Reductive subgroup schemes of a parahoric group scheme

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Local fields

- Notations:
  - $\mathcal{A}$: complete DVR
  - F: fractions of $\mathcal{A}$ – a "local field"
  - $k = \mathcal{A}/\pi\mathcal{A}$: residues – assumed to be *perfect*
- Examples:
  - "equicharacteristic": $\mathcal{A} = k[\![t]\!]$, $F = k(\!(t)\!)$
  - $[F : \mathbf{Q}_p] < \infty$, $\mathcal{A} =$ int. clos. of $\mathbf{Z}_p$ in F, $k = \mathcal{A}/\pi\mathcal{A} \simeq \mathbf{F}_q$.
  - k perfect of char. $p > 0$, $\mathcal{A} = W(k)$ "Witt vectors". Then F has char. 0.

Reductive groups over local fields

- Let G be a reductive algebraic group over a local field F.
- If $\mathcal{A}$ is the ring of integers of F, then G can be viewed as the *generic fiber* of various smooth affine $\mathcal{A}$-group schemes.
- For example, if G is *split* over F, then G is the generic fiber of a *split reductive* group scheme over $\mathcal{A}$.
- There are other natural group schemes $\mathcal{P}$ over $\mathcal{A}$ to with$\mathcal{P}_F = G$ – they were studied especially by Bruhat and Tits, and I'll call them the "parahoric group schemes" attached to G.
- In general a parahoric group scheme $\mathcal{P}$ is not reductive over $\mathcal{A}$, since its special fiber $\mathcal{P}_k$ is a linear algebraic group over k which need not be reductive.

Target theorem

Let G be a reductive group over a local field F, and suppose that G splits over an unramified extension of F. Let $\mathcal{P}$ be a parahoric group scheme with generic fiber G.

Theorem

There is a reductive $\mathcal{A}$-subgroup scheme $\mathcal{H} \subset \mathcal{P}$ for which $\mathcal{H}_F$ is a reductive subgroup of G containing a maximal torus of G, and $\mathcal{H}_k$ is a Levi factor for $\mathcal{P}_k$.

Remarks

1. In fact, we will be a bit more precise about "which $\mathcal{H}_F$ occur", below.
2. And we need to discuss Levi factors in more detail.

Outline

Levi factors

Suppose that k is a perfect field, and that H is a linear algebraic group over k.

- Since k is perfect, the unipotent radical R of H is defined over k.
- write $\pi : H \to H/R$ for quotient map.
- A closed k-subgroup $M \subset H$ is a *Levi factor* of H if $\pi_{|M} : M \to H/R$ is an isomorphism of algebraic groups.
- If H has a Levi factor M, then the multiplication map defines an isomorphism from the *semidirect product* to H:

$$M \ltimes R \xrightarrow{\sim} H$$

Levi factors: existence?

- If char. of k is 0, H has a Levi factor, and any two Levi factors are conjugate by an element of $H(k)$.
- Now suppose k has char. $p > 0$.
- Let G be a reductive group over k, and let V be a linear representation of G (over k).
- The cohom. gp $H^2(G, V)$ is a quotient of the space $Z^2(G, V)$ of all regular 2-cocycles $\alpha : G \times G \to V$.
- Given $\alpha \in Z^2(G, V)$, one may construct an algebraic group H_α fitting in a strictly exact seq

$$(\flat) \quad 0 \to V \to H_\alpha \to G \to 1$$

- H_α has a Levi factor $\iff$ $(\flat)$ is split $\iff$ $0 = [\alpha] \in H^2(G, V)$.

Previous results

Theorem (M)

Suppose that G is a reductive group over the local field F, and let $\mathcal{P}$ be a parahoric group scheme associated to G.

(a) *If G splits over an unramified extension of F, the special fiber $\mathcal{P}_k$ has a Levi factor.*

(b) *If G is split, any two Levi factors of $\mathcal{P}_k$ are conjugate by an element of $\mathcal{P}(k)$.*

(c) *If G splits over a tamely ramified extension of F, the geometric special fiber $\mathcal{P}_{\overline{k}}$ has a Levi factor, where $\overline{k}$ is an algebraic closure of k.*

Theorem (M)

If H is a linear algebraic group over k of characteristic $p > 0$, and if H_ℓ has a Levi factor for a finite galois extension $\ell \supset k$ for which $[\ell : k]$ is prime to p, then H has a Levi factor ("over k").

Motivation for our “target theorem”

- In some older work, DeBacker related certain data for G – e.g. rational nilpotent classes and certain maximal tori (“unramified maximal tori”) of G – to some data for the reductive quotients of the special fibers of various $\mathcal{P}$ for G.
- Some of his arguments can be simplified by working with a Levi factor M of $\mathcal{P}_k$, rather than $\mathcal{P}_k/R_u\mathcal{P}_k$.
- But it should be better yet to realize M as the special fiber of a reductive subgroup scheme.

Motivation for target theorem, continued

Theorem (M)

Let $\mathcal{H}$ be a standard reductive group scheme over $\mathcal{A}$, let $X \in \mathrm{Lie}(\mathcal{H}_k)$ be a nilpotent element. Then there is a nilpotent section $\widetilde{X} \in \mathrm{Lie}(\mathcal{H})$ and an $\mathcal{A}$-homomorphism $\phi : \mathbf{G}_m \to \mathcal{H}$ such that X is the image of $\widetilde{X}$, the centralizer $C_{\mathcal{H}}(\widetilde{X})$ is smooth over $\mathcal{A}$, ϕ_k is a cocharacter associated to X, and ϕ_F is a cocharacter associated to $X_F \in \mathrm{Lie}(\mathcal{H}_F)$.

Remarks

1. I've suppressed the defn of "stndrd"; for ss gps it amounts to requiring the char of both F and k to be very good for the (geom.) root system of $\mathcal{H}$.
2. Recall that "assoc cochars" play the role of $\mathfrak{sl}_2$-triples for reduc gps in pos char.
3. If $\mathcal{H} \subset \mathcal{P}$ is a reduc subgp scheme for which $\mathcal{H}_k \subset \mathcal{P}_k$ is a Levi, Thm gives a way to "lift" nilp elts and their assoc cochars from the reductive quotient of $\mathcal{P}_k$ to $\mathcal{H}_F \subset G_F$.

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The group scheme μ_n

Fix an integer $n \geq 1$ and let μ_n be the group scheme with coordinate ring $k[\mu_n] = k[T]/\langle T^n - 1\rangle$.

- if n is invertible in k, then μ_n is *smooth* and $\mu_n(\ell)$ is the group of n-th roots of unity in ℓ for an extension ℓ of k.
- if $n = p^n$ where p is the char. of k, note that the image u of $T - 1$ in $k[\mu_n]$ is *nilpotent*, hence μ_n is not smooth. In this case,
  - $\mu_n(\ell) = \{1\}$ for any field extension ℓ of k, and
  - $\mathrm{Lie}(\mu_n) \neq 0$.

μ-homomorphisms

Let G be a connected and reductive group over a field k, and let $\phi : \mu_n \to G$ be a homom of gp schemes over k.

- A second homom $\psi : \mu_m \to G$ is equivalent to ϕ provided $\exists N \in \mathbf{Z}$ such that $n \mid N$ and $m \mid N$, and that

$$\mu_N \to \mu_m \xrightarrow{\psi} G \quad \text{and} \quad \mu_N \to \mu_n \xrightarrow{\phi} G$$

  coincide.
- A μ-homomorphism with values in G is an equivalence class of some ϕ as above.

- Proposition

  If ϕ is a μ-homomorphism, there is a max'l k-torus T of G such that ϕ factors through T.

The centralizer of a μ-homomorphism

- If T is a split torus with character group $Y = X_*(T)$, the collection of μ-homoms with values in T identifies with $Y \otimes \mathbf{Q}/\mathbf{Z}$.
  (use that $\varinjlim \mathbf{Z}/n\mathbf{Z}$ identifies with $\mathbf{Q}/\mathbf{Z}$).
- Let T be a split maximal torus in G, let $\Phi \subset X^*(T)$ be the set of roots, and let $x \in Y \otimes \mathbf{Q}$. Then $\overline{x} \in Y \otimes \mathbf{Q}/\mathbf{Z}$ determines a μ-homom $\phi_{\overline{x}}$, and the connected centralizer M_x of the image of $\phi_{\overline{x}}$ in G is the reductive subgroup given by

  $$M_x = \langle T, U_\alpha \mid \alpha \in \Phi_x \rangle,$$

  where U_α is the root subgroup determined by the root α, and

  $$\Phi_x = \{\alpha \in \Phi \mid \langle \alpha, x \rangle \in \mathbf{Z}\}.$$

- Indeed, the action of $\phi_{\overline{x}}$ on U_α is trivial $\iff$ $\langle \alpha, x \rangle \in \mathbf{Z}$.

A description of Φ_x

- Let $V = Y \otimes \mathbf{Q}$, choose a postive definite W-invariant quadratic form on V
- For $x \in V$, the root subsystem Φ_x is independent of the W_{aff}-orbit of x.
- Let us suppose that G is split and simple with simple roots $\alpha_1, \ldots, \alpha_\ell$, and write $\alpha_0 = -\widetilde{\alpha}$.
- The roots $\widetilde{S} = \{\alpha_0, \ldots, \alpha_\ell\}$ label the walls of the lowest alcove A for the action of W_{aff} on V.

Description of Φ_x, continued

- Fix $\varpi_i \in V$ for which $\langle \alpha_i, \varpi_j \rangle = \delta_{i,j}$ "fundamental dominant coweights".
- Then A is the open simplex defined by 0 and the ϖ_i / n_i where $\widetilde{\alpha} = \sum_i n_i \alpha_i$.
- A point y in $\overline{A}$ thus has the form $y = \sum_{i=1}^n t_i \varpi_i / n_i$ where $0 \leq t_i$, $t_i \in \mathbf{Q}$ and $\sum_i t_i \leq 1$. The t_i are the *Kac coordinates* of the point y.

- Proposition

  Suppose $x \in \overline{A}$. The roots of $\widetilde{S}$ which label those walls of A containing x form a simple system of roots for Φ_x.

  - This amounts to the *Borel-de Siebenthal* description of the connected centralizer of a semisimple element.

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Chevalley systems

Suppose that G is *split* over F and fix a split maximal torus T. Write $\Phi \subset X^*(T)$ for the roots, and for $\alpha \in \Phi$, write U_α for the *root subgroup*.

- can choose a "Chevalley system" – i.e. a system of F-isomorphisms $x_\alpha : \mathbf{G}_a \to U_\alpha$ for each α with good properties.
- For each root α, there is an $\mathcal{A}$-forms $\mathcal{U}_\alpha$ of U_α for which

$$x_\alpha^{-1}(\mathcal{U}_\alpha(\mathcal{A})) = \mathcal{A} \subset F = \mathbf{G}_a(F).$$

- let $\mathcal{T}$ be the $\mathcal{A}$-split torus with generic fiber T.
- then $(\mathcal{T}, (\mathcal{U}_\alpha)_{\alpha\in\Phi})$ is a *schematic root datum* which determines a split reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ with $\mathcal{G}_F = G$.

Parahorics via schematic root data

- More generally, given $m \in \mathbf{Q}$ and a root α, there is an $\mathcal{A}$-form $\mathcal{U}_{\alpha,m}$ of U_α for which

$$x_\alpha^{-1}(\mathcal{U}_{\alpha,m}(\mathcal{A})) = v^{-1}([m,\infty)) \subset F$$

  where v is the valuation of F.
- e.g. $x_\alpha^{-1}(\mathcal{U}_{\alpha,1}(\mathcal{A})) = \pi\mathcal{A}$.
- Recall $V = X_*(T) \otimes \mathbf{Q}$.
- A point $x \in V$ determines a *schematic root datum* $(\mathcal{T}, (\mathcal{U}_{\alpha,\langle\alpha,x\rangle})_{\alpha\in\Phi})$.
- And thus x determines a smooth $\mathcal{A}$-group scheme $\mathcal{P}_x$ with generic fiber G.
- The $\mathcal{P}_x$ are the parahoric group schemes.

Levi factors

- The root system of the reductive quotient of $\mathcal{P}_{x,k}$ is Φ_x.
- $\mathcal{D} = (\mathcal{T}, (\mathcal{U}_\alpha)_{\alpha\in\Phi_x})$ is a schematic root datum for M_x, and hence determines a split reductive $\mathcal{A}$-group scheme $\mathcal{M}_x$ with generic fiber M_x.
- in a suitable sense, $\mathcal{D}' = (\mathcal{T}, (\mathcal{U}_{\alpha,\langle\alpha,x\rangle})_{\alpha\in\Phi_x})$ is conjugate to $\mathcal{D}$.
- It follows that $\mathcal{M}_x$ embeds in $\mathcal{P}_x$, hence the "target theorem" follows, at least when G is split.