Math 51 Spring 2022 - Final Exam - some review problems
Solutions

2022-05-01

1. Indicate which of the following best represents a *simplified guess* for a particular solution $p(t)$ to the non-homogeneous linear ODE:

$$(D-3)(D-1)x = te^{3t} + \cos(2t)$$

   a. $p(t) = k_1 te^{3t} + k_2 \cos(2t) + k_3 \sin(2t)$
   b. $p(t) = k_1 te^{3t} + k_2 \cos(2t)$
   c. $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t)$
   d. $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t) + k_4 \sin(2t)$

   Solution:

   correct response was d.

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2. Indicate which of the following represents the general solution to the homogeneous linear ODE $(D^2 - 2D + 2)^2 x = 0$.

   a. $h(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) + c_3 te^{-t}\cos(t) + c_4 te^{-t}\sin(t)$
   b. $h(t) = c_1 e^{t}\cos(t) + c_2 e^{t}\sin(t)$
   c. $h(t) = c_1 e^{t}\cos(t) + c_2 e^{t}\sin(t) + c_3 te^{t}\cos(t) + c_4 te^{t}\sin(t)$
   d. $h(t) = c_1 te^{t}\cos(t) + c_2 te^{t}\sin(t) + c_3 t^2 e^{t}\cos(t) + c_4 t^2 e^{t}\sin(t)$

   Solution:

   correct response was ~~a.~~ c.

   (The roots of the char poly are $\lambda = 1 \pm i$, which leads to solutions of the form $t^j \sin(t)$ and $t^j \cos(t)$.)

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3. The matrix $A = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix}$ has characteristic polynomial $\lambda^2 - 2\lambda + 2$ and thus its eigenvalues are $\lambda = 1 + i$ and $\lambda = 1 - i$.

   Which of the following is an eigenvector for A?

   a. A has no eigenvectors.

b. $\begin{bmatrix} 3-i \\ 2 \end{bmatrix}$

c. $\begin{bmatrix} 2 \\ -3+i \end{bmatrix}$

d. $\begin{bmatrix} 3+i \\ 2 \end{bmatrix}$

Solution:

correct response was d.

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4. Consider the linear system of ODEs

$$(\diamondsuit) \quad D\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

A third order linear ODE is *equivalent* to this system if for each of its solutions $x(t)$, the vector-valued function $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x'(t) \\ x''(t) \end{bmatrix}$ is a solution to $(\diamondsuit)$. Which of the following linear ODEs is equivalent to $(\diamondsuit)$?

a. $(D^3 - 2D^2 - D - 5)x = e^t$

b. $(D^3 - 5D^2 - D - 2)x = e^t$

c. $(D^3 + 2D^2 + D + 5)x = -e^t$

d. $(D^3 + 5D^2 + D + 2)x = -e^t$

Solution:

correct response was b.

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5. Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. $\lambda = 2$ is an eigenvalue of A with multiplicity two. The matrix $A - 2\mathbf{I}_3$ satisfies $(A - 2\mathbf{I}_3)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the generalized eigenvectors of A for $\lambda = 2$ are generated by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

Which of the following represents a solution $\mathbf{h}(t)$ to the system $D\mathbf{x} = A\mathbf{x}$ with the property that $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$?

a. $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

b. $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1+12t \\ 2 \\ 6 \end{bmatrix}$

c. $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1+t \\ 2 \\ 6 \end{bmatrix}$

d. No solution $\mathbf{h}(t)$ has the property that $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$.

Solution:

correct response was b.

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6. Consider the homogeneous system $(\diamondsuit)$ $\quad D\mathbf{x} = A\mathbf{x}$ where A is a 3×3 matrix, and let $\mathbf{h}_1(t), \mathbf{h}_2(t)$ be solutions to $(\diamondsuit)$. Which of the following statements is correct?

   a. $\mathbf{h}_1(0)$ and $\mathbf{h}_2(0)$ are *eigenvectors* for A.

   b. The system $(\diamondsuit)$ has exactly two solutions.

   c. If the vectors $\mathbf{h}_1(0), \mathbf{h}_2(0)$ are linearly independent, then the general solution to $(\diamondsuit)$ is given by $\mathbf{x}(t) = c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$.

   d. None of the above statements is correct.

   Solution:

   correct response was d.

   To see that **a.** is incorrect, consider solutions $e^{\lambda t}\mathbf{v}$ and $e^{\mu t}\mathbf{w}$ arising from eigenvectors $\mathbf{v}$ and $\mathbf{w}$.

   Then there is a solution $h(t) = e^{\lambda t}\mathbf{v} + e^{\mu t}\mathbf{w}$ but $h(0) = \mathbf{v} + \mathbf{w}$ which is not an eigenvector if $\lambda \neq \mu$.

   b. in incorrect. Indeed, all linear combinations $c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$ are solutions, so there are always *infinitely many solutions.*

   Finally, **c.** is incorrect because for a 3×3 system the general solution is generated by three solutions with linearly independent initial vectors; two solutions are not enough.

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7. The matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ has characteristic polynomial $\lambda(\lambda - 3)$ and hence has eigenvalues $\lambda = 0$ and $\lambda = 3$. An eigenvector for $\lambda = 0$ is given by $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and an eigenvector for $\lambda = 3$ is given by $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Find a particular solution $\mathbf{p}(t)$ for the system of linear ODEs

$$D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

Solution:

The general solution is generated by the solutions obtained from eigenvectors:

$$\mathbf{h}_1(t) = e^{0t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{h}_2(t) = e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

To find a particular solution, form the Wronskian matrix

$$W = \begin{bmatrix} -1 & e^{3t} \\ 1 & 2e^{3t} \end{bmatrix}$$

and notice that $\det W = -3e^{3t}$.

A particular solution has the form $\mathbf{p}(t) = c_1(t)\mathbf{h}_1(t) + c_2(t)\mathbf{h}_2(t)$, where the vector $\mathbf{c} = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$ satisfies the matrix equations

$$W\mathbf{c}' = \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

Using Cramer's Rule, we find that

$$c_1'(t) = \frac{\det \begin{bmatrix} 0 & e^{3t} \\ t & 2e^{3t} \end{bmatrix}}{-3e^{3t}} = \frac{-te^{3t}}{-3e^{3t}} = \frac{t}{3}.$$

$$c_2'(t) = \frac{\det \begin{bmatrix} -1 & 0 \\ 1 & t \end{bmatrix}}{-3e^{3t}} = \frac{-t}{-3e^{3t}} = \frac{te^{-3t}}{3}$$

Now we integrate to find $c_1(t)$ and $c_2(t)$:

$$c_1(t) = \int c_1'(t)dt = \frac{1}{3}\int tdt = \frac{t^2}{6} + A.$$

For c_2 we integrate by parts with $u = t, dv = e^{-3t}dt$:

$$c_2(t) = \int c_2'(t)dt = \frac{1}{3}\int te^{-3t}dt = \frac{1}{3}\left(\frac{-t}{3}e^{-3t} + \frac{1}{3}\int e^{-3t}dt\right) = \frac{-1}{9}e^{-3t}\left(t + \frac{1}{3}\right) + B$$

We may take $A = B = 0$ since we only seek a particular solution. This gives

$$\begin{aligned} \mathbf{p}(t) =& \frac{t^2}{6}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{-1}{9}e^{-3t}\left(t + \frac{1}{3}\right)e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ =& \frac{t^2}{6}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{-1}{9}\left(t + \frac{1}{3}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

8. Let $A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$.

The characteristic polynomial of A is $r^2 - 4r + 5$ so the eigenvalues of A are $\lambda = 2 \pm i$.

Moreover, $\mathbf{v} = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$ is an eigenvector for $\lambda = 2 + i$.

a. Find the general solution to $D\mathbf{x} = A\mathbf{x}$.

Solution:

The complex solution to (H) is

$$e^{2t}(\cos t + i \sin t)\begin{bmatrix} 2-i \\ 5 \end{bmatrix} = e^{2t}\begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + ie^{2t}\begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix}$$

so the real and imaginary parts of this expression generate the general solution

$$x(t) = C_1 e^{2t}\begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix}.$$

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b. Solve the initial value problem $D\mathbf{x} = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution:

The value at $t = 0$ of the general solution given above is

$$X(0) = C_1 e^0 \begin{bmatrix} 2\cos 0 + \sin 0 \\ 5\cos 0 \end{bmatrix} + C_2 e^0 \begin{bmatrix} -\cos 0 + 2\sin 0 \\ 5\sin 0 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix};$$

setting this equal to the desired initial condition yields the system of equations

$$\begin{aligned} 2C_1 - C_2 &= 1 \\ 5C_1 + 0C_2 &= 1 \end{aligned}$$

which can be solved by performing row operations on the augmented matrix

$$\begin{bmatrix} 2 & -1 & |1 \\ 5 & 0 & |1 \end{bmatrix},$$

or by using Cramer's Rule, or simply by noting that the second equation says $C_1 = \frac{1}{5}$, and substituting into the first equation yields $\frac{2}{5} - C_2 = 1$ or $C_2 = -\frac{3}{5}$.

Thus the desired solution of (H) is

$$X(t) = \frac{1}{5}e^{2t}\begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} - \frac{3}{4}e^{2t}\begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix} = e^{2t}\begin{bmatrix} \cos t - \sin t \\ \cos t - 3\sin t \end{bmatrix}.$$

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9. Solve the initial value problem $(4D^2 - 4D + 1)x = 0, \quad x(2) = x'(2) = e.$

Solution:

The polynomial $4r^2 - 4r + 1$ has root $r = 1/2$ with multiplicity 2. Thus the general solution is given by

$$x(t) = c_1 e^{t/2} + c_2 t e^{t/2}.$$

Note that

$$\begin{aligned} x'(t) = D[x(t)] &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} D[t] \\ &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} (D + 1/2)[t] \\ &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} (1 + t/2) \end{aligned}$$

Now, we need

$$e = x(2) = c_1 e + 2c_2 e$$

and

$$e = x'(2) = \frac{1}{2} e c_1 + 2ec_2$$

Thus we must solve the matrix equation

$$\begin{bmatrix} e & 2e \\ e/2 & 2e \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e \\ e \end{bmatrix}$$

This can be solved in several ways – e.g. by row operations on the augmented matrix, as follows:

$$\left[\begin{array}{cc|c} e & 2e & e \\ e/2 & 2e & e \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 4 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 1 \end{array}\right]$$

Thus $c_1 = 0$ and $c_2 = 1/2$ so that the solution to the initial value problem is given by

$$x(t) = \frac{te^{t/2}}{2}.$$

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10. Consider the matrix $B = \begin{bmatrix} 5 & -3 & -6 \\ 0 & 2 & 0 \\ 3 & -3 & -4 \end{bmatrix}$.

a. The vector $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector for B. What is the corresponding eigenvalue?

Hint: Compute the vector $B\mathbf{v}$ and compare with $\mathbf{v}$.

Solution:

The product $B\mathbf{v}$ is equal to

$$B\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2\mathbf{v}$$

so the eigenvalue is $\lambda = 2$.

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b. Find an eigenvector for B for the eigenvalue $\lambda = -1$.

Solution: Perform row operations on the matrix $B - (-1)\mathbf{I}_3 = B + \mathbf{I_3}$:

$$\begin{bmatrix} 6 & -3 & -6 \\ 0 & 3 & 0 \\ 3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Considering this echelon matrix, we see that an eigenvector for $\lambda = -1$ is given by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

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11. Laplace Transforms:

a. Compute the inverse Laplace tranform $\mathscr{L}^{-1}[F(s)]$ of the function $F(s) = \dfrac{3s^2 + s + 1}{(s+1)(s^2+2)}$.

Solution:

The partial fraction decomposition has the form

$$\frac{3s^2 + s + 1}{(s+1)(s^2+2)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 2};$$

combining over a common denominator and matching coefficients leads to

$$\begin{array}{lcccccc} s^2 \text{ terms}: & A & +B & & = & 3 \\ s \text{ terms}: & & B & +C & = & 1 \\ \text{constant terms}: & 2A & & +C & = & 1 \end{array}$$

We can solve the first (respectively, second) equation for A (respectively, C) in terms of B:

$$A = 3 - B$$
$$C = 1 - B$$

and substituting into the third equation yields

$$\begin{aligned}
(6-2B)+(1-B) &= 1 \\
-3B &= -6 \\
B &= 2 \\
A &= 1 \\
C &= -1
\end{aligned}$$

so

$$\frac{3s^2+s+1}{(s+1)(s^2+2)} = \frac{1}{s+1} + \frac{2s-1}{s^2+2}.$$

Then the inverse transform is

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{3s^2+s+1}{(s+1)(s^2+2)}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s+1}\right] + \mathscr{L}^{-1}\left[\frac{2s}{s^2+2}\right] - \mathscr{L}^{-1}\left[\frac{1}{s^2+2}\right] \\
&= e^{-t} + 2\cos t\sqrt{2} - \frac{1}{\sqrt{2}}\sin t\sqrt{2}.
\end{aligned}$$

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b. If x is a solution to $(D^2+D+1)x = 1$ with $x(0) = 0$ and $x'(0) = 1$, find an expression for $\mathscr{L}[x]$ as a function of s.

Solution:

By the first differentiation formula, applying the Laplace Transform to both sides of the problem yields

$$\begin{aligned}
\mathscr{L}D^2x + \mathscr{L}Dx + \mathscr{L}x &= \mathscr{L}1 \\
\{s^2\mathscr{L}x - sx(0) - x'(0)\} + \{s\mathscr{L}x - x(0)\} + \mathscr{L}x &= \mathscr{L}1 \\
s^2\mathscr{L}x - 1 + s\mathscr{L}x + \mathscr{L}x &= \frac{1}{s} \\
(s^2+s+1)\mathscr{L}x &= 1 + \frac{1}{s} = \frac{1+s}{s} \\
\mathscr{L}x &= \frac{1+s}{s(s^2+2+1)}
\end{aligned}$$

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12. Let $W = W(h_1(t), h_2(t))$ denote the *Wronskian matrix* of the functions $h_1(t) = e^{2t}$ and $h_2(t) = te^{2t}$. Which of the following represents the *determinant* of W?

   a. e^{4t}

   b. $(1+4t)e^{4t}$

   c. e^{2t}

   d. $(1+4t)e^{2t}$

Solution:

correct answer is c.

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13. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ in $\mathbf{R}^4$, and let $A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ be the 4×3 matrix whose columns are the $\mathbf{v}_i$. Which of the following statements is correct?

   a. The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are *linearly dependent.*

   b. Since $A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, the only solution to the equation $A\mathbf{w} = \mathbf{0}$ is $\mathbf{w} = \mathbf{0}$ so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are *linearly independent.*

   c. The equation $A\mathbf{w} = \mathbf{x}$ has a solution for every vector $\mathbf{x}$ in $\mathbf{R}^4$.

   d. The determinant of A is $\neq 0$.

   Solution:

   The correct response is b.

   Answer a. is incorrect because the vectors are independent.

   Answer c. is incorrect because the given equation has no solution when $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   Answer d. is incorrect because the determinant of a 4×3 (non-square!) matrix is not defined.

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14. Let A be an $n \times n$ matrix with constant coefficients a_{ij}, and let $\mathbf{E}(t)$ be a vector with n components. If $\mathbf{v}$ is any vector in $\mathbf{R}^n$, must there be a solution $\mathbf{x}(t)$ to the system of equations $D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$ for which $\mathbf{x}(0) = \mathbf{v}$?

   a. No, this conclusion is only guaranteed when the system is *homogeneous.*

   b. No, this conclusion is only guaranteed when the entries of the vector $\mathbf{E}(t)$ are *constant* functions of t.

   c. Yes, this conclusion is the content of the *Existence and Uniqueness Theorem for Solutions of Linear Systems.*

   d. No, this conclusion is only guaranteed when $\det A \neq 0$.

   Solution:

   correct response is c.

   The existence and uniqueness theorem applies for non-homogeneous systems (so a is incorrect) and applies so long as the entries of A and of $\mathbf{E}$ are *continuous* functions of t (so

b. is incorrect). Finally, the existence and uniqueness theorem is valid even when A has determinant 0 (so d. is incorrect).

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15. Consider the homogeneous system $(\diamondsuit)\quad D\mathbf{x} = A\mathbf{x}$ where A is a 3×3 matrix.

   a. If $\mathbf{h}(t)$ is a solution, must $\mathbf{h}(0)$ be an eigenvector for A? Why or why not?

Solution:

No, $\mathbf{h}(0)$ need not be an eigenvector. Suppose for example that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for A with eigenvalues λ, μ, and suppose that $\lambda \neq \mu$. Then $\mathbf{v} + \mathbf{w}$ is not an eigenvector.

Indeed, since $\lambda \neq \mu$ we know that $\mathbf{v}$ and $\mathbf{w}$ are *linearly independent.* Now, for any number β,

$$(A - \beta\mathbf{I})(\mathbf{v} + \mathbf{w}) = (\lambda - \beta)\mathbf{v} + (\mu - \beta)\mathbf{w}$$

Since $\lambda \neq \mu$, at least one of $\lambda - \beta$ or $\mu - \beta$ is non-zero, so the linear independence of $\mathbf{v}$ and $\mathbf{w}$ shows that $(A - \beta\mathbf{I})(\mathbf{v} + \mathbf{w})$ is non-zero. This shows that $\mathbf{v} + \mathbf{w}$ is not an eigenvector (for *any* eigenvalue β).

Now, the function

$$h(t) = e^{\lambda t}\mathbf{v} + e^{\mu t}\mathbf{w}$$

is a solution to $(\diamondsuit)$, and $h(0) = \mathbf{v} + \mathbf{w}$.

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   b. Show that the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are linearly dependent.

Solution:

We perform row operations on the matrix whose columns are given by these vectors:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting echelon matrix has 3 pivots and no free variables, the only solution $\mathbf{c}$ to the matrix equation

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$. (Alternatively, you could have obtained this conclusion by noting that the *determinant* of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ is equal to 0).

Thus the only coefficients satisfying the following equation

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \mathbf{0}$$

are $c_1 = c_2 = c_3 = 0$; this shows that the vectors are linearly independent.

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c. Let $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$ be solutions to $(\diamondsuit)$. Suppose that $\mathbf{h}_1(0) = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{h}_2(0) = \begin{bmatrix}-1\\2\\1\end{bmatrix}$, and $\mathbf{h}_3(0) = \begin{bmatrix}1\\2\\1\end{bmatrix}$ are the vectors from b. Do the solutions $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$ generate the general solution to $(\diamondsuit)$? Why or why not?

Solution:

Yes. Since A is a 3×3 matrix, one knows that three solutions $\mathbf{h}_1(t)$, $\mathbf{h}_2(t)$ and $\mathbf{h}_3(t)$ generate the general solution provided that the "initial vectors" $\mathbf{h}_1(0), \mathbf{h}_2(0), \mathbf{h}_3(0)$ are linearly independent; thus the result in part b. shows that the $\mathbf{h}_i(t)$ generate the general solution.

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16. A drug is absorbed by the body at a rate proportional to the amount of the drug present in the bloodstream after t hours. If there are $x(t)$ mg of the drug present in the bloodstream at time t, assume that the drug is absorbed at a rate of $0.5x(t)$ /hour. If a patient receives the drug intravenously at a constant rate of 3 mg/hour, to which of the following ODEs is $x(t)$ a solution?

a. $x'(t) = -0.5x(t) + 3$

b. $x'(t) = -0.5x(t); \quad x(0) = 3$

c. $x'(t) = 0.5x(0) + 3$

d. $x'(t) = .5x(t) - 3$

Solution:

correct response is a.

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17. You are given that a particular solution to

$$(\heartsuit) \quad (D^2 - 2D + 1)x = e^t$$

is $p(t) = \dfrac{t^2e^t}{2}$. Which of the following best represents the general solution to $(\heartsuit)$?

a. $c_1e^t + c_2te^t$.

b. $\dfrac{t^2e^t}{2} + c_1e^t + c_2te^t$.

c. $\dfrac{t^2e^t}{2} + ce^t$.

d. $\dfrac{t^2e^t}{2} + c_1e^t + c_2e^{-t}$.

Solution:

correct response was b.

18. Let $x_1(t)$ and $x_2(t)$ be solutions to the ODE $(t+1)x''+x'+x=0$. Suppose that $x_1(0)=x_2(0)$ and that $x_1'(0)=x_2'(0)$. Which of the following statements must be correct?

   a. $x_1(t)=x_2(t)$ for every t.

   b. Since the ODE is *normal* on the interval $(-1,\infty)$, we can conclude that $x_1(t)=x_2(t)$ for $-1<t<\infty$.

   c. No conclusion is possible because the existence and uniqueness theorem does not apply to this ODE.

   d. We can only conclude that $x_1(t)=x_2(t)$ for all t if we also assume that $x_1''(0)=x_2''(0)$.

   Solution:

   correct response was b. Indeed since the equation is of 2nd order and since it is normal on the interval $(-1,\infty)$, the existence and uniqueness theorem guarantees for any α,β that there is only one solution x which $x(0)=\alpha$ and $x'(0)=\beta$.

   Assertion a. need not be true since the ODE is not normal on $(-\infty,\infty)$.

   And assertion d. is incorrect – the existence and uniqueness theorem doesn't require a condition on the second derivative in this case.

19. Show that the functions

$$f_1(t)=e^t\cos(t),\quad f_2(t)=e^t\sin(t),\quad f_3(t)=e^t$$

   are linearly independent.

   You have been told that functions like this are independent. However, here we want you to demonstrate it directly in this case. You may use the *Wronskian test* (with all details needed to justify using it) or other, direct arguments from the definition.

   Solution:

   There are several possible strategies for solving this problem; here we list a few of them.

   First, you can use the Wronskian test. This requires computation of the first and second derivatives of the f_i, which is perhaps most easily done using the *exponential shift formula*.

   One finds:

$$D[e^t\cos(t)]=e^t(D+1)[\cos(t)]=e^t(\cos(t)-\sin(t))$$
$$D[e^t\sin(t)]=e^t(D+1)[\sin(t)]=e^t(\cos(t)+\sin(t))$$
$$D^2[e^t\cos(t)]=D[e^t(\cos(t)-\sin(t))]=e^t(D+1)[\cos(t)-\sin(t)]=-2e^t\sin(t).$$
$$D^2[e^t\sin(t)]=D[e^t(\cos(t)+\sin(t))]=e^t(D+1)[\cos(t)+\sin(t)]=2e^t\cos(t).$$

   Thus the Wronskian matrix is given by

$$W=W(f_1,f_2,f_3)=\begin{bmatrix} e^t\cos(t) & e^t\sin(t) & e^t \\ e^t(\cos(t)-\sin(t)) & e^t(\cos(t)+\sin(t)) & e^t \\ -2e^t\sin(t) & 2e^t\cos(t) & e^t \end{bmatrix}$$

Now, according to the Wronskian test, the functions will be linearly independent (on the interval $(-\infty, \infty)$) provided that $\det W(t_0)$ is non-zero for some t_0. If we take $t_0 = 0$, we find that

$$\det W\Big|_{t=0} = \det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = -1 + 2 = 1$$

Since this determinant is non-zero, the Wronskian test confirms the linear independence of f_1, f_2, f_3.

A second method of solving this problem just uses the *definition of linear independence.*

Suppose that c_1, c_2, c_3 are constants and that

$$c_1 e^t \cos(t) + c_2 e^t \sin(t) + c_2 e^t = 0.$$

To show that the functions are linearly independent, we must *argue* that $c_1 = c_2 = c_3 = 0$.

Factoring out the quantity e^t, our assumption shows that

$$e^t(c_1 \cos(t) + c_2 \sin(t) + c_3) = 0.$$

Since $e^t \neq 0$ for all t, we find that

$$c_1 \cos(t) + c_2 \sin(t) + c_3 = 0.$$

Now, since this equation holds for all times t, we may choose some particular values of t to find equations for the constants c_i.

When $t = 0$, we find that

$$0 = c_1 \cos(0) + c_2 \sin(0) + c_3 = c_1 + c_3.$$

When $t = \pi/2$, we find that

$$0 = c_1 \cos(\pi/2) + c_2 \sin(\pi/2) + c_3 = c_2 + c_3.$$

When $t = \pi$, we find that

$$0 = c_1 \cos(\pi) + c_2 \sin(\pi) + c_3 = -c_1 + c_3.$$

Now, we solve the system of equations

$$\begin{aligned} 0 &= c_1 + c_3 \\ 0 &= c_2 + c_3 \\ 0 &= -c_1 + c_3 \end{aligned}$$

Adding the first and third equation gives $0 = 2c_3$ so that $c_3 = 0$. Now the first equation shows that $c_1 = 0$ and the second shows that $c_2 = 0$.

Since we have argued that $c_1 = c_2 = c_3 = 0$, we conclude from the definition that f_1, f_2, f_3 are linearly independent.

20. Find the Laplace transform of the function

$$f(t) = \begin{cases} 1 & \text{for } t < 1, \\ t-1 & \text{for } 1 \le t < 2, \\ 1 & \text{for } t \ge 2. \end{cases}$$

Solution:

In order to be able to compute the Laplace transform, We first rewrite the function $f(t)$ using the *unit step functions.*

We have

$$\begin{aligned} f(t) &= 1 + u_1(t)\cdot(-1+(t-1)) + u_2(t)\cdot((-(t-1)+1) \\ &= 1 + u_1(t)\cdot(t-2) + u_2(t)\cdot(-t+2). \end{aligned}$$

Thus

$$\begin{aligned} \mathscr{L}[f(t)] &= \mathscr{L}[1 + u_1(t)\cdot(t-2) + u_2(t)\cdot(-t+2)] \\ &= \mathscr{L}[1] + \mathscr{L}[u_1(t)\cdot(t-2)] + \mathscr{L}[u_2(t)\cdot(-t+2)] \\ &= \mathscr{L}[1] + e^{-s}\mathscr{L}[(t+1)-2] + e^{-2s}\mathscr{L}[-(t+2)+2] \\ &= \mathscr{L}[1] + e^{-s}\mathscr{L}[t-1] + e^{-2s}\mathscr{L}[-t] \\ &= (1-e^{-s})\mathscr{L}[1] + e^{-s}\mathscr{L}[t] - e^{-2s}\mathscr{L}[t] \\ &= (1-e^{-s})\mathscr{L}[1] + (e^{-s}-e^{-2s})\mathscr{L}[t] \\ &= \frac{1-e^{-s}}{s} + \frac{e^{-s}-e^{-2s}}{s^2} \end{aligned}$$

21. Suppose $g(t)$ is the inverse Laplace transform of

$$F(s) = \frac{2se^{\pi s/2}}{(s^2+4)}.$$

Find $g\left(\frac{\pi}{4}\right)$.

Solution:

We use the *second shift formula* to find $g(t)$. Notice that if we set

$$f(t) = \mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos(2t)$$

then the second shift formula yields

$$\begin{aligned} g(t) = \mathscr{L}^{-1}[F(s)] &= 2\mathscr{L}^{-1}\left[e^{(\pi/2)s}\frac{s}{s^2+4}\right] \\ &= 2u_{\pi/2}(t)f(t-\pi/2) \end{aligned}$$

Thus $u_{\pi/2}(\pi/4) = 0$ so that $g(\pi/4) = 2u_{\pi/2}(\pi/4)\cdot f(\pi/4-\pi/2) = 0$.

15