MATH146 - PS2 DUE 2025-01-31

GEORGE MCNINCH

1. Consider the ring homomorphism $\phi : \mathbf{Q}[T] \to \mathbf{C}$ given by $\phi(f) = f(1+i)$.
   (a) Prove that $\ker \phi$ is the principal ideal $\langle T^2 - 2T + 2 \rangle$.
   (b) $\mathbf{Q}(1+i)$ is the subring $\{a + b(1+i) | a, b \in \mathbf{Q}\} \subset \mathbf{C}$. Conclude that $\mathbf{Q}(1+i)$ is isomorphic to $Q[T]/\langle T^2 - 2T + 2 \rangle$.
   (c) Explain why $\mathbf{Q}(1+i)$ is a *field*.
2. Let R be a PID and let $p \in R$ be irreducible. Let $I = \langle p^2 \rangle$ be the principal ideal generated by p^2.
   (a) Show that there is a surjective ring homomorphism $R/I \to R/\langle p \rangle$.
   (b) Show that the element $1 + p + I \in R/I$ is a unit in R/I.
3. Find $\gcd(T^3 - 1, 2T^2 - 3T + 1)$ in $\mathbf{Q}[T]$.
4. Consider the polynomial ring $\mathbf{Q}[T, S]$ in two indeterminants. By definition, this ring is just the iterated polynomial ring $\mathbf{Q}[T][S]$.
   (a) Explain why $\mathbf{Q}[T, S]$ is an integral domain.
   (b) Explain why the monomials $T^i S^j$ for $i, j \geq 0$ form a basis for $\mathbf{Q}[T, S]$ as a $\mathbf{Q}$-vector space.
   (c) Define the total degree of a monomial $T^i S^j$ to be $\operatorname{totdeg}(T^i S^j) = i + j$ and for a non-zero polynomial $f = \sum_{i,j \geq 0} a_{i,j} T^i S^j \in \mathbf{Q}[T, S]$ let the total degree of f be defined by
   $$\operatorname{totdeg}(f) = \max\{i + j \mid a_{i,j} \neq 0\}$$
   .
   Prove for $f, g \in \mathbf{Q}[T, S]$ that $\operatorname{totdeg}(fg) = \operatorname{totdeg}(f) + \operatorname{totdeg}(g)$.
   (d) Show that $\mathbf{Q}[T, S]$ is *not* a PID by showing that the ideal $\langle T, S \rangle$ is not principal. **Hint**: Use (c) to prove that there is polynomial $f \in \mathbf{Q}[T, S]$ for which $f \mid T$ and $f \mid S$.

Date: 2025-01-26 16:31:27 EST (george@valhalla).