Math 51

Var. of Param., Matrices

Summation Notation

$$\left(a_n(t)D^n + a_{n-1}(t)D^{n-1} + \cdots + a_1(t)D + a_0(t)\right)x$$
$$= \left(\sum_{i=0}^{n} a_i(t)D^i\right)x$$

$$1^2 + 2^2 + \cdots + n^2$$
$$= \sum_{i=1}^{n} i^2$$

Var. of Param.

Annihilators & undet. coef. applies only to

linear w/ const coef.

$$L(x) = E(t)$$

but $E(t)$ has to be a sum of products of

$$e^{\lambda t}, \quad t^k, \quad \cos t \text{ (or } \sin t\text{)}.$$

Var. of param.

$L(x)$ = linear but need not have const. coef.

$E(t)$ = need not be of special form.

Let $L(x) = (D^2 + b_1(t)D + b_0)x = g(t)$. (N)

Suppose h_1, h_2 generate the gen. sol. of $L(x) = 0$:

$$c_1 h_1(t) + c_2 h_2(t), \quad c_1, c_2 \in \mathbb{R}.$$

Assume that a particular sol to (N) is of the form

$$p(t) = c_1(t)\,h_1(t) + c_2(t)\,h_2(t),$$

$$= \sum_{i=1}^{2} c_i h_i$$

plug into (N):

$$\begin{aligned} Dp &= \sum (c_i h_i)' \\ &= \sum_{i=1}^{2} (c_i h_i' + c_i' h_i) \qquad \text{(product rule)} \\ &= \sum c_i h_i' + \sum c_i' h_i \end{aligned}$$

We make a simplifying assumption: $\boxed{\sum c_i' h_i = 0}$

So

$$\begin{aligned} Dp &= \sum c_i h_i' . \\ D^2 p &= \sum (c_i h_i')' \\ &= \sum c_i h_i'' + \sum c_i' h_i' \qquad \text{(product rule)} \end{aligned}$$

Therefore,

$$\begin{aligned} L(p) = (D^2 + b_1 D + b_0) p &= \sum c_i h_i'' + b_1 \sum c_i h_i' + b_0 \sum c_i h_i + \sum c_i' h_i' \\ &= \sum c_i \underbrace{(h_i'' + b_1 h_i' + b_0 h_i)}_{L(h_i) = 0} + \sum c_i' h_i' \\ &= \boxed{\sum c_i' h_i' = g(t)} . \end{aligned}$$

We end up with

$$\begin{cases} \sum c_i' h_i = 0 & \text{(simplifying assump.)} \\ \sum c_i' h_i' = g & \text{(solves (N))} \end{cases}$$

or

$$\boxed{\begin{cases} c_1' h_1 + c_2' h_2 = 0 \\ c_1' h_1' + c_2' h_2' = g . \end{cases}}$$ ← Remember this!

The det. of coef. matrix is

$$\det \begin{bmatrix} h_1 & h_2 \\ h_1' & h_2' \end{bmatrix} = W[h_1, h_2] := W .$$

Here, $W \neq 0$ because h_1, h_2 generate the sol of $L(x) = 0$.

By Cramer,

$$\boxed{c_1'} = \frac{\begin{vmatrix} 0 & h_2 \\ q & h_2' \end{vmatrix}}{W} = \boxed{\frac{-qh_2}{W}}$$

$$\boxed{c_2'} = \frac{\begin{vmatrix} h_1 & 0 \\ h_1' & q \end{vmatrix}}{W} = \boxed{\frac{qh_1}{W}},$$

Integrate to find c_1, c_2.

Then $p = c_1 h_1 + c_2 h_2$ will be a part. sol of (N).

The general sol of (N) is

$$k_1 h_1 + k_2 h_2 + p, \qquad k_1, k_2 \in \mathbb{R}.$$

Example. Solve $(D^2+1)x = \sec t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

Sol. $(D^2+1)x = (D+i)(D-i)x = 0$

has sol gen by e^{it}, e^{-it} or by

$$h_1 = \cos t, \quad h_2 = \sin t.$$

The Wronskian is

$$W[h_1, h_2] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1.$$

Need to solve

$$\begin{cases} c_1' \cos t + c_2' \sin t = 0 \\ -c_1' \sin t + c_2' \cos t = \sec t \end{cases}$$

By Cramer,

$$c_1' = \frac{\begin{vmatrix} 0 & \sin t \\ \sec t & \cos t \end{vmatrix}}{1} = -(\sec t)\sin t = -\frac{\sin t}{\cos t},$$

$$c_2' = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1.$$

Integrate:

$$c_1 = \int -\frac{\sin t}{\cos t}\, dt = \ln|\cos t| + k_1, \qquad \cos t > 0 \text{ on } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}).$$

$$c_2 = \int 1\, dt = t + k_2.$$

A particular solution $p(t) = c_1 h_1 + c_2 h_2$

$$= (\ln \cos t)\cos t + t \sin t.$$

The gen. sol to (N) is

$$\boxed{k_1 \cos t + k_2 \sin t + (\ln \cos t)\cos t + t \sin t.}$$

Matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} a & d & g & j \\ b & e & h & k \\ c & f & i & l \end{bmatrix} = \begin{bmatrix} \overset{a+2b+3c}{\bullet} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \underset{4g+5h+6i}{\bullet} & \cdot \end{bmatrix}$$

2×3 $\quad$ 3×4 $\quad$ 2×4

the same

Ex.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

2×3 $\quad$ 3×1

$$= \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2 + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} x_3.$$

- Multiplying a matrix A by a vector $\vec{x}$ is the same as taking a linear comb of the col. of A using x_1, x_2, x_3 as coef.

Converting a linear DE to a lin. system

Linear System.

$$x_1' = a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + E_1(t)$$
$$x_2' = a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + E_2(t)$$
$$x_3' = a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + E_3(t).$$

Ex. $(D^3 + a_2 D^2 + a_1 D + a_0)x = E(t).$

$$x''' + a_2 x'' + a_1 x' + a_0 x = E(t) \qquad (N)$$

① Define new var.

$$x_1 = x$$
$$x_2 = x' = x_1'$$
$$x_3 = x'' = x_2' \qquad \Rightarrow x_3' = x'''$$
$$x''' = -a_0 x - a_1 x' - a_2 x'' + E(t).$$
$$= -a_0 x_1 - a_1 x_2 - a_2 x_3 + E(t).$$

② Move x_1', x_2', x_3' to the left and x_1, x_2, x_3 to the right.

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = -a_0 x_1 - a_1 x_2 - a_2 x_3 + E(t)$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ E(t) \end{bmatrix}.$$

$$\vec{x}' = A\vec{x} + \vec{E}(t).$$