# Math087 - Review for Quiz 1

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A. *Summary*

The modeling techniques discussed so far in our course all focus on *optimization.* These techniques include:

1. *Calculus-based methods* for optimization of functions of one or more variables.
    - An important related issue is the *sensitivity* of the optimal solution to parameters involved in the model.
    - Finding optimal values subject to *constraints* can be carried out using the method of *Lagrange multipliers.*
2. Optimization of linear objective functions of several variables subject to linear inequality constraints; this is known as *linear programming.*
    - From a practical point-of-view, an important task in solving linear programs is identifying the objective function and constraints in a manner that is easy to provide as input to a computer.
    - Reformulation the problem by consideration of the *dual linear program* can provide useful insight about the model; the coefficients of a dual solution can often be viewed as *unit prices* for the quantities involved in the original constraints.
    - Formulating the objective function and constraints for a linear program by using a *network flow model* can be useful.
    - Occasionally one needs to find a solution to linear programming problem where some or all of the coefficients of the solution vector are required to be *integers.* Solving such *integer programming* problems can be challenging; we discussed a *branch-and-bound* strategy for solving such problems which leverages solutions to related *relaxed* linear programs.

B. *Problems and questions.*

1. Suppose that the temperature last Wednesday at a point $(x, y)$ in a national park is given by $T(x, y)$ degrees celcius. Here $x$ represents the distance east from a certain fixed location $(0, 0)$, and $y$ represents the distance north from this fixed location.

    The park is located in the mountains. The altitude above at the location $(x, y)$ is $h(x, y)$ meters.

    Consider the problem of finding the maximum temperature (on Wednesday) in the park – and a position where it occurs – at a given altitude of $a$ meters. Thus one needs to maximize the function $T(x, y)$ subject to the constraint $h(x, y) = a$.

a. To use the method of Lagrange multipliers to find the point $(x_0, y_0)$ at which $T(x, y)$ is maximal subject to $h(x, y) = a$, what equations must we solved? Choose one of the following:

(i) $\dfrac{\partial T}{\partial x} = \dfrac{\partial g}{\partial x}, \dfrac{\partial T}{\partial y} = \dfrac{\partial g}{\partial y}$ and $g(x, y) = a$.

(ii) $\dfrac{\partial T}{\partial x} = \lambda\dfrac{\partial g}{\partial x}, \dfrac{\partial T}{\partial y} = \lambda\dfrac{\partial g}{\partial y}$ for some real number $\lambda$.

(iii) $\dfrac{\partial T}{\partial x} = \lambda\dfrac{\partial g}{\partial x}, \dfrac{\partial T}{\partial y} = \lambda\dfrac{\partial g}{\partial y}$ for some real number $\lambda$, and $g(x, y) = a$.

b. If the position $(x, y) = (30-0.1a, 15-0.2a)$ maximizes the temperature $T(x, y)$ subject to the constraint $g(x, y) = a$, find the sensitivity of the $x$-coordinate of this solution to the quantity $a$ when $a = 100$ meters above sea level.
What are the *units* of the sensitivity figure?
*Recall that sensitivity is given by the formula* $S(x, a) = \dfrac{dx}{da} \cdot \dfrac{a}{x(a)}$.

c. Based on your answer to (b), finish the sentence:
"For the value $a = 100$ meters, a one percent change in $a$ results in a ________________ percent change in the $x$ coordinate of the location $(x, y)$ at the given altitude which had the highest temperature on Wednesday."

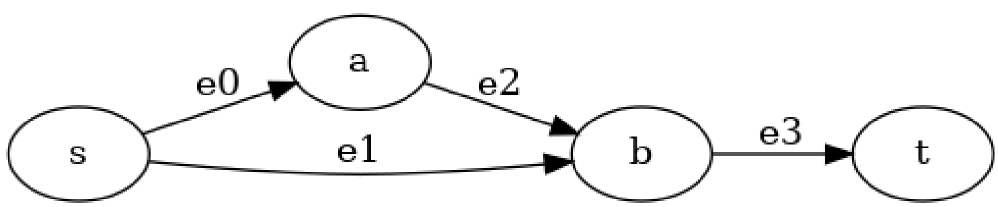
2. Consider a linear program $\mathscr{L}$ in *standard form*; say $\mathscr{L}$ is the linear program
`maximize` $\mathbf{c} \cdot \mathbf{x}$ where $\mathbf{c} = \begin{bmatrix} 10 & 20 & 1 & 1 \end{bmatrix}$
subject to costraints $A\mathbf{x} \leq \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$, and the non-negativity constraint $\mathbf{x} \geq \mathbf{0}$.

a. What are the inequalities that must be satisfied by the entries of the variable vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$?

b. What is the objective function of the *dual linear program* $\mathscr{L}'$?

c. What are the inequality constraints of the dual linear program $\mathscr{L}'$?

3. Consider the `max flow` linear program determined by the following *network flow diagram:*




The capacity `ci` through edge `ei` is given in the following table:

| edge | `e0` | `e1` | `e2` | `e3` |
|---|---|---|---|---|
| capacity | 10 | 15 | 20 | 5 |

We represent a *flow* as a vector $\mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$ where $f_i$ denotes the flow through edge `ei`.

a. What vector $\mathbf{c}$ determines the *objective function* for the `max flow` linear program?

b. What are the *inequality constraints* for the `max flow` linear program determined by this diagram?

c. What is the *conservation law* at the interior vertex `b`?

d. The equality constraints for the `max flow` linear program are given by the vector equation $B\mathbf{f} = \mathbf{0}$ for a suitable matrix $B$. What is $B$? (Remember that $B$ is determined by the conservation laws at the interior vertices).