PS 6 - Eigenvectors and Markov processes

Math087 - George McNinch

due 2025-03-09

1. Streaming

   A streaming service wants to model the annual growth rate of its subscribers. The data shows that the behavior of customers can be roughly predicted based on the duration of their subscription. Let p_n be the current number of subscribers who have had the service for less than a year, and p_o be the current number of subscribers who have had the service for more than a year (so in total there are $p_n + p_o$ subscribers). For convenience we will refer to the two types of subscriber as "new" and "old".

   Let $\mathbf{p} = \begin{bmatrix} p_n \\ p_o \end{bmatrix}$ be a vector recording the current subscriber population, and write $\mathbf{p}^*$ be the subscriber population in one year's time. Then the data shows that $\mathbf{p} = A\mathbf{p}$, where A is the matrix given by

   $$A = \begin{bmatrix} 1 & 0 \\ 0.8 & 0.5 \end{bmatrix}.$$

   This is, writing $\mathbf{p}^* = \begin{bmatrix} q_n \\ q_o \end{bmatrix}$, there will be q_n "new" users in one years time, and q_o "old" users.

   a. What percentage of current new users will still be subscribed by the start of the next year? How does this retention rate differ for old users?

   c. It turns out that A has eigenvalues $\lambda = 1$ and $\lambda = 0.5$. Find an eigenvector for the eigenvalue $\lambda = 1$ and use it to describe the long term behavior of the user population for this streaming service. In the long run, what will the ratio of new to old users be?

   d. Suppose that a different streaming service has the rate matrix

   $$B = \begin{bmatrix} 0.5 & 0 \\ 0.8 & 0.4 \end{bmatrix}.$$

   The matrix B has $\lambda = 0.5$ and $\lambda = 0.4$ as its eigenvalues. Using the eigenvector for the larger of the two eigenvalues, describe the long term behavior of this streaming service?

   e. Suppose that both streaming services (which we will refer to by their rate matrices A and B), start with the same population vector p this year, and that both charge $n for a yearly subscription. Write an expression (in terms of n, pa, pb) for the total income obtained by A and B after 4 years.

2. A game on a long street

   Suppose you're standing on a street with buildings labelled by the integers (specifically, you're in front of the building labelled 0, and suppose that the indices are increasing to the right). Suppose that every minute you flip a coin. If the coin is heads you walk right and if the coin is tails you walk left.

a. Explain why your position (i.e. the building you're in front of) as a function of time can be modeled as a Markov process.

b. Can the distance from where you started as a function of time be modeled by a Markov process?

c. Now suppose that every minute you flip two coins. If both are heads, you move right, if both are tails you move left and otherwise you stay put. Is your distance from where you started modeled by a Markov process in this scenario? How do you expect this to compare to the process described in part b?

d. For both experiments, compute the probability that you are standing on an odd number for minute $0, 1, 2, 3, 4$.

e. (Optional food for thought) Suppose your friend is playing the same game, but started at position -100. Do you think it is more likely that you two will eventually meet or that you two will never meet? Does this answer change when your friend starts at -1? How about -10000000?

3. Rain or shine

On Planet X, the weather is strangely predictable: the weather is always either sunny, rainy, foggy or snowy. If it rains today, its sunny tomorrow. If it is sunny today, its rainy tomorrow. If its foggy today, its not sunny tomorrow. Finally, the weather is never the same two days in a row. Apart from these rules, the weather is completely random, in that if e.g. its foggy today it is equally likely to be either rainy or snowy tomorrow. You live on Planet X and are trying to figure out what to wear this week, so you'd like to develop a model for the weather.

a. Explain why the weather can be modeled as a Markov process. Write out the transition matrix, and draw the corresponding finite state machine.

b. Check whether the conditions for the Perron-Frobenius theorem is satisfied for this problem (aperiodic and strongly connected). Explain your reasoning.

c. Do you expect power iteration to be effective for computing the greatest eigenvector of your transition matrix?

d. Find the eigenvalue decomposition for the transition matrix, and the associated eigenvectors. Explain why these values confirm your answer to part 2.

e. Suppose that the "weather rules" change so that if its sunny today, it is equally likely to be snowy or rainy tomorrow. Write out the new transition matrix, associated finite state machine, and determine whether the conditions for the Perron-Frobenius are satisfied. Compute the eigenvalue decomposition and compare to the previous set of eigenvalues.