Midterm Exam Solutions

Math 51 Fall 2021 – Tufts University – Z. Nitecki and G. McNinch

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I. Multiple Choice Problems (24 points)

1. (4pts) A drug is absorbed by the body at a rate proportional to the amount of the drug present in the bloodstream after t hours. If there are $x(t)$ mg of the drug present in the bloodstream at time t, assume that the drug is absorbed at a rate of $0.5x(t)$ /hour. If a patient receives the drug intravenously at a constant rate of 3 mg/hour, to which of the following ODEs is $x(t)$ a solution?

   a. $x'(t) = -0.5x(t) + 3$

   b. $x'(t) = -0.5x(t); \quad x(0) = 3$

   c. $x'(t) = 0.5x(0) + 3$

   d. $x'(t) = .5x(t) - 3$

   Solution:

   A.

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2. (4pts) Which of the following represents a linear ODE?

   a. $x \cdot x'' + x + 1 = \sin(t)$

   b. $t \cdot x'' + x^2 + 1 = \sin(t)$

   c. $t^2 \cdot x'' + (t+1) \cdot x + 1 = \sin(t)$

   d. $(D^2 + D + t)x^2 = \sin(t)$

   Solution:

   C.

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3. (4pts) Consider the Wronskian $W(t) = W(f_1, f_2, f_3)(t)$ of the functions

$$f_1(t) = 1, \qquad f_2(t) = 1 + t \qquad \text{and} \qquad f_3(t) = \ln(1 + t).$$

   Which of the following statements is most correct?

   a. The Wronskian is given by $W(t) = 1/(1 + t)^2$; since $W(1) = 1/4$ is non-zero, the functions are linearly independent on the interval $(-1, \infty)$.

b. Since $W(1) = 0$, the functions are linearly dependent on $(-1, \infty)$.

c. Since $W(t)$ is not defined on $(-\infty, \infty)$, the Wronskian test doesn't apply.

d. None of the above.

Solution:

A. or D.

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4. (4pts) Let $P(D)$ be a differential operator of order 4, and suppose that $h_1(t), h_2(t), h_3(t), h_4(t)$ are solutions to the homogeneous equation

$$(\heartsuit) \quad P(D)x = 0.$$

Suppose that

$$h_1(t) + h_2(t) + h_3(t) + h_4(t) = 0$$

for every t, $-\infty < t < \infty$.

Which of the following statements is most correct?

a. The general solution to $(\heartsuit)$ is given by

$$x(t) = c_1 h_1(t) + c_2 h_2(t) + c_3 h_3(t) + c_4 h_4(t).$$

b. The functions $h_1(t), h_2(t), h_3(t), h_4(t)$ are linearly dependent.

c. A particular solution to $(\heartsuit)$ has the form

$$q(t) = \int h_1(t) dt.$$

d. For some values of k_1, k_2 and k_3, the expression $q(t) = k_1 h_1(t) + k_2 h_2(t) + k_3 h_3(t)$ provides a particular solution to the ODE

$$P(D)x = e^t.$$

Solution:

B.

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5. (4pts) You are given that a particular solution to

$$(\heartsuit) \quad (D^2 - 2D + 1)x = e^t$$

is $p(t) = \dfrac{t^2 e^t}{2}$. Which of the following best represents the general solution to $(\heartsuit)$?

a. $c_1 e^t + c_2 t e^t$.

b. $\dfrac{t^2 e^t}{2} + c_1 e^t + c_2 t e^t$.

c. $\dfrac{t^2 e^t}{2} + ce^t$.

d. $\dfrac{t^2 e^t}{2} + c_1 e^t + c_2 e^{-t}$.

Solution:

B.

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6. (4pts) Let $x_1(t)$ and $x_2(t)$ be solutions to the ODE $(t+1)x'' + x' + x = 0$. Suppose that $x_1(0) = x_2(0)$ and that $x_1'(0) = x_2'(0)$. Which of the following statements is most correct?

a. $x_1(t) = x_2(t)$ for every t.

b. Since the ODE is normal on the interval $(-1, \infty)$, we can conclude that $x_1(t) = x_2(t)$ for $-1 < t < \infty$.

c. No conclusion is possible because the existence and uniqueness theorem does not apply to this ODE.

d. We can only conclude that $x_1(t) = x_2(t)$ for all t if we also assume that $x_1''(0) = x_2''(0)$.

Solution:

B.

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II. Partial Credit problems (75 points)

1. (15pts) Find the general solution to the differential equation

$$(t+1)x' = \frac{x}{t-1}, \quad t > 1.$$

Solution:

Separating variables, we are led to the integrals

$$(\clubsuit) \quad \int \frac{1}{x} dx = \int \frac{1}{(t+1)(t-1)} dt$$

To find the integral on the right, use the method of partial fractions. We must solve

$$\frac{1}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} = \frac{A(t-1) + B(t+1)}{(t+1)(t-1)}.$$

Thus, we require that

$$0t + 1 = 1 = A(t-1) + B(t+1) = (A+B)t + (B-A).$$

Comparing coefficients, we find that

$$0 = A + B$$
$$1 = -A + B$$

Adding the two equations, we find that $2B = 1$ so that $B = 1/2$ and then $A = -1/2$. Thus,

$$\frac{1}{(t+1)(t-1)} = \frac{-1}{2(t+1)} + \frac{1}{2(t-1)} = \frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right).$$

Returning now to the integrals ($\clubsuit$), we find that

$$\ln|x| = \frac{1}{2} \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt$$
$$= \frac{1}{2}\left(\ln(t-1) - \ln(t+1)\right) + C$$
$$= \ln(\sqrt{t-1}) - \ln(\sqrt{t+1}) + C$$

After taking the exponential of each side, we find that

$$x = k\sqrt{\frac{t-1}{t+1}}$$

for an arbitrary constant k.

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2. (15pts) Solve the initial value problem

$$(\clubsuit) \quad 2\frac{dx}{dt} - x = t \cdot e^t; \quad x(0) = 1$$

Solution:

We rewrite the equation in standard form: $\dfrac{dx}{dt} - \dfrac{x}{2} = \dfrac{t \cdot e^t}{2}$; thus the right-hand-side is $q(t) = \dfrac{te^t}{2}$.

We first find a solution to the corresponding homogeneous equation $x' = \dfrac{x}{2}$; one solution is $h(t) = e^{t/2}$.

Now search for a solution to ($\clubsuit$) of the form $x(t) = k(t)h(t)$ for an unknown function $k(t)$.

For a first order linear ODE in standard form, we know that

$$k'(t) = \frac{1}{h(t)} \cdot q(t) = e^{-t/2}\frac{te^t}{2} = \frac{te^{t/2}}{2}.$$

We now find $k(t)$ by integrating:

$$k(t) = \frac{1}{2}\int te^{t/2}dt.$$

We use integration by parts with $u = t$ and $dv = e^{t/2}dt$. Thus $du = dt$ and $v = 2e^{t/2}$, so that

$$k(t) = \frac{1}{2}\left(2te^{t/2} - 2\int e^{t/2}dt\right) = te^{t/2} - 2e^{t/2} + C.$$

Thus we find that the general solution to ($\clubsuit$) is given by

$$x(t) = k(t)h(t) = te^t - 2e^t + Ce^{t/2}.$$

To solve the initial value problem, we require that $x(0) = 1$, and we find that

$$1 = x(0) = -2 + C \implies C = 3.$$

Thus the solution to the IVP is

$$x(t) = te^t - 2e^t + 3e^{t/2}.$$

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3. (15pts) Use the exponential shift formula

$$P(D)[e^{\lambda t}y] = e^{\lambda t}P(D + \lambda)[y]$$

to compute the function $P(D)[f]$ in each of the following cases:

a. $P(D) = D^2 + D - 6$ and $f = t^2 e^{2t}$.

Solution: We have

$$P(D+2) = (D+2)^2 + (D+2) - 6$$
$$= D^2 + 4D + 4 + D + 2 - 6$$
$$= D^2 + 5D = (D+5)D$$

so

$$P(D)[e^{2t}] = e^{2t} P(D+2)[t^2]$$
$$= e^{2t}(D+5)D[t^2]$$
$$= e^{2t}(D+5)[2t]$$
$$= e^{2t}(2+10t)$$

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b. $P(D) = D^2 + 3$ and $f = e^t \cos(3t)$.

Solution:

$$P(D+1) = (D+1)^2 + 3 = D^2 + 2D + 1 + 3 = D^2 + 2D + 4$$

so

$$P(D)[e^t \cos t] = e^t P(D+1)[\cos 3t]$$
$$= e^t(D^2 + 2D + 4)[\cos(3t)]$$
$$= e^t(-9\cos(3t) + 2(-3\sin(3t)) + 4\cos(3t))$$
$$= e^t(-5\cos(3t) - 6\sin(3t)).$$

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4. (15pts) Show that the functions

$$f_1(t) = e^t \cos(t), \quad f_2(t) = e^t \sin(t), \quad f_3(t) = e^t$$

are linearly independent.

You have been told that functions like this are independent. However, here we want you to demonstrate it directly in this case. You may use the Wronskian test (with all details needed to justify using it) or other, direct arguments from the definition.

Solution:

There are several possible strategies for solving this problem; here we list a few of them.

First, you can use the Wronskian test. This requires computation of the first and second derivatives of the f_i, which is perhaps most easily done using the exponential shift formula.

One finds:
$$D[e^t \cos(t)] = e^t(D+1)[\cos(t)] = e^t(\cos(t) - \sin(t))$$
$$D[e^t \sin(t)] = e^t(D+1)[\sin(t)] = e^t(\cos(t) + \sin(t))$$
$$D^2[e^t \cos(t)] = D[e^t(\cos(t) - \sin(t))] = e^t(D+1)[\cos(t) - \sin(t)] = -2e^t \sin(t).$$

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$$D^2[e^t \sin(t)] = D[e^t(\cos(t) + \sin(t))] = e^t(D+1)[\cos(t) + \sin(t)] = 2e^t \cos(t).$$

Thus the Wronskian matrix is given by

$$W = W(f_1, f_2, f_3) = \begin{bmatrix} e^t \cos(t) & e^t \sin(t) & e^t \\ e^t(\cos(t) - \sin(t)) & e^t(\cos(t) + \sin(t)) & e^t \\ -2e^t \sin(t) & 2e^t \cos(t) & e^t \end{bmatrix}$$

Now, according to the Wronskian test, the functions will be linearly independent (on the interval $(-\infty, \infty)$) provided that $\det W(t_0)$ is non-zero for some t_0. If we take $t_0 = 0$, we find that

$$\det W \bigg|_{t=0} = \det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = -1 + 2 = 1$$

Since this determinant is non-zero, the Wronskian test confirms the linear independence of f_1, f_2, f_3.

A second method of solving this problem just uses the definition of linear independence.

Suppose that c_1, c_2, c_3 are constants and that

$$c_1 e^t \cos(t) + c_2 e^t \sin(t) + c_2 e^t = 0.$$

To show that the functions are linearly independent, we must argue that $c_1 = c_2 = c_3 = 0$.

Factoring out the quantity e^t, our assumption shows that

$$e^t(c_1 \cos(t) + c_2 \sin(t) + c_3) = 0.$$

Since $e^t \neq 0$ for all t, we find that

$$c_1 \cos(t) + c_2 \sin(t) + c_3 = 0.$$

Now, since this equation holds for all times t, we may choose some particular values of t to find equations for the constants c_i.

When $t = 0$, we find that

$$0 = c_1 \cos(0) + c_2 \sin(0) + c_3 = c_1 + c_3.$$

When $t = \pi/2$, we find that

$$0 = c_1 \cos(\pi/2) + c_2 \sin(\pi/2) + c_3 = c_2 + c_3.$$

When $t = \pi$, we find that

$$0 = c_1 \cos(\pi) + c_2 \sin(\pi) + c_3 = -c_1 + c_3.$$

Now, we solve the system of equations

$$0 = c_1 + c_3$$
$$0 = c_2 + c_3$$
$$0 = -c_1 + c_3$$

Adding the first and third equation gives $0 = 2c_3$ so that $c_3 = 0$. Now the first equation shows that $c_1 = 0$ and the second shows that $c_2 = 0$.

Since we have argued that $c_1 = c_2 = c_3 = 0$, we conclude from the definition that f_1, f_2, f_3 are linearly independent.

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5. (15pts) Solve the initial value problem

$$4x'' + 4x' - 3x = 0; \quad x(0) = 0, \quad x'(0) = 1.$$

Solution:

The characteristic polynomial factors as

$$4r^2 + 4r - 3 = (2r + 3)(2r - 1)$$

and thus has roots $\dfrac{-3}{2}$ and $\dfrac{1}{2}$.

It follows that the general solution to the indicated ODE is given by

$$x(t) = c_1 e^{-3t/2} + c_2 e^{t/2}$$

Note that

$$x'(t) = \frac{-3c_1}{2} e^{-3t/2} + \frac{c_2}{2} e^{t/2}.$$

We now require that

$$0 = x(0) = c_1 + c_2$$
$$1 = c'(0) = \frac{-3c_1}{2} + \frac{c_2}{2}$$

i.e.

$$0 = c_1 + c_2$$
$$2 = -3c_1 + c_2$$

Subtracting the first equation from the second gives $2 = -4c_1$ so that $c_1 = -1/2$, and then the first equation shows that $c_2 = -c_1 = 1/2$.

Thus the solution to the IVP is

$$x(t) = \frac{-e^{-3t/2}}{2} + \frac{e^{t/2}}{2}$$