LEVI DECOMPOSITIONS OF LINEAR ALGEBRAIC GROUPS AND NON-ABELIAN COHOMOLOGY

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To the memory of Gary Seitz (1943-2023)

Contents

Abstract. Let k be a field, and let G be a linear algebraic group over k for which the unipotent radical U of G is defined and split over k. Consider a finite, separable field extension ℓ of k and suppose that the group G_ℓ obtained by base-change has a *Levi decomposition* (over ℓ). We continue here our study of the question previously investigated in (McNinch 2013): does G have a *Levi decomposition* (over k)?

Using non-abelian cohomology we give some condition under which this question has an affirmative answer. On the other hand, we provide (another) example of a group G as above which has no Levi decomposition over k.

1. Introduction

Let k be a field, and let G be a linear algebraic group over k. Thus G is a group scheme which is smooth, affine and of finite type over k.

If k_{alg} denotes an algebraic closure of k, the *unipotent radical* of $G_{k_{\mathrm{alg}}}$ is its maximal connected, solvable, normal subgroup. The unipotent radical of G is defined over k if G has a k-subgroup R such that $R_{k_{\mathrm{alg}}}$ is the unipotent radical of $G_{k_{\mathrm{alg}}}$.

Definition 1.1. We say that G *satisfies condition **(R)*** if the unipotent radical $U = R_u(G)$ of G is *defined and split* over k. (See Definition 2.1 for the notion of *split* unipotent group).

Remark 1.2. If k is *perfect*, Then the unipotent radical U is defined over k by *galois descent*. Moreover, every connected unipotent group over a perfect field k is k-split; see Remark 2.2. Thus condition **(R)** holds for every linear algebraic group over a perfect field k.

If G satisfies condition **(R)**, we write $\pi : G \to G/U$ for the quotient morphism, and we say that G/U is the reductive quotient of G.

Definition 1.3. Suppose that G satisfies condition **(R)**. G has a *Levi decomposition* (over k) if

Date: 2024-07-01 10:51:43 EDT (george@valhalla).

(a) the unipotent radical of G is defined over k, and
(b) there is a closed k-subgroup scheme M of G such that the restriction of the quotient mapping determines an isomorphism

$$\pi_{|M} : M \xrightarrow{\sim} G/R.$$

The subgroup M is then a *Levi factor* of G.

Remark 1.4. If G has a Levi decomposition with Levi factor M, the product mapping

$$((u, m) \mapsto u \cdot m) : U \times M \to G$$

is an isomorphism of k-schemes; moreover, this map identifies G with the *semidirect product* $U \rtimes M$ as algebraic groups; see Proposition 2.12 for a precise statement.

In this paper, we are interested in the situation in which the field k has characteristic $p > 0$. We suppose that ℓ is a *finite, separable* field extension of k, and we suppose that G_ℓ has a Levi decomposition. In particular, the unipotent radical of G_ℓ is defined over ℓ; i.e. there is an ℓ subgroup $R_\ell \subset G_\ell$ which determines the geometric unipotent radical.

Since the extension $k \subset \ell$ is *separable*, it follows by *galois descent* that the unipotent radical of G is defined over k; i.e. there is a k-subgroup $R \subset G$ such that R_ℓ is obtained from R by extension of scalars.

Since G_ℓ has a Levi decomposition, there is an ℓ-subgroup $M_\ell \subset G_\ell$ which is a Levi factor. This leads to the question:

($\diamondsuit$) Under what conditions is there a k-subgroup M of G which is a Levi factor of G?

This question about descent of Levi factors was already considered in the paper (McNinch 2013) whose main result gave the following partial answer to ($\diamondsuit$):

Theorem 1.5. *Assume that ℓ is a finite,* galois *field extension of k with $\Gamma = \mathrm{Gal}(\ell/k)$, and that the linear algebraic group G_ℓ over ℓ obtained from G by* base-change *has a* Levi decomposition.

Suppose $|\Gamma|$ is not divisible by p. Then G has a Levi decomposition; in particular, there is a k-subgroup $M \subset G$ which is a Levi factor.

In the present paper, we describe some tools of non-abelian cohomology in Section 3, Section 4, and Section 5 and use these tools in Section 6 to articulate some partial answers to ($\diamondsuit$) which remain valid in the case where ℓ is any finite, separable extension of k. The main results are Theorem 6.5, Corollary 6.6, and Theorem 6.8.

We note that *some* hypotheses are definitely required to answer the question ($\diamondsuit$); Section 7 provides an example of an algebraic group G satisfying condition **(R)** for which G_ℓ has a Levi factor for some cyclic galois extension ℓ of degree p over k, but G has no Levi factor over k.

We should point out that our results are proved under relatively strong hypotheses; for example, Theorem 6.5 is proved under the assumption that the non-abelian cohomology set $H^1_{\mathrm{coc}}(M_\ell, U_\ell)$ is trivial and that the group $U_\ell^{M_\ell}$ is trivial – this is very far from the general case! On the other hand, every example known to the author of a group G satisfying **(R)** such that G_ℓ has a Levi factor but G has no Levi factor has the property that G is *connected.*

This suggests the following natural problem for which a solution would be desirable:

Problem 1.6. Let ℓ a finite, separable field extension of k and G a *connected* linear algebra group over k satisfying **(R)**. Either find a proof of the assertion "G_ℓ has a Levi factor implies that G has a Levi factor" or find an example of a group for which this condition fails.

Some motivation. To explain the original reason for interest in the question, let $\mathscr{A}$ be a complete discrete valuation ring (DVR) with field of fractions K and residue field k. Write G for a connected, reductive group over K and let $\mathscr{P}$ be a parahoric group scheme associated to G; see (Bruhat and Tits 1984). Thus $\mathscr{P}$ is a smooth, affine group scheme over $\mathscr{A}$, and the generic fiber $\mathscr{P}_K$ identifies with G.

The *special fiber* $\mathscr{P}_k$ of $\mathscr{P}$ is a linear algebraic group over k which in general is *not reductive.* We have been interested for some time in the question:

(♣) does $\mathscr{P}_k$ have a Levi decomposition?

We first gave a partial answer to (♣) in (McNinch 2010), Theorem A, where we proved that $\mathscr{P}$ has a Levi decomposition in case G_L is split reductive for an unramified extension L of K.

We later showed in (McNinch 2014a) that $\mathscr{P}_{k_{\text{sep}}}$ has a Levi decomposition provided that G_L is split reductive for some *tamely ramified* extension L of K. This result provided the original motivation for our interest in descent of Levi factors carried out in (McNinch 2013).

In the case where G splits over an unramified extension of K, we ultimately we gave a more satisfying construction of a Levi factor of $\mathscr{P}_k$ in (McNinch 2020) than that given in (McNinch 2010), and this was later used in (McNinch 2021) to answer some questions about nilpotent $G(K)$-orbits. However, this new construction does not apply when G is not split over any unramified extension of K. So far as I am aware, the question (♣) does not have a complete answer in that case.

2. Preliminaries

We fix an arbitrary field k. Throughout the paper, G will denote a linear algebraic group over k. Thus G is a group scheme which is smooth, affine, and of finite type over k.

If V is a linear representation of G, then for $i \geq 0$, $H^i(G,V)$ denotes the ith *(Hochschild) cohomology* group of V (Jantzen 2003).

Unipotent groups. Recall from (Springer 1998) §14 the following:

Definition 2.1. a connected, unipotent linear algebraic group U over k is said to be k-split provided that there is a sequence

$$1 = U_0 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = U$$

of closed, connected, normal k-subgroups of U such that $U_{i+1}/U_i \simeq \mathbf{G}_{a/k}$ for $i = 0, \cdots, m-1$, where $\mathbf{G}_a = \mathbf{G}_{a/k}$ is the *additive group.*

Remark 2.2. When k is *imperfect*, there are connected unipotent k-groups which are not k-split; for an example, see e.g. (Serre 2002) III.§2.1 Exercise 3. On the other hand, if k is perfect, every connected unipotent k-group is k-split. (Springer 1998), 14.3.10.

Proposition 2.3. *Let U be a k-split unipotent group. If V is a normal k-subgroup of U, then U/V is again a k-split unipotent group.*

Proof. The assertion follows from (Springer 1998), Theorem 14.3.8. □

A substantial reason for our focus on split unipotent groups is the following result of Rosenlicht:

Proposition 2.4. *Suppose that U is a connected, k-split unipotent subgroup of G and write $\pi : G \to G/U$ for the quotient morphism. Then there is a morphism of k-varieties*

$$\sigma : G/U \to G$$

which is a section to π – i.e. $\pi \circ \sigma$ is the identity. In particular, the mapping $\pi : G(k) \to (G/U)(k)$ on k-points is surjective.

Proof. See (Springer 1998) Theorem 14.2.6. □

Let U be a connected unipotent k-group and suppose that G acts on U by automorphisms of k-groups. Such an action is given by a suitable morphism $G \times U \to U$, or equivalently by a morphism of k-group functors $G \to \operatorname{Aut}(U)$.

Definition 2.5. If U is a vector group, the action of G on U is said to be *linear* if there is a G-equivariant isomorphism of algebraic groups $U \simeq \operatorname{Lie}(U)$.

Definition 2.6. A filtration

$$1 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{m-1} \subset U_m = U$$

by G-invariant closed k-subgroups U_i is a *linear filtration* for the action of G if U_{i+1}/U_i is a vector group on which G acts linearly for each $i = 0, \cdots, m-1$.

It is a *central linear filtration* if the image of U_{i+1} in U/U_i is central for each $i \geq 0$.

Theorem 2.7. *Assume that G is* connected *and that the unipotent radical R of G is defined and split over k.*

(a) There is a linear filtration of U for the action of G.
(b) There is a central linear filtration of U for the action of G.

Proof. (a) is the main result of (McNinch 2014b).

To see (b), we may clearly refine a linear filtration to arrange that $\operatorname{Lie}(U_i)/\operatorname{Lie}(U_{i+1})$ is an *irreducible* representation of G for each i. We claim that this refined filtration is central. We proceed by induction on the length m of the linear filtration. If $m = 1$ then U is abelian and the result is immediate.

Suppose now that $m > 1$ and that one knows that any linear filtration of length $< m$ for which the factors of consecutive terms form irreducible M-representations is central.

Now, the fixed points for the conjugation action of U on U_1 form a G-invariant subgroup scheme. Since $U_1 \simeq \operatorname{Lie}(U_1)$ is an irreducible G-representation, it follows that U acts *trivially* on U_1; thus U_1 is central in U. Now, it is clear that

$$1 \subset U_2 \cdot U_1/U_1 \subset \cdots \subset U_m \cdot U_1/U_1 = U/U_1 \tag{2.1}$$

forms a linear filtration of U/U_1 for the action of G for which the factors of consecutive terms form irreducible M-representations. Thus by induction (2.1) is a central linear filtration; this completes the proof. □

Remark 2.8. In the proof of Theorem 2.7, we constructed a central linear filtration by arranging that the action of M on each quotient U_{i+1}/U_i is irreducible. This condition is sufficient, but not necessary – in general, there are central linear filtrations for which $\operatorname{Lie}(U_{i+1})/\operatorname{Lie}(U_i)$ is a reducible M-representation for some i.

Group functors. By a k-group functor, we mean a functor G from the category of commutative k-algebras to the category of groups. A morphism of k-group functors is a *natural transformation* between the functors.

Of course, a k-group scheme G determines a k-group functor: to a k-algebra Λ one assigns the group

$$G(\Lambda) = \operatorname{Mor}(\operatorname{Spec}\Lambda, G_\Lambda) \tag{2.2}$$

of Λ-points of G. A k-group functor is said to be *representable* if it is isomorphic to a k-group scheme.

An important feature of *representable* k-group schemes is this: if G and H are k-group schemes, then it follows from the Yoneda Lemma that any homomorphism of k-group functors $G \to H$ is automatically a homomorphism of k-group schemes. Similarly, if X is a k-scheme, then a mapping of k-functors $G \to X$ is automatically a mapping of k-schemes.

We shall need to consider a few k-group functors that do not arise from algebraic groups. Here is an important example. Let H be any k-group functor. We consider the k-group functor $\operatorname{Aut}(H)$ which is defined for each commutative k-algebra Λ by the rule

$$\operatorname{Aut}(H)(\Lambda) = \operatorname{Aut}(H(\Lambda)), \quad \text{the group of automorphisms of } H(\Lambda). \tag{2.3}$$

Similarly one has the k-group functor of *inner automorphisms*

$$\operatorname{Inn}(Y)(\Lambda) = \operatorname{Inn}(H(\Lambda)), \quad \text{the group of inner automorphisms in } \operatorname{Aut}(H(\Lambda))\ . \tag{2.4}$$

Inclusion determines a homomorphism $\operatorname{Inn}(H) \to \operatorname{Aut}(H)$. If H is an algebraic group [1], then $\operatorname{Inn}(H)$ is a representable group scheme, isomorphic to the functor determined by the group scheme H/Z, where Z denotes the scheme theoretic center of H.

Now, the k-group functor $\operatorname{Out}(H)$ is defined for each Λ by the rule

$$\operatorname{Out}(H)(\Lambda) = \operatorname{Aut}(H)(\Lambda)/\operatorname{Inn}(H)(\Lambda)).$$

The quotient mappings $\operatorname{Aut}(H(\Lambda)) \to \operatorname{Aut}(H(\Lambda))/\operatorname{Inn}(H(\Lambda))$ determine a homomorphism of k-group functors

$$\Psi : \operatorname{Aut}(H) \to \operatorname{Out}(H) \tag{2.5}$$

and for each commutative k-algebra Λ we have a short exact sequence

$$1 \to \operatorname{Inn}(H)(\Lambda) \to \operatorname{Aut}(H)(\Lambda) \xrightarrow{\Psi(\Lambda)} \operatorname{Out}(H)(\Lambda) \to 1.$$

Definition 2.9. Let H and G be k-group functors. The H is a G-group provided that there is a morphism of k-group functors $G \to \operatorname{Aut}(H)$.

Definition 2.10. If H is a G-group, the *fixed-point sub-functor* H^G is defined by the functor

$$H^G(\Lambda) = \{h \in H(\Lambda) \mid \text{for all commutative } \Lambda\text{-algebra } \Lambda' \text{ and for all } x \in G(\Lambda')\}$$

Proposition 2.11. *If H and G are group schemes over k, then H^G is a closed subgroup scheme of H*

Proof. (Jantzen 2003) I.2.6. □

Suppose H is a G-group via the homomorphism of k-group functors

$$\alpha : G \to \operatorname{Aut}(H).$$

Then we can form the *semi-direct product* $H \rtimes_\alpha G$. If H and G are algebraic k-groups, then $H \rtimes_\alpha G$ is an algebraic k-group.

We record the following:

Proposition 2.12. *Let G be a linear algebra k-group satisfying condition **(R)** and write*

$$\pi : G \to M = G/U$$

for the quotient morphism, where U is the unipotent radical. Suppose that $s : M \to G$ is a group homomorphism that is a section to π. Then

$$(m \mapsto \operatorname{Inn} s(m)) : M \to \operatorname{Aut}(U)$$

[1] More precisely, if the group functor H is *represented by* a linear algebraic group.

determines a homomorphism of k-group functors, and the multiplication mapping

$$((u,m) \mapsto um) : U \times M \to G$$

induces an isomorphism

$$U \rtimes_\phi M \xrightarrow{\sim} G$$

of algebraic k-groups.

Galois cohomology. Write Γ for the absolute galois group of k: $\Gamma = \mathrm{Gal}(k_{\mathrm{sep}}/k), G(k_{\mathrm{sep}}))$

Let G be a k-group functor satisfying conditions (1), (2), and (3) of (Serre 2002) II.1.1. Then Γ acts continuously on the group $G(k_{\mathrm{sep}})$ and we may consider the galois cohomology set $H^1(k,G) := H^1(\Gamma, G(k_{\mathrm{sep}}))$ (Serre 2002), §5.1.

Proposition 2.13. *Let G be a k-group functor. The conditions of (Serre 2002) II.1.1 hold for G in the following cases:*

(a) G is represented by a linear algebraic group
(b) there is a k-vector space V such that G is isomorphic to the k-group functor determined by $\Lambda \mapsto V(\Lambda) = V \otimes_k \Lambda$.

Proof. (a) is (Serre 2002) II.1.1 Remark (2).

For (b), one can argue as in (McNinch 2013), §4.1. □

Proposition 2.14. *Let V be a vector space over k, viewed as a k-group functor. Then $H^1(k,V) = 1$*

Proof. This is essentially "additive Hilbert 90"; see for example (McNinch 2013). □

Proposition 2.15. *Let U be a connected, split unipotent algebraic group over k. Then the galois cohomology set satisfies $H^1(k,U) = 1$.*

Proof. The necessary tools are recalled in (McNinch 2004) Prop. 30. □

3. Non-abelian cohomology

Let A and G be linear algebraic k-groups and suppose that A is a G-group. Following (Demarche 2015) §2.1, we introduce the cohomology set $H^1_{\mathrm{coc}}(G,A)$ as follows. Let $Z^1_{\mathrm{coc}}(G,A)$ denote the set of regular maps $f : G \to A$ such that for each commutative k-algebra Λ and each $x, y \in G(\Lambda)$, the 1-cocycle condition

$$f(xy) = f(x) \cdot {}^x f(y) \tag{3.1}$$

holds. Two cocycles $f, f' \in Z^1_{\mathrm{coc}}(G,A)$ are *cohomologous* provided there is $u \in U(k)$ such that for each Λ and each $x \in G(\Lambda)$ we have

$$f(x) = u^{-1} \cdot f'(x) \cdot {}^x u.$$

This defines an equivalence relation on $Z^1_{\mathrm{coc}}(G,A)$ and we write $H^1_{\mathrm{coc}}(G,A)$ for the quotient set.

If B is a second G-group and if $\phi : A \to B$ is a morphism of k-group functors that is G-invariant, then ϕ induces a mapping

$$\phi_* : Z^1_{\mathrm{coc}}(G,A) \to Z^1_{\mathrm{coc}}(G,B)$$

by the rule $f \mapsto \phi \circ f$. This mapping respects the equivalence relation and thus induces a mapping

$$\phi_* : H^1_{\mathrm{coc}}(G,A) \to H^1_{\mathrm{coc}}(G,B).$$

One interpretation or application of this cohomology set arises from examination of the semidirect product $A \rtimes G$, or – see Remark 1.4 – to groups G together with a group homomorphism $s_0 : M \to G$ which is a section to the quotient mapping $\pi : G \to M$.

Definition 3.1. Consider the set of all homomorphisms of k-groups $M \to G$ which are sections to π; two such homomorphisms s, s' will be considered *equivalent* if there is $u \in U(k)$ such that $s = us'u^{-1}$. Then $\mathrm{Sect}(G \xrightarrow{\pi} M)$ denotes the quotient of the set of all such homomorphisms by this equivalence relation.

Proposition 3.2. *Write $\mu : G \times G \to G$ for the multiplication mapping. For a given homomorphism $s_0 : M \to G$ which is a section to π, the assignment*

$$f \mapsto \mu \circ (f, s_0)$$

where $(f, s_0) : M \to U \times G$ is the mapping $m \mapsto (f(m), s_0(m))$ determines a bijection

$$A_{s_0} : H^1_{\mathrm{coc}}(M, U) \to \mathrm{Sect}(G \xrightarrow{\pi} M).$$

Proof. The choice of s_0 determines an isomorphism of ℓ-groups $G_\ell \simeq U_\ell \rtimes M_\ell$; see Remark 1.4. Now the result follows from (Demarche 2015) Prop. 2.2.2. □

Remark 3.3. $H^1_{\mathrm{coc}}(G, A)$ is a *pointed set* – i.e. a set with a distinguished element. That distinguished element is the class of the trivial mapping $(x \mapsto 1) : G \to A$. In the bijection of Proposition 3.2 the section corresponding to the trivial class is s_0.

Remark 3.4. When Z is a *vector group* with a *linear* action of G, $H^1_{\mathrm{coc}}(G, Z)$ coincides with the usual Hochschild cohomology group $H^1(G, Z) \simeq H^1(G, \mathrm{Lie}(Z))$. In particular, in that case $H^1_{\mathrm{coc}}(G, Z)$ is a k-vector space.

Suppose now that $A = U$ is a split unipotent G-group and that $Z \subset U$ is a central k-subgroup that is G-invariant. Then U/Z is a split unipotent G-group, and there is a mapping

$$\Delta : H^1_{\mathrm{coc}}(G, U/Z) \to H^2(G, Z) \tag{3.2}$$

– where $H^2(G, Z)$ denotes the second Hochschild cohomology – defined as follows. First, use Rosenlicht's result Proposition 2.4 to choose a regular mapping $s : U/Z \to U$ which is a section to the quotient homomorphism $U \to U/Z$. Now for $\alpha = [f]$ with $f \in Z^1_{\mathrm{coc}}(G, A/Z)$, let $F = s \circ f$ so that $F : G \to U$ is a regular mapping.

For a commutative k-algebra Λ and for $g, h \in G(\Lambda)$, observe that

$$s(g)s(h)s(gh)^{-1} \in Z(\Lambda).$$

As in (Demazure and Gabriel 1970) II, Subsect. 3.2.3 – see also (McNinch 2010), §4.4 – the rule $(g, h) \mapsto s(g)s(h)s(gh)^{-1}$ determines a Hochschild 2-cocycle whose class in $H^2(G, Z)$ we denote $\Delta(\alpha)$.

Proposition 3.5. *Let U be a split unipotent G-group, and let Z be a central k-subgroup of U that is G-invariant. Write $i : Z \to U$ and $\pi : U \to U/Z$ for the inclusion and quotient mappings, respectively. Both Z and U/Z are G-groups.*

(a) the sequence of pointed sets

$$H^1(G, Z) \xrightarrow{i_*} H^1_{\mathrm{coc}}(G, U) \xrightarrow{\pi_*} H^1_{\mathrm{coc}}(G, U/Z) \xrightarrow{\Delta} H^2(G, Z)$$

is exact.

(b) If $(U/Z)^G = 1$ then i_* is injective.

Sketch. (a) The proof of the corresponding statement for cohomology of pro-finite groups given in (Serre 2002) I. §5.7 may be applied here *mutatis mutandum.* The main required adaptation is the definition (given above) of the mapping Δ (which required the existence of a regular section $U/Z \to U$).

For (b), suppose that $f_1, f_2 : G \to Z$ are cocycles and that $i_*([f_1]) = i_*([f_2])$. Thus f_1, f_2 are cohomologous in $Z^1_{\text{coc}}(G, U)$, so there is $u \in U(k)$ such that

$$f_1(x) = u^{-1} \cdot f_2(x) \cdot xu$$

for every commutative k-algebra Λ and every $x \in G(\Lambda)$. Passing to the quotient U/Z we see that $1 = u^{-1}xu$ so that the class of u lies in $(U/Z)^M(\Lambda)$. ☐

Remark 3.6. Assume that ℓ is a finite, galois extension of k with galois group $\Gamma = \text{Gal}(\ell/k)$. Then Γ acts on the galois cohomology $H^1(G_\ell, A_\ell)$ through its action on regular mappings $G_\ell \to A_\ell$.

If A is a vector group on which G acts linearly, then $H^1(G_\ell, A_\ell)$ identifies with $H^1(G, A) \otimes_k \ell$. In particular, in that case $H^1(G, A)$ may be identified with $H^1(G_\ell, A_\ell)^\Gamma = 1$.

This observation prompts several questions:

(a) Under what conditions is it true that $H^1_{\text{coc}}(G, U) = H^1_{\text{coc}}(G_\ell, U_\ell)^\Gamma$?
(b) Under what conditions is it true that the condition $H^1_{\text{coc}}(G, U) = 1$ is equivalent to the condition $H^1_{\text{coc}}(G_\ell, U_\ell) = 1$?

4. Extensions of algebraic groups

Let A and B be linear algebraic k-groups. An extension E of A by B is a linear algebraic k-group E together with a sequence

$$1 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 1. \tag{4.1}$$

where i and π are morphisms of algebraic groups over k, i determines an isomorphism of B onto $\ker \pi$, and the homomorphism π is faithfully flat.

Definition 4.1. Let E and E' be extensions of A by B. A *morphism of extensions* is a morphism $\phi : E \to E'$ of algebraic groups for which the diagram

$$\begin{tikzcd} 1 \arrow[r] & B \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & B \arrow[r, "i'"] & E' \arrow[r, "\pi"] & A \arrow[r] & 1 \end{tikzcd}$$

is commutative.

Observe that ϕ determines an isomorphism of algebraic groups $E \xrightarrow{\sim} E'$.

Let E be an extension of A by B. We write $\text{Autext}(E)$ for the group of automorphisms of E. Let Z be the (schematic) center of B. Since Z is characteristic in B, E acts on Z by conjugation. Since R acts trivially on Z, the action of E on Z factors through $A \simeq E/B$.

Let us write $Z^1(A, Z)$ for the group of *Hochschild 1-cocycles* as in (Demazure and Gabriel 1970), II. §3.1. Thus the elements of $Z^1(A, Z)$ are regular maps $\phi : A \to Z$ with the property that for each commutative k-algebra Λ and elements $g, h \in M(\Lambda)$ we have $\phi(gh) = \phi(g) \cdot g\phi(h)$.

The next result is a consequence of (Florence and Arteche 2020), Prop. 2.3. [2]

[2]In (Florence and Arteche 2020), M and R are *any* linear algebraic groups – i.e. the group schemes M and R are *smooth* but not necessarily *affine* over k. But we don't have an application for this generality so we only quote the linear (i.e. affine case) here.

Proposition 4.2. *There is a canonical group isomorphism*

$$Z^1(A,Z) \to \mathrm{Autext}(E) \tag{4.2}$$

given by $\phi \mapsto \mu \circ (\phi \circ \pi, \mathrm{id})$ *where* $\mu : E \times E \to E$ *is the multiplication on* E *and where* $\mathrm{Aut}(E)$ *denotes the group of automorphisms of the extension* (4.1).

Suppose that

$$1 \to B \to E \xrightarrow{\pi} A \to 1 \quad \text{and} \quad 1 \to B \to E' \xrightarrow{\pi'} A \to 1 \tag{4.3}$$

are two extensions of M by R, let ℓ be a finite, separable field extension of k, and suppose that the extensions

$$1 \to B_\ell \to E_\ell \xrightarrow{\pi} A_\ell \to 1 \quad \text{and} \quad 1 \to B_\ell \to E'_\ell \xrightarrow{\pi'} A_\ell \to 1$$

obtained by base-change are isomorphic (as extensions).

Theorem 4.3. *Assume that the center* Z *of* R *is a* vector group *and that the action of* M *on* Z *is* linear. *Then the extensions* (4.3) *are isomorphic over* k.

Proof. Write k_{sep} for a separable closure of k containing ℓ, and fix one of the extensions

$$(\clubsuit) \quad 1 \to B \to E \xrightarrow{\pi} A \to 1$$

from (4.3).

As in (Serre 2002), III.§1, one knows that there is a bijection

$$\mathscr{E}(k_{\mathrm{sep}}/k) \xrightarrow{\sim} H^1(k, \mathrm{Aut}(E)) := H^1(\mathrm{Gal}(k_{\mathrm{sep}}/k), \mathrm{Aut}(E)(k_{\mathrm{sep}})). \tag{4.4}$$

Here we write $\mathscr{E}(k_{\mathrm{sep}}/k)$ for the set of isomorphism classes of extensions of A by B over k which become isomorphic to the extension

$$(\clubsuit)_{k_{\mathrm{sep}}} \quad 1 \to B_{k_{\mathrm{sep}}} \to E_{k_{\mathrm{sep}}} \xrightarrow{\pi} A_{k_{\mathrm{sep}}} \to 1$$

after scalar extension to k_{sep}. For a field extension of k, we write $\mathrm{Autext}(E)(\ell) = \mathrm{Autext}(E_\ell)$ for the group of automorphisms of the extension $(\clubsuit)_\ell$. Finally, we write $H^1(\mathrm{Gal}(k_{\mathrm{sep}}/k), \mathrm{Aut}(E)(k_{\mathrm{sep}}))$ for the (non-abelian) Galois cohomology set for the group $\mathrm{Aut}(E)(k_{\mathrm{sep}})$.

Thus, the Theorem will follow if we argue that the Galois cohomology set

$$H^1(k, \mathrm{Aut}(E)) := H^1(\mathrm{Gal}(k_{\mathrm{sep}}/k), \mathrm{Aut}(E)(k_{\mathrm{sep}}))$$

is *trivial* – i.e. contains a unique element.

According to Proposition 4.2, for any field extension ℓ of k, the group of k_{sep}-automorphisms of the extension $(\clubsuit)_{k_{\mathrm{sep}}}$ is given by $\mathrm{Aut}(E_\ell) \simeq Z^1(A_\ell, Z_\ell)$. Now, by our assumption Z is a *vector group* with linear action of A; thus $Z^1(A_\ell, Z_\ell)$ is itself an ℓ-vector space, and

$$Z^1(A_\ell, Z_\ell) \simeq Z^1(A, Z) \otimes_k \ell.$$

It now follows from Proposition 2.14 that

$$H^1(k, \mathrm{Aut}(E)) \simeq H^1(k, Z^1(A, Z))$$

is trivial. ∎

5. Automorphisms and cohomology

Let A and G be a linear algebraic k-groups, and suppose that A is a G-group via the mapping

$$\phi : G \to \mathrm{Aut}(A).$$

Let Z denote the center of A as a group scheme. Then $\mathrm{Inn}(A) \simeq A/Z$ is an G-group via ϕ: for $h \in \mathrm{Inn}(A)(\Lambda)$ and $g \in G(\Lambda)$, we have [3] ${}^g h = \phi(g)a\phi(g)^{-1}$.

Denote by ϕ_0 the homomorphism of group functors

$$G \to \mathrm{Aut}(A) \xrightarrow{\Psi} \mathrm{Out}(A)$$

where $\Psi : \mathrm{Aut}(A) \to \mathrm{Out}(A)$ is the natural map of (2.5).

For two homomorphisms of group functors $\theta_1, \theta_2 : G \to \mathrm{Aut}(A)$ for which $\Psi \circ \theta_1 = \Psi \circ \theta_2 = \phi_0$ we say that θ_1 and θ_2 are equivalent if there is $a \in A(k)$ for which

$$\theta_1(g) = a^{-1}\theta_2(g)ga$$

and we write $\mathrm{Lift}(\phi_0)$ for the quotient of the set of all homomorphism $\theta : G \to \mathrm{Aut}(A)$ for which $\Psi \circ \theta = \phi_0$ by the equivalence relation just described.

Proposition 5.1. *For a 1-cocycle $f \in Z^1_{\mathrm{coc}}(G, \mathrm{Inn}(A))$, write Φ_f for the mapping*

$$(g \mapsto \Phi_f(g)) = f(g) \cdot \phi(g)) : G \to \mathrm{Aut}(A).$$

The assignment $f \mapsto \Phi_f$ determines a bijection

$$H^1_{\mathrm{coc}}(G, \mathrm{Inn}(A)) \to \mathrm{Lift}(\phi_0) \tag{5.1}$$

Proof. Suppose $\theta \in \mathrm{Lift}(\phi_0)$. Observe for each commutative k-algebra Λ and each $g \in G(\Lambda)$ that $\theta(g)\phi(g)^{-1} \in \mathrm{Inn}(A)(\Lambda)$. Thus we have a morphism of k-functors

$$g \mapsto \theta(g)\phi(g)^{-1} : G \to \mathrm{Inn}(A).$$

and a calculation confirms this assignment to be a 1-cocycle for the action of G on $\mathrm{Inn}(A)$ determined by ϕ.

On the other hand, for any 1-cocycle $f \in Z^1_{\mathrm{coc}}(G, M)$, the mapping $g \mapsto f(g) \cdot \phi(g)$ determines a morphism of group functors $G \to \mathrm{Aut}(A)$.

If $f_1 \in Z^1_{\mathrm{coc}}(G, \mathrm{Inn}(A)$ is cohomologous to f, then

$$f_1(g) = h^{-1}f(g)\,{}^g h$$

for some $h \in \mathrm{Inn}(U)(k)$.

We now observe that Φ_{f_1} is given by the rule

$$g \mapsto h^{-1}f(g)\,{}^g h \cdot \phi(g) = h^{-1}f(g)\phi(g)h\phi(g)^{-1}\phi(g) = h^{-1}f(g)\phi(g)h$$

so that $\Phi_{f_1}(g) = h^{-1}\Phi_f(g)h$. Thus Φ_f and Φ_{f_1} are *equivalent* elements of $\mathrm{Lift}(\phi_0)$.

It is straightforward to see that the assignment $f \mapsto (m \mapsto f(m) \circ \phi)$ respects the equivalence relations (on cocycles, and on liftings of ϕ_0).

This shows that the assignment $f \mapsto \Phi_f$ determines a well-defined and *surjective* mapping $H^1_{\mathrm{coc}}(G, \mathrm{Inn}(A)) \to \mathrm{Lift}(\phi_0)$. If $f_1, f_2 \in Z^1_{\mathrm{coc}}(G, A)$ and if Φ_{f_1} is equivalent to Φ_{f_2} a similar calculation as the one just given shows f_1 and f_2 to be *cohomologous*. Thus the assignment is indeed bijective, as required. $\square$

[3] The action of $g \in G(\Lambda)$ on $h \in \mathrm{Inn}(A)(\Lambda)$ is chosen so that for $a \in A(\Lambda)$ we have ${}^g h(\phi(g)a) = \phi(g)h(a)$.

Let

$$1 \to U \to E \xrightarrow{\pi} G \to 1$$

be an extension of algebraic k-groups where U is connected and k-split unipotent.

Write $\mathrm{Inn}_U : E \to \mathrm{Aut}(U)$ for the homomorhism of k-group functors given for each commutative k-algebra Λ and each $x \in E(\Lambda)$ by $\mathrm{Inn}_U(x) : u \mapsto x^{-1}ux$.

Proposition 5.2. *There is a unique homomorphism of k-group functors $\phi_0 : G \to \mathrm{Out}(U)$ such that $\phi_0 \circ \pi = \Psi \circ \mathrm{Inn}_U$ where $\Psi : \mathrm{Aut}(U) \to \mathrm{Out}(U)$ is the natural mapping (see (2.5)).*

Proof. Uniqueness is clear since π is the quotient mapping. By assumption, U is a connected, split unipotent k-group. By Rosenlicht's result – Proposition 2.4 – we may choose a regular mapping $s : G \to E$ which is a *section* to π.

For a commutative k-algebra Λ and $g \in G$ we define $\phi_0(g)$ to be the image of $\mathrm{Inn}_U(s(g))$ in $\mathrm{Out}(U)(\Lambda) = \mathrm{Aut}(U(\Lambda))/\mathrm{Inn}(U(\Lambda))$.

It is clear that ϕ_0 defines a natural transformation between the k-functors G and $\mathrm{Out}(U)$; it only remains to observe that $\phi_0 : G(\Lambda) \to \mathrm{Out}(U)(\Lambda)$ is a group homomorphism for each commutative k-algebra Λ.

A straightforward calcuation shows for $g_1, g_2 \in G(\Lambda)$ that

$$\phi_0(g_1)\phi_0(g_2) = \mathrm{Inn}_U(s(g_1)s(g_2)s(g_1g_2)^{-1}) \cdot \phi_0(g_1g_2) = \phi_0(g_1g_2)$$

since $s(g_1)s(g_2)s(g_1g_2)^{-1} \in U(\Lambda)$. This confirms that ϕ_0 is a homomorphism of k-group functors, as required. □

6. Descent of Levi factors

Let G be a linear algebraic group satisfying condition **(R)** of Definition 1.1. Writing U for the unpotent radical of G and $M = G/U$ for the reductive quotient, we view G as an extension of M by U:

$$1 \to U \to G \xrightarrow{\pi} M \to 1. \tag{6.1}$$

In this section, we are going to investigate some conditions for *descent of Levi factors*. We first formulate some assumptions and notation.

We are going to *assume throughout this section that U has a central linear filtration*

$$1 = Z_0 \subset Z_1 \subset \cdots Z_m = U$$

for the action of G; see Definition 2.6. Note that such a filtration always exists in case G is *connected*; see Theorem 2.7.

Proposition 6.1. *For each $n \geq 0$ the homomorphism of k-group group functors*

$$\phi_0 : M \to \mathrm{Out}(U)$$

of Proposition 5.2 determines an action *of M on the quotient Z_{n+1}/Z_n.*

Proof. Note that since Z_{n+1}/Z_n is abelian, $\mathrm{Out}(Z_{n+1}/Z_n) = \mathrm{Aut}(Z_{n+1}/Z_n)$. Now, the terms Z_n are G-invariant in U. Thus for each natural number n, ϕ_0 determines by restriction and passage to the quotient a homomorphism of k-group functors

$$\phi_{0|Z_{n+1}} : M \to \mathrm{Out}(Z_{n+1}/Z_n) = \mathrm{Aut}(Z_{n+1}/Z_n),$$

i.e. an action of M on Z_{n+1}/Z_n. □

Lemma 6.2. *Suppose that* $H^1(M, Z_{i+1}/Z_i) = 0$ *for each* $i = 0, \cdots, m-1$. *Then*

$$H^1_{\mathrm{coc}}(M, U) = 1 \quad \text{and} \quad H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1.$$

Proof. First observe that for a linear representation V of G, $H^1(G, V) = 0$ if and only if $H^1(G_\ell, V_\ell) = 0$. Now the result follows from Proposition 3.5. ☐

Remark 6.3. Viewing a finite dimensional linear representation V of M as an algebraic group, the fixed-point subgroup V^M of Definition 2.10 coincides with the vector group given by the M-fixed points on the linear representation V. In particular, if V is an *irreducible* representation of M, the group scheme V^M is trivial.

Lemma 6.4. *Suppose that* $(Z_{i+1}/Z_i)^M$ *is trivial for each* $i = 0, \cdots, m-1$. *Then* U^M *is the trivial group scheme.*

Proof. We proceed by induction on m, the length of the central linear filtration of U. If $m = 0$, $U = 1$ is itself trivial and the result is immediate.

Now suppose that $m > 0$ and that the result is known for connected and split unipotent M-groups having a central linear filtration of length $< m$. Thus by induction we know $(U/Z_1)^M$ to be trivial.

Now, U^M is contained in the kernel of the quotient mapping $U \to U/Z_1$, i.e. U^M is contained in Z_1. But then U^M is contained in $(Z_1)^M$ which is trivial; this completes the proof. ☐

Fix a finite separable field extension ℓ of k. We are going to suppose that G_ℓ has a Levi decomposition (over ℓ). Recall Definition 1.3; this means that there is an homomorphism of ℓ-groups $s_0 : M_\ell \to G_\ell$ which is a section to the quotient mapping $\pi : G_\ell \to M_\ell$.

Thus s_0 determines a class in $\mathrm{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$. The group G has a Levi decomposition over k if and only if there is a homomorphism $M \to G$ which is a section to π.

Clearly G has a Levi decomposition over any field extension of ℓ, so to establish the existence of a Levi decomposition over k, we may evidently pass to a separable extension field of ℓ and thus suppose that ℓ is finite and *galois* over k; write $\Gamma = \mathrm{Gal}(\ell/k)$ for the galois group.

Now, the action of Γ on ℓ-morphisms between k-schemes determines a Γ action on the set of homomorphisms $M_\ell \to G_\ell$ which are sections to π.

Theorem 6.5. *Suppose the following:*

(a) *G_ℓ has a Levi decomposition (over ℓ),*
(b) *the group scheme $U_\ell^{M_\ell}$ is trivial, and*
(c) *$H^1_{\mathrm{coc}}(M_\ell, U_\ell) = 1$.*

Then G has a Levi decomposition (over k).

Proof. By (a), we may choose a homomorphism $s : M_\ell \to G_\ell$ which is a section to π. By hypothesis (c) together with Proposition 3.2, the set $\mathrm{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$ contains a single element.

By the definition of $\mathrm{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$ it follows that every homomorphism $u : M_\ell \to G_\ell$ which is a section to π differs from s by conjugation with an element of $U(\ell)$.

For each $\gamma \in \Gamma$, we thus find an element $u_\gamma \in U(\ell)$ such that ${}^\gamma s = u_\gamma^{-1} \cdot s \cdot u_\gamma$.

We now contend that (♣): u_γ is a 1-cocycle on Γ with values in $U(\ell)$. Well, for $\gamma, \tau \in \Gamma$ we see that

$$ {}^{\gamma\tau}s = u_{\gamma\tau}^{-1} \cdot s \cdot u_{\gamma\tau} \tag{6.2}$$

while on the other hand

$$\begin{aligned} {}^{\gamma\tau}s &= {}^{\gamma}(u_\tau^{-1} \cdot s \cdot u_\tau) \\ &= {}^{\gamma}u_\tau^{-1} \cdot {}^{\gamma}s \cdot {}^{\gamma}u_\tau \\ &= {}^{\gamma}u_\tau^{-1} \cdot u_\gamma^{-1} \cdot s \cdot u_\gamma \cdot {}^{\gamma}u_\tau \end{aligned} \tag{6.3}$$

Now, condition (b) guarantees that $U_\ell^{M_\ell}$ is trivial, and it follows that the stabilizer in U_ℓ of the section s is trivial. Thus together (6.2) and (6.3) imply that

$$u_{\gamma\tau} = u_\gamma \cdot {}^{\gamma}u_\tau.$$

This confirms (♣). Since U is a split unipotent k-group, $H^1(k, U) = 1$; see Proposition 2.15. Thus there is $u \in U(\ell)$ such that

$$u_\gamma = u^{-1} \cdot {}^{\gamma}u \tag{6.4}$$

for each $\gamma \in \Gamma$; i.e. ${}^{\gamma}u = uu_\gamma$.

Now set $s_0 = u \cdot s \cdot u^{-1} \in \operatorname{Sect}(G_\ell \xrightarrow{\pi} M_\ell)$. We claim that s_0 is a k-homomorphism. It is enough to argue that s is fixed by the galois group Γ. For $\gamma \in \Gamma$ we note that

$$\begin{aligned} {}^{\gamma}s_0 &= {}^{\gamma}u \cdot s \cdot u^{-1} \\ &= {}^{\gamma}u \cdot {}^{\gamma}s \cdot {}^{\gamma}u^{-1} \\ &= u \cdot u_\gamma \cdot u_\gamma^{-1} \cdot s \cdot u_\gamma \cdot u_\gamma^{-1} \cdot u \\ &= usu^{-1} = s_0. \end{aligned}$$

Thus $s_0 : M \to G$ is a k-morphism which is a section to π; this shows that G has a Levi factor (over k) as required. □

Corollary 6.6. *Suppose the following:*

(a) G_ℓ has a Levi decomposition (over ℓ),
(bb) the group scheme $(Z_{i+1}/Z_i)^M$ is trivial for $i = 0, \cdots, m-1$, and
(cc) $H^1(M, Z_{i+1}/Z_i) = 0$ for $i = 0, \cdots, m-1$.

Then G has a Levi decomposition (over k).

Proof. Note that according to Lemma 6.4, condition (bb) implies hypothesis (b) of Theorem 6.5. Similarly, according to Lemma 6.2 (cc) implies hypothesis (c) of Theorem 6.5. Thus the result follows from Theorem 6.5. □

We now consider attempts to *descend* the action $M_\ell \to \operatorname{Aut}(U_\ell)$ determined by a choice of a Levi factor over ℓ. Following Proposition 5.1, this question involves the galois cohomology of $\operatorname{Inn}(U)$ rather than U.

We first prove the following:

Lemma 6.7. *Let A, B be linear algebraic groups, and suppose that B is an A-group via the homomorphism $\phi : A \to \operatorname{Aut}(B)$ of k-group functors. Let $b \in B(k)$ and consider the mapping $\phi_1 : A \to \operatorname{Aut}(B)$ given for each commutative k-algebra Λ and each $a \in A(\Lambda)$ by the rule $\phi_1(a) = \operatorname{Inn}(b)\phi_1(a)\operatorname{Inn}(b)^{-1}$. Then there is a k-isomorphism of extensions of A by B:*

$$B \rtimes_\phi A \simeq B \rtimes_{\phi_1} A.$$

Proof. Write $G = B \rtimes_\phi A$ for the semidirect product constructed using the action defined by ϕ. Now, the mapping $\phi : A \to \mathrm{Aut}(B)$ is defined by the composite

$$A \xrightarrow{a \mapsto (1,a)} G = B \rtimes_\phi A \xrightarrow{\mathrm{Inn}} \mathrm{Aut}(B)$$

and $\phi_1 : A \to \mathrm{Aut}(B)$ is defined by the composite

$$A \xrightarrow{a \mapsto (b,1)(1,a)(b,1)^{-1}} B \rtimes_\phi A \xrightarrow{\mathrm{Inn}} \mathrm{Aut}(B).$$

Write $s_1 : A \to G = B \rtimes_\phi A$ for the section given by

$$s_1(a) = (b,1)(1,a)(b,1)^{-1}$$

It now follows from Proposition 2.12 that the product mapping

$$((b,a) \mapsto b \cdot s_1(a)) : B \times A \to G$$

determines an isomorphism $B \rtimes_{\phi_1} A \xrightarrow{\sim} G = B \rtimes_\phi A$ of extensions, as required. □

Theorem 6.8. *Suppose the following:*

(a) *G_ℓ has a Levi decomposition (over ℓ),*
(b) *$\mathrm{Inn}(U_\ell)^{M_\ell}$ is trivial,*
(c) *the center Z of U is a vector group on which G acts linearly, and*
(d) *$H^1(M_\ell, \mathrm{Inn}(U_\ell)) = 1$.*

Then G has a Levi decomposition (over k).

Proof. The choice of a Levi factor M_ℓ of G_ℓ determines a homomorphism

$$\phi : M_\ell \to \mathrm{Aut}(U_\ell)$$

such that $\phi_0 = \Psi \circ \phi : M_\ell \to \mathrm{Out}(U_\ell)$.

Of course, the mapping $\phi_0 : M \to \mathrm{Out}(U)$ is a morphism of k-group functors. There is a natural action of the galois group Γ on $\mathrm{Aut}(U_\ell)$ and on$\mathrm{Out}(U_\ell)$ for which Ψ is equivariant. For any $\gamma \in \Gamma$ it follows that

$$\Psi \circ {}^\gamma\phi = \phi_0$$

i.e. in the notation of Proposition 5.1, ${}^\gamma\phi$ determines a class in $\mathrm{Lift}(\phi_0)$.

According to Proposition 5.1 there is a bijection $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell) \xrightarrow{\sim} \mathrm{Lift}(\phi_0)$. Since $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell)) = 1$ it follows that classes of the automorphisms ${}^\gamma\phi$ in $\mathrm{Lift}(\phi_0)$ all coincide; i.e. all ${}^\gamma\phi$ are *equivalent.*

By the definition of equivalence defining $\mathrm{Lift}(\phi_0)$, we find for each $\gamma \in \Gamma$ an element $h_\gamma \in \mathrm{Inn}(U)(\ell)$ such that

$${}^\gamma\phi = h_\gamma^{-1} \cdot \phi \cdot h_\gamma.$$

If $\gamma, \tau \in \Gamma$ we see that

$$ {}^{\gamma\tau}\phi = h_{\gamma\tau}^{-1} \cdot \phi \cdot h_{\gamma\tau}, \tag{6.5}$$

while on the other hand

$$\begin{aligned} {}^\gamma({}^\tau\phi) &= {}^\gamma(h_\tau^{-1} \cdot \phi \cdot h_\tau) \\ &= {}^\gamma h_\tau^{-1} \cdot {}^\gamma\phi \cdot {}^\gamma h_\tau \\ &= {}^\gamma h_\tau^{-1} \cdot h_\gamma^{-1}\phi \cdot h_\gamma \cdot {}^\gamma h_\tau. \end{aligned} \tag{6.6}$$

Now, applying condition (b) we see that the stabilizer in $\mathrm{Inn}(U)$ of the automorphism ϕ is trivial. Thus taken together (6.5) and (6.6) imply that

$$h_{\gamma\tau} = h_\gamma \, {}^\gamma h_\tau;$$

i.e. h_γ is a 1-cocycle on Γ with values in $\mathrm{Inn}(U)(\ell)$. Since U is connected and split unipotent, so is $\mathrm{Inn}(U)$; see Proposition 2.3. Thus $H^1_{\mathrm{coc}}(M_\ell, \mathrm{Inn}(U_\ell)) = 1$ by Proposition 2.15.

It follows that the cocycle h_γ is *trivial*. Thus there is $h \in \mathrm{Inn}(U)(\ell)$ such that for each $\gamma \in \Gamma$ we have

$$h_\gamma = h^{-1} \cdot {}^\gamma h$$

We now claim that the mapping $\phi_1 : M_\ell \to \mathrm{Aut}(U_\ell)$ defined by

$$\phi_1 = h \cdot \phi \cdot h^{-1}$$

is Γ-stable. For $\gamma \in \Gamma$ we have

$${}^\gamma\phi_1 = {}^\gamma(h \cdot \phi \cdot h^{-1}) = {}^\gamma h \cdot {}^\gamma \phi \cdot {}^\gamma h^{-1} = h h_\gamma \cdot h_\gamma^{-1} \phi h_\gamma \cdot h_\gamma^{-1} h^{-1} = \phi_1.$$

Thus ϕ_1 is Γ-stable and hence defines a morphism $\phi_1 : M \to \mathrm{Aut}(U)$ of k-group functors which we may use to define a semidirect product $G_1 = U \rtimes_{\phi_1} M$ over k. Since ϕ and ϕ_1 are conjugate by an element of $\mathrm{Inn}(U)(\ell)$ we see that

$$G_{1\ell} = U_\ell \rtimes_{\phi_1} M_\ell \simeq U_\ell \rtimes_\phi M_\ell = G_\ell. \tag{6.7}$$

We now observe that the isomorphism in (6.7) is even given by an isomorphism of *extensions* of M_ℓ by U_ℓ.

For this, first note that the center Z of U is a connected and split unipotent group; thus $H^1(\ell, Z) = 1$. It follows that the mapping $U(\ell) \to \mathrm{Inn}(U)(\ell)$ is surjective, so we may choose an element $u \in U(\ell)$ for which $\mathrm{Inn}(u) = h \in \mathrm{Inn}(U)(\ell)$. Thus we have

$$\phi_1 = \mathrm{Inn}(u) \cdot \phi \cdot \mathrm{Inn}(u)^{-1}.$$

It now follows from Lemma 6.7 that there is an isomorphism of extensions $U_\ell \rtimes_\phi M_\ell \simeq U_\ell \rtimes_{\phi_1} M_\ell$, so that $G_{1\ell} \simeq G_\ell$ as extensions of M_ℓ by U_ℓ.

According to Theorem 4.3, condition (c) implies that the extension G_ℓ has a unique k-form. Since G and G_1 are both k-forms of this extension, it follows that $G \simeq G_1$ are k-isomorphic extensions and in particular are k-isomorphic algebraic groups; since G_1 has a Levi factor over k, we conclude that G has a Levi factor over k as well. □

Remark 6.9. The reader should compare the results proved here which provide conditions for descent of Levi factors with the conditions stipulated in (McNinch 2010), Theorem 5.2. This older result gave a condition using vanishing of second cohomology H^2 which unconditionally guarantees the existence of a Levi factor. These newer results – e.g. Theorem 6.5 and Theorem 6.8 – instead give conditions using vanishing of (some form of) *first* cohomology to *descend* Levi factors over finite separable field extensions.

7. An example

In (McNinch 2013) §5 we gave an example of an extension

$$1 \to W \to E \to \mathbf{Z}/p\mathbf{Z} \to 1$$

with E commutative and W a commutative unipotent group of exponent p^2. The group E was constructed by *twisting*, and had the property that E_ℓ had a Levi factor for ℓ a galois extension of k but E had no Levi factor.

We conclude the present paper with another example of a linear algebraic group over k which has no Levi factor but acquires a Levi factor over a finite galois extension.

The example below gives a *non-commutative* extension of a finite abelian p-group by a connected, non-commutative unipotent group; in this case, the construction of the extension is perhaps slightly more straightforward.

Suppose that the characteristic of k is $p > 2$. We write $\mathscr{P}$ for the *additive polynomial* $X^p - X \in k[X]$ defining the *Artin-Schreier* mapping: for any commutative k-algebra Λ, this mapping $\mathscr{P} : \Lambda \to \Lambda$ is given by the rule $x \mapsto x^p - x$.

Recall that if $s \in k$ is not in the image of $\mathscr{P} : k \to k$ then the polynomial $F(X) = X^p - X - s \in k[X]$ is irreducible, and if α is a root of $F(X)$ in an extension field of k, then $\ell = k(\alpha)$ is a galois extension of k with $\mathrm{Gal}(\ell/k) \simeq \mathbf{Z}/p\mathbf{Z}$.

Let V be a vector space of dimension 2 over k with a basis e, f, and write $\beta : V \times V \to k$ for the unique non-degenerate symplectic form satisfying $\beta(e, f) = 1 = -\beta(f, e)$. Viewing $\mathscr{P} \circ \beta$ as a *factor system*,, we define a unipotent group H as an extension of V by $\mathbf{G}_a$; see (Serre 1988) VII.1.4. Explicitly, for a commutative k-algebra Λ we have

$$H(\Lambda) = \Lambda \times V \otimes_k \Lambda$$

with operation

$$(t, v) \cdot (s, w) = (t + s + \mathscr{P}(\beta(v, w)), v + w) = (t + s + \beta(v, w)^p - \beta(v, w), v + w)$$

for $v, w \in V \otimes \Lambda$ and $s, t \in \Lambda$.

Thus H is an extension

$$0 \to \mathbf{G}_a \xrightarrow[t \mapsto (0,t)]{i} H \xrightarrow[(v,t) \mapsto v]{} V \to 0. \tag{7.1}$$

This is a central extension, and it is non-split because H is non-abelian (the split extension is the direct product $\mathbf{G}_a \times V$). Write Z for the center of H; then $Z \simeq \mathbf{G}_a$ is the image of the mapping i of (7.1).

Fix $t \in k$ and let $V_{0,t} = \langle te, f \rangle \subset V$, so that $V_{0,t} \simeq (\mathbf{Z}/pZ)^2$. μ_t be the extension of V_0 by $Z \simeq \mathbf{G}_a$ defined by β (*not* by $\mathscr{P} \circ \beta$)). Thus there is an exact sequence

$$0 \to \mathbf{G}_a \to \mu_t \to V_{0,t} = (\mathbf{Z}/p\mathbf{Z})^2 \to 0$$

and the group operation is given by

$$(a, v) \cdot (b, w) = (a + b + \beta(v, w), v + w)$$

for $v, w \in V_{0,t}$ and $a, b \in \Lambda$. If $t \neq 0$, the extension μ_t is non-split because it is non-abelian.

Write E for the fiber product $E = H \times_{\mathbf{G}_a} \mu_t$; thus E is an extension of $V_{0,t} \simeq (\mathbf{Z}/p\mathbf{Z})^2$ by H. By the definition of the fiber product, there is a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_a \arrow[r] \arrow[d] & \mu_t \arrow[r] \arrow[d] & (\mathbf{Z}/p\mathbf{Z})^2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H \arrow[r] & H \times_{\mathbf{G}_a} \mu_t \arrow[r, "\pi"] & (\mathbf{Z}/p\mathbf{Z})^2 \arrow[r] & 0
\end{tikzcd}$$

Proposition 7.1. *If $X^p - X + t$ has no root in k, then the group $E = H \times_{\mathbf{G}_a} \mu_t$ has no Levi factor over k. If α is a root $X^p - X + t$ and $\ell = k(\alpha)$ then E_ℓ has a Levi factor.*

Sketch. We may represent elements of $E(k)$ as tuples (a, v, w) where $v \in V_{0,t}$, $w \in V$ and $a \in k$. We have

$$(a, v, w) \cdot (a', v', w') = (a + a' + \beta(v, v') + \mathscr{P}\beta(w, w)', v + v', w + w')$$

Now, any elements $\widetilde{e}, \widetilde{f}$ of $E(k)$ mapping to $te, f \in V_{0,t}$ via π must have the form $\widetilde{e} = (a, te, v)$ for some $v \in V$ and $a \in k$ and $\widetilde{f} = (b, f, w)$ for some $w \in V$ and $b \in k$.

We see that

$$\widetilde{e} \cdot \widetilde{f} = (a, te, v) \cdot (b, f, w) = (a + b + t + \mathscr{P}\beta(v, w), te + f, v + w)$$

while

$$\widetilde{f} \cdot \widetilde{e} = (b, f, w) \cdot (a, te, v) = (a + b + -t - \mathscr{P}\beta(v, w), te + f, v + w)$$

Since the characteristic of k is not 2, $\widetilde{e} \cdot \widetilde{f} = \widetilde{f} \cdot \widetilde{e}$ if and only if

$$0 = \mathscr{P}\beta(v, w) + t = \beta(v, w)^p - \beta(v, w) + t.$$

If $X^p - X + t$ has no root in k, it follows that the group $\langle \widetilde{e}, \widetilde{f} \rangle$ is non-abelian for any choice of $\widetilde{e}, \widetilde{f}$. This shows that E has no Levi factor.

On the other hand, E_ℓ always has a Levi factor since we may take $\widetilde{e} = (0, te, \alpha e)$ and $\widetilde{f} = (0, f, f)$; then $\langle \widetilde{e}, \widetilde{f} \rangle \simeq (\mathbf{Z}/p\mathbf{Z})^2$ so that $\langle \widetilde{e}, \widetilde{f} \rangle$ provides a Levi factor. □

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