H \subseteq G$ be a subgroup. The normalizer of H in G is the subgroup

$$N_G(H) = \{g \in H \mid \mathrm{Inn}_g H = H\};$$

it is the stabilizer of H in G for the conjugation action of G on the set of subgroups of G.

Notice that $G/N_G(H)$ is in bijection with the set of all conjugates $\{gHg^{-1} \mid g \in G\}$.

For the proof of the Theorem, we are going to use the following:

Proposition 4.1.4: Let P be a Sylow p-subgroup of G and let Q be any p-subgroup of G. Then

$$N_Q(P) = Q \cap P.$$

Proof: By definition, $N_Q(P) = Q \cap N_G(P)$, so we must show that $Q \cap N_G(P) = Q \cap P$.

Let $H = Q \cap N_G(P)$. Since $P \subseteq N_G(P)$, it is clear that $Q \cap P \subseteq H = Q \cap N_G(P)$. It remains to establish the reverse inclusion. Since $H \subseteq Q$ by definition, it only remains to prove that $H \subseteq P$.

For this, we first claim that PH is a p-subgroup of G. Assume for the moment that this claim has been established. Since PH contains P and since P is a p-subgroup of maximal possible order, we conclude that $P = PH$ and hence that $H \subseteq P$ as required.

Since $H \subseteq N_G(P)$, the product $PH = \{xh \mid x \in P \text{ and } h \in H\}$ is a subgroup of G. Moreover, we know that

$$|PH| = \frac{|P\|H|}{|P \cap H|};$$

see Corollary 2.6.5. Since $|P|$ and $|H|$ are powers of p, PH is a p-subgroup.

■

Finally, we now give:

Proof of Sylow's Theorem: Proposition 4.1.2 shows that $|E| \not\equiv 0 (\mathtt{mod}\ p)$. Thus there must be some $X \in E$ for which the orbit $G \cdot X$ satisfies $|G \cdot X| \not\equiv 0 (\mathtt{mod}\ p)$. If H is the *stabilizer* in G of X, there is of course a bijection between $G \cdot X$ and G/H. In particular,

$$|G/H| \not\equiv 0 (\mathtt{mod}\ p).$$

Since $|G| = |H| \cdot |G/H|$, conclude that p^m divides the order of H.

On the other hand, fix $x \in X$. We claim that $H \subseteq X \cdot x^{-1}$. Indeed, for $h \in H$, since h stabilizes X we have

$$hx = x' \text{ for some } x' \in X.$$

Then $h = x'x^{-1} \in X \cdot x^{-1}$ as required.

Concluding, we find that $|H| \leq |X \cdot x^{-1}| = |X| = p^m$ and thus $|H| = p^m$. In particular, H is a Sylow subgroup.

Now let H' be any p-subgroup of G and consider the action of H' on the quotient G/H determined by left-multiplication. Since $|G/H| = q$ is not divisible by p, Proposition 3.5.2 shows that $(G/H)^{H'} \neq \emptyset$. Suppose that the coset $gH \in G/H$ is fixed by H'. We claim that

$$H' \subset gHg^{-1}.$$

Indeed, since gH is fixed by H', we have

$$x \in H' \Rightarrow xgH = gH \Rightarrow g^{-1}xgH = H \Rightarrow g^{-1}xg \in H \Rightarrow x \in gHg^{-1}.$$

This confirms that $H' \subseteq gHg^{-1}$. Thus any p-subgroup of G is contained in a Sylow subgroup. This proves (b).

Applying the argument of the preceding paragraph to the case where H' is a Sylow subgroup we see that $H' = gHg^{-1}$; this shows that any two Sylow subgroups are conjugate, proving (a).

To prove (c), let P be a Sylow p-subgroup. Note that P acts by conjugation on the set of all Sylow p-subgroups of G. We choose Sylow p-subgroups $Q_1, Q_2, ..., Q_s$ which form a system of representatives of the P-orbits for this action. We may and will take $Q_1 = P$.

For $1 \leq i < s$, we write $\mathcal{O}_i = P \cdot Q_i = \{xQ_ix^{-1} \mid x \in P\}$ for the P-orbit of Q_i. Recall that $\mathcal{O}_i$ is in bijection with the quotient $P/N_P(Q_i)$ where $N_P(Q_i)$ is the *normalizer* of Q_i in P.

For $1 \leq i \leq s$ Proposition 4.1.4 shows that $N_P(Q_i) = Q_i \cap P$.

In particular, it follows that $N_P(Q_1) = P \cap P = P$ so that $|\mathcal{O}_1| = 1$. For all $2 \leq i \leq s$ we have $P \neq Q_i$ so that $N_P(Q_i) = Q_i \cap P \subsetneq P$. Thus $|\mathcal{O}_i| = [P : Q_i \cap P] > 1$ so that

$$|\mathcal{O}_i| \equiv 0 (\mathtt{mod}\ p).$$

Finally, the number r of Sylow p-subgroups satisfies

$$r = \sum_{i=1}^{s} |\mathcal{O}_i| = 1 + \sum_{i=2}^{s} |\mathcal{O}_i| \equiv 1 (\mathtt{mod}\ p)$$

which proves the first assertion of (c). The second assertion of (c) follows since

$$r = [G : N_G(P)]$$

and since $P \subseteq N_G(P)$.

■

For a finite group G and a prime number p, write $n_p = n_p(G)$ for the number of p-Sylow subgroups of G (this is the number r from Theorem 4.1.1).

Corollary 4.1.5:

Let $P \in \mathrm{Syl}_p(G)$. Then P is normal if and only if $n_p = 1$.

Proof: Indeed, P is normal if and only if $\mathrm{Inn}_g\, P = P$. Since all p-Sylow subgroups are conjugate, the result is immediate.

■

4.2. Applications of Sylow's Theorem

Definition 4.2.1:

Let G be a group. A subgroup H of G is characteristic if for every automorphism $\varphi : G \to G$, we have $\varphi(H) = H$.

A characteristic subgroup H is always normal, since H is invariant under all the inner automorphisms Inn_g for $g \in G$.

Example 4.2.2:

For a prime number p, let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$; we may identify the group G with the quotient of $\mathbb{Z} \times \mathbb{Z}$ by the subgroup $N = \langle (p,0), (0,p) \rangle$ Write $e = (1,0) + N$ and $f = (0,1) + N$ so that $e, f \in G$ are elements of order p and

$$G = \langle e, f \rangle = \langle e \rangle + \langle f \rangle \text{ and } \langle e \rangle \cap \langle f \rangle = \{0\}.$$

Since G is commutative, $H = \langle e \rangle$ is a normal subgroup of G. But H is not characteristic in G. Indeed, the automorphism $(a,b) \mapsto (b,a)$ of $\mathbb{Z} \times \mathbb{Z}$ induces an automorphism φ of $G = (\mathbb{Z} \times \mathbb{Z})/N$ for which

$$\varphi(e) = f \text{ and } \varphi(f) = e.$$

Thus $\varphi(H) \neq H$.

Definition 4.2.3:

A group G is simple if for any normal subgroup $N \subseteq G$, either $N = \{1\}$ or $N = G$.

Any group of prime order is simple. Below we are going to find the first example of a non-abelian simple group.

Throughout the remainder of this section, G denotes a finite group.

For a prime number p we write $n_p = n_p G)$ for the number of p-Sylow subgroups in G, and $\mathrm{Syl}_p = \mathrm{Syl}_p\, G)$ for the set of Sylow p-subgroups. Recall that

$$n_p \equiv 1 (\texttt{mod } p) \text{ and } p \mid |G/P| \text{ for } P \in \mathrm{Syl}_p(G)$$

.

Proposition 4.2.4:

a. If $n_p = 1$ for some prime p, then $P \in \mathrm{Syl}_p(G)$ is characteristic – and in particular normal – in G.

b. Suppose that $H \subseteq G$ is a normal subgroup, and suppose that $P \in \mathrm{Syl}_p(H)$ is a normal p-Sylow subgroup of H for some prime p. Then P is a normal subgroup of G.

Proof:

a. For any automorphism φ of G, $\varphi(P)$ is again a p-Sylow subgroup of G. Since $n_p = 1$, we then $\varphi(P) = P$ so that P is indeed characteristic.

b. For $g \in G$, Inn_g determines an automorphism of H. Thus by (a), $\mathrm{Inn}_g\, P = P$ which shows that P is normal in G.

■

Proposition 4.2.5:

Suppose that $|G| = pq$ for distinct prime numbers $p < q$.

a. $n_q = 1$ so that G has a normal Sylow q-subgroup.
b. If p does not divides $q-1$ then $n_p \equiv 1$ so that G has a normal Sylow p-subgroup. In this case, G is a cyclic group.

Proof:

a. By Theorem 4.1.1 we have $n_q \equiv 1 (\mathtt{mod}\ q)$ and $n_q \mid p$. Since $q > p$ it follows that $n_q = 1$.
b. Again we have $n_p \equiv 1 (\mathtt{mod}\ p)$ and $n_p \mid q$. Since q is prime, the only possibilities are $n_p = 1$ or $n_p = q$.

   If $n_p = q$ then $q \equiv 1 (\mathtt{mod}\ p)$ so that p divides $q-1$.

   If $n_p = 1$ then $G = PQ$ where P is a Sylow p-subgroup and Q is a Sylow q-subgroup. You will prove for homework that since P and Q are both normal in G, G is isomorphic to the direct product $P \times Q$. Thus

$$G \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \simeq \mathbb{Z}/pq\mathbb{Z}$$

   is indeed cyclic.

■

Proposition 4.2.6:

Suppose that the size of a p-Sylow subgroup of G is p. Then G contains exactly $(p-1) \cdot n_p$ elements of order p.

Proof: Indeed, suppose that $P, Q \in \mathrm{Syl}_p(G)$ with $P \neq Q$. Since $|P| = |Q| = p$ is prime we know that $P \cap Q = \{1\}$. Thus

$$\left| \bigcup_{P \in \mathrm{Syl}_p} P \setminus \{1\} \right| = \sum_{P \in \mathrm{Syl}_p} (p-1) = n_p(p-1).$$

■

Proposition 4.2.7:

Suppose that $|G| = 12$. Then either $n_3 = 1$ or G is isomorphic to the alternating group A_4, and in that case, $n_2 = 1$.

Proof:

We know that $n_3 \equiv 1 (\mathtt{mod}\ 3)$ and $n_3 \mid 4$. Thus n_3 is either 1 or 4.

To complete the proof we must show that if $n_3 = 4$ then G is isomorphic to A_4.

The group G acts by conjugation on the set $\Omega = \mathrm{Syl}_3$ of 3-Sylow subgroups, and $|\Omega| = 4$. This action determines a group homomorphism

$$\varphi : G \to S(\Omega) = S_4.$$

The kernel K of φ consists of all $g \in G$ which normalize each Sylow 3-subgroup. In particular for $P \in \Omega$, $K \subseteq N_G(P) = P$. Since K is normal, since P is not normal, and since $|P| = 3$, we conclude that $K = \{1\}$.

Thus φ is an isomorphism from G to its image in S_4.

Since $n_3 = 4$, Proposition 4.2.6 shows that G contains exactly 8 elements of order 3. On the other hand, there are exactly 8 three-cycles in S_4 and all are contained in A_4.

Thus the intersection of the image of φ with A_4 is a group containing at least 8 elements. Since both groups have order 12, they must coincide.

Finally, notice that $K = \langle (12)(34), (13)(24) \rangle$ is a normal 2-sylow subgroup of A_4; thus $n_2(A_4) = 1$.

■

Proposition 4.2.8:

Suppose that $|G| = 30$. Then $n_5 = 1$ so that G has a normal Sylow 5-subgroup.

Proof: Let $P \in \mathrm{Syl}_5$ and $Q \in \mathrm{Syl}_3$.

First suppose that neither P nor Q is normal in G. In that case, $n_5 = 6$ and $n_3 = 10$. By counting elements of order 3 and of order 5, it now follows from Proposition 4.2.6 that

$$|G| \geq 6 \cdot 4 + 10 \cdot 2 = 44 > 30.$$

This contradiction proves that at at least one of P or Q is normal in G.

Now if either P or Q is normal in G, then PQ is a subgroup of order 15. It follows from Proposition 4.2.5 that any group of order 15 is cyclic. Using Proposition 4.2.4 we then conclude that both P and Q are normal in G, and the result follows.

■

Proposition 4.2.9:

Suppose that $|G| = 60$. If $n_5 > 1$ then G is a simple group.

Proof: For any group of order 60, n_5 is either 1 or 6; thus we suppose $n_5 = 6$.

Let $P \in \mathrm{Syl}_5(G)$. If $N_G(P)$ is the normalizer of P, then $|G/N_G(P)| = 6$ so that $|N_G(P)| = 10$.

We proceed by contradiction; thus, we suppose that $\{1\} \neq H \subsetneq G$ is a normal subgroup of G.

First suppose that $5 \mid |H|$. Then H contains a Sylow 5-subgroup of G; since H is normal, H contains all six Sylow 5-subgroups of G. Counting elements of order 5 in H, it follows from Proposition 4.2.6 that

$$|H| \geq 6 \cdot 4 = 24.$$

Since the only divisor d of 60 with $d \geq 24$ is $d = 30$, we conclude that $|H| = 30$. Now Proposition 4.2.8 shows that H has a normal 5-Sylow subgroup $Q \in \mathrm{Syl}_5(H)$, and Proposition 4.2.4 shows that Q is normal in G. But this contradicts the assumption $n_5 > 1$.

This shows that

$$(\clubsuit) \quad G \text{ has no normal subgroup } H \text{ for which } 5 \mid |H|.$$

Thus we may suppose that $|H|$ is a divisor of $60/5 = 12$.

If $|H| = 12$, it follows from Proposition 4.2.7 that G has a normal Sylow p-subgroup for either $p = 2$ or $p = 3$. In view of Proposition 4.2.4, it follows that G has a normal subgroup of order 4 or 3.

If $|G| = 6$, then G has a normal Sylow 3-subgroup by Proposition 4.2.5.

Thus we may suppose that $|H|$ is one of $2, 3, 4$.

Write $\overline{G} = G/H$ for the quotient group, so that $\overline{G} = 30, 20$ or 15.

We claim in each case that $\overline{G}$ has a normal subgroup Q of order 5, i.e. that $n_5\left(\overline{G}\right) = 1$.

If $\left|\overline{G}\right| = 30$ this claim follows from Proposition 4.2.8. If $\left|\overline{G}\right| = 20$ note that $n_5 \mid 4$ and $n_5 \equiv 1 (\mathtt{mod}\ 5)$ shows that $n_5 = 1$. Finally, if $\left|\overline{G}\right| = 15$, then n_5 divides 3 and $n_5 \equiv 1 (\mathtt{mod}\ 5)$ again shows that $n_5 = 1$.

If $\pi : G \to \overline{G} = G/H$ is the quotient mapping, let $H_1 = \pi^{-1}(Q)$ be the inverse image of the normal subgroup Q of order 5. You will prove for homework that H_1 is a normal subgroup of G containing H; since $H_1/H \simeq Q$ it follows that $5 \mid H_1$. This contradicts $(\clubsuit)$ and completes the proof of the Proposition.

■

Corollary 4.2.10:

The alternating group A_5 is a simple group of order 60.

Proof: Indeed, the subgroups $\langle (1,2,3,4,5) \rangle$ and $\langle (1,3,2,4,5) \rangle$ are two distinct 5-Sylow subgroups of A_5 so that $n_5(A_5) > 1$. ■

4.3. Rings

Let R be a ring and recall that we only consider rings with identity.

Definition 4.3.1:

A left ideal I of R is an additive subgroup of R that is closed under multiplication on the leftby elements of R.

More precisely, for each $x \in I$ and each $a \in R$, we have $ax \in I$.

Remark 4.3.2:

a. There is an obvious related notion of right ideal.

b. If R is commutative, then I is a left ideal if and only if I is a right ideal, and in that case we simply call I an ideal.

c. For non-commutative R, we reserve the term ideal for an additive subgroup I which is both a left ideal and a right ideal. Sometimes we say that such an I is a two-sided ideal, for emphasis.

Let us now suppose that R is commutative.

Proposition 4.3.3:

a. Let I and J be ideals of R. The intersection $I \cap J$ is

a ideal.

a. More generally, if I_x is an ideal of R for each x in the index set X, then $\cap_{x\in X} I_x$ is an ideal of R.

Proof: Since b $\Rightarrow$ a, we prove b. Note that $0 \in I_x$ for each x so that the intersection is non-empty.

Let $a, b \in \bigcap_{x\in X} I_x$. We must show that $a - b$ is in the intersection. But for $x \in X$, $a, b \in I_x \Rightarrow a - b \in I_x$ so that indeed $a - b \in \bigcap_{x\in X} I_x$.

■

Proposition/Definition 4.3.4:

Let $S \subset R$ be a subset. The ideal generated by S, written $\langle S\rangle$ or RS is defined to be

$$\bigcap_{S\subseteq I} I,$$

the intersection taken over ideals of R containing S.

Proof: This intersection is an ideal by Proposition 4.3.3.

■

Definition 4.3.5:

For $a \in R$, the principal ideal generated by a is the ideal $\langle\{a\}\rangle$ and is denoted Ra or $\langle a\rangle$.

4.4. Ring homomorphisms and kernels

Definition 4.4.1:

If R and S are commutative rings, a function $f : R \to S$ is a ring homomorphism provided that is a homomorphisms of additive groups and that

a. $f(ab) = f(a)f(b)$ for every $a, b \in R$, and
b. $f(1_R) = 1_S$.

Proposition 4.4.2:

If $f : R \to S$ is a ring homomorphism, the kernel $\ker f$ is a ideal of R.

Proposition 4.4.3:

If I is an ideal of R, write $\pi : R \to R/I$ for the quotient homomorphisms (where R/I is the quotient additive group).

Then there is a unique ring structure on the quotient group R/I with the property that quotient mapping $\pi : R \to R/I$ is a ring homomorphism.

5. Week 5 [2025-09-29]

Parts of the material to be discussed this week are covered in the text [Dummit-Foote,"Abstract Algebra"]:

- quotient rings [Dummit-Foote] §7.3,7.4
- modules; products and direct sums [Dummit-Foote] §10.1,10.2,10.3
- [Dummit-Foote] doesn't use the language of categories.

5.1. Quotient rings

In this section, R will denote a ring (with identity as always, but not necessarily commutative).

By a (two-sided) ideal of R, we mean an additive subgroup I of R that is closed under multiplication with R on the left and on the right.

More precisely: I is an ideal if $\forall x \in I, \forall r \in R, rx \in I$ and $xr \in I$.

If I is an ideal, then R/I is an additive abelian group, and the quotient mapping $\pi : R \to R/I$ can be viewed as the mapping $\pi(r) = r + I$.

Theorem 5.1.1:

Let I be a two-sided ideal of R. Then the quotient group R/I has the structure of a ring with identity where the multiplication satisfies

$$(a + I)(b + I) = ab + I \text{ for } a, b \in R.$$

In particular, the quotient mapping $\pi : R \to R/I$ is a surjective ring homomorphism. If $I \neq R$, then $1_{R/I} \neq 0_{R/I}$.

Theorem 5.1.2 (First isomorphism theorem for rings):

Let $\varphi : R \to S$ be a surjective ring homomorphism. Recall that φ induces an isomorphism of additive groups $\overline{\varphi} : R/I \to S$ for which $\overline{\varphi}(a + I) = \varphi(a)$ for $a \in R$. Then $\overline{\varphi}$ is an isomorphism of rings.

Example 5.1.3:

Let R be a commutative ring. One checks that

$$S = \left\{ \begin{pmatrix} a & d \\ 0 & b \end{pmatrix} : a, b, c \in R \right\}$$

is a subring of $\mathrm{Mat}_3(R)$ which is not commutative if $1_R \neq 0_R$. The mapping

$$f : S \to R \times R \text{ given by } f\left(\begin{pmatrix} a & d \\ 0 & b \end{pmatrix} \right) = (a, b)$$

is a surjective ring homomorphism with kernel $K = \left\{ \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \mid \alpha \in R \right\}$. According to the theorem, f induces an isomorphism

$$S/K \simeq R \times R.$$

(Note that K is a two-sided ideal since $K = \ker f$. On the other hand, it is easy to check directly that K is a two-sided ideal of S.)

5.2. Categories

Definition 5.2.1: A category $\mathcal{C}$ consists of the following:

- a class $\mathrm{Ob}(\mathcal{C})$ of objects,
- a class $\mathrm{Mor}(\mathcal{C})$ of morphisms together with class functions

$$\mathrm{dom} : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}) \text{ and } \mathrm{codom} : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$$

  for domain and codomain.

  Denote by $\mathrm{Mor}(X,Y) = \mathrm{Mor}_{\mathcal{C}}(X,Y)$ the subclass of $\mathrm{Mor}(\mathcal{C})$ consisting of morphisms f in $\mathrm{Mor}(\mathcal{C})$ with $\mathrm{dom}(f) = X$ and $\mathrm{codom}(f) = Y$.

- for every three objects X, Y, Z there is a binary operation

$$(f,g) \mapsto g \circ f : \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) \to \mathrm{Mor}(X,Z)$$

This data is required to satisfy:

a. *associativity:* For f in $\mathrm{Mor}(X,Y)$, g in $\mathrm{Mor}(Y,Z)$ and h in $\mathrm{Mor}(Z,W)$ we have

$$(h \circ g) \circ f = h \circ (h \circ f).$$

b. *identity*: For every object Z, there is id_Z in $\mathrm{Mor}(Z,Z)$ such that every f in $\mathrm{Mor}(Z,X)$ satisfies $f \circ \mathrm{id}_Z = f$ and every g in $\mathrm{Mor}(X,Z)$ satisfies $\mathrm{id}_Z \circ f = f$.

Remark 5.2.2:

We often use function notation to represent morphisms – thus $f : A \to B$ denotes the morphism f in $\mathrm{Mor}(A,B)$. Be careful, though – in general, morphisms need not be functions.

Example 5.2.3: Here are some examples of categories.

a. The category Set of all sets, with morphisms given by functions.
b. The category Grp of all groups, with morphisms given by group homomorphisms.
c. The category Ab of all abelian groups, with morphisms given by group homomorphisms.
d. The category Top of topological spaces, with morphisms given by continuous functions.
e. The category Ring of rings with morphisms given by ring homomorphisms.

Definition 5.2.4:

Let $\mathcal{C}$ be a category. An object I of $\mathcal{C}$ is said to be initial if for each object X of $\mathcal{C}$ there is a unique morphism in $\mathrm{Mor}(I,X)$.

An object T of $\mathcal{C}$ is said to be terminal if for each object X of $\mathcal{C}$ there is a unique morphism in $\mathrm{Mor}(X,I)$.

Example 5.2.5:

a. The empty set is an initial object in Set. Every singleton set is a terminal object in Set.

b. The trivial group $\{1\}$ is both an initial and a terminal object in Grp

c. The trivial group $\{0\}$ is both an initial and a terminal object in Ab

Definition 5.2.6:

Let $\mathcal{C}$ be a category and let X and Y in $\mathrm{Ob}(\mathcal{C})$. Then X and Y are isomorphic provided that there are morphisms $f \in \mathrm{Mor}(X,Y)$ and $g \in \mathrm{Mor}(Y,X)$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

One says that f and g are isomorphisms between X and Y.

Proposition 5.2.7: Let $\mathcal{C}$ be a category.

a. If I, I' are initial objects, there is a unique isomorphism $I \to I'$.

b. If T, T' are terminal objects, there is a unique isomorphism $T \to T'$.

Proof: We prove a; the proof of b is essentially the same..

For a, since I is initial, there is a unique morphism f in $\mathrm{Mor}(I, I')$ and since I' is initial there is a unique morphism g in $\mathrm{Mor}(I', I)$.

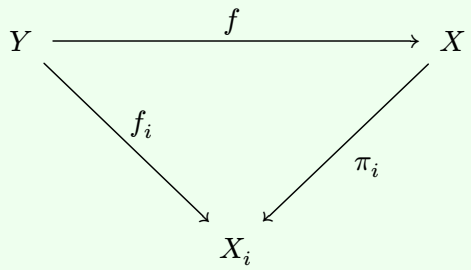
Now $f \circ g$ is a morphism in $\mathrm{Mor}(I', I')$. Since I' is initial, $\mathrm{id}_{I'}$ is the unique morhpism in $\mathrm{Mor}(I', I')$ and we conclude that $\mathrm{id}_{I'} = f \circ g$.

Similarly, $g \circ f$ is a morphism in $\mathrm{Mor}(I, I)$. Since I is initial, id_I is the unique morphism in $\mathrm{Mor}(I, i)$ and we conclude that $\mathrm{id}_I = g \circ f$. Thus we have proved that $f : I \to I'$ is the required unique isomorphism.

■

Definition 5.2.8:

Let $\mathcal{C}$ be a category, let I be an index set, and let X_i be an object of $\mathcal{C}$ for each $i \in I$. A product of the X_i is an object X of $\mathcal{C}$ together with morphisms $\pi_i : X \to X_i$ for $i \in I$ such that given any object Y of $\mathcal{C}$ toegether with morphisms $f_i : Y \to X_i$ there is a unique morphism $f : Y \to X$ such that $f_i = \pi_i \circ f$ for each $i \in I$; i.e. the diagram



commutes for each $i \in I$.

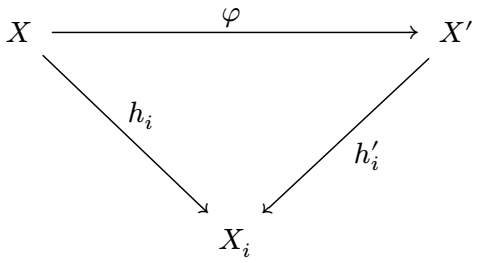
Proposition 5.2.9:

Let $\mathcal{C}$ be a category, let I an index set, and let X_i be objects of $\mathcal{C}$ for $i \in I$. If a product (X, π_i) of the X_i exists in $\mathcal{C}$ where $\pi_i : X \to X_i$ for $i \in I$, it is unique up to a unique isomorphism.

In other words, if $(X', \pi_{i'})$ is a second product in $\mathcal{C}$, there is a unique isomorphism $f : X \to X'$ in $\mathcal{C}$ with the property that $\pi_i = \pi_{i'} \circ f$.

Proof: We introduce a new category $\mathcal{D}$ depending on $\mathcal{C}$, I, the family X_i. An object of D is an object X of $\mathcal{C}$ together with morphisms $h_i : X \to X_i$ for each $i \in I$.

A morphism between objects (X, h_i) and (X', h'_i) of $\mathcal{D}$ is a morphism φ in $\mathrm{Mor}_{\mathcal{C}(X,X')}$ such that $h_i = h'_i \circ \varphi$; i.e. the diagram



commutes for each $i \in I$.

One checks that $\mathcal{D}$ is a category. It is then clear that to give a terminal object in $\mathcal{D}$ is the same as to give a product of the X_i in $\mathcal{C}$. Thus the uniqueness follows from Proposition 5.2.7.

■

Remark 5.2.10:

If the objects X_i for $i \in I$ have a product in the category $\mathcal{C}$, we write

$$\prod_{i \in I} X_i$$

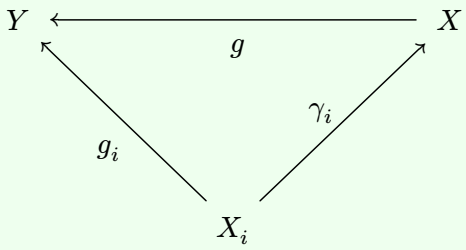
for the product, keeping in mind that the morphisms

$$\pi_j : \prod_{i \in I} X_i \to X_j$$

are part of the data determining a product.

Definition 5.2.11:

Let $\mathcal{C}$ be a category, let I be an index set, and let X_i be an object of $\mathcal{C}$ for each $i \in I$. A co-product of the X_i is an object X of $\mathcal{C}$ together with morphisms $\gamma_i : X_i \to X$ for $i \in I$ such that given any object Y of $\mathcal{C}$ toegether with morphisms $g_i : X_i \to Y$ there is a unique morphism $g : X \to Y$ such that $g_i = g \circ \gamma_i$ for each $i \in I$; i.e. the diagram



commutes for each $i \in I$.

Proposition 5.2.12:

Let $\mathcal{C}$ be a category, let I an index set, and let X_i be objects of $\mathcal{C}$ for $i \in I$. If a co-product (X, γ_i) of the X_i exists in $\mathcal{C}$ where $\gamma_i : X_i \to X$, it is unique up to a unique isomorphism.

In other words, if (X', γ_i') is a second co-product in $\mathcal{C}$, there is a unique isomorphism $f : X \to X'$ in $\mathcal{C}$ with the property that $\gamma_i = f \circ \gamma_i'$.

Remark 5.2.13: If the objects X_i for $i \in I$ have a co-product in the category $\mathcal{C}$, we write

$$\coprod_{i \in I} X_i$$

for the co-product, keeping in mind that the morphisms

$$\gamma_j : X_j \to \coprod_{i \in I} X_i$$

are part of the data determining a co-product.

5.3. Modules

Let R be a ring.

Definition 5.3.1:

A left R-module M is an additive abelian group M together with an operation of scalar multiplication $R \times M \to M$ satisfying

a. identity: $1 \cdot m = m$ for every $m \in M$.

b. associativity: $(ab)m = a(bm)$ for every $a, b \in R$ and $m \in M$.

c. bilinearity:

- $(a+b)m = am + bm$ for every $a, b \in R$ and $m \in M$
- $a(m+n) = am + an$ for every $a \in R$ and $m, n \in M$.

Remark 5.3.2:

There is a notion of right R-module M: for $r \in R$ and $m \in M$ the scalar multiplication is written $m \cdot r$ and this scalar multiplication must satisfy analogous of the conditions required for a left module. When R is commutative, any left module can be viewed as a right module – for $a \in R$ and $m \in M$ just define the right module action via $m \odot a = a \cdot m$ – and vice versa, so we may just speak of "R-modules" in this case.

Example 5.3.3:

a. If $R = F$ is a field, then the F-modules are precisely the F-vector spaces.

b. Any abelian group is a $\mathbb{Z}$-module, and vice-versa.

c. If R is a subring of some ring S, then S has the structure of an R-module.

d. Any ideal I of R is an R-module (in particular, I is an R-submodule of the R-module R).

Proposition 5.3.4: The data of an R-module M is equivalent to the data of an additive abelian group M together with a ring homomorphism $R \to \mathrm{End}_{\mathbb{Z}}(M)$, where $\mathrm{End}_{\mathbb{Z}}(M)$ is the ring of additive endomorphisms of M.

Definition 5.3.5:

If M and N are left R-modules, a function $\varphi : M \to R$ is a homomorphism of R-modules provided that

- φ is a homomorphism of additive groups, and
- $\varphi(rm) = r\varphi(m)$ for every $r \in R$ and every $m \in M$.

Remark 5.3.6:

a. If $R = F$ is a field, then a homomorphism of R-modules $\varphi : M \to N$ is the same as a linear map of vector spaces.

b. If A and B are abelian groups, a function $\varphi : A \to B$ is a group homomorphism if and only if it is a homomorphism of $\mathbb{Z}$-modules.

Definition 5.3.7: Let M be an R-module. By an R-submodule of M, we mean an additive subgroup $N \subseteq M$ such that $\forall x \in N, \forall r \in R, rx \in N$; i.e. such that $R \cdot N \subseteq N$.

Definition 5.3.8: If R is a commutative ring, there is a category R-mod whose objects are the R-modules and whose morphisms are the R-module homomorphisms.

5.4. The direct sum of R-modules.

Let I be an index set and suppose that M_i is an R-module for each $i \in I$.

Proposition/Definition 5.4.1: The direct sum $\bigoplus_{i\in I} M_i$ of the R-modules M_i is the set of all finitely supported functions $f : I \to \bigcup_{i\in I} M_i$ with the property that $f(j) \in M_j$ for $j \in I$.

a. $\bigoplus_{i\in I} M_i$ is an R-module with pointwise addition and scalar multiplication.

b. Define $\iota_j : M_j \to \bigoplus_{i\in I} M_i$ by setting $\iota_j(m)$ to be the finitely supported function on I whose support is $\{j\}$ and whose value at j is m.

   For each $j \in I$, the map ι_j is an R-module homomorphism.

Proof: The straightforward checks are left to the reader. ■

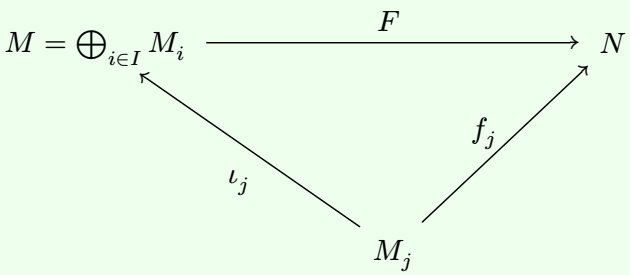
Proposition 5.4.2:

Let I be an index set and let M_i be an R-module for each $i \in I$. Write $M = \bigoplus_{i\in I} M_i$ and $\iota_j : M_j \to M$ $j \in I$ as in Proposition/Definition 5.4.1.

Then $M = \bigoplus_{i\in I} M_i$ together with the ι_j is a coproduct of the M_j in the category R-mod .

Recall that this means: Given any R-module N and R-module homomorphisms $f_j : M_j \to N$, there is a unique R-module homomorphism

$F : M \to N$ such that $f_j = F \circ \iota_j$.



Proof: Let N and $f_j : M_j \to N$ for each $j \in I$ be given.

We first prove uniqueness of the mapping F. Consider an element

$$m \in M = \bigoplus_{i \in I} M_i.$$

Since m has finite support, we see that there is a finite subset $J \subseteq I$ and for each $j \in J$ an element $m_j \in M_j$ for which

$$m = \sum_{j \in J} \iota_j(m_j).$$

Now we see that

$$(\heartsuit) \quad F(m) = \sum_{j \in I} (F \circ \iota_j)(m_j) = \sum_{j \in I} f_j(m_j).$$

This proves the uniqueness once we shows that ($\heartsuit$) defines an R-module homomorphism. But this follows from the definition of the R-module structure on $\bigoplus_{i \in I} M_i$ and the fact that the f_j are R-module homomorphisms. ■

5.5. Free modules

Let R be a ring.

Definition 5.5.1:

Let F be a left R-module, let B be a set and let $\beta : B \to F$ be a fuction. Then F is a free left R-module on β provided that for any R-module X and any function $j : B \to X$, there is a unique R-module homomorphism $\varphi : F \to X$ such that $j = \varphi \circ \beta$.

Proposition 5.5.2:

Suppose that F is a free R-module on $\beta : B \to F$. Then the function β is injective (i.e. one-to-one).

Proof: Let $b_1, b_2 \in B$ and suppose that $b_1 \neq b_2$. We must show that $\beta(b_1) \neq \beta(b_2)$. To this end, let $f : B \to R$ be the function defined by

$$f(b) = \begin{cases} 1 \text{ if } b = b_1 \\ 0 \text{ otherwise} \end{cases}.$$

Since F is a free R-module on β, there is an R-module homomorphism $\varphi : F \to R$ such that $\varphi \circ \beta = f$.

Then $\varphi(\beta(b_1)) = f(b_1) = 1$ while $\varphi(\beta(b_2)) = f(b_2) = 0$. Since $\varphi(\beta(b_1)) \neq \varphi(\beta(b_2))$ we must have $\beta(b_1) \neq \beta(b_2)$ as required. ■

We are going to argue that the free R-modules are precisely those R-moodules which have a basis.

Recall that for a set I and an additive abelian group A, the support of a function

$$f : I \to A \text{ is given by } \mathrm{Supp}(f) = \{i \in I \mid f(i) \neq 0_A\}.$$

Then f has finite support provided that $\mathrm{Supp}(f)$ is a finite set.

Here is the definition:

Definition 5.5.3:

If M is an R-module, a function $\beta : B \to M$ for some set B is an R-basis M provided that the following hold:

- β is linearly independent i.e. if $a : B \to R$ is a finitely supported function and if
$$\sum_{b \in B} a(b)\beta(b) = 0 \text{ then } a = 0.$$
- β spans M; i.e. for $x \in M$ there is a finitely supported function $a : B \to R$ such that
$$x = \sum_{b \in B} a(b)\beta(b).$$
(Observe that the sum is defined because a is finitely supported).

Notice that if $\beta : B \to M$ is an R-basis then every element $x \in M$ can be written in the form
$$x = \sum_{b \in B} a(b)\beta(b)$$
for a unique finitely supported function $a : B \to R$.

Remark 5.5.4: It might seem clumsy that we insist on defining bases etc. as functions from a set to an R-module. However, the reader will easily be able to pass back and forth between this language and other formulations.

For example, one says that an R-module F is free on a subset $B \subset F$ if it is free on the inclusion mapping $\iota : B \to F$. Similarly, one says that a subset $B \subset F$ is a basis if the inclusion mapping $\iota : B \to F$

Example 5.5.5:

In general, an R-module M need not have a basis. For example, for $n \in \mathbb{N}, n > 1$, the $\mathbb{Z}$-module $M = \mathbb{Z}/n\mathbb{Z}$ has no $\mathbb{Z}$-basis, since for any $x \in M$, $nx = 0$ but $0 \neq n \in \mathbb{Z}$. This shows that there can be no $\mathbb{Z}$ -linearly independent function from a non-empty set to M.

6. Week 6 [2025-10-06]

We continue the discussion of free modules.

Proposition 6.1:

Let B be a set and let $F = F(B, R)$ be the R-module consisting of all finitely supported maps $a : B \to R$. Consider the function $\beta_0 : B \to F(B, R)$ where $\beta_0(b)$ is the function

$$\beta_0(b)(b') = \begin{cases} 1 \text{ if } b = b' \\ 0 \text{ otherwise.} \end{cases}$$

a. Then F is a free R-module on β_0.
b. β_0 is an R-basis for F.

Proof: For any finitely supported function $a : B \to R$ we note that

$$(\heartsuit) \quad a = \sum_{b \in B} a(b)\beta_0(b).$$

a. To see that F is a free module on the indicated data, let N be an arbitrary R-module and let $\varphi : B \to N$ be any function. We must show that there is a unique R-module mapping $\Phi : F \to N$ such that

$$(*) \quad \Phi \circ \beta_0 = \varphi.$$

We first treat uniqueness. Thus we suppose that there is a linear mapping $\Phi : F \to N$ for which $\Phi \circ \beta = \varphi$.

Using $(\heartsuit)$, the R-linearity of Φ, and the requirement $(*)$, we see that

$$(\clubsuit) \quad \Phi(a) = \sum_{b \in B} a(b)\Phi(\beta_0(b)) = \sum_{b \in B} a(b)\varphi(b).$$

To complete the proof, it only remains to observe that the rule specified by $(\clubsuit)$ indeed determines an R-module homomorphism; this follows from the definition of the R-module structure on $F = F(B, R)$.

b. We first prove that β_0 is R-linearly independent. We suppose that $a : B \to R$ is a finitely supported function such that

$$\sum_{b \in B} a(b)\beta_0(b) = 0.$$

Then $(\heartsuit)$ shows that $a = 0$; this proves the linear independence.

Finally, we show that β_0 spans $F(B, R)$. Let $a \in F(B, R)$. Then $(\heartsuit)$ again shows that a has the required form.

■

Proposition 6.2: Let B be any set, consider for $b \in B$ the R-module $M_b = R$, and let C together with $\iota_b : R \to C$ be the coproduct (direct sum) of the modules $M_b = R$ for $b \in B$.

With notation β_0 as in Proposition 6.1, there is an R-module isomorphism

$$\Psi : C \to F(B, R)$$

such that for each $b \in B$,

$$(\Psi \circ \iota_b)(1) = \beta_0(b).$$

Proof: For each $b \in B$, there is an R-module homomorphism

$$f_b : R \to F(B, R) \text{ defined by } f_b(t) = t\beta_0(b) \text{ for } t \in R.$$

Using the defining property of the direct sum (coproduct), there is a unique R-module homomorphism $\Psi : C \to F(B, R)$ such that $(\Psi \circ \iota_b)(t) = f_b(t)$ for each $b \in B$. By the definition of the maps f_b we see that indeed $\Psi \circ \iota_b(1) = \beta_0(b)$.

To see that Ψ is an isomorphism, we construct the inverse homomorphism. For this, consider the function $j : B \to C$ defined by $j(b) = \iota_b(1)$. According to Proposition 6.1, $F(B, R)$ is a free R-module on β_0. Using the defining propert of a free R-module, we find a unique R-module homomorphism

$$\Phi : F(B, R) \to C \text{ such that } \Phi \circ \beta_0 = j.$$

We must now check that $\Phi \circ \Psi = \mathrm{id}_C$ and $\Psi \circ \Phi = \mathrm{id}_{F(B,R)}$.

Now,

$$(\Phi \circ \Psi \circ \iota_b)(1) = \Phi(\beta_0(b)) = j(b) = \iota_b(1)$$

so that for $\Phi \circ \Psi \circ \iota_b = \iota_b$. But according to the defining property of the direct sum (coproduct), the unique R-module homomomorphism $f : C \to C$ for which $f \circ \iota_b = \iota_b$ is $f - \mathrm{id}_C$. Thus we conclude that $\Phi \circ \Psi = \mathrm{id}_C$.

Similarly,

$$(\Psi \circ \Phi \circ \beta_0)(b) = \Psi(j(b)) = \Psi\Big(\iota_{b(1)}\Big) = \beta_0(b)$$

so that $\Psi \circ \Phi \circ \beta_0 = \beta_0$. But according to the defining property of free R-modules, if $f : F(B, R) \to F(B, R)$ is an R-module homomorphism for which $f \circ \beta_0 = \beta_0$ then $f = \mathrm{id}_{F(B,R)}$. This shows that $\Psi \circ \Phi = \mathrm{id}_{F(B,R)}$ and completes the proof that Φ and Ψ are inverse isomorphisms. ■

Corollary 6.3: If R is a commutative ring and if B is any set, there is an R-module F which is free on the set B.

Proof: Indeed, take $F = F(B, R)$. According to Proposition 6.1, F together with the mapping $\beta_0 : B \to F(B, R)$ determines a free R-module. ■

Theorem 6.4:

Let M be an R-module, let B be a set and let $\beta : B \to F$ be a function. For $b \in B$ consider the R-module homomorphism $\iota_b : R \to M$ given by $\iota_b(r) = r\beta(b)$.

The following are equivalent:

a. β is an R-basis for M

b. M is a free R-module on $\beta : B \to M$.

c. M together with the ι_b form a co-product of the R-modules $M_b = R$.

Proof: $(a \Rightarrow b)$: Let β be a basis; we show that M is free on β. Since β is a basis, we know that any $x \in M$ may be written uniquely in the form $x = \sum_{b \in B} a(b)\beta(b)$ for some $a \in F(B, R)$ where $F(B, R)$ is the R-module of all finitely supported functions $a : B \to R$.

Thus the assignment $x \mapsto a$ defines an isomorphism of R-modules $\Psi : M \to F(B, R)$; moreover, in the notation of Proposition 6.1, we see that $\Psi \circ \beta = \beta_0$. Now the fact that M is free on β follows at once from Proposition 6.1.

$(b \Rightarrow c)$: Suppose that M is free on β. We are going to argue that M together with the ι_b form a co-product of the modules $M_b = R$ in the category R-mod. Thus we suppose that N is any R-module and that $f_b : R \to N$ is an R-module map for each $b \in B$.

We form the function $\varphi : B \to N$ defined by $\varphi(b) = f_b(1)$.

We claim: $(\spadesuit)$ a linear map $\Phi : M \to N$ satisfies $\Phi \circ \beta = \varphi$ if and only if it satisfies

$$\Phi \circ \iota_b = f_b \text{ for all } b \in B.$$

Indeed, from definitions we have

$$\Phi \circ \beta = \varphi \Leftrightarrow \forall b \in B, (\Phi \circ \beta)(b) = \varphi(b) \Leftrightarrow \forall b \in B, (\Phi \circ \iota_b)(1) = f_b(1).$$

Now for each $b \in B$, the R-module homomorphisms $\Phi \circ \iota_b : R \to N$ and $f_b : R \to N$ are equal if and only if they agree at $1 \in R$. This proves the claim.

Since M is free on β, there is a unique linear mapping $\Phi : M \to N$ such that $\Phi \circ \beta = \varphi$. In view of $(\spadesuit)$ it follows that Φ is the unique linear map satsifying $\forall b \in B, \Phi \circ \iota_b = f_b$ as well. This proves that M is a coproduct of the $M_b = R$ as required.

$(c \Rightarrow a)$: Assume that (M, ι_b) is a co-product of the modules $M_b = R$ for $b \in B$. We must show that β is a basis.

According to Proposition 6.2, there is an R-module homomorphism $\Psi : M \to F(B, R)$ such that

$$(\Psi \circ \iota_b)(1) = \beta_0(b)$$

where $F(B, R)$ the is the R-module of finitely supported functions $B \to R$ and where β is the mapping defined in Proposition 6.1.

For $b \in B$, observe that $\iota_{b(1)} = \beta(b) \Rightarrow \Psi(\beta(b)) = \beta_0(b)$; thus $\Psi \circ \beta = \beta_0$.

On the other hand, according to Proposition 6.1, $F(B,R)$ is a free R-module on β_0. Apply the defining property of a free module – see Definition 5.5.1 – to the function $\beta : B \to M$ to obtain an R-module homomorphism $\Phi : F(B,R) \to M$ with the property that $\Phi \circ \beta_0 = \beta$.

We claim that the R-module homomorphisms Φ and Ψ are inverse to one another. Once this claim is established, we see that β_0 is a basis of $F(B,R)$ implies that $\beta = \Phi \circ \beta_0$ is a basis of M.

To prove the claim, first note that

$$\Phi \circ \Psi : M \to M$$

satisfies

$$\Phi \circ \Psi \circ \iota_b = \iota_b;$$

on the other hand, since M is a coproduct, id_M is the unique R-module map such that $\mathrm{id}_M \circ \iota_b = \iota_b$. Thus $\Phi \circ \Psi = \mathrm{id}_M$.

Finally note that

$$\Psi \circ \Phi : F(B,R) \to F(B,R)$$

satisfies

$$\Psi \circ \Phi \circ \beta_0 = \beta_0.$$

Since $F(B,R)$ is a free R-module on β_0, $\mathrm{id}_{F(B,R)}$ is the unique R-module map such that $\mathrm{id}_{F(B,R)} \circ \beta_0 = \beta_0$.

Thus $\Psi \circ \Phi = \mathrm{id}_{F(B,R)}$ as required. This completes the proof.

■

6.1. Algebras

We now return to rings.

Definition 6.1.1: If S is any ring, the center $Z(S)$ of S is the subring

$$Z(S) = \{r \in S \mid \forall x \in S, rx = xr\}$$

or R.

Observe that $Z(S)$ is a commutative ring.

Example 6.1.2: If F is a field, $n \in \mathbb{N}$, $n \geq 1$ and $S = \mathrm{Mat}_n(F)$, then $Z(S) = F.\mathrm{id} \simeq F$; in words, the only matrices which commute with every matrix are scalar multiples of the identity.

Definition 6.1.3: If R is a commutative ring, an R-algebra is a ring A together with a ring homomorphism $R \to Z(A)$.

Remark 6.1.4:

a. If A is an R-algebra, then A is an R-module. In particular, if F is a field, then any F-algebra is an F-vector space.

b. If F is a field, for any F-algebra A the mapping $F \to A$ is injective.

Definition 6.1.5: If A and B are R-aglebras, a homomorphism of R-algebras is a ring homomorphism $\varphi : A \to B$ such that φ is also a homomorphism of R-modules. In symbols:

$$\forall a \in A, \forall r \in R, \varphi(ra) = r\varphi(a).$$

Remark 6.1.6: Observe that for any R-algebra A there is a central subring that is the homomorphic image $\overline{R}$of R.

With this notation, for R-algebras A, B a ring homomorphism $\varphi : A \to B$ is a homomorphism of R-algebras $\Leftrightarrow$ the restriction of φ to the image `overline`$\{R\}$ in $Z(A)$ satisfies $\varphi(\overline{r}) = \overline{r}$ for $r \in R$.

Example 6.1.7:

a. Any commutative ring has the structure of a $\mathbb{Z}$-algebra.

6.2. Integeral Domains and prime ideals

Let R be a commutative ring.

Definition 6.2.1: An element $a \in R$ is said to be a zero-divisor if $a \neq 0$ and if $\exists b \in R, b \neq 0$ such that $ab = 0$.

Definition 6.2.2: R is an integral domain provided that R has no zero-divisors.

Example 6.2.3:

a. The ring $\mathbb{Z}$ of integers is an integral domain. For $n \in \mathbb{Z}$, the quotient ring $\mathbb{Z}/n\mathbb{Z}$ is an integral domain if and only if n is a prime number.

b. Any field is an integral domain.

Definition 6.2.4: An ideal $I \subset R$ is a prime ideal if $I \neq R$ and if

$$a, b \in R, a \cdot b \in I \Rightarrow a \in I \text{ or } b \in I.$$

Proposition 6.2.5: Let $I \subset R$ be an ideal. Then I is prime if and only if R/I is an integral domain.

Proof: $(\Rightarrow)$: Suppose that I is prime. We must show that R/I has no zero-divisors. To this end, let $\alpha \in R = /I$ be a non-zero element. Thus $\alpha = a + I$ for $a \in R$, $a \notin I$.

Now suppose that $\beta \in R/I$ and $\alpha \cdot \beta = 0$. To prove that α is not a zero divisor, we must argue that $\beta = 0$. Write $\beta = b + I$ for $b \in R$. To see that $\beta = 0$, we must arguet that $b \in I$. But $\alpha\beta = 0 \Rightarrow$ that $ab + I$ is zero in R/I which shows that $ab \in I$. Since I is prime and since $a \notin I$, conclude that $b \in I$ as required.

$(\Leftarrow)$: Suppose that R/I is an integral domain. To prove that the ideal I is prime, let $a, b \in R$ and suppose that $ab \in I$. We must show that $a \in I$ or $b \in I$, so suppose that $a \notin I$. We will argue that $b \in I$.

Write $\alpha = a + I, \beta = b + I \in R/I$. Then $ab \in I$ shows that $\alpha\beta = ab + I$ is zero in R/I. Moreover, $a \notin I$ shows that $\alpha \neq 0$ in R/I. Since R/I is an inteegral domain, we conclude that $\beta = 0$. Thus $b \in I$ as required. ■

Definition 6.2.6: An ideal I of R is maximal if $I \neq R$ and if any ideal J with $I \subseteq J \subseteq R$ satisfies $J = I$ or $J = R$.

Proposition 6.2.7: If $I \subset R$ is an ideal of R then I is a maximal ideal if and only if R/I is a field.

Proof: $(\Rightarrow)$: Suppose that I is a maximal ideal. To show that R/I is a field, let $\alpha \in R/I$, $\alpha \neq 0$. We must argue that α is a unit (see Definition 1.3.3). For this, write $\alpha = a + I$ for $a \in R$. Since $\alpha \neq 0$, $a \notin I$. Now consider the ideal $J = \langle I, a \rangle \subseteq R$. Since $a \notin I$, $J \neq I$. Since I is maximal, Definition 6.2.6 shows that $J = R$. Thus $1 \in \langle I, a \rangle$. This means that $1 = x + ab$ for some $x \in I$ and some $b \in R$. Setting $\beta = b + I$ we see that $\alpha\beta = ab + I = 1 + I$ so α is indeed a unit.

$(\Leftarrow)$: Suppose that R/I is a field. We must argue that I is a maximal ideal. If $\pi : R \to R/I$ is the quotient mapping, the assignment

$$J \mapsto \pi^{-1}(J)$$

determines a bijection between the ideals of R/I and the ideals of R containing I. Since the only ideals of the field R/I are 0 and R/I, the only ideals of R containing I are I and R. This proves that I is a maximal ideal of R as required. ■

Corollary 6.2.8: Any maximal ideal is prime.

Proof: Since any field is an integral domain, the Corollary follows from Proposition 6.2.7 and Proposition 6.2.5. ■

6.3. Monoid algebras

Definition 6.3.1: A monoid M is a set M equipped with a binary operation $M \times M \to M$ such that

- the operation is associative: $\forall x, y, z \in M, (xy)z = x(yz)$.
- there is a an identity element $1 \in M$ such that $\forall x \in M, 1x = x1 = x$.

Definition 6.3.2: If M and N are monoids, a function $f : M \to N$ is a monoid homomorphism if $f(xy) = f(x)f(y)$ for every $x, y \in M$.

Example 6.3.3:

a. Monoids are "not quite groups" – there is no requirement that elements have inverses.

b. An example of a monoid that is not a group is the set of natural numbers $\mathbb{N}$ under addition. More generally, for $n \in \mathbb{N}$, the set $\mathbb{N}^m$ is a monoid under addition.

c. If R is a ring, then $(R, \times)$ – i.e. the set R with the operation of multiplication – forms a monoid.

Proposition 6.3.4: Let S be any ring. Let M be a monoid, let

$$a, b : M \to Z(S)$$

be finitely supported functions, and let

$$e : M \to (S, \times)$$

be a monoid homomorphism. Then

$$(\clubsuit) \quad \left(\sum_{m \in M} a_m e(m)\right)\left(\sum_{m \in M} b_m e(m)\right) = \sum_{m \in M} c_m e(m)$$

where $c_m = \sum_{st=m} a_s b_t \in Z(S)$, the sum being taken over all elements $s, t \in M$ for which $st = m$.

Proof: Indeed, since the a and b take values in the center $Z(S)$ and since e is a monoid homomorphism, we see that

$$\left(\sum_{m\in M} a_m e(m)\right)\left(\sum_{n\in M} b_n e(n)\right) = \sum_{m\in M}\sum_{n\in M} a_m b_n \cdot e(m)\cdot e(n) = \sum_{m\in M}\sum_{n\in M} a_m b_n \cdot e(m\cdot n).$$

Now the result follows from the observation that

$$\sum_{m\in M}\sum_{n\in M} a_m b_n \cdot e(m\cdot n) = \sum_{p\in M}\sum_{st=p} a_s b_t \cdot e(p).$$

■

Given a monoid M, we now *define* an R-algebra which as R-module has a basis indexed by M with multiplication defined by (♣) from Proposition 6.3.4.

Proposition/Definition 6.3.5: Let R be a commutative ring and let M be a monoid. The monoid ring $R[M]$ over R is defined to be the free R-module with a basis $\{e(m) : m \in M\}$ with the multiplication defined as follows: each $\alpha, \beta \in R[M]$ may be written

$$\alpha = \sum_{m\in M} a_m e(m) \text{ and } \beta = \sum_{m\in M} b_m e(m)$$

where $a_m, b_m \in R$ and all but finitely many of the coefficients a_m and b_m are 0.

Define

$$\alpha\cdot\beta = \sum_{m\in M} c_m e(m)$$

where

$$c_m = \sum_{s\cdot t=m} a_s\cdot b_t, \text{the sum over } s,t\in M \text{ satisfying } s\cdot t = m.$$

With this multiplication, the R-module $R[M]$ is a R-algebra and the assignment $e : M \to R[M]$ is a monoid homomorhpism. The algebra $R[M]$ is commutative if and only if M is a commutative monoid. The homomorphism $R \to R[M]$ is given by $r \mapsto r\cdot e(1)$.

Proof sketch: First notice that under the indicated rule for multiplication, for $m, n \in M$, we have

$$(\blacklozenge) e(m)e(n) = e(mn).$$

In particular, e will be a monoid homomorphism $M \to (R[M], \times)$ once we show that $R[M]$ is a ring.

One first needs to argue that $R[M]$ is a ring.

First of all, the multiplication $(*)$ is well-defined. For this, notice that all but finitely many of the coefficients $c_{\vec{j}} \in R$ are 0, since that is true of a and b.

Now, $1 = e(1)$ is the multiplicative identity of S. Indeed, every element of $R[M]$ is an R-linear combination of elements of the form $e(m)$, so it suffices to prove that $e(1)$ acts as the identity on $e(m)$ for any $m \in M$. But

$$e(1)e(m) = e(1m) = e(m) \text{ and } e(m)e(1) = e(m1) = e(m).$$

Next, note that multiplication is associative. Indeed, let $\alpha, \beta, \gamma \in R[M]$ with

$$\alpha = \sum_{m \in M} a_m e(m), \beta = \sum_{m \in M} b_m e(m), \gamma = \sum_{m \in M} c_m e(m).$$

For $m \in M$, we define the coefficient $d_m = \sum_{stu=m} a_s b_t c_u \in R$. We claim that

$$(\alpha\beta)\gamma = \sum_{m \in M} d_m e(m) = \alpha(\beta\gamma).$$

To prove the first equality, just note that using (♦) we find

$$(\alpha\beta)\gamma = \left(\sum_m \sum_{st=m} a_s b_t e(m)\right)\left(\sum_n c_n e(n)\right) = \sum_{m,n} \sum_{st=m} a_s b_t c_n e(m)e(n) = \sum_m \sum_{stu=p} d_p e_p.$$

A similar calculation proves the second equality. This shows that the multiplication is associative.

Next we need to confirm that multiplication distributes over addition, i.e. that

$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma \text{ and } (\beta+\gamma)\alpha = \beta\alpha + \gamma\alpha \text{ for } \alpha, \beta, \gamma \in R[M].$$

Since every element of $R[M]$ is an R-linear combination of the elements $e(m)$ for $m \in M$, it is enough to confirm these statements when $\beta = r \cdot e(m)$ and $\gamma = s \cdot e(n)$ for $r, s \in R$ and $m, n \in M$.

Now, if $\alpha = \sum_p a_p e(p)$ for a finitely supported function $a : M \to R$, then by definition we have

$$\begin{aligned}
\alpha \cdot (r \cdot e(m) + s \cdot e(n)) &= \sum_p \left(\sum_{im=p} a_i r + \sum_{jn=p} a_j s \right) e(p) \\
&= \sum_p \left(\sum_{im=p} a_i r \right) + \sum_p \left(\sum_{jn=p} a_j s \right) e(p) \\
&= \alpha \cdot r \cdot e(m) + \alpha \cdot s \cdot e(n).
\end{aligned}$$

A similar calculation shows that $(r \cdot e(m) + s \cdot e(n))\alpha = r \cdot e(m)\alpha + s \cdot e(n)\alpha$. This confirms the distributive law.

Finally, it is straightforward to confirm that $r \mapsto re(1)$ defines a ring homomorphism $R \to Z(R[M])$ so that $R[M]$ is indeed an R-algebra. ■

Proposition 6.3.6: Let S be a ring and let $f : M \to (S, \times)$ be a monoid homomorphism.

a. Let $\iota : R \to Z(S)$ be a ring homomorphism. Then there is a unique ring homomorphism

$$\varphi : R[M] \to S \text{ such that } \varphi_{|R} = \iota \text{ and } \varphi(e(m)) = f(m).$$

b. If S is an R-algebra, there is a unique homomorphism of R-algebras

$$\psi : R[M] \to R \text{ such that } \psi(e(m)) = f(m).$$

Proof: b. is an immediate consequence of a.

To prove a., note that using the ring homomorphism ι, we may view S as an R-module. Now the function $f : M \to S$ determines a unique R-module homomorphism $\varphi : R[M] \to S$ such that $\varphi_{|\ R} = \iota$ and $\varphi \circ e = f$.

It only remainds to argue that φ is indeed a ring homomorphism; i.e. that φ preserves multiplication. In view of the definition of multiplication in $R[M]$, this follows from Proposition 6.3.4.■

6.4. The polynomial ring over R

For $m \in \mathbb{N}$, the polynomial ring over R in m variables is the monoid algebra constructed from the monoid $\mathbb{N}^m$. Let us write

$$\delta_j = (0, 0, ..., 0, 1, 0, ..., 0) \in \mathbb{N}^m$$

where 1 in the jth position.

Proposition 6.4.1: Let S be a ring, let $m \in \mathbb{N}$, and let $a_1, ..., a_m \in Z(S)$. The function

$$\varphi : \mathbb{N}^m \to (S, \times) \text{ given by } \varphi(\vec{i}) = a_1^{i_1} ... a_m^{i_m}$$

is the unique monoid homomorphism for which $\varphi\big(\delta_j\big) = T_j$ for $1 \leq j \leq m$, where

Proof: The uniqueness statement follows from the observation that

$$\vec{i} = \sum_{j=1}^{m} i_j \delta(j) \text{ so that } \varphi(\vec{i}) = \sum_{j=1}^{m} \varphi\big(\delta_j\big)^{i_j}.$$

Since the $a_1, ..., a_m$ are central in S, for $\vec{i}, \vec{j} \in \mathbb{N}^m$ we have

$$\varphi(\vec{i} + \vec{j}) = a_1^{i_1 + j_1} ... a_m^{i_m + j_m} = a_1^{i_1} ... a_m^{i_m} a_1^{j_1} ... a_m^{j_m} = \varphi(\vec{i}) \cdot \varphi(\vec{j})).$$

Thus φ is indeed a monoid homomorphism. ■

Let R be a commutative ring. Consider the additive monoid $\mathbb{N}^m$ for $m \in \mathbb{N}$

Definition 6.4.2: The polynomial ring over R in m variables is the monoid algebra of the additive monoid $\mathbb{N}^m$.

We write $R[T_1, ..., T_m]$ for the polynomial ring over R. For an element $\vec{i} \in \mathbb{N}^m$ we write $T_1^{i_1} ... T_m^{i_m}$ for the element $e(\vec{i})$ of the monoid algebra.

Remark 6.4.3:

a. Compare with [Dummit-Foote, Ch. 7.2]
b. Notice that the polynomial ring $R[T_1, ..., T_m]$ is a commutative R-algebra (since $\mathbb{N}^m$ is a commutative monoid).
c. Note for $\vec{i}, \vec{j} \in \mathbb{N}^m$ that $T^{\vec{i}} T^{\vec{j}} = T^{\vec{i} + \vec{j}}$.
d. The *variables* T_i of the polynomial ring correspond to the elements $e(\delta_i)$ of the monoid algebra, where $\delta_i = (0, 0, ..., 0, 1, 0, ..., 0) \in \mathbb{N}^m$ has 1 in the ith position.
e. Elements $f, g \in S = R[T_1, ..., T_n]$, the elements may be written uniquely in the form

$$f = \sum_{\vec{i}\in\mathbb{N}^m} a_{\vec{i}} T^{\vec{i}} \text{ and } g = \sum_{\vec{i}\in\mathbb{N}^m} b_{\vec{i}} T^{\vec{i}}$$

for coefficients $a_{\vec{i}}, b_{\vec{i}} \in R$ where all but finitely many of the $a_{\vec{i}}$ are 0, and all but finitely many of the $b_{\vec{i}}$ are 0.

The product of f and g is given by the formula

$$(*) \quad fg = \sum_{\vec{i}} \left(\sum_{\vec{s}+\vec{t}=\vec{i}} s_{\vec{s}} \cdot b_{\vec{t}} \right) T^{\vec{i}}.$$

Proposition 6.4.4: Let S be a ring and suppose that $a_1, ..., a_n \in Z(S)$.

a. If $\iota : R \to Z(S)$ is a ring homomorphism. Then there is a unique ring homomorphism

$$\varphi : R[T_1, ..., T_n] \to S$$

with the property that $\varphi_{|R} = \iota$ and $\varphi(T_i) = a_i$ for $1 \leq i \leq n$.

b. If S is an R-algebra, there is a unique R-algebra homomomorphism $\psi : R[T_1, ..., T_n] \to S$ such that $\psi(T_i) = a_i$ for $1 \leq i \leq n$.

Proof: First observe that according to Proposition 6.4.1 there is a unique monoid homomorphism $\gamma : \mathbb{N}^m \to (S, \times)$ with the property that $\gamma(\delta_j) = a_j$.

Now the existence of the required ring and algebra homomorphisms follows from Proposition 6.3.6. ■

6.5. Zorn's Lemma

Let $(A, \leq)$ be a partially ordered set; thus the relation $\leq$ is reflexive, antisymmetric and transitive.

Definition 6.5.1: Let $B \subseteq A$.

B is a chain in A if $\forall x, y \in A, x \leq y$ or $y \leq x$.

An element $u \in A$ is an upper bound for B if $\forall x \in B, x \leq u$.

An element $m \in A$ is a maximal in A if $\forall x \in A, m \leq x \Rightarrow x = m$.

Example 6.5.2:

a. Consider the partially ordered set $A = \texttt{cal}[P](X)$, the power set of some set X, where $\leq = \subseteq$ is set-containment.

An example of a chain is a collection of subsets X_i for $i \in \mathbb{Z}$ with

$$... \subseteq X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq ...$$

Any subset B of A has an upper bound, namely

$$u = \bigcup_{b\in B} b$$

is an upper bound for B.

In this case, A has a (unique) maximal element, namely X.

b. Let A be the set of proper subsets of $\mathbb{N}$. Then

$$\{0\} \subseteq \{0,1\} \subseteq \{0,1,2\} \subseteq \dots$$

describes a chain in A which has no upper bound in A.

On the other hand, A has (lots of) maximal elements, namely $\mathbb{N} \setminus \{n\}$ for each $n \in \mathbb{N}$.

Zorn's Lemma is the following statement:

If A is a non-empty partially ordered set in which every chain has an upper bound, then A has a maximal element.

The **Axiom of choice** is the statement: if I is a non-empty index set and if A_i is a set for each $i \in I$, then the cartesian product $\prod_{i\in I} A_i$ is non-empty. Thus, there is a choice function $f : I \to \cup_{i\in I} A_i$ with $f(j) \in A_j$ for each $j \in I$.

Theorem 6.5.3: Assuming usual axioms of set theory, Zorn's Lemma and the Axiom of Choice are logically equivalent.

Remark 6.5.4: In fact, the Axiom of Choice is independent of the axioms of set theory; namely the axioms of set theory together with the Axiom of choice are consistent, and the axioms of set theory together the negtation of the axiom of choice are consistent.

In this course, we use the axioms of set theory together with the axiom of choice.

6.6. Existence of maximal ideals.

Theorem 6.6.1: Let A be a ring (with identity). Then every proper ideal of A is contained in a maximal ideal.

Proof: Let $I \subset A$ be a proper ideal. Since $1 \notin I$, we know that $1 \neq 0$ and so $A \neq \{0\}$.

Let $\mathcal{S}$ be the set of proper ideals of R which contain I. Since $I \in \mathcal{S}$, $\mathcal{S} \neq \emptyset$. Moreover, $\mathcal{S}$ is partially ordered by inclusion.

If C is a chain in $\mathcal{S}$, define the ideal $J = J_C$ to be

$$J_C = \bigcup_{I\in C} I.$$

We first show that $J = J_C$ is an ideal of R. If $a, b \in J$, there are ideals A, B in C so that $a \in A$ and $b \in B$. Since C is a chain, either $A \subseteq B$ or $B \subseteq A$. In either case, $a - b \in J$. Since each $A \in C$ is closed under left and right multiplication by R, so is J. This proves that J is an ideal of R.

Notice that $I \subseteq J$ since $I \subseteq A$ for any $A \in C$. Finally we claim that J is a proper ideal. For this, suppose by way of contradiction that $J = R$. Then $1 \in J$ so that $1 \in I$ for some $I \in C$. In that case, $I = R$, contrary to the fact that $\mathcal{S}$ consists of proper ideals.

Note that J is an upper bound for the chain C. Thus we have confirmed that each chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$.

By Zorn's Lemma, the partially ordered set $\mathcal{S}$ has a maximal element, which is a maximal ideal of R containing I. ■

Example 6.6.2: Let k be a field, let $k[T_1, ..., T_n]$ be the polynomial ring over k, and let $a_1, ..., a_n \in k$.

7. Week 7 [2025-10-14]

7.1. The rank of a free R-module

Let R be a commutative ring.

Definition 7.1.1: If F is a free R-module on $\beta : B \to F$, the rank of the R-module F is the cardinality of B, denoted $\mathrm{rank}_R(F)$.

Theorem 7.1.2: Let F be an R-module and let $\beta_i : B_i \to F$ for $i = 1, 2$ be functions for sets B_1 and B_2. Suppose that F is a free R-module on β_1, and that F is a free R-module on β_2. Then the cardinality of B_1 is equal to the cardinality of B_2.

In particular, $\mathrm{rank}_R(F)$ is a well-defined cardinal.

Proof: We may suppose that $R \neq \{0\}$. According to Theorem 6.6.1, we may find a maximal ideal M of R. You showed in a homework problem that F is a free R-module on β_i if and only if F/MF is a free R/M-module on β_i.

Thus we have reduced the proof to the case in which $R = K$ is a *field* in this case, the result is known from *linear algebra*. ■

Remark 7.1.3: Let K be a field, let V be a vector space over K, and let $\beta_i : B_i \to V$ be bases for $i = 1, 2$.

To prove that the cardinality of B_1 is equal to that of B_2, one can proceed as follows.

In case $|B_1| = n$ and $|B_2| = m$, one uses the *Replacement Theorem* (see e.g. [Dummit-Foote, § 11.1 Theorem 3]) which states that for a suitable ordering $B_1 = \{a_1, \ldots, a_n\}$ and $B_2 = \{b_1, \ldots, b_m\}$ that $\{b_1, \ldots, b_k, a_{k+1}, \ldots, a_n\}$ is a basis for each $k \in \{1, 2, \ldots, m\}$. In particular, $n \geq m$. Using this result twice establishes $n = m$.

Finally, suppose that *a priori* only one of the basis is known to be finite. If $|B_1| = n$, one shows that any set of $n + 1$ vectors is linearly independent; this forces B_2 to be finite as well, so that $|B_1| = |B_2|$ by the Replacement Theorem.

It remains true that bases have the same cardinality even when they are infinite, though I won't give a proof here.

For vector spaces, the *rank* is known as *dimension* and one usually writes

$$\dim_F V = \mathrm{rank}_F V.$$

7.2. Principal Ideal Domains

Definition 7.2.1: A commutative ring A is a principal ideal domain (or PID) if A is an integral domain and if every ideal of A is a principal ideal.

Definition 7.2.2: Let A be a commutative ring and let $a, b \in A$.

- a divides b – in symbols, $a \mid b$ – provided that $\exists x \in A, b = ax$.
- a greatest common divisor of a and b is an element $c \in A$ such that
  - $c \mid a$ and $c \mid b$, and
  - if $d \mid a$ and $d \mid b$ for $d \in A$, then $d \mid c$.

We write $\gcd(a, b)$ for a greatest common divisor of a and b, provided it exists.

Proposition 7.2.3: Let A be a PID, let $a, b \in A$.

a. An element $d \in A$ is a greatest common divisor of a and b – i.e. $d = \gcd(a, b)$ – if and only if d is a generator for the ideal $\langle a, b\rangle$ – i.e. $\langle d\rangle = \langle a, b\rangle$.
b. Any $d = \gcd(a, b)$ has the form $d = xa + yb$ for suitable $x, y \in R$.
c. If d, d' are gcds of a and b, then $d' = ud$ for a unit $u \in R^\times$.

Proof:

a. If d is a generator for $\langle a, b\rangle$ then $a, b \in \langle d\rangle$ so that $d \mid a$ and $d \mid b$. Now suppose that $e \in A$ and that $e \mid a$ and $e \mid b$. It follows that $\langle a, b\rangle \subseteq \langle e\rangle$ so that $\langle d\rangle \subseteq \langle e\rangle$. We deduce that $e \mid d$ and conclude that $d = \gcd(a, b)$.

   On the other hand, suppose $d = \gcd(a, b)$. We prove that d is a generator for $\langle a, b\rangle$. By assumption we know $d \mid a$ and $d \mid b$. This shows that $\langle a, b\rangle \subseteq \langle d\rangle$; to prove that equality holds, it remains to show that $\langle d\rangle \subseteq \langle a, b\rangle$.

   Let e be a generator for $\langle a, b\rangle = \langle e\rangle$. Since $a, b \in \langle e\rangle$ deduce that $e \mid a$ and $e \mid b$. Since d is a gcd, conclude that $e \mid d$. This proves that $\langle d\rangle \subseteq \langle e\rangle = \langle a, b\rangle$ as required.

b. The assertion follows since a gcd is a generator for the ideal $\langle a, b\rangle$ by a.

c. In view of a., the assumption means that

$$\langle d\rangle = \langle a, b\rangle = \langle d'\rangle.$$

   Now $d \in \langle d'\rangle \Rightarrow d = d'\alpha$ for $\alpha \in R$ and $d' \in \langle d\rangle \Rightarrow d' = d\beta$ for $\beta \in R$.

   Now we see that $d = d'\alpha = d\alpha\beta$. Since A is an integral domain, we may cancel the factor of d and conclude that $1 = \alpha\beta$. Thus $\alpha, \beta \in R^\times$ and the proof of c. is complete.

■

Proposition 7.2.4: Let A be a PID. If P is a non-zero prime ideal of A, then P is a maximal ideal. Thus A/P is a field.

Proof: Let $P = \langle p\rangle$ be a non-zero prime ideal A, for $0 \neq p \in A$.

Let $I = \langle m\rangle$ be any ideal containing P. We must show that $I = P$ or $I = A$.

Since $P = \langle p\rangle \subseteq I = \langle m\rangle$, we know that $m \mid p$; i.e. $p = m\alpha$ for some $\alpha \in A$. Since $m\alpha \in P$ and since P is a prime ideal, either $m \in P$ or $\alpha \in P$.

If $m \in P = \langle p \rangle$ then $m = \beta p$ for some $\beta \in R$. Thus

$$p = m\alpha = p\alpha\beta.$$

Since A is an integral domain, conclude that $1 = \alpha\beta$ so that $\beta \in R^\times$. This implies that p and $m = \beta p$ generate the same ideal; i.e. $P = I$.

Finally, if $\alpha \in P = \langle p \rangle$ then $\alpha = p\beta$ for some $\beta \in R$. Now, $p = m\alpha = mp\beta$ and since A is an integral domain, we find that $1 = m\beta$. This shows that $m \in R^\times$ so that $I = \langle m \rangle = R$, as required. ■

We are are going to show in the next section that examples of PIDs are obtained from so-called Euclidean domains; these will include the integers $\mathbb{Z}$ and the ring $K[T]$ of polynomials where K is any field.

7.3. Euclidean domains

Let A be a commutative ring.

Definition 7.3.1: A function $N : A \to \mathbb{N} \cup \{-\infty\}$ is called a norm provided that $N(x) = -\infty \Leftrightarrow x = 0$.

Definition 7.3.2: A is called a Euclidean domain provided that A is an integral domain and there is a norm N on A such that for any pair of elements $a, b \in A$ with $b \neq 0$, $\exists q, r \in R$ such that

$$a = qb + r \text{ with } N(r) < N(b).$$

Proposition 7.3.3: Suppose that A is an integral domain and suppose that $a, b \in R$ with $a = qb + r$. Then

$$\langle a, b \rangle = \langle b, r \rangle$$

Proof: We must argue that $a \in \langle b, r \rangle$ and that $r \in \langle a, b \rangle$ But the equation $a = qb + r$ makes these assertions evident. ■

Algorithm 7.3.4 (*The Euclidean algorithm*): Let A be a Euclidean domain with norm N, and let $a, b \in A$ with $b \neq 0$.

Write $a = qb + r_0$ with $N(r_0) < N(b)$ so that $\langle a, b\rangle = \langle b, r_0\rangle$ by Proposition 7.3.3.

Now suppose for $m \in \mathbb{N}$ that $r_0, r_1, ..., r_m \in R$ have been chosen so that

$$(\clubsuit) \quad N(r_m) < N(r_{m-1}) < ... < N(r_1) < N(r_0) < N(b)$$

and so that

$$(\spadesuit) \quad \langle a, b\rangle = \langle b, r_0\rangle = \langle r_0, r_1\rangle = ... = \langle r_{m-1}, r_m\rangle.$$

Since A is a Euclidean domain, if $r_m \neq 0$ we may write

$$r_{m-1} = qr_m + r_{m+1} \text{ for } q, r_{m+1} \in A \text{ with } N(r_{m+1}) < N(r_m);$$

then $\langle a, b\rangle = \langle r_m, r_{m+1}\rangle$.

Continuing in this fashion, so long as $r_m \neq 0$ we may inductively construct a sequence of elements of R satisfying ($\clubsuit$) and ($\spadesuit$). Since $N(x) \in \mathbb{N}$ for $x \neq 0$, and since there can be no infinite decreasing sequence of natural numbers, ($\clubsuit$) shows that eventually we must have $r_n = 0$ for some $n \in \mathbb{N}$.

Thus $\langle a, b\rangle = \langle r_{n-1}, r_n\rangle = \langle r_{n-1}\rangle$ which shows that $\langle a, b\rangle$ is a principal ideal.

Proposition 7.3.5: Let A be a Euclidean domain. Then A is a principal ideal domain.

Proof: Let $0 \neq I \subseteq A$ be an ideal of A, and let $z \in I \setminus \{0\}$ be an element for which $N(z)$ is as small as possible.

We claim that $I = \langle z\rangle$. We must prove that $I \subseteq \langle z\rangle$. Let $x \in I$ and write

$$x = qz + r \text{ for } q, r \in A \text{ with } N(r) < N(z).$$

Then $r \in \langle x, z\rangle \subseteq I$. If $r \neq 0$, the condition $N(r) < N(z)$ contradicts the choice of z as an element for which $N(z)$ is minimal among $\{N(x) \mid x \in I \setminus \{0\}\}$. Conclude that $r = 0$ so that $x = qz \in \langle z\rangle$ as required. ■

Example 7.3.6: The ring $A = \mathbb{Z}$ of ordinary integers is a Euclidean domain. The norm is given by

$$N(x) = \begin{cases} |x| \text{ if } x \neq 0 \\ -\infty \text{ if } x = 0. \end{cases}$$

In particular, $\mathbb{Z}$ is a principal ideal domain.

7.4. The polynomial ring over a field.

Let K be a field and let $A = K[T]$ be the polynomial ring in a single variable over K. We are going to prove that A is a Euclidean domain.

Recall that an element f of $A = K[T]$ has the form

$$f = \sum_{i=0}^{N} a_i T^i \text{ for } a_i \in K$$

or

$$f = \sum_{i \in \mathbb{N}} a_i T^i \text{ where } a : \mathbb{N} \to K \text{ has finite support.}$$

Definition 7.4.1: The degree of a non-zero element $f = \sum_{i \in \mathbb{N}} a_i T^i$ – written $\deg(f)$ – is the maximum i for which $a_i \neq 0$. Put another way, $\deg(f)$ is the maximal element of the support of a.

We set $\deg(0) = -\infty$.

Proposition 7.4.2: If $f, g \in K[T]$ then $\deg(fg) = \deg(f) + \deg(g)$.

Proof: Write $f = \sum_{i \in \mathbb{N}} a_i T^i$ and $g = \sum_{i \in \mathbb{N}} b_i T^i$ for finitely supported functions $a, b : \mathbb{N} \to K$.

Then

$$fg = \sum_{i \in \mathbb{N}} \left(\sum_{s+t=i} a_s b_t \right) T^i$$

and the formula for the degree of fg follows at once. ■

Proposition 7.4.3: If $f, g \in k[T]$ are non-zero polynomials, then $\deg(f+g) \leq \max(\deg(f), \deg(g))$. If $\deg(f) \neq \deg(g)$ then equality holds.

Proof: Write $f = \sum_{i \in \mathbb{N}} a_i T^i$ and $g = \sum_{i \in \mathbb{N}} b_i T^i$ for finitely supported functions $a, b : \mathbb{N} \to K$. Then

$$f + g = \sum_{i \in \mathbb{N}} (a_i + b_i) T^i.$$

Thus $\deg(f) \leq n$ and $\deg(g) \leq m$ implies that $\deg(f+g) \leq n+m$ which proves the first assertion.

If $\deg(f) = n > m = \deg(g)$, then the coefficient of T^n in $f+g$ is just a_n so indeed

$$\deg(f+g) = n.$$

■

Theorem 7.4.4: $A = K[T]$ is a Euclidean domain for the norm $N = \deg : K[T] \to \mathbb{N} \cup \{-\infty\}$. In particular, $K[T]$ is a PID.

More precisely, if $f, g \in K[T]$ with $g \neq 0$. there exist *unique* elements $q, r \in K[T]$ with

$$f = qg + r \text{ and } N(r) < N(g).$$

Proof: To prove that A is a Euclidean domain and hence a PID, it suffices to prove the last assertion of the Theorem statement.

Let $f, g \in K[T]$ with $g \neq 0$. If $f = 0$, take $q = 0$ and $r = 0$ and observe that the required properties hold.

We now suppose that $f \neq 0$. If $\deg f < \deg g$, we may take $q = 0$ and $r = f$; then $N(r) < N(g)$ and $f = 0 \cdot g + r$.

Thus we may and will suppose that $\deg g \leq \deg f$. We now proceed by induction on $n = \deg f$.

If $\deg f = 0$, then f and g are both non-zero constant polynomials, say $f = a_0$ and $g = b_0$ with $a_0, b_0 \in K^\times$.

Then $f = a_0 = (a_0/b_0)b_0 + 0$ so we may take $q = a_0/b_0 \in K$ and $r = 0$ to satisfy the requirements.

Thus the result holds in the base case where $\deg f = 0$.

Now suppose that $m \in \mathbb{N}$ and that the result is known whenever f has degree $\leq m$.

Suppose that f has degree $m+1$ and that g has degree $\leq m+1$. Write

$$f = \sum_{i=0}^{m+1} a_i T^i \text{ and } g = \sum_{i=0}^{k} b_i T^i$$

where $a_{m+1} \neq 0$ and $b_k \neq 0$ and $k \leq m+1$.

Let

$$f' = f - (a_{m+1}/b_k)T^{m+1-k}g;$$

then $\deg f' \leq m$.

Applying the induction hypothesis, we may write

$$f' = qg + r \text{ where} \deg(r) < \deg(g) = k.$$

Now we find

$$f = f' + (a_{m+1}/b_k)T^{m+1-k}g = \big(q + (a_{m+1}/b_k)T^{m+1-k}\big)g + r$$

so that $f = q_1 g + r$ with $q_1 = (q + (a_{m+1}/b_k)T^{m+1-k}$ and $\deg(r) < \deg(g)$.

This proves that $K[T]$ is a Euclidean domain. It remains to prove the uniqueness assertion. So suppose that $f = qg + r = q_1 g + r_1$ with $\deg(r) < \deg(g)$ and $\deg(r_1) < \deg(g)$.

Then

$$(q - q_1)g = r_1 - r$$

so that g divides $r_1 - r$. Now, Proposition 7.4.3 shows that $\deg(r_1 - r) < \deg(g)$.

On the other hand, Proposition 7.4.2 shows that $\deg(g) + \deg(q - q_1) = \deg(r_1 - r)$. Since $\deg(r_1 - r) < \deg(g)$, this is only possible if $q - q_1 = r - r_1 = 0$.

■

Proposition 7.4.5: Let K be a field and let $g = \sum_{i=0}^{n} a_i T^i \in K[T]$ with $a_n \neq 0$. Then $A = K[T]/\langle g \rangle$ is a K-algebra with $\dim_K A = n$.

Proof: Write $I = \langle g \rangle$. We claim that $B = \{T^i + I \mid 0 \leq i < n\}$ forms a K-basis for $A = K[T]/I$.

For $\alpha \in A$, write $\alpha = f + I$ for some $f \in K[T]$. Now use Theorem 7.4.4 to write $f = qg + r$ for some $r \in K[T]$ with $\deg r < \deg g$.

Since $I = \langle g \rangle$, we see that $f - r \in I$ so that $\alpha = f + I = r + I$.

Since $\deg(r) < n$, clearly r is in the span of B. Thus we have proved that B spans $A = K[T]/I$ as a K-module.

To see that B is linearly independent over F, suppose that $(*)$ $\sum_{i=0}^{n-1} a_i T^i = 0$ for $a_i \in K$.

Put $h = \sum_{i=0}^{n-1} a_i \in K[T]\}$. Then $(*)$ implies that $h + I$ is the zero element of $K[T]/I$. Thus $h \in I$ so that $g \mid h$.

Since $\deg(g) = n$ and since $\deg h < n$, this is only possible if $h = 0$ in $K[T]$. Since $\{T^i \mid i \in \mathbb{N}\}$ is a K-basis for $K[T]$ we now conclude that $a_i = 0$ for each i. This confirms that B is linearly independent over K as required. ■

Example 7.4.6: Let K be a field and let $\alpha \in K$. Let $\varphi : K[T] \to K$ be the unique homomorphism of K-algebras for which $\varphi(T) = \alpha$; see Proposition 6.4.4.

We claim that $\ker \varphi = \langle T - \alpha \rangle$. First of all, $\varphi(T - \alpha) = 0$ so that $\langle T - \alpha \rangle \subseteq \ker \varphi$.

On the other hand, Proposition 7.4.5 shows that $K[T]/\langle T - \alpha \rangle$ is a K-algebra of dimension 1 as K-vector space. Thus the natural map $K \to K[T]/\langle T - \alpha \rangle$ is an isomorphism.

This shows that $\langle T - \alpha \rangle$ is a maximal ideal. Since $\langle T - \alpha \rangle \subseteq \ker \varphi$ we conclude that equality holds:

$$\langle T - \alpha \rangle = \ker \varphi.$$

Finally, if $\alpha, \beta \in K$ and $\alpha \neq \beta$ then $\langle T - \alpha \rangle \neq \langle T - \beta \rangle$.

Indeed, let $\varphi_\alpha : K[T] \to K$ be the evaluation homomorphism with $\varphi_{\alpha(T)} = \alpha$.

Notice that $\varphi_{\alpha(T-\beta)} = \alpha - \beta \neq 0$ since $\alpha \neq \beta$. Thus $T - \beta \notin \ker \varphi_\alpha = \langle T - \alpha \rangle$

7.5. The Chinese Remainder Theorem

Let R be a commutative ring.

Definition 7.5.1: Two ideals A and B of R are said to be comaximal if $A + B = R$.

Remark 7.5.2: If R_i for $i \in I$ are rings, the product $\prod_{i\in I} R_i$ in the category of rings is the product of the additive abelian groups together with a suitable multiplication. Identifying the product of the abelian groups as functions $f : I \to \cup_{i\in I} R_i$ for which $f(j) \in R_j$, the product of functions f, g in the ring $\prod_{i\in I} R_i$ is the *pointwise product* of f and g; thus for $j \in I$, $(fg)(j) = f(j)g(j)$ where the product is taken in R_j.

Remark 7.5.3: For ideals A, B of R recall that the product AB is the ideal generated by $\{ab \mid a \in A, b \in B\}$. We always have $AB \subseteq A \bigcap B$.

Proposition 7.5.4: Let A, B be ideals of R which are comaximal and consider the mapping

$$\Phi : R \to R/A \times R/B \text{ given by } \Phi(r) = (r + A, r + B).$$

Then Φ is a surjective ring homomorphism and $\ker \Phi = A \bigcap B = AB$.

Proof: Φ is the map to $R/A \times R/B$ determined by the mapping property of the product using the quotient maps $\pi_A : R \to R/A$ and $\pi_B : R \to R/B$; in particular, Φ is a ring homomorphism.

Since A and B are co-maximal, we may write $1 = a + b$ for some $a \in A$ and $b \in B$.

Let us show that Φ is surjective. Let $(\alpha, \beta) \in R/A \times R/B$. Thus $\alpha = s + A$ and $\beta = t + B$ for $s, t \in R$.

Let $r = sa + tb \in R$. Notice the following:

$$s = s1 = sa + sb \equiv sa \equiv r (\texttt{mod } B)$$

and

$$t = t1 = ta + tb \equiv tb \equiv r (\texttt{mod } A).$$

This shows that $\Phi(r) = (a + A, b + B) = (\alpha, \beta)$ so that Φ is surjective.

Finally, we argue that $\ker \Phi = AB$. For $a \in A$ and $b \in B$, notice that $\Phi(ab) = 0$; since AB is generated by such products, we see that $AB \subseteq \ker \Phi$.

Finally, notice that by the definition of the product, $\ker \Phi = \ker \pi_A \bigcap \ker \pi_B = A \bigcap B$.

It remains to argue that $A \bigcap B = AB$.

As was pointed out in Remark 7.5.3, we know that $AB \subseteq A \bigcap B$. To prove equality, suppose that $x \in A \bigcap B$.

Now note that $x = x \cdot 1 = xa + xb$. Since $x \in B$, $xa \in AB$. Since $x \in A$, $xb \in AB$. Thus $x = xa + xb \in AB$ and the proof is complete.

■

Theorem 7.5.5: Let $n \geq 1$ and let A_i be an ideal of R for each $i \in I_n = \{0, 1, ..., n-1\}$.

a. The mapping

$$\Phi : R \to \prod_{i \in I_n} R/A_i \text{ given by } \Phi(r) = \big(j \mapsto r + A_j\big)$$

is a ring homomorphism whose kernel is $\bigcap_{i \in I} A_i = A_0 \cap A_1 \cap ... \cap A_{n-1}$.

b. Assume that $\forall i, j \in I_n, i \neq j \Rightarrow A_i, A_j$ are comaximal. Then Φ is surjective and moreover

$$\bigcap_{i \in I} A_i \text{ is equal to the product } A_0 A_1 ... A_{n-1}.$$

Proof: First of all, the mapping $\Phi : R \to \prod_{i \in I} R/A_i$ is just precisely the mapping determined by the universal property of the product using the quotient maps $R \to R/A_j$ for each $j \in I$. Thus Φ is a ring homomorphism.

We give the proof by induction on n. When $n = 1$ the map Φ is just the quotient map $R \to R/A_0$ and there is nothing to prove in either a. or b.

Now suppose that $m > 0$ that $A_0, A_1, ..., A_{m+1}$ are ideals.

For the proof of (a), the induction hypothesis is that the homomorphism

$$\Phi_m : R \to \prod_{i \in I_m} R/A_i$$

has kernel $\bigcap_{i \in I_m} A_i$, and we must show that

$$\Phi_{m+1} : R \to \prod_{i \in I_{m+1}} R/A_i$$

has kernel $\bigcap_{i \in I_{m+1}} A_i$.

We may identify

$$\prod_{i \in I_{m+1}} R/A_i \text{ with } \left(\prod_{i \in I_m} R/A_i \right) \times R/A_m,$$

and then Φ_{m+1} is the mapping

$$(\Phi_m, \pi) : R \to \left(\prod_{i \in I_m} R/A_i \right) \times R/A_m$$

determined by Φ_m and the quotient map $\pi : R \to R/A_m$. Thus $\ker \Phi_{m+1} = \ker \Phi_m \bigcap \ker \pi$. So the induction hypothesis implies that

$$\ker \Phi_{m+1} = \ker \Phi_m \cap A_m = \left(\bigcap_{i \in I_m} A_i \right) \bigcap A_m = \bigcap_{i \in I_{m+1}} A_i.$$

Now the assertion of (a) follows by induction.

For the proof of (b), we suppose that $\forall i, j \in I_{m+1}, i \neq j \Rightarrow A_i + A_j = R$. The induction hypothesis is that Φ_m is surjective and that $\ker \Phi_m$ is the product $A_0 A_1 ... A_{m-1}$. We must prove that Φ_{m+1} is surjective and that $\ker \Phi_{m+1}$ is the product $A_0 A_1 ... A_m$.

Again identify

$$\prod_{i \in I_{m+1}} R/A_i \text{ with } \left(\prod_{i \in I_m} R/A_i \right) \times R/A_m$$

and Φ_{m+1} with

$$\left(\Phi_m, \pi_{A_m}\right) : R \to \left(\prod_{i \in I_m} R/A_i \right) \times R/A_m.$$

Put $B = \ker \Phi_m$. According to the induction hypothesis, we have

$$B = A_0 A_1 ... A_{m-1}.$$

For $0 \leq i \leq m-1$, A_i and A_m are comaximal, so we may write $1 = a_i + c_i$ for $a_i \in A_i$ and $c_i \in A_m$.

Then

$$1 = 1^m = \prod_{i=0}^{m-1} (a_i + c_i) = a_0 a_1 ... a_{m-1} + c \text{ for some } c \in A_m.$$

This shows that B and A_m are comaximal.

Now we may apply Proposition 7.5.4 to see that Φ_{m+1} is surjective and that

$$\ker \Phi_{m+1} = B \cap A_m = BA_m = (A_0 A_1 ... A_{m-1}) A_m$$

as required.

Thus the Theorem follows by induction. ■

Remark 7.5.6: If the ideal $A_i = Ra_i$ are all principal, then

$$A_0 A_1 ... A_{m-1} = Ra_0 a_1 ... a_{m-1}.$$

Example 7.5.7: Let K be a field. For $m \in \mathbb{N}$ let $\alpha_1, ..., \alpha_m \in K$ be *distinct* elements. We claim that

$$(\heartsuit) \quad K[T]/\langle (T-\alpha_1)(T-\alpha_2)...(T-\alpha_m) \rangle \xrightarrow{\sim} \prod_{i=1}^{m} K.$$

Indeed for $i \neq j$, $\langle T - \alpha_i \rangle$ and $\langle T - \alpha_j \rangle$ are distinct maximal ideals of $K[T]$ by Example 7.4.6. In partiuclar these ideals are co-maximal Now ($\heartsuit$) follows from Theorem 7.5.5.

8. Week 9 [2025-10-27]

8.1. Generation of algebras

Let R be a commutative ring and let A and an R-algebra.

If $x_1, x_2, ..., x_n \in A$ recall that there is an R-algebra homomorphism $\varphi : 1, T_2, ..., T_n] \to A$ with the property that $\varphi(T_i) = x_i$; see Proposition 6.4.4.

Definition 8.1.1: The R-subalgebra of A generated by $x_1, x_2, ..., x_n$ is defined to be the image of the algebra homomorphism φ, and it is denoted $R[x_1, x_2, ..., x_n]$.

Remark 8.1.2: Be careful: the notation $A = R[x_1, x_2, ..., x_n]$ looks a bit like the notation for the polynomial ring $R[T_1, T_2, ..., T_n]$, but A need not be isomorphic to a polynomial ring.

Example 8.1.3:

a. Let $i \in \mathbb{C}$ be a square root of -1. Then $\mathbb{Z}[i]$ is the $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by i.

   By definition $\mathbb{Z}[i]$ consists of the complex numbers $f(i)$ where $f \in \mathbb{Z}[T]$ is a polynomial.

   Since $T^2 + 1$ is a monic polynomial, an analogue of Theorem 7.4.4 shows for any $f \in \mathbb{Z}[T]$ that we may write

$$f = q \cdot (T^2 + 1) + r \text{ with } q, r \in \mathbb{Z}[T], \deg r < 2.$$

   Thus $f(i) = r(i)$ which shows that $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$.

   It is easy to argue that $\mathbb{Z}[i]$ is a free $\mathbb{Z}$-module with basis $\{1, i\}$.

b. You can describe $\mathbb{Z}\left[\sqrt{2}\right]$ or more generally $\mathbb{Z}\left[\sqrt{d}\right]$ as in a.

c. An $\alpha \in \mathbb{C}$ is *transcendental* over $\mathbb{Q}$ if it is not the root of any polynomial $f \in \mathbb{Q}[T]$. For example, the real numbers e and π are known to be transcendental.

   For α transcendental, the subalgebra $\mathbb{Z}[\alpha]$ is isomorphic to the polynomial ring $\mathbb{Z}[T]$.

   Indeed, the evaluation mapping

$$\mathbb{Z}[T] \to \mathbb{Z}[\alpha]$$

   is surjective by definition, and for transcendental α the kernel of this mapping is trivial.

8.2. Unique factorization

Let A be a commutative ring.

Definition 8.2.1: An element $a \in A$ is irreducible if $a \neq 0$, $a \notin A^\times$ and whenever $a = xy$ with $x, y \in A$ then either $x \in A^\times$ or $y \in A^\times$.

An element $a \in A$ is prime if the principle ideal $\langle a \rangle = A \cdot a$ is a prime ideal of A.

Elements $a, b \in A$ are associate if $\exists u \in A^\times$ for which $a = ub$.

Proposition 8.2.2: If A is an integral domain, a prime element of A is irreducible.

Proof: Let $a \in A$ be prime, and suppose that $a = xy$ for $x, y \in A$. Then $xy \in \langle a \rangle$; so the definition of prime ideal shows that one of x or y is in $\langle a \rangle$; without loss of generality, suppose $x \in \langle a \rangle$. To show that a is irreducible, we will now argue that $y \in A^\times$.

Since $x \in \langle a \rangle$, we have $a \mid x$ so that $x = az$ for some $z \in A$. Now we see that $a = xy = azy$. Since A is an integral domain, we find that $1 = zy$ so $y \in A^\times$, as required. ∎

Proposition 8.2.3: If A is a PID, then an element of A is prime if and only if it is irreducible.

Proof: Let $a \in A$. In view of Proposition 8.2.2, the Proposition will follow if we show a irreducible $\Rightarrow$ a prime.

So assume that a is irreducible. To show that $\langle a \rangle$ is prime, it suffices by Corollary 6.2.8 to show that $\langle a \rangle$ is a *maximal* ideal.

Suppose that M is an ideal of A and that $\langle a \rangle \subseteq M$. Since A is a PID, we know that $M = \langle m \rangle$ for some $m \in M$. Since $\langle a \rangle \subseteq \langle m \rangle$, we know that $m \mid a$. Thus $a = mx$ for some $x \in A$. Since $\langle a \rangle$ is a prime ideal containing mx, we conclude that either $m \in \langle a \rangle$ or $x \in \langle a \rangle$.

If $m \in \langle a \rangle$ then $\langle m \rangle \subseteq \langle a \rangle$ and we conclude that $M = \langle m \rangle = \langle a \rangle$.

If $x \in \langle a \rangle$ then $x = ay$ for some $y \in A$. Then $a = mx = may$. Thus $1 = my$ which shows that $m \in A^\times$ so that $M = A$.

This shows that the ideal $\langle a \rangle$ is maximal, and thus prime. So a is indeed a prime element. ∎

Example 8.2.4: Consider the ring $A = \mathbb{Z}\left[\sqrt{-5}\right]$.

Notice that the mapping $\alpha \mapsto \overline{\alpha}$ given for $\alpha = a + b \cdot \sqrt{-5}$ by $\overline{\alpha} = a - b \cdot \sqrt{-5}$ is a ring automorphism of A.

Now, the function $N : A \to \mathbb{Z}$ given by $N(\alpha) = \alpha \cdot \overline{\alpha}$ – i.e.

$$N\left(a + b \cdot \sqrt{-5}\right) = \left(a + b \cdot \sqrt{-5}\right)\left(a - b \cdot \sqrt{-5}\right) = a^2 + 5b^2$$

– is a monoid homomorphism. This means $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in A$.

Further note that if $\alpha \in A$ and $N(\alpha) = \pm 1$ then $\alpha \in A^\times$. Indeed, we see for such an element α that $\alpha^{-1} = \pm\overline{\alpha} \in A$.

We claim that A is not a PID. Indeed, first note that 3 is irreducible in A. Indeed, if $3 = \alpha\beta$ then $9 = N(\alpha)N(\beta)$. If $N(\alpha) = 9$ then $\beta \in A^\times$ and similarly if $N(\beta) = 9$ then $\alpha \in A^\times$.

On the other hand, there are no elements $\alpha \in A$ with $N(\alpha) = 3$. Indeed $N(\alpha) = a^2 + 5b^2$ and the equation $a^2 + 5b^2 = 3$ has no integer solutions. This proves the irreducibility of 3 in A.

On the other hand, 3 is not prime in A. Indeed,

$$9 = \left(2 + \sqrt{-5}\right) \cdot \left(2 - \sqrt{-5}\right) \in A \cdot 3 \text{ but } 2 \pm \sqrt{-5} \notin A \cdot 3$$

since

$$(\clubsuit) \quad A \cdot 3 = \left\{ 3x + 3y\sqrt{-5} \mid x, y \in \mathbb{Z} \right\}.$$

In particular, the ideal $I = \langle 3, 2 + \sqrt{-5} \rangle$ is not principle. Indeed, suppose that $I = A \cdot \alpha$. Then $\alpha \mid 3$ and $\alpha \mid 2 + \sqrt{-5}$.

Since 3 is irreducible, $\alpha = 3u$ where $N(u) = \pm 1$. But then $A \cdot \alpha = A \cdot 3$ so that $2 + \sqrt{-5} \in I = A \cdot 3$. But this is impossible by $(\clubsuit)$.

Definition 8.2.5: An integral domain A is a unique factorization domain (or UFD) if $\forall a \in A$ for which $a \neq 0$ and $a \notin A^\times$, we have the following:

a. There are finitely many irreducible elements $p_1, ..., p_n \in A$ such that $a = p_1 ... p_n$.
b. The expression in a. is unique up to associates: namely, if also $a = q_1 ... q_m$ for irreducible elements $q_j \in A$, then $n = m$ and – after possibly renumbering the factors – the elements p_i and q_i are associate for each $i = 1, 2, ..., n$.

Example 8.2.6:

a. Any field is a UFD (vacuously)
b. We will show shortly that any PID is a UFD.
c. And we will show that if A is a UFD, then $A[T]$ is a UFD. Thus by induction we see for example that $F[T_1, ..., T_d]$ is a UFD for any field and any $d \in \mathbb{N}$.
d. The ring $\mathbb{Z}\left[\sqrt{-5}\right]$ is not a UFD. Indeed, $6 = 2 \cdot 3 = \left(1 + \sqrt{-5}\right)\left(1 - \sqrt{-5}\right)$. We have seen before that 3 is irreducible. Similar arguments show that 2 is irreducible and that the elements $1 \pm \sqrt{-5}$ are irreducible. But 3 is not associate to either of $1 \pm \sqrt{-5}$.

Proposition 8.2.7: Let A be a UFD.

a. Let $p_i \in A$ be irreducibles for $i = 1, ..., n$. If p is irreducible and $p \mid p_1 ... p_n$ then for some $1 \leq i \leq n$, p and p_i are associates.
b. If $p \in A$ a non-zero non-unit, then p is prime if and only if it is irreducible.

Proof:

a. The assumption means that $px = p_1 ... p_n$ for some n. If x is a unit, the definition of UFD shows that $n = 1$ and p is associate with p_1. Otherwise, we may write $x = q_1 ... q_s$ for irreducibles $q_j \in A$. Now $pq_1 ... q_s = p_1 ... p_n$, and the definition of UFD shows that $s + 1 = n$ and for some i, p and p_i are associate. This completes the proof of a.

b. Let $p \in A$ be a non-zero non-unit. In view of Proposition 8.2.2, we must argue that if p is irreducible, then p is prime.

   Thus we suppose that $p \mid ab$ for $a, b \in A$. We must show that $p \mid a$ or $p \mid b$.

   If a is a unit, we see that $p \mid b$, and similarly if b is a unit, then $p \mid a$. Thus we may suppose that neither a nor b is a unit. We now find irreducible elements $p_1, ..., p_n$ and $q_1, ..., q_m$ in A such that $a = p_1 ... p_n$ and $b = q_1 ... q_m$. Thus

$$p \mid p_1 ... p_n \cdot q_1 ... q_m.$$

Now the result in part a. shows that p is associate to some p_i or some q_j. But it is then immediate that p divides a or p divides b. This completes the proof.

■

8.3. PIDs have unique factorization

Proposition 8.3.1: Let A be a PID. If

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq ... \subseteq I_n \subseteq ...$$

is a sequence of ideals of A, then $\exists N$ such that for $n \geq N$ we have $I_n = I_N$.

Proof: Let $I = \cup_{j \in \mathbb{N}} I_j$, and check that I is an ideal of A.

Since A is a PID, $I = A \cdot a$ for some $a \in A$. Now, by the definition of I, we see that $a \in I_j$ for some $j \in \mathbb{N}$. But then for any $\ell \geq j$ we have $I_\ell \subseteq I = A \cdot a \subseteq I_\ell$ so that $I = I_\ell = I_j$ as required.
■

Proposition 8.3.2: Let A be a PID and let $p \in A$ be prime.

a. If q is prime and $p \mid q$ then p and q are associate.
b. let $a_1, ..., a_m \in A$ and suppose that $p \mid a_1 ... a_m$. Then $p \mid a_j$ for some j.

Proof:

a. Write $px = q$ for some $x \in A$. Since q is prime, either $q \mid p$ or $q \mid x$. If $qy = x$ then $pqy = q$ so that $py = 1$; thus p is a unit, contrary to the assumption that p is prime.

   If $qy = p$ then $qyx = q$ which shows that $yx = 1$. Thus $y \in A^\times$ so that p and q are associate.
b. Proceed by induction on m. When $m = 1$ there is nothing to do. Now suppose that $n \in \mathbb{N}_{>0}$ and the result holds for n elements of A. Let $a_1, ..., a_{n+1} \in A$ and suppose that

$$p \mid a_1 ... a_n \cdot a_{n+1} = (a_1 ... a_n) \cdot a_{n+1}.$$

   We see that $(a_1 ... a_n) \cdot a_{n+1} \in A \cdot p$, a prime ideal. Thus, either $a_1 ... a_n \in A \cdot p$ or $a_{n+1} \in A \cdot p$.

   In the first case, $p \mid a_1 ... a_n$ and we may apply induction to learn that $p \mid a_j$ for some $1 \leq j \leq n$. And in the second case, $p \mid a_{n+1}$. Thus the conclusion of the Proposition holds in all cases.

■

Theorem 8.3.3: Let A be a PID. Then A is a UFD.

Proof: Let $a \in A$ be a non-zero non-unit.

We first argue that a may be expressed as a product of irreducible elements of A.

To give the proof, we proceed by contradiction. Thus, we suppose that A has no such factorization.

First note that a is not irreducible (otherwise, a can be written as a product of irreducibles)

Now set $a = a_0$ and write $a_0 = x \cdot y$ where neither x nor y is a unit. Now, at least one of x or y can not be written as a product of irreducibles; without loss of generality we supose it to be x and we set $a_1 = x$. Since y is not a unit, there is a proper inclusion of ideals $A \cdot a_0 \subset A \cdot a_1$.

Since a_1 can't be written as a product of irreducibles, we can repeat this argument with a_1 playing the role of a_0. Continuing in this manner, we produce a sequence of elements $a_0, a_1, a_2, ..., a_n, ...$ such that

$$A \cdot a_0 \subset A \cdot a_1 \subset ... \subset A \cdot a_n \subset ...$$

where each inclusion is proper.

Now, the existence of such a sequence contradicts Proposition 8.3.1. This contradiction proves that every element of A has a factorization as a product of irreducibles.

It remains to prove the uniqueness of factorization.

So, suppose that $a = p_1...p_n = q_1...q_m$ where p_i and q_j are irreducible in A and where $n \leq m$.

We proceed by induction on n. When $n = 1$, we have $p_1 = q_1...q_m$ and Proposition 8.3.2 shows for some j that p_1 and q_j are associate; i.e. $q_j = up_1$ for $u \in A^\times$. Reordering the terms we may suppose that $j = 1$.

Now

$$p_1 = up_1q_2...q_m \Rightarrow 1 = uq_2...q_m.$$

This shows that q_j is a unit for $j = 2, ..., m$. Since the q_j are primes, this means that $j = 1$. This completes the proof when $n = 1$.

Now assume the result is known for n, and suppose that

$$p_1p_2...p_{n+1} = q_1...q_m \text{ for some } m \geq n+1.$$

Then p_{n+1} divides $q_1...q_m$ and Proposition 8.3.2 shows that p_{n+1} is associate with some q_j. Reordering the factors, we may suppose that $j = m$; write $q_m = up_{n+1}$ for some $u \in A^\times$.

Thus

$$p_1...p_np_{n+1} = q_1...q_{m-1}up_m \Rightarrow p_1...p_n = uq_1...q_{m-1}.$$

Now it follows by the induction hypothesis that $n = m - 1$ and that after possibly re-ordering the q_i one knows that p_i and q_i are associate for all $i = 1, ..., n$. Since also p_{n+1} is associate with q_m, the result follows by induction. ■

Proposition 8.3.4: Let A be a UFD and let $a, b \in A \setminus \{0\}$. Suppose that

$$a = u \cdot p_1^{e_1} p_2^{e_2} ... p_n^{e_n} \text{ and } b = v \cdot p_1^{f_1} p_2^{f_2} ... p_n^{f_n}$$

where $u, v \in A^\times$, the p_i are irreducible elements of A which are pairwise not associate, and $e_i, f_j \in \mathbb{N}$.

Set

$$d = p_1^{\min(e_1,f_1)} p_2^{\min(e_2,f_2)} ... p_n^{\min(e_n,f_n)}.$$

Then d is a gcd of a, b.

In particular, gcds exists in UFDs.

Proof: The condition on the exponents of the p_i shows that $d \mid a$ and $d \mid b$.

Now suppose that c is any common divisor of a and b. and write

$$c = w q_1^{g_1} q_2^{g_2} ... q_m^{g_m}$$

for irreducibles q_j in A and a unit $w \in A^\times$. Fix $1 \leq i \leq m$. Since q_i divides c and hence divides a and b, and since irreducibles are prime in a UFD by Proposition 8.2.7, we see that p_i and q_j are associate for some j.

After replacing q_i be an associate (and possibly modifying $w \in A^\times$) we may suppose that $\{q_1, ..., q_m\} \subseteq \{p_1, ..., p_n\}$. Thus after re-indexing the irreducibles p_i, we may write

$$c = w p_1^{g_1} ... p_n^{g_n}.$$

Now $c \mid a \Rightarrow g_i \leq e_i$ for each i, and $c \mid b \Rightarrow g_i \leq f_i$ for each i. Thus $c \mid d$ which completes the proof that $d = \gcd(a, b)$. ■

Example 8.3.5: In general, for a UFD A it is not true that $\langle a, b \rangle = \langle \gcd(a, b) \rangle$.

We are going to show below that for a field F, the two-variable polynomial ring $F[T, U]$ is a UFD. Since T is irreducible in $F(U)[T]$, T is irreducible in $F[T, U]$. Similarly, U is irreducible in $F[T, U]$.

Thus $\gcd(T, U) = 1$. But $\langle T, U \rangle \neq F[T, U]$. To see this, note that for any $f \in \langle T, U \rangle$, evaluation at $T \mapsto 0$ and $U \mapsto 0$ gives $f(0, 0) = 0$. This shows that $1 \notin \langle T, U \rangle$.

8.4. Rings of fractions

Let A be a commutative ring.

When A is an integral domain, we would like to know that there is a field K containing A in the same way that $\mathbb{Q}$ contains $\mathbb{Z}$.

For this, we'll use a more general construction known as localization. Compare with [Dummit-Foote], § 7.5 and § 15.4.

Definition 8.4.1: A subset $S \subset A$ is said to be multiplicatively closed if $0 \notin S$, $1 \in S$, and $a, b \in S \Rightarrow a \cdot b \in S$.

Notice that S is a (multiplicative) moniod; in fact, it is a submonoid of $(A, \times)$.

Theorem 8.4.2: Let S be a multiplicatively closed subset of A. Then there is a commutative ring $A[S^{-1}]$ together with a ring homomorphism $\iota : A \to A[S^{-1}]$ such that $\iota(s) \in A[S^{-1}]^\times$ for each $s \in S$.

Every element of $A[S^{-1}]$ can be written in the form $\iota(a) \cdot \iota(s)^{-1}$ for some $a \in A$ and $s \in S$.

The ring $A[S^{-1}]$ satisfies the universal mapping property: if B is any commutative ring and if $\varphi : A \to B$ is a ring homomorphism such that $\varphi(s) \in B^\times$ for each $s \in S$, then there is a unique ring homomorphism $\psi : A[S^{-1}] \to B$ such that $\psi \circ \iota = \varphi$.

Proof: Consider the set $D = \{(a, s) \mid a \in A, s \in S\}$ and define a relation $\sim$ on D by the condition:

$$(a, s) \sim (b, t) \Leftrightarrow \exists x \in S, x(at - bs) = 0.$$

Note that if A is an integral domain, then

$$(a, s) \sim (b, t) \Leftrightarrow at = bs.$$

We now confirm that $\sim$ is an equivalence relation: reflexive and symmetric are immediate. To see that $\sim$ is transitive, let $(a, s), (b, t), (c, u) \in D$ and suppose that $(a, s) \sim (b, t)$ and $(b, t) \sim (c, u)$. We show that $(a, s) \sim (c, u)$. We know that $x(at - bs) = 0$ and $y(bu - ct) = 0$ for some $x, y \in S$. Multiplying the first equation by yu and the second by xs we find that

$$xatyu = xbsyu = yctsx.$$

We conclude that $au(xty) = cs(xty)$; since $xty \in S$, it follows that $(a, s) \sim (c, u)$.

Write $a/s = \dfrac{a}{s}$ for the equivalence class of the element $(a, s) \in D$. Note that for $t \in S$, we have $a/s = at/st$.

For $\alpha = a/s$ and $\beta = b/t$ we define

$$\alpha + \beta = \frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \text{ and } \alpha\beta = \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

Write Q for the set of equivalence classes in D for $\sim$; we claim that Q becomes a commutative ring with the indicated operations.

There are a number of things to check(!!):

a. the operations $+$ and $\times$ on Q are well-defined – i.e. they are independent of the chosen representatives of the equivalence classes.

b. Q is an abelian group under addition with identity element $\dfrac{0}{1} = \dfrac{0}{s}$ for any $s \in S$. The operation of taking additive inverses is given by

$$-\left(\frac{a}{b}\right) = \frac{-a}{b}.$$

c. multiplication in Q is associative, commutative and distributive. Moreover, $\frac{1}{1} = \frac{s}{s}$ for any $s \in S$ is a multiplicative identity.

The proofs of these assertions are straightforward and omitted.

Next, we define $\iota : A \to Q$ by

$$\iota(a) = \frac{a}{1} = \frac{as}{s} \text{ for any } s \in S.$$

One checks that this mapping is a ring homomorphism; again the proof is omitted.

Finally, let $\varphi : A \to B$ is a ring homomorphism for which $\varphi(s) \in B^\times$ for $s \in S$. We must find a unique $\psi : A[S^{-1}] \to B$ with the required property.

We first treat uniqueness: For $a/s \in A[S^{-1}]$, note that $a/s = a/1 \cdot 1/s$ so that

$$\psi(a/s) = \psi(a/1)\psi(1/s) = \psi(\iota(a)) \cdot \psi(s)^{-1} = \varphi(a)\varphi(s)^{-1}.$$

Thus it only remains to check that this formula yields a well-defined ring homomorhpism. The proof is routine. ■

Proposition 8.4.3: Let A be a commutative ring and let S a multiplicatively closed subset of A.

a. The kernel of the map $\iota : A \to A[S^{-1}]$ is given by

$$\ker \iota = \{a \in A \mid \exists s \in S, sa = 0\}.$$

b. If A is an integral domain, then ι is injective.

Definition 8.4.4: The ring constructed in Theorem 8.4.2 is called the ring of fractions of A for the multiplicatively closed subset S.

Proposition 8.4.5: If A is an integral domain and $S = A \setminus \{0\}$, then the ring of fractions $A[S^{-1}]$ is a field, called the field of fractions of A. We usually write $\mathrm{Frac}(A)$ for this field.

Proof: We only must confirm that any $\alpha \in A$ with $\alpha \neq 0$ is a unit. But α has the form $\frac{a}{s}$ for $a \in A$ and $s \in S$. And $\frac{a}{s} \neq 0$ implies that $a \neq 0$. Thus $a \in S = A \setminus \{0\}$ so that $\frac{s}{a} \in A[S^{-1}]$. Now it is straightforward to see that

$$\frac{a}{s} \cdot \frac{s}{a} = \frac{1}{1} = 1.$$

Thus α is a unit; this confirms that $A[S^{-1}]$ is a field. ■

Example 8.4.6:

a. We have $\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$.
b. Let K be a field. We write $K(T) = \mathrm{Frac}(K[T])$ for the field of fractions of the polynomial ring $K[T]$.

More generally, we write $K(T_1, T_2, ..., T_n) = \mathrm{Frac}(K[T_1, T_2, ..., T_n])$ for the field of fractions of the multi-variable polynomial ring $K[T_1, T_2, ..., T_n]$.

The ring of fractions satisfies the following universal mapping property:

Theorem 8.4.7: Let A an integral domain and S a multiplicatively closed subset of A. If B is any ring and $\varphi : A \to B$ any ring homomorhpism such that $\varphi(s) \in B^\times$ for each $s \in S$, then there is a unique ring homomorphism

$$\psi : A[S^{-1}] \to B$$

such that $\psi_{|A} = \varphi$.

Proof: We first prove uniqueness of ψ. Since any element α of $A[S^{-1}]$ may be written in the form $\alpha = a \cdot s^{-1}$ for $a \in A$ and $s \in S$ and since $\varphi(s)$ is a unit in B, we must have

$$(\clubsuit) \quad \psi(\alpha) = \psi(a)\psi(s^{-1}) = \psi(a)\psi(s)^{-1} = \varphi(a)\varphi(s)^{-1}.$$

Thus once we check that ($\clubsuit$) indeed defines a ring homomorphism, we see that there is only one possible definition of ψ.

It remains to check that ($\clubsuit$) indeed defines a ring homomorphism. First, we must confirm that this definition of ψ is well defined.

We suppose that $\frac{a}{s} = \frac{a'}{s'}$ in $A[S^{-1}]$; thus $as' = a's$. Applying φ we see that

$$\varphi(a)\varphi(s') = \varphi(a')\varphi(s)$$

so that

$$(\blacklozenge) \quad \varphi(a)\varphi(s)^{-1} = \varphi(a')\varphi(s')^{-1}.$$

Now, $\psi(a/s) = \varphi(a)\varphi(s)^{-1}$ and $\psi(a'/s') = \varphi(a')\varphi(s')^{-1}$ so ($\blacklozenge$) shows that $\psi(a/s) = \psi(a'/s')$.

It is now straightforward to check that ψ is additive and multiplicative.

For example

$$\begin{aligned}
\psi\left(\frac{a}{s} + \frac{a'}{s'}\right) &= \psi\left(\frac{as' + a's}{ss'}\right) \\
&= \varphi(as' + a's) \cdot \varphi(ss')^{-1} \\
&= \varphi(as')\varphi(ss')^{-1} + \varphi(a's)\varphi(ss')^{-1} \\
&= \varphi(a)\varphi(s)^{-1} + \varphi(a')\varphi(s')^{-1} \\
&= \psi\left(\frac{a}{s}\right) + \varphi\left(\frac{a'}{s'}\right).
\end{aligned}$$

■

Proposition 8.4.8: Let A be an integral domain and let S be a mult. closed subset of A.

- An ideal I of A, generates an ideal $I^e = A[S^{-1}] \cdot I$ of $A[S^{-1}]$
- An ideal J of $A[S^{-1}]$ determines an ideal J^c of A by the rule $J^c = \{x \in J \mid x \in A\} = A \cap J$.

a. For any ideal J of $A[S^{-1}]$ we have $J = (J^c)^e$.
b. For an ideal I of A we have

$$(I^e)^c = \{a \in A \mid sa \in I \text{ for some } s \in S\}.$$

c. The assignment $I \mapsto I^e$ determines a bijecttion

$$\{\text{prime ideals } I \text{ of } A \text{ with } I \cap S = \emptyset\} \to \{\text{prime ideals of } A[S^{-1}]\}$$

with inverse given by $J \mapsto J^c$.

Proof: Write $\iota : A \to A[S^{-1}]$ for the natural mapping $\iota(a) = a/1$.

a. It is true by definition that $(J^c)^e \subseteq J$. For the reverse inclusion, suppose that $a/s \in J$. Then $a/1 = s \cdot (a/s) \in J$. This shows that $a \in J^c$. Now $a/s = (a/1)(1/s) \in (I^e)^c$ as required.

b. Let $I' = \{a \in A \mid sa \in I \text{ for some } s \in S\}$. We first argue that $I' \subseteq (I^e)^c$.

If $a \in I'$ choose $s \in S$ with $sa = b \in I$. Then $a = \dfrac{b}{s} \in I^e$ which shows that $a \in (I^e)^c$.

On the other hand, let $a \in (I^e)^c$. Thus $a/1 = b/s$ for $b \in I$ and $s \in S$. Thus $sa = b \cdot 1 \in I$ which shows that $a \in I'$. This shows that $(I^e)^c \subseteq I'$ so that equality holds, as required.

c. Let's write

$$\mathcal{P} = \{\text{prime ideals } I \text{ of A with } I \cap S = \emptyset\} \text{ and } \mathcal{Q} = \{\text{prime ideals of } A[S^{-1}]\}$$

and let $e : \mathcal{P} \to \mathcal{Q}$ be the mapping $I \mapsto I^e$ and $c : \mathcal{Q} \to \mathcal{P}$ be the mapping $J \mapsto J^c$.

Part a. shows that $e \circ c = \mathrm{id}_{\mathcal{Q}}$. Now suppose that $I \in \mathcal{P}$.

We must argue that $(c \circ e)(I) = I$. According to b., we know that $(c \circ e)(I) = I'$ where $I' = \{a \in A \mid sa \in I \text{ for some } s \in S\}$.

We claim that since I is prime and $I \cap S = \emptyset$ that $I = I'$. Evidently $I \subseteq I'$ so we must prove the $I' \subseteq I$. Let $a \in I'$. By the definition of I' we know that $sa \in I$ for some $s \in S$. Since I is prime, either $s \in I$ or $a \in I$. But since $I \cap S = \emptyset$ we know that $s \notin I$ so that $a \in I$. This shows that $I = I'$, so that $(c \circ e)(I) = I$.

■

8.5. Localization at a prime ideal, and local rings

Let A be an integral domain and let P be a prime ideal of A. Write $S = A \setminus P$. Since P is prime, the subset S is multiplicatively closed in A.

We write A_P for the ring $A[S^{-1}]$ and call this ring the localization of A at P.

Proposition 8.5.1:
a. A_P is a local ring with unique maximal ideal $P^e = A_P \cdot P$.
b. There is a bijection between the prime ideals of A_P and the prime ideals of A contained in P.

Proposition 8.5.2: Let A be a local ring with unique maximal ideal $\mathfrak{m}$.
a. If $a \in A$, then $a \notin \mathfrak{m} \Leftrightarrow a \in A^\times$.
b. For $a \in \mathfrak{m}$, $1 + a \in A^\times$.

Example 8.5.3:

a. Let $A = \mathbb{Z}$ and let p be a prime number, so that $p\mathbb{Z}$ is a prime ideal. Then $S = \mathbb{Z} \setminus p\mathbb{Z}$ consists of all $z \in \mathbb{Z}$ for which p does not divide z.

   The local ring $\mathbb{Z}_{\langle p \rangle}$ may be identified with

$$\{m/n \in \mathbb{Q} \mid m, n \in \mathbb{Z} \text{ for which } p \nmid n\}$$

   and the unique maximal ideal is

$$p\mathbb{Z}_{\langle p \rangle} = \{m/n \in \mathbb{Q} \mid m, n \in \mathbb{Z}, p \mid m, p \nmid n\}.$$

b. Let K be a field and for $a \in K$ note that $\langle T - a \rangle$ is a maximal ideal of $K[T]$.

   The local ring $K[T]_{\langle T-a \rangle}$ may be identified with

$$\{f/g \in k(T) \mid f, g \in K[T], g(a) \neq 0\}$$

   and the unique maximal ideal is

$$T \cdot K[T]_{\langle T-a \rangle} = \{f/g \in k(T) \mid f, g \in K[T], f(a) = 0, g(a) \neq 0\}.$$

Proposition 8.5.4 (Nakayama's Lemma): Let A be a local ring with unique maximal ideal $\mathfrak{m}$. If M is a finitely generated A-module for which $\mathfrak{m} \cdot M = M$ then $M = 0$.

Proof: Let $n \in \mathbb{N}$ be the smallest number such that M is generated as A-module by n elements. To show that $M = 0$, we must show that $n = 0$.

We proceed by contradiction; thus we suppose $n > 0$ and we let $x_1, ..., x_n \in M$ be a system of A-module generators.

Since $\mathfrak{m} \cdot M = M$, there are elements $a_1, ..., a_n \in \mathfrak{m}$ such that

$$x_n = \sum_{i=1}^{n} a_i x_i.$$

But then

$$(1-a_n)x_n = \sum_{i=1}^{n-1} a_i x_i$$

and since $1-a_n \in A^\times$ by Proposition 8.5.2, we see that

$$x_n = \sum_{i=1}^{n-1} a_i (1-a_n)^{-1} x_i.$$

This shows that $x_1, ..., x_{n-1}$ generate M as A-module, contradicting the minimality of n. This contradiction proves that $n = 0$, and thus $M = 0$ as required. ■

8.6. Unique factorization in polynomial rings

Let A be a UFD. In this section we are going to prove that the polynomial ring $A[T]$ is a UFD as well.

Write K for the field of fraction and notice that $A[T] \subset K[T]$.

Now, $K[T]$ is a PID, hence in particular a UFD. We are going to use unique factorization in $K[T]$ together with unique factorization in A to deduce unique factorization in $A[T]$.

An important tool here is *Gauss' Lemma* comparing factorization in $A[T]$ with that in $K[T]$.

Recall that Proposition 8.3.4 we can compute gcds for a pair of elements of a UFD. If $a_0, a_1, ..., a_d \in A$ then we set

$$\gcd(a_0, ..., a_d) = \gcd(\gcd(a_0, ..., a_{d-1}), a_d).$$

Definition 8.6.1: Let $f \in A[T]$ be a non-zero polynomial and let $c = \gcd(\text{coefficients of } f)$. Then c is known as the content of f, denoted $\text{content}(f)$.

The polynomial f is primitive if $\text{content}(f)$ is a unit in A.

Proposition 8.6.2: Let $f \in A[T]$ be a non-zero polynomial, and write $c = \text{content}(f) \in A$. Then there is a primitive polynomial $g \in A[T]$ for which $f = c \cdot g$.

Proof: Write

$$f = \sum_{i=0}^{n} a_i T^i \text{ with } a_i \in A.$$

Then by definition $c = \gcd(a_0, a_1, ..., a_n)$. In particular, for each i, $c \mid a_i$ so we may write $a_i = cb_i$ for $b_i \in A$.

Set $g = \sum_{i=0}^{n} b_i T^i$ so that indeed $f = c \cdot g$.

Now observe that

$$\text{content}(g) = \gcd(b_0, b_1, ..., b_n) = \gcd\left(\frac{a_0}{c}, \frac{a_1}{c}, ..., \frac{a_n}{c}\right) = \frac{c}{c} = 1$$

so that g is primitive, as required. ■

Proposition 8.6.3: Let $p \in A$ be irreducible and consider the assignment

$$h \mapsto \overline{h} : A[T] \to (A/A \cdot p)[T]$$

defined for $h = \sum_{i=0}^{m} a_i T^i \in A[T]$ by

$$\overline{h} = \sum_{i=0}^{m} (a_i + A \cdot p) T^i = \sum_{i=0}^{m} \overline{a_i} T^i.$$

a. The assignment $h \mapsto \overline{h}$ is a ring homomorphism, and
b. For $h \in A[T]$, $\overline{h} = 0$ if and only if $p \mid \text{content}(h)$.

Proof:

a. In fact, the mapping $h \mapsto \overline{h}$ is the ring homomorphism given by Proposition 8.2.2 for the quotient mapping $\pi : A \to A/A \cdot p$ and the assignment $T \mapsto T$.
b. Just observe that $\overline{h} = 0$ if and only if $p \mid a_i$ for each i.

■

Proposition 8.6.4 (Gauss' Lemma): Let $f, g \in A[T]$ be non-zero polynomials.

a. $c(f \cdot g) = c(f) \cdot c(g)$.
b. In particular, if f and g are primitive, then $f \cdot g$ is primitive.

Proof: First note that a. is an immediate consequence of b. Indeed, write $f = cf_0$ and $g = dg_0$ for primitive $f_0, g_0 \in A[T]$. Then $fg = c \cdot d \cdot f_0 \cdot g_0$. By b. $f_0 \cdot g_0$ is primitive, and we see that $\text{content}(fg) = cd$ as required.

We now prove b. We proceed by contradiction. Thus, we suppose that f, g are primitve but that $f \cdot g$ is not primitive.

Write $d = \text{content}(f \cdot g)$. Since $f \cdot g$ is not primitive, there is an irreducible element $p \in A$ for which $p \mid d$.

Since p is irreducible, it is prime Proposition 8.2.7 so that $A \cdot p$ is a prime ideal. Thus $A/A \cdot p$ is an integral domain, and it follows that $(A/A \cdot p)[T]$ is an integral domain.

Now consider the homomorphism $h \mapsto \overline{h} : A[T] \to (A/A \cdot p)[T]$ of Proposition 8.6.3.

Since $p \mid \text{content}(fg)$ the proposition shows that $\overline{f \cdot g} = 0$. On the other hand, since f and g are primitive, the proposition shows that $\overline{f} \neq 0$ and $\overline{g} \neq 0$.

Since $\overline{f \cdot g} = \overline{f} \cdot \overline{g}$ and since $(A/A \cdot p)[T]$ is an integral domain, we have arrived at a contradiction. This proves that $\text{content}(f \cdot g) = 1$, and the proof is complete. ■

Corollary 8.6.5: Let $f \in A[T]$ be primitive. If $f = g \cdot h$ for non-constant polynomials $g, h \in K[T]$, there are non-zero elements $r, s \in K$ such that $r \cdot g \in A[T]$, $s \cdot h \in A[T]$ and $f = gh$.

In particular, if f is reducible in $K[T]$, then f is reducible in $A[T]$.

Proof: Use Proposition 8.6.2 to write

$$g = \left(\frac{x}{y}\right) g_0 \text{ and } h = \left(\frac{z}{w}\right) h_0 \text{ for primitive polys } g_0, h_0 \in A[T],$$

where $x, y, z, w \in R$ with $y \cdot w \neq 0$.

Then we find that

$$f = gh = \left(\frac{xz}{yw}\right) \cdot g_0 h_0$$

so that

$$ywf = xzg_0h_0.$$

Since f is primitive, $\text{content}(ywf) = yw$ Moreover, Proposition 8.6.4 shows that g_0h_0 is primitive, so that $\text{content}(xzg_0h_0) = xz$. It follows that yw and xz are associate; i.e. $ywu = xz$ for $u \in A^\times$.

Using cancellation in the integral domain A, we finally see that

$$ywf = ywug_0h_0 \Rightarrow f = (ug_0) \cdot h_0$$

which proves the result. ■

Corollary 8.6.6: Let A be a UFD with field of fractions K and let $f \in A[T]$. Suppose that $\text{content}(f) = 1$. Then f is irreducible in $K[T]$ if and only if f is irreducible in $A[T]$.

In particular, if $f \in A[T]$ is monic and irreducible in $F[T]$ then f is irreducible in $K[T]$.

Proof: $(\Rightarrow)$: By Corollary 8.6.5, we know that f reducible in $K[T]$ implies that f is reducible in $A[T]$.

$(\Leftarrow)$: Since the content of f is a unit, if $f = gh$ for $g, h \in A[T]$, then g, h are non-constant - i.e. $g, h \notin A$. But this factorization shows that f is reducible in $K[T]$. ■

Theorem 8.6.7: An integral domain A is a UFD if and only if $A[T]$ is a UFD.

Proof: $(\Leftarrow)$: Suppose $A[T]$ is a UFD. Viewing A as degree 0 polynomials in $A[T]$ and using the fact that $^\circ$ satisfies $\deg(f \cdot g) = \deg(f) + \deg(g)$, it is straightforward to see that A itself is a UFD.

$(\Rightarrow)$: Let $f \in A[T]$, and let $c = \text{content}(f) \in A$, so that $f = c \cdot f_0$ for a primitive polynomial $f_0 \in A[T]$. Such a factorization is unique up to multiplication by a unit in A. Since c can be factored uniquely into a product of irreducibles in A, it suffices to prove that f_0 can be factored uniquely into a product of irreducibles. Thus we may and will suppose that f is primitive.

Since $K[T]$ is a PID, f may be written as a scalar times a product of irreducibles in $K[T]$. By Corollary 8.6.5, there are polynomials $p_1, ..., p_n \in A[T]$ such that

$$f_= p_1 \cdots p_n$$

such that p_i is irreducible in $K[T]$. Now Corollary 8.6.6 shows that each p_i is irreducible in $A[T]$.

This completes the proof that any element of $A[T]$ has a factorization as a product of irreducibles.

We now prove that such a factorization is unique. Suppose that

$$f = p_1 ... p_m = q_1 ... q_n$$

for irreducibles $p_i, q_j \in A[T]$. Since f is primitive, p_i and q_j are primitive $\forall i, j$. According to Corollary 8.6.6, the p_i, q_j remain irreducible in $K[T]$. Since $K[T]$ is a PID hence a UFD, we conclude that $n = m$ and that after possibly re-ordering, p_i is associate with q_i in $K[T]$.

Since p_i and q_i are primitive, it follows that p_i is associate with q_i in $A[T]$ and the proof is complete. ■

Corollary 8.6.8: Let A be a UFD. For any $n \in \mathbb{N}$, $A[T_1, T_2, ..., T_n]$ is a UFD.

Proof: For $n \geq 1$, note that $A[T_1, T_2, ..., T_n] = A[T_1, T_2, ..., T_{n-1}][T_n]]$. Thus the result follows by induction on n. ■