ProblemSet 1 – Optimization

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1. An optimization question in auto manufacturing

An automobile manufacturer makes a profit of \$1,500 per unit on the sale of a certain car model. It is estimated that for every \$100 of rebate, the number of units of this model sold in a given month will increase by 15%.

a. What amount of rebate will maximize the manufacturers profit for the month? Model the question as a single-variable optimization problem.

b. Compute the *sensitivity* of your answer to the 15% assumption. Consider both the amount of rebate and the resulting profit.

c. Suppose that rebates actually generate only a 10% increase in sales per \$100. What is the effect? What if the response is somewhere between 10% and 15% per \$100 of rebate?

d. Under what circumstances would an offer of a rebate cause a reduction in profit?

2. Computing yields with multi-variate optimization

A chemist is synthesizing a compound. In the last step, she must dissolve her reagents in a solution with a particular `pH` level H, for $1.2 \le H \le 2.7$, and heated to a temperature T (in degrees Celsius), for $66 \le T \le 98$. Her goal is to maximize her percent yield as a percentage of the initial mass of the reagents.

The equation determining the percentage $F(H,T)$ is

$$F(H,T) = -0.038 \cdot T^2 - 0.223 \cdot T \cdot H - 10.982 \cdot H^2 + 7.112 \cdot T + 60.912 \cdot H - 328.898.$$

1. Find the optimal temperature and `pH` level in the allowed range.
2. Use `matplotlib` to produce a graph and a contour plot of the percentage of the powder function $F(H,T)$.

(To get a usable copy of your image, you can proceed in a few ways:

- if you produce the graph in colab you can right-click on the image and Save As a file on your file system.
- if you work in Python on your computer, you can save the image via a command like

  ```
  > g.savefig("my graph image.png")
  ```

3. Blood typing

Human blood is generally classified in the "ABO" system, with four blood types: `A`, `B`, `O`, and `AB`. These four types reflect six gene pairs (pairs of alleles), with blood type `A` corresponding to gene pairs `AA` and `AO`, blood type `B` corresponding to gene pairs `BB` and `BO`, blood type `O` corresponding to gene pair `OO`, and blood type `AB` corresponding to gene pair `AB`. Let p be the proportion of gene `A` in the population, let q be the proportion of gene `B` in the population, and let r be the proportion of gene `O` in the population. Observe that $p + q + r = 1$.

a. The Hardy-Weinberg principle implies that:

   > (♣) The quantities p, q, and r remain constant from generation to generation, as do the frequencies of occurrence of the different genotypes `AA`, `AO`, … .

   Assuming the validity of (♣), what is the probability that an individual has genotype `AA`? `BB`? `OO`? What is the probability of an individual having two different genes? Express your response using the quantities p, q and r.

b. Still assuming the valiidty of ($\clubsuit$), find the maximum percentage of the population that can have two different genes. Perform this computation in two different ways:

- directly maximize a function of only two variables
- use the method of Lagrange multipliers.

c. Explain in words what the Lagrange multiplier represents in the second computation of part (b).

4. Newton's method and root finding

a. microprocessors

One of the uses of Newton's method is in implementing division on microprocessors, where only addition and multiplication are available as primitive operations. To compute $x = a/b$, first the root of $f(x) = 1/x - b$ is found using Newton's method, then the fraction is computed with one last multiplication by a.

Find the Newton iteration needed to solve $f(x) = 0$ and explain why it is well-suited to this purpose. (**Note**: We are trying to approximate division, so we shouldn't actually use division functions implemented in `python`...)

b. experiments

Apply Newton's Method to compute $1/b$, where b is: (i) the last 3 digits of your student number; and (ii) the area code of your phone number. For these experiments, report the number of iterations required for the approximation to be consistent to 10 digits.
