1. **(Laplace transform from the definition)**
Let $f(t) = te^{2t}$. Calculate the Laplace transform $F(s) = \mathcal{L}[f(t)]$ directly from the definition and indicate the values of s for which the integral defining $F(s)$ converges.

Solution. From the definition,

$$F(s) = \int_0^\infty t e^{2t} e^{-st}\, dt$$

$$= \int_0^\infty \underbrace{t}_{u}\, \underbrace{e^{-(s-2)t}\, dt}_{dv} \qquad \left(\text{Set } u = t,\ v = -\frac{1}{s-2} e^{-(s-2)t}\right)$$

$$= \underbrace{t}_{u} \underbrace{\left(-\frac{1}{s-2}\right) e^{-(s-2)t}}_{v}\Big]_0^\infty + \underbrace{\int_0^\infty \frac{1}{s-2} e^{-(s-2)t}}_{v}\, \underbrace{dt}_{du}$$

(Integration by parts)

$$= -\frac{1}{(s-2)^2} e^{-(s-2)t}\Big]_0^\infty = -\frac{1}{(s-2)^2}(0-1)$$

$$= \frac{1}{(s-2)^2}, \quad s > 2.$$

(Need $s > 0$ so that the exponential $e^{-(s-2)t}$ vanishes at ∞.) □

2. **(Laplace transform)**
For each of the following functions, calculate its Laplace transform $F(s) = \mathcal{L}[f(t)]$ using the linearity of $\mathcal{L}$ together with the basic formulas summarized at the end of §5.2.

(a) $f(t) = 2t + e^{-4t} - 3\cos 5t$.

(b) $f(t) = e^{3t+2}$.

(c) $f(t) = (t+2)(t+3)$.

(a) $\mathcal{L}[2t + e^{-4t} - 3\cos 5t] = \boxed{\dfrac{2}{s^2} + \dfrac{1}{s+4} - \dfrac{3s}{s^2+25}}$.

(b) $\mathcal{L}[e^{3t} e^2] = e^2 \mathcal{L}[e^{3t}] = \boxed{\dfrac{e^2}{s-3}}$.

(c) $\mathcal{L}[t^2 + 5t + 6] = \dfrac{2}{s^3} + \dfrac{5}{s^2} + \dfrac{6}{s}$.

3. **(Inverse transform)**
For each of the following functions, calculate its inverse transform $f(t) = \mathcal{L}^{-1}[F(s)]$ using the linearity of $\mathcal{L}^{-1}$ together with the basic formulas summarized at the end of §5.2.

(a) $F(s) = \dfrac{1}{3s+1}$.

(b) $F(s) = \dfrac{2}{s^2+4} - \dfrac{10}{s^4} + \dfrac{1}{s}$.

(a) $\mathcal{L}^{-1}\left[\frac{1}{3s+1}\right] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s+\frac{1}{3}}\right] = \boxed{\frac{1}{3}e^{-\frac{1}{3}t}}$

(b) $\mathcal{L}^{-1}\left[\frac{2}{s^2+4} - \frac{10}{s^4} + \frac{1}{s}\right]$

$\mathcal{L}[\sin 2t] = \frac{2}{s^2+4}$,

$\mathcal{L}[t^3] = \frac{6}{s^4} \Rightarrow \mathcal{L}\left[-\frac{5}{3}t^3\right] = -\frac{5}{3}\frac{6}{s^4} = -\frac{10}{s^4}$.

$\mathcal{L}[1] = \frac{1}{s}$.

Hence,

$\mathcal{L}^{-1}\left[\frac{2}{s^2+4} - \frac{10}{s^4} + \frac{1}{s}\right] = \boxed{\sin 2t - \frac{5}{3}t^3 + 1}$

4. **(First differentiation formula)**
Use the first differentiation formula to find an expression for the Laplace transform $\mathcal{L}[x]$, where x is the solution of the given initial-value problem.

(a) $(D-1)x = e^{2t}, \quad x(0) = 2.$

(b) $(D^2-1)x = e^{2t}, \quad x(0) = 0, \ x'(0) = 1.$

(c) $(D^2+1)x = \cos 3t, \quad x(0) = 0, \ x'(0) = 3.$

(a) $\mathcal{L}[Dx - x] = s\mathcal{L}[x] - x(0) - \mathcal{L}[x] = \mathcal{L}[e^{2t}] = \frac{1}{s-2}$

$$(s-1)\mathcal{L}[x] = \frac{1}{s-2} + x(0) = \frac{1}{s-2} + 2$$

$$\boxed{\mathcal{L}[x] = \frac{1}{(s-1)(s-2)} + \frac{2}{s-1}}$$

(b) $\mathcal{L}[D^2x - x] = s^2\mathcal{L}[x] - s\,x(0) - x'(0) - \mathcal{L}[x]$

$$= \mathcal{L}[e^{2t}] = \frac{1}{s-2}$$

$$(s^2-1)\mathcal{L}[x] = \frac{1}{s-2} + s\,x(0) + x'(0) = \frac{1}{s-2} + 1.$$

$$\mathcal{L}[x] = \boxed{\frac{1}{(s^2-1)(s-2)} + \frac{1}{s^2-1}}$$

(c) $\mathcal{L}[D^2x + x] = s^2\mathcal{L}[x] - s\,x(0) - x'(0) + \mathcal{L}[x]$

$$= \mathcal{L}[\cos 3t] = \frac{s}{s^2+9}$$

$$(s^2+1)\mathcal{L}[x] = \frac{s}{s^2+9} + s\,x(0) + x'(0)$$

$$= \frac{s}{s^2+9} + 3.$$

$$\mathcal{L}[x] = \boxed{\frac{s}{(s^2+1)(s^2+9)} + \frac{3}{s^2+1}}.$$

5. **(Partial fraction decompostion)**

Find the inverse transform of $F(s) = \dfrac{s+4}{s^2+4s+3}$.

$$\frac{s+4}{s^2+4s+3} = \frac{s+4}{(s+1)(s+3)} = \frac{a}{s+1} + \frac{b}{s+3}$$

Clear the denominator by multiplying by $(s+1)(s+3)$:

$$s+4 = a(s+3) + b(s+1)$$

$$= (a+b)s + (3a+b)$$

Equating coefficients,

$$\begin{cases} a+b=1 & (1) \\ 3a+b=4 & (2) \end{cases}$$

$(2)-(1)$: $\quad 2a = 3 \Rightarrow a = \frac{3}{2}$

Plug into (1): $\quad \frac{3}{2} + b = 1 \Rightarrow b = -\frac{1}{2}$

Thus, $\quad F(s) = \boxed{\dfrac{\frac{3}{2}}{s+1} + \dfrac{-\frac{1}{2}}{s+3}}$.

6. **(Initial-value problem)**
Use the Laplace tranform to solve the initial-value problem:

$$(D^2+4)x = t, \quad x(0) = -1, \; x'(0) = 0.$$

Take the Laplace transform of both sides:

$$\mathcal{L}[D^2x] + \mathcal{L}[4x] = \mathcal{L}[t]$$

$$s^2\mathcal{L}[x] - s x(0) - x'(0) + 4\mathcal{L}[x] = \frac{1}{s^2}.$$

$$(s^2+4)\mathcal{L}[x] = \frac{1}{s^2} + s x(0) + x'(0)$$

$$= \frac{1}{s^2} - s$$

$$\mathcal{L}[x] = \frac{1}{s^2(s^2+4)} - \frac{s}{s^2+4}$$

Partial fraction decomposition:

$$\frac{1}{s^2(s^2+4)} = \frac{a}{s} + \frac{b}{s^2} + \frac{cs+d}{s^2+4}$$

Multiply by s^2:

$$\frac{1}{s^2+4} = as + b + \left(\frac{cs+d}{s^2+4}\right)s^2.$$

Plug in $s=0$: $b = 1/4$

Clear the denominator by multiplying by $s^2(s^2+4)$:

$$1 = as(s^2+4) + \frac{1}{4}(s^2+4) + (cs+d)s^2$$

$$= (a+c)s^3 + \left(\frac{1}{4}+d\right)s^2 + 4as + 1,$$

Equating coefficients gives

$$a+c=0 \quad \Rightarrow c=0$$

$$d+\frac{1}{4}=0 \Rightarrow d=-\frac{1}{4}.$$

$$4a=0 \Rightarrow a=0$$

Thus, $$\frac{1}{s^2(s^2+4)} = \frac{1}{4s} - \frac{1}{4(s^2+4)},$$

$$\mathcal{L}[x] = \frac{1/4}{s} - \frac{1/4}{s^2+4} - \frac{s}{s^2+4}$$

$$x = \boxed{\frac{1}{4} - \frac{1}{8}\sin 2t - \cos 2t}.$$