Homework 9 Solutions

Math 87

1 Spline interpolation

Between three points we need 2 polynomials. Let them be

$$\begin{aligned} g_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3, \\ g_2(x) &= b_0 + b_1(x-1) + b_2(x-1)^2 + b_3(x-1)^3. \end{aligned}$$

Then the value conditions become

$$\begin{aligned} 0 = g_1(0) &= a_0, \\ 2 = g_1(1) &= a_0 + a_1 + a_2 + a_3, \\ 2 = g_2(1) &= b_0, \\ 1 = g_2(2) &= b_0 + b_1 + b_2 + b_3. \end{aligned}$$

Next we have $g_1'(1) = g_2'(1)$ and $g_1''(1) = g_2''(1)$ resulting in

$$\begin{aligned} a_1 x + 2a_2 + 3a_3 &= b_1, \\ 2a_2 + 6a_3 &= 2b_2. \end{aligned}$$

Then free boundary conditions result in

$$g_1''(0) = 2a_2 = 0, \quad g_2''(2) = 2b_2 + 6b_3 = 0,$$

while clamped result in

$$g_1'(0) = a_1 = 0, \quad g_2'(2) = b_1 + 2b_2 + 3b_3 = 0.$$

So in a matrix form the system with clamped boundary conditions will look like

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix} * \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

and the system with free boundary conditions takes the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 \end{pmatrix} * \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

Using numpy to solve these we get for clamped case

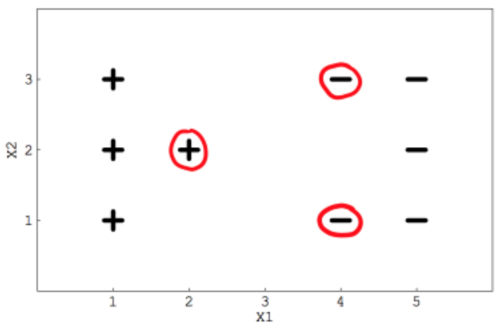
$$g_1(x) = 5.25x^2 - 3.25x^3, \quad g_2(x) = 2 + 0.75(x-1) - 4.5(x-1)^2 + 2.75(x-1)^3,$$

and for free case the polynomials comprising the spline are

$$g_1(x) = 2.75x - 0.75x^3, \quad g_2(x) = 2 + 0.5(x-1) - 2.25(x-1)^2 + 0.75(x-1)^3$$

2 Comprehending SVM

Suppose we have data drawn from two different populations, shown in the figure below as (+) and (−). Support vector machines are used to linearly separate such classes of data.



- The separating line should be vertical and equidistant from the (+) signs and the (−) signs.
- If the red circled plus sign is removed, the vertical line should move to the left.
- The supporting line would remain the same if all the red circled points are deleted.

3 Computing kernels

We may compute this by expanding

$$(\mathbf{x}.\mathbf{y} + c)^2 = (\mathbf{x}.\mathbf{y})^2 + 2c\mathbf{x}.\mathbf{y} + c^2 = (x_1y_1 + x_2y_2)^2 + 2cx_1y_1 + 2cx_2y_2 + c^2$$
$$= x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2 + 2cx_1y_1 + 2cx_2y_2 + c^2.$$

From this we see that the vectors

$$(x_1^2, \sqrt{2}x_1x_2, x_2^2, \sqrt{2c}x_1, \sqrt{2c}x_2, c)$$
$$(y_1^2, \sqrt{2}y_1y_2, y_2^2, \sqrt{2c}y_1, \sqrt{2c}y_2, c)$$

is a higher dimensional embedding that produces this kernel.