Formalization in Lean

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06/12/2025

Lean Basics

1. An **expression** in lean is essentially a mathematical expression converted into computer code, ex: numbers, functions, sets
2. Every expression in lean has a **type**. Examples of types:
   - natural numbers $\mathbb{N}$
   - functions $\mathbb{R} \to \mathbb{R}$
   - sets with elements in $\mathbb{R}$

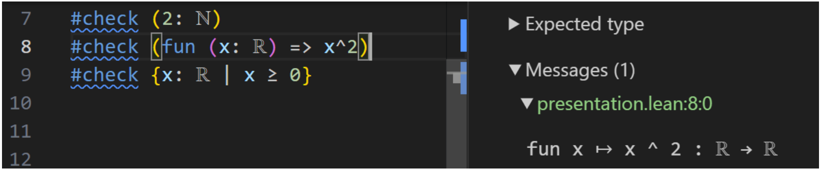
```
3    #check ℕ
4    #check ℝ → ℝ
5    #check Set ℝ
```

Lean Syntax

```
#check (2: ℕ)
#check (fun (x: ℝ) => x^2)
#check {x: ℝ | x ≥ 0}
```

```
▶ Expected type
▼ Messages (1)
  ▼ presentation.lean:7:0
    2 : ℕ
```

Lean Syntax


#check (2: ℕ)
#check (fun (x: ℝ) => x^2)
#check {x: ℝ | x ≥ 0}
▶ Expected type
▼ Messages (1)
▼ presentation.lean:8:0
fun x ↦ x ^ 2 : ℝ → ℝ

Lean Syntax

```
#check (2: ℕ)
#check (fun (x: ℝ) => x^2)
#check {x: ℝ | x ≥ 0}
```

▼ Messages (1)

▼ presentation.lean:9:0

```
{x | x ≥ 0} : Set ℝ
```

Propositions

1. A **proposition** is a mathematical statement converted to computer code
2. **Prop** is a type
3. Each proposition is an expression of type **Prop**
4. Examples of propositions:
   - $2 + 2 = 4$
   - $a < b$
   - $\forall x, \exists\, y$ such that $2x = y$

Proofs

Expressions of type P where P is of type *Prop* are proofs of P.

1. Expressions of type $1 < 2$ prove that $1 < 2$.
2. Expressions of type $x + y = y + x$ prove that $x + y = y + x$.

```
example: x+y=y+x :=
  add_comm x y
```

Lean proof example: $x * x = x^2$

```
example (x : ℤ) : x*x = x^2 := by
  rw [ pow_two ]
```

1 goal

```
x : ℤ
⊢ x * x = x ^ 2
```

pow_two definition: `⊢ ∀ {M : Type u_2} [inst : Monoid M] (a : M), a ^ 2 = a * a`

Lean proof example: $z * x * y * w = z * y * x * w$

```
example (x y z w : ℕ) : z * x * y * w = z * y * x * w := by
  rw [mul_assoc z x y]  Step 1
  rw [mul_assoc z y x]  Step 2
  rw [mul_comm x y]  Step 3
```

Step 1
```
x y z w : ℕ
⊢ z * (x * y) * w = z * y * x * w
```
Step 2
```
x y z w : ℕ
⊢ z * (x * y) * w = z * (y * x) * w
```
Step 3
```
x y z w : ℕ
⊢ z * (y * x) * w = z * (y * x) * w
```

```
example (x y z w : ℕ) : z * x * y * w = z * y * x * w := by
  ring
```

"Human" Proof of odd function times an odd function equals an even function

1. Let $f(x)$ and $g(x)$ be odd functions, so that $f(x) = -f(-x)$ and $g(x) = -g(-x)$. Let $h(x) = f(x) * g(x)$.
2. We want to show that $h(x)$ is even, or in other words, $h(x) = h(-x)$.
3. We have $h(x) = f(x) * g(x)$.
4. Using the definition of an odd function, $h(x) = -f(-x) * -g(-x) = f(-x) * g(-x)$.
5. We know $h(-x) = f(-x) * g(-x)$, so $h(x) = h(-x)$.
6. Thus, $h(x)$ is an even function.

Our definition of even and odd functions

```
def FnEven (f : ℝ → ℝ) : Prop :=
  ∀ x, f x = f (-x)

def FnOdd (f : ℝ → ℝ) : Prop :=
  ∀ x, f x = -f (-x)
```

Lean proof of odd function times an odd function equals an even function

```
example (of : FnOdd f) (og : FnOdd g) : FnEven fun x ↦ f x * g x := by
  intro x
  calc
    (fun x ↦ f x * g x) x = f x * g x :=rfl
    _ = -f (-x) * -g (-x) := by rw [of, og]
    _ = f (-x) * g (-x) := by ring
```

Original Goal

```
f g : ℝ → ℝ
of : FnOdd f
og : FnOdd g
⊢ FnEven fun x ↦ f x * g x
```

After intro x

```
f g : ℝ → ℝ
of : FnOdd f
og : FnOdd g
x : ℝ
⊢ (fun x ↦ f x * g x) x = (fun x ↦ f x * g x) (-x)
```

After of

```
f g : ℝ → ℝ
of : FnOdd f
og : FnOdd g
x : ℝ
⊢ -f (-x) * g x = -f (-x) * -g (-x)
```

After og

```
f g : ℝ → ℝ
of : FnOdd f
og : FnOdd g
x : ℝ
⊢ -f (-x) * -g (-x) = -f (-x) * -g (-x)
```

Alternative lean proof of odd function times an odd function equals an even function

```
0 example {f g} (of : FnOdd f) (og : FnOdd g) : FnEven fun x ↦ f x * g x := by
1   intro x₀
2   dsimp
3   rw[of, og]
4   exact neg_mul_neg (f (-x₀)) (g (-x₀))
```

0

```
1 goal
f g : ℝ → ℝ
of : FnOdd f
og : FnOdd g
⊢ FnEven fun x ↦ f x * g x
```

1

```
1 goal
f g : ℝ → ℝ
of : FnOdd f
og : FnOdd g
x₀ : ℝ
⊢ (fun x ↦ f x * g x) x₀ =
(fun x ↦ f x * g x) (-x₀)
```

2

```
1 goal
f g : ℝ → ℝ
of : FnOdd f
og : FnOdd g
x₀ : ℝ
⊢ f x₀ * g x₀ = f (-x₀) * g (-x₀)
```

3

```
1 goal
f g : ℝ → ℝ
of : FnOdd f
og : FnOdd g
x₀ : ℝ
⊢ -f (-x₀) * -g (-x₀) = f (-x₀) * g (-x₀)
```

Potential applications

1. Forms over finite fields
2. Quaternion algebra
3. Graph theory
4. The Grassmannian

What's the point?

1. The transformation from "human mathematics" to "formalized mathematics" removes inferences from proofs, continuously checking that proofs are correct
2. **mathlib** facilitates collaboration between mathematicians and allows proofs to be viewed by hundreds of scholars and students
3. Proofs can be continuously modified and maintained by other mathematicians rather than becoming incompatible with other developments in Lean
4. New methods to formalize a proof may have important pedagogical benefits

References

1. Browning, T., & Lutz, P. (2022). Formalizing Galois Theory. Experimental Mathematics, 31(2), 413-424.
2. Avigad, J. Buzzard, K. Lewis R. Y. Massot, P. (2020). *Mathematics in Lean*.

Thank you!

Special thanks to Dr. George McNinch and the REU