# week08-02–markov

March 2, 2025

## 1 George McNinch Math 87 - Spring 2024

## 2 Week 8

## 3 Stochastic matrices & Markov Chains

### 3.1 Probability, power iteration, and stochastic matrices

A vector $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T \in \mathbb{R}^n$ will be said to be a *probability vector* if all of its entries $v_i$ satisfy $v_i \geq 0$ and if

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \cdot \mathbf{v} = \sum_{i=1}^{n} v_i = 1.$$

Let $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{R}^{n \times n}$. We say that $A$ is a *stochastic matrix* if $a_{ij} \geq 0$ for all $i, j$ and if

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \cdot A = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix};$$

in words, $A$ is a stochastic matrix if each column of $A$ is a probability vector.

Notice that if $\mathbf{v}$ is a probability vector, and $A$ is a stochastic matrix, then $A\mathbf{v}$ is again a probability vector.

Indeed, by the definitions we have

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \cdot A \cdot \mathbf{v} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \cdot \mathbf{v} = 1$$

As a consequence if $A$ and $B$ are stochastic $n \times n$ matrices, then also $AB$ is stochastic. In particular, $A^m$ is stochastic for all $m \geq 0$.

### 3.2 Eigenvalues of stochastic matrices

**Proposition:** Let $A$ be a stochastic matrix.
**a)** $A$ has an eigenvector with eigenvalue 1.
**b)** Let $\lambda$ be any eigenvalue of a $A$. Then $|\lambda| \leq 1$.
**c)** If $\mathbf{w}$ is an eigenvector of $A$ with eigenvalue $\lambda$ satisfying $\lambda \neq 1$ then $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \mathbf{w} = 0$.

*Sketch:*

For **a)**, note that taking transposes and applying the definition, we find that

$$A^T \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T;$$

thus $\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ is an eigenvector for $A^T$ with eigenvalue 1. Since the matrices $A$ and $A^T$ have the same characteristic polynomial and hence the same eigenvalues, the assertion **a)** now follows.

Since all entries $a_{ij}$ of $A$ satisfy $0 \le a_{ij} \le 1$, assertion **b)** is a consequence of Gershgorin's Theorem.

### 3.3 Proof of c):

On one hand, we have

$$\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix} \lambda \mathbf{w} = \lambda \left( \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix} \mathbf{w} \right)$$

On the other hand, since $A$ is stochastic we have

$$\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix} A\mathbf{w} = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix} \mathbf{w};$$

since $A\mathbf{w} = \lambda\mathbf{w}$ and since $\mathbf{w} \ne \mathbf{0}$, we conclude that

$$\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix} \mathbf{w} = \lambda \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix} \mathbf{w}.$$

Since $\lambda \ne 1$ by assumption, this is only possible if $\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix} \mathbf{w} = 0$, as asserted.

### 3.4 Power iteration for stochastic matrices

Let $A$ be a stochastic matrix, and *suppose* that the eigenvalue $\lambda = 1$ has multiplicity one. This means that the *1-eigenspace* has dimension 1.

More concretely, this means that $A - \mathbf{I_n}$ has rank $n - 1$.

**Remark:** If $A$ has $n$ distinct eigenvalues, then the each eigenspace has dimension 1.

We have the following:

### 3.5 Corollary

Suppose that the stochastic matrix $A$ is diagonalizable, and that the *1-eigenspace* of $A$ has dimension 1. Let $\mathbf{v}$ be an eigenvector for $A$ with eigenvalue 1, and set $c = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix} \mathbf{v}$. Then $\mathbf{w} = \dfrac{\mathbf{v}}{c}$ is a probability vector, and

$$A^m \to B \quad \text{as } m \to \infty$$

for a stochastic matrix $B$. Each column of $B$ is equal to $\mathbf{w}$.

**Sketch:**

For $1 \le i \le n$, the $i$-th column of $B$ may be computed as

$$\lim_{m\to\infty} A^m \mathbf{e}_i$$

where $\mathbf{e}_i$ is the $i$-th standard basis vector.

Let $\mathbf{v} = \mathbf{v}_1, \dots, \mathbf{v}_n$ be linearly independent eigenvectors for $A$.

When $j > 1$, the eigenvalue for $\mathbf{v}_j$ is $< 1$ by assumption, and it follows from the preceding results that $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \cdot \mathbf{v}_j = 0$ for $j > 1$.

Fix $1 \le i \le n$ and consider the expression

$$\mathbf{e}_i = \sum_{j=1}^{n} c_j \mathbf{v}_j.$$

Since $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \mathbf{e}_i \neq 0$, it follows that $c_1 \neq 0$. Thus a result proved in the previous notebook shows that $\lim_{m \to \infty} A^m \mathbf{e}_i$ is a non-zero multiple of $\mathbf{w}$.

Since $B$ is stochastic, each column of $B$ is a probability vector, and must coincide with $\mathbf{w}$.

### 3.6 Markov Chains

Let's pause to recap our *state-machine* point-of-view.

We consider a system with a list of *states*. The system undergoes transitions, which we take to be given by probabilities.

We represent the system by a directed graph. Each state determines a node. A directed edge between two nodes $a \to b$ labeled with $p = p_{a,b}$ indicates that if the system is currently in state $a$, it will transform to state $b$ with probability $p$.

Thus for each node $a$, the sum of the probabilities on the edges $a \to b$ must be 1:

$$\sum_{(a \to b)} p_{a,b} = 1$$

The resulting matrix $P = (p_{a,b})_{a,b}$ has the property that its column-sums are all equal to 1. Thus $P$ is a *stochastic matrix*.

Let $G$ be the directed graph attached to our probabilistic state-machine as before. We will refer to $G$ as a *transition diagram*, and we call the *system* described by $G$ a *Markov chain*.

### 3.7 Diagram properties

Let $G$ be the transition diagram of a Markov chain.

**Definition:** $G$ is *strongly connected* if for each pair of nodes $a, b$, there is sequence of directed edges $e_1, \dots, e_m$ connecting $a$ to $b$.

**Remark:** If $P$ is the corresponding stochastic matrix, one often says that $P$ is *irreducible* when the transition diagram $G$ is *strongly connected*.

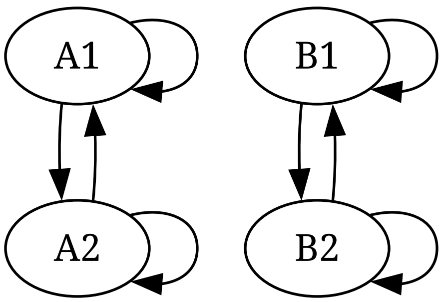
### 3.8 Example:

The following graph is not *strongly connected*.

[1]:
```
from graphviz import Digraph
from itertools import product
```

```
g = Digraph()

for i in ["A","B"]:
    for j in [1,2]:
        g.node(f"{i}{j}")
    for (j,k) in product([1,2],[1,2]):
        g.edge(f"{i}{j}",f"{i}{k}")

g
```

[1]:




### 3.9 Example:

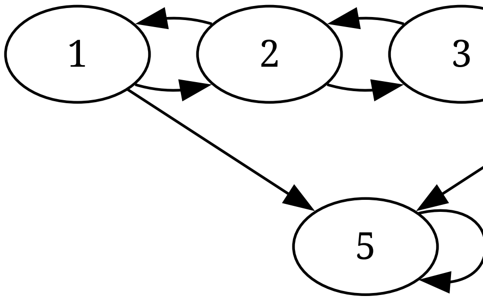
The following graph appears to be "connected" at least in some sense, but is not *strongly connected*.

Note that there is no path from the node 5 to the node 1, for example.

[2]:

```
h = Digraph()

I = [1,2,3,4]

with h.subgraph() as c:
    c.attr(rank="same")
    for i in I:
        c.node(f"{i}")
    for (j,k) in [(i,j) for (i,j) in product(I,I) if i == j+1 or i == j-1]:
        h.edge(f"{j}",f"{k}")

h.edge(f"1",f"5")
h.edge(f"4",f"5")
h.edge(f"5",f"5")

h
```

[2]:



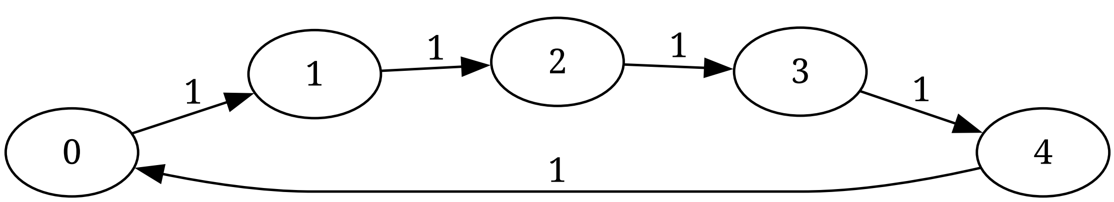

### 3.10 Cycles

A *cycle* of length $n$ in a transition diagram is a sequence $e_1, \ldots, e_n$ of edges for which that initial node of $e_1$ is equal to the terminal node of $e_n$.

Here is an example of a cycle of length 5:

[3]:
```
import numpy as np

def cycle(n=5,labels=None):
    if labels==None:
        labels= n*[1]
    cyc = Digraph()
    cyc.attr(rankdir='LR')
    I = list(range(n))

    for i in I:
        cyc.node(f"{i}")

    for i in I:
        cyc.edge(f"{i}",f"{np.mod(i+1,n)}",f"{labels[i]}")

    return cyc

cycle()
```

[3]:



### 3.11 Aperiodic

Given a transition diagram $G$, consider all possible cycles in $G$.

A transition diagram is said to be *aperiodic* if no integer $n > 1$ divides the length of each cycle.

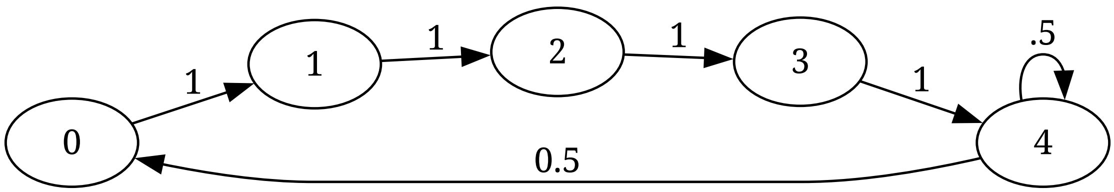
In other language, the diagram $G$ is aperiodic if the greatest common divisor of the lengths of the cycles in $G$ is equal to 1.

Example: The preceding graph $G$ with 5 nodes is not *aperiodic* since every cycle has length a multiple of 5.

Example: The following graph *is* aperiodic, since it contains a cycle of length 1.

[4]:
```
acycle = cycle(labels=[1,1,1,1,.5])
acycle.edge("4","4",".5")
acycle
```

[4]:




### 3.12 Theorem: (Perron-Frobenius)

Let $G$ be a transition diagram for a Markov chain, and suppose that $G$ is strongly connected and aperiodic. Let $P$ be the corresponding stochastic matrix. The multiplicity of the eigenvalue $\lambda = 1$ for $P$ is 1 – i.e.

$$\dim \operatorname{Null}(P - I_n) = 1.$$

All other eigenvalues $\lambda$ satisfy $|\lambda| < 1$.

There is a 1-eigenvector $\mathbf{v}$ which is a probability vector.

### 3.13 Corollary

**a)** $\lim_{m\to\infty} P^m$ is a matrix for which each column is equal to $\mathbf{v}$.
**b)** If $\mathbf{w}$ is a vector for which $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \mathbf{v} > 0$, then $\lim_{m\to\infty} P^m\mathbf{w}$ is a positive multiple of $\mathbf{v}$.

## 4 Financial market example

Consider the state of a financial market from week to week.

- by a *bull market* we mean a week of generally rising prices.
- by a *bear market* we mean a week of genreally declining prices.
- by a *recession* we mean a general slowdown of the economy.

Empirical observation shows for each of these three states what the probability of the state for the subsequent week, as follows:

| | *bull* | *bear* | *recession* |
|---|---|---|---|
| followed by bull | 0.90 | 0.15 | 0.25 |
| followed by bear | 0.075 | 0.80 | 0.25 |
| followed by recession | 0.025 | 0.05 | 0.50 |

In words, the first col indicates that if one has a bull market, then 90% of the time the next week is a bull market, 7.5% of the time the next week is a bear market, and 2.5% of the time the next week is in recession.

### 4.1 matrix

The matrix $A$ describing the state transformations is a stochastic matrix.

[8]:
```
ones = np.ones(3)

A = np.array([[0.90 , 0.15 , 0.25],
              [0.075, 0.80 , 0.25],
              [0.025, 0.05 , 0.50]])

ones @ A
```

[8]: `array([1.00, 1.00, 1.00])`

$A$ has 3 distinct eigenvalues:

[10]:
```
##
e_vals,e_vecs = npl.eig(A)

e_vals
```

[10]: `array([1.00, 0.74, 0.46])`

In particular, it follows that the 1-eigenspace of $A$ has dimension 1.

A 1-eigenvector is given by

[12]:
```
v = e_vecs[:,0]
v
```

[12]: `array([0.89, 0.45, 0.09])`

Rescaling `v` to make a probability vector, we indeed see that $A^m \to \begin{bmatrix} \mathbf{w} & \mathbf{w} & \mathbf{w} \end{bmatrix}$.

[29]:
```
float_formatter = "{:.4f}".format
np.set_printoptions(formatter={'float_kind':float_formatter})
```

```
w = (1/sum(v,0))*v

B=npl.matrix_power(A,200)

print(f"w = \n\n{w}\n\nA^200 = \n\n{B}")
```

```
w =

[0.6250 0.3125 0.0625]

A^200 =

[[0.6250 0.6250 0.6250]
 [0.3125 0.3125 0.3125]
 [0.0625 0.0625 0.0625]]
```

### 4.2 Interpretation:

Recall that $A$ describes the state transitions for a financial market.

The interpretation here means that *in the long run*, there is a 62.5 % chance of a bull market, a 31.25 % chance of a bear market, and a 6.25% chance of a recession.