Math 51 Problem Set 8 Solutions Spring 2022

1. Find the eigenvalues of A, and for each eigenvalue find a corresponding eigenvector. 5

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(*Hints*: To find the roots of a cubic (degree 3) polynomial, use the rational root test.)

Solution.

$$\begin{array}{ccc} & & \lambda \quad \lambda \quad \lambda \end{array}$$

$$p(\lambda) = \det(\lambda I - A) = \left| \begin{array}{ccc|cc} \lambda & -1 & -1 & \lambda & -1 \\ -1 & \lambda & -1 & -1 & \lambda \\ -1 & -1 & \lambda & -1 & -1 \end{array} \right|$$

$$\lambda^3 \quad -1 \quad -1$$

(the diagonal method for a 3×3 determinant)

$$= \lambda^3 - 1 - 1 - \lambda - \lambda - \lambda$$

$$= \lambda^3 - 3\lambda - 2.$$

By the rational root test, the possible rational roots are $\pm 1, \pm 2$.

$\lambda = -1 \Rightarrow (-1)^3 - 3(-1) - 2 = -1 + 3 - 2 = 0 \Rightarrow \lambda = -1$ is a root.

Therefore, $\lambda + 1$ is a factor. By long division,

$$\begin{array}{r|l} & \lambda^2 - \lambda - 2 \\ \hline \lambda + 1 & \lambda^3 \qquad - 3\lambda - 2 \\ & \underline{\lambda^3 + \lambda^2} \\ & -\lambda^2 - 3\lambda \\ & \underline{-\lambda^2 - \lambda} \\ & -2\lambda - 2 \\ & \underline{-2\lambda - 2} \end{array}$$

So $p(\lambda) = \lambda^3 - 3\lambda - 2 = (\lambda+1)(\lambda^2 - \lambda - 2)$

$$= (\lambda+1)(\lambda+1)(\lambda-2) = (\lambda+1)^2(\lambda-2).$$

$\boxed{\text{The eigenvalues are } -1, -1, 2.}$

For $\lambda = -1$, $(\lambda I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

iff $x + y + z = 0$

iff $z = -x - y$

iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$

$$\boxed{\text{The eigenvectors corresponding to } \lambda = -1 \text{ are generated by } \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.}$$

For $\lambda = 2$,

$$(\lambda I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row reduction

$$\xrightarrow[(-1)R_1]{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_2]{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} x = z \\ y = z \\ z = z \end{array} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\text{For } \lambda = 2, \text{ the eigenvectors are multiples of } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.}$$

2. Given a matrix A and an eigenvector of A, find

   (a) the eigenvalue λ to which $\mathbf{v}$ corresponds.

   (b) the associated solution of $D\mathbf{x} = A\mathbf{x}$.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Solution. (a) $A\vec{v} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$

The corresponding eigenvalue of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is $\boxed{-3}$.

(b) The associated solution of $D\vec{x} = A\vec{x}$ is

$$\boxed{\vec{x} = e^{-3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}}.$$

3. (a) Calculate the determinant of the **lower triangular matrix**

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 7 & -2 & -5 & 0 \\ -3 & 1 & 5 & 6 \end{bmatrix}.$$

   (b) What can you say about the determinant of a lower triangular matrix?

   (c) Find the eigenvalues of the matrix A in part (a).

   (d) What can you say about the eigenvalues of a lower triangular matrix?

   (Note. The same conclusions hold for an upper triangular matrix.)

(a) $\det A = 2 \cdot 3 \cdot (-5) \cdot 6 = \boxed{-180}$.

(b) The determinant of a lower triangular matrix is $\boxed{\text{the product of the diagonal entries}}$.

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 0 & 0 & 0 \\ -1 & \lambda-3 & 0 & 0 \\ -7 & 2 & \lambda+5 & 0 \\ 3 & -1 & -5 & \lambda-6 \end{vmatrix} = (\lambda-2)(\lambda-3)(\lambda+5)(\lambda-6)$

$\boxed{\text{eigenvalues: } 2, 3, -5, 6}$

(d) The eigenvalues of a lower triangular matrix are $\boxed{\text{the diagonal entries}}$

4. Use row reduction to decide whether the following three vectors are linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix}1\\4\\1\\3\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}-1\\2\\2\\1\end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 4 & -1 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -1 & 3 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{R_3 \to -R_3} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & -5 & 6 \\ 0 & -2 & 4 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 + 2R_2]{R_3 \to R_3 + 5R_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 \to -\frac{1}{6}R_3} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Although this matrix is not reduced, it already shows that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0 \Rightarrow c_3 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$. Hence, the three vectors are $\boxed{\text{linearly independent}}$.

5. Find the general solution of $D\mathbf{x} = A\mathbf{x}$, where A is the matrix $\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$.

Solution.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 3 \\ 3 & \lambda-1 \end{vmatrix} = \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2)$$

The eigenvalues are -2, 4.

For $\lambda = -2$, $\left[\lambda I - A \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] = \left[\begin{matrix} -3 & 3 \\ 3 & -3 \end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] \to \left[\begin{matrix} 1 & -1 \\ 0 & 0 \end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right]$

$\begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \Rightarrow$ Eigenvector $\vec{v} = \begin{bmatrix}1\\1\end{bmatrix} x_2 \Rightarrow$ Solution $e^{-2t}\begin{bmatrix}1\\1\end{bmatrix}$

For $\lambda = 4$, $\left[\lambda I - A \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] = \left[\begin{matrix} 3 & 3 \\ 3 & 3 \end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] = \left[\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right].$

$\begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix} \Rightarrow \vec{v} = \begin{bmatrix}-1\\1\end{bmatrix} x_2. \Rightarrow$ Solution $e^{4t}\begin{bmatrix}1\\-1\end{bmatrix}$

The general solution is $\boxed{\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1 e^{-2t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}1\\-1\end{bmatrix}}$. $\square$

6. Find all solutions of the given system of equations (if they exist). express your answer (1) as separate parametric equations for the variables and (2) as a linear combination of vectors.

(a)

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 + x_3 &= 4. \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 2x_2 + x_3 - x_4 - x_5 &= 2 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 - 2x_5 &= 1 \\ -x_1 - x_3 + 2x_4 + x_5 &= 0. \end{aligned}$$

(a) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & 2 & 5 \\ 1 & 2 & 1 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & 0 & 3 \end{array}\right]$

$\xrightarrow[R_2 \to \frac{1}{3}R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

pivot var free var

(1) $\boxed{\begin{aligned} x_1 &= -x_3 + 2 \\ x_2 &= 1 \\ x_3 &= x_3 \end{aligned}}$, where x_3 is the parameter (free variable).

(2) $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}.$

(b) $\left[\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 2 \\ 2 & 2 & 2 & -3 & -2 & 1 \\ -1 & 0 & -1 & 2 & 1 & 0 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 2 \\ 0 & -2 & 0 & -1 & 0 & -3 \\ 0 & 2 & 0 & 1 & 0 & 2 \end{array}\right]$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & -1 & -1 & 2 \\ 0 & -2 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

The last equation is $0 = -1$, which is impossible.

Hence, this system has $\boxed{\text{no solutions}}$ $\square$