

Problem Set week 13

Problem 1: Let A be a local ring with unique maximal ideal \mathfrak{m} , and write $k = A/\mathfrak{m}$ for the residue field of A .

Suppose that M is a free A -module of finite rank n . Let $\alpha_1, \dots, \alpha_n \in M/\mathfrak{m}M$ be a $k = A/\mathfrak{m}$ -basis for $M/\mathfrak{m}M$, and let $x_1, \dots, x_n \in M$ be elements for which $\alpha_i = x_i + \mathfrak{m}M$ for $i = 1, \dots, n$. Prove that x_1, \dots, x_n forms an A -basis for M .

Problem 2: Let A be a commutative ring and $P \subset A$ a prime ideal. Denote by A_P the *localization* of A at P ; thus A_P is a local ring with unique maximal ideal $\mathfrak{m} = P^e = P \cdot A_P$.

- Prove that $A_P/P \cdot A_P$ is isomorphic to the field of fractions of the integral domain A/P .
- If P is a maximal ideal of A conclude that the fields A/P and $A_P/P \cdot A_P$ are isomorphic.

Problem 3: Let A be a PID, $n \in \mathbb{N}$, let F be a free A -module of finite rank and let $\Phi \in \text{End}_A(F)$ be an A -linear endomorphism.

For a prime $p \in A$, write A_{pA} for the localization of A at pA . Note that Φ determines an A_{pA} -homomorphism

$$\text{id} \otimes \Phi : A_{pA} \otimes_A F \rightarrow A_{pA} \otimes_A F.$$

- Fix an A -basis \mathcal{B} for F . For $v \in F$, we write

$$v = \sum_{b \in \mathcal{B}} \alpha_b b \text{ for a function } \alpha : \mathcal{B} \rightarrow A$$

and we write $[v]_{\mathcal{B}} = \alpha \in A^{\mathcal{B}}$, where $A^{\mathcal{B}}$ is the module of all functions $\mathcal{B} \rightarrow A$ (recall that $|\mathcal{B}| = n < \infty$!)

Thus $v \mapsto [v]_{\mathcal{B}} : F \rightarrow A^{\mathcal{B}}$ is an isomorphism of A -modules where $A^{\mathcal{B}}$ is the module of

Show that there is a matrix $M \in \text{Mat}_{\mathcal{B} \times \mathcal{B}}(A)$ for which

$$[\Phi v]_{\mathcal{B}} = M \cdot [v]_{\mathcal{B}} \text{ for } v \in F.$$

- Let $d = \det(\Phi)$. Prove that $\text{id} \otimes \Phi$ is an isomorphism – i.e. is an *automorphism* of $A_{pA} \otimes_A F$ – if and only if $\gcd(d, p) = 1$.

In particular if $d \neq 0$, then $\text{id} \otimes \Phi$ is an automorphism for all but finitely many primes $p \in A$.

Problem 4: Let A be a commutative ring and let M, N, P be A -modules. Prove that there is an isomorphism

$$M \otimes_A (N \oplus P) \simeq (M \otimes_A N) \oplus (M \otimes_A P).$$

Problem 5: Let A be a PID and let $p, q \in A$. Write $d = \gcd(p, q) \in A$ for a greatest common divisor.

Prove that:

$$(A/pA) \otimes_A (A/qA) \simeq A/dA.$$

Problem 6: Let F be a field and let V, W be finite dimensional vector spaces over F .

Recall that the dual space V^* is the vector space $\text{Hom}_F(V, F)$.

- Show that $\dim_F V = \dim_F V^*$. **Hint:** exhibit a basis for V^* .
- Show that there is an isomorphism

$$V^* \otimes_A W \xrightarrow{\sim} \text{Hom}_F(V, W).$$

Hint: Use the mapping property of \otimes to define the indicated map. Basis considerations show that this map is surjective. Now compare the dimension of the domain and co-domain.

Let A be a commutative ring. Let M, M', N, N' be A -modules and let $\varphi : M \rightarrow N$ and $\varphi' : M' \rightarrow N'$ be A -module homomorphisms. There is a unique homomorphism of A -modules

$$\varphi \otimes \varphi' : M \otimes_A M' \rightarrow N \otimes_A N'$$

such that

$$(\varphi \otimes \varphi')(m \otimes m') = \varphi(m) \otimes \varphi'(m').$$

Problem 7: Let F be a field and let $\varphi : V \rightarrow W$ be a homomorphism of F -vector spaces (a “linear transformation”) and let X be an F -vector space.

If φ is injective, prove that $\text{id}_X \otimes \varphi : X \otimes_F V \rightarrow X \otimes_F W$ is injective.

Remark: This shows that the functor $X \otimes_F -$ is *exact* for a field F ; indeed, combine the preceding observation with the result proved in class that the functor $Y \otimes_A -$ is always right exact. In general, an A -module Y is said to be **flat** if $Y \otimes_A -$ is exact.

Problem 8: Let A be a commutative ring and let M be an A -module. If F is a free A -module on $\beta : \mathcal{B} \rightarrow F$, prove that $F \otimes_A M$ is isomorphic to $\bigoplus_{b \in \mathcal{B}} M$, a direct sum of copies of M indexed by \mathcal{B} .