MATH 51: HOMEWORK 4 SOLUTION

Homework exercise solution:

1. (a).

$$c_1 + c_2 t + c_3 t^2 = 0$$

Taking derivative on both sides, we have

$$c_2 + 2c_3 t = 0$$

Taking derivative again, we have

$$c_3 = 0$$

which also gives $c_2 = c_1 = 0$, therefore linearly independent.

(b). Notice that

$$\sin^2(t) = \frac{1-\cos 2t}{2}, \quad \cos^2(t) = \frac{1-\cos 2t}{2}$$

We have

$$c_1 \frac{1-\cos 2t}{2} + c_2 \frac{1+\cos 2t}{2} + c_3 \cos 2t = 0$$

Consider $c_1 = -1, c_2 = 1, c_3 = -1$, the functions are not linearly independent.

(c).

$$c_1 e^t + c_2 t e^t + c_3 e^{2t} + c_4 t e^{2t} = 0$$

Divide both sides by e^t, we have

$$c_1 + c_2 t + c_3 e^t + c_4 t e^t = 0$$

Taking derivative on both sides, we have

$$c_2 + c_3 e^t + c_4 e^t + c_4 t e^t = 0$$

Taking derivative on both sides, we have

$$c_3 e^t + 2c_4 e^t + c_4 t e^t = 0$$

Divide both sides by e^t, we have

$$c_3 + 2c_4 + c_4 t = 0$$

Taking derivative on both sides, we have

$$c_4 = 0$$

Plug back $c_4 = 0$, we'll have $c_1 = c_2 = c_3 = 0$, hence the functions are linearly independent.
(d).

$$c_1 e^{t^5} + c_2 e^{|t^5|} = 0$$

Consider $t = 0$, we have $c_1 + c_2 = 0$. Consider $t = -1$, then we have

$$c_1 e^{-1} + c_2 e = 0; \quad c_1 + c_2 e^2 = 0$$

which implies $c_1 = c_2 = 0$, the functions are linearly independent.

2. Taking derivative on $c_1 h_1 + \cdots + c_n h_n = 0$ for $n+1$ times we will have the only term remains is $n! c_n = 0$, which implies $c_n = 0$. After we have $c_n = 0$, plug in the $n-$derivative of $c_1 h_1 + \cdots c_n h_n$ would imply $c_{n-1} = 0$. Repeat this till we find $c_1 = 0$, which implies the polynomials are linearly independent.

3. Assume $h_1(t), ..., h_n(t)$ are linearly dependent, it implies that if

$$c_1 h_1 + c_n h_n = 0$$

then there exists some c_j, $1 \leq j \leq n$ such that $c_j \neq 0$. Therefore we have

$$c_j h_j = -\sum_{i \neq j}^{n} c_i h_i$$

taking derivative on both sides, we have

$$c_j h_j' = -\sum_{i \neq j}^{n} c_i h_i'$$

which is equivalent to

$$c_1 h_1' + \cdots c_n h_n' = 0$$

with at least one c nonzero, hence $h_1', ..., h_n'$ are linearly dependent as well.

4. (a).

$$P(r) = r^2 - 2r - 15 = 0; \quad r = -3, 5$$

The general solution is

$$x(t) = c_1 e^{-3t} + c_2 e^{5t}$$

(b).

$$P(r) = (r+3)^2 (r-5)^3$$

One root is -3 with multiplicity of 2 and one root is 5 with multiplicity of 3, hence the general solution is

$$x(t) = c_1 e^{-3t} + c_2 t e^{-3t} + c_3 e^{5t} + c_4 t e^{5t} + c_5 t^2 e^{5t}$$

5.

$$P(r) = r^3 + 3r^2 + 3r + 1 = 0$$

The root is -1 with multiplicity of 3, hence the general solution is

$$x(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t}$$
$$x'(t) = -c_1 e^{-t} - c_2 t e^{-t} + c_2 e^{-t} + 2c_3 t e^{-t} - c_3 t^2 e^{-t}$$
$$x''(t) = c_1 e^{-t} + c_2 t e^{-t} - c_2 e^{-t} + 2c_3 e^{-t} - 2c_3 t e^{-t} - 2c_3 t e^{-t} + c_3 t^2 e^{-t}$$
$$x(0) = c_1 = 0$$
$$x'(0) = -c_1 + c_2 = 0$$
$$x''(0) = c_1 - c_2 + 2c_3 = 0$$

Therefore, we have $c_1 = c_2 = c_3 = 0$, hence $x = 0$ is the solution.

6. (a).

$$Lf_1 = e^{2t}[(D+2)^2 + 3(D+2) - 2]1 = e^{2t}[D^2 + 7D + 8]1 = 8e^{2t}$$
$$Lf_2 = e^{2t}[D^2 + 7D + 8]t^3 = e^{2t}(8t^3 + 21t^2 + 6t)$$

(b).

$$Lf_1 = e^{-2t}[(D-2)^2 + 4(D-2) + 5](t \cos t) = e^{-2t}[D^2 + 1](t \cos t) = e^{-2t}(2 \sin t)$$

(End of Homework 2)