ProblemSet 2 – Linear programs

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1. A linear program

   Consider the optimization problem: find the $\max$ of $f(x,y) = x + 2y$ subject to the following constraints:

$$\begin{aligned} y &\le 9 \\ -y &\le -1 \\ 2x + y &\le 25 \\ -2x - y &\le -9 \\ -2x + y &\le 1 \\ 2x - y &\le 15. \end{aligned}$$

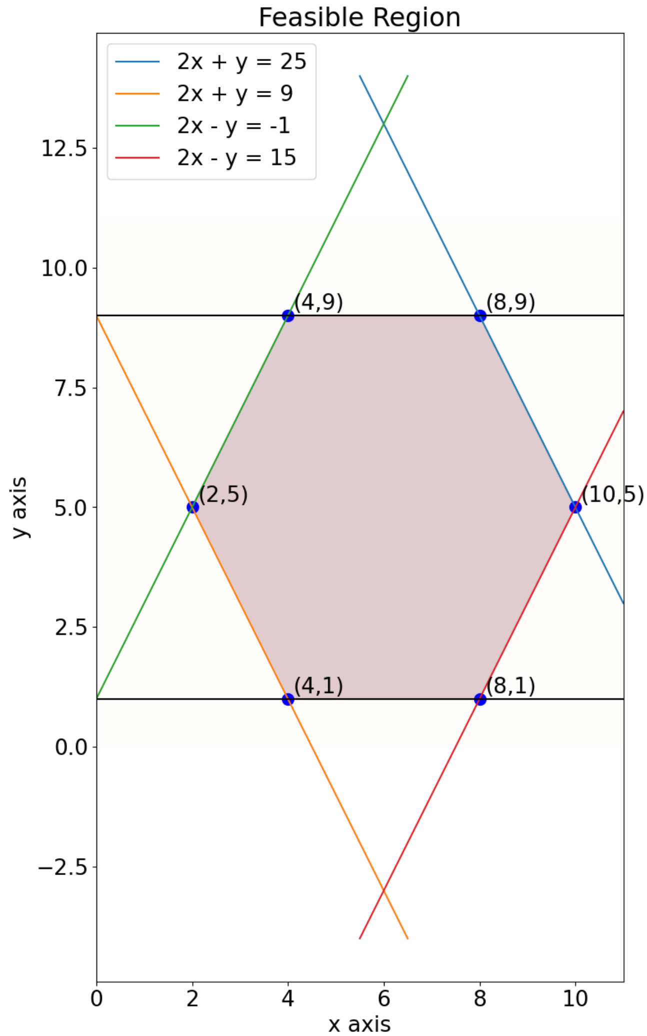
   a. Draw the feasible region. Label the boundary curves and corner points.

We can use `matplotlib` to draw a sketch of the feasible region, as follows. Of course, you can also just hand sketch it!

```
import matplotlib.pyplot as plt
import numpy as np
plt.rcParams.update({'font.size': 19})

# plot the feasible region
d = np.linspace(0,11,500)
e = np.linspace(0,11,500)

xx,yy = np.meshgrid(d,e)

fig,ax = plt.subplots(figsize=(15,15))
ax.imshow(((yy >= 1) &
           (yy <= 9) &
           (9 <= 2*xx + yy) &
           (2*xx + yy <= 25) &
           (-1 <= 2*xx - yy) &
           (2*xx - yy <= 15)
           ).astype(int),
          extent=(xx.min(),xx.max(),yy.min(),yy.max()),
          origin="lower",
          cmap="Reds",
          alpha = 0.2)

# plot the lines defining the constraints

def l1(x): return 25 - 2*x

def l2(x): return 9 - 2*x

def l3(x): return 2*x + 1

def l4(x): return 2*x - 15

x = np.linspace(5.5,11,500)
```

```python
ax.plot(x, l1(x) , label="2x + y = 25")

x = np.linspace(0,6.5,500)
ax.plot(x, l2(x) , label="2x + y = 9")

x = np.linspace(0,6.5,500)
ax.plot(x, l3(x), label = "2x - y = -1")

x = np.linspace(5.5,11,500)
ax.plot(x, l4(x), label = "2x - y = 15")

ax.axhline(y=1, color = "black")
ax.axhline(y=9, color = "black")

ax.legend()
ax.set_title("Feasible Region")
ax.set_xlabel("x axis")
ax.set_ylabel("y axis")

def ann_pt(x,y):
    "annotate the point (x,y)"
    s = f"({x},{y})"
    ax.annotate(s,xy=(x,y),xytext=(5,5),textcoords='offset points')

ax.scatter(8, 9,s=100,color="blue")
ann_pt(8,9)

ax.scatter(4,9,s=100,color="blue")
ann_pt(4,9)

ax.scatter(4,1,s=100,color="blue")
ann_pt(4,1)

ax.scatter(8,1,s=100,color="blue")
ann_pt(8,1)

ax.scatter(2,5,s=100,color="blue")
ann_pt(2,5)

ax.scatter(10,5,s=100,color="blue")
ann_pt(10,5)

fig.savefig("feasible.png")
```

which results in



b. Find the maximum value of f and the point where it occurs.

We evaluate the function f at each intersection point.

```
intersection_pts = [ (4,9), (8,9), (10,5), (8,1), (4,1), (2,5) ]

def f(x,y): return x + 2*y

## make a list of the pairs ( f(pt), pt ) and sort the list

a=[ ( f(pt[0],pt[1]) , pt ) for pt in intersection_pts ]

a.sort()
```

This yields

```
[(6, (4, 1)), (10, (8, 1)), (12, (2, 5)), (20, (10, 5)), (22, (4, 9)), (26,
      (8, 9))]
```

We see that the *maximum value of f* is 26, and this maximum occurs at $(x,y) = (8,9)$.

c. Verify your answer using `SciPy`.

We need to define the *objective function* `f` as a *vector* `c`, and we must define the `inequality constraints.`

```
import numpy as np
c = np.array([1,2])

Aub = np.array([ [0,1],
                 [0,-1],
                 [2,1],
                 [-2,-1],
                 [-2,1],
                 [2,-1]
                 ])

bub = np.array([ 9,
                -1,
                25,
                -9,
                1,
                15])
```

and now we run the `linprog` solver. Remember that since we want to *maximize* the objective function, we must use `-c` for the objective function!

```
from scipy.optimize import linprog
 linprog((-1)*c,A_ub=Aub,b_ub=bub)
```

This gives the result

```
        message: Optimization terminated successfully. (HiGHS Status 7:
        Optimal)
        success: True
         status: 0
            fun: -26.0
              x: [ 8.000e+00  9.000e+00]
            nit: 2
          lower:  residual: [ 8.000e+00  9.000e+00]
                 marginals: [ 0.000e+00  0.000e+00]
          upper:  residual: [       inf        inf]
                 marginals: [ 0.000e+00  0.000e+00]
          eqlin:  residual: []
                 marginals: []
        ineqlin:  residual: [ 0.000e+00  8.000e+00  0.000e+00  1.600e+01
                              8.000e+00  8.000e+00]
                 marginals: [-1.500e+00 -0.000e+00 -5.000e-01 -0.000e+00
                             -0.000e+00 -0.000e+00]
 mip_node_count: 0
 mip_dual_bound: 0.0
        mip_gap: 0.0
```

which confirms that the max of 26 occurs at the point `(8,9)` (see the fields `fun` and `x` in the result).

2. Bakers

A bakery wants to sell forty five Valentine's Day gift bags. They have decided to offer two types of bags:

- Bags of type A will contain four cupcakes and two cookies, and will be sold for \$12
- bags of type B will contain two cupcakes and five cookies, and will be sold for \$16

The bakery has 90 cookies and 115 cupcakes in total. Write the bakery's optimization problem as a linear program. Solve this to determine how many baskets of both types should be made. If a fractional solution is obtained, round down to whole number solutions. What is the maximum profit?

You may solve this by drawing the feasible region or using python.

We've written some code that assembles the objective function `obj`, and the constraint data `Aub` and `bub` in order to use `scipy.optimize.linprog` to solve the linear program.

```
import numpy as np
from scipy.optimize import linprog

bag_specs = { 'A' : { 'price': 12.0,
                      'cupcakes': 4,
                      'cookies': 2,
                      'bags': 1
                      },
              'B' : { 'price': 16.0,
                      'cupcakes': 2,
                      'cookies': 5,
                      'bags': 1
                      }
              }

resources = { 'cupcakes': 115.0,
              'cookies': 90.0,
              'bags': 45
              }


# objective function, as a vector
obj = np.array([ bags_specs[item]['price'] for item in bag_specs.keys() ])

# upper bound constraint matrix
Aub = np.array([ [ bag_specs[item][resource]
                   for item in bag_specs.keys()
                 ]
                for resource in resources.keys()
               ])

# upper bound constraint vector
bub = np.array([ resources[res] for res in resources.keys()])

# remember that this is a maximizing problem, so multiply the objective
        function by (-1)
#
res = linprog( (-1)*obj, A_ub = Aub, b_ub = bub)
```

This is perhaps overly complicated for a 2-variable problem, but it yields the following values:

```
obj
=> array([12., 16.])
```

```
Aub
=> array([[4, 2],
          [2, 5],
          [1, 1]])

bub
=> array([115.,  90.,  45.])

res
=>
        message: Optimization terminated successfully. (HiGHS Status 7:
         Optimal)
        success: True
         status: 0
            fun: -426.25
              x: [ 2.469e+01  8.125e+00]
            nit: 2
          lower:  residual: [ 2.469e+01  8.125e+00]
                 marginals: [ 0.000e+00  0.000e+00]
          upper:  residual: [       inf        inf]
                 marginals: [ 0.000e+00  0.000e+00]
          eqlin:  residual: []
                 marginals: []
        ineqlin:  residual: [ 0.000e+00  0.000e+00]
                 marginals: [-1.750e+00 -2.500e+00]
 mip_node_count: 0
 mip_dual_bound: 0.0
        mip_gap: 0.0
```

This result indicates that the max profit should be 426.25, obtained at the point $(24.69, 8.125)$. However, since we need to round to a whole number we find that we should make 24 or 25 A-type bags and 8 or 9 B type bags.

Let’s compute

(notice that the numpy operation `M @ v` means “multiply the matrix M and the vector v”. You could also do this via `np.matmul(M,v)`).

```
pp = [ [24,8], [24,9], [25,8], [25,9] ]

print("\n".join([ f"{p} => {Aub @ np.array(p)}    {Aub @ np.array(p) <= bub}"
      for  p in pp ]))

=>
[24, 8] => [112  88]   [ True  True]
[24, 9] => [114  93]   [ True False]
[25, 8] => [116  90]   [False  True]
[25, 9] => [118  95]   [False False]
```

This shows that, for example, making 24 A bags and 9 B bags requires 114 cupcakes and 93 cookies (which exceeds available resources).

Inspecting the list, we see that we should make 24 A bags and 8 B bags, yielding a profit of $416

```
obj @ np.array([24,8])
=>
416
```

3. A farmer owns 45 acres of land. This season, she will plant each acre with either wheat or corn.

Each acre of wheat yields $200 in seasonal profits, whereas each acre of corn yields $300 in seasonal profits. Each acre of wheat requires 3 workers and 2 tons of fertilizer, while each acre of corn requires 2 workers and 4 tons of fertilizer. The farmer has 100 workers and 120 tons of fertilizer available. Determine how many acres of wheat and corn need to be planted to maximize profits for the season. (Non-integer acreage values are allowed in the solution.)

We've written some code that assembles the objective function `obj`, and the constraint data `Aub` and `bub` in order to use `scipy.optimize.linprog` to solve the linear program.

```
import numpy as np
from scipy.optimize import linprog


## requirements per acre,
## as a (nested) python dictionary
## e.g. `crop_specs['corn']['fertilizer']` reflects the
## amount of fertilizer (in tons) required
## for an acre of corn
crop_specs = { 'wheat' : { 'profits': 200.0,
                           'workers': 3,
                           'fertilizer': 2,
                           'acres': 1
                         },
               'corn' : { 'profits': 300.0,
                          'workers': 2,
                          'fertilizer': 4,
                          'acres': 1
                        }
             }

resource_specs = { 'workers': 100.0,
                   'fertilizer': 120.0,
                   'acres': 45
                 }

# objective function, as a vector
obj = np.array([ crop_specs[c]['profits'] for c in crop_specs.keys() ])

# upper bound constraint matrix
# this matrix has a row for each `resource`,
# and the entries for that row are obtained from the `crop_specs`
Aub = np.array([ [ crop_specs[c][res]
                   for c in crop_specs.keys()
                 ]
                 for res in resource_specs.keys()
               ])

# upper bound constraint vector
bub = np.array([ resource_specs[res] for res in resource_specs.keys()])

res = linprog( (-1)*obj, A_ub = Aub, b_ub = bub)
```

This gives the following results:

```
obj
=>
[200. 300.]

Aub
=>
[[3 2]
```

```
 [2 4]
 [1 1]]

 bub
 =>
[100. 120.  45.]

res
=>
           message: Optimization terminated successfully. (HiGHS Status 7:
        Optimal)
        success: True
         status: 0
            fun: -10000.0
              x: [ 2.000e+01  2.000e+01]
            nit: 2
          lower:  residual: [ 2.000e+01  2.000e+01]
                 marginals: [ 0.000e+00  0.000e+00]
          upper:  residual: [       inf        inf]
                 marginals: [ 0.000e+00  0.000e+00]
          eqlin:  residual: []
                 marginals: []
        ineqlin:  residual: [ 0.000e+00  0.000e+00  5.000e+00]
                 marginals: [-2.500e+01 -6.250e+01 -0.000e+00]
 mip_node_count: 0
 mip_dual_bound: 0.0
        mip_gap: 0.0
```

The result shows that the maximal profit is $10,000 and it is achievd by planting 20 acres of corn and 20 acres of wheat.

This may seem slightly surprising, since after all we have 45 available acres.

Let’s investigate this solution a bit:

```
usage = Aub @ np.array([20,20])

for r in resources:
    print(f"{r} -> {usage[resources.index(r)]}")

=>
workers -> 100
fertilizer -> 120
acres -> 40
```

We see that this solutions requires all available of workers and all available tonnage of fertilizer, so it is indeed plausible that the maximum profit is achieved without use of the remaining 5 acres of land.