Homework 5

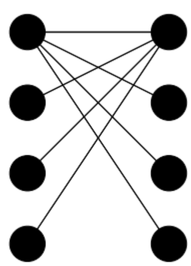
Math 87

October 2023

1 Matchmaker

It is best to model this problem as a bipartite graph matching problem, where one class of vertices represents quarries and the other represents processing plants, and edges represent rows.

1. There are many possible examples, for instance:



2. This answer will depend on your answer to part a. For the graph in our example, one solution would be to add a pair of edges from the bottom two vertices on the left to the bottom two vertices on the right.
3. The largest possible size of the edge set for a bipartite graph where $|U| = |W| = n$ is n^2.
4. The map $w : U \to W$ identifies which vertices in W may be matched to which vertices in W, hence the maximum number of distinct vertices $w(x)$ in the image correspond to a maximal matching. If w is a bijection, a perfect matching exists.

2 Growth predictions

1. From the matrix, we read that 80% of new users will still be subscribed next year.
2. Using the matrix and our initial population vector $\mathbf{p} = \begin{bmatrix} p_n \\ p_o \end{bmatrix}$, we may calculate the number of users next year as a function of the users this year: Observe that this corresponds to the first coordinate of the vector $A\mathbf{p}$, since this will correspond to the number of new users. We see that $A\mathbf{p} = \begin{bmatrix} p_n \\ 0.8p_n + 0.5p_o \end{bmatrix}$, and hence p_n new users will be recommended by a user this year.
3. Since we know $\lambda = 1$ is an eigenvalue, we may find the associated eigenvector $\mathbf{v}$ by solving $A\mathbf{v} = 1\mathbf{v} = \mathbf{v} := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, which is the system of equations $v_1 = v_1, 0.8v_1 + 0.5v_2 = v_2$. Solving for this, we obtain the solution $v_1 = \frac{5}{8}$, $v_2 = 1$. We normalize this to make this a probability

vector (which will give us percentages of population). The normalized vector is $v_1 =\sim .385$, $v_2 =\sim .615$. $\sim 38.5\%$ of the users are new users and 61.5% of users are old users.

4. Since the maximum eigenvalue is 0.5, $\lim_{n\to\infty} B^n \mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e. the service should lose all users.

5. $A^i\mathbf{p}$ and $B^i\mathbf{p}$ are vectors that give the populations in i years as a function of $\mathbf{p}$. Thus the expressions for the total number of subscribers for A and B over four years (as a function of $\mathbf{p}$) are

$$\mathbf{p} + A\mathbf{p} + A^2\mathbf{p} + A^3\mathbf{p}$$

$$\mathbf{p} + B\mathbf{p} + B^2\mathbf{p} + B^3\mathbf{p}.$$

Computing these, we obtain that the total users A has had is $6.5p_n + 1.875p_O$, and B has had $3.88p_n + 1.624p_O$. We then multiply these expressions by n to obtain the total revenue generated as a function of p_n, p_o, n.