Math 51 **Integrating Factors: General Case**

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First-order linear d.e. $x' + r(t)\,x = g(t)$

Step 1. Make sure that the differential equation is in standard form, with leading coefficient 1.

Step 2. Find an integrating factor $\boxed{\rho(t) = e^{\int r(t)\,dt}}$.

It has the property that

$$\rho(t)' = e^{\int r(t)\,dt}\, r(t) = \rho(t)\, r(t).$$

Step 3. Multiply the d.e. by the integrating factor:

$$\rho(t)\,x' + \underbrace{\rho(t)\, r(t)}_{\rho(t)'}\, x = \rho(t)\, g(t).$$

$$\underbrace{\qquad\qquad}_{(\rho(t)\,x)'} = \rho(t)\, g(t)$$

Integrating both sides gives

$$\rho(t)\, x = \int \rho(t)\, g(t)\, dt.$$

Dividing by $\rho(t)$:

$$\boxed{x = \rho(t)^{-1} \int \rho(t)\, g(t)\, dt.}$$