

Math146 - PS4 due 2025-02-23

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1. Find the minimal polynomial over \mathbf{Q} of the element $\omega = \exp(2\pi i/6) \in \mathbf{C}$.

Hint: First note that ω is a root of $\frac{T^6 - 1}{T - 1} = T^5 + T^4 + T^3 + T^2 + T + 1$. Does $T^3 - 1$ divide $T^6 - 1$?

2. Consider the field extension $E = \mathbf{Q}(\alpha)$ of \mathbf{Q} where the minimal polynomial of α is equal to $f(T) = T^2 - 3T - 3 \in \mathbf{Q}[T]$.
- a. Show that f is irreducible over \mathbf{Q} .
- b. Viewing E as a vector space over \mathbf{Q} , we know that $\{1, \alpha\}$ is a *basis* of E as a \mathbf{Q} -vector space. Let's write $\Phi : E \rightarrow \mathbf{Q}^2$ for the vector space isomorphism

$$s + t\alpha \mapsto \begin{bmatrix} s \\ t \end{bmatrix}.$$

Fix $\beta = a + b\alpha \in \mathbf{Q}(\alpha)$ for $a, b \in \mathbf{Q}$ and consider the \mathbf{Q} -linear transformation $\lambda_\beta : E \rightarrow E$ given by “multiplication by β ”. Thus

$$\lambda_\beta(\gamma) = \beta \cdot \gamma \quad \text{for } \gamma \in E = \mathbf{Q}(\alpha).$$

Find the 2×2 matrix $M = M_\beta \in \text{Mat}_{2 \times 2}(\mathbf{Q})$ with the property that

$$M \cdot \Phi(\gamma) = \Phi(\lambda_\beta(\gamma)).$$

Otherwise stated, the matrix M is determined by the condition

$$M \cdot \begin{bmatrix} s \\ t \end{bmatrix} = \Phi((a + b\alpha)(s + t\alpha)) \quad \text{for every } s, t \in \mathbf{Q}.$$

- c. Show that the assignment $\beta \mapsto M_\beta$ determines a homomorphism of rings

$$E \rightarrow \text{Mat}_{2 \times 2}(\mathbf{Q})$$

(For this, you need to show that $M_{\beta_1 + \beta_2} = M_{\beta_1} + M_{\beta_2}$ and that $M_{\beta_1 \cdot \beta_2} = M_{\beta_1} \cdot M_{\beta_2}$ for $\beta_i \in E$).

d. Using the quadratic formula, we see that the roots of f have the form

$$\frac{3 \pm \sqrt{3}}{2}.$$

Choosing $\alpha = \frac{3 + \sqrt{3}}{2}$ let us write $\bar{\alpha} = \frac{3 - \sqrt{3}}{2} = 3 - \alpha$.

For $\beta = s + t\alpha \in E$ with $s, t \in \mathbf{Q}$ put $\bar{\beta} = s + t\bar{\alpha} = (s + 3t) - t\alpha$.

Show that the assignment $\beta \mapsto \bar{\beta}$ determines an isomorphism of rings $E \rightarrow E$.

e. Show that $\beta \cdot \bar{\beta} \in \mathbf{Q}$ for every $\beta \in E$.

f. Verify that

$$\det M_{\beta} = \beta \cdot \bar{\beta}.$$

for every $\beta \in E$.

g. Write $\frac{1}{2 + \alpha} \in E$ in the form

$$\frac{1}{2 + \alpha} = s + t\alpha$$

for $s, t \in \mathbf{Q}$.