Predator-Prey system
(rabbit-fox)

$$\begin{cases} \frac{dr}{dt} = \alpha r - \beta rf = r(\alpha - \beta f) \\ \frac{df}{dt} = \gamma fr - \delta f = f(\gamma r - \delta) \end{cases}$$

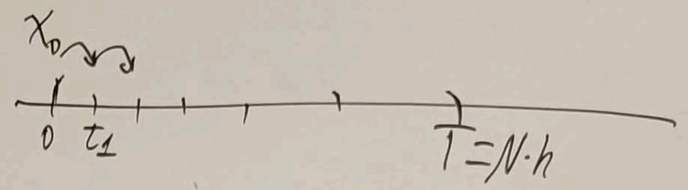
1) $(r, f) = (0, 0)$
2) $(r, f) = \left(\frac{\delta}{\gamma}, \frac{\alpha}{\beta}\right)$

$x(t)$

$$\frac{dx}{dt} = F(x,t), \quad x(0) = x_0$$

consider time-step $\Delta t = h$

$$t_i = \Delta t \cdot i = h \cdot i$$



$$\frac{dx}{dt} \approx \frac{x_{i+1} - x_i}{h}, \text{ where } x_i \approx x(t_i)$$

Single-step/single-stage methods

1) Explicit Euler method/scheme: (first order method)

$$\frac{x_{i+1} - x_i}{h} = F(x_i, t_i)$$
$$x_{i+1} = x_i + hF(x_i, t_i)$$

2) Implicit Euler (first order):

$$\frac{x_{i+1} - x_i}{h} = F(x_{i+1}, t_{i+1})$$
$$x_{i+1} - hF(x_{i+1}, t_{i+1}) = y_i$$

$$x(t_{i+1}) = x(t_i) + \frac{dx}{dt}(t_i) \cdot h + \frac{1}{2}\frac{d^2x}{dt^2}h^2 + O(h^3)$$

$$\frac{x(t_{i+1}) - x(t_i)}{h} = \frac{dx}{dt}(t_i) + \underbrace{\frac{1}{2}\frac{d^2x}{dt^2}h + O(h^2)}_{\text{Error}}$$
$$= F(x(t_i), t_i)$$

Approximation error

3) Midpoint or Crank-Nicholson (second order)

$$\frac{x_{i+1}-x_i}{h} = \begin{cases} \dfrac{F(x_{i+1},t_{i+1})+F(x_i,t_i)}{2} \\ F\left(\dfrac{x_{i+1}+x_i}{2}, \dfrac{t_{i+1}+t_i}{2}\right) \end{cases}$$

$$x_{i+1} = x(t_i+\tfrac{h}{2}) + \frac{dx}{dt}(t_i+\tfrac{h}{2})\cdot\frac{h}{2} + \cdots$$

$$x_i = x(t_i+\tfrac{h}{2}) + \frac{dx}{dt}(t_i+\tfrac{h}{2})(-\tfrac{h}{2}) + \cdots (-\tfrac{h}{2})^2$$

4) two-step explicit second order method

$$\frac{x_{i+1}-x_i}{h} = F(x_i^*, t_{i+\frac{1}{2}}) \qquad t_{i+\frac{1}{2}} = t_i + \frac{h}{2}$$

$$x_i^* \approx x_{i+\frac{1}{2}}$$

$$\frac{x_i^*-x_i}{h/2} = F(x_i, t_i)$$

5) Multi-stage Runge-Kutta method

4th order

$$k_1 = hF(x_i, t_i)$$
$$k_2 = hF(x_i+\tfrac{1}{2}k_1, t_{i+\frac{1}{2}})$$
$$k_3 = hF(x_i+\tfrac{1}{2}k_2, t_{i+\frac{1}{2}})$$
$$k_4 = hF(x_i+k_3, t_{i+1})$$

$$x_{i+1} = x_i + \frac{k_1+2k_2+2k_3+k_4}{6}$$

or

scipy.integrate.odeint