Differential Equations

Linear Systems of O.D.E's: General Properties

Solutions of Linear Systems

Fact: (Linearity of D)

1) (Sum)

$$D(\vec{x}_1 + \vec{x}_2) = D\vec{x}_1 + D\vec{x}_2$$

2) (Constant Multiple)

$$D(c\vec{x}) = cD\vec{x}$$

Thm: If $\vec{p}(t)$ is a particular solution of the non-homogeneous linear system

$$D\vec{x} = A\vec{x} + \vec{E}(t), \tag{S}$$

and if $\vec{H}(t)$ is the general solution of the related homogeneous equation

$$D\vec{x} = A\vec{x}, \tag{H}$$

then the general solution of (S) is

$$\vec{x} = \vec{H}(t) + \vec{p}(t).$$

Pf. To prove that our $\vec{x}$ is the general solution of (S), we need to show

(1) $\vec{H}(t) + \vec{p}(t)$ solves (S), and

(2) $\vec{H}(t) + \vec{p}(t)$ describes *all* solutions of (S).

For (1), we see that

$$\begin{aligned} D[\vec{H}(t) + \vec{p}(t)] &= D\vec{H}(t) + D\vec{p}(t) \\ &= A\vec{H}(t) + [A\vec{p}(t) + \vec{E}(t)] \\ &= A[\vec{H}(t) + \vec{p}(t)] + \vec{E}(t). \end{aligned}$$

For (2), suppose $D\vec{f}(t) = A\vec{f}(t) + \vec{E}(t)$. To show that

$$\vec{f}(t) = [\underline{\vec{f}(t) - \vec{p}(t)}] + p(\vec{t})$$

is described by $\vec{H}(t) + \vec{p}(t)$, we observe that

$$D[\underline{\vec{f}(t) - \vec{p}(t)}] = D\vec{f}(t) - D\vec{p}(t)$$
$$= A\vec{f}(t) + \vec{E}(t) - [A\vec{p}(t) + \vec{E}(t)] = A[\underline{\vec{f}(t) - \vec{p}(t)}],$$

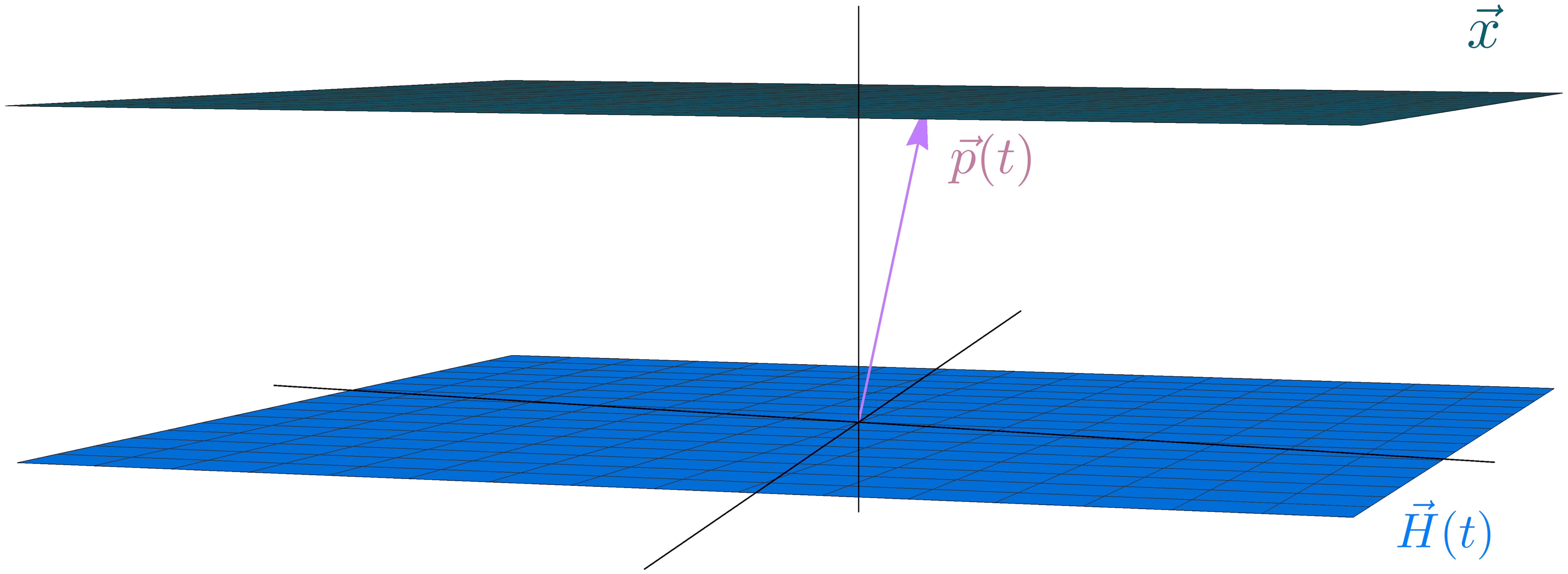
so $\underline{\vec{f}(t) - \vec{p}(t)}$ solves (H) and is indeed in the collection described by $\vec{H}(t)$. □

Ex: (§3.2) The general solution of

$$D\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 - 8e^t \end{bmatrix}$$

is

$$\vec{x} = c_1\begin{bmatrix} e^t \\ 3e^{3t} \end{bmatrix} + c_2\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + \begin{bmatrix} -2 + 2e^t \\ 2e^t \end{bmatrix}.$$



Homogeneous Solutions: the Wronskian

Suppose $\vec{x} = \vec{h}_1(t), \ldots, \vec{x} = \vec{h}_n(t)$ are solutions of (H) and that $c_1, \ldots, c_n$ are constants. Then

$$\begin{aligned} &D[c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)] \\ &= c_1 D\vec{h}_1(t) + \cdots + c_n D\vec{h}_n(t) \\ &= c_1 A\vec{h}_1(t) + \cdots + c_n A\vec{h}_n(t) \\ &= A[c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)]. \end{aligned}$$

Therefore,

Fact: If $\vec{h}_1(t), \ldots, \vec{h}_n(t)$ are solutions of (H), then any linear combination of these vector valued functions is also a solution of (H).

Q: If $D\vec{x} = A\vec{x}$ is an n-th order system and $\vec{h}_1(t)$, $\ldots, \vec{h}_n(t)$ are solutions, does the formula

$$\vec{x}(t) = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t) \tag{$*$}$$

describe the general solution of the homogeneous system?

Thm: (Existence and Uniqueness of Solutions of Linear Systems)

Suppose $D\vec{x} = A\vec{x} + \vec{E}(t)$ is an n-th
A1 order linear system of o.d.e.'s, where the entries $E_i(t)$ of $\vec{E}(t)$ and the coefficients a_{ij} of A are continuous on an interval I.
A2 Let t_0 be a point fixed in I.
A3 Then for any initial n-vector

$$\vec{v} = \vec{x}(t_0),$$

C there exists a unique solution of the system that is defined on I and that satisfies the initial condition.

A: We check if the collection $(*)$ is complete by trying to match every initial condition at t_0 with a function in the collection:

The following statements are equivalent.

- The general solution of $D\vec{x} = A\vec{x}$ is

$$\vec{x}(t) = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)$$

$$\Updownarrow$$

- For any fixed t_0 in I, we can match every initial condition

$$\vec{x}(t_0) = \vec{v}.$$

$$\Updownarrow$$

- For any fixed t_0 in I, the equation

$$c_1\vec{h}_1(t_0) + \cdots + c_n\vec{h}_n(t_0) = \vec{v}.$$

can be solved for $c_1, \ldots, c_n$ for all choices of $\vec{v}$.

$$\Updownarrow$$

- For any fixed t_0 in I, the equation

$$c_1\begin{bmatrix} h_{11}(t_0) \\ h_{21}(t_0) \\ \vdots \\ h_{n1}(t_0) \end{bmatrix} + \cdots + c_n\begin{bmatrix} h_{1n}(t_0) \\ h_{2n}(t_0) \\ \vdots \\ h_{n1}(t_0) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

can be solved for $c_1, \ldots, c_n$ for all choices of $v_1, \ldots, v_n$.

$$\Updownarrow$$

- For any fixed t_0 in I, the equations

$$\begin{aligned} c_1h_{11}(t_0) + {} & \cdots + c_nh_{1n}(t_0) = v_1 \\ c_1h_{21}(t_0) + {} & \cdots + c_nh_{2n}(t_0) = v_2 \\ & \vdots \\ c_nh_{n1}(t_0) + {} & \cdots + c_nh_{nn}(t_0) = v_n \end{aligned}$$

have a solution for $c_1, \ldots, c_n$ for all choices of $v_1, \ldots, v_n$.

(Cramer's $\Updownarrow$ Determinant Test)

· For any fixed t_0 in I,

$$\det \begin{bmatrix} h_{11}(t_0) & \dots & h_{1n}(t_0) \\ h_{21}(t_0) & \dots & h_{2n}(t_0) \\ \vdots & & \vdots \\ h_{n1}(t_0) & \dots & h_{nn}(t_0) \end{bmatrix} \neq 0.$$

Def: The <u>Wronskian</u> of $\vec{f}_1(t), \dots \vec{f}_n(t)$ (a number n of n-vector valued functions) is the function given by the determinant of the $n \times n$ matrix whose columns are $\vec{f}_1(t), \dots \vec{f}_n(t)$:

$$W\left[\vec{f}_1, \dots \vec{f}_n\right](t) = \det \begin{bmatrix} f_{11}(t) & \dots & f_{1n}(t) \\ f_{21}(t) & \dots & f_{2n}(t) \\ \vdots & & \vdots \\ f_{n1}(t) & \dots & f_{nn}(t) \end{bmatrix}.$$

Ex: The first determinant on this page is

$$W\left[\vec{h}_1, \dots \vec{h}_n\right](t_0).$$

We have shown:

Fact: Suppose $\vec{h}_1(t), \dots \vec{h}_n(t)$ are solutions of an n-th order linear homogeneous system

$$D\vec{x} = A\vec{x} \tag{H}$$

whose coefficients a_{ij} are continuous on an interval I, and let t_0 be any fixed point in I. Then

$\vec{h}_1, \dots, \vec{h}_n$ generate the general solution of (H) :

$$\vec{x}(t) = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)$$

$$\Updownarrow$$

$$W\left[\vec{h}_1, \dots \vec{h}_n\right](t_0) \neq 0.$$

Ex: Consider the homogeneous system

$$D\vec{x} = A\vec{x}, \tag{H}$$

where

$$\vec{x} = \begin{bmatrix} Q \\ I \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

Let

$$\vec{h}_1(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} \quad \text{and} \quad \vec{h}_2(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}.$$

Substitution into (H) shows

$$A\vec{h}_1(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = D\vec{h}_1(t)$$

and

$$A\vec{h}_2(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} -2e^{-2t} \\ 4e^{-2t} \end{bmatrix} = D\vec{h}_2(t),$$

so $\vec{h}_1(t)$ and $\vec{h}_2(t)$ are solutions of our second-order system (H).

The Wronskian of $\vec{h}_1(t)$ and $\vec{h}_2(t)$ is

$$W\left[\vec{h}_1, \vec{h}_2\right](t) = \det \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix}.$$

At $t_0 = 0$,

$$W\left[\vec{h}_1, \vec{h}_2\right](0) = \det \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = -1 \neq 0.$$

Since the Wronskian at our chosen t_0 is nonzero, $\vec{h}_1(t), \vec{h}_2(t)$ generate the general solution of (H):

$$\vec{x} = c_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix},$$

or

$$\begin{aligned} Q &= c_1 e^{-t} + c_2 e^{-2t} \\ I &= -c_1 e^{-t} - 2c_2 e^{-2t}. \end{aligned}$$

We can use the determinant and the Wronskian to show:

Fact: Suppose (H) is a homogeneous linear system of order n whose coefficients a_{ij} are continuous. Then the general solution of (H) has the form

$$\vec{x}(t) = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)$$

for a suitable choice of $\vec{h}_1(t), \ldots, \vec{h}_n(t)$. The general solution cannot be generated by fewer than n solutions.

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