

Math146 - Review for midterm 1

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- I. The exam will cover what is (currently) in sections 1 - 7 of the course lecture notes.
- II. You should be able to give careful statements for the definitions of the following terms:
- a a *commutative ring* R , a *field* F , an *integral domain* R , a *ring homomorphism* $f : R \rightarrow S$, an *ideal* I of a commutative ring R , a *principal ideal* of a commutative ring R , a *principal ideal domain*, the *quotient ring* R/I where I is an ideal of a commutative ring,
 - b an *irreducible element* of a commutative ring R , the greatest common divisor $\gcd(a, b)$ for elements $a, b \in R$ of a principal ideal domain R , a *unit* of a commutative ring, an *associate* of an element of a commutative ring, a 0-divisor of a commutative ring R
 - c the *field of fractions* F of an integral domain R , the *polynomial ring* $R[T]$ for a commutative ring R
- III. You should know the statements of the following results.
- a. The *first isomorphism theorem for rings*
 - b. the result that the *unique factorization property* holds in a PID
 - c. The *division algorithm* for the polynomial ring $F[T]$ where F is a field.
 - d. Eisenstein's irreducibility criterion
 - e. the Gauss Lemma and consequences
- IV. Be able to give examples of the following:
- a. an *integral domain* that is not a *principal ideal domain*
 - b. a field F and a polynomial $f \in F[T]$ such that f has no root in F but f is *reducible*.
 - c. A finite field F with exactly 9 elements. (**Hint:** Consider the field $\mathbf{F}_3 = \mathbf{Z}/3\mathbf{Z}$ of order 3, and find a polynomial of the form $p = T^2 - a \in \mathbf{F}_3[T]$ that is *irreducible*. How many elements are in the quotient $F[T]/\langle p \rangle$?)
- V. You should be able to write careful solutions to problems similar to the following:
1. Let F be a field and let $f, g \in F[T]$ be polynomials for which $\gcd(f, g) = 1$. Consider the mapping
$$\phi : F[T] \rightarrow F[T]/\langle f \rangle \times F[T]/\langle g \rangle$$
given by the rule $\phi(h) = (h + \langle f \rangle, h + \langle g \rangle)$.

- a. Show that $\ker \phi = \langle fg \rangle$ and that ϕ induces an isomorphism

$$\bar{\phi} : F[T]/\langle fg \rangle \xrightarrow{\sim} F[T]/\langle f \rangle \times F[T]/\langle g \rangle$$

- b. As a consequence, show that $\mathbf{Q}[T]/\langle T^7 - 1 \rangle$ is isomorphic to the direct product of two fields.
2. Let R be a PID, let $a_1, a_2, \dots, a_n \in R$ not all 0, and let $d = \gcd(a_1, a_2, \dots, a_n)$. Note that $\frac{a_i}{d} \in R$ for each i . Prove that $\gcd\left(\frac{a_1}{d}, \frac{a_2}{d}, \dots, \frac{a_n}{d}\right) = 1$
3. Show that $u = 2 + T + \langle T^3 \rangle$ is a unit in the quotient ring $\mathbf{Q}[T]/\langle T^3 \rangle$.
4. Let F be a field. Prove that \sqrt{T} is not in $F(T)$. (**Hint:** Suppose the contrary, namely that $\sqrt{T} = \frac{f}{g}$ for $f, g \in F[T]$. Explain why we find an equation $g^2 T = f^2$ in $F[T]$. Now apply unique factorization in the PID to deduce a contradiction).
5. If R is a PID and $p, q \in R$ are non-associate irreducible elements, compute $\gcd(p^2 q, pq^2)$?
6. Consider the field \mathbf{F}_5 with 5 elements.
- Prove that $T^2 - 3 \in \mathbf{F}_5[T]$ is irreducible.
 - Find $s, t \in \mathbf{F}_5$ so that $(s + t\gamma) \cdot (1 + \gamma) = 1$.
7. Be able to give the proof of the following results (taken from the notes).
- Let R be a PID and let $p \in R$ irreducible.
- If $p \mid ab$ for $a, b \in R$ then $p \mid a$ or $p \mid b$.
 - The quotient ring $R/\langle p \rangle$ is a field.