

Problem Set 7

Linear Systems of ODES; independence

Math 51 Fall 2021

due Monday 2022-03-07 at 11:59 PM

These problems relate to material of (Nitecki and Guterman 1992, secs. 3.2, 3.3, 3.4).

Reading for the Week of 2022-02-28

- §3.2: Linear Systems, Matrices, and Vectors
- §3.3: Linear Systems of ODES: general properties
- §3.4: Linear independence of vectors

Problems

1. For each of the following systems of ODEs, decide whether it is linear. For each linear system, do also the following:
 - indicate whether it is homogeneous
 - find a matrix A and a vector E such that the system can be rewritten in the form

$$D\mathbf{x} = A\mathbf{x} + E$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (or $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$).

$$(a) \quad \begin{cases} x' = ty - z \\ y' = -\frac{x}{t} - z + 1 \\ z' = -x - t^2y + z + 2t \end{cases} \quad (b) \quad \begin{cases} x' = 2x - 3y \\ y' = 3x^2y + y + 1 \end{cases} \quad (c) \quad \begin{cases} x' = 7x + 11y \\ y' = -2x + y \end{cases}$$

2. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and consider the non-homogeneous system

$$(\clubsuit) \quad D \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix}.$$

- a. Show that $\mathbf{h}_1(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$, $\mathbf{h}_2(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$ are solutions to the corresponding homogeneous system $D \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$.
- b. Show that $\mathbf{p}(t) = \begin{bmatrix} 0 \\ -t \end{bmatrix}$ is a particular solution to the (\clubsuit) .

- c. Show that the initial vectors $h_1(0)$ and $h_2(0)$ are linearly independent. Find the general solution to (♣).

3. Consider the linear ODE

$$(N) \quad (D-3)^2x = e^{3t} \quad \text{i.e.} \quad (D^2 - 6D + 9)x = e^{3t}.$$

- Find the equivalent linear system (S_N) of ODEs. Write this system in matrix form.
 - Note that the general solution to the homogeneous equation (H) $(D-3)^2x = 0$ is generated by $h_1(t) = e^{3t}$ and $h_2(t) = te^{3t}$. Find the corresponding vector solutions h_1 and h_2 to the homogeneous system (S_H).
 - Find a particular solution $p(t)$ to the equation $(D-3)^2x = e^{3t}$, and find the corresponding vector solution $p(t)$ to the system (S_N).
 - The general solution to (N) is given by $x(t) = p(t) + c_1h_1(t) + c_2h_2(t)$. What is the general solution to the system (S_N)?
4. Consider the the following matrices A and lists of vector-valued functions h_i . In each case, answer the following questions:
- Which of the functions h_i are solutions to the homogeneous equation $Dx = Ax$? Be sure to indicate how you reach your conclusion.
 - Consider the functions that are solutions. Do they generate the general solution to $Dx = Ax$? Why or why not?

a. $A = \begin{bmatrix} -3 & 8 \\ -3 & 7 \end{bmatrix}$; $h_1 = \begin{bmatrix} 2e^t \\ e^t \end{bmatrix}$, $h_2 = \begin{bmatrix} 2e^t - 4e^{3t} \\ e^t - 3e^{3t} \end{bmatrix}$, $h_3 = e^t \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

b. $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$; $h_1 = e^t \begin{bmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \\ 0 \end{bmatrix}$, $h_2 = e^t \begin{bmatrix} 2\cos(t) + 2\sin(t) \\ 4\cos(t) \\ e^{-3t} \end{bmatrix}$, $h_3 = \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \end{bmatrix}$.

5. Let

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ -5e^t \end{bmatrix}.$$

The formulas

$$(\clubsuit) \quad \begin{cases} x_1 = c_1 \cos(2t) + c_2 \sin(2t) + e^t \\ x_2 = -2c_2 \cos(2t) + 2c_1 \sin(2t) - e^t \end{cases}$$

describe a collection of solutions to the nonhomogeneous system $Dx = Ax + E$.

- Write the collection (♣) of solutions in the form $x = c_1h_1 + c_2h_2 + p$ where h_1 and h_2 are solutions to the homogeneous system $Dx = Ax$.
 - Decide whether the collection (♣) is complete.
6. Check the following list of vectors for linear independence:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. *Differential Equations: A First Course*. Saunders.