Signal processing

Tuesday, November 21, 2023 18:03

Fourier Transform

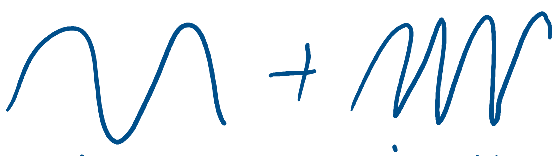
$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

depend on derivatives

$f: [0,1] \to \mathbb{R}$

$$\int_0^1 (f(x))^2 dx < \infty$$

$$f(x) = \sum_{k=0}^{\infty} a_k \sin(2\pi k x) + b_k \cos(2\pi k x)$$



$\sin 2\pi$ + $\sin 4\pi x$ $\to$ increasing frequency

$$A \sin(2\pi k (x - \varphi_k)) = a_k \sin(2\pi k x) + b_k \cos(2\pi k x)$$

$A_k = \sqrt{a_k^2 + b_k^2}$, φ_k - phase shift

Fourier transform?

$$a_k = \frac{1}{c}\int_0^1 f(x) \sin(2\pi k x) dx = a_k \int_0^1 \sin^2(2\pi k x) dx$$

$$b_k = \frac{1}{c}\int_0^1 f(x) \cos(2\pi k x) dx$$

$$\int_0^1 \sin(2\pi k x) \cos(2\pi m x) dx = 0$$

for all $m, k \in \mathbb{Z}$

$$\int_0^1 \sin(2\pi k x) \sin(2\pi m x)\, dx = 0$$

$\cos(\quad)\cos(\quad)$ if $k \neq m$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$b_k + i a_k = \frac{1}{l}\int_0^1 e^{i2\pi kx} f(x)\,dx$$

$$\mathcal{F}(f(x))(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{2\pi i \xi x} f(x)\,dx$$

$$\mathcal{F}^{-1}(\hat{f}(\xi))(x) = \int_{-\infty}^{\infty} e^{-2\pi i \xi x} \hat{f}(\xi)\,d\xi$$

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1) $\mathcal{F}(f'(x)) = \underbrace{2\pi i \xi}\,\hat{f}(\xi)$

$$\mathcal{F}: \left(f''(x) + a f'(x) + b f = g(x) \right)$$

$$-(2\pi\xi)^2 \hat{f} + 2\pi i \xi a \hat{f} + b\hat{f} = \hat{g}$$

$$f(x) = \mathcal{F}^{-1}\left(\frac{\hat{g}}{-(2\pi\xi)^2 + 2\pi i a \xi + b} \right)$$

$f(x_i)$

$\hat{f}(\xi_i)$

fast fourier transform

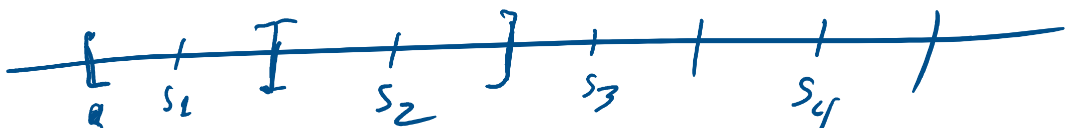
fft

2) $f(x)$, $g(x)$, $(f*g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$

$$\mathcal{F}(f*g) = \hat{f}\cdot\hat{g}, \quad f*g = \mathcal{F}^{-1}(\hat{f}\cdot\hat{g})$$

Look at local behavior

1) Short term Fourier-Transform



different FT on each interval

$$\hat{f}_1(\xi) = \int_{s_1-\alpha_1}^{s_1+\alpha_1} e^{2\pi i \xi x} f(x)\,dx$$

$$\hat{f}_i(\xi) = \int_{s_i-\alpha_i}^{s_i+\alpha_i} e^{2\pi i \xi x} f(x)\,dx$$

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2) Moving window FT

$$\psi(\xi, s) = \int_{-\infty}^{\infty} e^{2\pi i \xi x} f(x) g(x-s)\,dx$$

$$g(x) = \begin{cases} 1, & x \in [-\alpha, \alpha] \\ 0, & x \notin [-\alpha, \alpha] \end{cases}$$

$$g(x) = \begin{cases} 0, & x \notin [\alpha, \beta] \end{cases}$$

Consider $e^{2\pi i \xi x} g(x-s)$ as modified transform kernel

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3) Wavelets

mother wavelet $\psi(t)$: $\displaystyle\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty$

$$\psi_s(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{t}{s}\right)$$

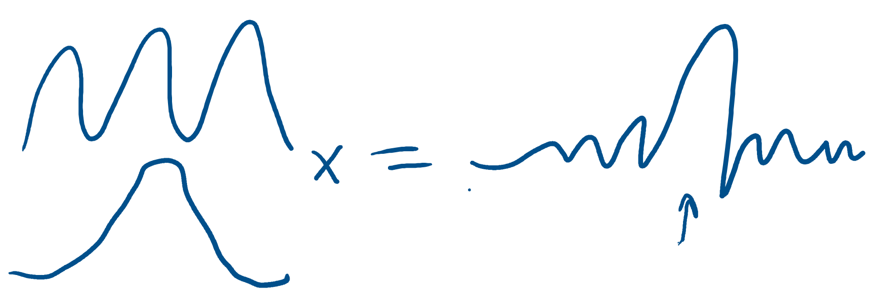
$$\Psi(f)(u,s) = \int_{-\infty}^{\infty} \psi_s(x-u) f(x)\, dx$$

$$f(x) = \frac{1}{C} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_s^*(x-u)\, \Psi(f)(s,u)\, ds\, du$$

complex conjugate
$f = \cos + i\sin$
$f^* = \cos - i\sin$

Morlet wavelet

$$\psi(x) = e^{i2\pi k x} \cdot e^{-x^2}$$



D.o.G. = Difference of Gaussians

DoG = Difference of Gaussians

$$\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{x^2}{2\sigma_1^2}} - \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{x^2}{2\sigma_2^2}}$$

$$\sigma_1 \neq \sigma_2$$