NILPOTENT ORBITS OVER GROUND FIELDS OF GOOD CHARACTERISTIC

GEORGE J. MCNINCH

ABSTRACT. Let X be an F-rational nilpotent element in the Lie algebra of a connected and reductive group G defined over the ground field F. Suppose that the Lie algebra has a non-degenerate invariant bilinear form. We show that the unipotent radical of the centralizer of X is F-split. This property has several consequences. When F is complete with respect to a discrete valuation with either finite or algebraically closed residue field, we deduce a uniform proof that $G(F)$ has finitely many nilpotent orbits in $\mathfrak{g}(F)$. When the residue field is finite, we obtain a proof that nilpotent orbital integrals converge. Under some further (fairly mild) assumptions on G, we prove convergence for arbitrary orbital integrals on the Lie algebra and on the group. The convergence of orbital integrals in the case where F has characteristic 0 was obtained by Deligne and Ranga Rao (1972).

CONTENTS

1. INTRODUCTION

Let F be a field and k an algebraically closed extension field. Denote by G a connected, reductive group defined over the ground field F, and suppose the characteristic of F to be good for G (see §2).

The geometric nilpotent orbits, i.e. the nilpotent orbits of the k-points of G in $\mathfrak{g} = \mathfrak{g}(k)$, are described by the Bala-Carter theorem; this result was proved for all good primes by Pommerening. Jens C. Jantzen has recently written a set of notes [J04] on the geometric nilpotent orbits of a reductive group; we refer to these notes

– and to their references – for background on many of the results mentioned in this introduction.

The study of arithmetic nilpotent orbits, i.e. the nilpotent orbits of the group of rational points $G(F)$ on the F-vector space $\mathfrak{g}(F)$, is more complicated for a general field F; the description of these orbits depends on Galois cohomology. One of the goals of this paper is to better understand the arithmetic nilpotent orbits.

In characteristic 0, or in large positive characteristic, the Bala-Carter theorem may be proved by appealing to $\mathfrak{sl}_2$-triples. To obtain a proof in any good characteristic, other techniques are required. Pommerening's proof eventually shows (after some case analysis) that one can associate to any nilpotent $X \in \mathfrak{g}$ a collection of cocharacters of G with favorable properties; see [J04]. Any cocharacter of G determines a parabolic subgroup, and it is a crucial result that each cocharacter associated with X determines the same parabolic subgroup P, which therefore depends only on X.

On the other hand, Premet has recently given a more conceptual proof of the Bala-Carter theorem. From the point of view of geometric invariant theory, a vector $X \in \mathfrak{g}$ is nilpotent precisely when its orbit closure contains 0; such vectors are said to be unstable. According to the Hilbert-Mumford criterion for instability there is a cocharacter ϕ of G such that X is unstable relative to the $\mathbf{G}_m$-action on $\mathfrak{g}$ corresponding to ϕ. A more precise form of the Hilbert-Mumford criterion was established by Kempf and by Rousseau; it yields cocharacters ϕ for which X is in a suitable sense optimally unstable relative to ϕ. Premet [Pre02] exploited these optimal cocharacters, together with an idea of Spaltenstein, to prove the Bala-Carter theorem in good characteristic.

Our first goal in this paper is to relate the associated cocharacters found by Pommerening with the optimal cocharacters found by Premet; this is done in Theorem 21 after some preliminaries in §3. We find that an associated cocharacter for X is optimal. An optimal cocharacter ϕ need not be associated to X, but it almost is if X is a weight vector for the torus $\phi(\mathbf{G}_m)$. In particular, the cocharacters associated with X determine the same parabolic subgroup P as the optimal cocharacters for X; P is called the instability parabolic (or instability flag) of X.

In a more general setting, Kempf exploited an important uniqueness property of optimal cocharacters to prove that the instability parabolic attached to an unstable F-rational vector is defined over F, in case F is perfect. In order to handle the case of an imperfect field in the special case of the adjoint representation, we invert this argument here. Since a maximal torus of G has at most one cocharacter associated to X, it suffices to find a maximal F-torus having a cocharacter associated to X; the rationality of the cocharacter then follows from Galois descent. We find such a torus under some assumptions on the separability of orbits; the assumption usually holds for all nilpotent orbits in good characteristic, at least when G is semisimple. The exception to keep in mind is the group SL_n with n divisible by the characteristic.

Since X has an F-cocharacter associated to it, we deduce more-or-less immediately that: (1) the instability parabolic P attached to X is defined over F – this had already been proved by the author using other techniques; (2) the unipotent radical $R_u(C)$ of the centralizer C of X is defined over F and is F-split; and (3) C has a Levi decomposition over F. See Theorem 28 and Corollary 29 for the latter two assertions.

When F is perfect, e.g. when char$F = 0$, all unipotent groups over F are split. See Remark 32 for an example of a non-split unipotent group.

In §6, we study the Galois cohomological consequences of the fact that $R_u(C)$ is F-split. Suppose that F is complete with respect to a non-trivial discrete valuation, and that the residue field is finite or algebraically closed. If each adjoint nilpotent orbit is separable, we prove that there are finitely many arithmetic nilpotent orbits. Our finiteness result improves one obtained by Morris [Mo88]. In *loc. cit.*, the finiteness was obtained for various forms of classical groups in good characteristic, and it was obtained for a general reductive group under the assumption $p > 4h - 4$ where h is the Coxeter number of G (note that by now the use of the term "very good prime" in *loc. cit.* §3.13 is non-standard).

Suppose that $\mathfrak{g}$ has a non-degenerate invariant bilinear form. This property guarantees that each geometric nilpotent orbit is separable. When the residue field of the complete field F is finite, it also guarantees that the centralizer of X in $G(F)$ is a unimodular locally compact group, so that the $G(F)$-orbit of X carries an invariant measure. When char$F = 0$, a result of Deligne and Rao asserts that this measure is finite for compact subsets of $\mathfrak{g}(F)$. In §8, we adapt the Deligne-Rao argument to the case where char$F > 0$. We first treat the case where X is nilpotent; here we need no additional assumptions. We obtain convergence for orbital integrals of unipotent conjugacy classes in $G(F)$ by invoking a result of Bardsley and Richardson which guarantees the existence of a "logarithm-like" map $G \to \mathfrak{g}$; these methods require some fairly mild assumptions on G which are valid, for example, when G is a Levi subgroup of a semisimple group in very good characteristic. Finally, we obtain the covergence for general adjoint orbits and conjugacy classes under a somewhat stronger additional assumption on the characteristic which guarantees that the Jordan decomposition is defined over F.

Our proof that $R_u(C)$ is F-split answers a question put to the author by D. Kazhdan; I thank him for his interest. I also thank S. DeBacker, S. Evens, J. C. Jantzen, R. Kottwitz, J-P. Serre, and T. Springer for useful conversations and comments regarding the manuscript.

2. GENERALITIES CONCERNING REDUCTIVE GROUPS

Recall that a homomorphism of algebraic groups $\varphi : A \to B$ is said to be an *isogeny* if it is surjective and has finite kernel. The isogeny φ will be said to be a *separable isogeny* if $d\varphi : \mathrm{Lie}(A) \to \mathrm{Lie}(B)$ is an isomorphism. The reader might keep the following example in mind as she reads the material in this section: for any $n \geq 1$, the isogeny $\varphi : \mathrm{SL}_{np/k} \to \mathrm{PGL}_{np/k}$ is not separable in characteristic p.

Throughout this section, G is a connected and reductive group defined over the infinite ground field F of characteristic p. The field k is an algebraically closed extension field of F.

2.1. **Good primes.** We first define the notions of *good* and *very good* primes for G. For a more thorough discussion of these notions, the reader is referred to [SS70, Hum, J04]. The reductive group G is assumed defined over F.

If G is quasisimple with root system R, the characteristic p of k is said to be bad for R in the following circumstances: $p = 2$ is bad whenever $R \neq A_r$, $p = 3$ is bad if $R = G_2, F_4, E_r$, and $p = 5$ is bad if $R = E_8$. Otherwise, p is good. [Here is a more intrinsic definition of good prime: p is good just in case it divides no coefficient of the highest root in R].

If p is good, then p is said to be very good provided that either R is not of type A_r, or that $R = A_r$ and $r \not\equiv -1 \pmod{p}$.

If G is reductive, the isogeny theorem [Spr98, Theorem 9.6.5] yields a – not necessarily separable – central isogeny $\prod_i G_i \times T \to G$ where the G_i are quasisimple and T is a torus. The G_i are uniquely determined by G up to isogeny, and p is good (respectively very good) for G if it is good (respectively very good) for each G_i.

The notions of good and very good primes are geometric in the sense that they depend only on G over an algebraically closed field. Moreover, they depend only on the isogeny class of the derived group (G, G).

A crucial fact is the following:

Lemma 1. *Let G be a quasisimple group in very good characteristic. Then the adjoint representation of G on $\mathrm{Lie}(G)$ is irreducible and self-dual.*

Proof. See [Hum, 0.13]. □

We also note:

Lemma 2. *Let $M \leq G$ be a reductive subgroup containing a maximal torus of G.*
  (1) *If p is good for G, then p is good for M.*
  (2) *Suppose that $p > \mathrm{rank}_{ss} G + 1$, where $\mathrm{rank}_{ss} G$ is the semisimple rank of $G_{/k}$ (the geometric semisimple rank). Then p is very good for M.*

Proof. For (1), see for instance [MS, Prop. 16]. Now an inspection when G is quasisimple yields (2). □

2.2. **Standard hypotheses.** We now recall the following *standard hypotheses* SH for G; cf. [J04, §2.9]:
  SH1 The derived group of G is simply connected.
  SH2 The characteristic of k is good for G.
  SH3 There exists a G-invariant non-degenerate bilinear form κ on $\mathfrak{g}$.

Definition 3. The reductive group G will be said to be *standard* if there is a separable isogeny between G and a reductive group H which satisfies the standard hypotheses SH. If G is an F-group, then G is F-standard if at least one such isogeny is defined over F.

Observe that SH3 is preserved under separable isogeny; thus any standard group has a non-degenerate invariant form κ on its Lie algebra. Moreover, if the standard group G is defined over a ground field F, we may (and will) suppose that κ is defined over F. Indeed, κ amounts to an isomorphism between the G-modules $\mathfrak{g}$ and $\mathfrak{g}^\vee$, so we need to find such an isomorphism defined over F. Since F is infinite, $\mathrm{Hom}_{G/F}(\mathfrak{g}(F), \mathfrak{g}^\vee(F))$ is a dense subset of $\mathrm{Hom}_G(\mathfrak{g}, \mathfrak{g}^\vee)$ and so the (non-empty, Zariski open) subset $\mathrm{Isom}_G(\mathfrak{g}, \mathfrak{g}^\vee)$ has an F-rational point.

Proposition 4. (a) *A semisimple group in very good characteristic is standard.*
   (b) *The group* $\mathrm{GL}(V)$ *is standard for any finite dimensional vector space* V.
   (c) *If* G *and* H *are standard, then* $G \times H$ *is standard.*
   (d) *The centralizer* M *of a semisimple element of a standard group is standard. Especially, a Levi subgroup of a standard group is standard.*

Proof. When G is simply connected, Lemma 1 implies that SH holds for G. If $\pi : G_{\mathrm{sc}} \to G$ is the simply connected covering isogeny, that same lemma implies that $d\pi$ is an isomorphism, hence that π is a *separable* isogeny; this proves (a) in general.

For (b), the only thing that needs verifying is SH3; for this, it is well-known that the trace form on $\mathfrak{gl}(V)$ is non-degenerate. Assertion (c) is straightforward.

For (d), note that the characteristic is good for M by Lemma 2. Let φ be a separable isogeny between G and a group $\widehat{G}$ satisfying SH. The Levi subgroup M of G is the connected centralizer of a suitable semisimple element $s \in G$; let $\hat{s} \in \widehat{G}$ correspond via φ to s (thus either $\hat{s} = \varphi(s)$ or $\hat{s} \in \varphi^{-1}(s)$). Then $\widehat{M} = C^o_{\widehat{G}}(\hat{s})$ is a Levi subgroup of $\widehat{G}$, and φ restricts to a separable isogeny between M and $\widehat{M}$. The assertion (d) now follows from the following lemma.   □

Lemma 5. *Let SH3 hold for* G, *let* $x \in G$ *be semisimple, and let* $M = C^o_G(x)$ *be the connected centralizer of* x. *Then* κ *restricts to a non-degenerate form on* $\mathfrak{m} = \mathrm{Lie}(M)$.

Proof. Write $\mathfrak{g} = \bigoplus_{\gamma \in k^\times} \mathfrak{g}_\gamma$ where for each $\gamma \in k^\times$, $\mathfrak{g}_\gamma$ is the γ-eigenspace of the (diagonalizable) map $\mathrm{Ad}(x)$. Since κ is invariant, the restriction $\kappa : \mathfrak{g}_\gamma \times \mathfrak{g}_{1/\gamma} \to k$ is a perfect pairing. The lemma now follows since $\mathfrak{m} = \mathfrak{g}_1$.   □

Let $X \in \mathfrak{g}$ and $g \in G$. When G is standard, the orbits of X and g are reasonably behaved:

Proposition 6. *Assume that* G *is standard. The geometric orbits of* $X \in \mathfrak{g}$ *and* $g \in G$ *are separable. In particular, if* $X \in \mathfrak{g}(F)$ *and* $g \in G(F)$, *the centralizers* $C_G(X)$ *and* $C_G(g)$ *are defined over* F.

Proof. Apply [SS70, I.5.2 and I.5.6] for the first assertion. The fact that the centralizers are defined over F then follows from [Spr98, Prop. 12.1.2].   □

In general, of course, the G-orbit of an F-rational element which is not separable need not be defined over F; such as orbit is defined over F if and only if it is defined over a separable closure of F.

3. The instability parabolic and nilpotent orbits

In this section, we are concerned with a connected, reductive group G over an algebraically closed field k whose characteristic is good for G.

As described in the introduction, our goal here is to relate the constructions given by Premet [Pre02] in his recent simplification of the Bala-Carter-Pommerening Theorem to constructions described in Jantzen's recent notes [J04]. The main result is Theorem 21.

3.1. **Length and cocharacters of G.** Fix for a moment a maximal torus T of G, and consider the lattice $X_*(T)$ of cocharacters of T. Fix a W-invariant positive definite, bilinear form β on $X_*(T) \otimes \mathbf{Q}$. Given any other torus $T' < G$, we may write $T' = \operatorname{Int}(g)T$, and one gets by transport of structure a W-invariant form β' on $X_*(T')$. Since β is W-invariant, β' is independent of the choice of the element g with $T' = \operatorname{Int}(g)T$.

The form β being fixed, there is a unique G-invariant function $(\phi \mapsto \|\phi\|)$: $X_*(G) \to \mathbf{R}_{\geq 0}$ with the property $\|\phi\| = \sqrt{\beta(\phi,\phi)}$ for $\phi \in X_*(T)$.

By a *length function* $\|\cdot\|$ on $X_*(G)$, we mean a G-invariant function $\phi \mapsto \|\phi\|$ associated with some positive definite bilinear form β on $X_*(T) \otimes \mathbf{Q}$ for some maximal torus T of G in the above sense. For the most part, the choice of T and β will be fixed and we will not refer to it.

For later use, we observe the following:

Lemma 7. *Suppose $\pi : G \to G'$ is a surjective homomorphism of reductive groups with central kernel. If $\|\cdot\|'$ is a given length function on $X_*(G')$, there is a length function $\|\cdot\|$ on $X_*(G)$ such that $\|\pi \circ \phi\|' = \|\phi\|$ for all $\phi \in X_*(G)$.*

Proof. Fix a maximal torus S' of G' and a positive definite form β' on $X_*(S') \otimes \mathbf{Q}$ giving rise to $\|\cdot\|'$. Under our assumptions on π, $S = \pi^{-1}S'$ is a maximal torus of G. Since $\pi_{|S} : S \to S'$ is a surjective map of tori, the image of the induced map $X_*(S) \to X_*(S')$ has finite index in $X_*(S')$. Moreover, π induces an isomorphism on Weyl groups $W = N_G(S)/S \xrightarrow{\sim} W' = N_{G'}(S')/S'$. Now choose a W-module splitting of the exact sequence

$$X_*(S) \otimes \mathbf{Q} \to X_*(S') \otimes \mathbf{Q} \to 0$$

so that $X_*(S) \otimes \mathbf{Q} \simeq K \oplus (X_*(S') \otimes \mathbf{Q})$ for some W-submodule K. Let β'' be a positive definite W-invariant bilinear form on K, and let $\beta = \beta'' \oplus \beta'$ be the corresponding form on $X_*(S) \otimes \mathbf{Q}$. One may then construct the length function on $X_*(G)$ using β, and the desired property is evident. $\square$

A similar observation is:

Lemma 8. *Let $G \subset G'$ be reductive groups and suppose that G contains a maximal torus of (G', G'). If $\|\cdot\|$ is a given length function on $X_*(G)$, one can choose a length function $\|\cdot\|'$ on $X_*(G')$ such that $\|\phi\|' = \|\phi\|$ for $\phi \in X_*(G)$.*

Sketch. Let S be a maximal torus of G containing a maximal torus of (G', G'), and suppose $S \subset S'$, with S' a maximal torus of G'. Then $N_G(S)$ normalizes S', and the map $N_G(S)/S \to N_{G'}(S')/S'$ is injective. The proof is now similar to that of the previous lemma. $\square$

3.2. **Cocharacters and parabolic subgroups.** Let ϕ be a cocharacter of G. Let

$$P = P(\phi) = \{g \in G \mid \lim_{t \to 0} \phi(t)g\phi(t^{-1}) \text{ exists}\}.$$

Then P is a parabolic subgroup of G [Spr98, 8.4.5]. For $i \in \mathbf{Z}$, let $\mathfrak{g}(i) = \mathfrak{g}(i; \phi)$ be the i-th weight space for $\phi(\mathbf{G}_m)$. Then

$$\operatorname{Lie}(P) = \mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$$

The unipotent radical of P is $U = \{g \in P \mid \lim_{t \to 0} \phi(t)g\phi(t^{-1}) = 1\}$; see [Spr98, 8.4.6 exerc. 5]. We have $\mathrm{Lie}(U) = \mathfrak{u} = \bigoplus_{i>0} \mathfrak{g}(i)$.

Note that $P(\phi) = P(n\phi)$ for any $n \in \mathbf{Z}_{\geq 1}$.

Lemma 9. *If $P = P(\phi)$ has unipotent radical U, then under the conjugation action, the torus $\phi(\mathbf{G}_m)$ has no fixed points $\neq 1$ on U.*

Proof. This is immediate from the above description of U. $\qquad\square$

Lemma 10. (1) *Let $X \in \mathfrak{g}(i; \phi) \subset \mathfrak{u}$ for $i \geq 1$, and let $u \in U$. Then*

$$\mathrm{Ad}(u)X = X + \sum_{j>i} X_j \quad \text{for } X_j \in \mathfrak{g}(j; \phi).$$

(2) *Let $u \in U$ and put $\psi = \mathrm{Int}(u) \circ \phi$. If $X \in \mathfrak{g}(i; \phi) \cap \mathfrak{g}(j; \psi)$ for some $i, j \geq 1$, then $i = j$ and $u \in C_P(X)$.*

Proof. For (1), choose a maximal torus T of P containing $\phi(\mathbf{G}_m)$, and choose a Borel subgroup B of P containing T. Let $R \subset X^*(T)$ be the roots, let $R^+ \subset R$ be the positive system of non-zero T-weights on $\mathrm{Lie}(B)$, and let $R_U \subset R^+$ be the T-weights on $\mathrm{Lie}(U)$; thus R_U consists of those $\alpha \in R$ with $\langle \alpha, \phi \rangle > 0$.

Let the homomorphisms $\mathscr{X}_\alpha : \mathbf{G}_a \to B$ parameterize the root subgroups corresponding to $\alpha \in R$. Then as a variety, U is the product of the images of the $\mathscr{X}_\alpha$ for $\alpha \in R_U$. So to prove (1), it suffices to suppose $u = \mathscr{X}_\beta(t)$ for $t \in \mathbf{G}_a$ and $\beta \in R_U$.

Since $\mathfrak{g}(i; \phi) = \sum_{\alpha \in R_U; \langle \alpha, \phi \rangle = i} \mathfrak{g}_\alpha$ for $i \geq 1$, it suffices to suppose that $X \in \mathfrak{g}_\alpha$ with $\langle \alpha, \phi \rangle = i > 0$. In fact, we may suppose that $X = d\mathscr{X}_\alpha(1)$. By the *Steinberg relations* [Spr98, Prop 8.2.3] we have for $s \in \mathbf{G}_a$

$$u\mathscr{X}_\alpha(s)u^{-1} = \mathscr{X}_\alpha(s) \cdot \prod_{\gamma \in R_U; \langle \gamma, \phi \rangle > i} \mathscr{X}_\gamma(c_\gamma(s))$$

for certain polynomial functions $c_\gamma(s)$. Differentiating this formula, we get

$$\mathrm{Ad}(u)X = X + \sum_{\gamma \in R_U; \langle \gamma, \phi \rangle > i} X_\gamma$$

with $X_\gamma \in \mathfrak{g}_\gamma$ as desired.

For (2), note that since $X \in \mathfrak{g}(j; \psi)$ we have $\mathrm{Ad}(u^{-1})X \in \mathfrak{g}(j; \mathrm{Int}(u^{-1}) \circ \psi) = \mathfrak{g}(j; \phi)$. On the other hand, since by assumption $X \in \mathfrak{g}(i; \phi)$, (1) shows that

$$\mathrm{Ad}(u^{-1})X = X + \sum_{l>i} X_l$$

with $X_l \in \mathfrak{g}(l; \phi)$. Since the component of $\mathrm{Int}(u^{-1})X$ of weight i for $\phi(\mathbf{G}_m)$ is the non-zero vector $X \in \mathfrak{g}(i; \phi)$, and since $\mathrm{Int}(u^{-1})X \in \mathfrak{g}(j; \phi)$, we see that $i = j$, and $\mathrm{Int}(u^{-1})X = X$. $\qquad\square$

Remark 11. Suppose the reductive group G is defined over the ground field F. If $\phi : \mathbf{G}_m \to G$ is a cocharacter defined over F, then $P(\phi)$ is an F-parabolic subgroup. Conversely, if P is an F-parabolic subgroup, then $P = P(\phi)$ for some cocharacter ϕ defined over F; for these assertions, see [Spr98, Lemma 15.1.2].

Remark 12. If the cocharacter $\phi : \mathbf{G}_m \to G$ is non-trivial, and $\phi(\mathbf{G}_m)$ is contained in the derived group of G, then $P(\phi)$ is a *proper* parabolic subgroup (indeed, the assumption means that $\langle \alpha, \phi \rangle < 0$ for some root α, hence $\mathfrak{g}_\alpha \not\subseteq \mathfrak{p}(\phi)$).

3.3. **Geometric invariant theory and optimal cocharacters.** If (ρ, V) is a rational representation of G, a vector $0 \neq v \in V$ is unstable if the orbit closure $\overline{\rho(G)v}$ contains 0. For an unstable v and a cocharacter ϕ of G, write $v = \sum_{i \in \mathbf{Z}} v_i$, where $v_i \in V(i; \phi)$ and $V(i; \phi)$ is the i-th weight space for $\phi(\mathbf{G}_m)$. We now define

$$\mu(v, \phi) = \min\{i \in \mathbf{Z} \mid v_i \neq 0\}.$$

A co-character ϕ of G is optimal for an unstable vector $v \in V$ if

$$\mu(v, \phi)/\|\phi\| \geq \mu(v, \psi)/\|\psi\|$$

for each cocharacter ψ of G. This notion of course depends on the choice of the length function $\|\cdot\|$ on $X_*(G)$. A co-character $\phi \in X_*(G)$ is primitive if there is no $(\psi, n) \in X_*(G) \times \mathbf{Z}_{\geq 2}$ with $\phi = n\psi$.

Proposition 13. *(Kempf* [K78]*, Rousseau) Let (ρ, V) be a rational representation of the reductive group G, and let $0 \neq v \in V$ be an unstable vector.*
   (1) *The function $\phi \mapsto \mu(v, \phi)/\|\phi\|$ on the set $X_*(G)$ attains a maximum value B; the cocharacters ϕ with $\mu(v, \phi)/\|\phi\| = B$ are the optimal cocharacters for v.*
   (2) *If ϕ and ψ are optimal cocharacters for v, then $P(\phi) = P(\psi)$.*

Let P be the common parabolic subgroup of (2). We then have:
   (3) *Let ϕ be an optimal cocharacter for v. For each $x \in P$, the cocharacter $\phi' = \mathrm{Int}(x) \circ \phi$ is optimal for v. Conversely, if ϕ and ϕ' are primitive optimal cocharacters for v, then ϕ and ϕ' are conjugate under P.*
   (4) *For each maximal torus T of P, there is a unique primitive $\phi \in X_*(T)$ which is optimal for v.*
   (5) *We have $\mathrm{Stab}_G(v) \subset P$.*

Write $P(v)$ for the parabolic subgroup of part (2) of the Proposition; it is known as the *instability flag* or the *instability parabolic*.

3.4. **Optimal cocharacters and central surjections.** Let $\pi : G \to G'$ be a surjective homomorphism between reductive groups with central kernel, and construct the length functions $\|\cdot\|$ on $X_*(G)$ and $\|\cdot\|'$ on $X_*(G')$ as in Lemma 7. Let (ρ, V) and (ρ', V') be rational representations of G and G' respectively, and let $f : V \to V'$ be a G-module homomorphism (for the pull-back G-module structure on V'). Suppose that every non-0 vector of $\ker f$ is semistable (i.e. not unstable). [We will consider precisely this setup in the proof of Proposition 16 below].

Lemma 14. *If $0 \neq v \in V$ is unstable, then $\phi \in X_*(G)$ is optimal for v if and only if $\phi' = \pi \circ \phi \in X_*(G')$ is optimal for $f(v)$.*

Proof. Let B be the maximal value of $\mu(v, \psi)/\|\psi\|$ for $\psi \in X_*(G)$, and let B' be the maximal value of $\mu'(f(v), \nu)/\|\nu\|'$ for $\nu \in X_*(G')$.

With ϕ, ϕ' as in the statement of the lemma, notice that we may find a $\phi(\mathbf{G}_m)$-submodule W of V such that $V \simeq \ker f \oplus W$ as $\phi(\mathbf{G}_m)$-modules. In particular,

f induces an isomorphism $f_{|W} : W \to f(V)$. By hypothesis, $\phi(\mathbf{G}_m)$ acts trivially on $\ker f$. If ϕ is optimal for v, then $v = \sum_{i>0} v_i$ with each $v_i \in W(i; \phi)$. Then $f(v) = \sum_{i>0} f(v_i)$ and it is clear that $\mu(v, \phi) = \mu'(f(v), \phi')$. Conversely, if ϕ' is optimal for $f(v)$, write $f(v) = \sum_{i>0} x_i$ with $x_i \in V'(i; \phi')$. Then v may be uniquely written $\sum_{i>0} y_i + z$ for certain $y_i \in W(i; \phi)$ with $f(y_i) = x_i$ and $z \in \ker f$. Since v is unstable, we must have $z = 0$ and it is clear again that $\mu(v, \phi) = \mu'(f(v), \phi')$.

Moreover, we have $\|\phi\| = \|\phi'\|'$. So the result will follow if we show that $B = B'$. By what was said above, we know that $B \le B'$. To show that equality holds, choose $\gamma \in X_*(G')$ which is optimal for $f(v)$. Since π is surjective with central kernel, there is, as in the proof of Lemma 7, an $n \in \mathbf{Z}_{\ge 1}$ and $\phi \in X_*(G)$ with $n\gamma = \pi \circ \phi$. Then applying the preceding considerations to $\phi' = n\gamma$ we get

$$B' = \mu'(f(v), \gamma)/\|\gamma\|' = \mu'(f(v), n\gamma)/\|n\gamma\|' = \mu(v, \phi)/\|\phi\| \le B.$$

Thus $B = B'$ and the lemma follows. $\qquad\square$

Remark 15. With notations as in the previous lemma, ϕ' may fail to be primitive when ϕ is primitive.

3.5. **Optimal cocharacters for nilpotent elements.** We are going to describe here a recent result of Premet giving a new approach to the classification of nilpotent orbits for G in good characteristic.

The first thing to notice is the following: for the adjoint representation of G, the unstable vectors are precisely the nilpotent elements. Indeed, that 0 lies in the closure of each nilpotent orbit is a consequence of the finiteness of the number of nilpotent orbits; see [J04, §2.10]. On the other hand, let $\chi : \mathfrak{g} \to \mathbf{A}^r$ be the *adjoint quotient map*; cf. [J04, §7.12, 7.13]. The fiber $\chi^{-1}(\mathbf{0})$ is precisely $\mathcal{N}$; see *loc. cit.* Proposition 7.13. If $X \in \mathfrak{g}$ is not nilpotent, then it is contained in a fiber $\chi^{-1}(b)$ with $b \ne \mathbf{0}$; since this fiber is closed and G-invariant, $0 \notin \overline{\mathrm{Ad}(G)X}$. This proves our observation. Given $X \in \mathfrak{g}$ nilpotent, the result of Kempf and Rousseau (Proposition 13) yields optimal cocharacters for X, and Premet [Pre02] used this fact to give a simple proof of the Bala-Carter-Pommerening Theorem.

To discuss Premet's work, we must recall some terminology. A nilpotent $X \in \mathfrak{g}$ is said to be distinguished provided that the connected center of G is a maximal torus of $C_G(X)$. A parabolic subgroup $P < G$ is distinguished if

$$\dim P/U = \dim U/(U, U) + \dim Z$$

where U is the unipotent radical of P, and Z is the center of G.

For $X \in \mathfrak{g}$ nilpotent, write $C = C_G(X)$ for the centralizer of X, and $P = P(X) = P(\phi)$ for the instability parabolic subgroup, where ϕ is any optimal cocharacter for X. Moreover, write U for the unipotent radical of P.

Proposition 16. *(Premet) Fix a length function on $X_*(G)$. If $X \in \mathfrak{g}$ is nilpotent, there is a cocharacter ϕ which is optimal for X with the following properties:*

(1) $X \in \mathfrak{g}(2; \phi)$.
(2) *The centralizer C_ϕ of $\phi(\mathbf{G}_m)$ in C is reductive, and $C = C_\phi \cdot R$ is a Levi decomposition, where $R = C \cap U = R_u(C)$.*

(3) *Choose a maximal torus S of C_ϕ, and let $L = C_G(S)$. Then X is distinguished in $\mathrm{Lie}(L)$, $P_L = P(\phi) \cap L$ is a distinguished parabolic subgroup of L, X is in the open (Richardson) orbit of P_L on its unipotent radical $U_L = U \cap L$, and $\phi(\mathbf{G}_m)$ lies in the derived subgroup of L.*

Note that in general neither C nor C_ϕ is connected; the assertion in (2) that C_ϕ is reductive is equivalent to: C_ϕ^o is reductive. This proposition was proved by Premet [Pre02, Theorem 2.3, Proposition 2.5, Theorem 2.7] under the additional assumption that G satisfies the *standard hypotheses* SH1–3 of §2. Premet used the validity of the result for this more restrictive class of groups G to deduce a proof of the Bala-Carter-Pommerening Theorem for any reductive group G in good characteristic. We will check here that the proposition itself is always true in good characteristic.

Proof of Proposition 16. Write $\| \cdot \|_G$ for the fixed length function on $X_*(G)$.

By [Spr98, 9.6.5], we may find a central isogeny $\pi : H \to G$ where $H = T \times \prod_i G_i$ and each G_i is a simply connected, quasisimple group in good characteristic. Since the characteristic is good, it follows from [Hum, 0.13] that each proper H submodule of $\bigoplus_i \mathrm{Lie}(G_i)$ is central in $\mathrm{Lie}(H)$. Thus $\ker d\pi$ is central, and so $d\pi$ induces a bijection $\mathcal{N}_H \to \mathcal{N}_G$ by [J04, §2.7]; here, $\mathcal{N}_G$ denotes the nilpotent variety of G, and $\mathcal{N}_H$ that of H. We get also that each non-0 vector in $\ker d\pi$ is a semisimple element of $\mathfrak{g}$, hence is semistable (in fact: stable).

We may choose a length function $\| \cdot \|_H$ on $X_*(H)$ compatible with $\| \cdot \|_G$ as in Lemma 7. We claim now that if the proposition holds for H with this choice of length function, then it holds for G. To prove this claim, let $X \in \mathcal{N}_G$, let $X' \in d\pi^{-1}(X)$ be the unique nilpotent preimage of X in $\mathrm{Lie}(H)$, and let $\phi' \in X_*(H)$ satisfy the conclusion of the proposition for H. Put $\phi = \pi \circ \phi' \in X_*(G)$. We will show that ϕ satisfies the conclusion of the proposition for G. Property (1) needs no comment. For (2), the only thing that must be verified is that C_ϕ is reductive. Since π restricts to a central isogeny $C' = C_H(X') \to C_G(X)$, it also restricts to a central isogeny $C'_{\phi'} \to C_\phi$. Since $C'_{\phi'}$ is reductive, C_ϕ is reductive, and (2) follows. Since π restricts to a central isogeny $P(\phi') \to P(\phi)$, the proof of (3) is similar. It only remains to see that ϕ' is optimal for X; in view of our choice of $\| \cdot \|_H$, this follows from Lemma 14. Our claim is proved.

Finally, we may find a reductive group M satisfying SH1–3 and an inclusion $H \subset M$ with $(M, M) = (H, H)$. Indeed, for each i such that $G_i = \mathrm{SL}_n$, let $G'_i = \mathrm{GL}_n$, otherwise let $G'_i = G_i$; then take $M = T \times \prod G'_i$ with the obvious inclusion $H \subset M$. As has already been remarked, Premet proved the proposition for M (for any choice of length function), and we claim that it is thus valid for H; this will complete our proof.

We may choose a length function $\| \cdot \|_M$ on $X_*(M)$ prolonging the length function $\| \cdot \|_H$ on $X_*(H)$ as in Lemma 8. Let $X \in \mathrm{Lie}(H)$ be nilpotent; regarding X as an element of $\mathrm{Lie}(M)$, we may find $\phi \in X_*(M)$ as in the statement of the proposition. According to (3), we have $\phi \in X_*(H)$. We are going to verify that ϕ satisfies the conclusion of the proposition for H and $X \in \mathrm{Lie}(H)$. Again, property (1) needs no further comment. For (2), note first that $M = H \cdot Z$ where Z denotes the center of M. Then $C_M(X) = C_H(X) \cdot Z$. Setting $C_\phi = C_M(X) \cap C_M(\phi(\mathbf{G}_m))$ and $C'_\phi =$

$C_H(X) \cap C_H(\phi(\mathbf{G}_m))$, we have $C_\phi = C'_\phi \cdot Z$. Thus the unipotent radical of C'_ϕ is a normal subgroup of C_ϕ; since C_ϕ is reductive, so is C'_ϕ. This suffices to verify (2). The verification of (3) is similar. Thus, it only remains to see that ϕ is optimal for X in H. Since $X_*(H) \subset X_*(M)$, optimality of ϕ for H follows at once. □

Remark 17. Let ϕ be as in the proposition. Since $X \in \mathfrak{g}(2; \phi)$, it is clear that either ϕ is primitive, or $\frac{1}{2}\phi \in X_*(G)$ is primitive (and again optimal for X).

3.6. **Cocharacters associated to nilpotent elements.** In this subsection, we again suppose that we have fixed a length function on $X_*(G)$.

Let $X \in \mathfrak{g}$ be nilpotent. A cocharacter $\phi : \mathbf{G}_m \to G$ is said to be associated with $X \in \mathfrak{g}$ if $\mathrm{Ad}(\phi(t))X = t^2 X$ for each $t \in \mathbf{G}_m$, and if ϕ takes values in the derived group of a Levi subgroup L of G for which $X \in \mathrm{Lie}(L)$ is distinguished.

Proposition 18.   (1) *There exists a cocharacter which is both optimal for and associated with X.*
   (2) *If the cocharacter ϕ is associated to X, then $\mathrm{Int}(g) \circ \phi$ is associated to X for each $g \in C_G(X)$. Conversely, any two cocharacters associated to X are conjugate by $C_G^o(X)$.*
   (3) *If ϕ is a cocharacter associated with the nilpotent X, then the parabolic subgroup $P(\phi)$ coincides with the instability parabolic $P(X)$.*

Proof. The optimal cocharacter found by Premet in Proposition 16 is associated with X (by (1) and (3) of that proposition). This proves (1). Assertion (2) follows from [J04, Lemma 5.3(b)]. With ψ as in (1) and ϕ as in (3), (2) implies that $\mathrm{Int}(g) \circ \phi = \psi$ is optimal for X for some $g \in C_G^o(X)$. By Proposition 13, we have $C_G^o(X) \subset P(X)$. Thus $P(X) = P(\psi) = P(\mathrm{Int}(g) \circ \phi) = P(\phi)$, whence (3). □

Remark 19. A proof of the existence of a cocharacter associated with X can be extracted from the work by Pommerening (which depends on some case-checking for exceptional types); see the overview in [J04, §4]. The proof given in [J04] of part (2) of the proposition is elementary: it does not depend on the existence of a cocharacter.

Write $C = C_G(X)$, let $P = P(X)$ denote the instability parabolic of X, and let U be the unipotent radical of P.

Corollary 20. *Let ϕ be associated with X, and let $R = R_u(C)$ be the unipotent radical of C.*
   (1) *The centralizer C_ϕ of $\phi(\mathbf{G}_m)$ in C is reductive, and $C = C_\phi \cdot R$.*
   (2) *$R = C \cap U = \{g \in C \mid \lim \phi(t)g\phi(t^{-1}) = 1\}$ and $\mathrm{Lie}(R) = \bigoplus_{i \geq 1} \mathrm{Lie}(C)(i; \phi)$.*

Proof. It follows from Premet's result Proposition 16 that 1 and 2 are valid for a particular cocharacter ϕ associated to X; the general case results from the conjugacy under $C_G^o(X)$ of associated cocharacters. □

Theorem 21. *Let $X \in \mathfrak{g}$ be nilpotent, and let ϕ be a cocharacter associated to X. Then ϕ is optimal for X. Conversely, suppose that $\psi \in X_*(G)$ is primitive, ψ is optimal for X, and $X \in \mathfrak{g}(m, \psi)$ for some $m \in \mathbf{Z}_{\geq 1}$. Then $m = 1$ or 2. If $m = 2$, then ψ is associated with X, if $m = 1$ then 2ψ is associated with X.*

Proof. Let ϕ_0 be a cocharacter which is both optimal for and associated with X as in Proposition 18(1).

Suppose first that ϕ is associated to X. By Proposition 18(2), ϕ is conjugate under $C_G^o(X)$ to ϕ_0. Since $C_G^o(X)$ is contained in the instability parabolic $P(X)$ by Proposition 13 (4), optimality of ϕ follows from Proposition 13(3).

Now suppose that ψ is primitive and optimal for X, and that $X \in \mathfrak{g}(m; \psi)$ as above. Let $P = P(X)$ be the instability parabolic, and let U be its unipotent radical. If ϕ_0 is primitive, write $\lambda = \phi_0$. Otherwise we put $\lambda = \frac{1}{2}\phi_0$. Thus in each case λ is primitive and optimal for X, and $X \in \mathfrak{g}(n; \lambda)$ with $n = 1$ if ϕ_0 is not primitive, and $n = 2$ if ϕ_0 is primitive.

By Proposition 13(3) ψ and λ are conjugate via P. By [Spr98, 13.4.2], the centralizer $C_G(\psi(\mathbf{G}_m))$ is a Levi subgroup of $P = P(\psi)$. It follows that ψ and λ are conjugate by an element $u \in U$. By Lemma 10(2), we see that $m = n$ and $u \in C_G(X)$. Applying Proposition 18(2) completes the proof. $\square$

Corollary 22. *Let S be a maximal torus of the instability parabolic P. There is at most one $\phi \in X_*(S)$ which is associated to X.*

Proof. Suppose $\phi, \phi' \in X_*(S)$ are associated to X. By the previous result, ϕ and ϕ' are optimal for X. If ψ denotes the unique primitive optimal cocharacter in $X_*(S)$ associated with X, then $\phi = n\psi$ and $\phi' = n'\psi$ for some $n, n' \in \mathbf{Z}_{\geq 1}$. Since $X \in \mathfrak{g}(2; \phi)$ and $X \in \mathfrak{g}(2; \phi')$, we see that $n = n' = 1$ or 2, and so $\phi = \phi'$. $\square$

Note that while the proof of the preceding corollary depends on the choice of the length function on $X_*(G)$, the conclusion is independent of that choice.

4. Rationality of associated cocharacters

If A is a linear algebraic group defined over the ground field F, we may always find a maximal torus of A which is defined over F; cf. [Spr98, 13.3.6]. Moreover, any two maximal tori of A are conjugate by an element of A^o [and even by an element of $A^o(F_{\mathrm{sep}})$]; [Spr98, Theorem 6.4.1, Prop. 13.3.1]. We will use these facts without further reference. In this section, $G = G_{/F}$ is a reductive group defined over F. We assume throughout that the characteristic of k is good for G.

4.1. **A separability lemma.** Let (ρ, V) be a linear representation for G, and let $0 \neq v \in V$. Make the following assumptions.

H1. Suppose that the G-orbit $\mathcal{O} = \rho(G)v$ is separable.

H2. Suppose that $\mathcal{O}$ contains $k^\times w = \{aw \mid a \in k^\times\}$ for each $w \in \mathcal{O}$.

Observe that H1 and H2 are geometric conditions; they only depend on G and V over k. Recall as well that a G-orbit $\mathcal{O}$ is separable just in case some (hence any) orbit map $(g \mapsto \rho(g)x) : G \to \mathcal{O}$ for a fixed $x \in \mathcal{O}$ has surjective differential at the identity of G.

Denote by ρ' the action $(g, [w]) \mapsto [\rho(g)w]$ of G on $\mathbf{P}(V)$, where $[w]$ denotes the line in V through w. Write $\mathcal{O}' = \rho'(G)[v]$.

Let $\mathcal{L}$ be the line bundle over $\mathbf{P}(V)$ corresponding to the invertible coherent sheaf $\mathscr{O}_{\mathbf{P}(V)}(1)$. The $\mathbf{G}_m$-bundle $\pi : V \setminus \{0\} \to \mathbf{P}(V)$ is the bundle $\mathcal{L}^\times$ obtained by discarding the zero section from $\mathcal{L}$. In view of H2, $\hat{\pi} = \pi_{|\mathcal{O}} : \mathcal{O} \to \mathcal{O}'$ is the pull-back of $\mathcal{L}^\times$ along the inclusion $i : \mathcal{O}' \to \mathbf{P}(V)$; in particular it is a (Zariski) locally trivial principal $\mathbf{G}_m$-bundle. It follows that $d\hat{\pi}_w : T_w\mathcal{O} \to T_{\hat{\pi}(w)}\mathcal{O}'$ is surjective for each $w \in \mathcal{O}$.

Lemma 23. *The orbit $\mathcal{O}' = \rho'(G)[v]$ is separable.*

Proof. Let $f : G \to \mathcal{O}$ be the orbit map $f(g) = \rho(g)v$, and let $f' : G \to \mathcal{O}'$ be $f'(g) = \rho'(g)[v]$. Since $\mathcal{O}$ is separable, $df_1 : T_1G \to T_v\mathcal{O}$ is surjective. Since $f' = \hat{\pi} \circ f$, and since $d\hat{\pi}_v$ is surjective, we deduce that df_1' is surjective. This proves the lemma. $\square$

Remark 24. The conclusion of the lemma is in general not true when H2 (or H1) doesn't hold. Consider the linear representation (ρ, V) of $G = \mathbf{G}_m$ where $V = k^2$ and ρ is given by $\rho(t)(a, b) = (t^{-1}a, t^{p-1}b)$. Each G-orbit on V is separable. However, $\rho'(t)[1 : 1] = [1 : t^p]$, hence the (open) orbit $\rho'(G)[1 : 1] \subset \mathbf{P}(V)$ is not separable (the orbit map has 0 differential).

Consider now the adjoint representation $(\rho, V) = (\mathrm{Ad}, \mathfrak{g})$ of G. According to [J04, §2.10, 2.11], condition H2 of §4.1 is valid for each nilpotent orbit (this holds even in bad characteristic; the only thing required is the finiteness of the number of nilpotent orbits over k. That finiteness is known by an uniform argument for good primes, and by case-checking (Holt–Spaltenstein) for bad primes).

The validity of condition H1 is discussed in Jantzen's notes [J04, §2.9]. For example, it is valid for the *standard groups* from § 2; see Proposition 6.

4.2. **Associated cocharacters over a ground field.**

Recall that the characteristic p is assumed to be good for the reductive F-group $G = G_{/F}$.

Fix $X \in \mathfrak{g}(F)$ nilpotent. We make the following assumption:

(4.1)   either F is perfect, or the G-orbit of X is separable.

Let $N = N(X) = \{g \in G \mid \mathrm{Ad}(g)X \in kX\}$. Thus N is the stabilizer of $[X] \in \mathbf{P}(\mathfrak{g})$ (see §4.1). If ϕ is an cocharacter of G associated with X, then $\phi \in X_*(N)$. Moreover, $\phi(\mathbf{G}_m)$ normalizes $C = C_G(X)$.

Lemma 25. *Let S be any maximal torus of N. Then there is a unique cocharacter in $X_*(S)$ associated with X.*

Proof. Fix a cocharacter ϕ associated to X; then $N = \phi(\mathbf{G}_m) \cdot C$ where $C = C_G(X)$; cf. [J04, §5.3]. Choose a maximal torus T of N with $\phi(\mathbf{G}_m) \subseteq T$. If S is another maximal torus of N, then $S = gTg^{-1}$ with $g \in N$. Writing $g^{-1} = \phi(a)h^{-1}$ with $a \in k^\times$ and $h \in C_G(X)$, we see that $gTg^{-1} = hTh^{-1}$. It follows that $\phi' = \mathrm{Int}(h) \circ \phi$ is a cocharacter of S; since h centralizes X, ϕ' is associated to X by Proposition 18(2).

Since $C_G(X) < P$ by Proposition 13, we have $N < P$. Thus S is contained in a maximal torus of P, and uniqueness of $\phi' \in X_*(S)$ then follows from Corollary 22. $\square$

If (4.1) holds, the discussion in §4.1 shows that the G-orbit of $[X] \in \mathbf{P}(\mathfrak{g})$ is separable; thus [Spr98, 12.1.2] implies that the group $N = N(X)$ is defined over F.

Theorem 26. *Let $X \in \mathfrak{g}(F)$ be nilpotent, and assume that (4.1) holds. Then there is cocharacter ϕ associated to X which is defined over F.*

Proof. Since N is defined over F, we may choose a maximal torus $S \subset N$ defined over F. Let $\phi \in X_*(S)$ be the unique cocharacter which is associated to X; see Lemma 25. It follows from [Spr98, 13.1.2] that ϕ is defined over a separable closure F_{sep} of F in k. We will show that ϕ is defined over F.

Since S is an F-torus, the Galois group Γ acts on $X_*(S)$: for $\gamma \in \Gamma$, $\psi \in X_*(G)$ and $t \in F_{\mathrm{sep}}$, one has

(4.2) $$(\gamma \cdot \psi)(t) = \gamma(\psi(\gamma^{-1}(t))).$$

We must show that ψ is fixed by each $\gamma \in \Gamma$. To do this, we show that $\gamma \cdot \psi$ is a cocharacter associated to X.

First, note that since $X = \gamma(X)$ we have

$$\mathrm{Ad}((\gamma \cdot \psi)(t))X = \gamma(\mathrm{Ad}(\psi(\gamma^{-1}(t)))X) = \gamma\left(\gamma^{-1}(t^2)X\right) = t^2 X.$$

Thu $X \in \mathfrak{g}(2; \gamma \cdot \psi)$, and it just remains to show that $\gamma \cdot \psi$ takes values in the derived group of some Levi subgroup M of G for which $X \in \mathrm{Lie}(M)$ is distinguished.

Since ϕ is itself associated with X, there is a Levi subgroup L of G such that $X \in \mathrm{Lie}(L)$ is distinguished, and such that $\phi(\mathbf{G}_m) < (L, L)$. Let $M = \gamma(L)$. Of course, M is again a Levi subgroup. Since $\gamma(X) = X$, we have $X \in \mathrm{Lie}(M)$. The equality $C_M(X) = \gamma(C_L(X))$ makes clear that X is distinguished in $\mathrm{Lie}(M)$. Moreover, $\gamma(L, L) = (M, M)$, so it is clear that $\gamma \cdot \phi(\mathbf{G}_m) < (M, M)$. This completes the proof that $\gamma \cdot \phi$ is associated to X.

Since $\gamma \cdot \phi \in X_*(S)$ and since ϕ is the unique cocharacter in $X_*(S)$ associated with X, we deduce $\phi = \gamma \cdot \phi$ and the theorem is proved. $\square$

5. The unipotent radical of a nilpotent centralizer

If A is a linear algebraic F-group, recall that the Galois cohomology set $H^1(F, A)$ is by definition $H^1(\Gamma, A(F_{\mathrm{sep}}))$ where F_{sep} is a separable closure of F, and $\Gamma = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ is the Galois group. The basic reference for Galois cohomology is [Ser97]; see also [Spr98, §12.3]. The set $H^1(F, A)$ classifies torsors (principal homogeneous spaces) of A over F. It can be defined as the equivalence classes for a suitable relation on the set $Z^1(F, A) = Z^1(\Gamma, A(F_{\mathrm{sep}}))$ of continuous 1-cocycles with values in $A(F_{\mathrm{sep}})$; especially, each $\alpha \in H^1(F, A)$ may be represented by an $a \in Z^1(F, A)$. When A is not Abelian, the set $H^1(F, A)$ is not in general a group, but it does have a distinguished element – so it is a "pointed set" – which we sometimes write as 1. Thus, the notation $H^1(F, A) = 1$ means that this set has one element.

Let G be a reductive F-group in good characteristic, and let $X \in \mathfrak{g}(F)$ be nilpotent. *Assume throughout this section that (4.1) holds for X.*

We begin by noting:

Proposition 27. *The instability parabolic $P(X)$ is defined over F.*

Proof. By Theorem 26, there is an F-cocharacter ϕ associated with X. Since $P(X) = P(\phi)$ by Proposition 18, $P(X)$ is defined over F by Remark 11. $\square$

This result was previously obtained in [Mc03, Theorem 15]. See [RR84, Theorem 2.3] for a related result.

If A is connected and unipotent (and defined over F), one says that A is F-split if there is a sequence of F-subgroups $1 = A_n \lhd A_{n-1} \lhd \cdots \lhd A_2 \lhd A_1 = A$ such that each quotient A_i/A_{i+1} is F-isomorphic to the additive group $\mathbf{G}_{a/F}$.

Theorem 28. *Write $C = C_G(X)$ for the centralizer of X, and let $R = R_u(C)$ be the unipotent radical. Then R is defined over F and is an F-split unipotent group.*

Proof. Let $P = P(X)$ be the instability parabolic of X. By Proposition 27, P is defined over F. Denote by U the unipotent radical of P; it is defined over F as well [Spr98, 13.4.2]. By Corollary 20(2), the unipotent radical of C is $R = C \cap U$, and $\mathrm{Lie}(R) = \mathrm{Lie}(C) \cap \mathrm{Lie}(U)$. Thus, it follows from [Spr98, 12.1.5] that R is defined over F.

By Theorem 26, we may find a cocharacter $\phi \in X_*(P)$ associated to X which is defined over F. Let S denote the image of ϕ; it is a 1-dimensional split F-torus. It is clear that S acts as a group of automorphisms of R. Since $R = U \cap C$, Lemma 9 implies that the F-torus S has no non-trivial fixed points on R. It now follows from [Spr98, Corollary 14.4.2] that R is an F-split unipotent group. $\square$

Corollary 29. *Let $C = C_G(X)$ be the centralizer of X. Choose a cocharacter ϕ associated with X and defined over F (Theorem 26). Then $C = C_\phi \cdot R$ is a Levi decomposition defined over F, where C_ϕ is as in Corollary 20 and $R = R_u(C)$.*

Proof. One knows that $C = C_\phi \cdot R$ is a Levi decomposition over k; the only thing to check is the rationality. The theorem shows that R is defined over F. Since $\phi(\mathbf{G}_m)$ is an F-torus, its centralizer in C is defined over F ([Spr98, 13.3.1]), whence the corollary. $\square$

Proposition 30. *Suppose that U is an F-split unipotent group.*
   (1) $H^1(F, U) = 1$, and if U is commutative, $H^i(F, U) = 1$ for all $i \geq 1$.
   (2) *If U is a normal subgroup of the F-group A, and $z \in Z^1(F, A)$, then $H^1(F, {}_zU) = 1$, where ${}_zU$ denotes the group obtained from U by twisting with z. If U is commuatative, $H^i(F, {}_zU) = 1$ for all $i \geq 1$.*

Proof. Since U has a filtration by normal F-subgroups such that each quotient is F isomorphic to $\mathbf{G}_{a/F}$, the first assertion follows from the additive version of Hilbert 90 [Ser97, II.1.2 Prop. 1] together with a long exact sequence argument.

Since U and ${}_zU$ are isomorphic over F_{sep} by construction, the second assertion follows from the fact that a unipotent F-group V is F-split if and only if $V_{/F_{\mathrm{sep}}}$ is F_{sep}-split; see [Spr98, 14.3.8]. $\square$

Remark 31. When F is not perfect, there are F-groups A whose unipotent radical $R_u(A)$ is not defined over F. Take for example the F-group $A = R_{E/F}\mathbf{G}_m$, where $F \subset E$ is a finite purely inseparable extension of degree p and $R_{E/F}$ is Weil's restriction of scalars functor. The unipotent radical of A has dimension $p - 1$, but is not defined over F. In fact, A is F-*reductive*; the maximal closed, connected, normal, unipotent F-subgroup of A is trivial.

Remark 32. Consider the field $F = \kappa((t))$ of formal series, where κ is any field of characteristic $p > 2$. Let $U \leq \mathbf{G}_a \times \mathbf{G}_a$ be the unipotent group F-group defined by

$$U = \{(y, z) \in \mathbf{G}_a \times \mathbf{G}_a \mid y^p - y = tz^p\}.$$

Then U is defined over F, and U is isomorphic over an algebraic closure $\overline{F}$ to $\mathbf{G}_{a/\overline{F}}$ (but not over F_{sep}). In fact, U is isomorphic with $\mathbf{G}_a$ over $F(t^{1/p})$. There is an exact sequence

$$F \times F \xrightarrow{(y,z) \mapsto y^p - y - tz^p} F \xrightarrow{\delta} H^1(F, U).$$

It is straightforward to verify that the equation $y^p - y = g$ has no solution $y \in F$ in case $v(g) < 0$ and $v(g) \not\equiv 0 \pmod{p}$, where v denotes the usual t-adic valuation on F. Since $v(tz^p) \equiv 1 \pmod{p}$ for any $z \in F^\times$, it follows that the elements $\{\delta(t^{-np+2}) \mid n \geq 1\}$ of $H^1(F, U)$ are all distinct. Thus, $H^1(F, U)$ is infinite; in particular, it is non-trivial. As a consequence, U is not F-isomorphic to $\mathbf{G}_{a/F}$ and so isn't F-split (this is [Ser97, II.§2.1 Exerc. 3]).

Proposition 33. *Let the F-split unipotent group U act on the F-variety X (by F-morphisms). Suppose $x, y \in X(F)$ are conjugate by $U(\overline{F})$. Assume:*

  (1) *The U-orbit of x is separable.*
  (2) *$U_x = \mathrm{Stab}_U(x)$ is F-split.*

Then x and y are conjugate via $U(F)$.

Proof. Let $\mathcal{O} \subset X$ be the orbit $U.x$. Then $\mathcal{O}$ is a locally closed subvariety of X defined over F. Since the orbit map $U \to \mathcal{O}$ is separable, the group U_x is smooth and there is a U-equivariant F-isomorphism $\mathcal{O} \simeq U/U_x$.

Thus there is an exact sequence of pointed sets

$$U_x(F) \to U(F) \to \mathcal{O}(F) \to H^1(F, U_x);$$

see [Spr98, 12.3.4] (and see the discussion in the beginning of §6 below). Since U_x is F-split, the latter set is trivial. Thus the orbit map $U(F) \to \mathcal{O}(F)$ is surjective, whence the proposition. $\square$

Recall our assumption (4.1) on the nilpotent element $X \in \mathfrak{g}(F)$.

Proposition 34. *Let ϕ be a cocharacter associated with X which is defined over F; cf. Theorem 26. Let $\mathfrak{u}$ be the Lie algebra of the unipotent radical U of $P = P(X)$, and let $\mathfrak{v} = \bigoplus_{i \geq 3} \mathfrak{g}(i; \phi) \subset \mathfrak{u}$. Then $\mathrm{Ad}(U(F))X = X + \mathfrak{v}(F)$.*

Proof. The group U acts on the F-variety $X + \mathfrak{v}$; see the proof of Lemma 10. Moreover, the stabilizer $U_X = C_U(X)$ is precisely the unipotent radical of $C_G(X)$; see Corollary 20. Especially, the U-orbit of X is separable, and U_X is F-split. Thus the result will follow from the previous proposition provided that $\mathrm{Ad}(U)X = X + \mathfrak{v}$; i.e. that the proposition holds over k.

Well, by [J04, Prop. 5.9(c)], we have $\overline{\mathrm{Ad}(P)X} = \bigoplus_{i \geq 2} \mathfrak{g}(2; \phi)$. Since the orbit of X is separable, the differential of the orbit map is surjective. Thus

(5.1) $\mathrm{ad}(X) : \mathfrak{g}(0) \to \mathfrak{g}(2)$ and $\mathrm{ad}(X) : \mathfrak{u} \to \mathfrak{v}$ are surjective.

From (5.1), we see that the orbit map $\rho : U \to X + \mathfrak{v}$ given by $u \mapsto \mathrm{Ad}(u)X$ is dominant. By a result of Rosenlicht, one knows that each U-orbit on the affine variety $X + \mathfrak{v}$ is closed (see [?MR50:4766, Prop. 2.5]). Since $\mathrm{Ad}(U)X$ is dense in the irreducible variety $X + \mathfrak{v}$, equality follows.   □

6. Galois cohomology and finiteness

Let $X \in \mathfrak{g}(F)$ and suppose that (4.1) holds. Since the centralizer $C = C_G(X)$ is smooth, there is a G-equivariant F-isomorphism $\mathcal{O} = \mathrm{Ad}(G)X \simeq G/C$. We have thus an exact sequence of pointed sets

(6.1) $$C(F) \to G(F) \to \mathcal{O}(F) \xrightarrow{\delta} H^1(F, C) \xrightarrow{\alpha} H^1(F, G);$$

see [Spr98, Prop. 12.3.4]. One should be cautious concerning such exact sequences. A sequence of pointed sets $A \xrightarrow{f} B \to 1$ is exact if and only if f is surjective. On the other hand, the sequence of pointed sets $1 \to A \xrightarrow{f} B$ can be exact even when f is not injective. Using techniques of "twisting" one shows that the $G(F)$-orbits in $\mathcal{O}(F)$ are in bijection with the kernel of $\alpha : H^1(F, C) \to H^1(F, G)$; see [KMRT, 28.2] or [Ser97, §5.4 Cor. 2].

Lemma 35. *Let A be a linear algebraic F-group, and suppose that R is a normal connected unipotent F-subgroup which is F-split.*

(1) *If $H^1(F, A/R)$ is finite, then $H^1(F, A)$ is finite.*
(2) *Suppose that an F-split torus S acts on A as a group of automorphisms, and assume that 1 is the only fixed point of S on R. Then the natural map $H^1(F, A) \to H^1(F, A/R)$ is bijective.*

Proof. Since R is F-split, $H^1(F, {}_zR) = 1$ for each $z \in Z^1(F, A)$ by Proposition 30. We have an exact sequence $1 \to H^1(F, A) \xrightarrow{f} H^1(F, A/R)$. Fix $\alpha \in H^1(F, A)$. By [Ser97, I.5 Cor. 2] the elements $\beta \in H^1(F, A)$ with $f(\alpha) = f(\beta)$ are in bijection with a certain quotient of $H^1(F, {}_aR)$, where $a \in Z^1(F, A)$ represents α. Thus f is injective and (1) follows.

Supposing now that a split torus S acts as in (2), we show that f is surjective. Let $R_1 = (R, R)$ be the derived group, and for $i > 1$, let $R_i = (R, R_{i-1})$. Then $R_n = \{1\}$ for some $n \geq 1$, and each R_i is normal in A. In particular, S acts without non-trivial fixed points on each R_i, so that each R_i is an F-split unipotent group by [Spr98, 14.4.2]. Moreover, each R_i/R_{i+1} is an F-split commutative unipotent group, by [Spr98, 14.3.12 exercise 2].

If we show that the natural map $H^1(F, A) \xrightarrow{f} H^1(F, A/R_{n-1})$ is a bijection, the result for R will follow by induction. Thus, we may suppose R to be F-split and *commutative*, and we must show that f is surjective. In this case, we have $H^2(F, {}_zR) = 1$ for all $z \in Z^1(F, A/R)$ by Proposition 30. Thus, we may apply [Ser97, I.5 Cor. to Prop 41] to see that f is surjective.   □

Lemma 36. *Suppose that F has cohomological dimension ≤ 1. Let A be a linear algebraic F-group, and suppose that the F-group A^o is reductive. Then the natural*

map

$$H^1(F, A) \xrightarrow{f} H^1(F, A/A^o)$$

is injective. The map f is bijective if moreover F is perfect.

Proof. Since F has cohomological dimension ≤ 1, a result of Borel and Springer [BS68, 8.6] implies tht $H^1(F, A^o) = 0$. Thus the exact sequence in Galois cohomology arising from the sequence

$$1 \to A^o \to A \to A/A^o \to 1$$

shows that $1 \to H^1(F, A) \xrightarrow{f} H^1(F, A/A^o)$ is exact. The proof that f is injective may then be found in the proof of [Ser97, III.2.4 Corollary 3]; note that F is assumed perfect in *loc. cit.* but this is not essential for the proof of injectivity [one just needs to use: if $b \in Z^1(F, A)$, the F-group $_bA^o$ is again connected and reductive and hence has trivial H^1 by the result of Borel–Springer]. This same result shows that f is bijective in case F is also perfect. $\square$

In the previous proof, the surjectivity of f when F is perfect depends on a result of Springer [Ser97, III.2.4 Theorem 3] concerning principal homogeneous spaces.

Recall that we suppose (4.1) to hold for the nilpotent $X \in \mathfrak{g}(F)$. The centralizer C is then defined over F, and hence the connected component C^o of C is defined over F as well. We write A_X for the component group C/C^o; it is a finite linear F-group.

Proposition 37. *Suppose that F has cohomological dimension ≤ 1. Then:*
  (1) *Each element of $A_X(F)$ can be represented by a coset gC^o with $g \in C(F)$.*
  (2) *The set of $G(F)$-orbits in $\mathcal{O}(F)$ identifies with a subset of $H^1(F, A_X)$.*

Proof. Let ϕ be a cocharacter associated to X which is defined over F. Then the F-split torus $S = \phi(\mathbf{G}_m)$ acts as a group of automorphisms of C and the only fixed point on the unipotent radical R of C is the identity. So Lemma 35 shows that the natural maps $H^1(F, C) \xrightarrow{g} H^1(F, C/R)$ and $H^1(F, C^o) \to H^1(F, C^o/R)$ are bijective.

Now, C^o/R is a connected, reductive F-group, so $H^1(F, C^o/R) = 1$ by the result of Borel–Springer [BS68, 8.6] cited in the previous Lemma. It follows that $H^1(F, C^o) = 1$. There is thus an exact sequence

$$1 \to C^o(F) \to C(F) \to A_X(F) \to 1$$

which proves (1).

It follows from [SS70, III.3.15] that each class in A_X can be represented by a semisimple element; thus $A_X \simeq (C/R)/(C^o/R)$. It follows from Lemma 36 that the map

$$H^1(F, C/R) \xrightarrow{f} H^1(F, A_X)$$

is injective.

If $f' : H^1(F, C) \to H^1(F, A_X)$ is induced by the quotient map $C \to A_X$, then $f' = f \circ g$. Since g is bijective, f' is injective. After the result of Borel–Springer already cited, we have $H^1(F, G) = 1$; thus [KMRT, Cor 28.2] implies that the map $\delta : \mathcal{O}(F) \to H^1(F, C)$ from (6.1) induces a bijection between $H^1(F, C)$ and the set of $G(F)$-orbits in $\mathcal{O}(F)$. Assertion (2) now follows. $\square$

Remark 38. With assumptions and notation as in the preceding proposition, if F is perfect one knows by Lemma 36 that the set of $G(F)$-orbits in $\mathcal{O}(F)$ identifies with $H^1(F, A_X)$. One might well wonder if this remains so when F is not assumed perfect (assuming (4.1) to hold, of course).

Let F be a field complete with respect to a non-trivial discrete valuation v, with $p = \mathrm{char}\, F$. By the residue field κ we mean the quotient of the ring of integers of F by its unique maximal ideal.

Proposition 39. *Suppose κ is finite or algebraically closed, and let A be a linear algebraic group over F. Suppose further that*

(1) *The unipotent radical of A is defined over F and is F-split.*
(2) *$|A/A^o|$ is invertible in F.*

Then $H^1(F, A)$ is finite.

Proof. Let $n = |A/A^o|$. Since n is invertible in F, the field F has only finitely many extensions of degree n in a fixed separable closure F_{sep}. Indeed, there is ≤ 1 unramified extension $F \subset F_m$ of degree m for each $m|n$. So we just need to show that the number of totally ramified extensions of $F_m \subset F'$ of degree n/m is finite. When the residue field is finite, this follows from Krasner's Lemma [Ser79, II.2 exer. 1,2] (since n/m is prime to p, the space of separable Eisenstein polynomials of degree n/m is compact). When the residue field is algebraically closed, apply [Ser79, IV, §2, Cor. 1, Cor. 3].

It follows that $H^1(F, A)$ is finite whenever A is a finite $\Gamma = \mathrm{Gal}(F_{\mathrm{sep}}/F)$-group whose order is prime to $\mathrm{char} F$. This is a variant of [Ser97, II.4.1 Prop. 8], and a proof is given in [Mo88, Lemma 3.11]. For completeness, we outline the argument here. We may find an open normal subgroup $\Gamma_o \lhd \Gamma$ which acts trivially on A (we suppose the profinite group Γ to act continuously on A). The open subgroups of Γ_o with index dividing n are finite in number (apply the conclusion of the previous paragraph to the fixed field $F' = (F_{\mathrm{sep}})^{\Gamma_o}$), and their intersection is an open normal subgroup Γ_1 of Γ. Every continuous homomorphism $\Gamma_o \to A$ vanishes on Γ_1, so the restriction map $H^1(\Gamma, A) \to H^1(\Gamma_1, A)$ is trivial. It then follows from [Ser97, §5.8 a)] that $H^1(F, A)$ identifies with $H^1(\Gamma/\Gamma_1, A)$, which is clearly finite.

Now suppose the proposition is proved in case A is connected. Since $_z A^o$ is connected for all $z \in Z^1(F, A)$, and since $H^1(F, A/A^o)$ is finite by the previous paragraph, we may apply [Ser97, I.5 Cor. 3] to the exact sequence $H^1(F, A^o) \to H^1(F, A) \to H^1(F, A/A^o)$ and deduce the proposition for general A. Since the unipotent radical of A is split, Lemma 35 shows moreover that we may suppose A to be connected and reductive.

If κ is algebraically closed, then by a result of Lang, F is a (C_1) field; see [Ser97, II.3.3(c)]. In particular, F has cohomological dimension ≤ 1; c.f. II.3.2 of *loc. cit.* So when A is connected and reductive, we have $H^1(F, A) = 1$ by the result of Borel–Springer cited in the proof of the previous proposition.

When κ is finite, the finiteness of $H^1(F, A)$ for A connected and reductive is a consequence of Bruhat-Tits theory; cf. [Ser97, III.4.3 Remark(2)]. $\square$

Theorem 40. *Suppose that F is complete for a non-trivial discrete valuation, and that the residue field κ of F is finite or algebraically closed. If (4.1) holds for the*

nilpotent element $X \in \mathfrak{g}(F)$, then $G(F)$ has finitely many orbits on $\mathcal{O}(F)$. In particular, if (4.1) holds for each nilpotent $X \in \mathfrak{g}$, the nilpotent $G(F)$ orbits on $\mathfrak{g}(F)$ are finite in number.

Proof. The Bala-Carter-Pommerening theorem implies that there are finitely many geometric nilpotent orbits; see [J04, §4]. So the final assertion follows from the first. Now, (6.1) shows that the first assertion follows once we know that $H^1(F, C)$ is finite, where $C = C_G(X)$. The order of the component group $A_X = C/C^o$ is invertible in F [SS70, 3.19] (this could also be deduced from the explicit results in [MS]). According to Theorem 28, the unipotent radical of C is defined over F and is F-split. Thus the theorem follows from the previous proposition. □

Remark 41.   (1) Theorem 40 was obtained by Morris [Mo88], under the assumption $p > 4h - 4$ where h denotes the Coxeter number of G. The main new contribution of the present work is application of Theorem 28.
   (2) Recall that (4.1) holds for each nilpotent $X \in \mathfrak{g}$ in case G is a *standard* reductive F-group; cf. Proposition 6.

7. An example: a non-quasisplit group of type C_2

In this section, we use Proposition 27 to study the arithmetic nilpotent orbits of a group of type C_2 which is not quasisplit over the ground field F (i.e. has no Borel subgroup defined over F). In case F is a local field of odd characteristic, we use some local class field theory to classify these orbits; we see especially that they are finite in number, as promised by Theorem 40.

Let Q be a division algebra with center F and $\dim_F Q = 4$ (one says that Q is a quaternion division algebra over F), and suppose that $\mathrm{char} F \neq 2$. There is a uniquely determined symplectic involution ι on Q; see for example [KMRT, §I.2.C].

Denote by $A = \mathrm{Mat}_2(Q)$, and let σ be the involution of A given by

$$\sigma \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \iota(\delta) & \iota(\beta) \\ \iota(\gamma) & \iota(\alpha) \end{pmatrix}.$$

Then σ is the adjoint involution determined by an isotropic hermitian form on a 2 dimensional Q-vector space; cf. [KMRT, I.4.A].

The algebra A together with the symplectic involution σ determine an F-form $G_{/F} = \mathrm{Iso}(A, \sigma)$ of Sp_4; we have

$$G(\Lambda) = \{g \in A \otimes_F \Lambda \mid g \cdot \sigma(g) = 1\}$$

for each commutative F-algebra Λ. The group G has no Borel subgroup over F (see [Spr98, 17.2.10]). There is a cocharacter $\phi = (t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})$ defined over F, and $P = P(\phi)$ is a minimal F-parabolic subgroup. By [Spr98, Theorem 15.4.6], and a little thought, all proper F-parabolic subgroups of G are conjugate by $G(F)$.

There are four geometric nilpotent orbits in $\mathfrak{sp}_4(F_{\mathrm{sep}})$; the corresponding conjugacy classes of instability parabolics are all distinct. So applying Proposition 27, we see that there is a unique non-0 geometric nilpotent orbit with an F-rational point.

For $0 \neq a \in \mathrm{Skew}(Q, \iota) = \mathrm{Skew}(Q) = \{x \in Q \mid x + \iota(x) = 0\}$, the element

$$X_a = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}(F) = \mathrm{Skew}(A, \sigma)$$

is nilpotent. If the field L splits Q, X_a has rank 2 in $\mathrm{Mat}_4(L) = A \otimes_F L$. It follows from the description of nilpotent orbits in $\mathfrak{sp}_4$ by partition that X_a lies in the *subregular orbit* $\mathcal{O}_{\mathrm{sr}} = \mathcal{O}$ (i.e. X_a acts with partition $(2, 2)$ on the natural symplectic module).

The preceding discussion shows that $\mathcal{O}$ is defined over F and has an F-rational point. Moreover, $\mathcal{O}(F)$ is precisely the set of nilpotent elements in $\mathfrak{g}(F)$.

Denote by M the subgroup $\left\{ \begin{pmatrix} x & 0 \\ 0 & \iota(x^{-1}) \end{pmatrix} \mid x \in \mathrm{GL}_{1,Q} \right\} < P$. Thus $M \simeq \mathrm{GL}_{1,Q}$, and since M is the centralizer of the image of the cocharacter ϕ, it is a Levi factor in P. Since a subregular nilpotent element lies in the Richardson orbit of its instability parabolic, it follows that *the arithmetic nilpotent orbits of $G(F)$ are in bijection with the $M(F)$ orbits on the nilradical of* $\mathrm{Lie}(P)$; by the nilradical we mean the Lie algebra of the unipotent radical of P. Moreover, P is the instability parabolic for each of the nilpotent elements X_a with $0 \neq a \in \mathrm{Skew}(Q, \iota)$, and one can even see that ϕ is a cocharacter associated with X_a.

The action of $M(F)$ on the nilradical of $\mathrm{Lie}(P)(F)$ identifies with the representation $(\rho, \mathrm{Skew}(Q))$ of $Q^\times = \mathrm{GL}_{1,Q}(F)$ given by

$$\rho(x)y = xy\iota(x) = \frac{1}{\mathrm{Nrd}(x)} xyx^{-1},$$

where $\mathrm{Nrd} : Q^\times \to F^\times$ is the reduced norm. So we seek a description of the $Q^\times$-orbits on $\mathrm{Skew}(Q)$. One easily sees that the function

$$\eta = (y \mapsto \mathrm{Nrd}(y)F^{\times 2}) : \mathrm{Skew}(Q)^\times \to F^\times/F^{\times 2}$$

is constant on $Q^\times$-orbits, so the essential problem is to find the $Q^\times$-orbits on the fibers of η. If $0 \neq y \in \mathrm{Skew}(Q)$, $F[y]$ is a maximal subfield of Q. It follows that $\eta(y)$ is not the trivial square class.

Let now F be the local field $\mathbf{F}_q((t))$ where $\mathbf{F}_q$ is the finite field having q elements, where q is odd. Then there is a unique division quaternion algebra Q over F [Ser79, ch. XIII]. The group of square classes $F^\times/F^{\times 2}$ is $\mathbf{F}_q^\times/\mathbf{F}_q^{\times 2} \times \mathbf{Z}/2 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$; cf. *loc. cit.* ch. V, Lemma 2.

We claim that the image of η consists in the non-trivial elements of $F^\times/F^{\times 2}$, and that each fiber of η is a single $Q^\times$-orbit. It will follow that there are 3 orbits of $Q^\times$ on $\mathrm{Skew}(Q)^\times$, and thus 3 non-zero arithmetic nilpotent $G(F)$-orbits on $\mathfrak{g}(F)$.

Let $x \in F^\times$ represent a non-trivial square class, and let $\Theta < F^\times$ be the subgroup generated by $F^{\times 2}$ and x. Then Θ is a closed subgroup of index 2 in $F^{\times 2}$ and so $\Theta = N_{L/F}(L^\times)$ for some quadratic extension L of F, by local class field theory [Ser79, Ch. XIV §6. Theorem 1]. We may write $L = F[\sqrt{a}]$ for some $a \in F^\times$ with $\mathrm{Nrd}(a) \in xF^{\times 2}$. By [Ser79, Ch. XIII §4. Cor. 3], L embeds in Q as a maximal subfield. Under any such embedding, $\sqrt{a}$ corresponds to an element $y \in \mathrm{Skew}(Q)$ with $\eta(y) = xF^{\times 2}$. This proves our first claim.

If $y_1, y_2 \in \mathrm{Skew}(Q)^\times$ and $\eta(y_1) = \eta(y_2)$, we show that y_1 and y_2 are conjugate under $Q^\times$. One knows $F[y_1] \simeq F[y_2]$ (as F-algebras), so we may suppose, by the Skolem-Noether Theorem, that $y_1 \in F[y_2] = L$. Since y_1, y_2 each have trace 0, $y_1 \cdot y_2^{-1} \in F^\times$. If $y_1 \cdot y_2^{-1} = N_{L/F}(\beta)$ for some $\beta \in L^\times$, then $y_1 = \rho(\beta)y_2$ and our claim holds. If $y_1 \cdot y_2^{-1} \notin N_{L/F}(L^\times)$, apply the Skolem-Noether theorem to find $\gamma \in Q^\times$ such that $y \mapsto \gamma y \gamma^{-1}$ is the non-trivial element of $\mathrm{Gal}(L/F)$. Then $\{1, \gamma\}$ is an L-basis of Q, and moreover, $\gamma^2 \in Z(Q) = F$, so Q identifies with the "cyclic F-algebra" (L, γ^2) [KMRT, §30.A]. Since Q is a division algebra, $\gamma^2 \notin N_{L/F}(L^\times)$; see [KMRT, Prop. 30.6]. Since $\mathrm{Trd}(\gamma) = 0$, we find

$$\rho(\gamma)y_2 = \frac{-1}{\mathrm{Nrd}(\gamma)}y_2 = \frac{1}{\gamma^2}y_2.$$

Since $[F^\times : N_{L/F}(F^\times)] = 2$, this implies $y_1 \cdot \rho(\gamma)y_2^{-1} = \frac{1}{\gamma^2}y_1 y_2^{-1} \in N_{L/F}(L^\times)$ and the claim follows.

Remark 42. If $0 \neq a \in \mathrm{Skew}(Q)$, the connected component of 1 in the centralizer $C = C_G(X_a)$ has dimension 1 and is isomorphic to the norm torus $\mathbf{G}_{m/L}^1 = \ker(N_{L/F} : R_{L/F}\mathbf{G}_m \to \mathbf{G}_m)$, where $L = F[a]$. Moreover, $[C : C^o] = 2$, C is non-abelian, and the non-trivial coset of C^o in C has no F-rational point. The above calculation shows that $|H^1(F, C)| = 3$ when $F = \mathbf{F}_q((t))$.

8. Orbital integrals

We now suppose that our field F is complete with respect to a non-trivial discrete valuation v, and that the residue field $\mathfrak{f}$ is finite. We suppose that the valuation satifies $v(t) = 1$ for a prime element $t \in F$; the normalized absolute value of $0 \neq a \in F$ is then the rational number $|a| = |\mathfrak{f}|^{-v(a)}$. If the characteristic of F is 0, we will have nothing new to say in this section. When F has characteristic $p > 0$, it is isomorphic to the field of formal power series $\mathfrak{f}((t))$.

If X is a smooth quasi-projective variety over F, then $X(F)$ is an analytic F-manifold. If ω is a non-vanishing regular differential form on X of top degree defined over F, it defines a measure $|\omega|$ on the locally compact topological space $X(F)$ in a well-known manner; see e.g. [PR94, §3.5].

Throughout this section, let G be a reductive group defined over F, and suppose that G is F-standard. Recall that all adjoint orbits and all conjugacy classes are thus known to be separable; cf. Proposition 6.

Since G is reductive, the representation of G on $\bigwedge^{\dim G}\mathfrak{g}$ is trivial (the restriction of this representation to a maximal torus of G is evidently trivial). Thus a left G-invariant differential form ω_G on G of top degree is also right invariant, so it defines a left- and right- Haar measure $|\omega_G|$ on the locally compact group $G(F)$.

Let $X \in \mathfrak{g}(F)$ or $x \in G(F)$, and let $\mathcal{O}$ be the geometric orbit of this element (thus $\mathcal{O} \subset \mathfrak{g}$ or $\mathcal{O} \subset G$), and let C be its centralizer. Since G is F-standard, Proposition 6 shows that C is defined over F.

Lemma 43. *There is a non-vanishing differential form τ of top degree on $\mathcal{O} \simeq G/C$ which is defined over F. Thus, $C(F)$ is unimodular.*

Proof. Since there is a G-invariant bilinear form on $\mathfrak{g}$ defined over F, the lemma follows from [SS70, 3.24, 3.27]. $\qquad\square$

Write $\mathcal{W} = \mathrm{Ad}(G(F))X$ when $X \in \mathfrak{g}(F)$, and write $\mathcal{W} = \mathrm{Int}(G(F))x$ when $x \in G(F)$.

Lemma 44. *$\mathcal{W}$ is an open submanifold of $\mathcal{O}(F)$, and is a locally closed subspace of $\mathfrak{g}(F)$ or of $G(F)$.*

Proof. $\mathcal{O}$ is a smooth variety defined over F, so $\mathcal{O}(F)$ is an analytic F-manifold. We have supposed that the orbit map $G \to \mathcal{O}$ is separable; in other words, this map has surjective differential at each $g \in G$. The inverse function theorem [Ser65, LG3.9] implies that $\mathcal{W}$ is open in $\mathcal{O}(F)$, whence the first assertion.

Now, $\mathcal{O}$ is Zariski-open in $\overline{\mathcal{O}}$, so $\mathcal{O}(F)$ is open in $\overline{\mathcal{O}}(F)$ in the F-topology. Thus $\mathcal{W}$ is open in $\overline{\mathcal{O}}(F)$, which shows that $\mathcal{W}$ is open in its closure $\overline{\mathcal{W}} \subset \overline{\mathcal{O}}(F)$. $\qquad\square$

For a topological space $\mathcal{X}$ we will write $C(\mathcal{X})$ for the algebra of **C**-valued continuous functions on $\mathcal{X}$, and $C_c(\mathcal{X})$ for the sub-algebra of compactly supported continuous functions.

With τ as in lemma 43, we obtain a $G(F)$-invariant measure dg^* on $G(F)/C(F)$. For $f \in C_c(\mathfrak{g}(F))$, respectively $f \in C_c(G(F))$, define the *orbital integral* of f over $\mathcal{W}$ to be

$$I_X(f) = \int_{G(F)/C(F)} f(\mathrm{Ad}(g)Y)dg^* \text{for } X \in \mathfrak{g}(F), \text{ and}$$

$$I_x(f) = \int_{G(F)/C(F)} f(gxg^{-1})dg^* \text{for } x \in G(F).$$

By construction, of course, we have $I_X(f), I_x(f) < \infty$ if $f_{|\mathcal{W}} \in C_c(\mathcal{W})$; this is so e.g. if $\mathcal{W}$ is *closed*. One is interested in the convergence of the integrals $I_X(f)$ in general; we will now investigate these integrals.

8.1. **Nilpotent case.** We first consider the integral $I_X(f)$ in the case where $X \in \mathfrak{g}(F)$ is nilpotent.

Theorem 45. *Let $X \in \mathfrak{g}(F)$ be nilpotent. Then $I_X(f) < \infty$ for each $f \in C_c(\mathfrak{g}(F))$.*

The theorem was proved by Deligne and by Ranga Rao [Rao72], in the case that F has characteristic 0. We show here how to adapt the original proof to the positive characteristic setting.

Let P be the instability F-parabolic subgroup determined by X. Fix a co-character ϕ associated to X and defined over F; cf. Theorem 26. We abbreviate $\mathfrak{g}(i; \phi)$ as $\mathfrak{g}(i)$ for $i \in \mathbf{Z}$, and we write $\mathfrak{w}_i = \mathfrak{g}(i)(F)$. Recall that ϕ determines a Levi factor $M = C_G(\phi(\mathbf{G}_m))$ of P which is defined over F.

Inspecting the argument given in [Rao72], one sees that $I_X(f) < \infty$ for $f \in C_c(\mathfrak{g}(F))$ if we establish the following:

R1. The $M(F)$-orbit of X is an open submanifold $\mathcal{V} \subset \mathfrak{w}_2$.

R2. The $P(F)$-orbit of X is $\mathcal{V} + \sum_{i \geq 3} \mathfrak{w}_i$.

R3. There is a non-negative function $\phi \in C(\mathfrak{w}_2)$ with $\phi(X) \neq 0$ and $\phi(\mathrm{Ad}(m)Y) = |\det(\mathrm{Ad}(m)_{\mathfrak{w}_1})|\phi(Y)$.

More precisely, suppose that R1–3 hold, let K be an open compact subgroup of $G(F)$ with the property $G(F) = K \cdot P(F)$ (that there should be such a K is a result of Bruhat–Tits; see e.g. [Ti79]), let dY and dZ be additive Haar measure respectively on $\mathfrak{w}_2$ and $\mathfrak{w}_{\geq 3} = \sum_{i \geq 3} \mathfrak{w}_i$, and put

$$\Lambda(f) = \int_{\mathfrak{w}_2 \oplus \mathfrak{w}_{\geq 3}} \phi(Y) f(Y + Z) dY \, dZ,$$

and

$$\overline{f}(Y) = \int_K f(\mathrm{Ad}(x)Y) dx, \quad Y \in \mathfrak{g}(F),$$

for $f \in C_c(\mathfrak{g}(F))$, dx denoting a Haar measure on K. Under our assumptions, it is proved in *loc. cit.* that

$$I_X(f) = c \cdot \Lambda(\overline{f}) \quad \text{for } f \in C_c(\mathfrak{g}(F)),$$

where $0 \neq c$ is a suitable constant; in particular, $I_X(f) < \infty$.

We first verify that conditions R1, R2 hold.

Proposition 46. *Let $\mathfrak{v}_+ = \sum_{i \geq 3} \mathfrak{w}_i$.*

(1) *The $M(F)$-orbit of X is an open submanifold $\mathcal{V} \subset \mathfrak{w}_2$.*

(2) *The $P(F)$-orbit of X is $\mathcal{V} + \mathfrak{v}_+$, an open submanifold of $\mathfrak{v}$.*

Proof. The orbit map $(m \mapsto \mathrm{Ad}(m)X) : M \to \mathfrak{g}(2)$ has surjective differential at 1 by (5.1) in the proof of Proposition 34, and hence has surjective differential at each $m \in M$ (by transport of structure). It follows as in the proof of Lemma 44 that $\mathcal{V}$ is open.

Now $P(F) = M(F) \cdot U(F)$; moreover, $\mathrm{Ad}(U(F))X = X + \mathfrak{v}_+$ by Proposition 34. Thus $\mathrm{Ad}(P(F))X = \mathrm{Ad}(M(F))(X + \mathfrak{v}_+) = \mathcal{V} + \mathfrak{v}_+$ as claimed. $\qquad\square$

The proposition implies R1 and R2. Turning to R3, note first that the non-degenerate form κ restricts to an $M(F)$-equivariant perfect pairing

$$\kappa : \mathfrak{w}_{-1} \times \mathfrak{w}_1 \to F$$

and hence an $M(F)$-equivariant perfect pairing

$$\mu = \bigwedge^d \kappa : \bigwedge^d \mathfrak{w}_{-1} \times \bigwedge^d \mathfrak{w}_1 \to F,$$

where $d = \dim \mathfrak{w}_1$ Note that when $d = 0$, μ is the multiplication pairing $F \times F \to F$. Also note for $m \in M(F)$ that $(\bigwedge^d \mathrm{Ad})(m)$ acts on $\bigwedge^d \mathfrak{w}_{\pm 1}$ as multiplication with $\det(\mathrm{Ad}(m)_{|\mathfrak{w}_1})^{\pm 1}$. Let $0 \neq \tau \in \bigwedge^d \mathfrak{w}_{-1}$.

Proposition 47. *(compare [Rao72, Lemma 2]) Define the function $\phi : \mathfrak{w}_2 \to \mathbf{R}_{\geq 0}$ by the rule*

$$\phi(Y) = |\mu(\tau, \bigwedge^d (\mathrm{ad}(Y))\tau)|^{1/2}.$$

Then $\phi(X) > 0$, and for each $m \in M(F)$ and $Y \in \mathfrak{w}_2$, we have

$$\phi(\mathrm{Ad}(m)Y) = |\det(\mathrm{Ad}(m)_{|\mathfrak{w}_1}| \phi(Y).$$

Proof. By Proposition 13 one knows that $C_G(X) \subset P$; thus $\mathrm{Lie}(C_G(X)) \subset \mathrm{Lie}(P)$. Since the orbit of X is separable, one knows that $\mathfrak{c}_\mathfrak{g}(X) = \mathrm{Lie}(C_G(X)) \subset \mathrm{Lie}(P)$. This implies that $\mathfrak{c}_\mathfrak{g}(X) \cap \mathfrak{g}(-1) = 0$, and so $\mathrm{ad}(X) : \mathfrak{g}(-1) \to \mathfrak{g}(1)$ is bijective. It follows that $\phi(X) > 0$.

We have the identity $\mathrm{ad}(\mathrm{Ad}(m)Y) = \mathrm{Ad}(m) \circ \mathrm{ad}(Y) \circ \mathrm{Ad}(m^{-1}) : \mathfrak{w}_{-1} \to \mathfrak{w}_1$. Thus $\bigwedge^d(\mathrm{ad}(\mathrm{Ad}(m)Y))\tau = \det(\mathrm{Ad}(m)_{|\mathfrak{w}_1})^2 \bigwedge^d(\mathrm{ad}(Y))\tau$. This implies the second assertion. $\square$

This proposition verifies R3, and in view of what was said before, completes the proof of Theorem 45.

8.2. **Jordan decomposition.** Let A be a linear algebraic group. If $x \in A$ recall that the Jordan decomposition of x is the expression $x = su$ with $s \in A$ semisimple, $u \in A$ unipotent, and $su = us$. It is a basic fact that each element has a Jordan decomposition, and that s and u are uniquely determined. Similar statements hold for the Jordan decomposition $X = S + N$ for $X \in \mathrm{Lie}(A)$ (where now N is nilpotent). In this section, we consider the question of when the Jordan decomposition of $x \in A(F)$ (and of $X \in \mathrm{Lie}(A)(F)$) is defined over F in the case when A is a reductive group. Of course, if $x = su \in A(F)$, we have $s \in A(F)$ if and only if $u \in A(F)$.

Proposition 48. *Suppose that* $p > \mathrm{rank}_{\mathrm{ss}} G + 1$.
(1) *Let* $g \in G(F)$, *and let* $g = su$ *be the Jordan decomposition of* g *with* $s, u \in G(\overline{F})$. *Then* $s, u \in G(F)$.
(2) *Let* $X \in \mathfrak{g}(F)$, *and let* $X = S + N$ *be the Jordan decomposition of* X *with* $S, N \in \mathfrak{g}(\overline{F})$. *Then* $S, N \in \mathfrak{g}(F)$.

Remark 49. Without our assumption on p, the proposition is false. Indeed, consider the group $G = GL_{p/F}$, let $f \in F[T]$ be a purely inseparable irreducible polynomial of degree p, and let $g \in \mathrm{GL}_p(F)$ and $X \in \mathfrak{gl}_p(F)$ be any elements having characteristic polynomial f. Then the semisimple part of each of these elements is the scalar matrix $\alpha \cdot I$ where α is the unique root of f in the algebraically closed extension k. In particular, this semisimple part is not F-rational.

We begin with a few lemmas.

Lemma 50. *Let A be a linear algebraic group defined over F.*
(1) *Let $x \in A(F)$ and let $x = su$ be the Jordan decomposition of x. Then $u \in A(F)$ if and only if $u \in A(F_{\mathrm{sep}})$.*
(2) *Let $X \in \mathrm{Lie}(A)(F)$ and let $X = S + N$ be the Jordan decomposition of X. Then $N \in A(F)$ if and only if $N \in A(F_{\mathrm{sep}})$.*

Put another way, the Jordan decomposition of an element is defined over F if and only if it is defined over F_{sep}.

Proof. We treat the case $x \in A(F)$; the Lie algebra version is similar. Suppose that $u \in A(F_{\mathrm{sep}})$ and let $\gamma \in \mathrm{Gal}(F_{\mathrm{sep}}/F)$. To see that $u \in A(F)$, it is enough to see that $u' = \gamma(u) = u$. But $x = \gamma(x) = \gamma(su) = s'u'$ (where $u' = \gamma(u)$). Since s' is semisimple and u' is unipotent, and since evidently $s'u' = u's'$, the fact that $u' = u$ follows from the unicity of the Jordan decomposition of x. $\square$

Lemma 51. *Let G be a semisimple group over F. Let $x \in G(F)$ have Jordan decomposition $x = su$, and suppose that s is contained in the center of G. Then $s, u \in G(F)$.*

Proof. In view of the previous lemma, we may as well suppose that F is separable closed. Since the center Z of G is a finite diagonalizable subgroup, $Z(\overline{F}) = Z(F)$ (recall we are assuming F to be separably closed). Since $s \in Z$, it follows that $s \in G(F)$ as desired. $\square$

Lemma 52. *Let A be a linear algebraic group over the algebraically closed field k and let $x \in A$ be semisimple. Then $C_A(x) = C_A(x^q)$ for any $q = p^n$. Similarly, let $X \in \mathrm{Lie}(A)$ be semisimple. Then $C_A(X) = C_A(X^{[q]})$, where $X \mapsto X^{[p]}$ denotes the p-operation on $\mathrm{Lie}(A)$.*

Proof. Since A has a faithful matrix representation, it suffices to prove the lemma for the group $A = \mathrm{GL}(V)$. Morever, the proof in the Lie algebra case is not essentially different, so we discuss only the case where $x \in A$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of x in k. Then $\lambda_1^q, \ldots, \lambda_m^q$ are the eigenvalues of x^q, and the lemma follows provided that the λ_i^q are all distinct. If $\lambda_i^q = \lambda_j^q$, then λ_i/λ_j is a q-th root of unity. Since k has characteristic p, it follows that $\lambda_i = \lambda_j$ so that $i = j$ as desired. $\square$

Proof of Proposition 48: We first prove (1). Since u is unipotent, $u^q = 1$ for some $q = p^n$. Since $su = us$, we have $g^q = s^q \in G(F)$. It follows from Lemma 52 that $C_G^o(s) = C_G^o(s^q) = C_G^o(g^q)$. Since g^q is F-rational and semisimple, $C = C_G^o(s)$ is a connected, reductive F-subgroup of G.

Let $C_1 = (C, C)$ be the derived group of C. Then C_1 is a semisimple subgroup of (G, G), and by Lemma 2 the prime p is *very good* for C_1.

Let $\overline{C} = C/Z$ be the corresponding adjoint group, and let $\pi : C \to \overline{C}$ be the canonical surjection. Since p is very good for C_1, Lemma 1 implies that the restriction $\phi = \pi_{|C_1}$ of π to C_1 is a separable isogeny $\phi : C_1 \to \overline{C}$.

Since p is good for G, it follows from [SS70, 3.15] that $u \in C = C_G^o(g^q)$; since s is contained in a maximal torus of G, we have also $s \in C$ so that $g \in C(F)$. Moreover, s is central in C. Consider the element $v = \pi(g) \in \overline{C}(F)$. It follows from [Spr98, 11.2.14] that the fiber $\phi^{-1}(v) \subset C_1$ is defined over F. That fiber must therefore contain a point rational over F_{sep}; thus, there is some $w \in C_1(F_{\mathrm{sep}})$ with $\phi(w) = v$. Let $w = s_1 u_1$ be the Jordan decomposition of w in C_1. An application of Lemma 51 shows that $u_1, s_1 \in C_1(F_{\mathrm{sep}})$.

We now have $w^{-1}g \in C(F_{\mathrm{sep}})$. But $\pi(w^{-1}g) = 1$ so that $w^{-1}g \in Z(F_{\mathrm{sep}})$. It follows that $u = u_1 \in C(F_{\mathrm{sep}})$. This shows that the Jordan decomposition $g = su$ is defined over F_{sep}. It now follows from Lemma 50 that $s, u \in C(F)$ as desired; this proves (1).

The proof of (2) is similar, though a bit easier. Let $X = S + N$ be the Jordan decomposition, and again find n large enough so that $N^{[q]} = 0$ where $q = p^n$. Since $[S, N] = 0$, we have $X^{[q]} = S^{[q]}$. Since $X^{[q]} \in \mathfrak{g}(F)$ is semisimple, its centralizer $C = C_G^o(X^{[q]})$ is a reductive F-subgroup. Again let $C_1 = (C, C)$. Arguing as before, one sees that the characteristic is very good for C_1. Since $\mathrm{Lie}(C_1)$ has no trivial submodules, one finds that $\mathrm{Lie}(C) = \mathrm{Lie}(C_1) \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the Lie algebra of the center of C. It follows that $N \in \mathrm{Lie}(C_1)$ and $S \in \mathfrak{z}$. The center of C is defined over F (e.g. since it is the kernel of the F-homomorphism $C \to C_{1,\mathrm{adj}}$). Thus, $\mathfrak{z}$ is defined over

F. Since also $\mathrm{Lie}(C_1)$ is defined over F, we deduce $\mathrm{Lie}(C)(F) = \mathrm{Lie}(C_1)(F) \oplus \mathfrak{z}(F)$. Since $X \in \mathrm{Lie}(C)(F)$, it follows that $N \in \mathrm{Lie}(C_1)(F)$ and $S \in \mathfrak{z}(F)$; the proof is now complete. $\qquad\square$

8.3. General orbital integrals on the Lie algebra.

We will now use Ranga Rao's argument [Rao72] to deduce the convergence of a general orbital integral in favorable cases.

Theorem 53. *Let G be an F-standard reductive group, let $X \in \mathfrak{g}(F)$ have Jordan decomposition $X = S+N$. If $p > \mathrm{rank}_{\mathrm{ss}} G+1$, then $I_X(f) < \infty$ for each $f \in C_c(\mathfrak{g}(F))$.*

Sketch. The proof is the same as that of Theorem 2 in [Rao72]. We outline the argument for the reader's convenience; for full details, refer to *loc. cit.*

We note first that Theorem 45 remains valid even when the reductive group G is not connected. This follows from the fact that the $G(F)$ orbit of X is the disjoint union of finitely many $G^o(F)$ orbits.

So fix $f \in C_c(\mathfrak{g}(F))$, consider the reductive F-group [1] $H = C_G(S)$ and note that $C = C_G(X)$ is the centralizer in H of N. Since $N \in \mathrm{Lie}(H)(F)$, it follows from Theorem 45, and the preceeding remarks, that we may define for $y \in G(F)$

$$(8.1) \qquad g(y) = \int_{H(F)/C(F)} f(\mathrm{Ad}(y)(S + \mathrm{Ad}(h)N))dh^*$$

where dh^* denotes the invariant measure on $H(F)/C(X)(F)$. Then g is continuous in y, satisfies $g(yh) = g(y)$ for $h \in H(F)$, and the argument in *loc. cit.* [2] shows that g has compact support in $G(F)/H(F)$. Thus,

$$\int_{G(F)/H(F)} g(y)dy^* = \int_{G(F)/H(F)} dy^* \int_{H(F)/C(F)} f(\mathrm{Ad}(yh)Z)dh^*$$

$$= \int_{G(F)/C(F)} f(\mathrm{Ad}(x)Z))dx^* = I_X(f)$$

is finite. $\qquad\square$

Notice that the proof only uses the assumption made on p to know that $S, N \in \mathfrak{g}(F)$, i.e. that the Jordan decomposition of X is defined over F.

8.4. Strongly standard groups.

We are going to prove the analogue for groups of Theorem 53; to do this, we require a somewhat stronger hypothesis on our reductive F-group G. We now explain this hypothesis. The field F is arbitrary.

Consider F-groups H which are direct products

$$(*) \quad H = H_1 \times S,$$

---

[1] In fact, under our assumptions on G, the centralizer $H = C_G(S)$ will also be connected – see e.g [SS70, 3.19]. However, we need to apply this argument for a proof of Theorem 61 below; in that setting the centralizer of the semisimple part of x will in general be disconnected, so that the argument described here is indeed necessary.

[2] It is assumed in [Rao72] that G is semisimple; the argument that the function g has compact support given in *loc. cit.* uses the fact that the adjoint representation is faithful. However, it is clear that one can use just any faithful linear representation of G, rather than the adjoint representation.

where S is an F-torus and H_1 is a connected, semisimple F-group for which the characteristic is very good. We say that the reductive F-group G is *strongly standard* if there exists a group H of the form $(*)$ and a separable F-isogeny between G and an F-Levi subgroup of H. Thus, G is separably isogenous to $M = C_H(S_1)$ for some F-subtorus $S_1 < H$; note that we do not require M to be the Levi subgroup of an F-rational parabolic subgroup. It is checked in [Mc, Proposition 2] that a strongly standard F-group G is F-standard in the sense of §2 of this paper. Note that any F-form of GL_n is strongly standard (see Remark 2 of *loc. cit.*) but that SL_n is strongly standard just in case $(n, p) = 1$.

8.5. **An algebraic analogue of the logarithm.** In characteristic 0, the convergence of unipotent orbital integrals (on the group) is deduced by Ranga Rao in [Rao72] using the exponential map from the Lie algebra to the group; of course, the exponential of a nilpotent element is always meaningful in this setting, and the existence of an open neighborhood of the nilpotent set on which the exponential converges is also required in *loc. cit.* When the characteristic of F is positive, the usual exponential map may well define an isomorphism between the nilpotent set and the unipotent set (at least if p is large) but this isomorphism will never extend to an open neighborhood of the nilpotent set in $\mathfrak{g}(F)$: the naive exponential of a semisimple element will never be defined.

To correct this problem, we require a construction used by Bardsley and Richardson. For the remainder of §8.5, F may be an arbitrary field of characteristic p.

Theorem 54. *Suppose that H_1 is a simply connected semisimple F-group in very good characteristic, and that G is an F-Levi subgroup of $H_1 \times S$ for some F-torus S. Let $\mathcal{U} \subset G$ be the unipotent variety, and let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent variety. Then there are G-stable, F-open sets $U \subset G$ and $V \subset \mathfrak{g}$ such that $\mathcal{U} \subset U$ and $\mathcal{N} \subset V$, and there is a G-equivariant morphism $\Lambda : G \to \mathfrak{g}$ such that*

(1) *Λ is defined over F,*
(2) *$\Lambda_{|\mathcal{U}} : \mathcal{U} \to \mathcal{N}$ is an isomorphism of varieties, and*
(3) *$\Lambda_{|U} : U \to V$ is surjective and étale.*

We will first prove a technical result.

Lemma 55. *Let H_1 be a simply connected semisimple F-group in very good characteristic. Then there is a semisimple F-representation (ρ, W) with the properties:*

BR1. *$d\rho : \mathfrak{h}_1 \to \mathfrak{gl}(W)$ is injective, and*
BR2. *there is an H_1-invariant F-subspace $\mathfrak{m} \subset \mathfrak{gl}(W)$ such that $\mathfrak{gl}(W) = \mathfrak{m} \oplus d\rho(\mathfrak{h}_1)$ and such that $1_W \in \mathfrak{m}$.*

Proof. If (ρ, W) is a semisimple F-representation of H_1, BR2 is a consequence of BR3:

BR3. The trace form $\kappa(X, Y) = \mathrm{tr}(d\rho(X) \circ d\rho(Y))$ on $\mathfrak{h}_1$ is non-degenerate.

Indeed, the trace form on $\mathfrak{gl}(W)$ is non-degenerate, and if BR3 holds, the first condition of BR2 holds with $\mathfrak{m} = d\rho(\mathfrak{h}_1)^{\perp}$. Since H_1 is semisimple, $d\rho(\mathfrak{h}_1)$ lies in $\mathfrak{sl}(W)$. Thus, 1_W is orthogonal to $d\rho(\mathfrak{h}_1)$ under the trace form and so lies in $\mathfrak{m}$.

When H_1 is *split*, it follows from [SS70, I.5.3] that there is a suitable semisimple F-representation for which BR1 and BR3 (and hence BR2) hold.

In general, we may choose a finite separable extension $F \subset E$ which splits H_1. The preceeding discussion yields an E-representation (ρ, W) satisfying BR1 and BR3.

By the adjoint property of the restriction of scalars functor, the E-homomorphism $\rho : H_{1/E} \to \mathrm{GL}(W)$ yields an F-homomorphism $\rho' : H_{1/F} \to R_{E/F} \mathrm{GL}(W)$; the latter group is a closed F-subgroup of $\mathrm{GL}(W_F)$, where $W_F = R_{E/F}(W)$ denotes the E-vector space W regarded as an F-vectorspace. Thus we may regard ρ' as an F-representation (ρ', W_F) of H_1.

We note that the F-representation (ρ', W_F) is semisimple. Indeed, extending scalars, there is an isomorphism of $H_{1/E}$-representations

$$(\rho' \otimes 1_E, W_F \otimes_F E) \simeq \bigoplus_{j=1}^{e} (\rho, W),$$

where $e = [E : F]$; since this scalar extension yields a semisimple representation, the original representation was already semisimple.

If $\phi : W \to W$ is any E-linear map, we have

$$\mathrm{tr}_{E/F}(\mathrm{tr}_E(\phi; W)) = \mathrm{tr}_F(\phi; W_F)$$

where $\mathrm{tr}_{E/F} : E \to F$ denotes the trace of the separable field extension E/F. If κ' is the form on $\mathfrak{h}_1$ determined by ρ', this shows that $\kappa' = \mathrm{tr}_{E/F} \circ \kappa$ on $\mathfrak{h}_1(F)$; since $\mathrm{tr}_{E/F}$ is non-0, κ' is nondegenerate on $\mathfrak{h}_1(F)$ and hence nondegenerate on $\mathfrak{h}_1$. This completes the proof of the lemma. $\qquad\square$

Proof of Theorem 54. The previous lemma gives a semisimple F-representation (ρ, W) of H_1 satisfying BR1, BR2 and BR3; we regard ρ as a representation of H with $\rho(S) = 1$.

We may now define a map $\Lambda : H \to \mathfrak{h}$ as follows. For $h \in H$, write $\rho(h) = (X, Y) \in \mathfrak{h} \oplus \mathfrak{m}$ and put $\Lambda(h) = X$. Evidently Λ is defined over F. Since $\mathfrak{m}$ is H-invariant, Λ is H-equivariant. Since $1_W \in \mathfrak{m}$ by BR2, $\Lambda(1) = 0$.

The fact that Λ satisfies condition (2) of the statement of the theorem follows from Corollary 9.3.4 of [BR85]; condition (3) follows from Theorem 6.2 in *loc. cit.* ("Luna's Fundamental Lemma"). This proves the theorem in case $H = G$.

To prove the result for G, recall that $G = C_H(S_1)$ for some F-torus $S_1 \leq H$. Thus $\mathfrak{g} = \mathfrak{c}_{\mathfrak{h}}(S_1)$ and it is clear that that $\Lambda_{|G} : G \to \mathfrak{g}$ satisfies conditions (1),(2) and (3) of the conclusion of the theorem. $\qquad\square$

Remark 56. Note that the group G in the statement of Theorem 54 is strongly standard. It is not clear to the author whether the theorem holds more generally for any strongly standard group, however. It holds for instance whenever $G = H$ is a semisimple group in very good characteristic such that the trace form of the adjoint representation ("Killing form") is non-degenerate. However, this latter condition is not always true; for instance, the trace form of the adjoint representation of $\mathrm{PSp}(V)$ is identically zero if $p \mid \dim V$.

Remark 57. The existence of an equivariant F-isomorphism $\mathcal{U} \simeq \mathcal{N}$ permits us to transfer to $\mathcal{U}$ a number of the results obtained in this paper for nilpotent elements. If $u = \Lambda^{-1}(X)$ for $X \in \mathcal{N}(F)$, then $C_G(u) = C_G(X)$. Moreover, the conjugacy class

of u is separable if and only if that is so of the orbit of X. In particular, it follows from Theorem 28 that the unipotent radical of $C_G(u)$ is F-split under the hypothesis that F is perfect or the conjugacy class of u is separable. In case all unipotent classes are separable and F is complete for a non-trivial discrete valuation with finite or algebraically closed residue field, it follows from Theorem 40 that there are only finitely many $G(F)$-orbits on $\mathcal{U}(F)$.

Note that the Bardsley-Richardson map Λ is not necessary; a result of T. Springer allows one to obtain an equivariant F-isomorphism $\mathcal{U} \simeq \mathcal{N}$ under milder hypotheses.

8.6. **Convergence of unipotent orbital integrals.** We now specialize again to the case where F is complete for a non-trivial discrete valuation and has finite residue field. Let G be a strongly standard F-group. We are going to prove the following theorem:

Theorem 58. *Let $u \in G(F)$ be unipotent. Then $I_u(f) < \infty$ for all $f \in C_c(G(F))$.*

We first suppose that $\hat{G}$ is a second strongly standard F-group and that $\pi : \hat{G} \to G$ is a separable isogeny. If $f \in C_c(G(F))$, we may define $\pi^*(f)$ by the rule:

$$\pi^*(f)(g) = f(\pi(g))$$

for $g \in \widehat{G}(F)$. Let $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{N}}$ be the unipotent and nilpotent varieties for $\widehat{G}$, and let $\mathcal{U}$ and $\mathcal{N}$ be those for G.

Lemma 59.　　(a) *Let $f \in C_c(G(F))$. Then $\pi^*(f) \in C_c(\widehat{G}(F))$.*
　　(b) *If $u \in G(F)$ there is a unique $\hat{u} \in \widehat{G}(F)$ such that $u = \pi(\hat{u})$. Moreover, let $f \in C_c(G(F))$. Then $I_u(f) < \infty$ if and only if $I_{\hat{u}}(\pi^*(f)) < \infty$.*

Proof. (a) The map $\pi^*(f)$ is evidently a continuous function on $\widehat{G}(F)$. The support of $\pi^*(f)$ is the inverse image under π of the support of f; since π is an open mapping and since f has compact support, this inverse image is contained in a compact set.

(b) The maps $\pi_{|\widehat{\mathcal{U}}} : \widehat{\mathcal{U}} \to \mathcal{U}$ is an equivariant F-isomorphism; see e.g. [Mc03, Lemma 27]; it is then clear that in fact $I_u(f) = I_{\hat{u}}(\pi^*(f))$.　　　　　　　　　□

Now suppose that G is an F-Levi subgroup of $H = H_1 \times S$ where H_1 is a simply connected semisimple F-group in very good characteristic, and S is an F-torus. Write $\mathcal{U}$ and $\mathcal{N}$ for the unipotent and nilpotent varieties for G, and denote by $\Lambda : G \to \mathfrak{g}$ the equivariant F-morphism given by Theorem 54. In particular, let U and V be as in the statement of that theorem.

Since the étale map $\Lambda_{|U}$ has finite fibers, one may define

$$\Lambda_*(f)(X) = \sum_{y \in \Lambda^{-1}(X)} f(y)$$

for any function $f \in C_c(G(F))$ whose support is contained in $U(F)$, and for any $X \in \mathfrak{g}(F)$.

Lemma 60. *Let $f \in C_c(G(F))$, and suppose the support of f is contained in $U(F)$. Then $\Lambda_*(f) \in C_c(\mathfrak{g}(F))$.*

Proof. The support of $\Lambda_*(f)$ is contained in the image under Λ of the support of f, hence $\Lambda_*(f)$ is compactly supported. The fact that $\Lambda_*(f)$ is continuous follows from the inverse function theorem [Ser65, LG3.9]. □

Proof of Theorem 58. It follows from definitions that G is separably isogenous to an F-group $\widehat{G}$ which is an F-levi subgroup of a group $H = H_1 \times S$ where H_1 is a semisimple, simply connected F-group in very good characteristic, and where S is an F-torus. According to Lemma 59, our theorem will follow if it is proved for $\widehat{G}$; thus we replace G by $\widehat{G}$.

We may now find an equivariant F-morphism $\Lambda : G \to \mathfrak{g}$ as in Theorem 54. With notation as in that theorem, one knows that the closure of the class $\mathrm{Int}(G(F))u$ is contained in $U(F)$. Thus, it suffices to consider only those f whose support is contained in $U(F)$. Let $X = \Lambda(u) \in \mathfrak{g}(F)$. Thus X is nilpotent and Λ defines an isomorphism between $\mathrm{Int}(G(F))u$ and $\mathrm{Ad}(G(F))X$. By Lemma 60 and Theorem 45 $I_X(\Lambda_*(f)) < \infty$. But it is clear that $I_u(f) = I_X(\Lambda_*(f))$, so the theorem is proved. □

8.7. Convergence of orbital integrals on $G(F)$.

Theorem 61. *Let G be a strongly standard reductive F-group, and assume that $p > \mathrm{rank}_{ss} G + 1$. Let $x \in G(F)$ and let $f \in C_c(G(F))$. Then $I_x(f) < \infty$.*

Sketch. This is deduced from Theorem 58 using Ranga Rao's argument [Rao72] in the same way that Theorem 53 was deduced from Theorem 45. □

Just as in Theorem 53, the proof only uses the assumption made on p to know that the Jordan decomposition of x is defined over F.

8.8. Convergence for more general groups.
One would hope that the hypothesis $p > \mathrm{rank}_{ss} G + 1$ made in Theorems 53 and 61 is unnecessary. It is indeed unnecessary in the following cases. Let A be a central simple F-algebra, and let G be the F-group $\mathrm{GL}_1(A)$, so that G is an inner form of the group $\mathrm{GL}_{n/F}$ where $\dim_F A = n^2$. The convergence of arbitrary orbital integrals is established in [DKV84] following arguments of R. Howe. See also [Lau98, Chapter 4] for a detailed argument in the case $G = \mathrm{GL}_{n/F}$.

REFERENCES

[BR85] Peter Bardsley and R. W. Richardson, *Étale slices for algebraic transformation groups in characteristic p*, Proc. London Math. Soc. (3) **51** (1985), no. 2, 295–317. MR86m:14034

[BS68] A. Borel and T. A. Springer, *Rationality properties of linear algebraic groups. II*, Tôhoku Math. J. (2) **20** (1968), 443–497. MR39#5576

[DKV84] P. Deligne, D. Kazhdan, and M.-F. Vignéras, *Représentations des algèbres centrales simples p-adiques*, Representations of reductive groups over a local field, 1984, pp. 33–117 (French).MR 86h:11044

[Hum] James E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Math. Surveys and Monographs, vol. 43, Amer. Math. Soc., 1995.

[J04] Jens Carsten Jantzen, *Nilpotent orbits in representation theory*, Lie Theory: Lie Algebras and Representations, 2004, pp. 1–211. Notes from Odense summer school, August 2000.

[KMRT] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, Amer. Math. Soc. Colloq. Publ., vol. 44, Amer. Math. Soc., 1998.

[K78]   George R. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), no. 2, 299–316. MR80c:20057

[Lau98] Gérard Laumon, *Cohomology of Drinfeld modular varieties. Part I*, Cambridge Studies in Advanced Mathematics, vol. 41, Cambridge University Press, Cambridge, 1996. Geometry, counting of points and local harmonic analysis. MR98c:11045a

[Mc03]   George J. McNinch, *Sub-principal homomorphisms in positive characteristic*, Math. Zeitschrift **244** (2003), 433–455. arXiv:math.RT/0108140.

[Mc]   ———, *Optimal* SL(2)-*homomorphisms* (2003). math.RT/0309385.

[MS]   George J. McNinch and Eric Sommers, *Component groups of unipotent centralizers in good characteristic*, J. Alg **260** (2003), 323–337. arXiv:math.RT/0204275.

[Mo88]   Lawrence Morris, *Rational conjugacy classes of unipotent elements and maximal tori, and some axioms of Shalika*, J. London Math. Soc. (2) **38** (1988), no. 1, 112–124. MR89j:22037

[PR94]   Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, 1994. English translation.

[Pre02] Alexander Premet, *Nilpotent orbits in good characteristic and the Kempf-Rousseau theory*, J. Alg **260** (2003), 338–366.

[RR84]   S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. J. (2) **36** (1984), no. 2, 269–291. MR85j:14017

[Rao72] R. Ranga Rao, *Orbital integrals in reductive groups*, Ann. of Math. (2) **96** (1972), 505–510. MR47#8771

[Ser65] Jean-Pierre Serre, *Lie algebras and Lie groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1965. MR36#1582

[Ser79] ———, *Local fields*, Grad. Texts in Math., vol. 67, Springer Verlag, 1979.

[Ser97] ———, *Galois cohomology*, Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author.MR 98g:12007

[Spr98] Tonny A. Springer, *Linear algebraic groups*, 2nd ed., Progr. in Math., vol. 9, Birkhäuser, Boston, 1998.

[SS70]   Tonny A. Springer and Robert Steinberg, *Conjugacy classes*, Seminar on algebraic groups and related finite groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), 1970, pp. 167–266. Lecture Notes in Mathematics, Vol. 131. MR42#3091

[St74]   Robert Steinberg, *Conjugacy classes in algebraic groups*, Springer-Verlag, Berlin, 1974. Notes by Vinay V. Deodhar, Lecture Notes in Mathematics, Vol. 366. MR50#4766

[Ti79]   Jacques Tits, *Reductive groups over local fields*, 1979, pp. 29–69.

PERSONAL ADDRESS: 1715 East LaSalle Ave, South Bend, Indiana 46617, USA

E-mail address: mcninchg@member.ams.org