Forms over a finite field

George McNinch

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1 Some references

- Simeon Ball - "Finite Geometries and Combinatorial Applications" (2015, Cambridge Univ Press)

  (available electronically via Tufts' Tisch Library)

- *Linear algebra and matrices : topics for a second course* Shapiro, Helene, 1954- Providence, Rhode Island : American Mathematical Society, 2015

- some notes of Bill Casselmann (UBC)

2 Statements

Here is an overview of the results I'd like to see formalized. For alternating forms, there is no need to put any hypotheses on the field. For symmetric forms, we'll consider finite fields.

Let V be a finite dimensional vector space over a (any!) field k and let $\beta : V \times V \to k$ be a bilinear form.

We write $V^\vee$ for the dual of V – i.e. the set $\operatorname{Hom}_k(V, k)$ of all linear maps $V \to k$. Then $V^\vee$ is again a vector space over k and $\dim V = \dim V^\vee$.

Notice that β determines a linear mapping

$$\Phi_\beta : V \to \operatorname{Hom}_k(V, k)$$

by the rule $v \mapsto \beta(v, -)$.

Thus for $v \in V$, $\Phi_\beta(v)$ is the linear mapping which for $w \in V$ satisfies

$$\Phi_\beta(v)(w) = \beta(v, w).$$

The form β is *non-degenerate* provided that the linear mapping Φ_β is an invertible.

If $e_1, \cdots, e_d$ is a basis for V, the *matrix of* β for this basis is the $d \times d$ matrix whose i, j entry is $\beta(e_i, e_j)$.

Lemma 2.0.1. *The following are equivalent for β:*

(1) β is non-degenerate

(2) $\beta(v, w) = 0$ for every $w \in V$ implies that $v = 0$.

(3) $\det M \neq 0$ where M is the matrix of β with respect to some (any) basis of V.

If $W \subset V$ is a subspace, we say that W is nondegenerate if the restriction $\beta_{|W}$ is a non-degenerate form on W.

(Notice!: we write $\beta_{|W}$ for the restriction, but really this means the restriction of the function β from $V \times V$ to $W \times W$).

Lemma 2.0.2. *Let W_1, W_2 be non-degenerate subspaces of V. Suppose that*

(1) $W_1 \cap W_2 = 0$.

(2) $\beta(W_1, W_2) = 0$ – i.e. $\beta(w_1, w_2) = 0$ for all $w_1 \in W_1$ and all $w_2 \in W_2$.

Then $W_1 + W_2$ is a non-degenerate subspace.

Suppose that W is a subspace of V. We say that Then W is said to be the *orthogonal sum* of the subspaces W_1, W_2 if $W = W_1 + W_2$ and if W_1 and W_2 satisfy the hypotheses of the previous Lemma.

2.1 Equivalence of forms

Let V_1, β_1 and V_2, β_2 be pairs each consisting of a vector space together with a bilinear form.

We say that V_1, β_1 is *isomorphic* to V_2, β_2 if there is an invertible linear mapping $\phi : V_1 \to V_2$ such that for every $x, y \in V_1$, we have

$$\beta_1(x, y) = \beta_2(\phi(x), \phi(y)).$$

We then say that ϕ is an *isomorphism.*

Lemma 2.1.1. *Suppose that that V_1, β_1 is* isomorphic *to V_2, β_2. Then β_1 is non-degenerate if and only if β_2 is non-degenerate.*

2.2 Alternating forms

We say that β is *alternating* (or *skew-symmetric*) if $\beta(x, x) = 0$ for every $x \in V$.

Lemma 2.2.1. *If β is alternating then $\beta(x, y) = -\beta(y, x)$ for each $x, y \in V$. If the characteristic of k is not 2, the converse also holds.*

Suppose that β is alternating. A 2 dimensional subspace W of V is said to be *hyperbolic* if W has a basis e, f such that $\beta(e, f) = 1$.

Note that a hyperbolic subspace is non-degenerate (use Lemma 2.0.1 and the fact that $\det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is non-zero).

More generally, a subspace W of dimension ≥ 2 is said to be *hyperbolic* if W is the orthogonal sum of subspaces W_1, W_2 where W_1 is hyperbolic of dimension 2, and W_2 is either itself hyperbolic or zero.

Lemma 2.2.2. *Suppose that V_1, β_1 and V_2, β_2 are spaces of vector spaces together with alternating forms β_i. If W_1 is a hyperbolic subspace of V_1 and if W_2 is a hyperbolic subspace of V_2 then $W_1, \beta_{1|W_1}$ and $W_2, \beta_{2|W_2}$ are isomorphic.*

Lemma 2.2.3. *If W is a hyperbolic subspace of V, then W is non-degenerate and* $\dim W$ *is even.*

Theorem 2.2.4. *Suppose that β is a non-degenerate alternating form on V. Then V is hyperbolic. In particular,* $\dim V$ *is even.*

Corollary 2.2.5. *Suppose for $i = 1, 2$ that V_i, β_i is a space V_i together with a non-degenerate alternating form β_i on V_i. If $\dim V_1 = \dim V_2$ then V_1, β_1 is isomorphic to V_2, β_2.*

2.3 Symmetric forms

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