Assignment 3

Math245
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1. Let $A = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and let $\alpha, \beta, \gamma \in A$.
   (a) Is the group $A/\langle \alpha, \beta \rangle$ finite for some α, β? Why or why not?
   (b) Give conditions under which the group $A/\langle \alpha, \beta, \gamma \rangle$ is finite?

   Hint: View the elements α, β, γ as vectors in $\mathbb{Q}^3$. From linear algebra, we know that if these vectors are linearly independent, they form a basis for $\mathbb{Q}^3$. What does this say about the subgroup of $\mathbb{Z}^3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ they generate?

2. For a group G and elements $x, y \in G$, the **commutator** of x and y is the element
$$[x, y] = xyx^{-1}y^{-1}.$$
   (a) Let $G' = \langle [x, y] \mid x, y \in G \rangle$ be the subgroup generated by all commutators. Prove that G' is a normal subgroup of G.
   (b) Prove that the quotient group G/G' is **abelian** (i.e., commutative).
   (c) Suppose that H is any abelian group and that $f : G \to H$ is a group homomorphism. Prove that $\forall x \in G', f(x) = 1$; i.e., that $G' \subseteq \ker f$.
   (d) Deduce that there is a homomorphism $\overline{f} : G/G' \to H$ for which $\overline{f} \circ \pi = f$ where $\pi : G \to G/G'$ is the quotient map.

   Remark: G/G' is known as the **abelianization** of G.

3. Let G be a finite p-group for some prime number p.
   (a) Prove that G has a central subgroup of order p.
   (b) **Remark**: This result is required for the proof of Corollary 3.4.5 in the notes, so you shouldn't quote that result for the proof! You can use Theorem 3.4.4 though.
   (c) Prove that the commutator subgroup G' (see previous problem) is not equal to G.

4. Let Q be the subgroup of $GL_2(\mathbb{C})$ generated by the matrices

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

   (a) Prove that Q is a group of order 8.

   (b) Prove that the center Z of Q has order 2 and that $Q/Z \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   (c) Prove that $Z = Q'$ where Q' is the commutator subgroup.

5. Let F be any field and for $n \in \mathbb{N}, n > 0$, let $G = GL_n(F)$. From linear algebra, we know that the determinant mapping

$$\det : G \to F^\times$$

is a group homomorphism, where $F^\times = F - \{0\}$ is the multiplicative group of the field.

Let $SL_n(F) = \ker(\det)$ be the subgroup $\{g \in GL_n(F) \mid \det(g) = 1\}$; it is known as the **special linear group**.

Prove that $GL_n(F)/SL_n(F) \cong F^\times$.

6. Let $n \in \mathbb{N}$ and let $D = \langle \sigma, \tau \rangle$ be the dihedral group of order $2n$ as in Example 2.6.6 in the notes. (So the order of σ is n.)

   (a) Suppose that $n = p$ is an odd prime number. Find a 2-Sylow subgroup P of D.

   (b) Suppose that $n = 2p$ where p is an odd prime number. Find a 2-Sylow subgroup P of D in this case.

   (c) In the two cases above, decide whether or not P is normal in D.