Final Exam

Math 51 Spring 2021 – Tufts University

2022-05-09

You may not use *calculators*, *books* or *notes* during the exam. All electronic devices (including your phones) must be *silenced* and *put away* for the duration of the exam.

After completing the exam, submit this exam booklet to the proctors. Your submission will be scanned and uploaded to *Gradescope* for marking; you do *not* need to take/upload images of your exam with a phone. You should write your name at the top of each page, as indicated (*especially* if you remove the staples from your exam booklet).

For the partial credit problems, always *show your work*. Try to fit this work in the *available space* if possible. If you scratch some work out, please make it clear what should be graded. There is a blank page at the back of your exam for use as scratch paper. If you need more space for a solution, please write clearly *in the indicated space* that your solution continues later.

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Please print your name, and sign your exam. With your signature, you pledge that you have neither given nor received assistance on this exam.

Name (printed): ________________________________

Signature: ________________________________

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Laplace transform formulas

definition

- $F(s) = \mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt$

basic formulas

$$
\begin{aligned}
\mathscr{L}[e^{\lambda t}] &= \frac{1}{s-\lambda} \quad \text{for } s > \lambda &, \quad \mathscr{L}^{-1}\left[\frac{1}{s-\lambda}\right] &= e^{\lambda t} \\
\mathscr{L}[t^n] &= \frac{n!}{s^{n+1}} \quad \text{for } s > 0 &, \quad \mathscr{L}^{-1}\left[\frac{1}{s^{n+1}}\right] &= \frac{t^n}{n!} \\
\mathscr{L}[\sin(\beta t)] &= \frac{\beta}{s^2+\beta^2} \quad \text{for } s > 0 &, \quad \mathscr{L}^{-1}\left[\frac{1}{s^2+\beta^2}\right] &= \frac{1}{\beta}\sin(\beta t) \\
\mathscr{L}[\cos(\beta t)] &= \frac{s}{s^2+\beta^2} \quad \text{for } s > 0 &, \quad \mathscr{L}^{-1}\left[\frac{s}{s^2+\beta^2}\right] &= \cos(\beta t)
\end{aligned}
$$

first differentiation formula:

- $\mathscr{L}[Dx] = s\mathscr{L}[x] - x(0)$,
- $\mathscr{L}[D^2x] = s^2\mathscr{L}[x] - sx(0) - x'(0)$,
- $\mathscr{L}[D^kx] = s^k\mathscr{L}[x(t)] - s^{k-1}x(0) - s^{k-2}x'(0) - \cdots - sx^{(k-2)}(0) - x^{(k-1)}(0)$ for $k \geq 1$.

first shift formula

- if $\mathscr{L}[f(t)] = F(s)$ then $\mathscr{L}[e^{at}f(t)] = F(s-a)$.
- $\mathscr{L}^{-1}[F(s)] = e^{at}\mathscr{L}^{-1}[F(s+a)]$
- $\mathscr{L}^{-1}[F(s-a)] = e^{at}\mathscr{L}^{-1}[F(s)]$

second differentiation formula

- $\mathscr{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\mathscr{L}[f(t)]$

second shift formula

- $\mathscr{L}[u_a(t)f(t)] = e^{-as}\mathscr{L}[f(t+a)]$.
- if $f(t) = \mathscr{L}^{-1}[F(s)]$ then $\mathscr{L}^{-1}[e^{-as}F(s)] = u_a(t)f(t-a)$.

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1. (10 points in total) Indicate your response to the following.

   (a) (2 pts) Consider the system of linear ODEs

   $$(\clubsuit) \quad D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$$

   where A is a 3×3 matrix with constant entries and where the entries of $\mathbf{E}(t)$ are continuous functions of t on the interval $(0, \infty)$. If $\mathbf{h}(t)$ and $\mathbf{k}(t)$ are *solutions* to $(\clubsuit)$ and if $\mathbf{h}(1) = \mathbf{k}(1)$, must it be true that $\mathbf{h}(t) = \mathbf{k}(t)$ whenever $0 < t$? *Circle your answer.*

Yes	**No**

   (b) (4 pts) Let A be an $n \times n$ matrix with an eigenvalue $\lambda = 2$ with *multiplicity 3.* Suppose that the vector $\mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^n$ is a solution to the equation $(A - 2\mathbf{I}_n)^3 \mathbf{x} = \mathbf{0}$. Give a formula for a solution $\mathbf{h}(t)$ to the homogeneous system of linear ODEs $D\mathbf{x} = A\mathbf{x}$ which satisfies $\mathbf{h}(0) = \mathbf{v}$.

   (c) (2 pts) Indicate whether the following statement is true or false: If $P(D)$ is a polynomial in D with constant coefficients, and if $h_1(t)$ and $h_2(t)$ are solutions to $P(D)x = e^t$, then $h_1(t) + h_2(t)$ is a solution to $P(D)x = 2e^t$. *Circle your answer.*

True	**False**

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(d) (2 pts) Indicate whether the following statement is true or false.

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ are vectors in $\mathbb{R}^3$ and if

$$\det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} = 0$$

then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are *linearly dependent. Circle your answer.*

True	**False**

2. (8 pts total)

(a) (3 pts) Let

$$f(t) = \begin{cases} e^{2t} & t < 1 \\ 0 & 1 \leq t \end{cases}$$

Re-write $f(t)$ using unit step functions.

(b) (5 pts) Find the Laplace transform $\mathscr{L}[u_2(t)e^{3t}]$.

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3. (15 pts in total)

(a) (5 pts) Compute $\mathscr{L}^{-1}\left[\dfrac{e^{-s}}{s}\right]$

(b) (10 pts) Compute $\mathscr{L}^{-1}\left[\dfrac{1}{s(s^2+4)}\right]$

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4. (10 pts) Let $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Find the general solution to the homogeneous system of ODEs $D\mathbf{x} = B\mathbf{x}$.

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5. (10 pts) Transform the following initial-value problem to an equation of the form $\mathscr{L}[x] = F(s)$; find $F(s)$. *You do not need to solve for x.*

$$(D^2 - 9)x = e^t + 1, \qquad x(0) = x'(0) = 0.$$

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6. (10 pts) Consider the ordinary differential equation

$$(\diamondsuit) \quad \frac{dx}{dt} - tx = e^{t^2/2}$$

Find the *general solution* $x(t)$ to $(\diamondsuit)$.

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7. (15 pts) Consider the ordinary differential equation

$$(\heartsuit) \quad (D^2 - 4)x = e^{2t} + e^{-2t}.$$

a. An annihilator of $e^{2t} + e^{-2t}$ is the operator $A(D) = (D-2)(D+2) = D^2 - 4$. A solution to $(\heartsuit)$ must be a solution to the homogeneous equation $A(D) \cdot (D^2 - 4)x = (D-2)^2(D+2)^2 x = 0$. Briefly explain why a *simplified guess* for a solution $p(t)$ to $(\heartsuit)$ is given by

$$p(t) = k_1 \cdot te^{2t} + k_2 \cdot te^{-2t}$$

b. Use the *exponential shift* formula to compute $(D^2 - 4)[p(t)] = (D^2 - 4)[k_1 \cdot te^{2t} + k_2 \cdot te^{-2t}]$.

(continue on the next page)

c. Use your answer to b) to find a particular solution $p(t)$ to $(\heartsuit)$.

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8. (12 pts) The matrix $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ has eigenvalues ± 2. An eigenvector for $\lambda = 2$ is given by $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and an eigenvector for $\mu = -2$ is given by $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

   (a) Write the general solution to the homogeneous system $D\mathbf{x} = A\mathbf{x}$.

   (b) Use the method of *variation of parameters* to find the general solution to the system of ODEs

$$D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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9. (10 pts) Solve the initial value problem

$$D(D^2 - 1)x = 0, \qquad x(0) = 0, x'(0) = 1, x''(0) = 0.$$

Hint: Don't use the Laplace transform. First find the *general solution* to $D(D^2 - 1)x = 0$.

End of exam