Readings for the Week of March 28, 2022

§3.6–10: Homogeneous linear systems (real, complex, double, multiple roots)

Problem Set 10 SOLUTIONS
(Due Monday, April 4, 2022, at 11:59pm.)

For a 10% penalty on your grade, you may hand in the problem set late, until Tuesday, April 5, 2022, 11:59 pm.

1. Let $A = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 1 & 3 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$, and $F = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.
Find the indicated matrices or explain that this cannot be done.

(a) $AB = \begin{bmatrix} 0 & 3 \\ 3 & 0 \\ 0 & 3 \\ 0 & 3 \end{bmatrix}$; (b) BA—size mismatch; (c) $AC = \begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 3 & 0 \\ 3 & 3 \end{bmatrix}$; (d) $A(B-C) = \begin{bmatrix} -3 & 3 \\ 3 & 0 \\ -3 & 3 \\ -3 & 0 \end{bmatrix}$;

(e) AF—size mismatch; (f) $BF = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$; (g) $FB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a generalized eigenvector of $\begin{bmatrix} -1 & 1 & 4 \\ -2 & 2 & 4 \\ -1 & 0 & 4 \end{bmatrix}$ for the eigenvalue 2.
Find the associated solution of $D\vec{x} = A\vec{x}$.

$$e^{2t} \left[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right]$$

3. Find the general solution of $D\vec{x} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \vec{x}$.

$\lambda = 2$ is a double eigenvalue, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are generalized eigenvectors:

$$c_1 e^{2t} \begin{bmatrix} 1-t \\ t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -t \\ 1+t \end{bmatrix}.$$

4. Find the general solution of $D\vec{x} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$.

Simple eigenvalue -1 with eigenvector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, double eigenvalue 1 with (obvious!!) eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and generalized eigenvector $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$: $c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} -t \\ -1 \\ 2 \end{bmatrix}$.

5. Find the general solution of $D\vec{x} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \vec{x}$.

$$c_1 e^t \begin{bmatrix} -\sin t \\ 2\sin t \\ \sin t + \cos t \\ 2\cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ -2\cos t \\ -(\sin t + \cos t) \\ 2\sin t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_4 e^{2t} \begin{bmatrix} t \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

6. Find the general solution of $\begin{aligned} x_1' &= 3x_1 - x_2 - 4x_4 \\ x_2' &= 3x_2 - 4x_4 \\ x_3' &= 2x_3 \\ x_4' &= x_2 - x_4 \end{aligned}$ as well as the specific solution satisfying $x_1(0) = x_2(0) = x_3(0) = 1$, $x_4(0) = 0$.

Write as a system: $D\vec{x} = \begin{bmatrix} 3 & -1 & 0 & -4 \\ 0 & 3 & 0 & -4 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \vec{x}$. 3, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and 2, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ are the obvious eigenvalue–eigenvector pairs; 1 is a double eigenvalue with generalized eigenvectors $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$:

General solution $c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \left[\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right] + c_4 e^t \left[\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -6 \\ -4 \\ 0 \\ -2 \end{bmatrix} \right]$.

When $t = 0$ this becomes $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. This gives $c_4 = 0$, $c_2 = 1$, $c_3 = 1$, then $c_1 = -1\ldots$

7. Suppose A is a square matrix. Show that if $(A - \lambda I)^2 \vec{v} = \vec{0}$ and $(A - \lambda I)\vec{v} \neq \vec{0}$, then $\vec{v}_1 = (A - \lambda I)\vec{v}$ is an eigenvector of A.
$(A - \lambda I)\vec{v}_1 = (A - \lambda I)(A - \lambda I)\vec{v} = (A - \lambda I)^2 \vec{v} = \vec{0}$; this means that $A\vec{v}_1 = \lambda \vec{v}_1$.