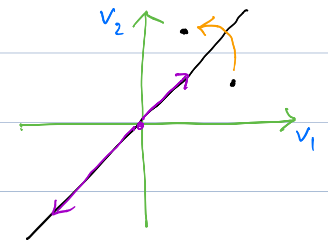
Math 51 Lecture 8

Eigenvalues, Eigenvectors, Row Reduction

Geometric Meaning of an Eigenvector

Multiplication by a 2×2 matrix A takes a vector $\vec{v}\in\mathbb{R}^2$ to another vector $A\vec{v}\in\mathbb{R}^2$:

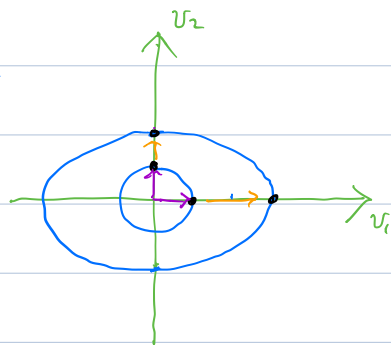
Ex.



$$A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$$

is reflection about the diagonal.

Ex.



$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 3v_2 \end{bmatrix}$$

expands by 3 in the v_1-direction and by 2 in the v_2-direction.

Under such a transformation, sometimes a line is fixed.

An eigenvector is a nonzero vector in a fixed line.

Def. An eigenvector of an $n\times n$ matrix A is a nonzero vector $\vec{v}\in\mathbb{R}^n$ s.t.

$$A\vec{v} = \lambda\vec{v} \quad \text{for some } \lambda\in\mathbb{R}$$

(λ could be 0.)

λ is called the eigenvalue corresponding to $\vec{v}$.

The eigenvalue λ is the scaling factor for the eigenvector $\vec{v}$.

To find an eigenvector, we solve

$$A\vec{v} = \lambda\vec{v} = \lambda I \vec{v}, \qquad I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = \text{identity matrix}.$$

$$\Leftrightarrow \quad (A - \lambda I)\vec{v} = \vec{0} \Leftrightarrow -(A-\lambda I)\vec{v} = \vec{0}$$

$$\Leftrightarrow \quad \boxed{(\lambda I - A)\vec{v} = \vec{0}}.$$

By Cramer's rule, $\exists$ nonzero sol $\vec{v}$ iff $\boxed{\det(\lambda I - A) = 0}$.

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1$$

$$= (\lambda+1)(\lambda-1) = 0$$

Eigenvalues $\lambda = -1, 1$

Eigenvectors:

$\lambda = 1$: $(I - A)\vec{v} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2 \Rightarrow \vec{v} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -1$: $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -v_1 - v_2 = 0 \Rightarrow v_1 = -v_2.$

$$\Rightarrow \vec{v} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = v_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This calculation shows that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has two

fixed lines through $\vec{0}$, one going through $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (the diagonal) and the other going through $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (the antidiagonal).

Solutions of $D\vec{x} = A\vec{x}$

$\boxed{\text{If } A \text{ has eigenvalue } \lambda \text{ with eigenvector } \vec{v}, \text{ then } \vec{x} = e^{\lambda t}\vec{v} \text{ is a solution of } D\vec{x} = A\vec{x}.}$

Why? We know $A\vec{v} = \lambda\vec{v}$.

$$\begin{aligned} D\vec{x} = (e^{\lambda t}\vec{v})' &= e^{\lambda t}\lambda\vec{v} = e^{\lambda t}A\vec{v} \quad (\text{because } A\vec{v} = \lambda\vec{v}) \\ &= A e^{\lambda t}\vec{v} \quad (\text{linearity}) \\ &= A\vec{x}. \end{aligned}$$

Therefore, $\vec{x} = e^{\lambda t}\vec{v}$ is a solution of $D\vec{x} = A\vec{x}$.

Geometrically a solution of the DE $D\vec{x} = A\vec{x}$ for a 2×2 matrix A is a curve $\vec{x}(t)$ in the plane.

The solution $\vec{x} = e^{\lambda t}\vec{v}$ is the straight line through the eigenvector $\vec{v}$.

Row Reduction

In solving a system of linear algebraic equations $B\vec{u} = \vec{h}$, Cramer's rule works 1) only for square systems ($n \times n$)

2) only if $\det B \neq 0$.

Thus, we can never use Cramer's rule to find eigenvectors since $\det(\lambda I - A) = 0$.

Ex. Solve

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & -1 & -3 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$u_1 + 2u_2 + 3u_3 = 1$
$u_1 - u_2 - 3u_3 + u_4 = 1$
$\vdots$

Three operations that do not change the solutions:

1) $R_i \to R_i + cR_j$.

2) $R_i \to cR_i$, $c \neq 0$

3) $R_i \leftrightarrow R_j$.

Use this (pivot) to kill everything else in its column

pivot = first nonzero entry of a nonzero row

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 1 & -1 & -3 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & -3 & -6 & 1 & 0 \\ 0 & -3 & -6 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & -3 & -6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

make pivot 1

Use this pivot to kill everything else in its column

pivots

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & \frac{2}{3} & 1 \\ 0 & 1 & 2 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the solutions are

$$
\begin{aligned}
u_1 \quad - u_3 + \tfrac{2}{3}u_4 &= 1 \quad \Rightarrow \quad u_1 = u_3 + \tfrac{2}{3}u_4 + 1 \\
u_2 + 2u_3 - \tfrac{1}{3}u_4 &= 0 \quad \Rightarrow \quad u_2 = -2u_3 + \tfrac{1}{3}u_4 \\
&\qquad\qquad\quad u_3 = u_3 \\
&\qquad\qquad\quad u_4 = u_4
\end{aligned}
$$

pivot variables

free variables

In matrix notation,

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} u_3 + \frac{2}{3}u_4 + 1 \\ -2u_3 + \frac{1}{3}u_4 \\ u_3 \\ u_4 \end{bmatrix} = \underbrace{u_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}}_{\text{general solution to the related homogeneous system}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \square
$$

Reduced matrix:

pivot variables

free variables

$$
\begin{array}{cccccccc|c}
u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & \\
\hline
0 & 0 & 1 & * & * & 0 & * & 0 & * \\
 & & 0 & & & 1 & * & 0 & * \\
 & & 0 & & & 0 & & 1 & * \\
 & & 0 & & & 0 & & 0 & 0
\end{array}
$$

staircase pattern

Pivot variables $\longleftrightarrow$ pivot columns

free variables $\longleftrightarrow$ nonpivot columns