

Math146 - Lecture notes

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1 Commutative rings

See [Stewart, chapter 16] ¹ for general results about commutative rings.

1.1 Definitions

Definition 1.1.1. A ring R is an additive abelian group together with an operation of multiplication $R \times R \rightarrow R$ given by $(a, b) \mapsto a \cdot b$ such that the following axioms hold:

- multiplication is *associative*
- multiplication *distributes* over addition: for every $a, b, c \in R$ we have ²

$$a(b + c) = ab + ac$$

and

$$(b + c)a = ba + ca$$

We say that the ring R is *commutative* if the operation of multiplication is commutative; i.e. if $ab = ba$ for all $a, b \in R$.

And we say that R has identity if multiplication has an identity, i.e. if there is an element $1_R \in R$ such that $a \cdot 1_R = 1_R \cdot a = a$ for every $a \in R$. ³

In the course, we will consider (almost?) exclusively rings which are commutative and have identity.

Here are some examples of commutative rings:

Example 1.1.2. (a) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

(b) if X is a set and if R is a commutative ring, the set X^R of all R -valued functions on X can be viewed as a commutative ring in a natural way.

1.2 Polynomial rings

If R is a commutative ring, the collection of all polynomials in the variable T having coefficients in R is denoted $R[T]$.

Notice that the set of *monomials* $S = \{T^i \mid i \in \mathbb{N}\}$ has the following properties:

(M1) every element of $R[T]$ is an R -linear combination of elements of S . This just amounts to the statement that every polynomial $f(T) \in R[T]$ has the form

$$f(T) = \sum_{i=0}^N a_i T^i$$

for a suitable $N \geq 0$ and suitable coefficients $a_i \in R$.

¹As noted in the course syllabus, Tisch library has an entry for this item here; click to find online access to the text *Galois Theory*, Ian Stewart. (CRC Press, 4th edition 2022).

²We often just denote multiplication by juxtaposition: i.e. we may write ab instead of $a \cdot b$ for $a, b \in R$

³Usually we write 1 for 1_R . The idea is that 1_R is the multiplicative identity of R . For example, the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the multiplicative identity 1_R of the matrix ring $R = \text{Mat}_2(\mathbb{R})$.

(M2) the elements of S are linearly independent i.e. if

$$\sum_{i=0}^N a_i T^i = 0 \quad \text{for} \quad a_i \in R,$$

then $a_i = 0$ for every i .

Polynomials in $R[T]$ can be added in a natural way. (This is just like adding vectors in a vector space).

And there is a product operation on polynomials, as follows:

if $f(T) = \sum_{i=0}^N a_i T^i$ and $g(T) = \sum_{i=0}^M b_i T^i$ then

$$f(T) \cdot g(T) = \sum_{i=0}^{N+M} c_i T^i \quad \text{where} \quad c_i = \sum_{s+t=i} a_s b_t.$$

Proposition 1.2.1. $R[T]$ is a commutative ring with identity.

2 Properties of rings

2.1 Ring Homomorphisms

Definition 2.1.1. If R and S are rings, a function $\phi : R \rightarrow S$ is called a *ring homomorphism* provided that

- (a) ϕ is a homomorphism of *additive groups*,
- (b) ϕ preserves multiplication; i.e. for all $x, y \in R$ we have $\phi(xy) = \phi(x)\phi(y)$, and
- (c) $\phi(1_R) = 1_S$.

Definition 2.1.2. The *kernel* of the ring homomorphism $\phi : R \rightarrow S$ is given by

$$\ker \phi = \phi^{-1}(0) = \{x \in R \mid \phi(x) = 0\};$$

thus $\ker \phi$ is just the kernel of ϕ viewed as a homomorphism of additive groups.

Here are some properties of the kernel:

- (K1) $\ker \phi$ is an additive subgroup of R
- (K2) for every $r \in R$ and every $x \in \ker \phi$ we have $rx \in \ker \phi$.

2.2 Ideals of a ring

For simplicity suppose that the ring R (and S) are *commutative* rings.

Definition 2.2.1. A subset I of R is an *ideal* provided that

- (a) I is an additive subgroup of R , and
- (b) for every $r \in R$ and every $x \in I$ we have $rx \in I$.

We sometimes describe condition (b) by saying that “ I is closed under multiplication by every element of R ”.

The proof of the following is immediate from definitions:

Proposition 2.2.2. *If $\phi : R \rightarrow S$ is a ring homomorphism, then $\ker \phi$ is an ideal of R .*

2.3 Quotient rings

Let R be a commutative ring and let I be an ideal of R .

Since I is a subgroup of the (abelian) additive group R , we may consider the quotient group R/I . Its elements are (additive) cosets $a + I$ for $a \in R$.

It follows from the definition of cosets that the $a + I = b + I$ if and only if $b - a \in I$.

The additive group can be made into a commutative ring by defining the multiplication as follows:

For $a + I, b + I \in R/I$ (so that $a, b \in R$), the product is given by

$$(a + I)(b + I) = ab + I.$$

In order to make this definition, one must confirm that this rule is well-defined. Namely, if we have equalities $a + I = a' + I$ and $b + I = b' + I$, we need to know that

$$(a + I)(b + I) = (a' + I)(b' + I).$$

Applying the definition, we see that we must confirm that

$$ab = I = a'b' + I.$$

For this, we need to argue that $a'b' - ab \in I$.

Since $a + I = a' + I$, we know that $a' - a = x \in I$ and since $b + I = b' + I$ we know that $b' - b = y \in I$.

Thus $a' = a + x$ and $b' = b + y$. Now we see that

$$a'b' = (a + x)(b + y) = ab + ay + xb + xy$$

Since I is an ideal, we see that $ay, xb, xy \in I$ hence $ay + xb + xy \in I$. Now conclude that $a'b' + I = ab + I$ as required.

It is now straightforward to confirm that the ring axioms hold for the set R/I with these operations.

Proposition 2.3.1. *If I is an ideal of the commutative ring R , then R/I is a commutative ring with the addition and multiplication just described.*

2.4 Principal ideals

Definition 2.4.1. If R is a commutative ring and $a \in R$, the *principal ideal generated by a* – written Ra or $\langle a \rangle$ – is defined by

$$Ra = \langle a \rangle = \{ra \mid r \in R\}.$$

Proposition 2.4.2. *For $a \in R$, Ra is an ideal of R .*

Example 2.4.3. Let $n \in \mathbb{Z}_{>0}$ and consider the principal ideal $n\mathbb{Z}$ of the ring \mathbb{Z} generated by $n \in \mathbb{Z}$.

As an additive group, $n\mathbb{Z}$ is the infinite cyclic group generated by n .

The quotient ring $\mathbb{Z}/n\mathbb{Z}$ is the finite commutative ring with n elements; these elements are precisely the *congruence classes* of integers modulo n .

2.5 Isomorphism Theorem

Theorem 2.5.1. *Let R, S be commutative rings with identity and let $\phi : R \rightarrow S$ be a ring homomorphism. Assume that ϕ is surjective (i.e. onto). Then ϕ determines an isomorphism $\bar{\phi} : R/I \rightarrow S$ where $I = \ker \phi$, where $\bar{\phi}$ is determined by the rule*

$$\bar{\phi}(a + I) = \phi(a) \quad \text{for } a \in R.$$

Proof. First, you must confirm that $\bar{\phi}$ is *well-defined*; i.e. that if $a + I = a' + I$ then $\bar{\phi}(a + I) = \bar{\phi}(a' + I)$.

Next, you must confirm that $\bar{\phi}$ is a ring homomorphism (this is immediate from the definition of ring operations on R/I).

Finally, you must confirm that $\ker \bar{\phi} = \{0\}$, where here 0 refers to the additive identity of the quotient ring R/I . This additive identity is of course the trivial coset $I = 0 + I \in R/I$. \square

2.6 A Homomorphism from the polynomial ring to the scalars

Let F is a field and let $a \in F$. consider the mapping

$$\Phi : F[T] \rightarrow F$$

given by $\Phi(f(T)) = f(a)$. Namely, applying Φ to a polynomial $f(T)$ results in the value $f(a)$ of $f(T)$ at a .

The definition of multiplication in $F[T]$ guarantees that Φ is a ring homomorphism.

3 Polynomials over a field and the division algorithm

3.1 Some general notions for commutative rings

Definition 3.1.1. If R is a commutative ring with 1 and if $u \in R$ we say that u is a *unit* - or that u is *invertible* - provided that there is $v \in R$ with $uv = 1$; then $v = u^{-1}$.

We write R^\times for the units in R .

A commutative ring R is a *field* provided that every non-zero element is invertible. Thus R is a field if $R^\times = R \setminus \{0\}$.

Proposition 3.1.2. *If R is a commutative, then R^\times is an abelian group (with operation the multiplication in R).*

For any commutative ring R and elements $a, b \in R$ we say that a **divides** b - written $a \mid b$ - if $\exists x \in R$ with $ax = b$.

Proposition 3.1.3. *For $a, b \in R$ we have $a \mid b$ if and only if $b \in \langle a \rangle$.*

Recall that we introduced the principal ideal $\langle a \rangle = aR$ for any commutative ring R and any $a \in R$. In fact, given $a_1, \dots, a_n \in R$ we can consider the ideal

$$\langle a_1, \dots, a_n \rangle = \sum_{i=1}^n a_i R$$

defined as

$$\langle a_1, \dots, a_n \rangle = \left\{ \sum_{i=1}^n r_i a_i \mid r_i \in R \right\}.$$

It is straightforward to check that $\langle a_1, \dots, a_n \rangle$ is indeed an ideal of R .

Definition 3.1.4. A non-zero element $a \in R$ is said to be a *0-divisor* provided that there is $0 \neq b \in R$ with $ab = 0$.

Example 3.1.5. Let n be a composite positive integer, so that $n = ij$ for integers $i, j > 0$. Consider the elements $[i] = i + n\mathbf{Z}, [j] = j + n\mathbf{Z}$ in the quotient ring $\mathbf{Z}/n\mathbf{Z}$.

Then $[i]$ and $[j]$ are both non-zero since $0 < i, j < n$ so that $n \nmid i$ and $n \nmid j$. But $[i] \cdot [j] = [n] = 0$ so that $[i]$ and $[j]$ are 0-divisors of the ring $\mathbf{Z}/n\mathbf{Z}$.

Definition 3.1.6. A commutative ring R is said to be an *integral domain* provided that it has no zero-divisors.

Example 3.1.7. (a) Any field is an integral domain.

(b) The ring \mathbf{Z} of integers is an integral domain.

(c) Any subring of an integral domain is an integral domain.

For example, the ring $\mathbf{Z}[i] = \{a + bi \mid a, b \in \mathbf{Z}\}$ of gaussian integers is an integral domain.

(d) $\mathbf{Z}/n\mathbf{Z}$ is not an integral domain whenever n is composite.

(e) If R and S are commutative rings, the direct product $R \times S$ is *never* an integral domain.

Indeed, the elements $(1, 0)$ and $(0, 1)$ are 0-divisors.

Lemma 3.1.8. (Cancellation) *Let R be an integral domain and let $a, b, c \in R$ with $c \neq 0$. If $ac = bc$ then $a = b$.*

Proof. The equation $ac = bc$ implies that $ac - bc = 0$ so that $(a - b)c = 0$ by the distributive property. Since R has no zero divisors and since $c \neq 0$ by assumption, conclude that $a - b = 0$ i.e. that $a = b$. \square

Proposition 3.1.9. *Let R be an integral domain and let $d, d' \in R \setminus \{0\}$. If $\langle d \rangle = \langle d' \rangle$ then d and d' are associate.*

Proof. Since $d \in \langle d' \rangle$ we may write $d = xd'$ and since $d' \in \langle d \rangle$ we may write $d' = yd$. Now we see that $d = xd' = xyd$. Since $d \neq 0$ cancellation (Lemma 3.1.8) implies that $xy = 1$. Thus $x, y \in R^\times$ and indeed d, d' are associate. \square

3.2 An important result on polynomial rings

Proposition 3.2.1. *Let R and S be rings, let $\phi : R \rightarrow S$ be a ring homomorphism, and let $\alpha \in S$ be an element. There is a unique ring homomorphism*

$$\Psi : R[T] \rightarrow S$$

such that $\Psi(T) = \alpha$ and such that $\Psi|_R = \phi$.

Proof. Let $f, g \in R[T]$, say

$$f = \sum_{i=0}^n a_i T^i \quad \text{and} \quad g = \sum_{i=0}^m b_i T^i$$

be elements of $R[T]$.

To see that Ψ is an additive homomorphism, note that $f + g = \sum_{i=0}^{\max(n,m)} (a_i + b_i) T^i$ so that

$$\Psi(f + g) = \sum_{i=0}^{\max(n,m)} (a_i + b_i) \alpha^i = \sum_{i=0}^n a_i \alpha^i + \sum_{i=0}^m b_i \alpha^i = \Psi(f) + \Psi(g)$$

Similarly, to see that Ψ is multiplicative, note that $fg = \sum_{i=0}^{n+m} c_i T^i$ where $c_i = \sum_{s+t=i} a_s b_t$. Now,

$$\Psi(fg) = \sum_{i=0}^{n+m} \phi(c_i) \alpha^i = \left(\sum_{i=0}^n \phi(a_i) \alpha^i \right) \left(\sum_{i=0}^m \phi(b_i) \alpha^i \right) = \Psi(f) \cdot \Psi(g)$$

\square

3.3 The degree of a polynomial

Let F be a field and consider the ring of polynomials $F[T]$.

Definition 3.3.1. The *degree* of a polynomial $f = f(T) \in F[T]$ is defined to be $\deg(f) = -\infty$ if $f = 0$, and otherwise $\deg(f) = n$ where

$$f = \sum_{i=0}^n a_i T^i \quad \text{with each } a_i \in F \text{ and } a_n \neq 0.$$

We have some easy and familiar properties of the degree function:

Proposition 3.3.2. *Let $f, g \in F[T]$.*

(a) $\deg(fg) = \deg(f) + \deg(g)$.

(b) $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$ and equality holds if $\deg(f) \neq \deg(g)$.

(c) $f \in F[T]^\times$ if and only if $\deg(f) = 0$. In particular, $F[T]^\times = F^\times$.

Corollary 3.3.3. *For a field F , the polynomial ring $F[T]$ is an integral domain.*

Proof. Let $f, g \in F[T]$ and suppose that $fg = 0$. We must argue that either $f = 0$ or $g = 0$. \square

Proposition 3.3.4. *Let $f, g \in F[T]$. If $g \neq 0$ and $\deg g < \deg f$ then $[g] = g + \langle f \rangle$ is a non-zero element of $F[T]/\langle f \rangle$.*

3.4 The division algorithm

Theorem 3.4.1. *Let F be a field, and let $f, g \in F[T]$ with $0 \neq g$. Then there are polynomials $q, r \in F[T]$ for which*

$$f = qg + r$$

and $\deg r < \deg g$.

Proof. First note that we may suppose f to be non-zero. Indeed, if $f = 0$, we just take $q = r = 0$. Clearly $f = qg + r$, and $\deg(r) = -\infty < \deg(g)$ since g is non-zero.

We now proceed by induction on $\deg(f) \geq 0$.

For the base case in which $\deg(f) = 0$, we note that $f = c$ is a constant polynomial; here $c \in F^\times$.

If $\deg(g) = 0$ as well, then $g = d \in F^\times$ and then $c = (c/d)d + 0$ so we may take $q = c/d$ and $r = 0$. Now $\deg(r) = -\infty < \deg(g)$ as required.

If $\deg(g) > 0$, we simply take $q = 0$ and $r = f$: we then have $f = 0 \cdot g + f$ and $\deg(f) = 0 < \deg(g)$ as required.

We have now confirmed the Theorem holds when $\deg(f) = 0$.

Proceeding with the induction, we now suppose $n > 0$ and that the Theorem holds whenever f has degree $< n$. We must prove the Theorem holds when f has degree n .

Since f has degree n , we may write $f = a_n T^n + f_0$ where $a_n \in F^\times$ and $f_0 \in F[T]$ has $\deg(f_0) < n$.

Let us write $g = \deg(g)$; we may write $g = b_m T^m + g_0$ where $b_m \in F^\times$ and $g_0 \in F[T]$ has $\deg(g_0) < m$.

If $n < m$ we take $q = 0$ and $r = f$ to find that $f = qg + r$ and $\deg(r) < \deg(g)$.

Finally, if $m \leq n$ we set

$$f_1 = f - (a_n/b_m)T^{n-m}g = a_n T^n + f_0 - \left(\frac{a_n}{b_m} b_m T^n + \frac{a_n}{b_m} T^{n-m} g_0 \right) = f_0 - \frac{a_n}{b_m} T^{n-m} g_0.$$

We have $\deg(f_0) < n$ by assumption, and $\deg\left(\frac{a_n}{b_m} T^{n-m} g_0\right) < n$ by the Proposition together with the fact that $\deg(g_0) < m$.

Thus $\deg(f_1) < n$. Now we apply the induction hypothesis to write

$$f_1 = q_1 g + r_1 \quad \text{with } \deg(r_1) < \deg(g).$$

Finally, we have

$$f = f_1 + (a_n/b_m)T^{n-m}g = q_1 g + r_1 + (a_n/b_m)T^{n-m}g = (q_1 + (a_n/b_m)T^{n-m})g + r_1$$

so we have indeed written $f = qg + r$ in the required form. \square

Corollary 3.4.2. *Let F be a field and let $f \in F[T]$. For $a \in F$, there is a polynomial $q \in F[T]$ for which*

$$f = q(T - a) + f(a).$$

Corollary 3.4.3. *For $f \in F[T]$ an element $a \in F$ is a root of the polynomial f if and only if $T - a \mid f$ in $F[T]$. In particular, if $d = \deg(f)$, f has no more than d distinct roots in F .*

Proof. The first statement is clear from Corollary 3.4.2. Now consider the distinct roots $\alpha_1, \dots, \alpha_e \in F$ of f . Then $T - \alpha_1$ divides f so that $f = (T - \alpha_1)f_1$ for some $f_1 \in F[T]$. Since α_2 is a root of f we see that

$$0 = f(\alpha_2) = (\alpha_2 - \alpha_1)f_1(\alpha_2)$$

which shows that α_2 is a root of f_1 since $\alpha_1 \neq \alpha_2$. Thus we find that

$$f = (T - \alpha_1)(T - \alpha_2)f_2$$

for some $f_2 \in F[T]$. Continuing in this way we find that $\prod_{i=1}^e (T - \alpha_i)$ divides f , so that $e \leq \deg f$ by Proposition 3.3.2. \square

4 Ideals of the polynomial ring

4.1 Ideals of the polynomial ring $F[T]$

Corollary 4.1.1. *Let F be a field and let I be an ideal of the ring $F[T]$. Then I is a principal ideal; i.e. there is $g \in I$ for which*

$$I = \langle g \rangle = g \cdot F[T].$$

Proof. If $I = \{0\}$ ⁴ the results is immediate. Thus we may suppose $I \neq 0$.

Consider the set $\{\deg(g) \mid 0 \neq g \in I\}$. This is a non-empty set of natural numbers, hence it contains a minimal element by the **well-ordering principle**.

Choose $g \in I$ such that $\deg(g)$ is this minimal degree; we claim that $I = \langle g \rangle$.

Clearly $\langle g \rangle \subseteq I$. To complete the proof, it remains to establish the inclusion $I \subseteq \langle g \rangle$. Let $f \in I$ and use the **Division Algorithm** to write $f = qg + r$ for $q, r \in F[T]$ with $\deg r < \deg g$.

Observe that $f - qg \in I$ so that $r \in I$. Since $\deg r < \deg g$ conclude that $r = 0$. This shows that $f = qg \in \langle g \rangle$ as required, completing the proof. \square

Let F be a field, $F[T]$ be the ring of polynomials with coefficients in F , let $f, g \in F[T]$ be polynomials which are not both 0.

Definition 4.1.2. The **greatest common divisor** $\gcd(f, g)$ of the pair f, g is a monic polynomial d such that

(a) $d \mid f$ and $d \mid g$,

(b) if $e \in F[T]$ satisfies $e \mid f$ and $e \mid g$, then $e \mid d$.

Remark 4.1.3. If d, d' are two gcds of f, g then $d \mid d'$ and $d' \mid d$. In particular, $\deg(d) = \deg(d')$ and $d' = \alpha d$ for some $\alpha \in F^\times$. It is then clear that there is no more than one monic polynomial satisfying i. and ii.

Proposition 4.1.4. *Let $f, g \in F[T]$ not both 0 ⁵.*

(a) $\langle f, g \rangle$ is an ideal. According to the previous

corollary, there is a monic polynomial $d \in F[T]$ with

$$\langle d \rangle = \langle f, g \rangle.$$

Then $d = \gcd(f, g)$

(b) In particular, $d = \gcd(f, g)$ may be written in the form $d = uf + vg$ for $u, v \in F[T]$.

Proof. For a., write $I = \langle f, g \rangle = \langle d \rangle$. Since $f, g \in I$, the definition of $\langle d \rangle$ shows that $d \mid f$ and $d \mid g$.

Now suppose that $e \in F[T]$ and that $e \mid f$ and $e \mid g$. Then $f, g \in \langle e \rangle$ which shows that $\langle f, g \rangle \subseteq \langle e \rangle$.

But this implies that $\langle d \rangle \subset \langle e \rangle$ so that $e \mid d$ as required. Thus we see that d is indeed equal to $\gcd(f, g)$.

Since $d \in \langle d \rangle = \langle f, g \rangle$, assertion b. follows from the definition of $\langle f, g \rangle$. \square

⁴We will write simply 0 for the ideal $\{0\}$.

⁵Note that f, g are not both 0 if and only if the ideal $\langle f, g \rangle$ is not 0.

4.2 Principal ideal domains (PIDs)

Definition 4.2.1. An integral domain R is said to be a **principal ideal domain** (abbreviated PID) provided that every ideal I of R has the form

$$I = \langle a \rangle \quad \text{for some } a \in R;$$

i.e. provided that every ideal of R is principal.

Example 4.2.2. (a) The ring \mathbf{Z} of integers is a PID.

(b) For any field F , the ring $F[T]$ of polynomials is a PID - this follows from the Corollary to the division algorithm, above.

(c) The rings $\mathbf{Z}[i]$ and $\mathbf{Z}[\sqrt{2}]$ are PIDs - to see this one can argue that these rings are Euclidean domains and then one proves that any Euclidean domain is a PID.

4.3 PIDs and greatest common divisors

Let R be a PID.

The results about gcd in the polynomial ring proved in Section 4.1 actually hold in the generality of the PID R . We quickly give the statements:

Definition 4.3.1. Let $a, b \in R$ such that $\langle a, b \rangle \neq 0$. A gcd of a and b is an element $d \in R$ such that

- (i) $d \mid a$ and $d \mid b$ (in words: “ d is a common divisor of a and b ”)
- (ii) if $e \mid a$ and $e \mid b$ then $e \mid d$. (in words: “any common divisor of a and b divides d ”)

Lemma 4.3.2. *If R is a PID and if d and d' are gcds of a and b then d and d' are associates.*

Proof. This follows from Proposition 3.1.9 □

Proof. Using the definition of gcd we see that $d \mid d'$ and $d' \mid d$. Thus $d' = dv$ and $d = d'u$ for $u, v \in R$.

This shows that $d' = dv = d'uv$. Using cancellation, find that $1 = uv$ so that $u, v \in R^\times$. □

Remark 4.3.3. This definition of course covers the cases when $R = \mathbf{Z}$ and when $R = F[T]$. The main thing to point out is that when $R = \mathbf{Z}$, there is a unique **positive** gcd for any pair $a, b \in \mathbf{Z}$ and when $R = F[T]$ there is a unique **monic** gcd for any pair $f, g \in F[T]$.

For a general PID there need not be a natural choice of gcd, so for $x, y \in R$ we can only speak of $\gcd(x, y)$ up to multiplication by a unit of R .

Proposition 4.3.4. *Let R be a PID and let $x, y \in R$ with $\langle x, y \rangle \neq 0$.*

(a) *Since R is a PID, we may write find $d \in R$ with*

$$\langle d \rangle = \langle x, y \rangle.$$

Then $d = \gcd(x, y)$.

(b) *In particular, $d = \gcd(x, y)$ may be written in the form $d = ux + vy$ for $u, v \in R$.*

To prove Proposition 4.3.4 proceed as in the proof of Proposition 4.1.4.

Proposition 4.3.5. *Let R be a PID and let $a, b \in R$ not both 0. Put $d = \gcd(a, b)$, so that $\frac{a}{d}, \frac{b}{d} \in R$. Then $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.*

Proof. According to Proposition 4.3.4 (b), we may write $d = ax + by$ for suitable $x, y \in R$. Since $d \mid a$ we know that $\frac{a}{d} \in R$; similarly $\frac{b}{d} \in R$. We now see that

$$d = d\frac{a}{d}x + d\frac{b}{d}y = d\left(\frac{a}{d}x + \frac{b}{d}y\right);$$

now applying *cancellation* – i.e. Lemma 3.1.8 – we conclude that

$$1 = \frac{a}{d}x + \frac{b}{d}y.$$

This shows that $1 \in \left\langle \frac{a}{d}, \frac{b}{d} \right\rangle$, the ideal generated by $\frac{a}{d}$ and $\frac{b}{d}$. But this implies that $R \subset \left\langle \frac{a}{d}, \frac{b}{d} \right\rangle$ so that $\langle 1 \rangle = R = \left\langle \frac{a}{d}, \frac{b}{d} \right\rangle$. According to Proposition 4.3.4 this proves that $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ as required. \square

5 Prime elements and unique factorization

5.1 Irreducible elements

Let R be a principal ideal domain.

Definition 5.1.1. A non-zero element $p \in R$ is said to be *irreducible* provided that $p \notin R^\times$ and whenever $p = xy$ for $x, y \in R$ then either $x \in R^\times$ or $y \in R^\times$.

Remark 5.1.2. Assume that $p, a \in R$ with p irreducible. Then either $\gcd(p, a) = 1$ or $\gcd(p, a) = p$.

Proposition 5.1.3. $p \in R$ is irreducible if and only if (\clubsuit): whenever $a, b \in R$ and $p \mid ab$ then either $p \mid a$ or $p \mid b$.

Proof. (\Rightarrow): Assume that p is irreducible, suppose that $a, b \in R$ and that $p \mid ab$. We must show that $p \mid a$ or $p \mid b$.

For this, we may as well suppose that $p \nmid a$; we must then prove that $p \mid b$. Since $p \nmid a$, we see that $\gcd(a, p) = 1$ by the Remark above. Then $ua + vp = 1$ for elements $u, v \in R$.

Now we see that

$$b = 1 \cdot b = (ua + vp) \cdot b = uab + vpb.$$

Since $p \mid ab$ we see that $p \mid uab + vpb$ which proves that $p \mid b$, as required.

(\Leftarrow): Assume that condition (\clubsuit) holds for p . We must show that p is irreducible. For this, assume $p = xy$ for $x, y \in R$; we must show that either $x \in R^\times$ or $y \in R^\times$.

Since $p = xy$, in particular $p \mid xy$ and we may apply (\clubsuit) to conclude without loss of generality that $p \mid x$.

Write $x = pa$. We now see that $p = xy = pay$; by cancellation, find that $1 = ay$ so that $y \in R^\times$. We conclude that p is irreducible, as required. \square

Remark 5.1.4. For any integral domain R , we can speak of *irreducible elements* defined as in Definition 5.1.1. And we can speak of *prime elements*, where an element $p \in R$ is *prime* if it satisfies condition (\clubsuit) of Proposition 5.1.3. In this language, Proposition 5.1.3 shows that in a PID, an element is prime iff it is irreducible.

Corollary 5.1.5. Let R be a PID, let $p, a_1, \dots, a_n \in R$ with p prime, and suppose that $p \mid a_1 a_2 \cdots a_n = \prod_{i=1}^n a_i$. Then $p \mid a_i$ for some $1 \leq i \leq n$.

Example 5.1.6. Let F a field and let $f \in F[T]$ be a non-constant polynomial; i.e. $\deg(f) > 0$. If f is reducible there are polynomials $g, h \in F[T]$ for which $f = gh$ and $\deg(g), \deg(h) > 0$.

Example 5.1.7. If $f \in F[T]$ is reducible (i.e. not irreducible) then the quotient ring $F[T]/\langle f \rangle$ is not an integral domain.

Indeed, write $f = gh$ for $g, h \in F[T]$ non-units. Thus $\deg f > \deg g, \deg h > 0$ by Proposition 3.3.2. According to Proposition 3.3.4, the classes $[g], [h] \in F[T]$ are non-zero, but $[g] \cdot [h] = [f] = 0$. Thus $F[T]/\langle f \rangle$ has zero divisors and is not an integral domain.

5.2 Unique factorization in a PID

The Fundamental Theorem of Arithmetic says that any integer $n > 1$ may factored uniquely as a product of primes. This result holds for any PID, as follows:

Theorem 5.2.1. Let R be a PID, let $0 \neq a \in R$, and suppose that a is not a unit.

- (a) There are irreducible elements $p_1, p_2, \dots, p_n \in R$ such that $a = p_1 \cdot p_2 \cdots p_n$.
- (b) if $q_1, \dots, q_m \in R$ are irreducibles such that $a = q_1 \cdots q_m$ then $n = m$ and – after possibly reordering the q_i – there are units $u_i \in R^\times$ for which $q_i = u_i p_i$ for each i .

Proof. We first prove (a). For this, we first prove the following claim:

(*) : if the conclusion of (a) fails, there is a sequence of elements $a_1, a_2, \dots \in R \setminus R^\times$ with the property that for each $i \geq 1$ we have: (i) $a_{i+1} \mid a_i$ and (ii) a_{i+1} and a_i are not associate.

To prove (*), let $x_1 = a$. Now suppose we have found elements a_1, a_2, \dots, a_n such that for each $1 \leq i \leq n$ conditions (i) and (ii) hold, and such that the conclusion of (a) fails for a_n . In particular, a_n is reducible, so we may write $a_n = xy$ with $x, y \in R$ and $x, y \notin R^\times$. Without loss of generality, we may suppose that the conclusion of (a) fails for x and we set $a_{n+1} = x$. By construction, $a_{n+1} \mid a_n$; moreover a_{n+1} and a_n are not associates. Thus we have proved by induction that (*) holds.

To prove (a), we will now show that (*) leads to a contradiction.

Let $\{a_i\}$ be a sequence of elements as in (*) and let I be given by

$$I = \bigcup_{i \geq 1} \langle a_i \rangle.$$

Since

$$\langle a_1 \rangle \subset \langle a_2 \rangle \subset \langle a_3 \rangle \subset \cdots$$

it is straightforward to see that I is an ideal. Since R is a PID, we may write $I = \langle d \rangle$ for some $d \in R$. By the definition of I , we may find an index N for which $d \in \langle a_j \rangle$ for each $j \geq N$.

Fix $j \geq N$. We may write $d = x \cdot a_j$ for $x \in R$.

On the other hand, $\langle a_j \rangle \subseteq \langle d \rangle$, we may write $a_j = y \cdot d$ for $y \in R$.

We now see that $d = x \cdot a_j = xyd$ so that $x, y \in R^\times$ by cancellation (Lemma 3.1.8). Thus d and a_j are *associates* so that $\langle d \rangle = \langle a_j \rangle$. In particular, we have proved that

$$\langle d \rangle = \langle a_N \rangle = \langle a_{N+1} \rangle = \langle a_{N+2} \rangle = \cdots$$

contradicting the assumption (ii) that a_{j+1} and a_j are not associates. This contradiction proves (a).

We now prove (b). We are given an equality

$$p_1 \cdots p_n = q_1 \cdots q_m$$

with p_i, q_j irreducible and $n, m \geq 1$.

We proceed by induction on the minimum $\min(n, m)$, and without loss of generality we suppose that $n \leq m$ so that $n = \min(n, m)$.

In case $n = 1$, our assumption is $p_1 = q_1 \cdots q_m$. Applying Corollary 5.1.5 we see that $p_i \mid q_j$ for some $1 \leq j \leq m$. Since p_i and q_j are irreducible, we see that $q_j = u \cdot p_1$ for some unit $u \in R^\times$. Thus

$$p_1 = u \cdot p_1 \cdot \prod_{i \neq j} q_i.$$

Applying cancellation (Lemma 3.1.8) we see $u \cdot \prod_{i \neq j} q_i = 1$ so that $q_i \in R^\times$ for $i \neq j$. Thus $m = 1$ and p_1 and q_1 are associates, as required. This confirms the base-case of the induction.

Now suppose that $n > 1$ and that the result is known when the element has an expression as a product of $< n$ irreducibles.

Thus we have

$$p_1 \cdots p_n = q_1 \cdots q_m$$

and $m \geq n$. Now $p_n \mid q_1 \cdots q_m$ and as before we see for some $1 \leq j \leq m$ that $q_j = up_n$ for a unit $u \in R^\times$. Without loss of generality we may suppose that $j = m$. We find

$$p_1 \cdots p_{n-1} \cdot p_n = u \cdot p_n \cdot q_1 \cdots q_{m-1}$$

Applying cancellation (Lemma 3.1.8) we find that

$$p_1 \cdots p_{n-1} = uq_1 \cdots q_{m-1}$$

Replacing q_1 by the irreducible uq_1 , we can view the right-hand side as a product of $m - 1$ irreducibles. Since $m - 1 \geq n - 1$ we may apply the induction hypothesis to find that $m - 1 = n - 1$ and that after re-ordering we have p_i associate to q_i for $1 \leq i \leq m - 1$. Since p_n and q_m are associate as well, this proves (b). \square

6 The Field of fractions of an Integral Domain

Recall Example 3.1.7 that any subring of a field is an integral domain. We now want to argue that the *converse* to this statement is true, as well. Namely, an integral domain R is a subring of a field. In fact, we are essentially going to give a *construction* of such a field from R .

Let's fix an integral domain R . To confirm the suggested converse to the above Corollary, we must construct a field F and an inclusion $i : R \subset F$.

Of course, if we have such a mapping i , then for any $0 \neq b \in R$, the element $i(b)$ is non-zero in F and hence $i(b)^{-1} = \frac{1}{i(b)}$ should be an element of F (even though $i(b)^{-1}$ is possibly not an element of R). For any $a \in R$ we should be able to multiply $i(a)$ and $\frac{1}{i(b)}$ in F to form the *fraction* $\frac{i(a)}{i(b)}$. If we choose to identify R with the image $i(R)$, we might simply write $\frac{a}{b} = \frac{i(a)}{i(b)}$ for this *fraction*.

So if the field F *exists*, it must contain all fractions $\frac{a}{b}$ for $a, b \in R$ with $0 \neq b$.

In fact, we are going to construct a field F by formally introducing such fractions.

Consider the set $W = \{(a, b) \mid a, b \in R, b \neq 0\}$ and define a relation \sim on the set W by the condition

$$(a, b) \sim (s, t) \iff at = bs.$$

This relation is motivated by the observation that for *fractions* in a field F we have

$$\frac{a}{b} = \frac{s}{t} \iff at = bs.$$

One needs to check the following:

Proposition 6.0.1. \sim defines an equivalence relation on W .

Proof. We must confirm properties of \sim :

(*reflexive*) if $(a, b) \in W$, then $ab = ba \implies (a, b) \sim (a, b)$.

(*symmetric*) if $(a, b), (s, t) \in W$ then

$$(a, b) \sim (s, t) \implies at = bs \implies sb = ta \implies (s, t) \sim (a, b).$$

(*transitive*) Let $(a, b), (s, t), (u, v) \in W$ and suppose that $(a, b) \sim (s, t)$ and $(s, t) \sim (u, v)$. The assumptions mean that $at = bs$ and $sv = tu$.

Multiplying the equation $at = bs$ by v on each side, we see that

$$atv = bsv \implies atv = btu \implies (av)t = (bu)t;$$

since $t \neq 0$ and since the cancellation law holds in an integral domain – see Lemma 3.1.8, conclude $av = bu$. Hence $(a, b) \sim (u, v)$ which confirms the transitive law.

□

We are now going to show that the fractions - i.e. the equivalence classes in W - form a field. We define $Q = Q(R)$ to be the set of equivalence classes of W under the equivalence relation \sim .

We write $\frac{a}{b} = [(a, b)]$ for the equivalence class of $(a, b) \in W$. Thus Q is the set of (formal) fractions of elements of R , and

$$\frac{a}{b} = \frac{s}{t} \iff (a, b) \sim (s, t) \iff at = bs$$

It remains to argue that Q has the structure of a field. To do this, we must define binary operations $+$ and \cdot on the set Q and check that they satisfy the correct axioms.

Define addition of fractions: for $a, b, s, t \in R$ with $b, t \neq 0$,

$$(\clubsuit) \quad \frac{a}{b} + \frac{s}{t} = \frac{at + bs}{bt}.$$

And define multiplication of fractions:

$$(\diamond) \quad \frac{a}{b} \cdot \frac{s}{t} = \frac{as}{bt}.$$

Theorem 6.0.2. *For an integral domain R , the set $Q(R)$ of fractions of R forms a field with the indicated addition and multiplication.*

Sketch of proof. What must be checked??

- must first confirm that (\clubsuit) is *well-defined*! i.e. if $a', b', s', t' \in R$ with $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{s}{t} = \frac{s'}{t'}$, we must check that $\frac{a}{b} + \frac{s}{t} = \frac{a'}{b'} + \frac{s'}{t'}$; i.e. that

$$\frac{at + bs}{bt} = \frac{a't' + b's'}{b't'}.$$

This is straightforward if a bit tedious.

- One readily checks that $0 = \frac{0}{1}$ is an identity for the binary operation $+$ on Q .
- One readily checks that $+$ is commutative for Q .
- One readily checks that $\frac{-a}{b}$ is an additive inverse for $\frac{a}{b}$.
- With some more effort, one confirms that $+$ is *associative* on Q ; i.e. for $\alpha, \beta, \gamma \in Q$

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

Thus $(Q, +)$ is an abelian group. Now consider the operation \diamond of multiplication.

- must again confirm that (\diamond) is *well-defined*! i.e. if $a', b', s', t' \in R$ with $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{s}{t} = \frac{s'}{t'}$, we must check that $\frac{a}{b} \cdot \frac{s}{t} = \frac{a'}{b'} \cdot \frac{s'}{t'}$; i.e. that

$$\frac{as}{bt} = \frac{a's'}{b't'}.$$

- One readily checks that $1 = \frac{1}{1}$ is an identity for the binary operation \cdot on Q .
- One readily checks that \cdot is commutative for Q .
- With some more effort, one confirms that \cdot is *associative* on Q ; i.e. for $\alpha, \beta, \gamma \in Q$

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

- Next, one must confirm the *distributive law*: for $\alpha, \beta, \gamma \in Q$,

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma.$$

Phew! □

Remark 6.0.3. Despite the details of the preceding proof, all that is happening is confirming properties of operations of fractions that you have used since grade-school...

Now, we want to emphasize a crucial property of the field of fractions of an integral domain.

Let $Q(R)$ be the field constructed above, and note that there is a natural ring homomorphism $i : R \rightarrow Q(R)$ given by $r \mapsto i(r) = \frac{r}{1}$ for $r \in R$. This homomorphism is one-to-one: indeed, if $\frac{r}{1} = 0 = \frac{0}{1}$, then $r \cdot 1 = 0 \cdot 1 \implies r = 0$. Thus, we may identify R with a subring of $Q(R)$.

Proposition 6.0.4. *Let R be an integral domain, let $\phi : R \rightarrow S$ be any ring homomorphism, and suppose that for all $0 \neq d \in R$, $\phi(d) \in S^\times$ - i.e. $\phi(d)$ is a unit in S . Then there is a unique homomorphism $\tilde{\phi} : Q(R) \rightarrow S$ with the property that $\tilde{\phi}|_R = \phi$.*

Proof. Let $x \in Q(R)$ be any element. Thus $x = \frac{a}{b} = \frac{a}{1} \cdot \frac{1}{b}$ for $a, b \in R$ with $b \neq 0$.

Let's first argue that uniqueness of $\tilde{\phi}$. If $\tilde{\phi}$ is a ring homomorphism, then

$$1 = \tilde{\phi}(1) = \tilde{\phi}(b \cdot \frac{1}{b}) = \phi(b)\tilde{\phi}(\frac{1}{b}) \implies \tilde{\phi}(\frac{1}{b}) = \phi(b)^{-1}$$

Since $\tilde{\phi}$ is a ring homomorphism, we must have

$$(\clubsuit) \quad \tilde{\phi}(x) = \tilde{\phi}(\frac{a}{1})\tilde{\phi}(\frac{1}{b}) = \phi(a) \cdot \phi(b)^{-1}$$

which confirms the uniqueness.

It now only remains to check that the rule (\clubsuit) determines a ring homomorphism, which is straightforward. □

Example 6.0.5. The field of rational functions

Let F be a field, and consider $R = F[T]$ the ring of polynomials. This is an integral domain, and its field of fractions $Q(R)$ is usually written $F(T)$ and is known as the field of rational functions over F .

Note that

$$F(T) = \left\{ \frac{f}{g} \mid f, g \in F[T], g \neq 0 \right\};$$

thus elements of $F(T)$ are fractions $\frac{f}{g}$ whose numerator and denominator are *polynomials*; we usually call such expressions *rational functions*.

7 Irreducible polynomials over a field

7.1 Fields as quotient rings

Proposition 7.1.1. *Let R be a PID and let $p \in R$ be an irreducible element. Then the quotient ring $A = R/\langle p \rangle$ is a field.*

Proof. Let $\alpha \in A$ be non-zero. To prove that A is a field, we must show that α has a multiplicative inverse. Thus α has the form $h + \langle p \rangle$ and since $\alpha \neq 0$ we know that $p \nmid h$. Since p is irreducible, Remark 5.1.2 shows that $\gcd(p, h) = 1$.

Thus according to Proposition 4.3.4 there are elements $x, y \in R$ for which

$$1 = xp + yh$$

Let $\beta = y + \langle p \rangle \in A$. Then

$$\alpha\beta = yh + \langle p \rangle = 1 + \langle p \rangle$$

since $yh \equiv 1 \pmod{p}$. Thus β is the multiplicative inverse of α in A . \square

Example 7.1.2. • $\mathbf{Z}/p\mathbf{Z}$ is a field for a prime number p .

As a special case of Proposition 7.1.1, we have:

Corollary 7.1.3. *Let F be a field and let f be an irreducible polynomial in $F[T]$. Then $A = F[T]/\langle f \rangle$ is a field.*

For small degree polynomials, one can confirm irreducibility just by considering roots, as follows:

Proposition 7.1.4. *Let F be a field and let $f \in F[T]$ be a polynomial with $\deg(f) \leq 3$. If f has no root in F then f is irreducible.*

Proof. Suppose that f is reducible, say $f = gh$ with $\deg(g), \deg(h) > 0$. Since $\deg(f) \leq 3$ and since $\deg(g) + \deg(h) = \deg(f)$ by Proposition 3.3.2, we see that at least one of g or h must have degree 1; without loss of generality we suppose $\deg(g) = 1$.

Thus $g = aT + b$ for $a, b \in F$ with $a \neq 0$. Set $\alpha = \frac{-b}{a} \in F$ and observe that $f(\alpha) = g(\alpha)h(\alpha) = 0$; thus f has the root $\alpha \in F$. \square

Example 7.1.5. Let p be a prime number. Then the polynomial $T^2 - p \in \mathbf{Q}[T]$ is *irreducible*. In particular,

$$\mathbf{Q}(\sqrt{p}) = \mathbf{Q}[T]/\langle T^2 - p \rangle$$

is a field.

7.2 The rational roots test

Theorem 7.2.1. *Let R be a PID with field of fractions F and let $f \in R[T]$, say*

$$f = a_0 + a_1T + \cdots + a_nT^n$$

with $a_i \in R$ and $a_n \neq 0$.

If $\alpha = \frac{x}{y} \in F$ is a root of f for $x, y \in R$ and $y \neq 0$ and $\gcd(x, y) = 1$ then $x \mid a_0$ and $y \mid a_n$.

Proof. Since α is a root of f we have the equation

$$0 = f(\alpha) = a_0 + a_1 \left(\frac{x}{y}\right) + \cdots + a_n \left(\frac{x}{y}\right)^n = \sum_{i=0}^n a_i \left(\frac{x}{y}\right)^i$$

in the field F . Multiplying by the non-zero element $y^n \in R$ we find the equation

$$0 = a_0 y^n + a_1 x y^{n-1} + \cdots + a_n x^n = \sum_{i=0}^n a_i x^i y^{n-i}$$

in R .

Thus we see that

$$a_0 y^n = -(a_1 x y^{n-1} + \cdots + a_n x^n) = - \sum_{i=1}^n a_i x^i y^{n-i} = -x \sum_{i=1}^n a_i x^{i-1} y^{n-i}$$

which shows that $x \mid a_0 y^n$. Since $\gcd(x, y) = 1$ also $\gcd(x, y^n) = 1$. Now conclude that $x \mid a_0$.

Similarly, we see that

$$a_n x^n = - \sum_{i=0}^{n-1} a_i x^i y^{n-i} = -y \sum_{i=0}^{n-1} a_i x^i y^{n-i-1}$$

which shows that $y \mid a_n x^n$. Since $\gcd(x^n, y) = 1$ we conclude that $y \mid a_n$ as required. \square

Remark 7.2.2. Let $f = \sum_{i=0}^n a_i T^i \in R[T]$ as in the statement of Theorem 7.2.1. According to the theorem, to find a root of f in the field of fractions F of R , we must consider all fractions $\alpha = \frac{x}{y}$ where $\gcd(x, y) = 1$, where x is a divisor of a_0 and where y is a divisor of a_n .

Writing $a_0 = p_1 p_2 \cdots p_n$ and $a_n = q_1 q_2 \cdots q_m$ for irreducibles p_i and q_j , we see that it is possible in principle to make a list of all possible α and then check for each candidate whether or not α is a root of f .

Example 7.2.3. Consider the polynomial $f = T^3 - 3T^2 + 2T - 6 \in \mathbf{Z}[T]$. For any root $\alpha = \frac{x}{y} \in \mathbf{Q}$ with $\gcd(x, y) = 1$ we must have that $x \mid 6$ and $y \mid 1$. Thus according to Theorem 7.2.1, the possible rational roots are $\alpha = \pm 1, \pm 2, \pm 3, \pm 6$.

Notice that if $x \in \mathbf{R}$ is negative, then $f(x) < 0$. Thus the possible rational roots are simple $\alpha = 1, 2, 3, 6$. We notice that $f(1) = -6$, $f(2) = -6$ and $f(3) = 0$. Using the division algorithm we see that

$$T^3 - 3T^2 + 2T - 6 = (T^2 + 2)(T - 3)$$

It is now clear that 6 is not a root and that $T^2 + 2$ is irreducible. We f has exactly one rational root, namely $\alpha = 3$.

7.3 The Gauss Lemma

Let R be a PID with field of fractions F . The polynomial ring $R[T]$ is the subring of $F[T]$ consisting of polynomials whose coefficients lie in R . In particular $R[T]$ is itself an integral domain.

Remark 7.3.1. Note that in the case where R is *already* a polynomial ring $F[X]$, we introduce a *new* variable T different from X .

Definition 7.3.2. The *content* $\text{content}(f)$ of the element $f = \sum_{i=0}^N a_i T^i \in R[T]$ where $a_i \in R$ is defined to be

$$\text{content}(f) = \gcd(a_0, a_1, \dots, a_N).$$

We say that the polynomial $f \in R[T]$ is *primitive* if $\text{content}(f) = 1$.

Lemma 7.3.3. *Let $f \in R[T]$ be a non-zero polynomial and let $c = \text{content}(f) \in R$. Then f may be written $f = cf_0$ where $f_0 \in R[T]$ is primitive.*

Proof. Write $f = \sum_{i=0}^n a_i T^i$ with $a_i \in R$. Then by definition we have $c = \gcd(a_0, \dots, a_n)$. Note that $c \mid a_i$ for each i ; we write $b_i = \frac{a_i}{c} \in R$.

We set $f_0 = \sum_{i=0}^n b_i T^i \in R[T]$ and notice that

$$c \cdot f_0 = \sum_{i=0}^n c \cdot b_i T^i = \sum_{i=0}^n a_i T^i = f$$

as required. Finally,

$$\text{content}(f_0) = \gcd(b_0, \dots, b_n) = \gcd\left(\frac{a_0}{c}, \dots, \frac{a_n}{c}\right) = 1$$

by Proposition 4.3.5. Thus f_0 is indeed primitive. \square

Lemma 7.3.4. *Let $p \in R$ be irreducible and consider the assignment*

$$h \mapsto \bar{h} : R[T] \rightarrow (R/\langle p \rangle)[T]$$

defined as follows: for $h = \sum_{i=0}^N c_i T^i \in R[T]$ with $c_i \in R$, the polynomial $\bar{h} \in (R/\langle p \rangle)[T]$ is given by

$$\bar{h} = \sum_{i=0}^N [c_i] T^i$$

where $[c_i] = c_i + pR$ is the class of c_i modulo pR .

(a) *This assignment is a ring homomorphism.*

(b) *For $h \in R[T]$, $\bar{h} = 0$ if and only if $p \mid \text{content}(h)$.*

Proof. (a) follows from Proposition 3.2.1. For (b), just observe that $\bar{h} = 0$ if and only if $p \mid c_i$ for every i . \square

Proposition 7.3.5. (“The Gauss Lemma”) *If $f, g \in R[T]$ are primitive, then the product fg is primitive.*

Proof. Suppose on the contrary that there are primitive polynomials $f, g \in R[T]$ for which fg is not primitive. Writing $d = \text{content}(fg)$ for the content of the product, we know that $\langle d \rangle \neq R$ so that d is divisible by some prime $p \in R$.

Consider the ring homomorphism $h \mapsto \bar{h}$ of Lemma 7.3.4.

Now, $p \mid \text{content}(fg) \implies 0 = \overline{fg} = \bar{f} \cdot \bar{g}$. Since R/pR is a field, the ring $(R/pR)[T]$ is an integral domain, so we may conclude that either $\bar{f} = 0$ or $\bar{g} = 0$.

But according to Lemma 7.3.4 (b), $\bar{f} = 0 \implies p \mid \text{content}(f)$ and $\bar{g} = 0 \implies p \mid \text{content}(g)$. This contradicts our assumption that $1 = \text{content}(f) = \text{content}(g)$. Thus indeed $\text{content}(fg) = 1$. \square

Theorem 7.3.6. Suppose that $f \in R[T]$ is a primitive polynomial, and that $g, h \in K[T]$ are polynomials for which $f = gh$ in $K[T]$. Then there are polynomials $g_1, h_1 \in R[T]$ with $\deg g = \deg g_1$ and $\deg h = \deg h_1$ for which $f = g_1 h_1$ in $R[T]$.

Proof. Using Lemma 7.3.3, we may write $g = \frac{x}{y}g_1$ and $h = \frac{z}{w}h_1$ where $g_1, h_1 \in R[T]$ are primitive and $x, y, z, w \in R$ with $y, w \neq 0$. We now see that

$$(\heartsuit) \quad yw \cdot f = xz \cdot g_1 h_1.$$

Since f is primitive, notice that $yw = \text{content}(yw f)$. Moreover, the Gauss Lemma – i.e. Proposition 7.3.5 – shows that $g_1 h_1$ is primitive; thus, we have $\text{content}(xz g_1 h_1) = xz$.

It follows that

$$\langle yw \rangle = \langle xz \rangle$$

i.e. that $(\clubsuit) \quad u \cdot yw = xz$ for a unit $u \in R^\times$ – see Proposition 3.1.9.

But then (\heartsuit) and (\clubsuit) together show that $yw \cdot f = u \cdot yw \cdot g_1 h_1$ and now the cancellation law Lemma 3.1.8 in the integral domain $R[T]$ implies $f = (u g_1) \cdot h_1$ which proves the Theorem. \square

7.4 Eisenstein's irreducibility criterion

Theorem 7.4.1. Let $p \in R$ be irreducible, and let

$$f = \sum_{i=0}^n a_i T^i \in R[T], \quad (\text{where } a_i \in R, \ 0 \leq i \leq n)$$

be a polynomial with $a_n \neq 0$. Suppose that $p \nmid a_n$, that $p \mid a_i$ for $0 \leq i \leq n-1$ and that $p^2 \nmid a_0$. Then f is irreducible when viewed as an element of $F[T]$.

Proof. Let $c = \text{content}(f)$. Then $c \not\equiv 0 \pmod{p}$ since $p \nmid a_n$. Observe now that the polynomial $\tilde{f} = \frac{1}{c}f \in R[T]$ still satisfies the assumptions of the Theorem. Since \tilde{f} is irreducible in $K[T]$ if and only if the same is true for f , it suffices to prove the Theorem when $f = \tilde{f}$ is primitive.

Now, according to Theorem 7.3.6 the irreducibility of $f \in F[T]$ will follow once we show that if $f = gh$ for $g, h \in R[T]$ then either $\deg g = 0$ or $\deg h = 0$. So suppose $f = gh$ for $g, h \in R[T]$.

Consider the ring homomorphism $f \mapsto \bar{f} : R[T] \rightarrow (R/pR)[T]$ as in Lemma 7.3.4. Assumptions on the coefficients a_i show $\bar{f} = \bar{g}\bar{h}$ to be a non-zero multiple of T^n . Using unique factorization in the principal ideal domain $(R/pR)[T]$, it follows that \bar{g} is a non-zero multiple of T^i and \bar{h} is a non-zero multiple of T^j where $i + j = n$ and $0 \leq i, j \leq n$. Moreover $i = \deg g$ and $j = \deg h$.

Now the Theorem follows since if $i, j > 0$ then p divides the constant term of both g and h , and then $p^2 \mid a_0$ contradicting our assumption. \square

Example 7.4.2. (a) Let p be a prime integer, let $n \geq 1$ and let $f = T^n - p$. Then Theorem 7.4.1 shows that $f \in \mathbf{Q}[T]$ is irreducible.

(b) Let K be a field and consider the ring $K[X]$ of polynomials over K . The field of fractions of $K[X]$ is the field $F = K(X)$ of rational functions.

Let $n \geq 1$ and consider the polynomial $f = T^n - X \in F[T] = K(X)[T]$. Then f is irreducible in $K(X)[T]$ by Theorem 7.4.1.

7.5 Irreducibility of certain cyclotomic polynomials

For a prime number p consider the polynomial

$$F(T) = F_p(T) = \frac{T^p - 1}{T - 1} = T^{p-1} + T^{p-2} + \cdots + T + 1 \in \mathbf{Q}[T].$$

Applying the change of variables $U = T - 1$ we see that

$$\begin{aligned} F(U+1) &= \frac{(U+1)^p - 1}{(U+1) - 1} = \frac{\sum_{i=1}^p \binom{p}{i} U^i}{U} \\ &= \frac{U^p + \binom{p}{p-1} U^{p-1} + \cdots + \binom{p}{2} U^2 + \binom{p}{1} U}{U} \\ &= U^{p-1} + \binom{p}{p-1} U^{p-2} + \cdots + \binom{p}{2} U + p \end{aligned}$$

In particular, $g(U) = F(U+1) = \sum_{i=0}^{p-1} c_i U^i \in \mathbf{Q}[U]$ has degree $p-1$ and the coefficients are given by the formulae

$$c_i = \binom{p}{i+1}, \quad 0 \leq i \leq p-1.$$

Proposition 7.5.1. *For a prime number $p > 0$, the polynomial*

$$F(T) = \frac{T^p - 1}{T - 1} = T^{p-1} + T^{p-2} + \cdots + T + 1 \in \mathbf{Q}[T]$$

of degree $p-1$ is irreducible.

Proof. Clearly $F(T) \in \mathbf{Q}[T]$ is irreducible if and only if $g(U) \in \mathbf{Q}[U]$ is irreducible. Now, $g(U) \in \mathbf{Z}[U]$ since binomial coefficients $\binom{n}{m}$ are always integers. We are going to apply Eisenstein's criteria to show the irreducibility of $g(U)$. For this, we first note that $c_{p-1} = 1$ is not divisible by p and that $c_0 = p$ is divisible by p but not by p^2 .

The irreducibility will now follow from Theorem 7.4.1 once we argue that $(\clubsuit) : p \mid \binom{p}{i}$ for each $1 \leq i \leq p-1$.

To prove (\clubsuit) just note that

$$\binom{p}{i} = \frac{p!}{i!(p-i)!}.$$

Since $0 < i < p$, neither $i!$ nor $(p-i)!$ is divisible by p . On the other hand

$$p! = p \cdot (p-1) \cdot (p-2) \cdots 2 \cdot 1$$

is divisible by p .

Since one knows that $\binom{p}{i} \in \mathbf{Z}$, unique factorization Section 5.2 implies that $p \mid \binom{p}{i}$ as required. \square

Example 7.5.2. For example, $f(T) = T^4 + T^3 + T^2 + T + 1 \in \mathbf{Q}[T]$ is an irreducible since $f(T) = \frac{T^5 - 1}{T - 1}$ and since $p = 5$ is prime.

8 Some recollections of Linear Algebra

Let F be a field. Much of what you learned in a course on linear algebra remains valid for vector spaces over F and not just for vector spaces over \mathbf{R} or \mathbf{C} .

8.1 Vector Spaces

Definition 8.1.1. A *vector space* over F is an additive abelian group V together with a mapping

$$F \times V \rightarrow V$$

denoted by

$$(\alpha, v) \mapsto \alpha v$$

called *scalar multiplication* that is required to satisfy several axioms:

(VS1) the multiplicative identity $1 = 1_F \in F$ satisfies $1 \cdot v = v$ for all $v \in V$.

(VS2) scalar multiplication is associative: for all $\alpha, \beta \in F$ and all $v \in V$, we have $\alpha(\beta v) = (\alpha\beta)v$.

(VS3) scalar multiplication distributes over addition in V : for all $\alpha, \beta \in F$ and for all $v, w \in V$, we have

$$\alpha \cdot (v + w) = \alpha \cdot v + \alpha \cdot w$$

and

$$(\alpha + \beta) \cdot v = \alpha v + \beta v.$$

You should compare these requirements with axioms you may have seen in a course in linear algebra. The present list is probably shorter – that is because one needs axioms governing the behavior of addition, which we have handled by requiring V to be an additive abelian group.

8.2 Linear Transformations, subspaces and quotient vector spaces

Definition 8.2.1. Let V be a vector space over F . A subset $W \subset V$ is called a **subspace** (or more precisely, an F -subspace) provided that

- (a) W is an additive subgroup of V , and
- (b) W is closed under scalar multiplication by F – i.e.

$$\alpha w \in W \quad \text{for all } \alpha \in F \text{ and all } w \in W.$$

Definition 8.2.2. If V and W are vector spaces over F , a function $T : V \rightarrow W$ is a *linear transformation* (or more precisely, an F -linear transformation) if

- (a) T is a homomorphism of additive groups $V \rightarrow W$, and
- (b) T commutes with scalar multiplication – i.e. $T(\alpha v) = \alpha T(v)$ for all $\alpha \in F$ and all $v \in V$.

Definition 8.2.3. If V, W are vector spaces, a linear transformation $T : V \rightarrow W$ is an *isomorphism* if there is a linear transformation $S : W \rightarrow V$ such that $T \circ S = 1_W$ and $S \circ T = 1_V$.

If T is an isomorphism, one says that V and W are isomorphic vector spaces.

Proposition 8.2.4. *Let V, W be F -vector spaces and let $T : V \rightarrow W$ be a linear transformation. Then T is an isomorphism if and only if T is bijective.*

Proof. Suppose that T is bijective. Then we know that T is an isomorphism of additive groups, and hence there is an inverse isomorphism $S : W \rightarrow V$. It only remains to show that S is a linear transformation (rather than simply a group homomorphism).

So let $\alpha \in F$ and $w \in W$. Since T is onto, we may write $w = T(v)$ for some $v \in V$. Now,

$$S(\alpha w) = S(\alpha T(v)) = S(T(\alpha v)) = 1_W(\alpha v) = \alpha v = \alpha S(T(v)) = \alpha S(w).$$

On the other hand, if T is an isomorphism, then the inverse isomorphism S is an inverse function to T so in particular T is one-to-one and onto. \square

Proposition 8.2.5. *If $T : V \rightarrow W$ is a linear transformation, then*

- (a) $\ker(T)$ is a subspace of V , and
- (b) the image $T(V) = \{T(v) \mid v \in V\}$ is a subspace of W .

Proof. Exercise! \square

Proposition 8.2.6. *Let W be a subspace of the F -vector space V . The quotient group V/W has the structure of an F -vector space, and the natural quotient mapping $\pi : V \rightarrow V/W$ given by $\pi(v) = v + W$ is an F -linear transformation.*

Proof. We must define a scalar multiplication for the additive group V/W . Given $\alpha \in F$ and an element $v + W \in V/W$, define

$$\alpha \cdot (v + W) = (\alpha v) + W.$$

We must confirm that this rule is independent of the choice of coset representative v for $v + W$. Thus, we must suppose that

$$v + W = v' + W$$

and we must show that $\alpha \cdot (v + W) = \alpha \cdot (v' + W)$ i.e. that $\alpha v + W = \alpha v' + W$.

The assumption that $v + W = v' + W$ means that $v - v' \in W$. Since W is a F -subspace, we find that $\alpha(v - v') \in W$ and using the distributive law we conclude that $\alpha v - \alpha v' \in W$. This shows that $\alpha v + W = \alpha v' + W$ as required. This proves that we've given a well-defined operation of scalar multiplication.

It now remains to check that the associative and distributive laws hold for this operation. Since these properties hold for the scalar multiplication in V , the verification is straightforward; details are left to the reader. \square

Proposition 8.2.7. *If $T : V \rightarrow W$ is a linear transformation, there is an isomorphism $\tilde{T} : V/\ker(T) \rightarrow T(V)$ given by $\tilde{T}(v + \ker T) = T(v)$ for $v \in V$.*

Proof. The first isomorphism theorem for groups tells us that the rule \tilde{T} is an isomorphism of groups. In view of @prop:inv-iso, it remains to argue that \tilde{T} is a linear transformation.

Thus, let $\alpha \in F$ and $x \in V/\ker T$. We may write $x = v + \ker T$ for some $v \in V$. Now, by definition we have

$$\alpha x = \alpha(v + \ker T) = \alpha v + \ker T.$$

Thus, since T is a linear transformation we find the following:

$$\tilde{T}(\alpha x) = \tilde{T}(\alpha v + \ker T) = T(\alpha v) = \alpha T(v) = \alpha \tilde{T}(v + \ker T).$$

This confirms that \tilde{T} commutes with scalar multiplication and is thus a linear transformation. \square

8.3 Bases and dimension

You are probably familiar with the notions of *spanning set* and of *linear independence*. One issue to be aware of is how to handle possibly-infinite sets in this setting.

To quote from Michael Artin's algebra text (Artin 2011):

In algebra it is customary to speak only of linear combinations of finitely many vectors. Therefore, the span of an infinite set S must be interpreted as the set of those vectors V which are linear combinations of finitely many elements of S ...

Definition 8.3.1. If $S \subset V$ is a set of elements, the span of S is defined to be

$$\text{span}(S) = \left\{ \sum_{i=1}^r a_i x_i \mid r \in \mathbf{Z}_{\geq 0}, a_i \in F, x_i \in V (1 \leq i \leq r) \right\}$$

It is clear that $\text{span}(S)$ is a *subspace* of V .

Definition 8.3.2. A subset $S \subset V$ of the vector space V is said to be *linearly independent* if whenever $n \in \mathbf{Z}_{\geq 0}$, whenever $x_1, \dots, x_n \in V$ are *distinct* elements of V , and whenever $\alpha_1, \dots, \alpha_n \in F$ then

$$\sum_{i=1}^n \alpha_i x_i = 0 \implies \alpha_j = 0 \quad \text{for each } 1 \leq j \leq n.$$

Remark 8.3.3. We say that the vector space is *finitely generated* if there is a *finite* set $S \subset V$ for which $V = \text{span}(S)$. In fact, V is then *finite dimensional* (see Definition 8.3.6 below).

Definition 8.3.4. Let V be a vector space over the field F . A *basis* for V is a subset $S \subset V$

- (a) S spans V ; i.e. $V = \text{span}(S)$, and
- (b) S is linearly independent.

Proposition 8.3.5. Let V be an F -vector space.

- (a) There is a basis \mathcal{B} for V .
- (b) If $W \subset V$ is a subspace of V , and if \mathcal{C} is a basis for W , there is a basis \mathcal{B} for V with $\mathcal{C} \subseteq \mathcal{B}$.
- (c) If $V = \text{span}(S)$ then there is a basis of V contained in S .
- (d) If $S \subset V$ is a linearly independent subset, there is a basis of V containing S .
- (e) Any two bases of V have the same cardinality.

Proof. When V is finitely generated, results (a)-(e) can be found in (Hoffman and Kunze 1971), §2.2 and 2.3, and in (Friedberg, Insel, and Spence 2002) §1.6.

For the general case of (a)-(d) see (Friedberg, Insel, and Spence 2002) §1.7.

A proof of (e) in case \mathcal{B}_1 and \mathcal{B}_2 are *infinite* bases for V requires the Schroeder-Bernstein Theorem; we won't need this result in the course. \square

Definition 8.3.6. If V is a vector space with basis \mathcal{B} , the *dimension* of V

- written $\dim V$ or $\dim_F V$ - is equal to the cardinality of the set \mathcal{B} .

It follows from Proposition 8.3.5 (e) that the dimension of V doesn't depend on the choice of basis.

Proposition 8.3.7. *Let V, W be F -vector spaces, let \mathcal{B} be a basis for V , and let $x_b \in W$ for each $b \in \mathcal{B}$. Then there is a unique linear transformation $T : V \rightarrow W$ such that $T(b) = x_b$ for each $b \in \mathcal{B}$.*

Example 8.3.8. Let $F[T]$ be the polynomial ring over the field F . Then $F[T]$ is in particular a vector space over F with countably infinite basis given by $\{T^i \mid i \geq 0\}$.

The linear independence of this basis precisely means that if $f = \sum_{i=0}^N a_i T^i \in F[T]$ for $a_i \in F$, then $f = 0$ if and only if all $a_i = 0$.

Proposition 8.3.9. *Let $T : V \rightarrow W$ be a linear transformation of F -vector spaces with $\dim V < \infty$. Then*

$$\dim_F V = \dim_F T(V) + \dim_F \ker(V).$$

9 Field extensions

Definition 9.0.1. Let F and E be fields and suppose that $F \subset E$ is a *subring*. We say that F is a *subfield* of E and that E is a *field extension* of F .

Throughout this discussion, let $F \subseteq E$ be an extension of fields.

9.1 Algebraic extensions of fields

Definition 9.1.1. An element $\alpha \in E$ is said to be *algebraic* over F provided that there is some polynomial $0 \neq f \in F[T]$ for which α is a root – i.e. for which $f(\alpha) = 0$.

If α is not algebraic over F , we say that α is *transcendental* over F .

Example 9.1.2. • it is a fact that $\pi, e \in \mathbb{R}$ are transcendental over \mathbb{Q} .

- Of course, π, e are algebraic over \mathbb{R} .
- Any element $\alpha = a + bi \in \mathbb{C}$ (for $a, b \in \mathbb{R}$) is algebraic over \mathbb{R} . Indeed, α is a root of the polynomial

$$\begin{aligned} f(T) &= (T - \alpha)(T - \bar{\alpha}) \\ &= T^2 - 2\operatorname{Re}(\alpha)T + |\alpha|^2 \\ &= T^2 - 2aT + (a^2 + b^2) \in \mathbb{R}[T] \end{aligned}$$

where $\operatorname{Re}(\alpha) = a$ denotes the *real part* of the complex number α .

9.2 The minimal polynomial

Proposition 9.2.1. *Let $\alpha \in E$ and suppose that α is algebraic over F . Then there is a unique monic irreducible polynomial $p \in F[T]$ for which α is a root.*

Moreover,

(a) *p is the monic polynomial of smallest degree for which α is a root.*

(b) *if $f \in F[T]$ is any polynomial with $f(\alpha) = 0$, then $p \mid f$.*

Proof. Let $I = \{f \in F[T] \mid f(\alpha) = 0\}$. It is straightforward to check that it is an additive subgroup, and it is closed under multiplication with any polynomial in $F[T]$; thus I is an ideal of $F[T]$.

Since α is algebraic, $I \neq \{0\}$. Thus I coincides with the principal ideal $I = \langle p \rangle$ for some monic $0 \neq p \in F[T]$, and p is the unique monic element of smallest degree in I .

It only remains to argue that p is irreducible. Suppose that $f, g \in F[T]$ and that $p \mid fg$. We need to argue that $p \mid f$ or $p \mid g$. Well, since $fg = pq$ for $q \in F[T]$, we see that

$$0 = (pq)(\alpha) = (fg)(\alpha) = f(\alpha) \cdot g(\alpha).$$

Since $f(\alpha), g(\alpha)$ are elements of the field E , the only way their product can be 0 is for at least one factor to be zero – i.e. either $f(\alpha) = 0$ or $g(\alpha) = 0$. But then either $f \in I$ or $g \in I$ and thus $p \mid f$ or $p \mid g$. \square

Corollary 9.2.2. *Let $\alpha \in E$. If $p \in F[T]$ is irreducible and monic, and if $p(\alpha) = 0$, then p is the minimal polynomial of α over F .*

Definition 9.2.3. Let $\alpha \in E$ be algebraic over F .

- The irreducible polynomial p of the proposition is known as the *minimal polynomial* of α over F .
- The *degree* of α over F is defined to be the degree of the minimal polynomial p .

Example 9.2.4. An element $\alpha \in F$ has degree 1 over F , since it is the root of the irreducible degree 1 polynomial $T - \alpha \in F[T]$.

Example 9.2.5. Consider the complex number $z = a + bi \in \mathbb{C}$ with $a, b \in \mathbb{R}$. Then z has degree ≤ 2 over \mathbb{R} , and that degree is 2 if and only if $b \neq 0$.

Indeed, if $b = 0$, then $z = a \in \mathbb{R}$ is a root of $T - a \in \mathbb{R}[T]$ so z has degree 1 over \mathbb{R} . Otherwise, z is a root of

$$p = (T - z)(T - \bar{z}) = T^2 - 2aT + (a^2 + b^2) \in \mathbb{R}[T].$$

Since p has roots z, \bar{z} , it has no real roots; since it has degree 2, p is irreducible over \mathbb{R} . Now the Corollary shows that p is the minimal polynomial of z .

Example 9.2.6. Let F be a field and let $F(X)$ be the field of fractions $Q(F[X])$ of the polynomial ring $F[X]$.

$F(X)$ is often called the field of rational functions over F ; its elements have the form

$$\frac{f}{g} = \frac{f(X)}{g(X)} \quad \text{for } f, g \in F[X]$$

Then the element $X \in F(X)$ is *transcendental* over F .

Indeed, given any non-zero polynomial $f(T) \in F[T]$, we wonder: is $f(X) = 0$? and of course, the answer is “no” because $f(X)$ is just the polynomial $f(T)$ after the substitution $T \mapsto X$.

In particular, the degree of X over F is undefined (or we could define it to be ∞).

Example 9.2.7. Consider the field $F = \mathbb{Q}(\sqrt{2})$ defined by adjoining to \mathbb{Q} a root of $T^2 - 2$. We identify F with a subfield of \mathbb{R} .

Consider the polynomial $p(T) = T^4 - 2$ and write $\alpha = 2^{1/4}$ for the positive real root of $p(T)$. Since $p \in \mathbb{Q}[T]$ is irreducible, α has degree 4 over \mathbb{Q} .

On the other hand, α has degree 2 over F . Indeed, note that in $F[T]$,

$$p(T) = T^4 - 2 = (T^2 - \sqrt{2})(T^2 + \sqrt{2}).$$

Since α is a root of $T^2 - \sqrt{2} \in F[T]$, the degree of α over F is ≤ 2 . To see that equality holds, we must argue that $T^2 - \sqrt{2}$ is irreducible over F .

To establish this irreducibility, we will argue that $T^2 - \sqrt{2}$ has no root in F .

A typical element of F has the form $x = a + b\sqrt{2}$ for $a, b \in \mathbb{Q}$.

Suppose that

$$(\diamond) \quad \sqrt{2} = x^2 = (a + b\sqrt{2})^2 = a^2 + 2b^2 + 2ab\sqrt{2}.$$

But then comparing coefficients we see that $a^2 + 2b^2 = 0$ and $2ab = 1$.

Now

$$a^2 + 2b^2 = 0 \implies a = b = 0 \implies 2ab \neq 1.$$

Thus the assumption (\diamond) is impossible and so

$$T^2 - \sqrt{2} \in F[T] = \mathbb{Q}(\sqrt{2})[T]$$

is indeed irreducible.

We repeat for emphasis:

- the minimal polynomial of α over \mathbb{Q} is $T^4 - 2$ and has degree 4,
- the minimal polynomial of α over $\mathbb{Q}(\sqrt{2})$ is $T^2 - \sqrt{2}$ and has degree 2.

9.3 Generation of extensions and primitive extensions

Definition 9.3.1. Let $S \subset E$ be a subset. The smallest subfield of E containing F and S is denoted by $F(S)$. If $S = \{u_1, u_2, \dots, u_n\}$ is a finite set, we often write $F(S) = F(u_1, \dots, u_n)$ for this field.

If $E = F(u_1, \dots, u_n)$ we say that the elements u_i *generate* the extension E of F .

If $n = 1$, the extension $F(u) = F(u_1)$ of F is said to be a *primitive extension* (or sometimes: a *simple extension*).

Remark 9.3.2. Remark: Note that $F(S)$ is equal to the intersection

$$F(S) = \bigcap_{K \in \mathcal{E}} K$$

of the collection

$$\mathcal{E} = \{K \subset E \mid K \text{ a subfield of } E \text{ containing } F \text{ and } S\}.$$

Since the intersection of subfields is again a subfield (check!), the notation $F(S)$ is meaningful.

Remark 9.3.3. Note that by definition

$$F(u_1, u_2, \dots, u_n) = F(u_1, u_2, \dots, u_{n-1})(u_n).$$

So to “describe” the extension $F \subset F(u_1, \dots, u_n)$ we can focus on describing primitive extensions. Given a description of primitive extensions, we can first describe the extension $F \subset F(u_1)$ of F , next we can describe the extension $F(u_1) \subset F(u_1)(u_2)$ of $F(u_1)$, and so on.

Proposition 9.3.4. *Let $\alpha \in E$.*

a. If α is algebraic over F with minimal polynomial $p \in F[T]$ over F , then

$$F(\alpha) \simeq F[T]/\langle p \rangle,$$

where α identifies with $T + \langle p \rangle$.

In particular, $F(\alpha)$ has as an F -basis the elements

$$1, \alpha, \dots, \alpha^{n-1}$$

where $n = \deg p = \deg \alpha$.

b. If α is transcendental over F , then $F(\alpha) \simeq F(T)$ where $F(T)$ is the field of fractions of the polynomial ring $F[T]$.

Proof. Construct the homomorphism

$$\phi : F[T] \rightarrow E \quad \text{such that } \phi|_F \text{ is the identity, and } \phi(T) = \alpha.$$

We are going to argue in both case (a) and (b) that ϕ induces the desired isomorphism.

First consider case (a). Suppose that α is algebraic with minimal polynomial p . The previous Proposition now shows that $\ker \phi = \langle p \rangle$.

Since p is irreducible, the quotient $F[T]/\langle p \rangle$ is a *field*. According to the first isomorphism theorem, ϕ induces an isomorphism between $F[T]/\langle p \rangle$ and its image K . Thus $K \subset E$ is a subfield containing F and α , so by definition $F(\alpha) \subset K$.

On the other hand, α identifies with the class $T + \langle p \rangle$, and so we've seen that the elements $1, \alpha, \dots, \alpha^{n-1}$ form an F -basis for K viewed as a vector space over F . Now, any subfield K_1 of E containing F and α must contain all F -linear combinations of the elements α^i ; thus $K \subset K_1$ and this proves that

$$K \subset F(\alpha) = \bigcap_{K_1 \in \mathcal{E}} K_1.$$

We now conclude that $K = F(\alpha)$ as required.

Now consider case (b). The condition that α is transcendental is equivalent to the requirement that $\ker \phi = \{0\}$.

Thus for any non-zero polynomial $f \in F[T]$, $\phi(f) = f(\alpha)$ is a non-zero element of $F(\alpha)$. In particular, $f(\alpha)^{-1} \in E$.

Now the *defining property* of the field of fractions gives a unique ring homomorphism $\tilde{\phi} : F(T) \rightarrow E$ for which $\tilde{\phi}|_{F[T]} = \phi$.

Since $F(T)$ is a field, $\tilde{\phi}$ is one-to-one, and its image is a subfield of E containing α . On the other hand, any subfield of E containing α must contain the image of $\tilde{\phi}$ and statement (b) follows at once. \square

Example 9.3.5. For any transcendental number $\gamma \in \mathbb{R}$, the subfield $\mathbb{Q}(\gamma)$ of \mathbb{R} is isomorphic to the field $\mathbb{Q}(T)$ of rational functions.

In particular, Proposition 9.3.4 shows that there is an isomorphism $\mathbb{Q}(e) \simeq \mathbb{Q}(\pi)$.

Remark 9.3.6. Here is a question we'll answer in an upcoming lecture. As before, let $F \subset E$ be a field extension.

If $\alpha, \beta \in E$ are algebraic over F , is $\alpha + \beta$ algebraic over F ? How about $\alpha \cdot \beta$?

Example 9.3.7. Let $E = \mathbf{Q}[T]/\langle T^3 - 2 \rangle$ and let $\gamma = T + \langle T^3 - 2 \rangle$. Of course, $E \simeq \mathbf{Q}(\sqrt[3]{2})$ and under this isomorphism, γ is mapped to $\sqrt[3]{2}$. Put another way, γ is a root of $T^3 - 2$ in F .

We recall that since $T^3 - 2$ has degree 3, E has dimension 3 as a \mathbf{Q} -vector space, and $\{1, \gamma, \gamma^2\}$ is a \mathbf{Q} -basis for E .

For an element $\alpha = a + b\gamma + c\gamma^2$ consider the \mathbf{Q} -linear mapping

$$\lambda_\alpha : E \rightarrow E$$

given by the left multiplication with α ; i.e. by the rule $\lambda_\alpha(\beta) = \alpha \cdot \beta$ for $\beta \in E$.

We are going to compute the *matrix* of λ_α in the above basis for E . For this, note that the choice of basis determines a linear isomorphism $\phi : E \rightarrow \mathbf{Q}^3$ given by $\phi(s + t\gamma + u\gamma^2) = \begin{bmatrix} s \\ t \\ u \end{bmatrix}$.

So we are looking for a 3×3 matrix $M = M_\alpha$ with the property that

$$\phi(\lambda_\alpha(\beta)) = M \cdot \phi(\beta).$$

- $\lambda_\alpha(1) = \alpha$ so that $\phi(\lambda_\alpha(1)) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. This is the first column of M .
- $\lambda_\alpha(\gamma) = \alpha\gamma = a\gamma + b\gamma^2 + c\gamma^3 = a\gamma + b\gamma^2 + 2c = 2c + a\gamma + b\gamma^2$ so that $\phi(\lambda_\alpha(\gamma)) = \begin{bmatrix} 2c \\ a \\ b \end{bmatrix}$.
This is the second column of M .
- $\lambda_\alpha(\gamma^2) = \alpha\gamma^2 = a\gamma^2 + b\gamma^3 + c\gamma^4 = a\gamma^2 + 2b + 2c\gamma = 2b + 2c\gamma + a\gamma^2$ so that $\phi(\lambda_\alpha(\gamma^2)) = \begin{bmatrix} 2b \\ 2c \\ a \end{bmatrix}$.
This is the third column of M .

Thus

$$M = M_\alpha = M_{a+b\gamma+c\gamma^2} = \begin{bmatrix} a & c & 2b \\ b & a & 2c \\ c & b & a \end{bmatrix}$$

We claim for $\alpha_1, \alpha_2 \in E$ that $M_{\alpha_1+\alpha_2} = M_{\alpha_1} + M_{\alpha_2}$ and $M_{\alpha_1 \cdot \alpha_2} = M_{\alpha_1} \cdot M_{\alpha_2}$. Since M_α is the matrix determined by the linear transformation λ_α , our claim will follow if we just observe that $\lambda_{\alpha_1} + \lambda_{\alpha_2} = \lambda_{\alpha_1+\alpha_2}$ and $\lambda_{\alpha_1} \circ \lambda_{\alpha_2} = \lambda_{\alpha_1 \cdot \alpha_2}$ (where \circ denotes the *composition* of linear transformations). But for $\beta \in E$ notice that $\lambda_{\alpha_1} \circ \lambda_{\alpha_2}(\beta) = \lambda_{\alpha_1}(\alpha_2\beta) = \alpha_1\alpha_2\beta = \lambda_{\alpha_1\alpha_2}(\beta)$; the other verification is similarly straightforward.

This proves that $\alpha \mapsto M_\alpha$ determines a *ring homomorphism*

$$E \rightarrow \text{Mat}_{3 \times 3}(\mathbf{Q})$$

Consider the element $1 + \gamma \in E$ and notice that $M_{1+\gamma} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Now, we can compute the inverse matrix $M_{1+\gamma}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ which we recognize

as the matrix $M_{(1-\gamma+\gamma^2)/3}$.

Thus we see that

$$\frac{1}{1+\gamma} = \frac{1}{3}(1-\gamma+\gamma^2)$$

\

9.4 The degree of a field extension

Definition 9.4.1. We write $[E : F] = \dim_F E$ and say that $[E : F]$ is the *degree* of the extension $F \subset E$.

If E is *not* a finite dimensional vector space over F , then $[E : F] = \dim_F E = \infty$.

Proposition 9.4.2. *Let $\alpha \in E$. Then α is algebraic over F if and only if $[F(\alpha) : F] < \infty$.*

Remark 9.4.3. If α is transcendental, the cardinality of an F -basis for $F(\alpha)$ fails to be countable if F is uncountable. Indeed, you can show that the elements

$$\left\{ \frac{1}{T-a} \in F(T) \mid a \in F \right\}$$

are linearly independent.

Proposition 9.4.4. *Let E be an extension of the field F and let $\alpha \in E$. The following are equivalent:*

- a. α is algebraic over F .*
- b. the primitive extension $F(\alpha)$ is a finite extension of F .*
- c. $\alpha \in E_1$ for some subfield $E_1 \subset E$ with $F \subset E_1$ which is a finite extension of F .*

Proof. a. \implies b: If α is algebraic, let $d = \deg \alpha$ be the degree of α over F . We have seen that $1, \alpha, \dots, \alpha^{d-1}$ form an F -basis for $F(\alpha)$, so $[F(\alpha) : F] = d$ and thus $F(\alpha)$ is indeed a finite extension of F .

b. \implies c: This is immediate; just take $E_1 = F(\alpha)$.

c. \implies a.: Assume $\dim_F E_1 = d$. Since $\alpha \in E_1$ and E_1 is a field, also $\alpha^i \in E_1$ for all $i \in \mathbb{Z}_{\geq 0}$. Since E_1 has dimension d over F , it follows from linear algebra that the $d+1$ elements

$$1, \alpha, \dots, \alpha^{d-1}, \alpha^d$$

are linearly dependent over F . Let $c_0, c_1, \dots, c_d \in F$ not all zero be such that

$$\sum_{i=0}^d c_i \alpha^i = 0$$

and consider the *polynomial*

$$f(T) = \sum_{i=0}^d c_i T^i \in F[T].$$

Since not all of the coefficients c_i are 0, $f(T) \neq 0$. Since $f(\alpha) = 0$, we have proved that α is algebraic over F as required. \square

Proposition 9.4.5. *Let $F \subset E \subset K$ be fields where K is a finite extension of E and E is a finite extension of F . Then K is a finite extension of F and moreover:*

$$[K : F] = [K : E] \cdot [E : F].$$

Proof. Let

$$a_1, \dots, a_N \in E \quad \text{be an } F\text{-basis for } E$$

and let

$$b_1, \dots, b_M \in K \quad \text{be an } E\text{-basis for } K$$

Multiplying in the field K , we consider the elements $a_s b_t$, and we assert:

$$\mathcal{B} = \{a_s b_t \mid 1 \leq s \leq N, 1 \leq t \leq M\} \quad \text{is an } F\text{-basis for } K$$

- \mathcal{B} spans K over F : indeed, let $x \in K$. We must express x as a linear combination of the vectors \mathcal{B} .

Since the $\{b_t\}$ span K over E , we may write

$$x = u_1 b_1 + \cdots + u_M b_M \quad \text{for } u_t \in E.$$

Since the $\{a_s\}$ span E over F , for each $1 \leq t \leq M$ we may write

$$u_t = v_{1,t} a_1 + \cdots + v_{N,t} a_N \quad \text{for } v_{s,t} \in F$$

Now

$$x = \sum_{t=1}^M u_t b_t = \sum_{t=1}^M \left(\sum_{s=1}^N v_{s,t} a_s \right) b_t = \sum_{1 \leq s \leq N, 1 \leq t \leq M} v_{s,t} \cdot a_s b_t$$

- \mathcal{B} is linearly independent over F .

Suppose that

$$0 = \sum_{1 \leq s \leq N, 1 \leq t \leq M} v_{s,t} \cdot a_s b_t = \sum_{t=1}^M \left(\sum_{s=1}^N v_{s,t} a_s \right) b_t$$

for coefficients $v_{s,t} \in F$.

Now use the fact that $\{b_t\}$ are linearly independent over E to conclude for each $1 \leq t \leq M$ that

$$0 = \sum_{s=1}^N v_{s,t} a_s$$

For any $1 \leq t \leq M$, use the fact that $\{a_s\}$ are linearly independent over F to conclude for each $1 \leq s \leq N$ that $v_{s,t} = 0$.

□

Corollary 9.4.6. *Let E be a finite extension of F . If $\alpha \in E$ then the degree of α over F is a divisor of $[E : F]$:*

$$\deg_F(\alpha) \mid [E : F].$$

Proof. Apply Proposition 9.4.5 to the tower of field extensions

$$F \subset F(\alpha) \subset E$$

to deduce that

$$[E : F] = [E : F(\alpha)] \cdot [F(\alpha) : F]$$

and the result follows since $[F(\alpha) : F] = \deg_F \alpha$.

□

9.5 Examples of finite extensions

Example 9.5.1. $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

The polynomials $T^2 - 2, T^2 - 3 \in \mathbb{Q}[T]$ are known to be irreducible over \mathbb{Q} (can you give a quick argument?)

We claim that $T^2 - 3$ remains irreducible over $\mathbb{Q}(\sqrt{2})$ –i.e. that $T^2 - 3 \in \mathbb{Q}(\sqrt{2})[T]$ is irreducible.

If we verify the claim, it follows that

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$$

and thus

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] \cdot [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4$$

as required.

Let's now prove the claim. Since $T^2 - 3$ has degree 2, the irreducibility will follow provided we argue that $T^2 - 3$ has no root in $\mathbb{Q}(\sqrt{2})$.

So: suppose that $3 = (a + b\sqrt{2})^2$ for $a, b \in \mathbb{Q}$. Thus

$$3 + 0 \cdot \sqrt{2} = 3 = a^2 + 2b^2 + 2ab\sqrt{2}$$

and comparing coefficients we find that

$$3 = a^2 + 2b^2 \quad \text{and} \quad 0 = 2ab.$$

Now $2ab = 0 \implies a = 0$ or $b = 0$ and the equation $3 = a^2 + 2b^2$ is then impossible (since neither 3 nor 3/2 is a square in \mathbb{Q}). This completes the proof that $T^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$.

Example 9.5.2. $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 4$.

To prove the claim, we argue that

$$\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3});$$

the assertion then follows from the previous example.

Write $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. To confirm this equality, first note that trivially we have

$$K \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$$

so it is enough to argue

$$\sqrt{2}, \sqrt{3} \in K.$$

(Why?)

In fact, it is easy to see that $\sqrt{2} \in K \iff \sqrt{3} \in K$ (since $\sqrt{2} + \sqrt{3} \in K$ by construction!).

So it only remains to argue e.g. that $\sqrt{3} \in K$.

Let's observe that

$$\frac{1}{\sqrt{2} + \sqrt{3}} = \frac{1}{\sqrt{2} + \sqrt{3}} \cdot \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3} - \sqrt{2}}{1} \in K$$

and since K is a field,

$$\frac{1}{\sqrt{2} + \sqrt{3}} + \sqrt{2} + \sqrt{3} = (\sqrt{3} - \sqrt{2}) + (\sqrt{2} + \sqrt{3}) = 2\sqrt{3} \in K$$

so indeed $\sqrt{3} \in K$.

The preceding calculation confirms (for example) that $\sqrt{2}$ may be written in the form

$$\begin{aligned}\sqrt{2} &= a + b\alpha + c\alpha^2 + d\alpha^3 \\ &= a + b(\sqrt{2} + \sqrt{3}) + c(\sqrt{2} + \sqrt{3})^2 + d(\sqrt{2} + \sqrt{3})^3\end{aligned}$$

for some coefficients $a, b, c, d \in \mathbb{Q}$, though we'd need to do some work to find a, b, c, d .

9.6 Algebraic extensions

Let $F \subset E$ be any extension of fields. We are going to argue that

$$E_{\text{alg}} = \{u \in E \mid u \text{ is algebraic over } F\}$$

is a subfield of E .

For example, this requires us to know that if $x, y \in E_{\text{alg}}$ then $x - y \in E_{\text{alg}}$. It is not completely clear how to find an algebraic equation satisfied by $x - y$, so we use a more indirect argument.

Our main tool is the following:

Lemma 9.6.1. *Let $\alpha, \beta \in E$ be algebraic. Then $[F(\alpha, \beta) : F]$ is a finite extension. In particular, $\alpha \pm \beta$ and $\alpha \cdot \beta$ are algebraic over F ; if $0 \neq \alpha$, then also $\alpha^{-1} = \frac{1}{\alpha}$ is algebraic over F .*

Proof. Indeed, β is algebraic over F hence algebraic over $F(\alpha)$ so

$$[F(\alpha, \beta) : F(\alpha)] < \infty$$

since $F(\alpha, \beta) = F(\alpha)(\beta)$.

Since α is algebraic over F , $[F(\alpha) : F] < \infty$ and thus

$$[F(\alpha, \beta) : F] = [F(\alpha, \beta) : F(\alpha)] \cdot [F(\alpha) : F]$$

is finite. The result now follows from Proposition 9.4.4. \square

Corollary 9.6.2. *Let E be an extension field of F . The set of all elements of E which are algebraic over F forms a subfield E_{alg} of E .*

Proof. We first observe that E_{alg} is an additive subgroup of E . For this, note that $0 \in E_{\text{alg}}$ so it just remains to show that if $x, y \in E_{\text{alg}}$ then $x - y \in E_{\text{alg}}$. But this statement follows from the Lemma 9.6.1.

It now remains to argue that E_{alg} is closed under multiplication and contains the inverse of its non-zero elements. These statements again follow from Lemma 9.6.1. \square

Definition 9.6.3. An extension field E of F is *algebraic* over F if each element of E is algebraic over F .

Proposition 9.6.4. *Every finite extension of fields is algebraic.*

Proof. Let $F \subset E$ be a finite extension and let $\alpha \in E$ be an arbitrary element of E . Since $[F(\alpha) : F]$ is a divisor of $[E : F]$, $[F(\alpha) : F]$ is finite and hence α is algebraic by Proposition 9.4.4. This shows that E is algebraic over F as required. \square

Lemma 9.6.5. *Let $F \subset E$ be an algebraic extension, and let $\alpha_1, \dots, \alpha_n \in E$. Then*

$$[F(\alpha_1, \dots, \alpha_n) : F] < \infty.$$

Proof. Proceed by induction on $n \geq 1$.

First consider the case $n = 1$. Since E is algebraic over F , $\alpha = \alpha_1$ is algebraic over F and $[F(\alpha) : F]$ is finite by previous results.

Now suppose $n > 1$ and write $E_i = F(\alpha_1, \dots, \alpha_i)$ for $1 \leq i \leq n$. The induction hypothesis is then: $[E_i : F] < \infty$ for $i < n$. Note that $E_n = E_{n-1}(\alpha_n)$, and – since α_n is algebraic over F – α_n is algebraic over E_{n-1} . Thus

$$[E_n : E_{n-1}] = [E_{n-1}(\alpha_n) : E_{n-1}] < \infty$$

by Proposition 9.4.4 and it follows by induction that

$$[E_n : F] = [E_n : E_{n-1}] \cdot [E_{n-1} : F] < \infty$$

as required. \square

Proposition 9.6.6. *Let E be an algebraic extension of F and let K be an algebraic extension of E . Then K is an algebraic extension of F .*

Proof. Let $\alpha \in K$. We must argue that α is algebraic over F . Since α is algebraic over E , it is the root of some polynomial

$$f(T) = a_0 + a_1T + a_2T^2 + \dots + a_NT^N \quad a_i \in E.$$

Now, form the extension $E_1 = F(a_0, a_1, \dots, a_N)$. Since E is algebraic over F , all a_i are algebraic over F . It follows from Lemma 9.6.5 that $[E_1 : F] < \infty$. Since α is algebraic over E_1 we know that $[E_1(\alpha) : E_1] < \infty$ by Proposition 9.4.4. It now follows that

$$[E_1(\alpha) : F] = [E_1(\alpha) : E_1][E_1 : F] < \infty$$

so that α is algebraic over F by Proposition 9.6.4. \square

9.7 Another example

Consider the field $K = \mathbb{Q}(T)$ where T is transcendental over \mathbb{Q} . It follows from Theorem 7.4.1 that

$$X^n - T - a \in K[X] = \mathbb{Q}(T)[X]$$

is irreducible for $n = 2, 3$ for any $a \in \mathbb{Q}$.

These irreducibility statements mean that

$$[K(\sqrt{T-a}) : K] = 2 \quad \text{and} \quad [K(\sqrt[3]{T-a}) : K] = 3$$

(or writing everything out in full detail, that

$$[\mathbb{Q}(T, \sqrt{T-a}) : \mathbb{Q}(T)] = 2 \quad \text{and} \quad [\mathbb{Q}(T, \sqrt[3]{T-a}) : \mathbb{Q}(T)] = 3.)$$

Lemma 9.7.1. $K(\sqrt{T-a}, \sqrt[3]{T-a}) = \mathbb{Q}(T, \sqrt{T-a}, \sqrt[3]{T-a})$ has degree 6 over $K = \mathbb{Q}(T)$.

Proof. Let $L = K(\sqrt{T-a}, \sqrt[3]{T-a})$. The claim will follow if we show that

$$(\clubsuit) \quad [L : K(\sqrt{T-a})] = 3$$

since then

$$[L : K] = [L : K(\sqrt{T-a})] \cdot [K(\sqrt{T-a}) : K] = 3 \cdot 2 = 6.$$

Now, (\clubsuit) follows if we argue that $f(X) = X^3 - T - a \in K(\sqrt{T-a})[X]$ is irreducible; since f has degree 3, it suffices to argue that f has no root in $K(\sqrt{T-a})$.

But were $\alpha \in K(\sqrt{T-a})$ a root of f , we know that α has degree 3 over K . But this is impossible since

$$\alpha \in K(\sqrt{T-a}) \implies \deg_K \alpha \mid [K(\sqrt{T-a}) : K] = 2.$$

This completes the proof that f is irreducible over $K(\sqrt{T-a})$ and thus the Lemma is verified. \square

10 Constructible real numbers

As an example of the utility of field theory, we are going to describe a field-theory-based answer to a “geometric-constructions/geometric” question about numbers. Loosely put, we are going to answer the question: “can one trisect an angle using ruler and compass?”

10.1 Ruler and compass constructions

As a starting point, we are given two points at *unit distance* in the Euclidean plane.

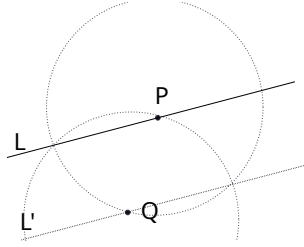
Given any two distinct known points P and Q , one can construct:

- the *line* through P and Q (this uses a *straightedge*)
- the circle with center P which passes through Q (this uses a *compass*)

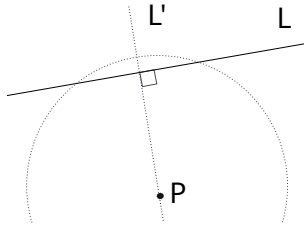
One views the points of intersection of lines and circles that have been constructed as *constructible* (i.e. known) points.

Here are some useful constructions that we are going to use without further argumentation:

Lemma 10.1.1. (\clubsuit) *Given a point P on a line L , and a second point Q not on L , we can construct a line L' parallel to L passing through Q .*

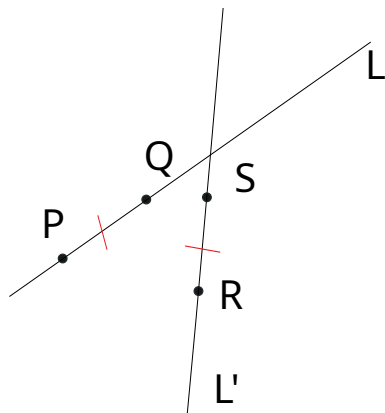


Lemma 10.1.2. (\heartsuit) *Given a line L and a point P not lying on L , one can construct a line L' containing P and perpendicular to L .*



Lemma 10.1.3. (\spadesuit) *Given two points $P \neq Q$ on a line L , a second line L' , and a point R on L' , we can construct a point S on L' such that*

$$|\overline{PQ}| = |\overline{RS}|.$$



10.2 Constructions

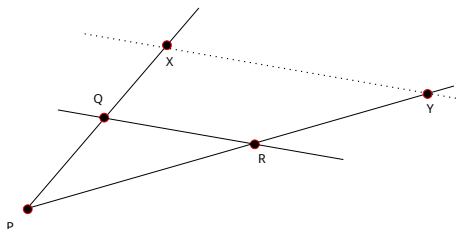
Definition 10.2.1. A real number r is *constructible* if one can construct a line segment of length $|r|$ using straightedge and compass.

Proposition 10.2.2. *The set of constructible real numbers forms a subfield $C \subset \mathbb{R}$.*

Sketch of proof. First, use Lemma 10.1.3 to show that C forms an additive subgroup of \mathbb{R} .

To argue that C is closed under multiplication, proceed as follows:

- Given positive constructible numbers y, z, w construct a diagram with points P, Q, R, Y as follows with $|PQ| = z$, $|PR| = w$ and $|PY| = y$.



- Now use (\clubsuit) to construct the line through Y parallel to the line through Q and R .
- Writing X for the (constructible) point of intersection of the indicated lines, write $x = |PX|$ and notice that $x/y = z/w$.
- Now let $a, b > 0$ be constructible and let $y = a$, $z = b$ and $w = 1$; the above argument shows that $x = yz = ab$ is constructible.

Similar arguments give the constructibility of a/b where $a, b > 0$ are constructible. \square

Let's observe that according to the Proposition, every rational number is constructible.

We may and will suppose that the points $(1, 0)$ and $(0, 1)$ in the plane are constructible. In particular, the coordinates r, s of any constructible point $P = (r, s)$ are constructible real numbers.

10.3 Lines and Circles over a field

Of course, any line may be described as the set of solutions to an equation

$$aX + bY + c = 0$$

for $a, b, c \in \mathbb{R}$, and any circle may be described as the solutions to an equation

$$X^2 + Y^2 + aX + bY + c = 0$$

for $a, b, c \in \mathbb{R}$.

If F is a subfield of \mathbb{R} , a *line over F* means a line with equation $aX + bY + c = 0$ where $a, b, c \in F$.

Similarly, a *circle over F* means a circle with equation

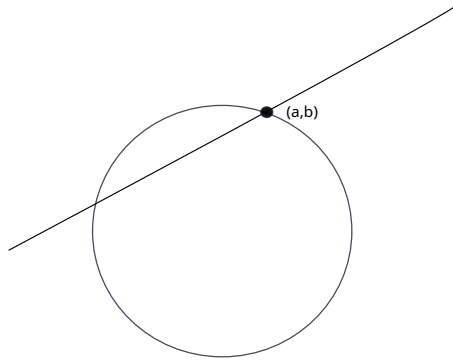
$$X^2 + Y^2 + aX + bY + c = 0 \quad \text{where } a, b, c \in F.$$

Lemma 10.3.1. • If the points $P \neq Q$ both have coordinates in F , the line through P and Q is a line over F .

- If C is circle for which both the radius and the coordinates of its center are all in F , then C is a circle over F .

Constructing points via ruler and compass amounts to finding the intersections of lines and circles. We record the following fact about such intersections:

Proposition 10.3.2. Let $F \subset \mathbb{R}$ be a subfield. The coordinates of the points of intersection of lines over F and circles over F belong to the field $F(\sqrt{u})$ for some $u \in F$.



If in this diagram the line and the circle are “over F ”, the conclusion is that $a, b \in F(\sqrt{u})$ for a suitable $u \in F$.

10.4 Characterizing constructible numbers

Using Proposition 10.3.2, we can give an important characterization of constructible real numbers:

Theorem 10.4.1. $u \in \mathbb{R}$ is constructible \iff there are $u_1, \dots, u_n \in \mathbb{R}$ such that:

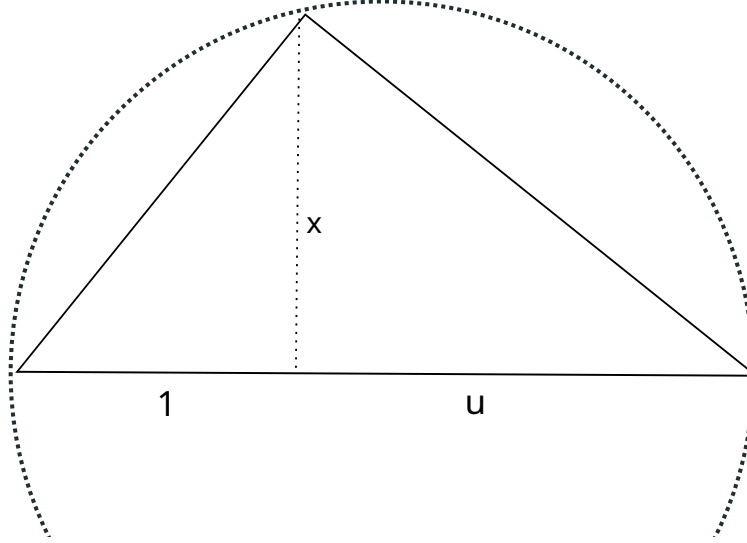
- $u_1^2 \in \mathbb{Q}$,

b. $u_i^2 \in \mathbb{Q}(u_1, \dots, u_{i-1})$ for $2 \leq i \leq n$, and

c. $u \in \mathbb{Q}(u_1, \dots, u_n)$.

Proof. \Rightarrow : This follows from the Proposition.

\Leftarrow : Use the following: if F is any subfield of the field of constructible numbers, then \sqrt{u} is constructible for each positive $u \in F$. For this, construct a circle of diameter $1 + u$, and a line perpendicular to the diameter, intersecting the diameter 1 unit from the west pole:



Then $x = \sqrt{u}$. □

Corollary 10.4.2. *If u is a constructible real number, then u is algebraic over \mathbb{Q} and $\deg(u)$ is a power of 2.*

10.5 Angle trisection

Lemma 10.5.1. a. *For any angle θ , we have the following identities:*

$$4 \cos^3(\theta) - 3 \cos(\theta) - \cos(3\theta) = 0.$$

b. *Let $\alpha = \cos\left(\frac{\pi}{9}\right)$. α is a root of the irreducible polynomial*

$$f(T) = 8T^3 - 6T - 1 \in \mathbb{Q}[T].$$

In particular, the degree of α over \mathbb{Q} is 3.

c. α is not a constructible number.

Proof. Recall the trigonometric identities:

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta) \tag{10.1}$$

and

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta). \tag{10.2}$$

Taking $\alpha = \beta$ we get

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

and

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha).$$

For a real number θ , we find that “double angle formula”

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos^2(\theta) - (1 - \cos^2(\theta)) \\ &= 2\cos^2(\theta) - 1\end{aligned}$$

This shows that

$$2\cos^2(\theta) - \cos(2\theta) - 1 = 0 \tag{10.3}$$

To prove (a), let $\alpha = 2\theta$ and $\beta = \theta$ in (10.2); we get

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta + \theta) \\ &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\ &= (2\cos^2(\theta) - 1)\cos(\theta) - (2\sin(\theta)\cos(\theta))\sin(\theta) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta)\sin^2(\theta) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta)(1 - \cos^2(\theta)) \\ &= 4\cos^3(\theta) - 3\cos(\theta).\end{aligned}$$

This shows that $4\cos^3(\theta) - 3\cos(\theta) - \cos(3\theta) = 0$ as required.

We now prove (b). If $\theta = \frac{\pi}{9}$, then of course $\cos(3\theta) = \frac{1}{2}$, so (a) shows θ to be a root of the equation $4T^3 - 3T - \frac{1}{2} \in \mathbb{Q}[T]$. Multiplying this polynomial by 2 gives $8T^3 - 6T - 1$ and we can use the *rational roots test* Theorem 7.2.1 to confirm the that this polynomial has no root in \mathbf{Q} and is thus irreducible in $\mathbb{Q}[T]$.

Now (c) follows from Corollary 10.4.2, since $3 \nmid 2^m$ for any $m \geq 1$. \square

Theorem 10.5.2. *It is impossible to find a general construction for trisecting an angle.*

Proof. Since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, one can construct points $Q = \frac{1}{2}(1, \sqrt{3})$, $P = (0, 0)$, $R = (1, 0)$ and then $\angle QPR$ is $\frac{\pi}{3}$.

We claim that one can't construct further points S, T such that the $\angle QPS$, $\angle SPT$ and $\angle TPR$ are all equal.

Indeed, if it were so, the coordinates of T would be $(\cos\left(\frac{\pi}{9}\right), \sin\left(\frac{\pi}{9}\right))$, and then $\cos\left(\frac{\pi}{9}\right)$ would be a constructible number, contrary to Lemma 10.5.1. \square

11 Splitting fields

11.1 The notion of a splitting field

Let F be a field and consider a polynomial

$$f = a_0 + a_1T + \cdots + a_nT^n \in F[T]$$

of degree $n \geq 1$.

Definition 11.1.1. If E is an extension field of F , we say that f *splits over* E provided that there are elements $r_1, \dots, r_n \in E$ such that

$$f = (T - r_1)(T - r_2) \cdots (T - r_n) = \prod_{i=1}^n (T - r_i) \in E[T].$$

Definition 11.1.2. If f splits over the field extension E of F , and if $r_1, \dots, r_n \in E$ are the roots of f , we say that E is a *splitting field* for f over F if moreover $E = F(r_1, \dots, r_n)$.

Thus a splitting field E is somehow a minimal field extension over which f splits.

Example 11.1.3. $E = \mathbb{Q}(i)$ is a splitting field over \mathbb{Q} for the polynomial $f = T^2 - 2T + 2$ since

$$f = (T - 1 - i)(T - 1 + i) \in \mathbb{Q}(i)[T]$$

and since $\mathbb{Q}(i) = \mathbb{Q}(1 + i, 1 - i)$.

Theorem 11.1.4. *Let $f \in F[T]$ has degree $n \geq 1$. Then there exists a splitting field E for f over F with $[E : F] \leq n!$.*

Proof. Proceed by induction on $n \geq 1$. The result holds when $n = 1$, since then f splits over $E = F$.

Now suppose that the result is known for all fields F and all polynomials of degree $\leq n - 1$.

Now, choose an irreducible factor p of f in $F[T]$, say of degree $d \leq n$. Choose a root of p in some field extension of F , and consider the field $K = F(\alpha)$. We know that $[K : F] = [F(\alpha) : F] = d = \deg p$.

Since α is a root of p , it is also a root of f ; thus by the *remainder theorem*, we may write

$$f = (T - \alpha) \cdot g \quad \text{for } g \in K[T] \text{ with } \deg g = n - 1.$$

Now use the *induction hypothesis* to construct a splitting field E for g over K with $[E : K] \leq (n - 1)!$.

Thus $E = K(r_2, \dots, r_n)$ and

$$g = \prod_{i=2}^n (T - r_i) \in E[T].$$

We now have

$$f = (T - \alpha) \cdot g = (T - \alpha) \cdot \prod_{i=2}^n (T - r_i) \in E[T];$$

thus, f splits over E . Moreover, $E = K(r_2, \dots, r_n) = F(\alpha, r_2, \dots, r_n)$ which confirms that E is a splitting field of f over F .

Finally, note that

$$[E : F] = [E : K][K : F] \leq (n - 1)! \cdot d \leq n!$$

since $d \leq n$. □

11.2 More examples of splitting fields

11.2.1 Fourth root of 2

The field $E = \mathbb{Q}(i, \sqrt[4]{2})$ is a splitting field for $f = T^4 - 2$ over \mathbb{Q} , and $[E : \mathbb{Q}] = 8$.

First, if we write $\alpha = \sqrt[4]{2}$ for the *real* fourth root of 2, the roots of f are precisely $\pm\alpha, \pm i\alpha$. Indeed,

$$(T - \alpha)(T + \alpha)(T - i\alpha)(T + i\alpha) = (T^2 - \sqrt{2})(T^2 + \sqrt{2}) = f.$$

Now, $E = \mathbb{Q}(i, \sqrt[4]{2}) = \mathbb{Q}(\pm\alpha, \pm i\alpha)$.

Finally, to see that $[E : \mathbb{Q}] = 8$, first note that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ since $T^4 - 2$ is irreducible over \mathbb{Q} .

Now $\alpha \in \mathbb{R} \implies \mathbb{Q}(\alpha) \subset \mathbb{R}$, so $\mathbb{Q}(\alpha)$ does not contain a root of $T^2 + 1$. Thus $T^2 + 1$ is irreducible over $\mathbb{Q}(\alpha)$

This shows that

$$[E : \mathbb{Q}] = [E : \mathbb{Q}(\alpha)] \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}] = 2 \cdot 4 = 8.$$

11.2.2 Transcendental extension

$E = \mathbb{C}(X, \sqrt[4]{X+1})$ is a splitting field over $\mathbb{C}(X)$ for $T^4 - (X+1)$, and $[E : \mathbb{C}(X)] = 4$.

11.2.3 Finite field example

Let $F = \mathbb{Z}_7$ be the field with 7 elements.

Let's describe the splitting field for $f = T^3 - 3 \in F[T]$ over F .

First, note that the cubes mod 7 are as follows:

In particular, $f = T^3 - 3$ has no root in $F = \mathbb{Z}_7$. So if α denotes a root of f in some extension field, then $F(\alpha)$ is a degree 3 extension of F .

Now let's notice that the multiplicative order of (the class of) 2 in \mathbb{Z}_7^\times is 3: indeed $2^3 = 8 \equiv 1 \pmod{7}$ but $2, 2^2 \not\equiv 1 \pmod{7}$. So we can observe that also 2α and 4α are roots of $T^3 - 2$.

Thus

$$f = (T - \alpha)(T - 2\alpha)(T - 4\alpha) \in F(\alpha)[T] = \mathbb{Z}_7(\alpha)[T]$$

This shows that $F(\alpha) = \mathbb{Z}_7(\alpha)$ is a splitting field over F of $f = T^3 - 3$.

Observe that $|F(\alpha)| = 7^3 = 343$; elements of $F(\alpha)$ all have the form

$$a + b\alpha + c\alpha^2 \quad a, b, c \in \mathbb{Z}_7.$$

11.2.4 Uniqueness of splitting fields

We are going to argue that a splitting field for a polynomial f over F is *essentially unique*.

Let us first make an observation: if $\theta : F \rightarrow F_1$ is an isomorphism of fields, then θ may be extended to an isomorphism

$$\theta : F[T] \rightarrow F_1[T]$$

with the property that $\theta(T) = T$. Note that polynomials satisfy

$$p \in F[T] \text{ is irreducible} \iff \theta(p) \in F_1[T] \text{ is irreducible.}$$

Lemma 11.2.1. *Let $\theta : F \rightarrow F_1$ be an isomorphism of fields, let $E = F(u)$ where u is algebraic over F with minimal polynomial $p \in F[T]$, and let $p_1 = \theta(p)$. If v is a root of p_1 in an extension field of F_1 , there is a unique way of extending θ to an isomorphism $\phi : F(u) \rightarrow F_1(v)$ subject to the conditions:*

- $\phi(u) = v$
- $\phi|_F = \theta$.

This diagram might be useful for visualizing the situation:

$$\begin{array}{ccc} F(u) & \xrightarrow{\phi} & F_1(v) \\ \uparrow & \circ & \uparrow \\ F & \xrightarrow{\theta} & F_1 \end{array}$$

Proof. We first observe that ϕ is uniquely determined by the indicated conditions. Indeed, $F(u)$ is spanned as F -vector space by elements of the form u^i , and since ϕ is a ring homomorphism it must satisfy $\phi(u^i) = v^i$.

To see the existence of ϕ , first note that there are isomorphisms $F[T]/\langle p \rangle \xrightarrow{\gamma} F(u)$ and $F_1[T]/\langle p_1 \rangle \xrightarrow{\gamma_1} F_1(v)$ with

$$\psi(T + \langle p \rangle) = u \quad \text{and} \quad \gamma_1(T + \langle p_1 \rangle) = v$$

arising from the definition of minimal polynomial.

Now, consider the ring homomorphism $F[T] \xrightarrow{\theta} F_1[T] \xrightarrow{\pi_1} F_1[T]/\langle p_1 \rangle$ where π_1 is the quotient mapping. The mapping $\gamma_1 \circ \pi_1 \circ \theta : F[T] \rightarrow F_1(v)$ is onto and has kernel $\langle p \rangle$; according to the first isomorphism theorem, $\gamma_1 \circ \pi_1 \circ \theta$ induces an isomorphism

$$\Psi : F[T]/\langle p \rangle \rightarrow F_1(v)$$

satisfying $\Psi(T + \langle p \rangle) = v$.

Now $\phi = \psi^{-1} \circ \Psi$ satisfies $\phi(u) = v$ and $\phi|_F = \theta$ as required. \square

Proposition 11.2.2. *Let E be a splitting field over F for $f \in F[T]$, let $\theta : F \rightarrow F_1$ be a field isomorphism, and let $g = \theta(f) \in F_1[T]$. Let E_1 be a splitting field for g over F_1 . Then there is an isomorphism $\phi : E \rightarrow E_1$ such that $\phi|_F = \theta$.*

Proof. We use induction on $n = \deg f$. If $n = 1$, then $E = F$, $E_1 = F_1$ and we can simply take $\phi = \theta$.

Now suppose that $n > 1$ and that the result holds for all field F and all polynomials of degree $< n$.

Let $p \in F[T]$ be an irreducible factor of f , so that $q = \theta(p)$ is an irreducible factor of g .

Since f splits over E , also p splits over E . Choose a root $u \in E$ of p . Thus $F \subset F(u) \subset E$.

Choose also a root $v \in E_1$ of q , so that $F_1 \subset F_1(v) \subset E_1$.

According to the preceding Lemma, there is an isomorphism $\hat{\theta} : F(u) \rightarrow F_1(v)$ such that $\hat{\theta}|_F = \theta$ and such that $\hat{\theta}(u) = v$.

Write

$$f = (T - u)s \in F(u)[T] \quad \text{for } s \in F(u)[T]$$

and

$$g = (T - v)s_1 \in F_1(v)[T] \quad \text{for } s_1 \in F_1(v)[T]$$

Now, E is a splitting field for s over $F(u)$ and E_1 is a splitting field for s_1 over $F_1(v)$. And since $\theta(f) = g$ and $\hat{\theta}(u) = v$ it is easy to see that $\hat{\theta}(s) = s_1$.

Thus the induction hypothesis gives an isomorphism $\phi : E \rightarrow E_1$ such that $\phi|_{F(u)} = \hat{\theta}$. This isomorphism ϕ has the required properties. \square

We find the following theorem as an immediate consequence:

Theorem 11.2.3. *Let $f \in F[T]$ be a polynomial with $\deg f > 0$. If E and E_1 are splitting fields for f over F , there is an isomorphism $\phi : E \rightarrow E_1$ such that $\phi(a) = a$ for each $a \in F$ – i.e. such that $\phi|_F$ is the identity mapping.*

Proof. In the Proposition, just take θ to be the identity map! \square

Remark 11.2.4. Observe that the proof of Proposition 11.2.2 requires us to prove the statement involving θ , even though in Theorem 11.2.3 we are interested in only in the case $\theta = \text{id}$.

11.2.5 Example: automorphisms of a splitting field

The ideas behind the results Proposition 11.2.2 and Theorem 11.2.3 will be really important as we start talking about Galois theory. So, it seems useful to first do a non-trivial example.

Let's give an example of automorphisms of a splitting field.

Let's fix a prime number p , consider the polynomial $f = T^3 - p \in \mathbb{Q}[T]$, and let E be a splitting field for this polynomial over \mathbb{Q} .

The Theorem 11.2.3 tells us that any splitting field of f over \mathbb{Q} is isomorphic to E . Let's try to understand what this statement could mean about automorphisms of E .

First, let's make some observations. Notice that if β and β' are roots of f , then $\left(\frac{\beta}{\beta'}\right)^3 = 1$ i.e. $\frac{\beta}{\beta'}$ is a root of $T^3 - 1$. Moreover, $\frac{\beta}{\beta'} = 1$ if and only if $\beta = \beta'$.

Let's exclude the “trivial” cube root of unity; observe that

$$\frac{T^3 - 1}{T - 1} = T^2 + T + 1 \in \mathbb{Q}[T]$$

has roots $\omega, \omega^2 \in \mathbb{C}$ where

$$\omega = \exp\left(\frac{2\pi i}{3}\right) = \cos\left(\frac{2\pi i}{3}\right) + i \sin\left(\frac{2\pi i}{3}\right) \in \mathbb{C};$$

Notice that $\omega \neq 1$ and $\omega^3 = 1$ so viewed as an element of the group \mathbb{C}^\times , ω has order 3.

Neither ω nor ω^2 is rational, so $T^2 + T + 1$ is *irreducible* over \mathbb{Q} .

We can now construct a splitting field E of f over \mathbb{Q} *abstractly*. Take $E = \mathbb{Q}(\alpha, \omega)$ where α is a root of $T^3 - p$ and ω is a root of $T^2 + T + 1$.

First notice that

$$E = \mathbb{Q}(\alpha, \omega) = \mathbb{Q}(\alpha, \omega\alpha, \omega^2\alpha)$$

so that E is a splitting field. Now notice that $\deg_{\mathbb{Q}} \alpha = 3$ and $\deg_{\mathbb{Q}} \omega = 2$ so $T^2 + T + 1$ remains irreducible over $\mathbb{Q}(\alpha)$. Thus we may conclude that

$$[E : \mathbb{Q}] = [\mathbb{Q}(\alpha, \omega) : \mathbb{Q}(\alpha)] \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}] = 6.$$

Now observe that this argument actually shows that if we fix *any* root β of f in E , and *any* root ζ of $T^2 + T + 1$ in E then

$$f = (T - \beta)(T - \zeta\beta)(T - \zeta^2\beta).$$

E.g. if we choose $\zeta = \omega^2$ and $\beta = \omega\alpha$, then

$$f = (T - \beta)(T - \zeta\beta)(T - \zeta^2\beta) = (T - \omega\alpha)(T - \omega^2(\omega\alpha))(T - \omega^4(\omega\alpha))$$

since

$$\{\omega\alpha, \omega^2(\omega\alpha), \omega^4(\omega\alpha)\} = \{\omega\alpha, \omega^3\alpha, \omega^5\alpha\} = \{\omega\alpha, \alpha, \omega^2\alpha\}.$$

The thing to take home from all this is that there are some choices to be made in describing the roots of f . In this case, you could pin things down more precisely e.g. by taking for α the “real” cube root of P and for ω the complex root of $T^2 + T + 1$ which is in “quadrant 2”. But a more systematic way of keeping track of choices is through study of automorphisms of the splitting field E .

Notice that α and $\beta = \omega\alpha$ are roots of the irreducible polynomial $T^3 - p \in \mathbb{Q}[T]$. Thus, there is an isomorphism of fields

$$\theta : \mathbb{Q}(\alpha) \rightarrow \mathbb{Q}(\beta)$$

such that θ is the identity on \mathbb{Q} and $\theta(\alpha) = \beta = \omega\alpha$.

Notice that $\theta(T^2 + T + 1) = T^2 + T + 1$ is irreducible over $\mathbb{Q}(\alpha)$ and over $\mathbb{Q}(\beta)$.

Now, Lemma 11.2.1 tells us that there is an isomorphism

$$\Theta : \mathbb{Q}(\alpha, \omega) \rightarrow \mathbb{Q}(\beta, \zeta)$$

such that $\Theta|_{\mathbb{Q}(\alpha)} = \theta$ – i.e. for which $\Theta(\alpha) = \beta$ – and for which $\Theta(\omega) = \zeta$.

This Θ is an isomorphism between splitting fields of f . Since we took $\beta = \omega\alpha$ and $\zeta = \omega^2$, we have

$$E = \mathbb{Q}(\alpha, \omega) = \mathbb{Q}(\beta, \zeta)$$

so in fact $\Theta : E \rightarrow E$ is an *automorphism* of E .

Note that Θ is not the identity mapping on the roots of f :

$$(\Theta(\alpha), \Theta(\omega\alpha), \Theta(\omega^2\alpha)) = (\omega\alpha, \zeta\omega\alpha, \zeta^2\omega\alpha) = (\omega\alpha, \alpha, \omega^2\alpha).$$

Also note that upon restriction to $\mathbb{Q}(\omega)$, $\Theta|_{\mathbb{Q}(\omega)}$ is *complex conjugation*, since

$$\Theta(\omega) = \omega^2 = \bar{\omega}.$$

12 Finite fields

12.1 The prime subfield of a field

First let's recall for any field F that there is always a ring homomorphism

$$\mathbb{Z} \rightarrow F$$

for which $n \mapsto n \cdot 1_F$.

Proposition 12.1.1. *Let F be a field.*

- a. *If the homomorphism $\mathbb{Z} \rightarrow F$ is one-to-one, then F contains a copy of the field \mathbb{Q} of rational numbers.*
- b. *If the homomorphism $\mathbb{Z} \rightarrow F$ is not one-to-one, then F contains a copy of the field $\mathbb{Z}/p\mathbb{Z}$ for some prime number p .*

Remark 12.1.2. In case a., we say that F has *characteristic 0*. Note in that case that the *additive order* of any non-zero element of F is ∞ .

In case b., we say that F has *characteristic p* . In that case, the additive order of any non-zero element of F is p .

Definition 12.1.3. The *prime subfield* of F is the smallest subfield containing the image of the homomorphism $\mathbb{Z} \rightarrow F$; thus when F has characteristic 0, the prime subfield identifies with \mathbb{Q} , and when F has characteristic $p > 0$, the prime subfield identifies with $\mathbb{Z}/p\mathbb{Z}$.

Proof of the Proposition. If the homomorphism $\phi : \mathbb{Z} \rightarrow F$ is injective, it maps non-zero elements of \mathbb{Z} to *invertible* elements of F . Thus by the *defining property* of the field of fractions $\mathbb{Q} = Q(\mathbb{Z})$, the homomorphism ϕ extends to a homomorphism $\hat{\phi} : \mathbb{Q} \rightarrow F$ and thus F indeed contains a copy of \mathbb{Q} .

On the other hand, if the homomorphism ϕ is not one-to-one, let

$$\{0\} \neq n\mathbb{Z} = \ker \phi.$$

The *First Isomorphism Theorem* now implies that the image of ϕ is a subring of F isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Since F is a field, this subring must be an integral domain; thus, we see that $n = p$ must be a prime number. \square

12.2 Finite fields

We've met some finite fields already, namely $\mathbb{Z}/p\mathbb{Z}$ for a prime number p .

We've can construct finite extensions of $\mathbb{Z}/p\mathbb{Z}$ to get fields F for which $|F|$ is not prime. Let's first make an observation about $|F|$, as follows:

Proposition 12.2.1. *Let F be a finite field. Then F has characteristic $p > 0$ for some prime number p . The number of elements of F is p^m for some whole number $m \geq 1$.*

Proof. Since \mathbb{Q} is not finite, the previous proposition shows that F must have characteristic $p > 0$ for a prime number p .

Write $F_0 \subset F$ where F_0 is the prime subfield; thus $F_0 \simeq \mathbb{Z}/p\mathbb{Z}$.

Now, F may be viewed as an F_0 -vector space. A basic theorem in linear algebra says that F must have a *basis* \mathcal{B} as an F_0 -vector space. Since F is finite, this basis must be finite; say $|\mathcal{B}| = m$.

Write $\mathcal{B} = \{b_1, b_2, \dots, b_m\}$. Then an element x of F may be written uniquely in the form

$$x = t_1 b_1 + t_2 b_2 + \dots + t_m b_m$$

for $t_i \in F_0$. Since $F_0 \simeq \mathbb{Z}/p\mathbb{Z}$, there are p choices for each t_i ; this shows that the number of elements of F is

$$|F| = p \cdot p \cdot \dots \cdot p = p^m$$

as required. \square

12.3 Finite fields as splitting fields over the prime field

Proposition 12.3.1. *Let F be a finite field with p^m elements for some prime number p . Then F is the splitting field over the prime subfield $F_0 \simeq \mathbb{Z}/p\mathbb{Z}$ of the polynomial*

$$T^{p^m} - T \in F_0[T].$$

Proof. Since F has p^m elements, the multiplicative group F^\times has $p^m - 1$ elements. This means that every element $x \in F^\times$ satisfies the condition

$$x^{p^m-1} = 1.$$

It is then immediate that every element $x \in F$ satisfies

$$x^{p^m} = x.$$

Put another way, every element of F is a root of the polynomial

$$f = T^{p^m} - T \in F_0[T].$$

Since f can have no more than p^m roots in an extension field, it follows that F contains all roots of f . Since F is generated by these roots, F is a splitting field for f over F_0 . \square

Remark 12.3.2. The proof shows that the identity

$$f = T^{p^m} - T = \prod_{\alpha \in F} (T - \alpha)$$

holds in $F[T]$.

Corollary 12.3.3. *Two finite fields F and E are isomorphic if and only if $|F| = |E|$.*

Proof. If F and E are isomorphic, there is a one-to-one onto function $\phi : F \rightarrow E$ and thus $|F| = |E|$.

On the other hand, if $|F| = |E|$, we know that $|F| = p^m$ and $|E| = q^n$ for some primes p, q and some $m, n \geq 1$. By unique factorization of integers, $p = q$ and $m = n$. Now the Proposition shows that E, F are splitting fields of $T^{p^m} - T$ over $\mathbb{Z}/p\mathbb{Z}$.

Now the existence of an isomorphism $F \simeq E$ is a consequence of the uniqueness of splitting fields. \square

12.4 Some examples of finite fields

We are going to argue (below) that for each prime power $q = p^n$, there is a field of that order. Thus from what we've already seen there is exactly one field of order q , up to isomorphism.

The computer algebra system `sagemath` knows how to compute finite fields. For example, we can ask to describe the field of $19^2 = 361$ elements.

```
H.<a>=FiniteField(19^2)
a.minpoly()

-> x^2 + 18*x + 2
```

The output here tells us that

$$H = \mathbb{Z}_{19}[T]/\langle T^2 + 18T + 2 \rangle.$$

We can construct larger finite fields and ask about `subfields`:

```
G.<z>=FiniteField(19^6)
z.minpoly()
->
x^6 + 17*x^3 + 17*x^2 + 6*x + 2
-----

G.subfields()
->
[(Finite Field of size 19,
  Ring morphism:
    From: Finite Field of size 19
    To:   Finite Field in z6 of size 19^6
    Defn: 1 |--> 1),
 (Finite Field in z2 of size 19^2,
  Ring morphism:
    From: Finite Field in z2 of size 19^2
    To:   Finite Field in z6 of size 19^6
    Defn: z2 |--> 18*z6^5 + 9*z6^4 + 5*z6^3 + 2*z6^2 + 12*z6 + 7),
 (Finite Field in z3 of size 19^3,
  Ring morphism:
    From: Finite Field in z3 of size 19^3
    To:   Finite Field in z6 of size 19^6
    Defn: z3 |--> 13*z6^5 + 10*z6^4 + 2*z6^3 + 15*z6^2 + 7*z6 + 18),
 (Finite Field in z6 of size 19^6,
  Ring morphism:
    From: Finite Field in z6 of size 19^6
    To:   Finite Field in z6 of size 19^6
    Defn: z6 |--> z6)]
```

The output here tells us that the field G of order $19^6 = 47045881$ – i.e. roughly forty seven million elements – has exactly 4 subfields: G , a subfield of order 19^3 , a subfield of order 19^2 and a subfield of order 19.

Here **sage** has found an element z for which

$$G = \mathbb{Z}_{19}(z) \simeq \mathbb{Z}_{19}[T]/\langle T^6 + 17 \cdot T^3 + 17 \cdot T^2 + 6 \cdot T + 2 \rangle,$$

The subfield

$$\mathbb{Z}_{19}(13 \cdot z^5 + 10 \cdot z^4 + 2 \cdot z^3 + 15 \cdot z^2 + 7 \cdot z + 18)$$

has order $19^3 = 6859$.

The subfield

$$\mathbb{Z}_{19}(18 \cdot z^5 + 9 \cdot z^4 + 5 \cdot z^3 + 2 \cdot z^2 + 12 \cdot z + 7)$$

has order $19^2 = 361$.

Let's pause and ask **sagemath** to compute the non-squares in \mathbb{Z}_{19} :

```
F.<a>=FiniteField(19)
squares = [ x^2 for x in F]
nonSquares = [x for x in F if not(x in squares)]
len(nonSquares)
->
9
```

This output tells us that there are 9 elements $a \in \mathbb{Z}_{19}$ for which $T^2 - a$ is **irreducible**. Those elements are:

```
nonSquares
->
[2, 3, 8, 10, 12, 13, 14, 15, 18]
```

Notice that since up to isomorphism there is a unique field of order 19^2 . It follows that

$$\mathbb{Z}_{19}(\sqrt{2})$$

must contain a square root of each of these **nonSquares**, e.g. 13 and 8 (to pick two at random. . .) **sagemath** can find them!:

```
F= FiniteField(19)
R.<T>=PolynomialRing(F)
E.<a> = F.extension(T^2 - 2)
[x for x in E if x^2==2]
->
[a, 18*a]
-----
[x for x in E if x^2==13]
->
[4*a, 15*a]
-----
[x for x in E if x^2==8]
->
[2*a, 17*a]
```

This makes clear for example that

$$\mathbb{Z}_{19}(\sqrt{13}) = \mathbb{Z}_{19}(4\sqrt{2}) = \mathbb{Z}_{19}(\sqrt{2}).$$

In fact, we can get a full list of irreducible polynomials:

```
irred = [T^2 + a*T + b for a in F for b in F if (T^2+a*T+b).is_irreducible()]
len(irred)
->
171
```

The output tells us that there are 171 monic irreducible quadratic polynomials in $\mathbb{Z}_{19}[T]$.
Let's look at a few:

```
irred[0:11]
[T^2 + 1,
 T^2 + 4,
 T^2 + 5,
 T^2 + 6,
 T^2 + 7,
 T^2 + 9,
 T^2 + 11,
 T^2 + 16,
 T^2 + 17,
 T^2 + T + 2,
 T^2 + T + 3]
```

Let's notice

```
def polroots(p):
    return [x for x in E if p(x)==0]
[irred[10],
 polroots(irred[10])]
->
[T^2 + T + 3, [a + 9, 18*a + 9]]
```

The output shows that the two roots of $T^2 + T + 3$ in $\mathbb{Z}_{19}(\sqrt{2})$ are

$$9 + \sqrt{2} \quad \text{and} \quad 9 + 18\sqrt{2} = 9 - \sqrt{2}.$$

(We could actually get those from the quadratic formula...)

This makes clear that $\mathbb{Z}_{19}(\sqrt{2})$ is a splitting field for $T^2 + T + 3$.

In fact, we know that $\mathbb{Z}_{19}(\sqrt{2})$ is a splitting field for all 171 polynomials p in the list `irred`.

12.5 Existence of a finite field of any prime-power order

Let p be a prime number. You have probably seen the following results:

Lemma 12.5.1. *For $x, y \in \mathbb{Z}$, we have:*

a. $x^p \equiv x \pmod{p}$

b. $(x + y)^p \equiv x^p + y^p \pmod{p}$.

Slightly more general statements are true for elements of any field of characteristic $p > 0$, as follows:

#+beginlemma Let F be a field of char. $p > 0$, let $x, y \in F$, and let $n \in \mathbb{Z}_{>0}$. Then:

a. $(x + y)^{p^n} = x^{p^n} + y^{p^n}$.

b. $\{x \in F \mid x^{p^n} = x\}$ is a subfield of F .

Proof. For $0 < i < p$, the binomial coefficients $\binom{p}{i} = \frac{p!}{i! \cdot (p-i)!}$ satisfy the congruence

$$\binom{p}{i} \equiv 0 \pmod{p}.$$

Indeed, p divides the numerator $p!$ but p does not divide the denominator $i! \cdot (p-i)!$ and the result follows since the quotient is integral.

This implies that

$$(x + y)^p = \sum_{i=0}^p \binom{p}{i} x^i y^{p-i} = x^p + y^p$$

for elements $x, y \in F$. To prove a., proceed by induction on $n \geq 1$. The case $n = 1$ was just handled. Assuming the result is valid for $n - 1$, we see that

$$(x + y)^{p^n} = ((x + y)^{p^{n-1}})^p = (x^{p^{n-1}} + y^{p^{n-1}})^p = x^{p^n} + y^{p^n};$$

we used the induction hypothesis for the second equality, and the “base case” $n = 1$ for the final equality. This proves a.

For b., write

$$F_1 = \{x \in F \mid x^{p^n} = x\}.$$

To see that F_1 is an additive subgroup of F , first note that $0 \in F_1$. Now, the result from a. shows that if $x, y \in F_1$ then $x - y \in F_1$.

Next we argue that F_1 is closed under multiplication. This follows since if $x, y \in F_1$ then

$$(xy)^{p^n} = x^{p^n} y^{p^n} = xy.$$

Finally, if $x \in F_1$ is non-zero, then

$$1 = 1^{p^n} = (x \cdot x^{-1})^{p^n} = x^{p^n} x^{-p^n} = x x^{-p^n}$$

which shows that $(x^{-1})^{p^n} = x^{-p^n} = x^{-1}$ hence $x^{-1} \in F_1$. □

Remark 12.5.2. If m, n are positive integers for which $n = qm$, then $T^m - 1$ is a divisor of $T^n - 1$ in the polynomial ring $\mathbb{Z}[T]$. Indeed, note that

$$\frac{u^q - 1}{u - 1} = u^{q-1} + u^{q-2} + \cdots + u + 1$$

so that

$$\begin{aligned} f(T) &= \frac{T^n - 1}{T^m - 1} = \frac{(T^m)^q - 1}{T^m - 1} \\ &= (T^m)^{q-1} + (T^m)^{q-2} + \cdots + T^m + 1 \\ &= T^{m(q-1)} + T^{m(q-2)} + \cdots + T^m + 1 \in \mathbb{Z}[T] \end{aligned}$$

In particular, for an element y of a ring R , we conclude that $y^m - 1$ divides $y^n - 1$. Indeed, $y^n - 1 = f(y) \cdot (y^m - 1)$.

Proposition 12.5.3. *Let F be a field with p^n elements. Each subfield of F has p^m elements for some divisor m of n . Conversely, for each divisor $m \mid n$, there exists a unique subfield of F having p^m elements.*

Proof. Let F_0 be the prime subfield of F . Any subfield E of F must contain F_0 and must have p^m elements, where $m = [E : F_0]$. Since

$$n = [F : F_0] = [F : E][E : F_0] = [F : E] \cdot m$$

we conclude that m must be a divisor of n .

Conversely, let m be a divisor of n . Then $p^m - 1$ is a divisor of $p^n - 1$ by the *Remark* above.

Applying the *Remark* a second time, we see that the polynomial $g(T) = T^{p^m-1} - 1$ is a divisor of $h(T) = T^{p^n-1} - 1$ in the polynomial ring $F_0[T]$.

Since F is the splitting field of $T \cdot h(T)$ over F_0 , it must contain all p^m distinct roots of $T \cdot g(T)$.

Now, part b. of the *Lemma* implies that the roots of $T \cdot g(T)$ form a subfield E of F . Any other subfield having order p^m must be a splitting field of $T \cdot g(T)$ and so it must coincide with E . This completes the proof. \square

Lemma 12.5.4. *Let F be a field of char. $p > 0$. If $n \in \mathbb{Z}_{>0}$ and $n \not\equiv 0 \pmod{p}$ then $T^n - 1$ has no repeated roots in any extension field of F . Put another way, if E denotes a splitting field of $T^n - 1$ over F , then*

$$T^n - 1 = \prod_{i=1}^n (T - \alpha_i)$$

for n distinct elements $\alpha_i \in E$.

Proof. Let c be a root of $T^n - 1$ in a splitting field E . Using “long division”, we compute that

$$f = \frac{T^n - 1}{T - c} = T^{n-1} + cT^{n-2} + c^2T^{n-3} + \cdots + c^{n-2}T + c^{n-1}$$

so that $T^n - 1 = f \cdot (T - c)$. To prove the Lemma, we must show that f is not divisible by $T - c$. By the remainder theorem, it is sufficient to prove that $f(c) \neq 0$. But we have:

$$f(c) = c^{n-1} + cc^{n-2} + c^2c^{n-3} + \cdots + c^{n-2}c + c^{n-1} = n \cdot c^{n-1}$$

and the result follows since $n1_F \neq 0$ and $c \neq 0$. \square

Theorem 12.5.5. *For every prime p and every positive integer n , there is a field \mathbb{F}_q with $q = p^n$ elements, and any field of order q is isomorphic to \mathbb{F}_q .*

Proof. The uniqueness has already been proved; it remains to argue the *existence* of \mathbb{F}_q for $q = p^n$.

Let F be the splitting field of the polynomial $T^{p^n} - T$ over $\mathbb{Z}/p\mathbb{Z}$. The previous Lemma shows that $T^{p^n} - T$ has p^n distinct roots. By an earlier Lemma, these roots form a *subfield* of F , so we conclude that F consists exactly in these roots. Thus $|F| = p^n$ as required. \square

Remark 12.5.6. For a prime power q , some texts write $\text{GF}(q)$ for the field we write \mathbb{F}_q . The symbol GF seems to stand for “Galois Field”.

12.6 Some examples: fields of order 4 and 8

There are 4 monic polynomials of degree 2 over the field \mathbb{F}_2 of two elements. Of these, only one is irreducible, namely

$$T^2 + T + 1.$$

Thus

$$\mathbb{F}_4 \simeq \mathbb{F}_2(\alpha)$$

where $\deg \alpha = 2$ and $\alpha^2 = \alpha + 1$. Notice that

$$T^2 + T + 1 = (T + \alpha)(T + \alpha + 1).$$

There are 8 monic polynomials of degree 3 over \mathbb{F}_2 . Of these, only two are irreducible:

```
H = FiniteField(2)
R.<T>=PolynomialRing(H)
[T^3 + a*T^2 + b*T + c
 for a in H
 for b in H
 for c in H
 if (T^3+a*T^2+b*T + c).is_irreducible()]
->
[T^3 + T + 1, T^3 + T^2 + 1]
```

Thus $\mathbb{F}_8 = \mathbb{F}_2(\beta)$ where $\deg \beta = 3$ and $\beta^3 = \beta + 1$.

And indeed $\mathbb{F}_2(\beta)$ is a splitting field for *both* the irreducible polynomials of degree 3:

```
HH.<b>=FiniteField(8)
RR.<T>=PolynomialRing(HH)
[RR(T^3+T+1).factor(),
 RR(T^3+T^2+1).factor()]
->
[(T + b) * (T + b^2) *p (T + b^2 + b),
 (T + b + 1) * (T + b^2 + 1) * (T + b^2 + b + 1)]
```

12.7 The multiplicative group of a finite field

We begin our discussion of the multiplicative group of a finite field with a Lemma from group theory:

Lemma 12.7.1. *Let G be a finite abelian group (written multiplicatively). If $a \in G$ is an element of maximal order in G , then the order of every element of G is a divisor of the order $o(a)$ of a .*

Proof. Let $x \in G$ be any element different from 1. If $o(x) \nmid o(a)$ then in the prime factorizations of $o(x)$ and $o(a)$ we can find a prime p that occurs to a higher power in $o(x)$ than in $o(a)$.

Write $o(a) = p^\alpha n$ and $o(x) = p^\beta m$ where $\alpha < \beta$ and $p \nmid n, p \nmid m$.

Now $o(a^{p^\alpha}) = n$ and $o(x^m) = p^\beta$, so the orders of a^{p^α} and x^m are relatively prime. It follows that the order of the product $a^{p^\alpha} \cdot x^m$ is equal to the product of the orders of the elements, i.e. to np^β . But this exceeds $o(a)$ contrary to the hypothesis. \square

Theorem 12.7.2. *Let F be any field. Any finite subgroup of the multiplicative group F^\times is cyclic.*

Proof. Let H be a finite subgroup of F^\times and let $a \in H$ be an element with maximal order. Write $N = o(a)$. The preceding lemma shows that $o(x) \mid N$ for all $x \in H$. Thus, every element of H is a root of the polynomial $T^N - 1$.

It follows that $|H| \leq N$. Since the cyclic group $\langle a \rangle$ has order N , conclude that $H = \langle a \rangle$. \square

Corollary 12.7.3. \mathbb{F}_q^\times is a cyclic group of order $q - 1$ for any prime power $q = p^n$.

Corollary 12.7.4. *For any prime power $q = p^n$, there is $\alpha \in \mathbb{F}_q$ for which $\mathbb{F}_q = \mathbb{F}_p(\alpha)$. In words: each finite field is a primitive extension of its prime subfield.*

Proof. Let β be a generator for the cyclic group \mathbb{F}_q^\times . Then

$$\langle \beta \rangle \subset \mathbb{F}_p(\beta) \implies \mathbb{F}_q = \mathbb{F}_p(\beta).$$

\square

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