Quaternion algebras

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1 Some references

- Some notes of Keith Conrad (UConn)
- first chapter of Gille-Szamuely - “Central Simple Algebras and Galois Cohomology”

2 Quaternion algebras, defined

If k is a field, an *algebra* A over k is a k-vector space A together with operations $+ : A \times A \to A$ and $\cdot : A \times A \to A$ which satisfy the axioms of a *ring*.

Here, we are going to insist that the algebra A be finite dimensional as a k-vector space, and that there is a multiplicative identity element $1 \in A$.

Given a field ℓ containing k (a “field extension of k”) we can form an ℓ-algebra A_ℓ by *extension of scalars*. (Really, this is the tensor product: $A_\ell = A \otimes_k \ell$).

The algebra A is said to be *central simple* over k if for some field extension ℓ of k and for some $n \in \mathbb{N}$, the ℓ-algebra A_ℓ is isomorphic as ℓ-algebras to $\operatorname{Mat}_n(\ell)$, the algebra of $n \times n$ matrices over ℓ.

Now, a *quaternion algebra* is a central simple algebra Q over k with $\dim Q = 4$. Thus for some field extension ℓ of k, the ℓ-algebra Q_ℓ is isomorphic to $\operatorname{Mat}_2(k)$

3 A description of quaternion algebras

A quaternion algebra Q over k can be described in a explicit manner. The case where k has characteristic 2 is slightly different and I'll omit it here, so suppose that k has characteristic $\neq 2$.

Given $a, b \in k$ non-zero elements, we define the k-algebra $(a, b)_k$ to be the k-vector space with basis $1, i, j, ij$ where the multiplication satisfies

$$i^2 = a, j^2 = b, ij = -ji$$

Theorem 3.0.1. *Suppose that k does not have characteristic 2. If Q is a quaternion algebra over k, there are non-zero elements $a, b \in k$ for which $Q \simeq (a, b)_k$.*

If $\alpha = s + ti + uj + vij \in (a, b)_k$ for $s, t, u, v \in k$, the conjugate $\overline{\alpha}$ is defined to be

$$\overline{\alpha} = s - ti - uj - vij$$

Proposition 3.0.2. *The assignment $N : (a, b)_k \to k$ given by $N(\alpha) = \alpha \cdot \overline{\alpha} = s^2 - at^2 - bu^2 + abv$ defines a non-degenerate quadratic form on the vector space $(a, b)_k$.*

We call this quadratic form N the *norm* – or the *norm form* – of the quaternion algebra $(a,b)_k$.

Theorem 3.0.3. *The quaternion algebra $(a,b)_k$ is a division algebra if and only if the norm N does not vanish at any nonzero element of $(a,b)_k$; i.e. $N(\alpha)=0 \implies \alpha=0$.*

4 Associated conics

Associated with the quaternion algebra $(a,b)_k$ is the conic $C=C(a,b)$ which is the set of solutions to the equation $ax^2+by^2=z^2$ in the projective plane $\mathbb{P}^2$. In turn, we can consider the field of rational functions $k(C)$ on this conic; it is the field of fractions of the algebra $k[x,y]/\langle ax^2+by^2-1\rangle$. One sometimes calls $k(C)$ the "function field of C".

We may now state an important theorem due to Witt:

Theorem 4.0.1. *Let $Q_1=(a_1,b_1)_k$ and $Q_2=(a_2,b_2)_k$ be quaternion algebras over k, and let C_1 and C_2 be the associated conics. The algebra Q_1 and Q_2 are isomorphic if and only if the the function fields $k(C_1)$ and $k(C_2)$ are isomorphic.*

In particular, Witt's theorem shows that two quaternion algebras are isomorphic if and only if the associated conics are isomorphic as algebraic curves.