Tufts University - Department of Mathematics
Math 87 Homework 1
Due: Thursday, Sept 14

1 Incentivizing gym memberships

A high-end gym wishes to offer a New Year's special to all customers who sign up for a year long membership before February 1st. The gym typically charges $150 a month and is trying to decide how much of a discount to offer. It is estimated that for every $100 off the yearly price, the number of gym memberships purchased will increase by 15%.

1. How much of a discount will maximize the gyms profits on this special? Model the question as a single-variable optimization problem.

2. Compute the sensitivity of the optimal discount and the corresponding profit to the 15% assumption.

3. Suppose that the special only generates a 10% increase in sales per $100. What is the effect?

4. Under what circumstances would an offer of a special cause a reduction in profit (your answer should be quantitative)?

2 Computing yields with multivariate optimization

A chemist is synthesizing a compound. In the last step, she must dissolve her reagents in a solution with a particular pH level H, for $1.2 \leq H \leq 2.7$, and heated to a temperature T (in degrees Celsius), for $66 \leq T \leq 98$. Her goal is to maximize her percent yield as a percentage of the initial mass of the reagents. The equation determining the percentage $F(H, T)$ is

$$F(H,T) = -0.038T^2 - 0.223TH - 10.982H^2 + 7.112T + 60.912H - 328.898.$$

1. Find the optimal temperature and pH level in the allowed range.

2. Use matplotlib to produce a graph and a contour plot of the percentage of the powder function $F(H, T)$.
   (To get a usable copy of your image, you can proceed in a few ways:

   - if you produce the graph in colab you can right-click on the image and Save As a file on your file system.
   - if you work in Python on your computer, you can save the image via a command like
     >>> g.savefig("my graph image.png")

3 Hardy, Weinberg, Lagrange

Human blood is generally classified in the "ABO" system, with four blood types: A, B, O, and AB. These four types reflect six gene pairs (genotypes), with blood type A corresponding to gene pairs AA and AO, blood type B corresponding to gene pairs BB and BO, blood type O corresponding to gene pair OO, and blood type AB corresponding to gene pair AB. Let p be the proportion of gene A in the population, q be the proportion of gene B in the population, and r be the proportion of gene O in the population. Note that $p + q + r = 1$.

1. The Hardy-Weinberg principle states that p, q, and r are fixed from generation to generation, as are the frequencies of the different genotypes. Under this assumption, what is the probability that an individual has genotype AA? BB? OO? What is the probability of an individual having two different genes?

2. Find the maximum percentage of the population that can have two different genes under the Hardy-Weinberg principle in two different ways, by directly maximizing a function of only two variables and by using the method of Lagrange multipliers.

3. Can you say what the Lagrange multiplier represents in the above example?