

week14-01-springs

April 14, 2025

1 George McNinch Math 87 - Spring 2025

2 Week 14

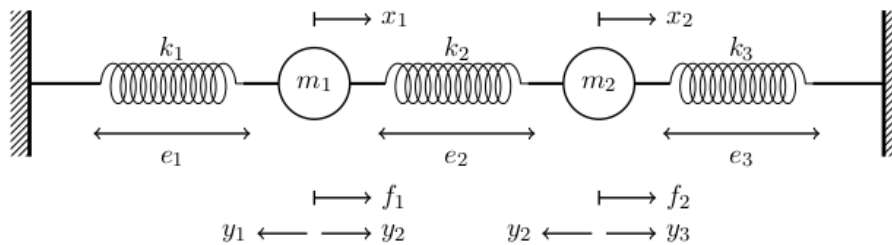
3 Spring networks

4 Modeling spring networks

Examples of “Structural modeling” that will ultimately involve *least squares* approximation.

4.1 One-dimensional models

Let’s consider a linear network of 3 springs and 2 masses:



Here are the variables:

- f_j = applied load or force to mass (in N = Newtons), for $j = 1, 2$
- k_i = spring constant (in N/m = Newtons per meter), for $i = 1, 2, 3$
- e_i = elongation of spring i from equilibrium (in m = meters)
- x_j = displacement of mass j from equilibrium (in m = meters)
- y_i = restoring force on spring i (in N = Newtons)

The “inputs” are the applied forces f_j which cause the masses to move, resulting in elongation of springs.

We’ll take “movement to the right” to be *positive*, and a stretch as *positive* elongation.

Thus we have the equations:

$$e_1 = x_1, \quad e_2 = x_2 - x_1, \quad e_3 = -x_2.$$

(This third equation reflects the fact that spring 3 compresses when m_2 moves to the right.)

Let's put this in matrix form:

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B\mathbf{x}.$$

Now, let's recall that according to [Hooke's Law](#), the elongation of the spring causes a restoring force on the mass, determined by the *spring constant* $k_i > 0$. Thus we get equations

$$y_j = k_j e_j \quad \text{for } j = 1, 2, 3.$$

In matrix form, these equations read:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = K\mathbf{e}.$$

Combining these equations gives

$$\mathbf{y} = K\mathbf{e} = KB\mathbf{x}.$$

```
[15]: import numpy as np
import numpy.linalg as la

import sympy as sp

def sbv(i,n):
    return np.array([1 if j == i else 0 for j in range(n)])

def makeK(kar):
    n = len(kar)
    return np.array([kar[i]*sbv(i,n) for i in range(n)])

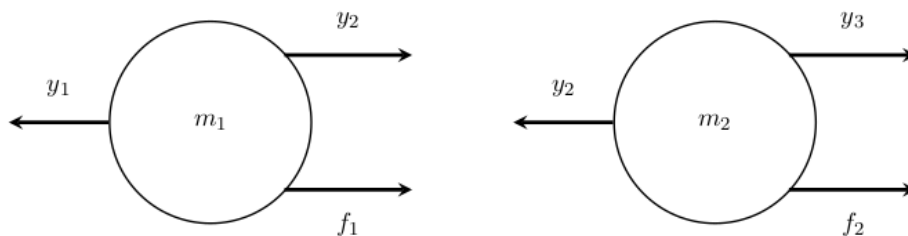
def makeB(m,n):
    return np.array([(-1)*sbv(i-1,m) + sbv(i,m) for i in range(n)])

k1, k2, k3 = sp.symbols('k1 k2 k3')

K = makeK([k1,k2,k3])
B = makeB(2,3)
(K,B)
```

```
[15]: (array([[k1, 0, 0],
              [0, k2, 0],
              [0, 0, k3]], dtype=object),
       array([[ 1,  0],
              [-1,  1],
              [ 0, -1]]))
```

Next, we assume that the system is at rest after the loads are applied (i.e. the forces f_i).



Looking at the diagram, we see that the following equations must hold:

(The first diagram gives:)

$$\begin{aligned} y_1 &= y_2 + f_1 \implies \\ y_1 - y_2 &= f_1 \end{aligned}$$

(The second diagram gives:)

$$\begin{aligned} y_2 &= y_3 + f_2 \implies \\ y_2 - y_3 &= f_2 \end{aligned}$$

In matrix form this reads

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

i.e.

$$B^T \mathbf{y} = \mathbf{f}$$

Combined with our earlier equation

$$\mathbf{y} = K\mathbf{e} = KB\mathbf{x}$$

we now see

$$B^T KB\mathbf{x} = \mathbf{f}.$$

```
[2]: A=B.transpose() @ K @ B
A
```

```
[2]: array([[k1 + k2, -k2],
          [-k2, k2 + k3]], dtype=object)
```

Thus we have

$$A = B^T KB = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}.$$

```
[16]: def findDisplacements(kar,far):
        # kar = array of spring constants
        # far = array of initial forces.
        m = len(far)
        n = len(kar)
        B = makeB(m,n)
        K = makeK(kar)
        A = B.transpose() @ K @ B
        f = np.array(far)
        return la.solve(A,f)

        # Let's find the displacements for spring constants `k = [1,1,1]`
        # and forces `f = [3,-3]`

        findDisplacements([1,1,1],[3,-3])
```

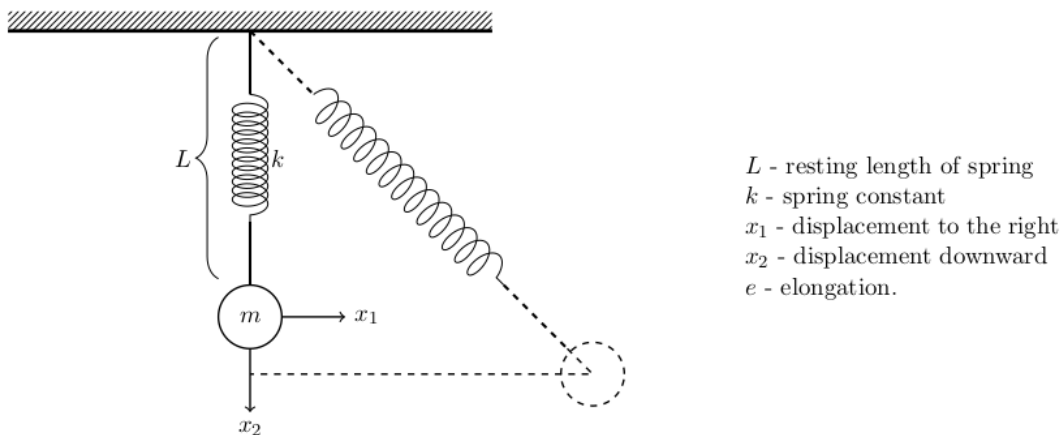
```
[16]: array([ 1., -1.])
```

```
[4]: findDisplacements([1,1,1],[3,-2])
```

```
[4]: array([ 1.33333333, -0.33333333])
```

4.2 Two dimensional models

Now let's allow the mass to move in two dimensions:



We see in this case that the elongation e satisfies

$$e = \sqrt{x_1^2 + (L + x_2)^2} - L$$

Since this does not express a linear relationship between e and the displacements x_1, x_2 , we can't express this relationship using a matrix.

But we can *linearize*. Recall that the linearization (first-order Taylor polynomial) about $t = 0$ of

the function $y = \sqrt{1+t}$ is given by

$$(\clubsuit) \quad \sqrt{1+t} \approx 1 + \frac{t}{2} + O(t^2).$$

Let's use this linearization to rewrite the expression for e given above.

We first rewrite

$$\begin{aligned} x_1^2 + (L + x_2)^2 &= x_1^2 + L^2 + 2Lx_2 + x_2^2 \\ &= L^2 \left(\frac{x_1^2}{L^2} + 1 + \frac{2x_2}{L} + \frac{x_2^2}{L^2} \right) \\ &= L^2 \left(1 + \frac{2x_2}{L} + \frac{x_1^2}{L^2} + \frac{x_2^2}{L^2} \right) \end{aligned}$$

so that

$$\begin{aligned} e &= \sqrt{x_1^2 + (L + x_2)^2} - L \\ &= \sqrt{L^2 \left(1 + \frac{2x_2}{L} + \frac{x_1^2}{L^2} + \frac{x_2^2}{L^2} \right)} - L \\ &= L \sqrt{1 + \frac{2x_2}{L} + \frac{x_1^2}{L^2} + \frac{x_2^2}{L^2}} - L \end{aligned}$$

Now taking $t = \frac{2x_2}{L} + \frac{x_1^2}{L^2} + \frac{x_2^2}{L^2}$ the approximation (\clubsuit) gives

$$\begin{aligned} e &\approx L \left(1 + \frac{1}{2}t \right) - L \\ &= L \left(1 + \frac{1}{2} \frac{2x_2}{L} + \frac{x_1^2}{L^2} + \frac{x_2^2}{L^2} \right) - L \\ &= x_2 + \frac{x_1^2 + x_2^2}{2L} \end{aligned}$$

Now, this is of course still not a linear relationship between e and x_1, x_2 . Note that the approximation (\clubsuit) depends on the assumption that $t = \frac{2x_2}{L} + \frac{x_1^2}{L^2} + \frac{x_2^2}{L^2} \approx 0$.

If we suppose that the displacements x_1, x_2 are small compared to the resting length L of the spring, then $x_1^2 + x_2^2$ is even smaller compared to L , so making one more approximation, we eliminate the quadratic term and so we get

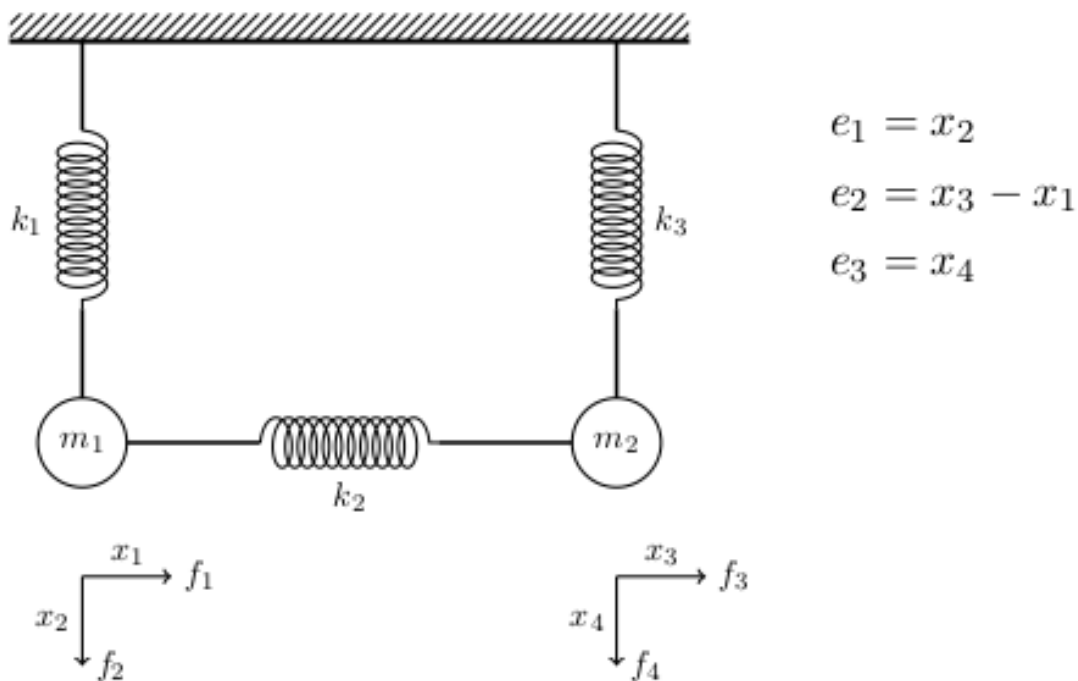
$$e \approx x_2.$$

Remark The approximation $e \approx x_2$ is equivalent to a [small-angle approximation](#). It essentially says that the horizontal displacements are negligible in the elongation, but vertical displacements are important.

Remark This assumption is of course “**wrong**”, but can still be useful. Especially, it is now *linear*

4.3 Spring networks

Let's consider a network of springs in the two-dimensional setting.



The *linearization* discussed in the previous section says that the elongation is determined by the displacement in the “1 dimensional direction of the spring”.

We get the matrix equation

$$\mathbf{e} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = B\mathbf{x}.$$

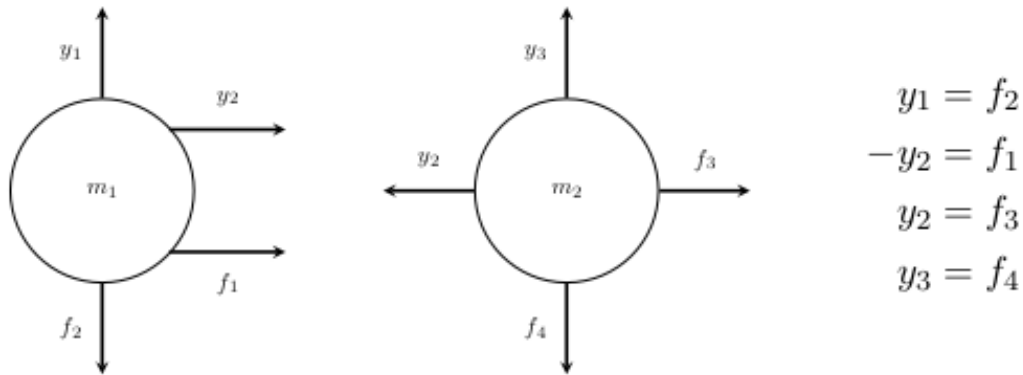
i.e.

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As before, Hooke's Law gives

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = K\mathbf{e}.$$

We see again that the equations to balance the forces come from the transposed matrix B^T :



$$\begin{aligned} y_1 &= f_2 \\ -y_2 &= f_1 \\ y_2 &= f_3 \\ y_3 &= f_4 \end{aligned}$$

This gives

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{y} = B^T \mathbf{y} = \mathbf{f}.$$

Thus

$$B^T K B \mathbf{x} = \mathbf{f}.$$

```
[73]: B = np.array([[0,1,0,0],[-1,0,1,0],[0,0,0,1]])

K=makeK([1,1,1])          ## springs of equal strength
f = np.array([0,1,0,1])    ## downward forces only

A = B.transpose() @ K @ B

try:
    la.solve(A , f)
except:
    print(f"the matrix B^T K B = \n\n{A}\n\n is singular")
```

the matrix $B^T K B =$

```
[[ 1  0 -1  0]
 [ 0  1  0  0]
 [-1  0  1  0]
 [ 0  0  0  1]]
```

is singular

```
[69]: # numpy confirms that the null space has dimension 1

la.matrix_rank(A)
```

[69]: 3

so we just need to find a non-zero vector in the null space

now, the Null space of A can be obtained from the *singular value decomposition* of A.

`la.svd(A)` returns a tuple (U , S, Vh) where e.g. U is a unitary matrix

`U,S,Vh = la.svd(A)`

notice that the last column (i.e. the fourth column, or the column in pos 3) of U is in the null space of A:

```
[75]: nullA = U[:,3]
      A @ nullA
```

[75]: array([0., 0., 0., 0.])

```
[76]: # now we just need a solution to A @ x = f
      # let's use `lstsq` to get one

      x,_,_,_ = la.lstsq(A,f,rcond=None)
      (x,nullA)
```

```
[76]: (array([0., 1., 0., 1.]),
      array([0.70710678, 0.          , 0.70710678, 0.          ]))
```

Thus we see that the general solution to

$$A\mathbf{x} = \mathbf{f}$$

is given by

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 1 \\ t \\ 1 \end{bmatrix}.$$

What is the physical meaning of the solutions given by the null space of A? i.e. the solutions

$$t\mathbf{x}_n = t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

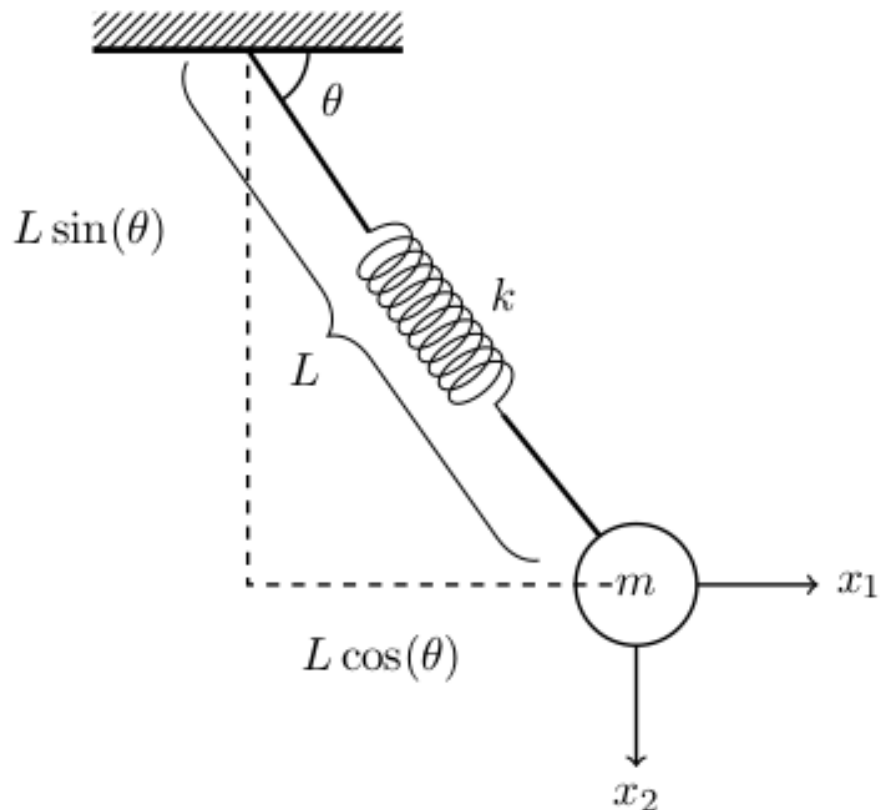
Well, since $A\mathbf{x}_n = \mathbf{0}$, we see that a non-zero displacement can result from application of *zero force* (!).

This is clearly “wrong”, and is a consequence of our linearization, **but** these displacement are referred to as *unstable modes*, and they do correspond to some real physical phenomena –

see e.g. [Tacoma bridge collapse \(1940\)](#)

5 Stabilizing the unstable modes

One way to try to stabilize the unstable modes is to add a diagonal spring:



For the indicate angle θ , the elongation is given by

$$e = \sqrt{(L \cos(\theta) + x_1)^2 + (L \sin(\theta) + x_2)^2}$$

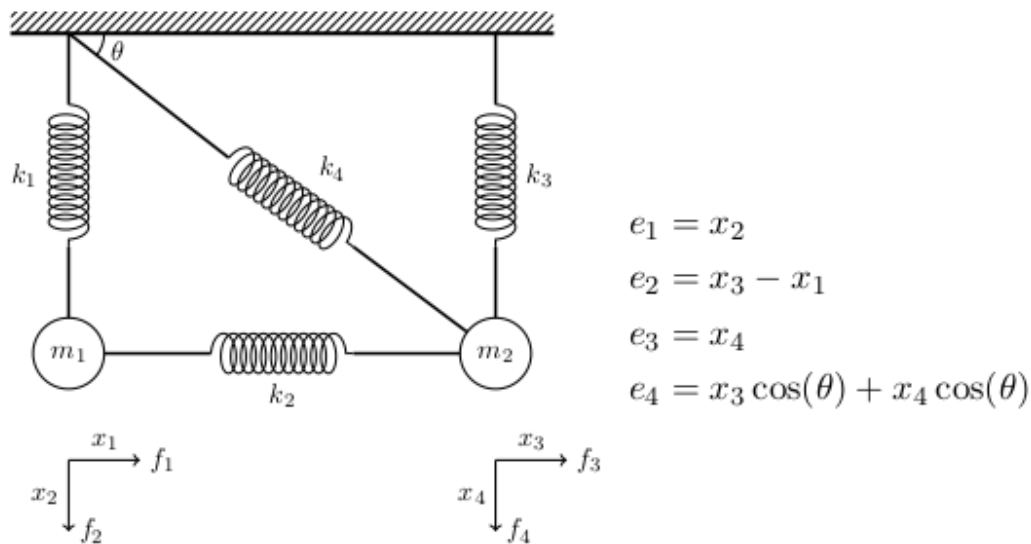
Now our linearization assumption gives

$$e \approx x_1 \cos(\theta) + x_2 \sin(\theta).$$

Observe that under the linearization, the elongation is again in the “spring direction”.

Note e.g. if $\theta = 0$ then $e = x_1$ and if $\theta = \frac{\pi}{2}$ then $e = x_2$ (as before).

Now return to our unstable network, and add a diagonal spring:

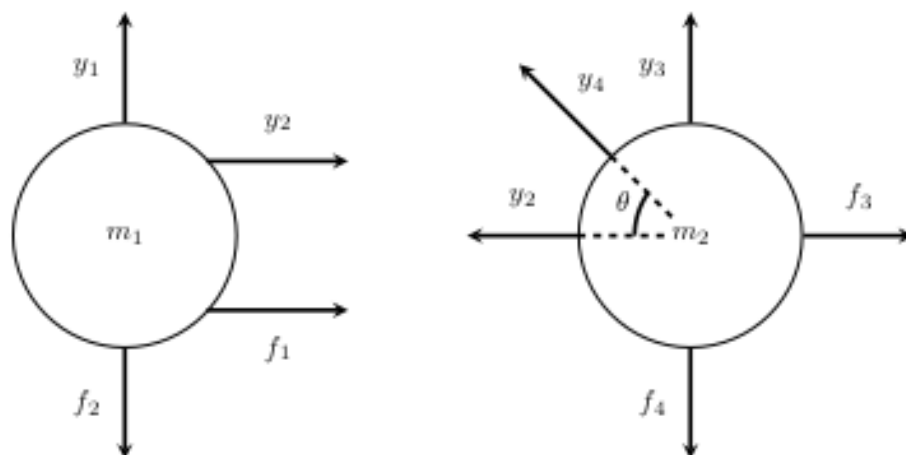


The new parameter here is the elongation e_4 .

Now we see that $\mathbf{e} = B\mathbf{x}$ where

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \end{bmatrix}.$$

Let's inspect the force balancing conditions:



As before, we see that

$$B^T \mathbf{y} = \mathbf{f}$$

so that

$$B^T K B \mathbf{x} = \mathbf{f}.$$

```
[82]: # let's suppose theta = pi/3
theta = np.pi/3
BB = np.array([[0,1,0,0],[-1,0,1,0],[0,0,0,1],[0,0,np.cos(theta),np.
    ↪sin(theta)]])

K=makeK([1,1,1,1])      ## springs of equal strength
f = np.array([0,1,0,1])  ## downward forces only

AA = BB.transpose() @ K @ BB

## AA is non-singular!
la.matrix_rank(AA)
```

[82]: 4

So there are no *unstable modes* for this system. Non-zero displacements \mathbf{x} are only determined by non-zero force vectors \mathbf{f} .

[]: