Math 51 Spring 2022 - Final Exam - some review problems

2022-04-27

- Differential equations via integration (§ 1.1)
- Separation of variables (§ 1.2)
- Linear Differential Equations (§ 1.3)
- Existence & Uniqueness; Linear ODEs (§ 1.6, 2.2)
- Cramer's Rule and the Wronskian (§ 2.3 & App. A)
- Linear Independence (§ 2.4)
- Const coeff linear ODEs (real roots) (§ 2.5)
- Const coeff linear ODEs (complex roots) (§ 2.6)
- Non-homog linear ODEs via undetermined coeffs (§ 2.7)
- Non-homog linear ODEs via variation of parameters (§ 2.8)
- Linear Systems (§ 3.2, 3.3)
- Linear systems and independence (§ 3.4)
- Eigenvalues, Eigenvectors (§ 3.5)
- Row Reduction (§ 3.6)
- Homogeneous linear systems (real roots) (§ 3.7)
- Homogeneous linear systems (complex roots) (§ 3.8)
- Homogeneous linear systems (double roots) (§ 3.9)
- Homogeneous linear systems (higher multiplicity roots) (§ 3.10)
- Non-homogeneous Systems (§ 3.11)
- The Laplace transform $\mathcal{L}$ and initial value problems (§ 5.2, 5.3)
- Properties of $\mathcal{L}$ and $\mathcal{L}^{-1}$ (§ 5.4)
- Piecewise functions (§ 5.5)
- ~~Convolution (§ 5.6)~~

1. Indicate which of the following best represents a *simplified guess* for a particular solution $p(t)$ to the non-homogeneous linear ODE:

$$(D-3)(D-1)x = te^{3t} + \cos(2t)$$

   a. $p(t) = k_1 te^{3t} + k_2 \cos(2t) + k_3 \sin(2t)$

   b. $p(t) = k_1 te^{3t} + k_2 \cos(2t)$

   c. $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t)$

   d. $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t) + k_4 \sin(2t)$

2. Indicate which of the following represents the general solution to the homogeneous linear ODE $(D^2 - 2D + 2)^2 x = 0$.

   a. $h(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t) + c_3 te^{-t} \cos(t) + c_4 te^{-t} \sin(t)$

b. $h(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t)$

c. $h(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t) + c_3 t e^t \cos(t) + c_4 t e^t \sin(t)$

d. $h(t) = c_1 t e^t \cos(t) + c_2 t e^t \sin(t) + c_3 t^2 e^t \cos(t) + c_4 t^2 e^t \sin(t)$

3. The matrix $A = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix}$ has characteristic polynomial $\lambda^2 - 2\lambda + 2$ and thus its eigenvalues are $\lambda = 1 + i$ and $\lambda = 1 - i$.

   Which of the following is an eigenvector for A?

   a. A has no eigenvectors.

   b. $\begin{bmatrix} 3 - i \\ 2 \end{bmatrix}$

   c. $\begin{bmatrix} 2 \\ -3 + i \end{bmatrix}$

   d. $\begin{bmatrix} 3 + i \\ 2 \end{bmatrix}$

4. Consider the linear system of ODEs

$$(\diamondsuit) \quad D\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

   A third order linear ODE is *equivalent* to this system if for each of its solutions $x(t)$, the vector-valued function $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x'(t) \\ x''(t) \end{bmatrix}$ is a solution to $(\diamondsuit)$. Which of the following linear ODEs is equivalent to $(\diamondsuit)$?

   a. $(D^3 - 2D^2 - D - 5)x = e^t$

   b. $(D^3 - 5D^2 - D - 2)x = e^t$

   c. $(D^3 + 2D^2 + D + 5)x = -e^t$

   d. $(D^3 + 5D^2 + D + 2)x = -e^t$

5. Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. $\lambda = 2$ is an eigenvalue of A with multiplicity two. The matrix $A - 2\mathbf{I}_3$ satisfies $(A - 2\mathbf{I}_3)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the generalized eigenvectors of A for $\lambda = 2$ are generated by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

   Which of the following represents a solution $\mathbf{h}(t)$ to the system $D\mathbf{x} = A\mathbf{x}$ with the property that $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$?

a. $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

b. $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1+12t \\ 2 \\ 6 \end{bmatrix}$

c. $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1+t \\ 2 \\ 6 \end{bmatrix}$

d. No solution $\mathbf{h}(t)$ has the property that $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$.

6. Consider the homogeneous system $(\diamondsuit)$ $\quad D\mathbf{x} = A\mathbf{x}$ where A is a 3×3 matrix, and let $\mathbf{h}_1(t), \mathbf{h}_2(t)$ be solutions to $(\diamondsuit)$. Which of the following statements is correct?

   a. $\mathbf{h}_1(0)$ and $\mathbf{h}_2(0)$ are *eigenvectors* for A.

   b. The system $(\diamondsuit)$ has exactly two solutions.

   c. If the vectors $\mathbf{h}_1(0), \mathbf{h}_2(0)$ are linearly independent, then the general solution to $(\diamondsuit)$ is given by $\mathbf{x}(t) = c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$.

   d. None of the above statements is correct.

7. The matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ has characteristic polynomial $\lambda(\lambda - 3)$ and hence has eigenvalues $\lambda = 0$ and $\lambda = 3$. An eigenvector for $\lambda = 0$ is given by $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and an eigenvector for $\lambda = 3$ is given by $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   Find a particular solution $\mathbf{p}(t)$ for the system of linear ODEs

$$D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

8. Let $A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$.

   The characteristic polynomial of A is $r^2 - 4r + 5$ so the eigenvalues of A are $\lambda = 2 \pm i$.

   Moreover, $\mathbf{v} = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$ is an eigenvector for $\lambda = 2 + i$.

   a. Find the general solution to $D\mathbf{x} = A\mathbf{x}$.

   b. Solve the initial value problem $D\mathbf{x} = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

9. Solve the initial value problem $(4D^2 - 4D + 1)x = 0, \quad x(2) = x'(2) = e$.

10. Consider the matrix $B = \begin{bmatrix} 5 & -3 & -6 \\ 0 & 2 & 0 \\ 3 & -3 & -4 \end{bmatrix}$.

a. The vector $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector for B. What is the corresponding eigenvalue?

Hint: Compute the vector $B\mathbf{v}$ and compare with $\mathbf{v}$.

b. Find an eigenvector for B for the eigenvalue $\lambda = -1$.

11. Laplace Transforms:

a. Compute the inverse Laplace tranform $\mathcal{L}^{-1}[F(s)]$ of the function $F(s) = \dfrac{3s^2 + s + 1}{(s+1)(s^2+2)}$.

b. If x is a solution to $(D^2 + D + 1)x = 1$ with $x(0) = 0$ and $x'(0) = 1$, find an expression for $\mathcal{L}[x]$ as a function of s.

12. Let $W = W(h_1(t), h_2(t))$ denote the *Wronskian matrix* of the functions $h_1(t) = e^{2t}$ and $h_2(t) = te^{2t}$. Which of the following represents the *determinant* of W?

a. e^{4t}

b. $(1 + 4t)e^{4t}$

c. e^{2t}

d. $(1 + 4t)e^{2t}$

13. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ in $\mathbf{R}^4$, and let $A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ be the 4×3 matrix whose columns are the $\mathbf{v}_i$. Which of the following statements is correct?

a. The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are *linearly dependent*.

b. Since $A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, the only solution to the equation $A\mathbf{w} = \mathbf{0}$ is $\mathbf{w} = \mathbf{0}$ so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are *linearly independent*.

c. The equation $A\mathbf{w} = \mathbf{x}$ has a solution for every vector $\mathbf{x}$ in $\mathbf{R}^4$.

d. The determinant of A is $\neq 0$.

14. Let A be an $n \times n$ matrix with constant coefficients a_{ij}, and let $\mathbf{E}(t)$ be a vector with n components. If $\mathbf{v}$ is any vector in $\mathbf{R}^n$, must there be a solution $\mathbf{x}(t)$ to the system of equations $D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$ for which $\mathbf{x}(0) = \mathbf{v}$?

a. No, this conclusion is only guaranteed when the system is *homogeneous*.

b. No, this conclusion is only guaranteed when the entries of the vector $\mathbf{E}(t)$ are *constant* functions of t.

c. Yes, this conclusion is the content of the *Existence and Uniqueness Theorem for Solutions of Linear Systems.*

d. No, this conclusion is only guaranteed when $\det A \neq 0$.

15. Consider the homogeneous system $(\diamondsuit) \quad D\mathbf{x} = A\mathbf{x}$ where A is a 3×3 matrix.

a. If $\mathbf{h}(t)$ is a solution, must $\mathbf{h}(0)$ be an eigenvector for A? Why or why not?

b. Show that the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are linearly dependent.

c. Let $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$ be solutions to $(\diamondsuit)$. Suppose that $\mathbf{h}_1(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{h}_2(0) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{h}_3(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are the vectors from b. Do the solutions $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$ generate the general solution to $(\diamondsuit)$? Why or why not?

16. A drug is absorbed by the body at a rate proportional to the amount of the drug present in the bloodstream after t hours. If there are $x(t)$ mg of the drug present in the bloodstream at time t, assume that the drug is absorbed at a rate of $0.5x(t)$ /hour. If a patient receives the drug intravenously at a constant rate of 3 mg/hour, to which of the following ODEs is $x(t)$ a solution?

a. $x'(t) = -0.5x(t) + 3$

b. $x'(t) = -0.5x(t); \quad x(0) = 3$

c. $x'(t) = 0.5x(0) + 3$

d. $x'(t) = .5x(t) - 3$

17. You are given that a particular solution to

$$(\heartsuit) \quad (D^2 - 2D + 1)x = e^t$$

is $p(t) = \dfrac{t^2e^t}{2}$. Which of the following best represents the general solution to $(\heartsuit)$?

a. $c_1e^t + c_2te^t$.

b. $\dfrac{t^2e^t}{2} + c_1e^t + c_2te^t$.

c. $\dfrac{t^2e^t}{2} + ce^t$.

d. $\dfrac{t^2e^t}{2} + c_1e^t + c_2e^{-t}$.

18. Let $x_1(t)$ and $x_2(t)$ be solutions to the ODE $(t+1)x'' + x' + x = 0$. Suppose that $x_1(0) = x_2(0)$ and that $x_1'(0) = x_2'(0)$. Which of the following statements is most correct?

a. $x_1(t) = x_2(t)$ for every t.

b. Since the ODE is *normal* on the interval $(-1, \infty)$, we can conclude that $x_1(t) = x_2(t)$ for $-1 < t < \infty$.

c. No conclusion is possible because the existence and uniqueness theorem does not apply to this ODE.

d. We can only conclude that $x_1(t) = x_2(t)$ for all t if we also assume that $x_1''(0) = x_2''(0)$.

19. Show that the functions

$$f_1(t) = e^t \cos(t), \quad f_2(t) = e^t \sin(t), \quad f_3(t) = e^t$$

are linearly independent.

You have been told that functions like this are independent. However, here we want you to demonstrate it directly in this case. You may use the *Wronskian test* (with all details needed to justify using it) or other, direct arguments from the definition.

20. Find the Laplace transform of the function

$$f(t) = \begin{cases} 1 & \text{for } t < 1, \\ t-1 & \text{for } 1 \le t < 2, \\ 1 & \text{for } t \ge 2. \end{cases}$$

21. Suppose $g(t)$ is the inverse Laplace transform of

$$F(s) = \frac{2se^{\pi s/2}}{(s^2+4)}.$$

Find $g\left(\frac{\pi}{4}\right)$.