

Assignment 2

Question 1: Let G and H be group.

- Suppose that $S \subseteq G$ and assume that $G = \langle S \rangle$. Let $f : G \rightarrow H$ be a function. Show that f is a homomorphism provided that $f(xy) = f(x)f(y)$ for each $x, y \in S$.
- Let $S_1, S_2 \subseteq G$ be subsets and let $H_i = \langle S_i \rangle$ for $i = 1, 2$. Suppose that $\forall x \in S_1$ and $\forall y \in S_2$ we have $xyx^{-1} \in H_2$. Prove that H_1 normalizes H_2 .

Question 2: Let $n \in \mathbb{N}, n > 0$ and consider the group $S = S(\mathbb{Z}/n\mathbb{Z})$ of permutations of the set $\mathbb{Z}/n\mathbb{Z}$.

- For $x \in \mathbb{Z}/n\mathbb{Z}$, recall that the additive order of x is a divisor of n .

Describe the *cycle structure* of the element $\sigma \in S$ defined by the rule $\sigma(z) = z + \overline{m}$. What is the order of σ ?

- Suppose that $n = p$ is a prime number, and let $k \in \mathbb{Z}$ with $\gcd(k, p) = 1$. Thus the class \overline{k} of k in $\mathbb{Z}/p\mathbb{Z}$ lies in the group $(\mathbb{Z}/p\mathbb{Z})^\times$ of *units*. The multiplicative order of \overline{k} is a divisor of $p - 1$.

Describe the *cycle structure* of the element $\tau \in S$ defined by the rule $\tau(z) = \overline{k} \cdot z$. What is the order of τ ?

Question 3: Let G be the group of invertible 2×2 matrices with entries in $F = \mathbb{Z}/p\mathbb{Z}$ for a prime number p ; the group operation is given by matrix multiplication.

- Show that $|G| = (p^2 - 1)(p^2 - p)$.
- Show that $T = \left\{ \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix} \mid t, s \in F^\times \right\}$ is a subgroup of G . Here F^\times denotes the multiplicative group of invertible elements of $F = \mathbb{Z}/p\mathbb{Z}$. Also show that

$$T \text{ is isomorphic to } F^\times \times F^\times$$

- Show that $U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in F \right\}$ is a subgroup of G isomorphic to the additive group F .
- Show that T normalizes U . Find the order of the group $B = TU$.
- A line in F^2 is by definition a linear subspace of dimension 1. For any non-zero vector v , the set $Fv = \text{Span}(v)$ is a line. Note that G acts in a natural way on the set of lines in F^2 .
If we write $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the standard basis of F^2 , show that B is the stabilizer of the line Fv .
- Show that G acts transitively on the set of lines in F^2 .
- Conclude that the set of lines in F^2 is in bijection with the set G/B . How many lines are there in F^2 ?

Question 4: Let G be a group and let Ω be a G -set. If $x, y \in \Omega$ and $x = gy$ for some $g \in G$, prove that the stabilizers $G_x = \text{Stab}_G(x)$ and $G_y = \text{Stab}_G(y)$ are *conjugate*. More precisely, show that

$$G_x = g \cdot G_y \cdot g^{-1}$$

Question 5: Let G be a group. G acts on itself by conjugation: for $g, x \in G$, the action of g on x is given by $\text{Inn}_g x = gxg^{-1}$.

- Prove that the assignment $g \mapsto \text{Inn}_g$ determines a group homomorphism $G \rightarrow \text{Aut}(G)$ where $\text{Aut}(G)$ is the group of automorphisms of G .
- Let $Z = \{g \in G \mid \forall x \in G, gx = xg\}$ be the *center* of G . Prove that $Z = \ker \text{Inn}$.

For the action of G on itself by conjugation, the stabilizer $\text{Stab}_G(x)$ of $x \in G$ is usually written $C_G(x)$ and is called the *centralizer* of x in G . Note that

$$C_G(x) = \{y \in G \mid yxy^{-1} = x\} = \{y \in G \mid yx = xy.\}$$

Question 6: Let $I = I_n$ be a finite set with n elements, and let $S = S_n = S(I_n)$ be the group of permutations of I . Recall that $|S| = n!$.

- Prove that there are $(n-1)!$ n -cycles in S . **Hint:** If the elements of I are written $I = \{a_1, a_2, \dots, a_n\}$, then the n -cycles (a_1, a_2, \dots, a_n) and $(a_2, a_3, \dots, a_n, a_1)$ are *equal*.
- Prove that if σ is an n -cycle in S , then $C_S(\sigma) = \langle \sigma \rangle$