PS 5 - Matching & Finite state machines

Math087 - George McNinch

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1. A mining company needs to set up supply line between some quarries –labeled by the set U – and some processing plants –labeled by the set W. The company is interested in finding *perfect matchings* of quarries to plants, i.e. each quarry should have a supply line to a unique processing plant, such that no quarry or plant is on two supply lines. Conveniently, there are as many quarries as there are processing plants – i.e. $|U| = |W|$. Less conveniently, supply lines need to travel along roads. Let E be the set of roads. We assume that each road in E starts at a quarry and travels directly to a processing plant, and that the roads do not intersect.

   a. Suppose that |U| = |W| = 4 and that there are seven roads total. Construct an example where no perfect matching is possible.

      Suggestion: for this problem and the rest, we suggest you model this set up as a bipartite graph.

   b. If no perfect matching exists, the company would like to know what is the maximal number of quarry/plant pairs such that no quarry or plant is on two supply chains (call this a maximal matching). Compute this for your example in a). How many additional roads must be made in order for a perfect matching to exist?

   c. If $|U| = |W| = n$, what is the largest possible size of E? Give your answer as an expression in n.

   d. Suppose that for each $x \in U$, there is exactly one road involving x, and write this edge as $x \to w(x)$ for some $w(x) \in W$. Explain why the size of a maximum matching is equal to the number of distinct nodes $\{w(x)\}$.

   e. (Optional) Suppose that |U | = |W | = n and let m = |E|. Can you always find a configuration of roads such that a perfect matching is impossible? Put differently, is there a minimal m such that a perfect matching will exist no matter how the roads are placed?

2. A streaming service wants to model the annual growth rate of its subscribers. The data shows that they can roughly predict the behavior of customers based on how long they have been users of the service: Let p_n be the current number of subscribers who have had the service for less than a year, and p_o be the current number of subscribers who have had the service for more than a year (so in total there are $p_n + p_o$ subscribers). For convenience we will refer to the two types of subscriber as “new” and “old”.

   Let $\mathbf{p} = \begin{bmatrix} p_n \\ p_o \end{bmatrix}$ be a vector recording the current subscriber population, and write $\mathbf{p}^*$ be the subscriber population in one year’s time. Then the data shows that $\mathbf{p} = A\mathbf{p}$, where A is the matrix given by

$$A = \begin{bmatrix} 1 & 0 \\ 0.8 & 0.5 \end{bmatrix}.$$

This is, writing $\mathbf{p}^* = \begin{bmatrix} q_n \\ q_o \end{bmatrix}$, there will be q_n "new" users in one years time, and q_o "old" users.

a. What percentage of current new users will still be subscribed by the start of the next year? How does this retention rate differ for old users?

c. It turns out that A has eigenvalues $\lambda = 1$ and $\lambda = 0.5$. Find an eigenvector for the eigenvalue $\lambda = 1$ and use it to describe the long term behavior of the user population for this streaming service. In the long run, what will the ratio of new to old users be?

d. Suppose that a different streaming service has the rate matrix

$$B = \begin{bmatrix} 0.5 & 0 \\ 0.8 & 0.4 \end{bmatrix}.$$

The matrix B has $\lambda = 0.5$ and $\lambda = 0.4$ as its eigenvalues. Using the eigenvector for the larger of the two eigenvalues, describe the long term behavior of this streaming service?

e. Suppose that both streaming services (which we will refer to by their rate matrices A and B), start with the same population vector p this year, and that both charge \$n for a yearly subscription. Write an expression (in terms of n, pa, pb) for the total income obtained by A and B after 4 years.