Math 51

Laplace Transforms
3. More Tools

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There are a few more tools for calculating Laplace transforms of the basic functions which arise in the context of constant-coefficient o.d.e.'s.

Exponential Factors

We already know the Laplace transform of exponential functions like e^{at}. However, complex characteristic roots, as well as real roots of higher multiplicity, involve such functions as factors in more involved expressions. These can be handled via the following calculation.

Suppose we know how to transform a function $f(t)$, but need to transform $e^{at}f(t)$. By definition

$$\mathcal{L}\left[e^{at}f(t)\right] = \int_0^\infty e^{-st}e^{at}f(t)\ dt = \int_0^\infty e^{-(s-a)t}f(t)\ dt$$

which is the formula for $\mathcal{L}[f(t)]$, except that s has been replaced by $s-a$. So we can write

$$\boxed{\textbf{First Shift Formula}: \text{ If } \mathcal{L}[f(t)] = F(s), \text{ then } \mathcal{L}\left[e^{at}f(t)\right] = F(s-a).}$$

Note that this includes the formula for $\mathcal{L}\left[e^{at}\right]$ which we found earlier, by letting $f(t) = 1$.

A more useful application is

Example: $\mathcal{L}\left[t^3e^{2t}\right]$:

Since $F(s) = \mathcal{L}\left[t^3\right] = \frac{6}{s^4}$,

$$\mathcal{L}\left[t^3e^{2t}\right] = F(s-2) = \frac{6}{(s-2)^4}.$$

Example: $\mathcal{L}\left[e^{-2t}\sin 3t\right]$:

Since $\mathcal{L}[\sin 3t] = \frac{3}{s^2+9}$,

$$\mathcal{L}\left[e^{-2t}\sin 3t\right] = \frac{3}{(s-2)^2+9} = \frac{3}{s^2-4s+13}$$

First Shift Formula for Inverse Transforms

We can reverse-engineer the First Shift Formula to detect an exponential factor in $f(t)$ when we know $F(s)$. Suppose we recognize $F(s)$ as the formula for a known transform, except that $s-a$ has been substituted for s. We can undo this by replacing s in the formula for $F(s)$ with $s+a$:

First Shift Formula for $\mathcal{L}^{-1}$:
$\mathcal{L}^{-1}[F(s)] = e^{at}\mathcal{L}^{-1}[F(s+a)]$.

Example: $\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2}\right]$:

$\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2}\right] = 3e^{-t}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = 3te^{-t}$.

Example: $\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2+9}\right]$:

$\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2+9}\right] = e^{-t}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] = e^{-t}\sin 3t$.

Example: Solve the initial-value problem
$(D^2-4D+5)x + 2(t+1)e^t, \quad x(0) = x'(0) = 0.$

Taking the Laplace transform of both sides, we have
$(s^2-4s+5)\mathcal{L}[x] = \frac{2}{(s-1)^2} + \frac{2}{s-1} = \frac{2s}{(s-1)^2}$, or
$\mathcal{L}[x] = \frac{2s}{(s-1)^2(s^2-4s+5)}$.
The partial fraction decomposition
$\frac{2s}{(s-1)^2(s^2-4s+5)} = \frac{A}{(s-1)^2} + \frac{B}{(s-1)} + \frac{Cs+E}{s^2-4s+5}$ leads to the equations

$$\begin{array}{rrrrl} & B & +C & & =0 \\ A & -5B & -2C & +E & =0 \\ -4A & +9B & +C & -2E & =2 \\ 5A & -5B & & +E & =0. \end{array}$$

Reducing the augmented matrix of this system

$$\left[\begin{array}{rrrr|r} 0 & 1 & 1 & 0 & 0 \\ 1 & -5 & -2 & 1 & 0 \\ -4 & 9 & 1 & -2 & 2 \\ 5 & -5 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$\mathcal{L}[x] = \frac{2}{(s-1)^2} + \frac{2}{(s-1)} + \frac{-2s+5}{s^2-4s+5}$
so
$x = \mathcal{L}^{-1}\left[\frac{2}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s-1)}\right] + \mathcal{L}^{-1}\left[\frac{-2s+5}{s^2-4s+5}\right]$.
The first two inverse transforms are

$$\mathcal{L}^{-1}\left[\frac{2}{(s-1)^2}\right] = e^t \mathcal{L}^{-1}\left[\frac{2}{s^2}\right] = 2te^t$$
$$\mathcal{L}^{-1}\left[\frac{2}{s-1}\right] = e^t \mathcal{L}^{-1}\left[\frac{2}{s}\right] = 2e^t$$

while the third is

$$\mathcal{L}^{-1}\left[\frac{-2s+5}{(s-2)^2+1}\right]$$
$$= e^{2t}\mathcal{L}^{-1}\left[\frac{-2(s+2)+5}{s^2+1}\right] = e^{2t}\mathcal{L}^{-1}\left[\frac{-2s+1}{s^2+1}\right]$$
$$= -2e^{2t}\cos t + e^{2t}\sin t$$

so our solution is
$x(t) = (2t+2)e^t + e^{2t}(-2\cos t + \sin t)$.

Factors of t^k:
We know that roots of higher multiplicity lead to our basic solution functions multiplied by powers of t, so it is natural to try to understand the effect of such multipliers on Laplace transforms. This turns out to be related to the derivative of the transform of a function: by defintion

$$\frac{d}{ds}\mathcal{L}[f(t)] = F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\ dt;$$

in the context that we are working in, the derivative can be taken inside the integral sign:

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)\ dt$$
$$= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right]\ dt$$
$$= \int_0^\infty -te^{-st} f(t)\ dt$$

In other words,

$$\frac{d}{ds}\mathcal{L}\left[f(t)\right] = \mathcal{L}\left[(-t)f(t)\right].$$

Taking successive derivatives of this, we obtain the relation

$$\frac{d^k}{ds^k}\left[\mathcal{L}\left[f(t)\right]\right] = \mathcal{L}\left[(-t)^k f(t)\right].$$

It is more useful in the form

Second Differentiation Formula:
$\mathcal{L}\left[t^k f(t)\right] = (-1)^k \frac{d^k}{ds^k}\left[\mathcal{L}\left[f(t)\right]\right].$

Example:

$$\begin{aligned}
&\mathcal{L}\left[t^2 \sin 3t\right] \\
&\quad = (-1)^2 \frac{d^2}{ds^2}\left[\frac{3}{s^2+9}\right] = \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right] \\
&\quad = \frac{18(s^2-3)}{(s^2+9)^3}.
\end{aligned}$$

Example: $\mathcal{L}\left[te^{3t}\sin 2t\right]$:
We can parse this into two steps:

1. By the First Differentiation Formula, the exponential factor e^{3t} will result in a substitution:
   if $F(s) = \mathcal{L}\left[t\sin 2t\right]$, then
   $\mathcal{L}\left[te^{3t}\sin 2t\right] = F(s-3)$.

2. By the Second Differentiation Formula,

$$\begin{aligned}
F(s) = \mathcal{L}\left[t\sin 2t\right] &= -\frac{d}{ds}\mathcal{L}\left[\sin 2t\right] \\
&= -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] \\
&= \frac{4s}{(s^2+4)^2}
\end{aligned}$$

So

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$$\begin{aligned}\mathcal{L}\left[te^{3t}\sin 2t\right] &= F(s-3) \\ &= \frac{4(s-3)}{[(s-3)^2+4]^2} \\ &= \frac{4s-12}{(s^2-6s+13)^2}.\end{aligned}$$

Implication for Inverse Transforms: The straighforward translation of the Second Differentiation Formula to a version for the inverse Laplace Transform is

Second Differentiation Formula Inverse Version:

$$\mathcal{L}^{-1}\left[\frac{d^k}{ds^k}\left[F(s)\right]\right] = (-1)^k t^k \mathcal{L}^{-1}\left[F(s)\right].$$

It is a bit awkward to see how to "solve" this equation for $F(s)$ if we are simply given an expression in s inside the inverse Laplace transform on the left. A more efficient way to handle such problems is via convolution, explored in §5.6 of the book (which we aren't covering in this course).

However, this version (especially with $k = 1$) can guide us toward finding a solution by means of an appropriate integration–or more likely, by recognizing our expression as the derivative of another, more familiar one. We illustrate with one example.

Example: Solve the initial-value problem

$$(D^2 - 2D + 2)x = e^t \cos t, \quad x(0) = x'(0) = 0.$$

The Laplace transform of both sides (using the First Shift Formula on the right side) reads

$$(s^2 - 2s + 2)\mathcal{L}\left[x\right] = \frac{s-1}{(s-1)^2+1}.$$

Noting that the factor on the left is also $(s-1)^2 + 1$, we see that we need to solve

$$x(t) = \mathcal{L}^{-1}\left[\frac{s-1}{\{(s-1)^2+1\}^2}\right].$$

We can simplify the problem by means of the First Shift Formula:

$$\mathcal{L}^{-1}\left[\frac{s-1}{\{(s-1)^2+1\}^2}\right] = e^t\mathcal{L}^{-1}\left[\frac{s}{\{s^2+1\}^2}\right].$$

We note that the fraction inside the inverse Laplace transform on the right is easy to integrate:

$$\int \frac{s}{\{s^2+1\}^2}ds = \int \frac{\frac{1}{2}d(s^2+1)}{(s^2+1)^2} = -\frac{1}{2}\frac{1}{s^2+1}$$

(This is integration by substitution with u = s^2 +1)

or

$$\frac{s}{(s^2+1)^2} = -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

from which, by the inverse version of the Second Differentiation Formula, we can conclude that

$$\mathcal{L}^{-1}\left[\frac{s}{\{s^2+1\}^2}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[-\frac{d}{ds}\left[\frac{s}{s^2+1}\right]\right] = \frac{t}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \frac{t}{2}\sin t.$$

Applying the First Shift Formula, we find that the solution to our initial-value problem is

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left[\frac{s-1}{\{(s-1)^2+1\}^2}\right] \\ &= e^t\mathcal{L}^{-1}\left[\frac{s}{\{s^2+1\}^2}\right] \\ &= \frac{t}{2}e^t\sin t. \end{aligned}$$