Central subalgebras of the centralizer of a nilpotent element

George McNinch

Department of Mathematics
Tufts University

Special session in Athens GA - March 2016

Contents

Outline

Introduction

- This talk reports on a joint paper with Donna Testerman, EPFL, which will appear in Proc. AMS (already available online)
- Let G a "standard" reductive alg gp over the field k.
- Let $X \in \mathrm{Lie}(G)$ nilpotent.
- There is an "optimal" parabolic subgroup P with $X \in \mathrm{Lie}(R_u P)$; P has Levi factor $M = C_G(\mathrm{im}(\phi))$.

Main result

Suppose X is *even*. In that case, $\dim C_G(X) = \dim M$.

Theorem (M.-Testerman)

If X is even, $\dim Z(C_G(X)) \geq \dim Z(M)$. [Where $Z(-)$ means "the center of -"].

- In fact, Lawther-Testerman already proved that equality holds (for G semisimple). Their methods were "case-by-case".
- The argument I'll describe here is more direct.
- Reason for interest: let the unipotent u correspond to X via a Springer isomorphism. In char. $p > 0$, one has in general no well-behaved exponential map, but one might still hope to embed u in a "nice" abelian connected subgroup. $Z(C_G(X))^0 = Z(C_G(u))^0$ is a starting point.

Outline

Modules over a Dedekind domain

- Let A be a *Dedekind domain* – e.g. a *principal ideal domain*.
- For a maximal ideal $\mathfrak{m} \subset A$ and an A-module N, write $k(\mathfrak{m}) = A/\mathfrak{m}$, and $N(\mathfrak{m}) = N/\mathfrak{m}N = N \otimes_A k(\mathfrak{m})$,
- let K be the field of fractions of A and write $N_K = N \otimes_A K$.
- Let M be a fin. gen A-module. Then $M = M_0 \oplus M_{\text{tor}}$ where M_{tor} is torsion and M_0 is projective.

Homomorphisms (notation)

- Let $\phi : M \to N$ be an A-module homom where M and N are f.g. projective A-modules.
- let $P = \ker \phi$ and $Q = \operatorname{coker} \phi$.
- write $Q = Q_0 \oplus Q_{\text{tor}}$ as before.
- M/P is torsion free and thus projective, so for any max'l ideal $\mathfrak{m}$, we may view $P(\mathfrak{m})$ as a subspace of $M(\mathfrak{m})$.
- Write $\phi(\mathfrak{m}) : M(\mathfrak{m}) \to N(\mathfrak{m})$ for $\phi \otimes 1_{k(\mathfrak{m})}$.

Fibers of a kernel

Recall $\phi : M \to N$, $P = \ker\phi$, and $Q = \operatorname{coker}\phi$.

Theorem

(a) $P(\mathfrak{m}) \subset \ker\phi(\mathfrak{m})$, *with equality* $\iff$ $Q_{\text{tor}} \otimes k(\mathfrak{m}) = 0$.

(b) $P(\mathfrak{m}) = \ker\phi(\mathfrak{m})$ *for all but finitely many* $\mathfrak{m}$.

- Pf of (a) uses the following fact: for a finitely generated A-module M

$$(\clubsuit) \quad \operatorname{Tor}^1_A(M, k(\mathfrak{m})) \simeq M_{\text{tor}} \otimes k(\mathfrak{m})$$

.

- For (b), one just notes that Q_{tor} has *finite length*.
- If one knows that $\dim_{k(\mathfrak{m})} \ker\phi(\mathfrak{m})$ is equal to a constant d for all $\mathfrak{m}$ in some infinite set Γ of prime ideals, then $d = \dim_K \ker\phi(K)$.

Fibers of the center of an A-Lie algebra

- Let L be a Lie algebra over A which is f.g. projective as A-module.
- Let $Z = \{X \in L \mid [X, L] = 0\}$ be the center of L.

Theorem

(a) *L/Z is torsion free.*

(b) $\dim_{k(\mathfrak{m})} Z(\mathfrak{m})$ *is constant.*

(c) *For each maximal $\mathfrak{m} \subset A$, $Z(\mathfrak{m}) \subset \mathfrak{z}(L(\mathfrak{m}))$, and equality holds for all but finitely many $\mathfrak{m}$.*

- Here $\mathfrak{z}(L(\mathfrak{m}))$ means the center of the $k(\mathfrak{m})$-Lie algebra $L(\mathfrak{m})$.
- The result essentially follows from the result for kernels.

Center example

- Let $A = k[T]$ for alg. closed k, and identify maximal ideals of A with elements in k.
- let $L = Ae + Af$, with e and f an A-basis where $[e, f] = T \cdot f$.
- Now $Z(L) = 0$, and $\mathfrak{z}(L(t)) = 0$ for $t \neq 0$.
- But $L(0)$ is abelian, i.e $\mathfrak{z}(L(0)) = L(0)$.

Outline

Overview

Fibers of modules over a Dedekind domain

Application: the center of nilpotent centralizer

Nilpotent elements

Let G standard reductive group and $X \in \mathrm{Lie}(G)$ nilpotent.

- There are cocharacters $\phi : \mathbf{G}_m \to G$ associated to X defined using *Geom Inv Theory* which play the role of the semisimple element in an $\mathfrak{sl}(2)$-triple.
- any such ϕ determines the same parabolic P by the condition:

$$\mathrm{Lie}(P) = \sum_{i \geq 0} \mathrm{Lie}(G)(\phi; i)$$

.

- X is *even* if $\mathrm{Lie}(G)(\phi; i) \neq 0 \implies i \in 2\mathbf{Z}$.

Smoothness results

Keep the above assumptions: G standard, X nilpotent. Write $\mathfrak{g} = \mathrm{Lie}(G)$.

- $C_G(X)$ is a smooth group scheme.
- $Z(C_G(X))$ is a smooth group scheme [M.-Testerman '09].
- The centralizer $C_G(u) = C_G(X)$ has a Levi factor $B = C_G(u)^S$ where S is the image of the cochar ϕ.
- One knows that

$$\mathrm{Lie}(Z(C_G(u))) = \mathfrak{z}(\mathrm{Lie}(C_G(u))^{\mathrm{Ad}(B)} = \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(X)) \cap \mathfrak{g}^{\mathrm{Ad}(B)}$$

- In particular, to prove the main result, it is enough to argue that $\dim \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(X)) \cap \mathfrak{g}^{\mathrm{Ad}(B)} \geq \dim \mathfrak{z}(\mathrm{Lie}(M))$.
- (This reduction requires to know: the center of the standard reductive group M is smooth!)

Centralizers

Let ε be a Springer isomorphism, and $u = \varepsilon(X) \in G(k)$ unip. element.

- Let $A = k[\mathbf{G}_m] = k[T, T^{-1}]$; so A is Dedekind. Identify maximal ideals of A with points $t \in \mathbf{G}_m = k^\times$.
- Consider $\mathfrak{g} = \mathrm{Lie}(G)$ and $L = \mathfrak{g} \otimes_k A$.
- View the cocharacter ϕ as an element of $G(A)$, and consider $u \cdot \phi \in G(A)$. Via Ad, this element acts A-linearly on L.
- Form the Lie algebra $D = \ker(\mathrm{Ad}(u \cdot \phi) - 1_L) \subset L$.
- (in char. 0, one can instead take $A = k[T]$ and $D = \ker(\mathrm{ad}(X + T \cdot H))$

Proposition

Assume that X is even.

- $D(1) = \mathfrak{c}_{\mathfrak{g}}(u) = \mathfrak{c}_{\mathfrak{g}}(X)$.
- *For almost all $1 \neq t \in k^\times$, the algebra $D(t)$ identifies with $\mathfrak{c}_{\mathfrak{g}}(\phi(t)u)$ which in turn is conjugate to $\mathfrak{g}(\phi; 0) = \mathrm{Lie}(M)$.*

Center of the centralizer

Keep the notation from the previous slide.

- Write Z for the center of the A-Lie algebra D.
- And write $H = \mathfrak{g}^B \otimes A \subset L$.
- Ultimately, must argue that

  $$(Z \cap H)(1) \subset \mathfrak{z}(\mathfrak{c}_{\mathfrak{g}}(X)) \cap \mathfrak{g}^B$$

  while for almost all $t \neq 1$,

  $$(Z \cap H)(t) = Z(t) = \mathfrak{c}_{\mathfrak{g}}(\phi(t)u).$$

- This implies the "main result".