

Problem Set 5

Let R be a commutative ring (with identity...).

Question 1:

- a. Let M be an R -module, and let I be an index set, and for $i \in I$, let $M_i \subseteq M$ for $i \in I$ be an R -submodule. Write $\sum_{i \in I} M_i$ for the sum of submodules M_i . Thus $\sum_{i \in I} M_i$ coincides with the *submodule generated by the M_i* .

For a finitely supported function $(*) f : I \rightarrow \bigcup_{i \in I} M_i$ for which $f(j) \in M_j$ for each $j \in I$, note that $\Sigma(f) = \sum_{i \in I} f(i)$ is a well-defined element of M .

Prove that $\sum_{i \in I} M_i = \{\Sigma(f) \mid f \text{ a finitely supported function satisfying } (*)\}$

- b. Assume that $M = \sum_{i \in I} M_i$ and that for a finitely supported function f satisfying $(*)$, we have $\Sigma(f) = 0 \Rightarrow f(i) = 0$ for each $i \in I$. Prove that $M \simeq \bigoplus_{i \in I} M_i$.

One says in this case that M is the *internal direct sum* of the submodules M_i .

- c. Let $X_1, X_2 \subseteq M$ be R -submodules. Suppose that $M = X_1 + X_2$ and that $X_1 \cap X_2 = 0$. Prove that $M \simeq X_1 \oplus X_2$.

One says in this case that M is the *internal direct sum* of X_1 and X_2 .

Question 2:

Let I be an index set and let M_i be an R -module for each $i \in I$. Let M be the set of all functions $f : I \rightarrow \bigcup_{i \in I} M_i$ with such that $f(j) \in M_j$ for each $j \in I$. For $j \in I$ let

$$\pi_j : M \rightarrow M_i \text{ be the mapping } \pi_j(f) = f(j).$$

Then M is an R -module in a natural way, and π_j is an R -module homomorphism for each j .

- a. Explain why (M, π_j) forms a *product* of the M_i in the category $\text{mod}(R)$.

We usually write $M = \prod_{i \in I} M_i$ for this R -module.

- b. Suppose that I is a finite set, let $(\prod_{i \in I} M_i, \pi_i)$ be a product of the M_i , and let $(\bigoplus_{i \in I} M_i, \iota_i)$ be a coproduct of the M_i . Show that there is an isomorphism

$$\Phi : \bigoplus_{i \in I} M_i \rightarrow \prod_{i \in I} M_i$$

of R -modules such that for $i, j \in I$ we have

$$\pi_j \circ \Phi \circ \iota_i = \begin{cases} \text{id if } i = j \\ 0 \text{ otherwise.} \end{cases}$$

Question 3: (question 3 removed – will appear next week!)

Question 4: Let M, N be free R -modules. Thus there is some set B and function $\beta : B \rightarrow M$ such that M is a free R -module $\beta : B \rightarrow M$, and similarly for N .

Prove that $M \oplus N$ is a free R -module.

Question 5: Let M be an R -module, let $I \subseteq R$ be an ideal. Assume that $ax = 0$ for each $a \in I$ and each $x \in M$. Show that M has the structure of an R/I -module.

Question 6: Let I be an ideal of R and let M be an R -module.

Let IM be the R -submodule of M generated by the set

$$\{ax \mid a \in I, x \in M\}.$$

- Prove that the R -module M/IM has the structure of an R/I -module. (Use [Question 5](#).)
- If M is a free R -module on $\beta : B \rightarrow M$, prove that M/IM is a free R/I -module on

$$\beta' = \pi \circ \beta : B \rightarrow M/IM$$

where $\pi : M \rightarrow M/IM$ is the quotient morphism, where IM is the R -submodule $IM = \{ax \mid a \in I, x \in M\}$ of M .