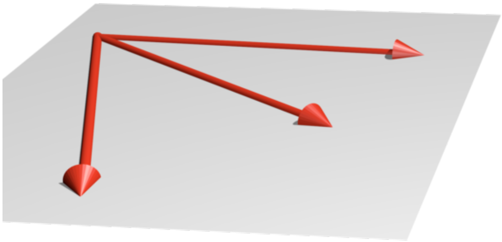
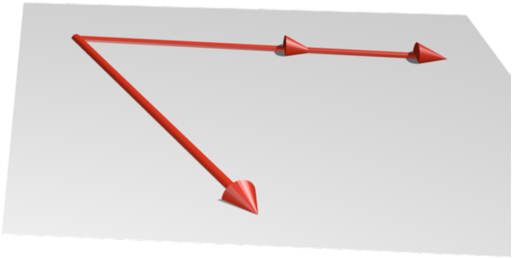
Differential Equations

Linear Independence of Vectors





Initial Vectors

Recall: For an n-th order homogenenous linear system

$$D\vec{x} = A\vec{x}, \tag{H}$$

· $\vec{h}_1, \dots, \vec{h}_n$ generate the general solution of (H) :

$$\vec{x}(t) = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)$$
$$\Updownarrow$$

· For any fixed t_0, we can always solve for $c_1, \dots, c_n$ in

$$c_1\vec{h}_1(t_0) + \cdots + c_n\vec{h}_n(t_0) = \vec{v}.$$

The second statement is about the initial vectors $\vec{h}_1(t_0), \dots, \vec{h}_n(t_0)$. We will show the statement is equivalent to the linear independence of the n initial vectors.

Ex: Consider the system $D\vec{x} = A\vec{x}$, where

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Substitution will show that

$$\vec{h}_1(t) = \begin{bmatrix} (1+t)e^{-t} \\ -e^{-t} \\ -(1+t)e^{-t} \end{bmatrix}, \ \vec{h}_2(t) = \begin{bmatrix} (1-t)e^{-t} \\ e^{-t} \\ -(1-t)e^{-t} \end{bmatrix},$$

and

$$\vec{h}_3(t) = \begin{bmatrix} (3+t)e^{-t} \\ -e^{-t} \\ -(3+t)e^{-t} \end{bmatrix}$$

are solutions of this third-order system. The Wronskian of $\vec{h}_1, \vec{h}_2, \vec{h}_3$ at $t_0 = 0$ is

$$W\left[\vec{h}_1, \vec{h}_2, \vec{h}_3\right](0) = \det \begin{bmatrix} 1 & 1 & 3 \\ -1 & 1 & -1 \\ -1 & -1 & -3 \end{bmatrix} = 0,$$

so $\vec{h}_1, \vec{h}_2, \vec{h}_3$ *do not* generate the general solution. The initial vectors are

$$\vec{h}_1(0) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \ \vec{h}_2(0) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{h}_3(0) = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}.$$

Note that $\vec{h}_3(0) = 2\vec{h}_1(0) + \vec{h}_2(0)$ implies

$$2\vec{h}_1(0) + \vec{h}_2(0) - \vec{h}_3(0) = \vec{0}.$$

We can check that

$$\vec{k}(t) = \begin{bmatrix} \left(2 - t^2\right) e^{-t} \\ 2te^{-t} \\ t^2 e^{-t} \end{bmatrix}$$

is also a solution of $D\vec{x} = A\vec{x}$ and that $\vec{h}_1(t), \vec{h}_2(t)$ and $\vec{k}(t)$ do generate the general solution. The initial vector of $\vec{k}(t)$ at $t_0 = 0$ is

$$\vec{k}(0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Def: The n-vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent if there exist constants $c_1, \ldots, c_k$ not all zeros such that

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}. \qquad (*)$$

The vectors are linearly independent if the only constants for which the equality $(*)$ holds are

$$c_1 = c_2 = \cdots = c_k = 0.$$

Ex: Check for independence:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Sln: We solve for c_1, c_2, c_3 in

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}. \qquad (*)$$

to see if the only solution is $c_1 = c_2 = c_3 = 0$.

The left side of $(*)$ is

$$c_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + 2c_3 \\ -c_1 + c_2 \\ -c_1 - c_2 \end{bmatrix}.$$

What values of c_1, c_2, c_3 give the right side $\vec{0}$? We solve

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 0 && (1) \\ -c_1 + c_2 \quad &= 0 && (2) \\ -c_1 - c_2 \quad &= 0. && (3) \end{aligned}$$

Now (1) + (3) shows $2c_3 = 0$, so $c_3 = 0$. Similarly, (2) + (3) and (2) − (3) force $c_1 = c_2 = 0$. Hence, the only solution of $(*)$ is

$$c_1 = c_2 = c_3 = 0$$

and the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

Note that they are the initial vectors at t_0

$$\vec{h}_1(t_0),\ \vec{h}_2(t_0),\ \vec{k}(t_0)$$

of $\vec{h}_1, \vec{h}_2, \vec{k}$ which *do* generate the general solution of the third order system in the previous example.

Ex: Check for independence:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}.$$

Sln: A typical linear combination of these vectors looks like

$$\begin{aligned} & c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} c_1 + c_2 + c_3 + 2c_4 \\ c_2 + 2c_3 - 2c_4 \\ 2c_4 \end{bmatrix}. \end{aligned}$$

Equating this and $\vec{0}$ gives

$$\begin{aligned} c_1 + c_2 + c_3 + 2c_4 &= 0 \\ c_2 + 2c_3 - 2c_4 &= 0 \\ 2c_4 &= 0. \end{aligned}$$

Then c_1, c_2, c_3, c_4 solve the equations if and only if

$$\begin{aligned} c_4 &= 0 \\ c_2 &= -2c_3 + 2c_4 = -2c_3 \\ c_1 &= -c_2 - c_3 - c_4 = -(-2c_3) - c_3 = c_3. \end{aligned}$$

Hence, for any number c_3,

$$\begin{cases} c_1 = & c_3 \\ c_2 = & -2c_3 \\ c_3 = & c_3 \\ c_4 = & 0 \end{cases}$$

is a solution and the equations have infinitely many solutions for c_1, c_2, c_3, c_4. By definition, the vectors are linearly dependent.

In general, to check for the independence of $\vec{v}_1, \ldots, \vec{v}_k$, we solve

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

for $c_1, \ldots, c_k$. The equation always has at least one solution $c_1 = \cdots = c_k = 0$, and the vectors are independent if and only if this is the *only* solution.

Independence of n n-Vectors

· The n-vectors

$$\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{bmatrix}, \ \dots, \ \vec{v}_n = \begin{bmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{nn} \end{bmatrix}$$

are linearly independent.

$$\Updownarrow$$

· The vector equation

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$$

has a unique solution $c_1 = \cdots = c_n = 0$.

$$\Updownarrow$$

· The system

$$\begin{aligned} c_1 v_{11} + c_2 v_{12} + \cdots + c_n v_{1n} &= 0 \\ c_1 v_{21} + c_2 v_{22} + \cdots + c_n v_{2n} &= 0 \\ &\vdots \\ c_1 v_{n1} + c_2 v_{n2} + \cdots + c_n v_{nn} &= 0 \end{aligned}$$

has a unique solution $c_1 = \cdots = c_n = 0$.

(Cramer's $\Updownarrow$ determinant test)

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$$\det \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ & & \vdots & \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix} \neq 0$$

Thus, we have:

Fact: The n-vectors $\vec{v}_1, \dots, \vec{v}_n$ are independent if and only if the $n \times n$ matrix whose columns are $\vec{v}_1, \dots, \vec{v}_n$ has nonzero determinant.

Independence of n Initial n-Vectors

If the n-vectors $\vec{v}_1, \ldots, \vec{v}_n$ are initial vectors,

$$\vec{v}_1 = \vec{h}_1(t_0), \ \ldots, \ \vec{v}_n = \vec{h}_n(t_0)$$

then

$$\det \begin{bmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ & & \vdots & \\ v_{n1} & v_{n2} & \ldots & v_{nn} \end{bmatrix} = W[\vec{h}_1, \vec{h}_2, \ldots, \vec{h}_n](t_0).$$

We summarize our work in Section 3.3 and Section 3.4 into the following fact about the equivalence between linear independence and generating the general solution, for n solutions of an n-th order homogeneous linear system.

Fact: Let $D\vec{x} = A\vec{x}$ be an nth-order homogeneous linear system of o.d.e's whose coefficients a_{ij} are continuous on an interval I. Suppose $\vec{h}_1(t), \ldots, \vec{h}_n(t)$ are solutions of $D\vec{x} = A\vec{x}$ and t_0 is a fixed value of t in I. The following are equivalent:

- The solutions $\vec{h}_1, \ldots, \vec{h}_n$ generate the general solution of $D\vec{x} = A\vec{x}$.

$$\Updownarrow$$

- $W[\vec{h}_1, \vec{h}_2, \ldots, \vec{h}_n](t_0) \neq 0.$

$$\Updownarrow$$

- The initial vectors $\vec{h}_1(t_0), \ldots, \vec{h}_n(t_0)$ are linearly independent.