Carefully PRINT your full name: Solutions

Math 51	Differential Equations Exam 2 (~~100~~ 90 points) + 10 bonus	April 11, 2022 noon–1:20 p.m.

There are seven problems on the exam.

You may not use calculators, books or notes during the exam. All electronic devices (including your phones) must be silenced and put away for the duration of the exam.

After finishing your exam, you will submit this exam booklet. We will scan your submission and upload it to Gradescope for marking (you do not need to take images of your exam). You should write your name at the top of each page, as indicated (especially if you remove the staples from your exam booklet).

For the partial credit problems, always show your work. Try to fit this work in the available space if possible. There is a blank page at the back of your exam for use as scratch paper. If you need more space for a solution, please write clearly in the indicated space that your solution continues later.

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Please sign the pledge below. With your signature, you pledge that you have neither given nor received assistance on this exam.

Signature:

1. (14 points) True-False and Multiple Choice.

   (a) (2 pts.) True or False (circle your choice). Any set of vectors that includes the zero vector $\mathbf{0}$ is linearly dependent.

   (b) (2 pts.) True or False (circle your choice). If $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$ are solutions of the nonhomogeneous system $D\vec{\mathbf{x}} = A\vec{\mathbf{x}} + \vec{\mathbf{E}}(t)$, then $\vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_2$ is a solution of the related homogeneous system $D\vec{\mathbf{x}} = A\vec{\mathbf{x}}$.

   (c) (2 pts.) True or False (circle your choice). Assume that all the functions in this question are differentiable. Let $\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_n$ be solutions of the *nonhomogeneous* linear system $D\vec{\mathbf{x}} = A\vec{\mathbf{x}} + \vec{\mathbf{E}}(t)$ on an interval I and let t_0 be a point in I. Then $\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_n$ generate the general solution of the given system if and only if the Wronskian $W[\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_n](t_0) \neq 0$.

   only for homogeneous systems

   (d) (2 pts.) True or False (circle your choice). Two vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are linearly dependent if and only if one of them is a constant multiple of the other.

   (e) (2 pts.) True or False (circle your choice). Three vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$ are linearly dependent if and only if some vector is a constant multiple of another vector.

   (f) (2 pts.) Which of the following formulas for $\vec{\mathbf{x}}$ gives the general solution to the linear homogeneous ode $(D^2+1)^3\vec{\mathbf{x}} = \cancel{\vec{\mathbf{0}}}$? Circle your choice.

   A. $c_1 \cos t + c_2 \sin t$
   B. $c_1 t^2 \cos t + c_2 t^2 \sin t$
   C. $c_1 t^2 \cos t + c_2 t^2 \sin t + c_3 t \cos t + c_4 t \sin t + c_5 \cos t + c_6 \sin t$
   D. None of the above.

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(g) (2 pts.) For which of the following expressions for $E(t)$ does the method of undetermined coefficients **not** apply when solving the linear nonhomogeneous ode $Lx = E(t)$? Circle your choice.

$$3t^4, \qquad \sin t, \qquad 2t^3e^{-4t}\cos 5t, \qquad \ln t.$$

A. Only $3t^4$.
B. Only $\sin t$.
C. Only $2t^3e^{-4t}\cos 5t$.
D. Only $\ln t$.
E. The method of undetermined coefficients does not apply for at least two of the four functions.
F. The method of undetermined coefficients applies for all four functions.

2. (16 points) Short-Answer Questions.

(a) (3 pts.) The matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

has eigenvalue 1 with multiplicity 3. Find three linearly independent generalized eigenvectors. No explanation is required. (*Hint*: Approached correctly, this problem does not require any computation.)

Since three linearly independent generate $\mathbb{R}^3$, every vector in $\mathbb{R}^3$ is a generalized eigenvector. Thus, any set of three linearly independent vectors in $\mathbb{R}^3$ are generalized eigenvectors, for example,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) (5 pts.) Write down an annihilator of smallest possible order with real coefficients for the function $3e^t + 2te^{-t} + \sin t$. No work or explanation is required.

$$(D-1)(D+1)^2(D^2+1)$$

(c) (3 pts.) Suppose

$$\begin{array}{rcrcrcl} 2u_1 & + & 5u_2 & + & 3u_3 & = & a \\ u_1 & - & 2u_2 & + & u_3 & = & b \\ u_1 & + & 4u_2 & - & u_3 & = & c, \end{array}$$

where it is given that the determinant of the coefficient matrix is nonzero. Write down the formula for u_3 in terms of determinants. Do not evaluate the determinants.

$$u_3 = \frac{\begin{vmatrix} 2 & 5 & a \\ 1 & -2 & b \\ 1 & 4 & c \end{vmatrix}}{\begin{vmatrix} 2 & 5 & 3 \\ 1 & -2 & 1 \\ 1 & 4 & -1 \end{vmatrix}}$$

(d) (5 points) Given that $\vec{\mathbf{h}}_1(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ and $\vec{\mathbf{h}}_2(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ are solutions of $D\vec{\mathbf{x}} = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, determine whether or not the general solution is $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{h}}_1(t) + c_2\vec{\mathbf{h}}_2(t)$. Show your work.

$\vec{h}_1(t), \vec{h}_2(t)$ generate the general solution

iff $\quad W[\vec{h}_1, \vec{h}_2] = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = e^{-t}e^{-2t}\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = e^{-3t}(-1) \neq 0$

for some t.

Yes, the general solution is $\vec{x}(t) = c_1\vec{h}_1(t) + c_2\vec{h}_2(t)$.

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3. (12 points)

(a) Convert the differential equation

$$x''' - e^t x'' - 4tx + x = e^{2t}$$

into a linear system of three equations in three unknowns x_1, x_2, x_3.

(b) Write the linear system in the form $D\vec{\mathbf{x}} = A(t)\vec{\mathbf{x}} + \vec{\mathbf{E}}(t)$ for some matrix $A(t)$ and vector $\vec{\mathbf{E}}(t)$.

(a) Let $x_1 = x$

$x_2 = x' = x_1'$

$x_3 = x'' = x_2'$.

Then $x_3' = x''' = e^t x'' + (4t-1)x + e^{2t}$

$= e^t x_3 + (4t-1)x_1 + e^{2t}$.

Thus

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= (4t-1)x_1 + e^t x_3 + e^{2t}. \end{aligned}}$$

(b)

$$\boxed{D\vec{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4t-1 & 0 & e^t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix}}.$$

4. (10 points) Suppose $3+2i, 3-2i$ are eigenvalues of the 2×2 matrix A with corresponding eigenvectors $\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}$, respectively. Write down two linearly independent real solutions of $D\vec{\mathbf{x}} = A\vec{\mathbf{x}}$. Show your work and simplify your answers.

A complex solution is $e^{(3+i2)t}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$$= e^{3t}(\cos 2t + i\sin 2t)\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= e^{3t}\begin{bmatrix} -\sin 2t + i\cos 2t \\ \cos 2t + i\sin 2t \end{bmatrix}$$

$$= e^{3t}\begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} + i\, e^{3t}\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}.$$

real solutions:

$$\boxed{e^{3t}\begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix}, \quad e^{3t}\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}}.$$

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5. (15 points) Find the general solution of $4x'' - 4x' + x = \frac{8}{t^2}e^{t/2}$ for $t > 0$, given that two solutions to the associated homogeneous equation are $\cancel{\vec{\mathbf{h}}_1}\ h_1 = e^{t/2}$, $\cancel{\vec{\mathbf{h}}_2}\ h_2 = te^{t/2}$.

First put the equation in standard form with leading coefficient 1:

$$x'' - x' + \frac{1}{4}x = \frac{2}{t^2}e^{t/2}.$$

The general solution to the homogeneous equation is

$$x(t) = c_1 e^{t/2} + c_2 t e^{t/2}.$$

Vary the parameters c_1, c_2.

We need to solve

$$c_1' h_1 + c_2' h_2 = 0$$
$$c_1' h_1' + c_2' h_2' = \frac{2}{t^2}e^{t/2}.$$

$$\begin{bmatrix} e^{t/2} & te^{t/2} \\ \frac{1}{2}e^{t/2} & e^{t/2} + \frac{1}{2}te^{t/2} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{t^2} \end{bmatrix} e^{t/2}$$

$$\begin{bmatrix} 1 & t \\ \frac{1}{2} & 1 + \frac{1}{2}t \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2/t^2 \end{bmatrix}.$$

$$c_1' = \frac{\begin{vmatrix} 0 & t \\ 2/t^2 & 1 + \frac{1}{2}t \end{vmatrix}}{1} = -\frac{2}{t}, \quad c_2' = \begin{vmatrix} 1 & 0 \\ \frac{1}{2} & 2/t^2 \end{vmatrix} = 2/t^2$$

$$c_1 = -2\ln t, \quad c_2 = -\frac{2}{t}.$$

A particular solution is $p(t) = -2(\ln t)e^{t/2} - \frac{2}{\cancel{t}}\cancel{t}e^{t/2}$.

Since $-2e^{t/2}$ solves the homogeneous equation, we don't need it. The general solution is

$$\boxed{x = c_1 e^{t/2} + c_2 t e^{t/2} - 2(\ln t)e^{t/2}}.$$

Check: $(D^2 - D + \frac{1}{4})[-2(\ln t)e^{t/2}]$

$= -2e^{t/2}\left((D+\frac{1}{2})^2 - (D+\frac{1}{2}) + \frac{1}{4}\right)[\ln t]$ (exp. shift)

$= -2e^{t/2} D^2[\ln t]$

$= -2e^{t/2} D[\frac{1}{t}] = -2e^{t/2}(-\frac{1}{t^2})$

$= \frac{2}{t^2}e^{t/2}.$ ✓

An alternate method.

By variation of parameters,

$x(t) = c_1(t)e^{t/2} + c_2(t)te^{t/2}$

$= (c_1(t) + c_2(t)t)e^{t/2}$

$= f(t)e^{t/2}$,

i.e., we may absorb $c_1(t) + c_2(t)t$ into a single function $f(t)$. Then

$(D^2 - D + \frac{1}{4})[f(t)e^{t/2}]$

$= e^{t/2}\left((D+\frac{1}{2})^2 - (D+\frac{1}{2}) + \frac{1}{4}\right)[f(t)]$

$= e^{t/2} D^2[f(t)] = e^{t/2}f''(t) = \frac{2}{t^2}e^{t/2}.$

Thus,

$f''(t) = \frac{2}{t^2}$,

$f'(t) = -\frac{2}{t}$

$f(t) = -2\ln t.$

A particular solution is $-2(\ln t)e^{t/2}$.

The general solution is

$x(t) = \boxed{c_1 e^{t/2} + c_2 t e^{t/2} - 2(\ln t)e^{t/2}}, \quad c_1, c_2 \in \mathbb{R}.$

6. (13 points) The matrix A below has an eigenvalue λ and a generalized eigenvector $\vec{\mathbf{v}}$ as follows:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \lambda = 1, \qquad \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find the general solution of $D\vec{\mathbf{x}} = A\vec{\mathbf{x}}$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2.$$

Thus, $\lambda = 1$ is a double eigenvalue.

eigenvector: $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = x_1 \\ x_2 = 0 \end{matrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector.

A solutions is $e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Since there are two linearly independent generalized eigenvectors in $\mathbb{R}^2$, any vector is a generalized eigenvector, say $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then a second solution is

$$e^t (\vec{v} + t(A - I)\vec{v})$$

$$= e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

$$= e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = e^t \begin{bmatrix} 1+2t \\ 0 \end{bmatrix}.$$

The general solution is

$$\vec{x} = \boxed{c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1+2t \\ 0 \end{bmatrix}}.$$

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7. (10 points) Make a *simplified* guess for a particular solution of the differential equation

$$(D+2)^3(D^2+1)^2\,\vec{\mathbf{x}} = te^{-2t} + \sin t.$$

Do not solve for the coefficients.

An annihilator of $te^{-2t} + \sin t$ is

$$(D+2)^2(D^2+1).$$

Hence,

$$(D+2)^2(D^2+1)(D+2)^3(D^2+1)^2\,\vec{x} = \vec{0}.$$

$$(D+2)^5(D^2+1)^3\,\vec{x} = 0.$$

The solutions are

$$e^{-2t},\ te^{-2t},\ \ldots,\ t^4e^{-2t},\ \cos t,\ \sin t,\ t\cos t,\ t\sin t,\ t^2\cos t,\ t^2\sin t.$$

Deleting those that solve the homogeneous equation, we are left with

$$t^3e^{-2t},\ t^4e^{-2t},\ t^2\cos t,\ t^2\sin t.$$

A simplified guess for a particular solution is

$$\boxed{c_1 t^3 e^{-2t} + c_2^4 t\, e^{-2t} + c_3 t^2\cos t + c_4^2 t \sin t.}$$