# Final Exam *Solutions*

Math 51 Spring 2021 – Tufts University

2022-05-09

1. (10 points in total) Indicate your response to the following.

   (a) (2 pts) Consider the system of linear ODEs

   $$(\clubsuit) \quad D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$$

   where $A$ is a $3 \times 3$ matrix with constant entries and where the entries of $\mathbf{E}(t)$ are continuous functions of $t$ on the interval $(0, \infty)$. If $\mathbf{h}(t)$ and $\mathbf{k}(t)$ are *solutions* to $(\clubsuit)$ and if $\mathbf{h}(1) = \mathbf{k}(1)$, must it be true that $\mathbf{h}(t) = \mathbf{k}(t)$ whenever $0 < t$?

   **Solution:**

   Yes; this results from the *Existence and Uniqueness Theorem* for systems of linear ODEs.

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   (b) (4 pts) Let $A$ be an $n \times n$ matrix with an eigenvalue $\lambda = 2$ with *multiplicity 3*. Suppose that the vector $\mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^n$ is a solution to the equation $(A - 2\mathbf{I}_n)^3 \mathbf{x} = \mathbf{0}$. Give a formula for a solution $\mathbf{h}(t)$ to the homogeneous system of linear ODEs $D\mathbf{x} = A\mathbf{x}$ which satisfies $\mathbf{h}(0) = \mathbf{v}$.

   **Solution:**

   $$\mathbf{h}(t) = e^{2t}\left(\mathbf{v} + t(A - 2\mathbf{I}_n)\mathbf{v} + \frac{t^2}{2}(A - 2\mathbf{I}_n)^2\mathbf{v}\right)$$

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   (c) (2 pts) Indicate whether the following statement is true or false: If $P(D)$ is a polynomial in $D$ with constant coefficients, and if $h_1(t)$ and $h_2(t)$ are solutions to $P(D)x = e^t$, then $h_1(t) + h_2(t)$ is a solution to $P(D)x = 2e^t$.

   **Solution:**

   This statements is *true*; indeed, if $h(t) = h_1(t) + h_2(t)$ then

   $$P(D)[h(t)] = P(D)[h_1(t) + h_2(t)] = P(D)[h_1(t)] + P(D)[h_2(t)] = e^t + e^t = 2e^t$$

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   (d) (2 pts) Indicate whether the following statement is true or false.

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ are vectors in $\mathbb{R}^3$ and if

$$\det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} = 0$$

then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are *linearly dependent*.

**Solution:**

This statement is *true*. Since the determinant of $B = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$ is 0, one knows that the equation

$$B\mathbf{x} = \mathbf{0}$$

has a non-zero solution $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ and thus $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$, which confirms the linear dependence.

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2. (8 pts total)

   (a) (3 pts) Let

$$f(t) = \begin{cases} e^{2t} & t < 1 \\ 0 & 1 \le t \end{cases}$$

   Re-write $f(t)$ using unit step functions.

   **Solution:**

   $f(t) = e^{2t} - u_1(t) \cdot e^{2t}$.

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   (b) (5 pts) Find the Laplace transform $\mathscr{L}[u_2(t)e^{3t}]$.

   **Solution:**

   $\mathscr{L}[u_2(t)e^{3t}] = e^{-2s}\mathscr{L}[e^{3(t+2)}] = e^{-2s}e^6\mathscr{L}[e^{3t}] = \dfrac{e^{-2s+6}}{s-3}$.

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3. (15 pts in total)

   (a) (5 pts) Compute $\mathscr{L}^{-1}\left[\dfrac{e^{-s}}{s}\right]$

   **Solution:**

   Notice that $f(t) = \mathscr{L}^{-1}\left[\dfrac{1}{s}\right] = 1$.

   Thus the second shift formula shows that

$$\mathscr{L}^{-1}\left[\frac{e^{-s}}{s}\right] = u_1(t)f(t-1) = u_1(t).$$

(b) (10 pts) Compute $\mathscr{L}^{-1}\left[\dfrac{1}{s(s^2+4)}\right]$

**Solution:**

We first solve the *partial fractions* problem:

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

We need

$$1 = A(s^2+4) + (Bs+C)s = As^2 + 4A + Bs^2 + Cs = (A+B)s^2 + Cs + 4A.$$

The augmented matrix corresponding to the system of equations obtained by *equating coefficients* is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1/4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

We see that

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1/4 \\ -1/4 \\ 0 \end{bmatrix}$$

and thus

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

Now,

$$\mathscr{L}^{-1}\left[\frac{1}{s(s^2+4)}\right] = \frac{1}{4}\left(\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right]\right) = \frac{1}{4}\left(1 - \cos(2t)\right)$$

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4. (10 pts) Let $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Find the general solution to the homogeneous system of ODEs $D\mathbf{x} = B\mathbf{x}$.

**Solution:**

The characteristic polynomial is given by

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} = \lambda^2 - 2$$

so the eigenvalues of $B$ are $\pm\sqrt{2}$.

To find an eigenvector for $\lambda = \sqrt{2}$, consider

$$B - \sqrt{2}\mathbf{I} = \begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & -1-\sqrt{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -(1+\sqrt{2}) \\ 0 & 0 \end{bmatrix}.$$

Since an eigenvector is a solution to $(B - \sqrt{2}\mathbf{I})\mathbf{x} = 0$, we find that

$$\mathbf{v}_+ = \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$$

is an eigenvector for $\lambda = \sqrt{2}$.

A similar calculation shows that

$$\mathbf{v}_- = \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$$

is an eigenvector for $\lambda = -\sqrt{2}$.

We now find solutions to $D\mathbf{x} = B\mathbf{x}$ using these eigenvectors:

$$\mathbf{h}_+(t) = e^{t\sqrt{2}}\mathbf{v}_+ = e^{t\sqrt{2}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$$

$$\mathbf{h}_-(t) = e^{-t\sqrt{2}}\mathbf{v}_- = e^{-t\sqrt{2}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$$

Since $\mathbf{v}_+$ and $\mathbf{v}_-$ are *linearly independent*, the general solution to $D\mathbf{x} = B\mathbf{x}$ is given by

$$c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t) = c_1 e^{t\sqrt{2}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} + c_2 e^{-t\sqrt{2}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}.$$

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5. (10 pts) Transform the following initial-value problem to an equation of the form $\mathscr{L}[x] = F(s)$; find $F(s)$. *You do not need to solve for $x$.*

$$(D^2 - 9)x = e^t + 1, \qquad x(0) = x'(0) = 0.$$

**Solution:**

Using the *first differentiation formula together with the fact that $x(0) = x'(0) = 0$, we see that $\mathscr{L}[D^2 x] = s^2\mathscr{L}[x]$. Thus applying $\mathscr{L}$ to each side of the ODE yields

$$(s^2 - 9)\mathscr{L}[x] = \mathscr{L}[e^t + 1] = \frac{1}{s-1} + \frac{1}{s}.$$

We now find that

$$\mathscr{L}[x] = \frac{1}{(s-1)(s^2-9)} + \frac{1}{s(s^2-9)};$$

i.e. $F(s) = \dfrac{1}{(s-1)(s^2-9)} + \dfrac{1}{s(s^2-9)}$.

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6. (10 pts) Consider the ordinary differential equation

$$(\diamondsuit) \quad \frac{dx}{dt} - tx = e^{t^2/2}$$

Find the *general solution* $x(t)$ to $(\diamondsuit)$.

**Solution:**

We first find a solution to the homogeneous equation $\dfrac{dx}{dt} = tx$. Separating variables, this equation yields

$$\frac{dx}{x} = tdt \implies \int \frac{dx}{x} = \int tdt \implies \ln|x| = \frac{t^2}{2} + C$$

so that the general solution to this homogeneous equation is

$$x(t) = k \cdot e^{t^2/2}$$

for an arbitrary constant $k$.

We choose the homogeneous $h(t) = e^{t^2/2}$.

Now, to solve the non-homogeneous equation, we use *variation of parameters*. Thus we seek solutions of the form $k(t)h(t)$ for some function $k(t)$.

We know that

$$k'(t) = h(t)^{-1} \cdot e^{t^2/2} = e^{-t^2/2}e^{t^2/2} = 1.$$

Thus $k(t) = \int 1dt = t + C$ and we find the general solution to $(\diamondsuit)$:

$$x(t) = k(t)h(t) = (t + C)e^{t^2/2} = te^{t^2/2} + Ce^{t^2/2}.$$

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7. (15 pts) Consider the ordinary differential equation

$$(\heartsuit) \quad (D^2 - 4)x = e^{2t} + e^{-2t}.$$

a. An annihilator of $e^{2t} + e^{-2t}$ is the operator $A(D) = (D-2)(D+2) = D^2 - 4$. A solution to $(\heartsuit)$ must be a solution to the homogeneous equation $A(D) \cdot (D^2 - 4)x = (D-2)^2(D+2)^2 x = 0$. Briefly explain why a *simplified guess* for a solution $p(t)$ to $(\heartsuit)$ is given by

$$p(t) = k_1 \cdot te^{2t} + k_2 \cdot te^{-2t}$$

**Solution:**

The general solution to the homogeneous equation $A(D)(D^2 - 4)\mathbf{x} = \mathbf{0}$ is generated by the functions $e^{2t}, te^{2t}, e^{-2t}, te^{-2t}$.

Among these, the functions $e^{2t}$ and $e^{-2t}$ are solutions to the homogeneous equation corresponding to $(\heartsuit)$ and thus won't play a role in solving $(\heartsuit)$.

So there is a solution to $(\heartsuit)$ which is a linearly combination of the remaining functions, namely

$$k_1 \cdot te^{2t} + k_2 \cdot te^{-2t}$$

and we may use this for our simplified guess.

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b. Use the *exponential shift* formula to compute $(D^2-4)[p(t)] = (D^2-4)[k_1 \cdot te^{2t} + k_2 \cdot te^{-2t}]$.

**Solution:**

We have

$$(D^2-4)[te^{2t}] = e^{2t} \cdot ((D+2)^2 - 4)[t] = e^{2t}(D^2+4)[t] = 4e^{2t}$$

and

$$(D^2-4)[te^{-2t}] = e^{2t} \cdot ((D-2)^2 - 4)[t] = e^{-2t}(D^2-4)[t] = -4e^{-2t}.$$

Thus

$$(D^2-4)[k_1 \cdot te^{2t} + k_2 \cdot te^{-2t}] = 4k_1 e^{2t} - 4k_2 e^{-2t}.$$

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c. Use your answer to b) to find a particular solution $p(t)$ to (♡).

**Solution:**

To find a particular solution of the form

$$p(t) = k_1 \cdot te^{2t} + k_2 \cdot te^{-2t}$$

we must have

$$(D^2-4)[p(t)] = e^{2t} + e^{-2t}$$

According to the result in (b), we need

$$4k_1 e^{2t} - 4k_2 e^{-2t} = e^{2t} + e^{-2t}.$$

Comparing coefficients, we see that

$$4k_1 = 1 \quad \text{and} \quad -4k_2 = 1.$$

Thus $k_1 = 1/4$ and $k_2 = -1/4$, so a particular solution is given by

$$p(t) = \frac{1}{4}\left(te^{2t} - te^{-2t}\right)$$

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8. (12 pts) The matrix $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\pm 2$. An eigenvector for $\lambda = 2$ is given by $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and an eigenvector for $\mu = -2$ is given by $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(a) Write the general solution to the homogeneous system $D\mathbf{x} = A\mathbf{x}$.

**Solution:**

The general solution is given by

$$x(t) = c_1 e^{2t}\mathbf{v} + c_2 e^{-2t}\mathbf{w} = c_1 e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

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(b) Use the method of *variation of parameters* to find the general solution to the system of ODEs

$$D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

**Solution:**

We form the Wronskian matrix $W$ whose columns are the generators for the general homogeneous solution from (a):

$$W = \begin{bmatrix} 2e^{2t} & -2e^{-2t} \\ e^{2t} & e^{-2t} \end{bmatrix}$$

We seek a particular solution of the form

$$\mathbf{p}(t) = c_1(t)e^{2t}\mathbf{v} + c_2(t)e^{-2t}\mathbf{w}.$$

To find $c_1(t)$ and $c_2(t)$, we must solve the matrix equation

$$W \cdot \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We use *Cramer's Rule.* First notice that

$$\det W = 2 - (-2) = 4.$$

Now

$$c_1'(t) = \frac{\begin{bmatrix} 1 & -2e^{-2t} \\ 1 & e^{-2t} \end{bmatrix}}{4} = \frac{3e^{-2t}}{4} \implies c_1(t) = \int \frac{3e^{-2t}}{4} dt = \frac{-3}{8} e^{-2t} + C_1$$

and

$$c_2'(t) = \frac{\begin{bmatrix} 2e^{2t} & 1 \\ e^{2t} & 1 \end{bmatrix}}{4} = \frac{e^{2t}}{4} \implies c_2(t) = \int \frac{e^{2t}}{4} dt = \frac{1}{8} e^{2t} + C_2$$

By choosing $C_1 = C_2 = 0$, we find the particular solution

$$\mathbf{p}(t) = \frac{-3}{8} e^{-2t} e^{2t} \mathbf{v} + \frac{1}{8} e^{2t} e^{-2t} \mathbf{w} = \frac{1}{8}(-3\mathbf{v} + \mathbf{w}) = \frac{1}{8}\left(-3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -1/4 \end{bmatrix}.$$

We conclude that the general solution to the non-homogeneous euqation is given by

$$\begin{bmatrix} -1 \\ -1/4 \end{bmatrix} + c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

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9. (10 pts) Solve the initial value problem

$$D(D^2 - 1)x = 0, \qquad x(0) = 0, x'(0) = 1, x''(0) = 0.$$

**Hint:** Don't use the Laplace transform. First find the *general solution* to $D(D^2 - 1)x = 0$.

**Solution:**

Since the roots of the characteristic polynomial are $0, 1, -1$, the general solution is given by

$$x(t) = c_1 + c_2 e^t + c_3 e^{-t}.$$

Notice that

$$x'(t) = c_2 e^t - c_3 e^{-t}$$

and that

$$x''(t) = c_2 e^t + c_2 e^{-t}$$

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Evaluating at $t = 0$ now gives equations:

$$0 = x(0) = c1 + c_2 + c_3$$

$$1 = x'(0) = c_2 - c_3$$

$$0 = x''(0) = c_2 + c_3$$

To solve this system of equations, we consider the corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

We now see that

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Thus the solution to the initial value problem is

$$x(t) = \frac{1}{2}\left(e^t - e^{-t}\right).$$