Reductive subgroups of a reductive algebraic group over a local field

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1 Overview

- These notes concern recent work of the speaker
  McNinch (2021) and McNinch (2020) on reductive groups over a local field.
- Ultimately this work originated from attempts to give a different perspective on construction(s) of (DeBacker 2002).
- These notes will be posted at https://gmcninch-tufts.github.io/math/ (just google for "McNinch Tufts math" if you'd like to find them…)
- I'd like to thank the organizers of this Special Session on Cohomology, Representation Theory, and Lie Theory for the invitation to speak. Bummer we couldn't be in Mobile.
  - I'm going to talk about Lie theory. The questions considered are relevant for (some types of) representation theory. And cohomology is at least playing a back-story….

    Nevertheless, I realize that my talk is not exactly at the barycenter of the topics one might have expected in this session, so thanks for your patience!

2 Reductive groups and certain subgroups

- Let F be a field of characteristic $p \geq 0$, let G be a reductive group over F, and let μ_n be the group scheme of n-th roots of unity, for $n \geq 2$.
- Proposition: If $\phi : \mu_n \to G$ is a homomorphism, then the image of ϕ is contained in a maximal torus of G.
- when $p \mid n$, note that μ_n is not a smooth group scheme. When $n = p$, the image of ϕ amounts to $X \in \mathrm{Lie}(G)$ with $X^{[p]} = X$.
- There is a natural notion of equivalence for such homomorphisms; we call the equivalence classes "μ-homomorphisms" and denote them as $\phi : \mu \to G$.

3 μ-homomorphisms to a split torus

Proposition: If T is a split torus over F with co-character group $Y = X_*(T)$, there is a bijection $\overline{x} \mapsto \phi_{\overline{x}}$

$$Y \otimes \mathrm{Q}/\mathrm{Z} = V/Y \xrightarrow{\sim} \{\mu\text{-homomorphisms } \mu \to T\}$$

where $x \in V = Y \otimes \mathrm{Q}$.

4 sub-systems and sub-groups

- let $\phi : \mu \to G$ be a μ-homomorphism with image in a split torus T, corresponding to the class of $x \in Y \otimes \mathrm{Q} = V$ in V/Y.
- the centralizer $C^0_G(\phi)$ of the image of ϕ is a subsystem subgroup of G
- if G is split and T a maximal split torus, and if Φ denotes the roots of G in $X^*(T)$, the root system of $C^0_G(\phi)$ is given by $\Phi_x = \{\alpha \in \Phi \mid \langle \alpha, x\rangle \in \mathrm{Z}\}$.
- Φ_x is the root subsystem determined by the Borel-de Siebenthal procedure from the extended Dynkin diagram of G.
- we refer to the reductive subgroups of G that arises as connected centralizers of homomorphisms $\phi : \mu \to G$ as subgroups of type $C(\mu)$.

5 Local fields

- Let K be a local field, by which I mean the field of fractions of a complete DVR $\mathcal{A}$
- write $\mathrm{k} = \mathcal{A}/\pi\mathcal{A}$ for the residue field.
- e.g. $\mathcal{A}$ could be the completion of the ring of integers O_{L} of a number field L at some non-zero prime ideal $\mathfrak{p}$.

  Then $[\mathrm{K} : \mathrm{Q}_p] < \infty$ where $p\mathrm{Z} = \mathrm{Z} \cap \mathfrak{p}$.
- or $\mathcal{A}$ could be the completion of the local ring $\mathcal{O}_X$ where X is an (smooth, geometrically irreducible) algebraic curve over k.

  Then $\mathrm{K} \simeq \ell((t))$ where $[\ell : \mathrm{k}] < \infty$.
- we assume throughout that the char. of the residue field k is $p > 0$.

6 Reductive groups and splitting fields

- Let G be a connected and reductive group over the local field K.
- can always find a finite, separable extension $\mathrm{K} \subset \mathrm{L}$ such that G_{L} is split.
- Recall that for a finite separable extension $\mathrm{k} \subset \ell$ of the residue field, there is a unique extension – called an unramified extension – $\mathrm{K} \subset \mathrm{L}$ for which the "residue field of L" is ℓ and $[\mathrm{L} : \mathrm{K}] = [\ell : \mathrm{k}]$.
- We suppose that $(\diamondsuit) : G$ splits over an unramified extension of K – i.e. that the group G_{L} obtained via base-change is split for a suitable unramified extension $\mathrm{K} \subset \mathrm{L}$.

7 Unramified groups

- One says that $(\clubsuit) : G$ is an unramified group over K if there is a reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ for which $G = \mathcal{G}_{\mathrm{K}}$.
- Of course, if G is split over K, it is a fundamental fact – essentially, the existence theorem for a reductive group scheme over $\mathcal{A}$ corresponding to a given root datum – that there is a split reductive "Chevalley group scheme" $\mathcal{G}$ over $\mathcal{A}$ with $G = \mathcal{G}_{\mathrm{K}}$.
- Any unramified group splits over an unramified extension – i.e. $(\clubsuit) \implies (\diamondsuit)$ – but the converse is not true in general.

8 Parahoric group schemes

- The parahoric group schemes attached to G are certain affine, smooth group schemes $\mathcal{P}$ over $\mathcal{A}$ having generic fiber $\mathcal{P}_{\mathrm{K}} = G$.
- We just said that G is unramified over K if there is a reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ with $G = \mathcal{G}_{\mathrm{K}}$. Such a group scheme $\mathcal{G}$ is a parahoric group scheme.
- But in general, parahoric group schemes $\mathcal{P}$ are not reductive over $\mathcal{A}$, even for split G. In particular, the special fiber $\mathcal{P}_{\mathrm{k}}$ need not be a reductive group over the residue field k.

9 Levi factors of the special fiber of a parahoric

Suppose that G splits over an unramified extension of K, and let $\mathcal{P}$ a parahoric attached to G.

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Studied Levi decompositions of $\mathcal{P}_{\mathrm{k}}$ in (McNinch 2010), (McNinch 2014), (McNinch 2020).

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Theorem (McNinch 2020) There is a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ such that:

a. $\mathcal{M}_{\mathrm{K}}$ is a reductive subgroup of G of type $C(\mu)$, and
b. $\mathcal{M}_{\mathrm{k}}$ is a Levi factor of the special fiber $\mathcal{P}_{\mathrm{k}}$.

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Remarks:

- note that $R_u\mathcal{P}_{\mathrm{k}}$ is defined and split over k, even if k is imperfect. (Thus $\mathcal{P}_{\mathrm{k}}$ has a Levi decomposition over k).
- parahorics are determined up to $G(\mathrm{K})$-conjugacy by $x \in V = Y \otimes \mathrm{Q}$, and $\mathcal{M}_{\mathrm{K}}$ is the centralizer of $\phi_{\overline{x}}$. Here $Y = X_*(S)$ for a max'l split torus S in G.

10 Main result on nilpotent elements

- Let G be a reductive group over the local field K, and suppose that G splits over an unramified extension.
- Write p for the char. of the residue field k of K, and s'pose $p > 2h - 2$ where $h = h(G)$ is the Coxeter number of G (i.e. the sup of the Coxeter numbers of simple components of $G_{\overline{\mathrm{K}}}$.)
- Let $X \in \mathrm{Lie}(G)$ be a nilpotent element.

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Theorem: (McNinch 2021) There is a K-subgroup $M \subset G$ such that:

a. M is a reductive subgp of type $C(\mu)$ containing a maximal K-torus of G which is unramified.
b. M is an unramified reductive group over K
c. $X \in \mathrm{Lie}(M) \subset \mathrm{Lie}(G)$ and X is geometrically distinguished for M.

11 Primary tool

- let G be reductive over K, suppose that G splits over unramif. ext, and let $\mathcal{P}$ be a parahoric for G.
- Choose reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ as in earlier Theorem – thus $\mathcal{M}_{\mathrm{k}}$ is a Levi factor of $\mathcal{P}_{\mathrm{k}}$.
- Suppose that $p = \mathrm{char}(\mathrm{k}) > 2h - 2$ as before.

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Theorem: (McNinch 2021) Let $X_0 \in \operatorname{Lie}(\mathcal{P}_{\mathrm{k}}/R_u\mathcal{P}_{\mathrm{k}}) = \operatorname{Lie}(\mathcal{M}_{\mathrm{k}})$ be nilpotent.

a. there is a nilpotent section $\mathcal{X} \in \operatorname{Lie}(\mathcal{M})$ lifting X_0 which is balanced for $\mathcal{M}$ – i.e. $C_{\mathcal{M}_{\mathrm{k}}}(\mathcal{X}_{\mathrm{k}} = X_0)$ and $C_{\mathcal{M}_{\mathrm{K}}}(\mathcal{X}_{\mathrm{K}})$ are smooth of the same dimension.
b. Moreover, $\mathcal{X}$ is balanced for $\mathcal{P}$ – i.e. the centralizers $C_{\mathcal{P}_{\mathrm{k}}}(\mathcal{X}_{\mathrm{k}})$ and $C_{\mathcal{P}_{\mathrm{K}}}(\mathcal{X}_{\mathrm{K}})$ are smooth of the same dimension.

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- I view this as an alternative version of the lifting Theorem of (DeBacker 2002).

Remarks:

- The Main Theorem above is deduced from the Primary Tool in part via the observation that any nilpotent X may be placed in $\operatorname{Lie}(\mathcal{M}) \subset \operatorname{Lie}(\mathcal{P})$ for some parahoric $\mathcal{P}$.
- in order to control e.g. the dimensions of the centralizers of $\mathcal{X}_{\mathrm{k}}$ and $\mathcal{X}_{\mathrm{K}}$, we actually place $\mathcal{X}$ in the image of an $\mathcal{A}$-homomorphism $\mathrm{SL}_{2/\mathcal{A}} \to \mathcal{M}$ and use the representation theory of SL_2 (which is well-behaved since $p > 2h - 2$).

  The techniques used for this construction build on earlier work of McNinch (2005) on optimal SL_2-homomorphisms.

Bibliography

DeBacker, Stephen. 2002. "Parametrizing Nilpotent Orbits via Bruhat-Tits Theory." Annals of Mathematics. Second Series 156 (1): 295–332. https://doi.org/10.2307/3597191.

McNinch, George. 2005. "Optimal SL(2) Homomorphisms." Commentarii Mathematici Helvetici. A Journal of the Swiss Mathematical Society 80 (2): 391–426. https://doi.org/10.4171/CMH/19.

———. 2010. "Levi Decompositions of a Linear Algebraic Group." Transformation Groups 15 (4): 937–64. https://doi.org/10.1007/s00031-010-9111-8.

———. 2014. "Levi Factors of the Special Fiber of a Parahoric Group Scheme and Tame Ramification." Algebras and Representation Theory 17 (2): 469–79. https://doi.org/10.1007/s10468-013-9404-4.

———. 2020. "Reductive Subgroup Schemes of a Parahoric Group Scheme." Transformation Groups 25 (1): 217–49. https://doi.org/10.1007/s00031-018-9508-3.

———. 2021. "Nilpotent Elements and Reductive Subgroups over a Local Field." Algebras and Representation Theory 24: 1479–1522. https://doi.org/10.1007/s10468-020-10000-2.