Homework 3

Math 87

September 27 2023

1 Work the fertilizer

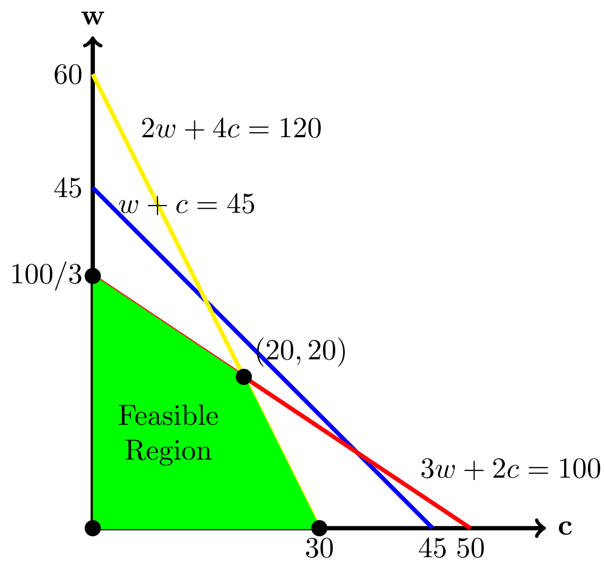
A farmer owns 45 acres of land. They are planning to plant each acre with either wheat or corn. Each acre of wheat yields $200 in profits, whereas each acre of corn yields $300 in profits. Each acre of wheat requires 3 workers and 2 tons of fertilizer. Each acre of corn requires 2 workers and 4 tons of fertilizer. The farmer has 100 workers and 120 tons of fertilizer available.

1. **Write down and solve the primal problem to determine how many acres of wheat and corn need to be planted to maximize profits. What is the maximum profit? Non-integer values are allowed. You may solve this by drawing the feasible region or using python.**

   We can rewrite this as a linear program as follows. Let w be the number of acres of wheat planted and c be the acres of corn. We know that the profit, $p = 200w + 300c$. We also know that the number of workers is bounded by $3 * w + 2 * c \leq 100$, the fertilizer is bounded by $2w + 4c \leq 120$, and the total acreage is bounded by $c + w \leq 45$. Of course, we can only have positive acreage. Thus, the linear program is:

$$\begin{aligned} \text{Maximize } & 200w + 300c \\ \text{subject to } & w + c \leq 45 \\ & 3w + 2c \leq 100 \\ & 2w + 4c \leq 120 \\ & c, w \geq 0 \end{aligned}$$

We can sketch the feasible region as follows:



The vertices of the feasible set are $(0,0)$, $(30,0)$, $(20,20)$, and $(0,100/3)$. The function values at these points are $p(0,0) = \$0$, $p(30,0) = \$9000$, $p(20,20) = \$10,000$, and $p(0,100/3) = \$6,666.67$. So, the maximum profit is achieved at $(20,20)$, and its value is $p(1,0) = \$10,000$.

Alternatively, using SciPy's `linprog` with input $c = -[200, 300]$, $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 45 \\ 100 \\ 120 \end{bmatrix}$, we get max profit of 9999.999999457514 at point $(20,20)$.

Primal Linear Program

```
from scipy.optimize import linprog
import numpy as np


c = np.array([200,300])
A = np.array([[1,1],[3,2], [2,4]])
b = np.array([45,100,120])

res=linprog((-1)*c, A_ub=A, b_ub=b)
print(res)
```

2. **Write down and solve the dual linear program. You may solve this by drawing the feasible region or using python.** The corresponding dual program is:

$$\begin{aligned} &\text{Minimize} \quad 45y_L + 100y_W + 120y_F \\ &\text{subject to} \\ &\qquad y_L + 3y_W + 2y_F \geq 200 \\ &\qquad y_L + 2y_W + 4y_F \geq 300 \\ &\qquad y_L, y_W, y_F \geq 0 \end{aligned}$$

or, in matrix form:

$$\text{Minimize} \quad \begin{bmatrix} 45 & 100 & 120 \end{bmatrix} \cdot \begin{bmatrix} y_L \\ y_W \\ y_F \end{bmatrix}$$

subject to

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_L \\ y_W \\ y_F \end{bmatrix} \geq \begin{bmatrix} 200 \\ 300 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This will be difficult to plot with 3 variables, so using `linprog` is better. The greater than zero constraints are built in to the python function, so we can eliminate those from the matrix. We can run `linprog` with $c = [45, 100, 120]$, $A = -\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ and $b = -\begin{bmatrix} 200 \\ 300 \end{bmatrix}$. The optimal solution occurs at $y_L = 0$, $y_W = 25$, and $y_F = 62.5$ to give a minimum of 9999.999999606973.

Dual Linear Program

```
from scipy.optimize import linprog
import numpy as np


c = np.array([45,100, 120])
A = np.array([[1, 3, 2],[1, 2 , 4]])
b = np.array([200, 300])

res=linprog(c,A_ub=-A,b_ub=-b)
print(res)
```

3. **Explain the meaning of the dual variables, objective function, and the constraints in the dual problem.**

The dual program seeks to minimize the unit price per resource (which we can think of as the cost to produce the crops) given constraints on how much the farmer can earn per each acre of crop. The dual variables y_L, y_W, y_F are the unit prices per acre, per worker, and ton of fertilizer, respectively. Therefore, the objective function is to minimize the cost for 45 acres at price y_L per acre, the cost of 100 workers at price y_W per worker, and the cost of 120 tons of fertilizer at price y_F per ton, giving us the goal to minimize cost $45y_L + 100y_W + 120y_F$. Next, the constraints tell what the minimum price the farmer must receive for each crop. The first constraint describes the wheat. With 1 acre of land at price y_L, 3 workers at price y_W, and 2 tons of fertilizer at price y_F, the farmer must receive at least \$200, giving $y_L + 3y_W + 2y_F \geq 200$. Similarly, for corn, With 1 acre of land at price y_L, 2 workers at price y_W, and 4 tons of fertilizer at price y_F, the farmer must receive at least \$300, giving $y_L + 2y_W + 4y_F \geq 300$.

2 Hope for the best...

You've decided to build a doomsday shelter under your house. You have a barrel which can store seven gallons of food, and you decide to fill it with rice and dried beans. You estimate that each gallon of beans provides enough nutrition for approximately 9 days of meals, whereas each gallon of rice only provides around 5 days (Are these at all realistic? Asking for my doomsday shelter). Each gallon of beans costs \$12 and each gallon of rice costs \$5. You have \$60 to spend, and would like to calculate how many gallons of rice and beans to buy in order to maximize the number of days your food stores will last (fractional purchases are allowed).

1. **Write this problem as a dual linear program.**

   First, we write this as a primal linear program:

$$\begin{aligned} \text{Maximize} \quad & 9x_1 + 5x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \leq 7 \\ & 12x_1 + 5x_2 \leq 60 \\ & x_1, x_2 \geq 0 \end{aligned}$$

   We identify the dual linear program by interchanging the roles of the constraint right-hand sides with the coefficients in the objective function, and transposing the coefficients in the constraints:

$$\begin{aligned} \text{Minimize} \quad & 7y_1 + 60y_2 \\ \text{subject to} \quad & \\ & y_1 + 12y_2 \geq 9 \\ & y_1 + 5y_2 \geq 5 \\ & y_1, y_2 \geq 0 \end{aligned}$$

2. **Find the solutions to both the primal and the dual linear programs by plotting the feasible sets. Confirm that both the strong duality theorem and complementary slackness are satisfied. Write out the dual prices for each of your primal constraints.**

   Plot these by hand or through a software package. The strong duality theorem says that the minimum of the dual is equal to the maximum of the primal; both are 49.2857, so strong duality is satisfied.

   The complementary slackness conditions are that:

$$\begin{aligned} (7 - x_1 - x_2)y_1 &= 0 \\ (60 - 12x_1 - 5x_2)y_2 &= 0 \\ (y_1 + 12y_2 - 9)x_1 &= 0 \\ (y_1 + 5y_2 - 5)x_2 &= 0 \end{aligned}$$

   The optimal solutions to the primal problem are $x_1 = 3.571$ and $x_2 = 3.428$. Plugging these in to the first pair of constraints show that complementary slackness is satisfied for these. Similarly, the second pair of constraints are satisfied by the optimal solutions of dual variables. Thus, complementary slackness is satisfied.

3. **Suppose you can increase the size of your barrel to hold an additional c gallons of food. Does the dual price for this modified constraint provide an accurate prediction of the increase in the primal objective function (i.e. the number of days of nutrition)? Answer this question for $c = 1, 2, 4, 6$.**

   This problem asks to consider the modified linear program:

$$\begin{aligned} \text{Maximize} \quad & 9x_1 + 5x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \leq 7 + c \\ & 12x_1 + 5x_2 \leq 60 \\ & x_1, x_2 \geq 0 \end{aligned}$$

   The dual price for the first constraint at the optimal solution is 2.1428, hence if the dual price provides an accurate prediction of the increase in primal objective function, the optimal amounts would be:

   - ($c = 1$) $49.29 + 1 * 2.14 = 51.43$
   - ($c = 2$) $49.29 + 2 * 2.14 = 53.57$
   - ($c = 4$) $49.29 + 4 * 2.14 = 57.85$
   - ($c = 6$) $49.29 + 6 * 2.14 = 62.13$

   Computing the optimal solutions for the actual modified programs, we get:

   - ($c = 1$) $49.29 + 1 * 2.14 = 51.43$
   - ($c = 2$) $49.29 + 2 * 2.14 = 53.57$
   - ($c = 4$) $49.29 + 4 * 2.14 = 57.85$
   - ($c = 6$) $49.29 + 6 * 2.14 = 60$

   Thus, we see that for $c = 1, 2, 4$ the dual price provide an accurate prediction, whereas for $c = 6$ it does not.