Math 51

Laplace Transforms
4. Functions Defined in Pieces

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Switching Formulas

Imagine a country with a current population of 25 million, a net birth rate (births minus deaths) of 3%, and a steady net immigration of 200,000 individuals per year. After one year immigration rate starts to rise by 25,000 more immigrants each year. Two years later the borders are tightened and immigration is reduced to 100,000 per year; what will be the population after five years?

Setting $x(t)$ to be the population (in hundreds of thousands of individuals) after t years, we get

$$x' = .03x + \begin{cases} 2 & \text{for } 0 \le t < 1, \\ 2 + 0.25(t-1) & \text{for } 1 \le t < 3, \\ 1 & \text{for } 3 \le t, \end{cases} \qquad x(0) = 250.$$

To model sudden changes in the "external" term of a linear o.d.e., we use the following to switch between different formulas.[1]

Definition: The unit step function is defined for $\alpha \ge 0$ by

$$u_\alpha(t) = \begin{cases} 0 & \text{for } 0 \le t < \alpha, \\ 1 & \text{for } t \ge \alpha. \end{cases}.$$

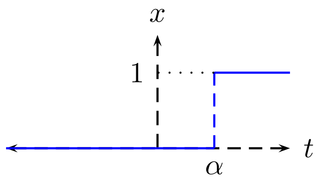


Figure 1: Graph of $u_\alpha(t)$

[1] This formal approach was pioneered by the electrical engineer, mathematician and physicist Oliver Heaviside (1850-1925).

This function acts as an "on" switch: for any function $f(t)$, the function $\tilde{f}(t) = u_\alpha(t)\, f(t)$ is given by

$$\tilde{f}(t) = u_\alpha(t) = \begin{cases} 0 & \text{for } t < \alpha, \\ f(t) & \text{for } t \geq \alpha. \end{cases}$$

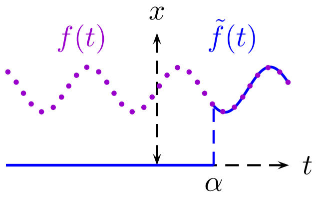


Figure 2: Graph of $\tilde{f}(t) = u_\alpha(t) f(t)$

Let us see how to express the "external" term (immigration) in the population model we considered at the beginning of this video:

$$E(t) = \begin{cases} 2 & \text{for } 0 \leq t < 1, \\ 2 + 0.25(t-1) & \text{for } 1 \leq t < 3, \\ 1 & \text{for } 3 \leq t. \end{cases}$$

We begin with the first formula

$$E(t) = 2 \text{ for } t < 1;$$

at the first interface time $t = 1$, we need to start adding in the extra term $0.25(t-1)$:

$$E(t) = 2 + u_1(t)\,[0.25(t-1)];$$

this works until $t = 3$, at which time we need to *turn off* the preceding formula and *turn on* the next formula:

$$E(t) = 2 + u_1(t)\,[0.25(t-1)] + u_3(t)\,[1 - \{2 + 0.25(t-1)\}].$$

Transforming Functions Defined in Pieces: To transform the function $f(t)$ “cut off” before $t = \alpha > 0$,

$$\tilde{f}(t) = u_\alpha(t) f(t)$$

we look at the definition:

$$\begin{aligned}
\mathcal{L}\left[\tilde{f}(t)\right] &= \int_0^\infty e^{-st} \tilde{f}(t)\, dt \\
&= \int_0^\infty e^{-st} u_\alpha(t) f(t)\, dt \\
&= \int_0^\alpha 0\, dt + \int_\alpha^\infty e^{-st} f(t)\, dt.
\end{aligned}$$

The last integral looks like the definition of $\mathcal{L}[f(t)]$, except that the integral starts at $t = \alpha$ instead of $t = 0$. Making the change of variable $t = \tau + \alpha$, and remembering that s acts as a constant during this integration, we can rewrite it as

$$\begin{aligned}
\mathcal{L}\left[\tilde{f}(t)\right] &= \int_0^\infty e^{-s(\tau+\alpha)} f(\tau + \alpha)\ d\tau \\
&= e^{-s\alpha} \int_0^\infty e^{-s\tau} f(\tau + \alpha)\ d\tau.
\end{aligned}$$

Recalling that τ is only a placeholder in this integral, we can replace it with t and see that the integral is the Laplace transform of the “translated” function $f(t + \alpha)$. We obtain

Second Shift Formula: for $\alpha > 0$,
$\mathcal{L}[u_\alpha(t)\, f(t)] = e^{-\alpha s} \mathcal{L}[f(t + \alpha)]$.

Second Shift for Inverse Transforms

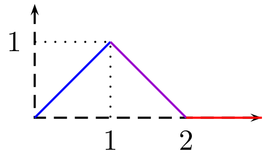
This is the first time that we see an exponential appearing in the formula for a transform $F(s)$. This shift formula tells us that if we see an exponential factor in the transform of a function, it came from transforming some function $f(t)$ multiplied by $u_\alpha(t)$, and the rest of the function is the transform of a translate of $f(t)$ by α. Unpacking this, we get

Second Shift Formula, Inverse Version: If $F(s) = \mathcal{L}[f(t)]$, then

$$\mathcal{L}^{-1}\left[e^{-\alpha s} F(s)\right] = u_\alpha(t)\, f(t-\alpha)\,.$$

On the next slides, we consider some Laplace transform calculations involving $u_\alpha(t)$.

Example: Consider the function $f(t)$ whose graph is shown below:



It is easy to tell from the graph that this function is defined by

$$f(t) = \begin{cases} t \text{ for } t \leq 1, \\ 2 - t \text{ for } 1 < t \leq 2, \\ 0 \qquad \text{ for } 2 < t \end{cases} .$$

In terms of step functions this can be written as

$$\begin{aligned} f(t) &= t + u_1(t)\,[(2-t) - t] + u_2(t)\,[0 - (2-t)] \\ &= t + u_1(t)\,[2-2t] + u_2(t)\,[t-2]. \end{aligned}$$

The transform of the first term is

$$\mathcal{L}[t] = .\frac{1}{\text{s^2}}\,.$$

The transform of the second term is

$$\begin{aligned} &\mathcal{L}[u_1(t)\,[2-2t]] \\ &\quad = e^{-s}\mathcal{L}[2 - 2(t+1)] = e^{-s}\mathcal{L}[-2t] \\ &\quad = e^{-s}\left[- \quad \frac{2}{s^2}\right] \end{aligned}$$

and the transform of the third term is

$$\begin{aligned} &\mathcal{L}[u_2(t)\,[t-2]] \\ &\quad = e^{-2s}\mathcal{L}[(t+2) - 2] = e^{-2s}\mathcal{L}[t] \\ &\quad = e^{-2s}\left[\frac{1}{s^2} \quad \right] \end{aligned}$$

so

$$\mathcal{L}[f(t)] = \frac{1}{\text{s^2}} + e^{-s}\left[\; - \quad \frac{2}{s^2}\right] + e^{-2s}\left[\frac{1}{s^2} \quad \right].$$

Example: Find the inverse transform of

$$F(s) = \frac{se^{-s}}{s^2+9} + \frac{e^{-2s}}{s+1}.$$

The inverse transform of the first term is

$$\mathcal{L}^{-1}\left[\frac{se^{-s}}{s^2+9}\right] = \mathcal{L}^{-1}\left[e^{-s}\mathcal{L}\left[\cos 3t\right]\right]$$

$$= u_1(t)\left[\cos 3(t-1)\right] = \begin{cases} 0 & \text{for } t \le 1, \\ \cos(3t-3) & \text{for } t > 1. \end{cases}$$

The inverse transform of the second term is

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s+1}\right] = \mathcal{L}^{-1}\left[e^{-2s}\mathcal{L}\left[e^{-t}\right]\right]$$

$$= u_2(t)\left[e^{-(t-2)}\right]$$

$$= \begin{cases} 0 & \text{for } t \le 2, \\ e^{2-t} & \text{for } t > 2. \end{cases}$$

So

$$\mathcal{L}^{-1}\left[F(s)\right] = u_1(t)\left[\cos 3(t-1)\right] + u_2(t)\left[e^{-(t-2)}\right]$$

$$= \begin{cases} 0 & \text{for } t \le 1, \\ \cos(3t-3) & \text{for } 1 < t \le 2, \\ \cos(3t-3) + e^{2-t} & \text{for } t > 2. \end{cases}$$

Finally, we work through one o.d.e. involving functions defined in pieces to see how the process works from beginning to end.

Example: Find the solution of

$$(D+1)^2 x = \begin{cases} 0 & \text{for } t \le 3, \\ t-3 & \text{for } t > 3, \end{cases} \qquad x(0) = 0, x'0 = -3.$$

The right hand side is written in step function notation as $u_3(t)\left[t-3\right]$:

$$(D+1)^2 x = u_3(t)\left[t-3\right], \quad x(0) = 0, x'(0) = -3.$$

Applying the Laplace transform to

$$(D+1)^2 x = u_3(t)\,[t-3], \quad x(0)=0, x'(0)=-3$$

yields

$$\begin{aligned} s^2\mathcal{L}[x] + 3 + 2s\mathcal{L}[x] + \mathcal{L}[x] &= e^{-3s}\mathcal{L}[(t] \\ (s^2+2s+1)\mathcal{L}[x] &= -3 + \frac{e^{-3s}}{s^2} \\ \mathcal{L}[x] &= -\frac{3}{(s+1)^2} + \frac{e^{-3s}}{s^2(s+1)^2} \\ x &= \mathcal{L}^{-1}\left[-\frac{3}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2(s+1)^2}\right]. \end{aligned}$$

The First Shift Formula applied to the first term yields

$$\mathcal{L}^{-1}\left[-\frac{3}{(s+1)^2}\right] = e^{-t}\mathcal{L}^{-1}\left[\frac{-3}{s^2}\right] = -3te^{-t}.$$

To evaluate the second term, we need to find the partial fraction decomposition of $\frac{1}{s^2(s+1)^2}$, which has the form

$$\frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^2} + \frac{E}{s+1}.$$

Combining these terms over a common denominator and matching coefficients, we need to solve

$$\begin{array}{rrrrl} & B & +C & & =0 \\ A & +2B & +C & +E & =0 \\ 2A & +B & & & =0 \\ A & & & & =1 \end{array}$$

leading to

$$A=1, \quad B=-2, \quad C=2, \quad E=1$$

so

$$
\begin{aligned}
\frac{1}{s^2(s+1)^2} &= \frac{1}{s^2} - \frac{2}{s} + \frac{2}{s+1} + \frac{1}{(s+1)^2} \\
&= \mathcal{L}\left[t - 2 + 2e^{-t} + te^{-t}\right].
\end{aligned}
$$

This means our second term is

$$
\begin{aligned}
&\mathcal{L}^{-1}\left[e^{-3s}\mathcal{L}\left[t - 2 + 2e^{-t} + te^{-t}\right]\right] \\
&= u_3(t)\left[(t-3) - 2 + 2e^{-(t-3)} + (t-3)e^{-(t-3)}\right] \\
&= u_3(t)\left[t - 5 + (t-1)e^{3-t}\right],
\end{aligned}
$$

so the solution to our initial value problem is

$$
\begin{aligned}
x(t) &= -3te^{-t} + u_3(t)\left[t - 5 + (t-1)e^{3-t}\right] \\
&= \begin{cases} -3te^{-t} & \text{for } t \leq 3 \\ t - 5 + e^{-t}\left(e^3(t-1) - 3t\right) & \text{for } t > 3. \end{cases}
\end{aligned}
$$