Formalization in Lean

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Outline

What is Formalization?

"Expressing mathematics (objects, arguments) in a format that a computer can handle and interact with rigorously." - Kevin Buzzard

Benefits of Formalization

- Finding Mistakes
- Allowing large scale collaboration
  - Equational theories project
  - Public formalization projects
  - Base productivity shrunk by a large constant factor
- Pedagogical benefits

Lean works using Type Theory

- Every object has an associated *Type*
  - $\sqrt{2} : \mathbb{R}$
  - $1 : \mathbb{N}$
  - $\mathrm{id} : \mathbb{Q} \to \mathbb{Q}$
  - $\mathbb{N} : \mathrm{Type}$
- Definitions must be created with types specified.

```
def foo (a: ℕ): ℕ := 2 * a

#eval foo 3      -- 6
```

- Types can be omitted with implicit notation.

```
def foo' {α β: Type} (f: α → β) (a: α): β := f a

#eval foo' foo 2 -- 4
```

Type Classes in Lean

- Implicit {} notation allows you to avoid giving types explicitly.
- Type classes allows you to have much more inferred from context

```
import Mathlib

open Module

variable {F V: Type} [Field F] [AddCommGroup V] [Module F V] [FiniteDimensional F V]

noncomputable example (W: Subspace F V): W ≃ₗ[F] Dual F (Dual F W) := evalEquiv F W
```

```
def Module.evalEquiv
      (R : Type u_3) (M : Type u_4) [CommSemiring R] [AddCommMonoid M]
      [Module R M] [IsReflexive R M] :
   M ≃ₗ[R] Dual R (Dual R M)
```
source

The bijection between a reflexive module and its double dual, bundled as a `LinearEquiv`.

▶ Equations

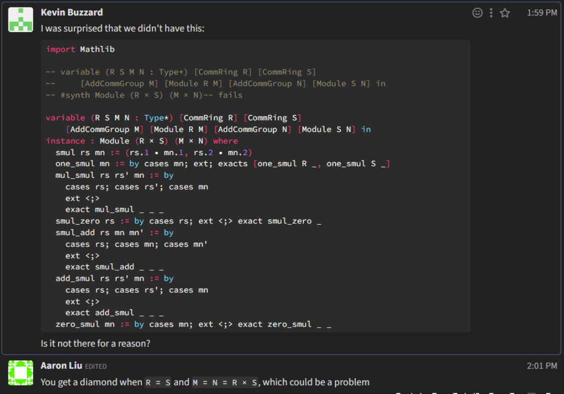
Issues with Type Classes

- Hard to see whats going on
- Slow compile times

```
import Mathlib.Tactic

open Module
open LinearMap
open LinearMap (BilinForm)

variable  {k V :Type} [Field k]
  [AddCommGroup V] [Module k V] [Module.Finite k V]

example : Ring (Module.End k V) := inferInstance        -- baseline; this succeeds

example : Ring (Module.End k (BilinForm k V)) := inferInstance -- fails

example : Algebra k (Module.End k (BilinForm k V)) := inferInstance -- succeeds?!
```

Diamonds

- Suppose A has an instance for B and C, which both have instances for D. Then A has two instances for D. Which to choose?



Kevin Buzzard 1:59 PM

I was surprised that we didn't have this:

```
import Mathlib

-- variable (R S M N : Type*) [CommRing R] [CommRing S]
--     [AddCommGroup M] [Module R M] [AddCommGroup N] [Module S N] in
-- #synth Module (R × S) (M × N)-- fails

variable (R S M N : Type*) [CommRing R] [CommRing S]
    [AddCommGroup M] [Module R M] [AddCommGroup N] [Module S N] in
instance : Module (R × S) (M × N) where
  smul rs mn := (rs.1 • mn.1, rs.2 • mn.2)
  one_smul mn := by cases mn; ext; exacts [one_smul R _, one_smul S _]
  mul_smul rs rs' mn := by
    cases rs; cases rs'; cases mn
    ext <;>
    exact mul_smul _ _ _
  smul_zero rs := by cases rs; ext <;> exact smul_zero _
  smul_add rs mn mn' := by
    cases rs; cases mn; cases mn'
    ext <;>
    exact smul_add _ _ _
  add_smul rs rs' mn := by
    cases rs; cases rs'; cases mn
    ext <;>
    exact add_smul _ _ _
  zero_smul mn := by cases mn; ext <;> exact zero_smul _ _
```

Is it not there for a reason?

Aaron Liu EDITED 2:01 PM

You get a diamond when `R = S` and `M = N = R × S`, which could be a problem

Proofs in Lean are guaranteed to be correct.

But what if your definitions are wrong?

Orthogonal Complement

- We say V is the internal direct sum of W_1 and W_2 with respect to a bilinear form $B : V \times V \to F$ if
  - W_1 and W_2 are *orthogonal*, i.e. that $B(w_1, w_2) = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$,
  - $W_1 + W_2 = V$,
  - $W_1 \cap W_2 = 0$.
- This doesn't force W_2 orthogonal to W_1! I.e, the form is only to be block upper triangular rather than block diagonal.
- Explicitly checking examples in Lean, and immediately proving compatibility results with similar theorems helps avoid this

Forced Implementation Choice

- Exact definition p of an object not viewed with much importance in math, as we can just prove $p \iff q$ and then use q everywhere
- Lean forces us to pick a privileged definition
- A basis in Lean internally is an F-linear isomorphism $V \simeq F^{\alpha}$
- F-linear isomorphisms are themselves implemented as maps $V \to W$ and $W \to V$ which are inverses.
- In the REU, I formalized Hyperbolic bilinear forms using a basis and a predicate, whereas the Mathlib one uses an isomorphism to $V^* \times V$
- Allowing computations or not

Canonical Isomorphism

- There is an obvious isomorphism $X \times (Y \times Z) \simeq (X \times Y) \times Z$ to where we just write $=$ directly.
- Often in algebra we just write $=$ for canonical isomorphism or even for definitions (via universal properties), like with $A \otimes_R B$.
- Limits practical freedom and forces people to be more explicit in choices

Quadratic Forms Extension by Scalars

- A given quadratic form $\phi : V \to F$ can be extended to one $A \otimes_F V \to A$ for F-algebras A.
- The Lean implementation of this goes through Bilinear forms, forcing $2 \neq 0$ in F.
- Fairly easy to find what is exactly needed/true for definitions from Mathlib
  - Improvement to traditional textbooks

Quadratic Forms Extension by Scalars

```
variable (R A) in
/-- The tensor product of two quadratic maps injects into quadratic maps on tensor products.

Note this is heterobasic; the quadratic map on the left can take values in a module over a larger
ring than the one on the right. -/
def tensorDistrib :
    QuadraticMap A M₁ N₁ ⊗[R] QuadraticMap R M₂ N₂ →ₗ[A] QuadraticMap A (M₁ ⊗[R] M₂) (N₁ ⊗[R] N₂) :=
  letI : Invertible (2 : A) := (Invertible.map (algebraMap R A) 2).copy 2 (map_ofNat _ _).symm
  -- while `letI`s would produce a better term than `let`, they would make this already-slow
  -- definition even slower.
  let toQ := BilinMap.toQuadraticMapLinearMap A A (M₁ ⊗[R] M₂)
  let tmulB := BilinMap.tensorDistrib R A (M₁ := M₁) (M₂ := M₂)
  let toB := AlgebraTensorModule.map
      (QuadraticMap.associated : QuadraticMap A M₁ N₁ →ₗ[A] BilinMap A M₁ N₁)
      (QuadraticMap.associated : QuadraticMap R M₂ N₂ →ₗ[R] BilinMap R M₂ N₂)
  toQ ∘ₗ tmulB ∘ₗ toB
```

Ended up using the external framework of theorems instead

Lean

Current Issues

- Strict Definitions
- *Dependent* Type Theory
- Universe issues (rarely)
- Manually using type classes not automatically instantiated to avoid diamonds
- Mathlib having large gaps

To be Improved On:

- Tooling
  - Better Tactics
  - AI Usage
- The Math

Contributing to Mathlib

- Code to be ported to Mathlib over time, cleaning portions at a time
- In the form of “pull requests”
- Results to be ported over:
  - Bilinear Form Isometries (Isomorphisms)
  - Notion of degree over polynomial modules, and the surrounding theory
  - Quotients of bilinear and quadratic forms
  - Surrounding theory for Hyperbolic Spaces
  - Compatibility of quadratic and bilinear forms and degree notions with extensions of scalars by the polynomial ring
  - Cassels-Pfister Theorem: The values taken by the extension of a quadratic map $\phi : V \to F$ to $V(X) \to F(X)$ that are in $F[X]$ are taken by the extension $V[X] \to F[X]$.

Current State

- The Symmetric algebra of a vector space has been formalized, however the grading on it has not (though being actively discussed).
  - This blocks progress on constructing extension by scalars in characteristic 2
- Fermat's Last Theorem: Current effort to formalize led by Kevin Buzzard
- The polynomial Freiman–Ruzsa conjecture was proved in 2023 by Tim Gowers, Ben Green, Freddie Manners, and Terry Tao, and has been formalized in Lean.

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