Forms over a finite field

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1 Some references

- Simeon Ball - "Finite Geometries and Combinatorial Applications" (2015, Cambridge Univ Press)

  (available electronically via Tufts' Tisch Library)

- *Linear algebra and matrices : topics for a second course* Shapiro, Helene, 1954- Providence, Rhode Island : American Mathematical Society, 2015

- some notes of Bill Casselmann (UBC)

2 Statements

Let V be a finite dimensional vector space over a (any!) field k and let $\beta : V \times V \to k$ be a bilinear form.

We write $V^\vee$ for the dual of V – i.e. the set $\mathrm{Hom}_k(V, k)$ of all linear maps $V \to k$. Then $V^\vee$ is again a vector space over k and $\dim V = \dim V^\vee$.

Notice that β determines a linear mapping

$$\Phi_\beta : V \to \mathrm{Hom}_k(V, k)$$

by the rule $v \mapsto \beta(v, -)$.

Thus for $v \in V$, $\Phi_\beta(v)$ is the linear mapping which for $w \in V$ satisfies

$$\Phi_\beta(v)(w) = \beta(v, w).$$

The form β is *non-degenerate* provided that the linear mapping Φ_β is an invertible.

If $e_1, \cdots, e_d$ is a basis for V, the *matrix of* β for this basis is the $d \times d$ matrix whose i, j entry is $\beta(e_i, e_j)$.

Lemma 2.0.1. *The following are equivalent for β:*

(1) β is non-degenerate

(2) $\beta(v, w) = 0$ for every $w \in V$ implies that $v = 0$.

(3) $\det M \neq 0$ where M is the matrix of β with respect to some (any) basis of V.

If $W \subset V$ is a subspace, we say that W is nondegenerate if the restriction $\beta_{|W}$ is a non-degenerate form on W.

(Notice!: we write $\beta_{|W}$ for the restriction, but really this means the restriction of the function β from $V \times V$ to $W \times W$).

Lemma 2.0.2. *Let W_1, W_2 be non-degenerate subspaces of V. Suppose that*

(1) $W_1 \cap W_2 = 0$.

(2) $\beta(W_1, W_2) = 0$ – i.e. $\beta(w_1, w_2) = 0$ for all $w_1 \in W_1$ and all $w_2 \in W_2$.

Then $W_1 + W_2$ is a non-degenerate subspace.

Suppose that W is a subspace of V. We say that Then W is said to be the *orthogonal sum* of the subspaces W_1, W_2 if $W = W_1 + W_2$ and if W_1 and W_2 satisfy the hypotheses of the previous Lemma.

2.1 Equivalence of forms

Let V_1, β_1 and V_2, β_2 be pairs each consisting of a vector space together with a bilinear form.

We say that V_1, β_1 is *isomorphic* to V_2, β_2 if there is an invertible linear mapping $\phi : V_1 \to V_2$ such that for every $x, y \in V_1$, we have

$$\beta_1(x, y) = \beta_2(\phi(x), \phi(y)).$$

We then say that ϕ is an *isomorphism*.

Lemma 2.1.1. *Suppose that that V_1, β_1 is* isomorphic *to V_2, β_2. Then β_1 is non-degenerate if and only if β_2 is non-degenerate.*

2.2 Alternating forms

We say that β is *alternating* (or *skew-symmetric*) if $\beta(x, x) = 0$ for every $x \in V$.

Lemma 2.2.1. *If β is alternating then $\beta(x, y) = -\beta(y, x)$ for each $x, y \in V$. If the characteristic of k is not 2, the converse also holds.*

Suppose that β is alternating. A 2 dimensional subspace W of V is said to be *hyperbolic* if W has a basis e, f such that $\beta(e, f) = 1$.

Note that a hyperbolic subspace is non-degenerate (use Lemma 2.0.1 and the fact that $\det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is non-zero).

More generally, a subspace W of dimension ≥ 2 is said to be *hyperbolic* if W is the orthogonal sum of subspaces W_1, W_2 where W_1 is hyperbolic of dimension 2, and W_2 is either itself hyperbolic or zero.

Lemma 2.2.2. *Suppose that V_1, β_1 and V_2, β_2 are spaces of vector spaces together with alternating forms β_i. If W_1 is a hyperbolic subspace of V_1 and if W_2 is a hyperbolic subspace of V_2 then $W_1, \beta_{1|W_1}$ and $W_2, \beta_{2|W_2}$ are isomorphic.*

Lemma 2.2.3. *If W is a hyperbolic subspace of V, then W is non-degenerate and* $\dim W$ *is even.*

Theorem 2.2.4. *Suppose that β is a non-degenerate alternating form on V. Then V is hyperbolic. In particular,* $\dim V$ *is even.*

Corollary 2.2.5. *Suppose for $i = 1, 2$ that V_i, β_i is a space V_i together with a non-degenerate alternating form β_i on V_i. If $\dim V_1 = \dim V_2$ then V_1, β_1 is isomorphic to V_2, β_2.*

2.3 Symmetric forms

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