Worksheet for week 10 - some solutions

Math 51 Fall 2021

2022-04-01

Concerning problem 6.

Write $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & B^t \end{bmatrix}$ where

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and B^t is the *transpose* of B.

Since A is *block upper triangular*, the char. poly of A is given by

$$\det(A - \lambda I) = \det \begin{bmatrix} B - \lambda I & C \\ 0 & B^t - \lambda I \end{bmatrix} = \det(B - \lambda I)\det(B^t - \lambda I) = \det(B - \lambda I)^2.$$

(This last equality holds because $\det M = \det M^t$ for any square matrix M and because $(M - \lambda I)^t = M^t - \lambda I$).

Now, $\det(B - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2.$

Thus the characteristic polynomial of A is $(\lambda^2 - 2\lambda + 2)^2$. In particular, using the quadratic formula we see that A has eigenvalues $1 + i$ (with multiplicity 2) and $1 - i$ (with multiplicity 2).

If we now try to find eigenvectors, we will find that there is only one linearly independent $1 + i$ eigenvector, and only 1 linearly independent $1-i$ eigenvector. So to find the general solution $Dx = Ax$ we find generalized eigenvectors.

Thus we need to find solutions to $(A - (1+i))^2 x = 0$. We start by computing $(A - (1+i))^2$ and putting the result in echelon form using row operations:

$$(A-(1+i))^2 = \begin{bmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 1 & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix} \cdot \begin{bmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 1 & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix} = \begin{bmatrix} -2 & -2i & 1 & 0 \\ 2i & -2 & -2i & -1 \\ 0 & 0 & -2 & 2i \\ 0 & 0 & -2i & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 2i & -1 & 0 \\ 2 & 2i & -2 & i \\ 0 & 0 & 2 & -2i \\ 0 & 0 & 2 & -2i \end{bmatrix} \sim \begin{bmatrix} 2 & 2i & -1 & 0 \\ 0 & 0 & -1 & i \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 2i & -1 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2i & 0 & -i \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We find now that solutions to $(A-(1+i))^2x=0$ are generated by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} i/2 \\ 0 \\ i \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

– i.e. $\mathbf{v}_1$ and $\mathbf{v}_2$ generate the generalized eigenvectors for $\lambda = 1+i$.

Let's compute

$$(A-(1+i))\mathbf{v}_1 = \begin{bmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 1 & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix} \begin{bmatrix} i/2 \\ 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ i/2 \\ 0 \\ 0 \end{bmatrix}$$

and

$$(A-(1+i))\mathbf{v}_2 = \begin{bmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 1 & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix} \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $\mathbf{v}_2$ is an *actual* eigenvector.

We now find complex solutions

$$\mathbf{h}_1(t) = e^{(1+i)t}\left(\begin{bmatrix} i/2 \\ 0 \\ i \\ 1 \end{bmatrix} + t\begin{bmatrix} 1/2 \\ i/2 \\ 0 \\ 0 \end{bmatrix}\right) \quad = \frac{e^{(1+i)t}}{2}\cdot\begin{bmatrix} i+t \\ it \\ 2i \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{h}_2(t) = e^{(1+i)t}\begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

To find the general solution to $Dx = Ax$, it remains to find the *real* and *imaginary* parts of $\mathbf{h}_1(t)$ and of $\mathbf{h}_2(t)$.

We have

$$\mathbf{h}_1(t) = \frac{e^{(1+i)t}}{2}\cdot\begin{bmatrix} i+t \\ it \\ 2i \\ 2 \end{bmatrix} = \frac{e^t}{2}(\cos(t)+i\sin(t))\begin{bmatrix} i+t \\ it \\ 2i \\ 2 \end{bmatrix} = \frac{e^t}{2}\begin{bmatrix} -\sin(t)+i\cos(t)+t(\cos(t)+i\sin(t)) \\ t(-\sin(t)+i\cos(t)) \\ 2(-\sin(t)+i\cos(t)) \\ 2(\cos(t)+i\sin(t)) \end{bmatrix}$$
$$= \frac{e^t}{2}\begin{bmatrix} -\sin(t)+t\cos(t) \\ -t\sin(t) \\ -2\sin(t) \\ 2\cos(t) \end{bmatrix} + \frac{ie^t}{2}\begin{bmatrix} \cos(t)+t\sin(t) \\ t\cos(t) \\ 2\cos(t) \\ 2\sin(t) \end{bmatrix}$$

$$\mathbf{h}_2(t) = e^{(1+i)t}\begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} = e^t(\cos(t)+i\sin(t))\begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} = e^t\begin{bmatrix} \sin(t)-i\cos(t) \\ \cos(t)+i\sin(t) \\ 0 \\ 0 \end{bmatrix} = e^t\begin{bmatrix} \sin(t) \\ \cos(t) \\ 0 \\ 0 \end{bmatrix} + ie^t\begin{bmatrix} -\cos(t) \\ \sin(t) \\ 0 \\ 0 \end{bmatrix}$$

Thus the general solution to $Dx = Ax$ is given by

$$c_1e^t\begin{bmatrix} -\sin(t)+t\cos(t) \\ -t\sin(t) \\ -2\sin(t) \\ 2\cos(t) \end{bmatrix} + c_2e^t\begin{bmatrix} \cos(t)+t\sin(t) \\ t\cos(t) \\ 2\cos(t) \\ 2\sin(t) \end{bmatrix} + c_3e^t\begin{bmatrix} \sin(t) \\ \cos(t) \\ 0 \\ 0 \end{bmatrix} + c_4e^t\begin{bmatrix} -\cos(t) \\ \sin(t) \\ 0 \\ 0 \end{bmatrix}$$