Homework 4

Math 87

October 2023

1 One man's trash

Answer:

The first step is to observe that this problem may be phrased as a constrained integer programming problem:

$$\begin{aligned} \text{Maximize} \quad & 5000x_1 + 600x_2 + 3500x_3 + 6000x_4 \\ \text{subject to} \quad & \\ & 24x_1 + 76x_2 + 43x_3 + 754x_4 \leq 800 \\ & 75.5x_1 + 2.7x_2 + 3.3x_3 + 6.7x_4 \leq 85 \end{aligned}$$

where x_1 denotes Crate A, x_2 denotes Crate B, x_3 denotes Crate C and x_4 denotes Crate D. As such, x_i is either zero or one for each i.

We first solve the relaxed linear program,

$$\begin{aligned} \text{Maximize} \quad & 5000x_1 + 600x_2 + 3500x_3 + 6000x_4 \\ \text{subject to} \quad & \\ & 24x_1 + 76x_2 + 43x_3 + 754x_4 \leq 800 \\ & 75.5x_1 + 2.7x_2 + 3.3x_3 + 6.7x_4 \leq 85 \\ & 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3, 4. \end{aligned}$$

The solution of the relaxed linear program is given by $\vec{x} = (0.996, 0, 1, 0.97)$ and optimal value 14313.

Since x_1 is the closest non-integer to an integer, we branch on the value of x_1, and resolve two relaxed linear programs, one constrained with $x_1 = 0$ and one with $x_1 = 1$. To do this, we modify the above code by changing the bounds constraints for x_1 to have the same upper and lower bounds.

With $x_1 = 0$, the solution of the relaxed linear program is $\vec{x} = (0, 0.04, 1, 1)$ with optimal value 9524. With $x_1 = 1$, the solution of the relaxed linear program is $\vec{x} = (1, 0, 1, 0.93)$ with optimal value 14052. Since the latter value is much greater, we branch on this partial solution's only non-integer variable, x_4.

With $x_1 = 1$ and $x_4 = 0$, we achieve an integer solution to the relaxed linear program of $\vec{x} = (1, 1, 1, 0)$ with optimal value 9100. This isn't large enough to allow us to prune any branches, so we continue. With $x_1 = 1$ and $x_4 = 1$, the solution of the relaxed linear program is $\vec{x} = (1, 0, 0.51, 1)$ with optimal value 12791. Since this is then the largest value that we encounter, we branch on its non-integer variable, x_3.

With $x_1 = 1$, $x_4 = 1$, and $x_3 = 0$, the solution of the relaxed linear program is $\vec{x} = (1, 0.29, 0, 1)$ with optimal value 1117.4. With $x_1 = 1$, $x_4 = 1$, and $x_3 = 1$, the relaxed linear program is infeasible. The largest objective functional up to now is with $x_1 = 1$, $x_3 = 0$, and $x_4 = 1$, so we branch on x_2 here.

With $\vec{x} = (1, 0, 0, 1)$, the objective function value is 11000. This allows us to prune the branch with $x_1 = 0$ and no other constraints, since its best possible value is only 9524. We can also discard the previously discovered integer solution of $\vec{x} = (1, 1, 1, 0)$ that had optimal value 9100. We next check $\vec{x} = (1, 1, 0, 1)$, but find this is infeasible.

Since there are no more branches left to check (all have been pruned or found infeasible), we conclude the solution to the optimal solution under these integer constraints is when $\vec{x} = (1, 0, 0, 1)$, giving value 11000.

2 Max flow

We may solve this by setting up and solving the dual linear program for this flow problem (see notes). We write d_{nm} for the dual variable associated to an edge traveling from vertex n to vertex m, and write p_i for dual variable associated to vertex i. In other words, d_{nm} come from the inequality constraints (flow cannot be greater than capacity) and p_i come from the equality constraints (flow must be conserved at non-source or sink vertices). The program is:

Maximize:

$$16d_{01} + 13d_{02} + 10d_{12} + 4d_{21} + 12d_{13} + 9d_{32} + 14d_{24} + 7d_{34} + 20d_{35} + 4d_{45}$$

Subject to:

- $d_{01} + p_1 \geq 1$
- $d_{02} + p_2 \geq 1$
- $d_{12} - p_1 - p_2 \geq 0$
- $d_{21} - p_1 - p_2 \geq 0$
- $d_{13} - p_1 - p_3 \geq 0$
- $d_{32} - p_2 - p_3 \geq 0$
- $d_{14} - p_2 - p_4 \geq 0$
- $d_{34} - p_3 - p_4 \geq 0$
- $d_{35} - p_3 \geq 0$
- $d_{45} - p_4 \geq 0$
- $d_{01}, d_{02}, d_{12}, d_{21}, d_{13}, d_{32}, d_{14}, d_{35}, d_{45} \geq 0$

It is easiest to solve this using linsolve in Python. The optimal solution to this program yields the minimum cut, in the sense that a nonzero values for d_{ij} corresponds to edge ij being included in the minimum cut. By strong duality, the value of the min cut (i.e. summing up all capacities on these edges) gives the max flow for this problem. Solving the above problem shows that the min cut is given by $e_{13} = 1 \to 3$, $e_{43} = 4 \to 3$, $e_{45} = 4 \to 5$ and hence the max flow is $12 + 7 + 4 = 23$.