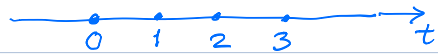
Math 51 Lecture 11 4/12/2021

The Laplace Transform

Motivation

The Laplace transform is the continuous analogue of a power series

$$\sum_{n=0}^{\infty} f(n)\, x^n .$$



Analogue:

$$\int_0^{\infty} f(t)\, x^t \, dt$$

(n is replaced by t and the sum becomes an integral)

Replace x by e^{-s} (to make the integral vanish at ∞)

$$\int_0^{\infty} f(t)\, e^{-st}\, dt ,$$

(e^{-st} will vanish at ∞ if $s > 0$)

Def. $\boxed{\mathcal{L}[f](s) = \int_0^{\infty} f(t)\, e^{-st}\, dt}$.

Examples.

1) $\mathcal{L}[e^{\lambda t}] = \dfrac{1}{s-\lambda}$, $s > \lambda$

2) $\mathcal{L}[1] = \dfrac{1}{s}$, $s > 0$ (set $\lambda = 0$)

3) $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$, $s > 0$

4) $\mathcal{L}[\cos \beta t] = \dfrac{s}{s^2+\beta^2}$, $s > 0$

5) $\mathcal{L}[\sin \beta t] = \dfrac{\beta}{s^2+\beta^2}$, $s > 0$.

Derivation 1) $\mathcal{L}[e^{\lambda t}] = \int_0^{\infty} e^{\lambda t} e^{-st}\, dt = \int_0^{\infty} e^{-(s-\lambda)t}\, dt$

$$= -\frac{1}{(s-\lambda)} e^{-(s-\lambda)t} \Big]_0^{\infty}$$

positive for $s > \lambda$

$$= -\frac{1}{s-\lambda}\left[e^{-(s-\lambda)\infty} - e^0 \right]$$

$$= -\frac{1}{s-\lambda}[0 - 1] \qquad (s > \lambda)$$

$$= \frac{1}{s-\lambda} .$$

First Differentiation Formula

$$\boxed{\mathcal{L}[Dx] = s\,\mathcal{L}[x] - x(0)}, \qquad s > 0$$

$$\mathcal{L}[D^2x] = s^2\mathcal{L}[x] - s\,x(0) - x'(0), \qquad s > 0$$

$$\mathcal{L}[D(Dx)] = s\,\mathcal{L}[Dx] - x'(0)$$

$$= s^2\mathcal{L}[x] - s\,x(0) - x'(0)$$

$$\mathcal{L}[D^3x] = s^3\mathcal{L}[x] - s^2x(0) - s\,x'(0) - x''(0)$$

$$\vdots$$

Initial-Value Problem

Ex. Solve $(D^2-1)x = e^{2t}$, $x(0) = x'(0) = 0$.

Solution. Take the Laplace transf. of both sides:

$$\mathcal{L}[D^2x] - \mathcal{L}[x] = \mathcal{L}[e^{2t}]$$

$$s^2\mathcal{L}[x] - \mathcal{L}[x] = \frac{1}{s-2}$$

$$(s^2-1)\,\mathcal{L}[x] = \frac{1}{s-2}$$

$$\mathcal{L}[x] = \frac{1}{(s^2-1)(s-2)} = \frac{1}{(s+1)(s-1)(s-2)}$$

Partial Fraction Decomp. (P.F.D.)

$$\frac{1}{(s+1)(s-1)(s-2)} = \frac{a}{s+1} + \frac{b}{s-1} + \frac{c}{s-2}$$

$$= \frac{a(s-1)(s-2) + b(s+1)(s-2) + c(s+1)(s-1)}{(s+1)(s-1)(s-2)}$$

$$\vdots$$

$$= \frac{(a+b+c)s^2 + (-3a-b)s + (2a-2b-c)}{(s+1)(s-1)(s-2)}$$

$$\begin{cases} a+b+c = 0 \\ -3a-b = 0 \\ 2a-2b-c = 1 \end{cases}$$

Solve.

How the book finds the P.F.D., not good.

Better way: Use simple poles.

Poles of a Functions

$$F(s) = \frac{1}{(\sin s)(s^2+1)(s-1)(s-3)^2}$$

zero at $s=0$ but not a pole

not a pole because never 0

simple pole at $s=1$

double pole at $s=3$

Def. A pole of a function $F(s)$ is a number a where $F(a) = \pm\infty$ due to the presence of $(s-a)^k$ in the denominator.

$s=a$ is a simple pole if $k=1$.

Use Simple Poles to Find P.F.D. (Trade secret)

$$\frac{1}{(s+1)(s-1)(s-2)} = \frac{a}{s+1} + \frac{b}{s-1} + \frac{c}{s-2}$$

simple pole at $s=-1$

To get a, multiply by $s+1$:

$$\frac{1}{(s-1)(s-2)} = a + \frac{b}{s-1}(s+1) + \frac{c}{s-2}(s+1).$$

Plug in $s=-1$.

$$\frac{1}{(-2)(-3)} = a + \frac{b}{s-1}\cdot 0 + \frac{c}{s-2}\cdot 0 = a$$

$$a = \frac{1}{6}$$

To get b, multiply by $s-1$:

$$\frac{1}{(s+1)(s-2)} = \frac{a}{s+1}(s-1) + b + \frac{c}{s-2}(s-1)$$

Evaluate at $s=1$:

$$\frac{1}{2(-1)} = 0 + b + 0 \Rightarrow b = -\frac{1}{2}$$

To get c, same thing $\Rightarrow$ $c = \frac{1}{3}$.

Back to DE:

$$\mathcal{L}[x] = \frac{1}{(s^2-1)(s-2)} = \frac{\frac{1}{6}}{s+1} + \frac{-\frac{1}{2}}{s-1} + \frac{\frac{1}{3}}{s-2}.$$

$$\boxed{x = \frac{1}{6}e^{-t} - \frac{1}{2}e^{t} + \frac{1}{3}e^{2t}}.$$

By the existence and uniqueness theorem for o.d.e., this is the unique sol to the init-value problem.

Double Poles

$$\frac{1}{(s-1)^2(s-5)} = \frac{a}{s-1} + \frac{b}{(s-1)^2} + \frac{c}{s-5}$$

double pole at $s=1$

Multiply by $(s-1)^2$:

$$\frac{1}{s-5} = a(s-1) + b + \frac{c}{s-5}(s-1)^2$$

Evaluate at $s=1$:

$$\frac{1}{-4} = 0 + b + 0 \Rightarrow b = -\frac{1}{4}.$$

Since 5 is a simple pole, multiply by $s-5$:

$$\frac{1}{(s-1)^2} = \frac{a}{s-1}(s-5) + \frac{b}{(s-1)^2}(s-5) + c$$

Evaluate at $s=5$:

$$\frac{1}{16} = 0 + 0 + c \Rightarrow c = \frac{1}{16}$$

The method of poles will not give you a, but once you have $b = -\frac{1}{4}$ and $c = \frac{1}{16}$, you can solve for a.

$$\frac{1}{(s-1)^2(s-5)} = \frac{a}{s-1} + \frac{-\frac{1}{4}}{(s-1)^2} + \frac{\frac{1}{16}}{s-5}$$

Since this equation is true for all s, we can simply plug in any value of s to solve for a.

Plug in $s = 0$:

$$\frac{1}{+5} = +a + \frac{1}{4} + \frac{1}{80}$$

$$16 = 80a + 20 + 1$$

$$80a = 16 - 21 = -5 \Rightarrow a = -\frac{5}{80} = -\frac{1}{16}. \qquad \square$$