Homework 1-Optimization

Math 87

January 2022

1. **A high-end gym wishes to offer a New Year's special to all customers who sign up for a year long membership before February 1st. The gym typically charges \$150 a month and is trying to decide how much of a discount to offer. It is estimated that for every \$100 off the yearly price, the number of gym memberships purchased will increase by 15%.**

   (a) **How much of a discount will maximize the gyms profits on this special? Model the question as a single-variable optimization problem.**

   The relevant parameters are:

   - p = profit per membership = \$150 $*$ 12 = 1800
   - r = discount size = \$100
   - e = profit increase per discount (i.e. the effectiveness of the discount) = 0.15
   - N_0 = the initial number of units sold without discount

   The main variable is, n_r = the optimal number of \$100 discounts to offer. The following equation models total profit:

   $$P(n_r) = N_0(p - n_r * r)(1 + n_r * e)$$

   (b) **Compute the sensitivity of the optimal discount and the corresponding profit to the 15% assumption.**

   It is clear that N_0 does not affect the value of optimal n_r, so without loss of generality we can assume $N_0 = 1$.

   Then

   $$\frac{dP}{dn_r} = (p - n_r \cdot r) \cdot e - r \cdot (1 + n_r \cdot e) = 0$$

   and optimal

   $$n_r = (pe - r)/(2re) = \frac{17}{3} \approx 5.67$$

   Then sensitivity of optimal number of \$100 discounts n_r to effectiveness e is

   $$S = \frac{dn_r}{de} \cdot \frac{e}{n_r} = \left(\frac{1}{2e^2}\right)\frac{e}{n_r} = \frac{1}{2en_r} \approx 0.588$$

   (c) **Suppose that the special only generates a 10% increase in sales per \$100. What is the effect?**

   Repeating the above analysis, one sees that the optimal value for n_r in this case is 4, and the sensitivity of this becomes 1.25.

(d) **Under what circumstances would an offer of a special cause a reduction in profit (your answer should be quantitative)?**

The special would cause a reduction in profit if the effectiveness was low enough. The offer of discounts causing reduction in profit is equivalent to derivative of profit with respect to number of discounts being negative when $n_r = 0$, because then applying any discount would decrease the objective. Let us look back at that derivative and consider its value when $n_r = 0$:

$$\frac{dP}{dn_r} = (p - n_r \cdot r) \cdot e - r \cdot (1 + n_r \cdot e)|_{n_r=0} = pe - r < 0.$$

Thus when rebate efficiency is $e < r/p = 1/18 \approx 0.0555\%$ it would not be profitable to offer any rebate.

2. **A chemist is synthesizing a compound. In the last step, she must dissolve her reagents in a solution with a particular pH level H, for $1.2 \leq H \leq 2.7$, and heated to a temperature T (in degrees Celsius), for $66 \leq T \leq 98$. Her goal is to maximize her percent yield as a percentage of the initial mass of the reagents. The equation determining the percentage $F(H,T)$ is**

$$F(H,T) = -0.038T^2 - 0.223TH - 10.982H^2 + 7.112T + 60.912H - 328.898.$$

(a) **Find the optimal temperature and pH level in the allowed range.**

To find optimal values let us consider partial derivatives:

$$\begin{aligned} \tfrac{\partial F}{\partial H}(H,T) = &\quad -0.223T - 21.964H + 60.912 = 0 \\ \tfrac{\partial F}{\partial T}(H,T) = &\quad -0.076T - 0.223H + 7.112 = 0 \end{aligned}$$

One can solve this using elementary methods to obtain: $T = 88.0651, H = 1.87914$

(b) **Use matplotlib to produce a graph and a contour plot of the percentage of the powder function $F(H,T)$.**

Below is some sample code for producing such a figure: First, we import the necessary packages, and then define a function that will allow us to draw graphs with our desired parameters (it is not strictly necessary to define a function but doing so will allow you more versatility if you re-use code on future assignments):

```
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
import numpy as np

# https://matplotlib.org/mpl_toolkits/mplot3d/tutorial.html
# https://matplotlib.org/3.3.1/gallery/mplot3d/surface3d.html

def draw_graph(f,x,y,x0,y0,elev_azim=[]):
    X,Y = np.meshgrid(x,y)

    fig = plt.figure(figsize=(len(elev_azim)*10,10))
    for idx,(e,a) in enumerate(elev_azim,start=1):
```

```
            ax = fig.add_subplot(1,len(elev_azim),idx,projection='3d',

            elev=e,azim=a)

            ax.plot_wireframe(X,Y,f(X,Y))

            ax.plot(x,y0*np.ones(y.shape), zs= f(x,y0), color="red", linewidth=3)
            ax.plot(x0*np.ones(x.shape),y, zs= f(x0,y), color="red", linewidth=3)

        return fig

H=np.linspace(1.2,2.7,101)
T=np.linspace(66,98,101)

F= lambda h,T: -0.038*T**2 - 0.223*T*h - 10.982*h**2 + 7.112*T

+ 60.912*h - 328.898

draw_graph(F,
                x=H,
                y=T,
                x0=1.87914,
                y0=88.0651,
                elev_azim=[(45,20)])

# countour plot

x,y = np.meshgrid(H,T)

figc = plt.figure(figsize=(10,7))
axc = figc.add_subplot()
axc.contourf(x,y,F(x,y),levels=20
                , extend='both')
axc.scatter(1.87914,88.0651,marker="X")
    return fig
```

3. (15 pts) **Human blood is generally classified in the "ABO" system, with four blood types: A, B, O, and AB. These four types reflect six gene pairs (genotypes), with blood type A corresponding to gene pairs AA and AO, blood type B corresponding to gene pairs BB and BO, blood type O corresponding to gene pair OO, and blood type AB corresponding to gene]pair AB. Let p be the proportion of gene A in the population, q be the proportion of gene B in the population, and r be the proportion of gene O in the population. Note that $p + q + r = 1$.**

   (a) **The Hardy-Weinberg principle states that p, q, and r are fixed from generation to generation, as are the frequencies of the different genotypes.**

Under this assumption, what is the probability that an individual has genotype AA? BB? OO? What is the probability of an individual having two different genes?

Since the probability of a single gene being A is p, the probability that an individual has 2 A genes, as required for the AA genotype, is just p^2. Similarly, the probability of genotype BB is q^2, and the probability of genotype OO is r^2.

The probability of an individual having two different genes is, simply, the probability that they are of none of genotypes AA, BB, or OO; this is $1 - p^2 - q^2 - r^2$.

(b) **Find the maximum percentage of the population that can have two different genes under the Hardy-Weinberg principle in two different ways, by directly maximizing a function of only two variables and by using the method of Lagrange multipliers.**

This question asks us to consider the maximum possible value of $f(p, q, r) = 1 - p^2 - q^2 - r^2$, subject to the constraint that $p + q + r = 1$. For the first approach, we write $r = 1 - p - q$, to obtain $g(p, q) = 1 - p^2 - q^2 - (1 - p - q)^2$ as a function of two variables that we can maximize directly. The second approach will use the constraint $h(p, q, r) = p + q + r = 1$.

For the first approach, we compute

$$\begin{aligned} g(p,q) &= 1 - p^2 - q^2 - (1 - 2p - 2q + p^2 + 2pq + q^2) \\ &= 2p + 2q - 2p^2 - 2pq - 2q^2 \\ \frac{\partial g}{\partial p} &= 2 - 4p - 2q \\ \frac{\partial g}{\partial q} &= 2 - 2p - 4q \end{aligned}$$

Setting the two partial derivatives equal to zero gives the solution $p = q = \frac{1}{3}$ which, in turn, gives $r = \frac{1}{3}$. Checking boundary values shows that this gives the maximum value for $f(p, q, r)$, with $f(1/3, 1/3, 1/3) = 2/3$.

For the second approach, we compute

$$\begin{aligned} f(p,q,r) &= 1 - p^2 - q^2 - r^2 \\ \frac{\partial f}{\partial p} &= -2p,\ \frac{\partial f}{\partial q} = -2q,\ \frac{\partial f}{\partial r} = -2r. \end{aligned}$$

Thus, the only local max/min value of f occurs when $p = q = r = 0$, but this does not satisfy the constraint. Using Lagrange multipliers, then, we have the equations $\nabla f = \lambda \nabla h$ plus the constraint $h(p, q, r) = 1$, giving

$$\begin{aligned} -2p &= \lambda \\ -2q &= \lambda \\ -2r &= \lambda \\ p + q + r &= 1. \end{aligned}$$

Adding the first three equations gives $-2p - 2q - 2r = 3\lambda$, but we know from the fourth equation that $p + q + r = 1$. Thus, $\lambda = -2/3$, and $p = q = r = 1/3$ gives the optimal value. (It is easy to check that this is a maximum value and not a minimum.)

(c) **Can you say what the Lagrange multiplier represents in the above example?**

Thinking about it as a "shadow price", this would represent how much the probability of having two different genes would change if the total probability of having any one gene changes. Of course this isn't really possible, because the sum of probabilities always equals 1. But it would say if the probability of having an A, B, or O gene went down (maybe because there was a new gene discovered), then the likelihood of having two distinct genes would go up!