week02-01–multivariable-optimization

January 23, 2025

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2 Week 2: Multi-variable Optimization

2.1 Optimization of functions of several variables

Consider a function $f(x, y)$ of two variables. You learned in Calculus 3 (*vector calculus*) how to search for the points (x, y) at which f assumes its maximum and minimum value. Let's briefly recapitulate this story.

Recall that for a function of a single variable, critical points are those points for which the tangent line is horizontal. In the single variable case, the criteria depends instead on the *tangent plane*.

Recall that the surface defined by $z = f(x, y)$ can be *parameterized* by $\mathbf{r}(x, y) = (x, y, f(x, y))$. So a **normal vector** to this surface at a point $P = (x_0, y_0, f(x_0, y_0))$ on the surface is given by the *cross product*

$$\left(\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right)_P.$$

Alternatively, consider $F(x, y, z) = z - f(x, y)$ so that the surface is defined by $F = 0$. Then the gradient ∇F defines a normal vector at each point P, where

$$(\nabla F)_P = \left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}\right)_P.$$

Both points of view can be useful, but here they lead to the same formula. At the point $P = (x_0, y_0, f(x_0, y_0))$ one has a normal to the surface defined by $z = f(x, y)$ given by

$$\mathbf{n}|_P = \left(\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}\right)_P \times \left(\mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}\right)_P = \left(\mathbf{k} - \frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j}\right)_P$$

Now, the **tangent plane** at P to the surface $z = f(x, y)$ is just the plane orthogonal to this normal vector $\mathbf{n}_P$. Thus, the tangent plane at P is horizontal – parallel to the x, y-plan – just in case this normal vector points in the $\mathbf{k}$ direction – i.e. provided that

$$(\clubsuit) \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = 0 \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = 0$$

Mimicking the one variable case, we say that the points (x_0, y_0) for which the tangent plane to the surface $P = (x_0, y_0, f(x_0, y_0))$ is horizontal are the *critical points*.

So we find the critical points by simultaneously solving the equations (♣).

There is a *second derivative test* which gives information about the "max/min status" of these critical points.

To use this test, consider the matrix of second partial derivatives

$$M(x_0, y_0) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} \Bigg|_{(x_0,y_0)}.$$

For a reasonable class of functions, the "mixed partials" $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ coincide.

Remember that the *determinant* of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$.

So, the *determinant* of M is the expression

$$D = D(x_0, y_0) = \left(\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y} \right]^2 \right) \Bigg|_{(x_0,y_0)}$$

Theorem: (Second derivative test) > Suppose that (x_0, y_0) is a critical point.
> a. If $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)} < 0$, then $f(x, y)$ has a relative maximum at (x_0, y_0).
> b. If $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2}\Big|_{(x_0,y_0)} > 0$, then $f(x, y)$ has a relative minimum at (x_0, y_0).
> c. If $D < 0$, then $f(x, y)$ has a saddle point at (x_0, y_0).
> d. If $D = 0$, the second derivative test is inconclusive.

Let's examine some *graphs* in order to see examples of critical opints & relative max/mins.

We are going to use the `python` library matplotlib to draw graphs.

[12]:
```
import matplotlib.pyplot as plt
import numpy as np

# https://matplotlib.org/mpl_toolkits/mplot3d/tutorial.html
# https://matplotlib.org/3.3.1/gallery/mplot3d/surface3d.html

def draw_graph(f,x,y,x0,y0,elev_azim=[]):
    X,Y = np.meshgrid(x,y)
    fig = plt.figure(figsize=(20,20))

    for idx,(e,a) in enumerate(elev_azim,start=1):

        ax = fig.add_subplot(1,len(elev_azim),idx,projection='3d',elev=e,azim=a)
        ax.plot_wireframe(X,Y,f(X,Y))
```

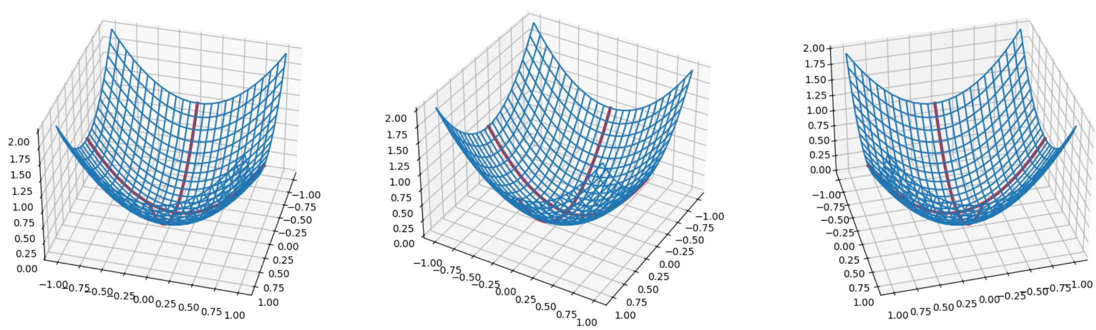
```
        ax.plot(x,y0*np.ones(y.shape), zs= f(x,y0), color="red", linewidth=3)
        ax.plot(x0*np.ones(x.shape),y, zs= f(x0,y), color="red", linewidth=3)

    return fig
```

Now we define some functions `f(x)` and `g(x)` and use the function= **`draw_graph`** that we just defined to render an image of their graphs.

[13]:
```
def f(x,y):
    return x**2 + y**2

af=draw_graph(f,
              x=np.linspace(-1,1,25),
              y=np.linspace(-1,1,25),
              x0=0,
              y0=0,
              elev_azim=[(35,15),(35,30),(35,75)])
```



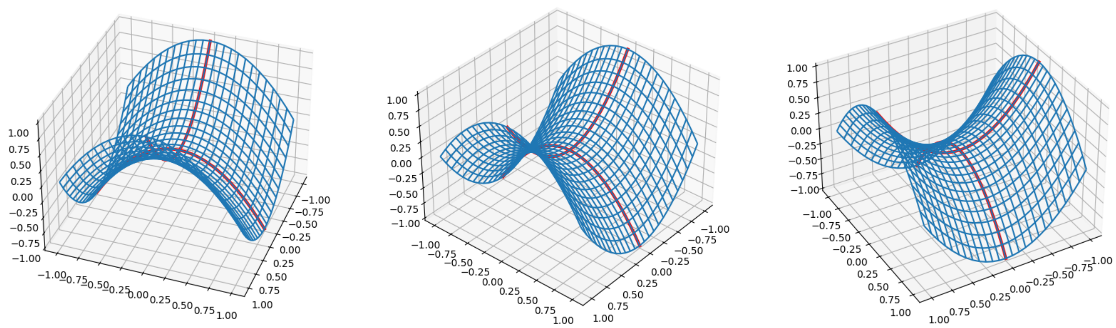
Note that $(0,0)$ is a critical point for $f(x) = x^2 + y^2$. Moreover,

$$D(x,y) = \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4.$$

Since $D(0,0) > 0$ and $\dfrac{\partial^2 f}{\partial x^2} = 2 > 0$, the second derivative test shows that the function f has a **`relative min`** at $(0,0)$ (& this is confirmed by the viewing the image).

[4]:
```
def g(x,y):
    return x**2 - y**2

ag=draw_graph(g,
              x=np.linspace(-1,1,25),
              y=np.linspace(-1,1,25),
              x0=0,
              y0=0,
              elev_azim=[(35,20),(35,40),(35,60)])
```



Again, $(0,0)$ is the only critical point for $g(x,y)=x^2-y^2$. Moreover,

$$D(x,y)=\det\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}=-4.$$

Since $D(0,0)<0$, the second derivative test shows that the function g has a `saddle point` at $(0,0)$ (& this is confirmed by examining the graph).

2.2 Example: Television manufacturer

A company that makes TV sets sells two models: a 19” set and a 21” set.

Their annual production costs are \$ 195 per TV for the 19” model and \$ 225 per TV for the 21” model, plus \$ 400,000 per year in fixed costs.

They expect to sell their production to a single wholesaler who will pay a base price of \$ 339 per 19” TV and $399 per 21” TV. The wholesaler recieves a volume discount calculated as 1¢ per 19” TV + 0.3¢ per 21” TV for the 19” models and 1¢ per 21” TV + 0.4¢ per 19” TV for the 21” models.

How many 19” and 21” TVs should be produced to maximize the profits?

Let’s go through our modeling procedure. Let’s set up the problem and ask the right questions. What are our variables? - $s=\#$ of 19” TVs produced - $t=\#$ of 21” TVs produced - $p=$ selling price of each 19” TV - $q=$ selling price of each 21” TV - $C=$ cost of production - $R=$ total revenue of sales - $P=$ total profit

What do we *know* to start with??

- $p(s,t)=339-0.01s-0.003t$ dollars
- $q(s,t)=399-0.004s-0.01t$ dollars
- $R(s,t)=ps+qt=339s-0.01s^2-0.003st+399t-0.004st-0.01t^2$

  $=339s+399t-0.01s^2-0.01t^2-0.007st \quad$ dollars
- $C(s,t)=400,000+195s+225t$ dollars

- $P(s,t) = R(s,t) - C(s,t)$

  $= -400,000 + 144s + 174t - 0.01s^2 - 0.01t^2 - 0.007st$ dollars

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Of course, our goal is to maximize profit – i.e. to find (s_0, t_0) for which $P(s_0, t_0)$ is at a maximum.

According to the discussion above, we should compute the partial derivitives of P and simultaneously solve the equations

$$0 = \frac{\partial P}{\partial s} = \frac{\partial P}{\partial t}$$

So we need to solve the equations:

$$\frac{\partial P}{\partial s} = 144 - 0.02s - 0.007t = 0$$

$$\frac{\partial P}{\partial t} = 174 - 0.02t - 0.007s = 0$$

This amounts to solving the matrix equation

$$\begin{bmatrix} 0.02 & 0.007 \\ 0.007 & 0.02 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 144 \\ 174 \end{bmatrix}$$

which we can solve using *row reduction* on the corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 0.02 & 0.007 & 144 \\ 0.007 & 0.02 & 174 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & .35 & 7200 \\ 1 & 2.857 & 24857.14 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & .35 & 7200 \\ 0 & 2.507 & 17657.14 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & .35 & 7200 \\ 0 & 1 & 7043.135 \end{array}\right] \sim \left[\begin{array}{c} 1 \\ 0 \end{array}\right.$$

We find that the function P has exactly one critical point which occurs at $(s_0, t_0) = (4735, 7043)$.

Alternatively, we can use `numpy` to solve the matrix equation, as follows:

[5]:
```
A= np.array([[.02,.007],[.007,.02]])
b=np.array([144,174])
np.linalg.solve(A,b)
```

[5]:
```
array([4735.04273504, 7042.73504274])
```

Now, let's apply the second derivative test to investigate this critical point.

The matrix of second derivatives is

$$\begin{bmatrix} \frac{\partial^2 P}{\partial s^2} & \frac{\partial^2 P}{\partial s \partial t} \\ \frac{\partial^2 P}{\partial s^2} & \frac{\partial^2 P}{\partial s \partial t} \end{bmatrix} = \begin{bmatrix} -0.02 & -0.007 \\ -0.007 & -0.02 \end{bmatrix}$$

which has determinant $(0.02)^2 - (.007)^2 > 0$.

Since $\dfrac{\partial^2 P}{\partial s^2} = -0.02 < 0$, the second derivative test shows that P has is local maximum at (s_0, t_0), and we conclude that profit is maximized there.

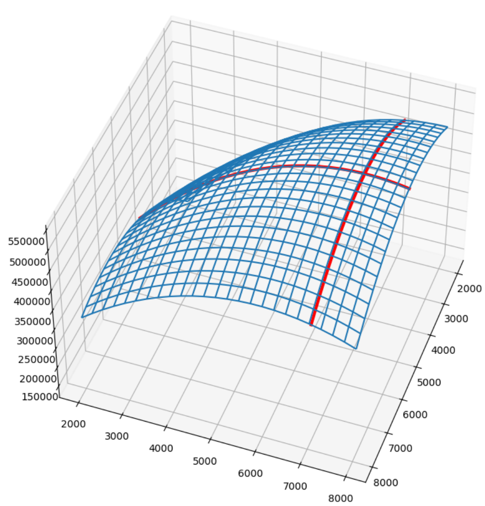
(Technically, we should check for minima "on the boundary" but in this case that would be the point $(0, 0)$ which clearly doesn't maximize P).

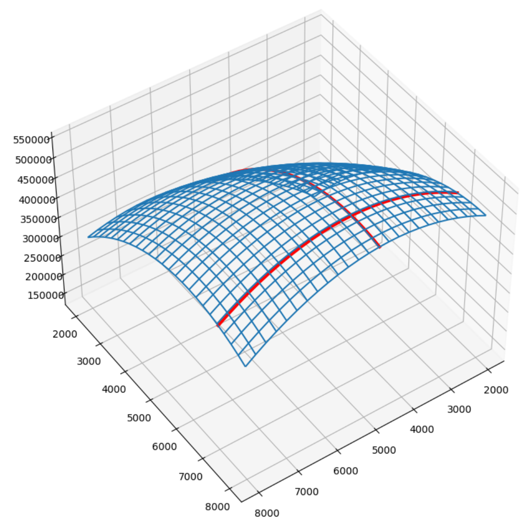
Let's produce a (or a few…) graph(s) to confirm our work:

[6]:
```
s = np.linspace(2000,8000,25)
t = np.linspace(2000,8000,25)


def p(s,t):
    return -400000 + 144*s + 174*t - 0.01*s**2 - 0.01*t**2 - .007*s*t

a=draw_graph(p,
             x=s,
             y=t,
             x0=4735,
             y0=7043,
             elev_azim=[(45,20),(45,55)])
```





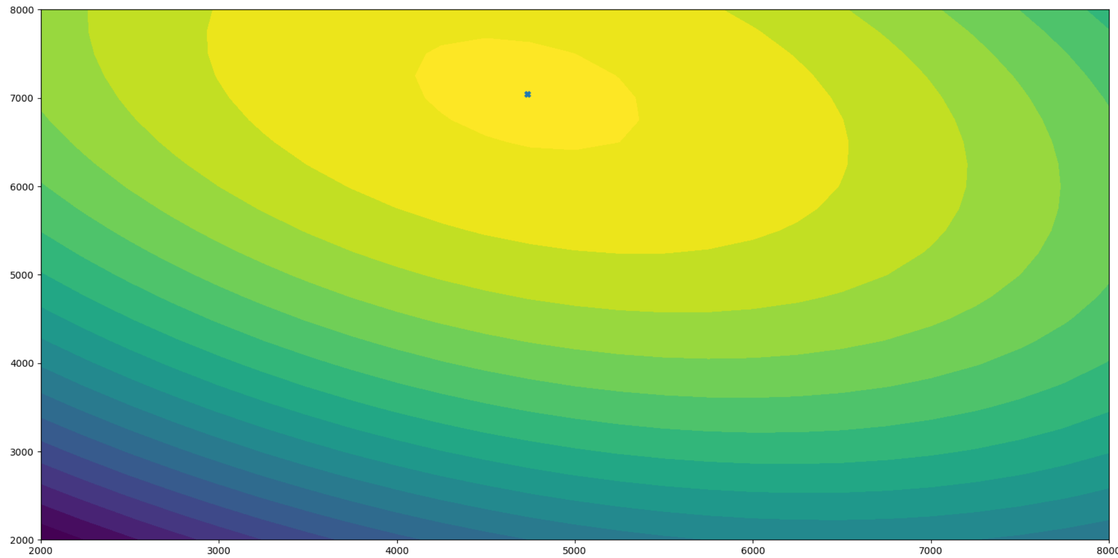
[7]:
```
# countour plot

S,T = np.meshgrid(s,t)

figc = plt.figure(figsize=(20,10))
axc = figc.add_subplot()
axc.contourf(S,T,p(S,T),levels=20
             , extend='both')
```

```
axc.scatter(4735,7043,marker="X")
```

[7]: <matplotlib.collections.PathCollection at 0x7fbd905d93f0>



2.3 Sensitivity Analysis (the television example, continued)

Just as in the single-variable case, we should be able to perform *Sensitivity Analysis* for our optimization problems.

Let's start by picking a parameter we want to change.

Consider the *Price Elasticity* parameter, a for 19" TVs. Thus, a is the amount the selling price of the 19" TVs decreases per 19" TV sold (due to the volume discount).

In the original description of the problem, this is described as:

> volume discount calculated as 1¢ per 19" TV

Thus, we considered above the case $a = 0.01$.

Rewriting our parameters using a, we see that the selling price of 19" TVs is given by $p(s,t) = 339 - as - 0.003t$ dollars.

In turn, rewriting the profit equation with a, we obtain:

$$P(s,t) = 144s + 174t - as^2 - 0.01t^2 - 0.007st - 400000$$

We now look for optimal values $s = s(a)$ and $t = t(a)$ depending on a.

We need to solve the system:

$$\begin{cases} 0 &= \dfrac{\partial P}{\partial s} = 144 - 2as - 0.007t \\ 0 &= \dfrac{\partial P}{\partial t} = 174 - .02t - 0.007s \end{cases}$$

Solving this system, find that $s = s(a) = \dfrac{144 - 0.007t}{2a}$ so that

$$174 - 0.02t - 0.007 \cdot \frac{144 - 0.007t}{2a} = 0$$

We now find that

$$t = 8,700 - \frac{581,700}{40,000a - 49}$$

and

$$s = \frac{1,662,000}{40,000a - 49}$$

Now we check the sensitivity:

$$S(s,a) = \frac{ds}{da} \cdot \frac{a}{s} \quad \text{and} \quad S(t,a) = \frac{dt}{da} \cdot \frac{a}{t}$$

Thus

$$S(s,a) = \frac{66,480,000,000}{(40,000a - 49)^2} \cdot \frac{40,000a^2 - 49a}{1,662,000}$$

and

$$S(t,a) = \frac{23,268,000,000}{(40,000a - 40)^2} \cdot \frac{40,000a^2 - 49a}{8,700 \cdot (40,000a - 49) - 581,700}$$

The sensitivity near our guess of $a = 0.01$ is thus

$$S(s, 0.01) \approx -1.1 \quad \text{and} \quad S(t, 0, 01) \approx 0.2$$

Interpretation: If the price elasticity increases by 10% (i.e. the warehouse receives a bigger bulk discount) the optimal value of s decreases by 11% and the optimal value of t increases by 2.7%