

Problem Set week 11

Problem 1: Let A be a ring and let $(\heartsuit) \quad 0 \rightarrow X \xrightarrow{j} Y \xrightarrow{\psi} Z \rightarrow 0$ be a short exact sequence of A -modules.

Prove that the following are equivalent:

- a. There is an isomorphism $\varphi : X \oplus Z \xrightarrow{\sim} Y$ such that $\psi \circ \varphi = \pi_2$ and $\varphi^{-1} \circ j = \iota_1$ where

$$\pi_2 : X \oplus Z \rightarrow Z \text{ is } (x, z) \mapsto z, \text{ and } \iota_1 : X \rightarrow X \oplus Z \text{ is } x \mapsto (x, 0).$$

- b. There is a **section** to ψ ; i.e., there is an A -module homomorphism $\sigma : Z \rightarrow Y$ such that

$$\psi \circ \sigma = \text{id}_Z.$$

- c. There is a **retract** of j ; i.e. there is an A -module homomorphism $\rho : Y \rightarrow X$ such that

$$\rho \circ j = \text{id}_X.$$

The short exact sequence (\heartsuit) is said to be *split exact* if it satisfies these equivalent conditions.

Problem 2: Let

$$0 \rightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \rightarrow 0$$

be a short exact sequence of A -modules.

Prove that for an A -module W , the sequence

$$(\clubsuit) \quad 0 \rightarrow \text{Hom}_A(W, X) \xrightarrow{f \mapsto \iota \circ f} \text{Hom}_A(W, Y) \xrightarrow{g \mapsto \pi \circ g} \text{Hom}_A(W, Z)$$

is *exact*.

NB. A sequence of A -modules and A -module homomorphisms is *exact* if for each interior term

$$\dots S \xrightarrow{f} T \xrightarrow{g} Y \dots$$

of the sequence, $\ker g = \text{im } f$. In the sequence (\clubsuit) , $\text{Hom}_A(W, Y)$ is not an interior term.

Remark: The exactness of (\clubsuit) for each short exact sequence is often expressed by saying that the functor $\text{Hom}_A(W, -)$ is *left-exact*.

So your task is to check exactness of (\clubsuit) at the interior terms $\text{Hom}_A(W, X)$ and $\text{Hom}_A(W, Y)$.

Problem 3: An A -module P is said to be **projective** if for every surjective homomorphism of A -modules $\pi : M \rightarrow N$ and every A -module homomorphism $f : P \rightarrow N$, there is an A -module homomorphism $\alpha : P \rightarrow M$ such that $\pi \circ \alpha = f$.

- Prove that any free A -module is projective.
- Prove that a module P is projective if and only if P is a direct summand of a free module; i.e. if there is a free A -module F and an isomorphism of A -modules $F \simeq P \oplus Q$ for some A -module Q .

Problem 4: Prove that the following are equivalent:

- P is projective
- for every short exact sequence of A -modules $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ the sequence of A -modules

$$0 \rightarrow \operatorname{Hom}_A(P, X) \rightarrow \operatorname{Hom}_A(P, Y) \rightarrow \operatorname{Hom}_A(P, Z) \rightarrow 0$$

is exact.

Remark: Thus P is projective if and only if the left-exact functor $\operatorname{Hom}_A(P, -)$ is exact (in the sense that $\operatorname{Hom}_A(P, -)$ carries short exact sequences to short exact sequences.).

Problem 5: Let A be a commutative ring, let $B = A \times A$, and let $\pi : B \rightarrow A$ be the projection ring homomorphism given by $\pi(x, y) = x$.

Using π , we can view A as a B -module. Prove that A is a projective B -module that is not a free B -module.

Problem 6: Let K be a field and let $a, b \in K \setminus \{0, 1\}$ with $a \neq b$. Let $q = T(T - a)(T - b) \in K[T]$ and consider the polynomial $f = S^2 - q \in K[S, T]$.

You proved in the previous homework set that:

- $A = K[S, T]/\langle f \rangle$ is an integral domain. Write s, t for the images of S, T in the quotient ring A ; thus $s^2 = q(t) = t(t - a)(t - b)$ in A .
- The map $K[T] \rightarrow k[t]$ for which $T \mapsto t$ is an isomorphism.
- A is a free $K[t]$ -module on a basis $\{1, s\}$.

- a. Define $N : A \rightarrow K[t]$ by the rule $N(\alpha) = N(f + gs) = f^2 - g^2 \cdot q(t)$ for $\alpha = f + gs$ with $f, g \in K[t]$.

Then N is multiplicative (i.e. it is a monoid homomorphism) and $\alpha \in A$ is a unit in A if and only if $N(\alpha) \in K[t]^\times = K^\times$.

- b. Write $\mathfrak{m} = \langle s, t \rangle$ for the ideal of A generated by s and t . Show that the ideal \mathfrak{m} is not principal.

Hint: Suppose to the contrary that $\alpha = f + g \cdot s \in A$ is a generator for \mathfrak{m} , for $f, g \in k[t]$. Then $\alpha \mid t \Rightarrow N(\alpha) \mid t^2$. Now use a degree argument to show that $g = 0$.

- c. Show that \mathfrak{m} is a maximal ideal of A .

Remark 1: Essentially the same arguments show that the ideals

$$\langle s, t - a \rangle \text{ and } \langle s, t - b \rangle$$

are maximal and not principal.

Remark 2: The field of fractions $F = \text{Frac}(A) = K(s, t)$ of A is the *field of rational functions* on the *elliptic curve* over K defined by the cubic equation $S^2 = T(T - a)(T - b)$.

If we had consider instead the quadratic equation $S^2 = T(T - a)$, the analogous ideal $\mathfrak{m} = \langle s, t \rangle$ is principal (!).

Problem 7: Keep the notations of the previous problem. We are going to show that \mathfrak{m} is a projective A -module.

- First, explain why the condition that \mathfrak{m} is not a principal ideal implies that \mathfrak{m} is not a free A -module.
- Explain why $1 = u \cdot t + v \cdot (t - a)(t - b)$ for some $u, v \in k[t]$.
- Show for every $z \in \mathfrak{m}$ that $\frac{(t-a)(t-b)z}{s} \in A$; note that *a priori* $\frac{(t-a)(t-b)z}{s}$ is an element of the field of fractions of A .
- Consider the surjective A -module homomorphism

$$\pi : A^2 \rightarrow \mathfrak{m} \text{ given by } \pi \begin{pmatrix} x \\ y \end{pmatrix} = xs + yt.$$

Show that $\sigma : \mathfrak{m} \rightarrow A^2$ given by the rule

$$\sigma(z) = \begin{pmatrix} v(t-a)(t-b)z/s \\ uz \end{pmatrix}$$

is a *section* to π (where u, v are as in (b) above).

- Conclude using problem (1) that \mathfrak{m} is isomorphic to a direct summand of the free module A^2 .