ProblemSet 4 – Integer programs and max flow **solutions**

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1. One man's trash

   You are at a yard sale, and have spied four crates of goods. You've estimated the value of each crate; these values are listed as **actual value** in the following table.

   The owner has no idea what these items are worth, and is selling them for far less than they are worth; the prices being asked are listed as **sales price** in the following table).

   Finally, the *weight* of each of the crates is listed in the table, as well.

crate	actual value	sales price	weight in kg
A	\$ 5000	\$ 24	75.5
B	\$ 600	\$ 76	2.7
C	\$ 3500	\$ 43	3.3
D	\$ 6000	\$ 754	6.7

   You realize that you can purchase these crates and sell them at a much higher mark up. However, you walked to the yard sale and can only buy what you can carry on your person.

   You have 800 dollars, and you and your friend together can carry an estimated 85 kg.

   Fortunately, you have identified this as an integer programming problem!

   a. Describe an integer linear program which models the situation (think carefully about what values the variables can take).

      The integer program problem is:

      consider a vector `x = [A, B, C, D]` of variables, where `A` represents the purchase of the crate `A`, `B` represents the purchase of the crate `B`, etc.

```
maximize

value = actual_value - purchase_price
```

where

```
actual_value = [ 5000, 600, 3500, 6000 ] . [ A, B, C, D].T
             = 5000 A + 600 B + 3500 C + 6000 D


purchase_price = [ 75.7, 27, 3.3, 6.7 ] . [ A, B, C, D].T
               = 75.7 A + 27 B + 3.3 C + 6.7 D
```

subject to:

- `[ 24, 76, 43, 754 ] . x.T <= 800` (i.e. `24 A + 76 B + 43 C + 754 D <= 800`),
- `[ 75.5, 27, 3.3, 6.7] . x.T <= 85` (i.e. `75.5 A + 27 B + 3.3 C + 6.7 D <= 85`), and
- `A,B,C,D` in `[ 0,1 ]`

b. Use the branch and bound algorithm to find the optimal solution, explaining your choices for which variables to branch on and where to prune the tree.

We first solve the *relaxed* linear program. Note that we specify `0 <= A,B,C,D <=` using the `bounds` argument to `linprog`.

```python
import numpy as np
from scipy.optimize import linprog

actual_value = np.array([ 5000, 600, 3500, 6000 ])
sales_price  = np.array([ 24, 76, 43, 754 ])
weight       = np.array([ 75.5, 27, 3.3, 6.7 ])

obj = actual_value - sales_price

bounds = 4*[(0,1)]

Aub = np.array([sales_price,weight])
bub = np.array([800,85])

res = linprog((-1)*obj, A_ub = Aub, b_ub = bub,bounds=bounds)

pprint({ 'obj_value': res.fun, 'solution': res.x})
=>
{'obj_value': -13512.872801068243,
'solution': array([0.99583731, 0.        , 1.        , 0.97228104])}
```

We are now going to branch; we'll use the following code.

```
# use the python pretty-printer
from pprint import pprint

# make the jth standard basis vector of length 'size'
def sbv(j,size):
   return np.array([1.0 if i == j else 0.0 for i in range(size)])

# record the data for the linear program as a dictionary, for ease of passage
lp = { 'obj': obj,
       'Aub': Aub,
       'bub': bub,
       'bounds': bounds
       }

def branch(specs,lp):
    n = len(lp["obj"])

    # each spec is a dictionary {"var": a, "value": b}

    # first, lookup the indices of the variable for each spec
    crates = ['A','B','C','D']
    indices = [ crates.index(spec['var']) for spec in specs ]

    # now create equality constraints for the "specs"
    Aeq = np.array([sbv(index,4) for index in indices])
    beq = np.array([spec["value"] for spec in specs])

    result = linprog((-1)*lp["obj"],
                     bounds = lp["bounds"],
                     A_ub=lp["Aub"],
                     b_ub=lp["bub"],
                     A_eq = Aeq,
                     b_eq = beq)

    if result.success:
        return {"obj_value": (-1)*result.fun,
                "solution": result.x}
    else:
        return "lin program failed"
```

Since the values of `B` and `C` are already integers in the relaxed solution, and since `1-A < 1-D`, we first branch on `A`.

- `A=0`: We solve the linear program specifying `A=0`.

  ```
  res_A0 = branch([{'var': 'A', 'val': 0}],lp)
  ```

```
pprint(res_A0)
=>
{'obj_value': 8723.684210526317,
'solution': array([-0.        ,  0.03947368,  1.        ,  1.        ])}
```

- `A=1`: We solve the linear program specifying `A=1`.

  ```
  res_A1 = branch([{'var': 'A', 'val': 1}],lp)
  pprint(res_A1)
  =>
  {'obj_value': 13287.50746268657,
  'solution': array([1.        , 0.        , 1.        , 0.92537313])}
  ```

Since the `obj_value` is larger for `A=1`, we branch below that value on `D`.

- `A=1, D=0`

  ```
  res_A1_D0 = branch([{'var': 'A', 'val': 1},
                      {'var': 'D', 'val': 0}],lp)
  pprint(res_A1_D0)
  =>
  {'obj_value': 8553.325925925925,
  'solution': array([ 1.        ,  0.22962963,  1.        , -0.        ])}
  ```

- `A=1, D=1`

  ```
  res_A1_D1 = branch([{'var': 'A', 'val': 1},
                      {'var': 'D', 'val': 1}],lp)
  pprint(res_A1_D1)
  =>
  {'obj_value': 11990.697674418605,
  'solution': array([1.        , 0.        , 0.51162791, 1.        ])}
  ```

The largest objective value so far is `A=1,D=1` so we branch below that node, on the variable `C`.

- `A=1, D=1, C=0`

  ```
  res_A1_D1_C0 = branch([{'var': 'A', 'val': 1},
                         {'var': 'D', 'val': 1},
                         {'var': 'C', 'val': 0}],lp)
  pprint(res_A1_D1_C0)
  =>
  {'obj_value': 10276.34074074074,
  'solution': array([ 1.       ,  0.1037037, -0.       ,  1.       ])}
  ```

- `A=1, D=1, C=1`

  ```
  res_A1_D1_C1 = branch([{'var': 'A', 'val': 1},
                         {'var': 'D', 'val': 1},
                         {'var': 'C', 'val': 1}],lp)
  ```

```
pprint(res_A1_D1_C1)
=>
'lin program failed'
```

This linear program is infeasible, so we **prune**.

Now we branch on `B` below `A=1, D=1, C=0`:

- `A=1, D=1, C=0, B=0`

  ```
  res_A1_D1_C0_B0 = branch([{'var': 'A', 'val': 1},
                            {'var': 'D', 'val': 1},
                            {'var': 'C', 'val': 0},
                            {'var': 'B', 'val': 0}],lp)
  pprint(res_A1_D1_C0_B0)
  =>
  {'obj_value': 10222.0, 'solution': array([ 1., -0., -0.,  1.])}
  ```

- `A=1, D=1, C=0, B=1`

  ```
  res_A1_D1_C0_B1 = branch([{'var': 'A', 'val': 1},
                            {'var': 'D', 'val': 1},
                            {'var': 'C', 'val': 0},
                            {'var': 'B', 'val': 1}],lp)
  pprint(res_A1_D1_C0_B1)
  =>
  'lin program failed'
  ```

  This linear program is infeasible, so we **prune** this node.

Now we pause to inspect our results so far. We have found an integral solution `A=1, D=1, C=0, B=0` with `obj_value = 10222`.

This objective value exceeds `8724` which is the `obj_value` at `A=0`. So we prune at `A=0`

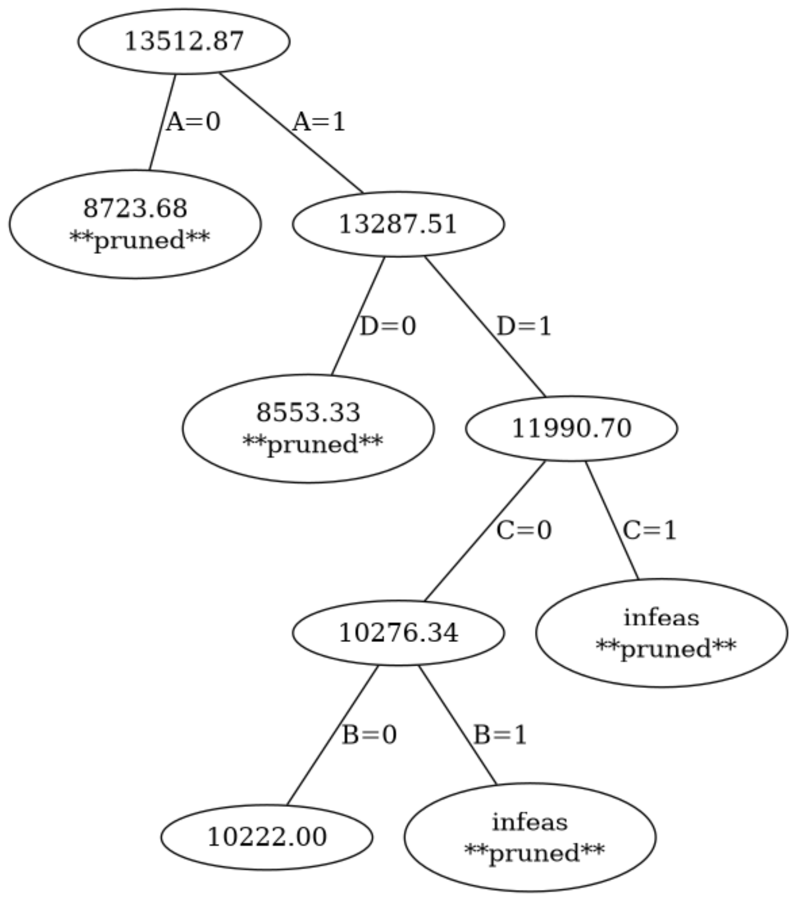
And this objective value exceeds `8553` which is the `obj_value` at `A=1,D=0`. So we prune at `A=1,D=0`.

So we find that the optimal integral solution to the linear program is to buy the `A`-crate and the `D`-crate, getting a real value of `10,222` from the purchase.

c. Draw the branch and bound tree for your solution.

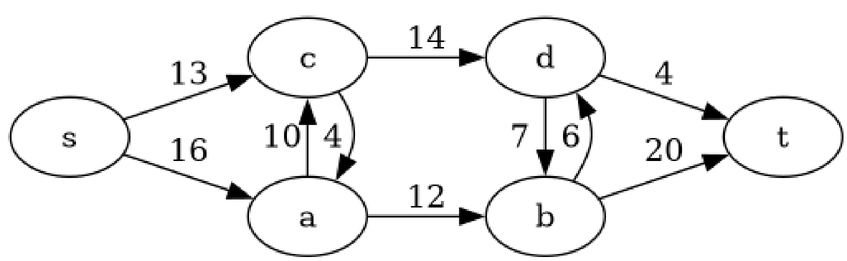
```
from graphviz import Graph

nodes = { 0: f"{(-1)*res.fun:.2F}",
          1: f"{res_A0['obj_value']:.2f}",
          2: f"{res_A1['obj_value']:.2f}",
          3: f"{res_A1_D0['obj_value']:.2f}",
          4: f"{res_A1_D1['obj_value']:.2f}",
```

```
          5: f"{res_A1_D1_C0['obj_value']:.2f}",
          6: "infeas",
          7: f"{res_A1_D1_C0_B0['obj_value']:.2f}",
          8: "infeas"
          }

pruned = [1,3,6,8]

def describe(n):
    if n in pruned:
        return f"{nodes[n]}\n **pruned**"
    else:
        return f"{nodes[n]}"

dot = Graph()
dot.filename='PS4--tree'
dot.format='png'


for n in range(9):
    dot.node(f"{n}",describe(n))

dot.edge('0','1','A=0')
dot.edge('0','2','A=1')
dot.edge('2','3','D=0')
dot.edge('2','4','D=1')
dot.edge('4','5','C=0')
dot.edge('4','6','C=1')
dot.edge('5','7','B=0')
dot.edge('5','8','B=1')

dot.render()
```



(**Hint**: Note that you should use `linprog` to solve the relaxed linear program, initially with your variables constrained between 0 and 1).

2. Consider the following directed graph (see below for the code producing this graph).



   a. Find the `minimum cut` value for this weighted directed graph.

      (Recall that this means to consider all possible partitions of the nodes into an s-group and a t-group.

To identify such a partition, it is enough to indicate the s-group. For example, s, a and c together form a possible s-group. The edge-cuts required to form this s-group involve the edges $c \to d$ and $a \to b$; thus the cut-value for this partition is $14 + 12 = 26$.

Make a list all possible s-groups and indicate the corresponding cut-values. Remember that we are only interested in partitions that arise "from cuts" – thus, an s-group should be "connected". And remember that cut value only involve capacities of edges $u \to v$ where u is in the s-group and v is in the t-group (you would not also count the capacity of an edge $v \to u$ if it exists).

We make a list whose elements are pairs `( s-groups, edge-cut values )` for all possible `s`-groups:

```
pp = [ ( ['s'], [13, 16] ),
       ( ['s', 'a'], [13, 10, 12] ),
       ( ['s', 'c'], [16, 4, 14] ),
       ( ['s', 'a', 'c'], [12, 14] ),
       ( ['s', 'a', 'b'], [13, 10, 7, 20] ),
       ( ['s', 'a', 'b', 'c'], [14, 6, 20] ),
       ( ['s', 'a', 'c', 'd'], [12, 7, 4] ),
       ( ['s', 'a', 'b', 'c', 'd'], [4,20])
      ]

# sum the cut-values, and sort by the results
def min_cut(data):
    aa = [ (s, sum(cuts) ) for  (s,cuts) in data]
    aa.sort(key=lambda x: x[1])
    return aa

pprint(min_cut_data)
=>
[(['s', 'a', 'c', 'd'], 23),
 (['s', 'a', 'b', 'c', 'd'], 24),
 (['s', 'a', 'c'], 26),
 (['s'], 29),
 (['s', 'c'], 34),
 (['s', 'a'], 35),
 (['s', 'a', 'b', 'c'], 40),
 (['s', 'a', 'b'], 50)]
```

The result shows that the `min-cut` is achieved by considering the `s`-group `['s','a','c','d']`. For this, you must cut the edge `a -> b`, `d->b`, and `d->t`; these have value 12, 7, 4. This shows that the `min-cut` value is 23.

b. By strong duality, you now know the `maximum flow` value for the graph. Does strong duality tell you how to find a flow which achieves this value? Why or why not?

   Strong duality tells us that the `max-flow` is 23. But It doesn't tell us how to find the flow `f` that achieves `|f| = 23`.

c. Suppose that the capacity on the edge $d \to t$ is increased from its current value of 4. By how much must this capacity change in order to change the `min cut` configuration?

   Since the `min-cut` is achieved with the group `['s','a','c','d']`, the edge `d -> t` is involved in the edge cuts. So increasing the edge weight will change the configuration.

   To see what happens, we can create a function to modify the specified edge-weight:

```
def pp1(x):
  return [ ( ['s'], [13, 16] ),
           ( ['s', 'a'], [13, 10, 12] ),
           ( ['s', 'c'], [16, 4, 14] ),
           ( ['s', 'a', 'c'], [12, 14] ),
           ( ['s', 'a', 'b'], [13, 10, 7, 20] ),
           ( ['s', 'a', 'b', 'c'], [14, 6, 20] ),
           ( ['s', 'a', 'c', 'd'], [12, 7, 4+x] ),
           ( ['s', 'a', 'b', 'c', 'd'], [4+x, 20])
           ]

for c in range(1,5):
  print(f"{c} -> {min_cut(pp1(c))[0]}")
=>
1 -> (['s', 'a', 'c', 'd'], 24)
2 -> (['s', 'a', 'c', 'd'], 25)
3 -> (['s', 'a', 'c'], 26)
4 -> (['s', 'a', 'c'], 26)
```

   We see that raising the indicated edge weight by 1 or 2 (i.e. to a value of 5), the `min-cut` is achieved with the same `s`-group, but once the edge weight is increased by 3 the `min-cut` is achieved by a different choice of `s`-group.

d. Would increasing the capacity for the edge $b \to t$ from its current value of 20 affect the maximum flow value? Why or why not?

   No, because the edge `b -> t` is not involved in the `min-cut` configuration.

e. Explain whether or not the following statement seems reasonable, and why: "The `min cut` configuration (i.e. the edges involved in the `min cut`) reflects the maximal bottleneck in the system."

Here is the code used to produce the graph; you can download it here.

```
from graphviz import Digraph

dg = Digraph()
#dg = Digraph(engine='neato')

dg.attr(rankdir='LR')

V = [ 's', 'a', 'b', 'c', 'd', 't' ]

weights = { ('s','a'):   16,
            ('s','c'):   13,
            ('c','a'):    4,
            ('a','c'):   10,
            ('a','b'):   12,
            ('c','d'):   14,
            ('d','b'):    7,
            ('b','d'):    6,
            ('b','t'):   20,
            ('d','t'):    4
           }

c.node('t')

with dg.subgraph() as c:
    c.attr(rank='same')
    for x in ['a','c']:
        c.node(x)

with dg.subgraph() as c:
    c.attr(rank='same')
    for x in ['b','d']:
        c.node(x)

c.node('s')

for (f,t) in weights.keys():
    wt = f"{weights[(f,t)]}"
    dg.edge(f,t,wt)

dg.filename='PS4--graph'
dg.format='png'
dg.render()
```