Homework 11

Math 87

Due: December 11 2023

1 Visualizing Lotka-Volterra

Many dynamical systems (such as the Lotka-Volterra predator-prey model) do not have known closed form solutions. Hence, scientists interested in such systems are often more interested in studying properties of the set of ALL solutions. In this exercise, you will use Python to construct a phase portrait, i.e. a convenient way of representing solutions and their trajectories. For this, you will need to import the following modules:

```
import numpy
from scipy import integrate
import matplotlib.pyplot as plt
```

1. For concreteness, recall from lecture that the Lotka-Volterra model describing the evolution of the population of rabbits r and foxes f as a function of time is given by the differential equations:
$$\frac{\partial r}{\partial t} = \alpha r - \beta r f = r(\alpha - \beta f)$$
$$\frac{\partial f}{\partial t} = -\gamma f + \delta r f = f(\delta r - \gamma).$$
Give an intuitive explanation of why these equations provide a reasonable model for predator-prey populations.

2. Write a function dX/dt that takes as input some chosen values $\alpha, \beta, \gamma, \delta$ and an initial population vector X and produces the gradient vector $[\frac{\partial r}{\partial t}, \frac{\partial f}{\partial t}]$. What does this gradient vector evaluated at a some fixed (r, f) tell you? (**For simplicity, you may choose $\alpha, \beta, \gamma, \delta$ and "hardcode" it into the function, so that the only input to the function is the inintial condition.**

3. Suppose you were to integrate these two differential equations over a time interval. Fix a starting time (say, $t = 0$) and consider the value of these integrals as functions of t. What information do these curves give about the system?

4. The scipy.integrate module is a tool you may use to integrate ODEs. The function integrate.odeint takes as input a differential equation, an initial condition and an array representing "times." Using this, plot the integrals of these functions from times 0 to 15, with inital conditions 10 rabbits and 5 foxes. Make sure that enough samples are plotted so that the curves look "continuous." What do you notice about the curves? Repeat this step and plot the outcome with two other initial conditions.

5. Observe that the vectors $[\frac{\partial r}{\partial t}, \frac{\partial f}{\partial t}]$ define a vector field on $\{(r, f)|r \geq 0, f \geq 0\}$, i.e. on the set of possible solutions. Produce a plot of this vector field on the region $0 \leq r \leq 60$, $0 \leq f \leq 40$ (you may use larger or smaller region of your choice if it illustrates the problem better). Hints:

   - Start by producing two one-dimensional arrays, for instance one interpolating from 0 to 60 and the other interpolating from 0 to 40 (interpolating with, for instance, 20-30 increments). You may then use the np.meshgrid function to obtain the x and y coordinates for all points in the "square" defined by the arrays. (The bounds you use should allow for good visibility of the fixed points, which corresponds to your choice of parameters)
   - Apply the function $[\frac{\partial r}{\partial t}, \frac{\partial f}{\partial t}]$ to each point in your grid, which produces a vector for each point. Normalize each vector so that it has length 1.
   - Use the pyplot.quiver function to plot the vector field computed this way.

6. What do you notice about the vector field? In particular, how does the vector field explain or relate to your observations in part 4? From the plot can you guess where the fixed points of the system are?

7. What are some unrealistic assumptions made in the Votka-Loterra model?

8. Finally, suppose that there was in fact a second "super" predator present in the environment (e.g. humans who hunt both foxes and rabbits). Write out a system of differential equations that would describe the Lotka-Volterra model for this situation.