Homework 7

Math 87

1 You can't handle the truth!

1. We may model this problem as a binomial random variable with the probability of success equal to $p = 0.6$. With this in mind, the probability of a tie is $\binom{12}{6}0.7^6 0.3^6 = 0.0792$.
2. Since the judge's vote is independent of the jury, the probability of a tie AND the judge voting correctly is the product of the two probabilities, i.e. $0.0792 * 0.8 = 0.0634$.
3. The probability that at least 7 members vote correctly is $\sum_{k=7}^{12}\binom{12}{k}0.7^k 0.3^{12-k}$. We then add the probability from part 2 to this to obtain the probability a correct verdict is reached, which gives a probability of 0.9455.
4. This is the probability of an incorrect verdict $(1 - 0.9455 = 0.0545)$ times the probability that the convicted was actually innocent (0.1), which gives 0.0055.
5. $2,000,000 * 0.0055 = 11000$ people.
6. Example code:

Likelihood

```
import math
def likelihood(p):
    C=np.arange(7,13)
    c=0
    for i in C:
        c=c+math.comb(12,i)*p**i*(1-p)**(12-i)
    tie=math.comb(12,6)*p**6*(1-p)**6
    return (1-(c+tie*0.8))*.1
```

7. The following pairs give (likelihood, expected number of people):
   - $p = 0.5$: $(0.043232421875000001, 86464)$
   - $p = 0.6$: $(0.019352812257280007, 38705)$
   - $p = 0.7$: $(0.0054450422218799924, 10890)$
   - $p = 0.8$: $(0.0007003561164799833, 1400)$
   - $p = 0.9$: $(1.483906347999886 \cdot 10^{-5}, 29)$
8. It is unlikely that a real jury will be independent of one another and that a judge's vote will be independent of the jury. It is even unrealistic that a jury will all have the same voting habits, as these will no doubt be informed by individual biases of each jury member, as well as factors about the person on trial (e.g. race, occupation etc.)

2 The Germinator

1. After k days in the lab, the population of bacteria is going to be:

$$p_k = p_{k-1} + 0.0635 p_{k-1} + m \tag{1}$$

2. To solve this recurrence relation, we first determine the steady state, of the form:

$$\bar{p} = -\frac{m}{r} = (1+r)\bar{p} + m,$$

where $r = 0.0635$. This then lets us write the term C_k, defined as the difference of the k-th step of the recurrence and the steady state:

$$C_k = p_k - \bar{p} = (1+r)p_{k-1} + m - ((1+r)\bar{p} + m) = (1+r)C_{k-1} = C_0(1+r)^k.$$

The resultant recurrence relation can be solved as:

$$p_k = -\frac{m}{r} + (p_0 + \frac{m}{r}(1+r)^k \tag{2}$$

$$= -\frac{m}{0.0635} + (500 + \frac{m}{0.0635})(1.0635)^k \tag{3}$$

3. We are looking for the day where we will have the needed amount of bacteria in the lab. Let's call this number g, letting us express:

$$g = -\frac{m}{0.0635} + (500 + \frac{m}{0.0635})(1.0635)^k.$$

This can be rewritten as:

$$\frac{g + \frac{m}{0.0635}}{500 + \frac{m}{0.0635}} = 1.0635^k \quad \text{or} \quad \log(\frac{g + \frac{m}{0.0635}}{500 + \frac{m}{0.0635}}) = k\log(1.0635),$$

leading to:

$$k = \frac{\log(g + \frac{m}{0.0635}) - \log(500 + \frac{m}{0.0635})}{\log 1.0635} \tag{4}$$

4. This final expression can be written as a python function of the form:

Sampling

```
def bact(g, r, N, m):
    return np.log((g + (m/r))/(N + (m/r)))/np.log(1 + r)
```

Which lets us determine the number of days needed to grow a colony of a desired size, as presented in Table 1.

Number of bacteria	Bacteria added/day	Time
200,000	100	74.3 days
	250	62.2 days
2,000,000	100	3.6 months
	250	3.2 months

Table 1: Time it takes for a sample of $N = 500$ bacteria growing at a rate of $r = 0.0635$ to grow into the desired size.