week08-01–eigen

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1 George McNinch Math 87 - Spring 2025

2 Week 8

3 Power-iteration & eigenvalues

4 Eigenvalues & power-iteration

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. Our goal is to understand the *eventual behavior* of powers of A; i.e. the matrices A^m for $m \to \infty$.

4.1 Example: Diagonal matrices

Let's look at a simple example. Consider the following matrix:

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

In this case, it is easy to understand the powers of A; indeed, we have

$$A^m = \begin{bmatrix} \lambda_1^m & 0 & 0 & 0 \\ 0 & \lambda_2^m & 0 & 0 \\ 0 & 0 & \lambda_3^m & 0 \\ 0 & 0 & 0 & \lambda_4^m \end{bmatrix}$$

4.2 example, continued

Observe that if $|\lambda| < 1$, then $\lambda^m \to 0$ as $m \to \infty$. So e.g. if $|\lambda_i| < 1$ for $i = 1, 2, 3, 4$, then

$$A^m \to \mathbf{0} \quad \text{as} \quad m \to \infty.$$

If $\lambda_1 = 1$ and $|\lambda_i| < 1$ for $i = 2, 3, 4$, then

$$A^m \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, if $|\lambda_i| > 1$ for some i, then $\lim_{m\to\infty} A^m$ doesn't exist, because $\lambda_i^m \to \pm\infty$ as $m \to \infty$.

Of course, "most" matrices aren't diagonal, or at least not *literally*.

4.3 Eigenvalues and eigenvectors

Recall that a number $\lambda \in \mathbb{R}$ is an *eigenvalue* of the $n \times n$ matrix A if there is a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ for which

$$A\mathbf{v} = \lambda\mathbf{v};$$

$\mathbf{v}$ is then called an *eigenvector*.

If A is diagonal – e.g. if

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

– it is easy to see that each standard basis vector $\mathbf{e}_i$ is an eigenvector, with corresponding eigenvalue λ_i (the (i, i)-the entry of A).

4.4 Eigenvectors

Now suppose that A is an $n \times n$ matrix, that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are eigenvectors for A, and that $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues. Write

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$$

for the matrix whose columns are the $\mathbf{v}_i$.

Theorem 0: P is invertible if and only if the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.

Theorem 1: If the eigenvalues $\lambda_1, \dots, \lambda_n$ are *distinct*, then the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent, and in particular, the matrix P is invertible.

4.5 Diagonalizable matrices

Theorem 2: If the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent – equivalently, if the matrix P is invertible – then

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

i.e. $P^{-1}AP$ is the diagonal matrix $n \times n$ matrix whose diagonal entries are $\lambda_1, \dots, \lambda_n$.

Because of **Theorem 2**, one says that the $n \times n$ matrix A is *diagonalizable* if it has n linearly independent eigenvectors.

Thus if we are willing to replace our matrix by the *conjugate* matrix $P^{-1}AP$, then for A diagonalizable, for some purposes "we may as well suppose that A is diagonal" (though of course that statement is imprecise!).

4.6 Finding eigenvalues

One might wonder "how do I find eigenvalues"? The answer is: the eigenvalues of A are the roots of the *characteristic polynomial* $p_A(t)$ of A, where:

$$p_A(t) = \det(A - t \cdot \mathbf{I_n}).$$

Proposition: The characteristic polynomial $p_A(t)$ of the $n \times n$ matrix A has degree n, and thus A has no more than n distinct eigenvalues.

Remark: The eigenvalues of A are complex numbers which in general may fail to be real numbers, even when A has only real-number coefficients.

4.7 Tools for finding eigenvalues

`python` and `numpy` provides tools for finding eigenvalues. Let's look at the following example:

Example: Consider the matrix

$$A = \left(\frac{1}{10}\right) \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

[1]:
```python
import numpy as np
import numpy.linalg as npl


float_formatter = "{:.2f}".format
np.set_printoptions(formatter={'float_kind':float_formatter})


A = (1/10)*np.array([[1,1,0,0],[0,2,2,0],[0,3,3,1],[0,0,1,2]])

A
```

[1]:
```
array([[0.10, 0.10, 0.00, 0.00],
       [0.00, 0.20, 0.20, 0.00],
       [0.00, 0.30, 0.30, 0.10],
       [0.00, 0.00, 0.10, 0.20]])
```

Let's find the eigenvectors/values of this 4×4 matrix A; we'll use the function `eig` found in the python module `numpy.linalg`:

[3]:
```
(e_vals,e_vecs) = npl.eig(A)
[e_vals,e_vecs]
```

[3]:
```
[array([0.10, 0.52, -0.02, 0.20]),
 array([[1.00, -0.12, -0.47, -0.30],
        [0.00, -0.51, 0.56, -0.30],
        [0.00, -0.81, -0.62, -0.00],
        [0.00, -0.25, 0.28, 0.90]])]
```

[8]:
```
e_vecs[1,:]
```

[8]:
```
array([0.00, -0.51, 0.56, -0.30])
```

The function `eig` returns a "list of np arrays". This first array contains the eigenvalues, and the second contains the a matrix whose *columns* are the eigenvectors.

We've assigned the first component of the list to the variable `e_vals` and the second to `e_vecs`.

To get the individual eigenvectors, we need to slice the array `e_vecs`.

For example, to get the 0-th ("first"!) eigenvector, we can use

`e_vecs[:,0]`

Here, the argument `:` indicates that the full range should be used in the first index dimension, and the argument `0` indicates the the second index dimension of the slice is `0`. Thus `numpy` returns the array whose entries are `e_vecs[0,0]`, `e_vecs[1,0]`, `e_vecs[2,0]`, `e_vecs[3,0]`.

Let's confirm that this is really an eigenvector with the indicated eigenvalue:

[10]:
```
v = e_vecs[:,3]
[v,A @ v,e_vals[3]*v, (A @ v - e_vals[3] * v < 1e-7).all()]
```

[10]:
```
[array([-0.30, -0.30, -0.00, 0.90]),
 array([-0.06, -0.06, -0.00, 0.18]),
 array([-0.06, -0.06, -0.00, 0.18]),
[array([1.00, 0.00, 0.00, 0.00]),
 array([0.10, 0.00, 0.00, 0.00]),
 array([0.10, 0.00, 0.00, 0.00]),
 np.True_]
```

Let's check *all* of the eigenvalues:

[4]:
```
import pprint

def check(A):
    e_vals,e_vecs = npl.eig(A)

    def check_i(i):
```

```
        lam = float(e_vals[i])
        v= e_vecs[:,i]
        return { "lambda": lam,
                 "A.v": A @ v,
                 "lambda.v": lam*v,
                 "match?": np.abs(A @ v - lam*v < 1e-6).all()
               }
    return [ check_i(i) for i in range(len(e_vals)) ]
```

[5]:
```
pprint.pp(check(A))
```

```
[{'lambda': 0.1,
  'A.v': array([0.10, 0.00, 0.00, 0.00]),
  'lambda.v': array([0.10, 0.00, 0.00, 0.00]),
  'match?': np.True_},
 {'lambda': 0.5192582403567257,
  'A.v': array([-0.06, -0.26, -0.42, -0.13]),
  'lambda.v': array([-0.06, -0.26, -0.42, -0.13]),
  'match?': np.True_},
 {'lambda': -0.019258240356725218,
  'A.v': array([0.01, -0.01, 0.01, -0.01]),
  'lambda.v': array([0.01, -0.01, 0.01, -0.01]),
  'match?': np.True_},
 {'lambda': 0.19999999999999998,
  'A.v': array([-0.06, -0.06, -0.00, 0.18]),
  'lambda.v': array([-0.06, -0.06, -0.00, 0.18]),
  'match?': np.True_}]
```

Let's observe that A has 4 distinct eigenvalues, and is thus diagonalizable. Moreover, every eigenvalue λ of A satisfies $|\lambda| < 1$. Thus, we conclude that $A^m \to \mathbf{0}$ as $m \to \infty$.

And indeed, we confirm that:

[6]:
```
res=[(npl.matrix_power(A,j) - np.zeros((4,4)) < 1e-7*np.ones((4,4))).all() for
 ↪j in range(50)]

j = res.index(True)   ## find the first instance in the list of results

print(f"A^{j} == 0")
```

```
A^24 == 0
```

4.8 Eigenvalues and power iteration.

Theorem 3: Let A be a diagonalizable $n \times n$, with n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. As before, write

$$P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

a) Suppose $|\lambda_i| < 1$ for all i. Then $A^m \to \mathbf{0}$ as $m \to \infty$.

b) Suppose that $\lambda_1 = 1$, and $|\lambda_i| < 1$ for $2 \le i \le n$. Any vector $\mathbf{v} \in \mathbb{R}^n$ may be written

$$\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{v}_i.$$

If $c_1 \neq 0$, then

$$A^m \mathbf{v} = c_1 \mathbf{v}_1 \quad \text{as} \quad m \to \infty.$$

If $c_1 = 0$ then

$$A^m \mathbf{v} = \mathbf{0} \quad \text{as} \quad m \to \infty.$$

4.9 Proof:

For **a)**, note that $P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Which shows that

$$(P^{-1}AP)^m = \operatorname{diag}(\lambda_1, \dots, \lambda_n)^m = \operatorname{diag}(\lambda_1^m, \dots, \lambda_n^m) \to \mathbf{0} \quad \text{as } m \to \infty.$$

Let's now notice that

$$(P^{-1}AP)^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}AAP = P^{-1}A^2P$$

and more generally

$$(P^{-1}AP)^m = P^{-1}A^mP \quad \text{for } m \ge 0.$$

We now see that

$$P^{-1}A^mP \to \mathbf{0} \quad \text{as } m \to \infty$$

so that

$$A^m \to P \cdot \mathbf{0} \cdot P^{-1} = \mathbf{0} \quad \text{as } m \to \infty$$

4.10 Proof of b):

Recall that $\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{v}_i$.

For $i > 1$, **a)** shows that

$$A^m \mathbf{v}_i \to \mathbf{0} \quad \text{as } m \to \infty.$$

while

$$A^m \mathbf{v}_1 = \mathbf{v}_1 \quad \text{for all } m.$$

The preceding discussion now shows that

$$A^m \mathbf{v} = \sum_{i=1}^{n} c_i A^m \mathbf{v}_i \mapsto c_1 \mathbf{v}_1$$

and **b)** follows at once.

4.11 Corollary

Suppose that A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$, that $\lambda_1 = 1$, and that $|\lambda_i| < 1$ for $i = 2, ..., n$. Let $\mathbf{v_1}$ be a 1-eigenvector for A.

Then

$$A^m \to B \quad \text{as } m \to \infty$$

for a matrix B with the property that each column of B is either $\mathbf{0}$ or some multiple of $\mathbf{v_1}$.

Indeed: the ith column of B can be found by computing

$$(\heartsuit) \quad \lim_{m \to \infty} A^m \mathbf{e}_i$$

where $\mathbf{e}_i$ is the ith standard basis vector.

We've seen above that $(\heartsuit)$ is either 0 or a multiple of $\mathbf{v}$, depending on whether or not the coefficient c_1 in the expression

$$\mathbf{e}_i = \sum_{j=1}^{n} c_j \mathbf{v}_j$$

is zero.

4.12 Examples revisited: population growth & aging

Recall from last week our finite-state machine describing population & aging.

We considered a population of organisms described by:

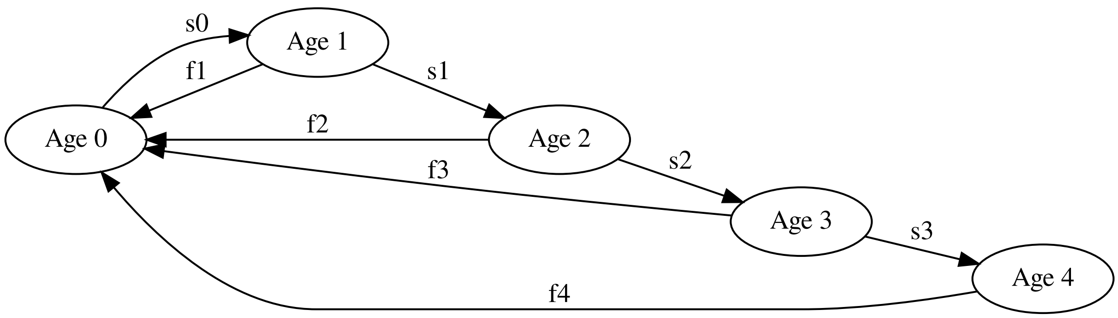
[1]:
```
from graphviz import Digraph
pop = Digraph("pop")
pop.attr(rankdir='LR')

p = list(range(5))
with pop.subgraph() as c:
#    c.attr(rank='same')
    for i in p:
        c.node(f"Age {i}")

for i in p:
    if i+1 in p:
        pop.edge(f"Age {i}",f"Age {i+1}",f"s{i}")
    if i != 0:
        pop.edge(f"Age {i}","Age 0",f"f{i}")

pop
```

[1]:



We suppose that $s_7 = 0$, so that the life-span of the organism in question is ≤ 8 time units.

If the population at time t is described by $\mathbf{p}^{(t)} = \begin{bmatrix} p_0 & p_1 & \cdots & p_7 \end{bmatrix}^T$ then the population at time $t+1$ is given by

$$\mathbf{p}^{(t+1)} = \begin{bmatrix} \sum_{i=0}^{7} f_i p_i & s_0 p_0 & \cdots & s_6 p_6 \end{bmatrix}^T = A\mathbf{p}^{(t)}$$

where

$$A = \begin{bmatrix} f_0 & f_1 & f_2 & \cdots & f_6 & f_7 \\ s_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_6 & 0 \end{bmatrix}.$$

4.13 parameters

Previously, we considered this model for two different sets of parameters:

```
fA = [.30,.50,.35,.25,.25,.15,.15,.5]
sA = [.30,.60,.55,.50,.30,.15,.05,0]
```

and

```
fB = [.50,.70,.55,.35,.35,.15,.15,.5]
sB = [.40,.70,.55,.50,.35,.15,.05,0]
```

[8]:
```
import numpy as np


float_formatter = "{:.2f}".format
np.set_printoptions(formatter={'float_kind':float_formatter})

def sbv(index,size):
    return np.array([1.0 if i == index else 0.0 for i in range(size)])

## note
## bv("b",["a","b","c"])
## >> np.array([0,1,0])
```

```
ones = np.ones(8)


def onePowers(f,s,iter=20,skip=1):
    # create the "all ones vector" of the appropriate length
    ones = np.ones(len(f))

    # create the matrix `A` -- initial row is the vector `f`; subsequent rows␣
 ↪are multiples of
    # standard basis vectors

    A = np.concatenate([ [f], [ s[i]*sbv(i,len(f)) for i in range(len(sA))] ],␣
 ↪axis = 0)

    # computes the product of `ones` and the jth power of the matrix `A`
    # returns the results as a `dictionary` with key `j` and value
    # `ones @ A^j`

    s ={ j : ones @ np.linalg.matrix_power(A,j)
         for j in range(0,iter,skip) }

    return s


def A(f=[],s=[]):
    return np.concatenate([ [f], [ s[i]*sbv(i,len(f)) for i in range(len(s))]␣
 ↪], axis = 0)


def abs_eig_vals(f,s):
    e_val,_ = npl.eig(A(f,s))
    return [ float(np.abs(x)) for x in e_val ]
```

[9]:
```
fA = [.30,.50,.35,.25,.25,.15,.15,.5]
sA = [.30,.60,.55,.50,.30,.15,.05]
pprint.pp(abs_eig_vals(fA,sA))
```

[10]:
```
fB = [.50,.70,.55,.35,.35,.15,.15,.5]
sB = [.40,.70,.55,.50,.35,.15,.05]
pprint.pp(abs_eig_vals(fB,sB))
```

```
[1.0104953810383637,
 0.36497808499800904,
 0.36497808499800904,
 0.3748508468483046,
 0.3748508468483046,
 0.1787639416420966,
```

```
 0.17288710686695213,
 0.17288710686695213]
```

[11]:
```
(len(fB),len(sB))
```

[11]:
```
(8, 7)
```

Let's look at one more example, now where the organisms have a max life-span of 4 time units (for simplicity!)

Let's consider

```
fC = [0, .2, .49559, 0.4]
sC = [.98, .96, .9]
```

[12]:
```
fC = [0.000, .2, .49559, 0.399]
sC = [.9799, .96, .9]

pprint.pp(abs_eig_vals(fC,sC))

e_vals,e_vecs = npl.eig(A(fC,sC))

pprint.pp(e_vals)
pprint.pp(e_vecs)
```

```
[0.9999969131763253, 0.7508930550032085, 0.7508930550032085, 0.5991196464539855]
array([ 0.99999691+0.j        , -0.20043863+0.72364683j,
       -0.20043863-0.72364683j, -0.59911965+0.j        ])
array([[ 0.5298541 +0.j        ,  0.23531368-0.22372896j,
         0.23531368+0.22372896j,  0.19595556+0.j        ],
       [ 0.51920563+0.j        , -0.3633377 -0.2180027j ,
        -0.3633377 +0.2180027j , -0.32049834+0.j        ],
       [ 0.49843895+0.j        , -0.14460272+0.52206151j,
        -0.14460272-0.52206151j,  0.51355085+0.j        ],
       [ 0.44859644+0.j        ,  0.64928822+0.j        ,
         0.64928822-0.j        , -0.7714582 +0.j        ]])
np.complex128(0.9999969131763253+0j)
```

[15]:
```
# note that the 0-th eigenvalue is essentially 1
pprint.pp(e_vals[0])

# and the corresponding eigenvector is
pprint.pp(e_vecs[:,0])
```

```
np.complex128(0.9999969131763253+0j)
array([0.5298541 +0.j, 0.51920563+0.j, 0.49843895+0.j, 0.44859644+0.j])
```

Remark: Observe that these results purport to be complex numbers – but they have 0 imaginary part e.g. the `0-th` eigenvalue is `0.9999969131763253+0j` which is roughly the real number 1.

However `e_vals[1]` is genuinely a complex number: `-0.20043863336116968+0.723646829819877j` or about `-.2 + .72j`

[16]:
```
e_vals[1]
```

[16]:
```
np.complex128(-0.20043863336116968+0.723646829819877j)
```

4.14 Explainer

In each case, the matrix A has distinct eigenvalues (in case `C` there are two eigenvalues with the same absolute value, but they are complex and distinct from one another!) Thus A is diagonalizable in each case.

For the parameters `fA,sA` all eigenvalues of A have absolute value < 1. This confirms our previous conclusion that

$$A^m \to \mathbf{0} \quad \text{as } m \to \infty$$

For the parameters `fB,sB` there is an eigenvalue of A which has abolute value $1.01 > 1$ (actually, this 1.01 *is* the eigenvalue). Thus A^m has no limiting value as $m \to \infty$.

Finally, the parameters `fC,sC` yield an eigenvalue of A which is very close to 1.

4.15 `fC,sC`

In this setting, note that the corresponding 1-eigenvector is

`w=[0.5298541, 0.51920563, 0.49843895, 0.44859644]`

Let's normalize the vector `w` by dividing by the sum of its components

(recall that any non-zero multiple of a λ-eigenvector is again a λ-eigenvector!)

[13]:
```
w=np.array([0.5298541, 0.51920563, 0.49843895, 0.44859644])
ww = (1/np.sum(w))*w
ww
```

[13]:
```
array([0.27, 0.26, 0.25, 0.22])
```

Thus the components of `ww` sum to 1. They represent *probabilities*.

We conclude that the expected longterm population distribution in this case is:

Age 0	Age 1	Age 2	Age 3
27 %	26 %	25 %	22 %