Carefully PRINT your full name:

Math 51	Differential Equations Exam 2 (100 points)	April 11, 2022 noon–1:20 p.m.

There are 6 problems on the exam.

You may not use calculators, books or notes during the exam. All electronic devices (including your phones) must be silenced and put away for the duration of the exam.

After finishing your exam, you will submit this exam booklet. We will scan your submission and upload it to Gradescope for marking (you do not need to take images of your exam). You should write your name at the top of each page, as indicated (especially if you remove the staples from your exam booklet).

For the partial credit problems, always show your work. Try to fit this work in the available space if possible. There is a blank page at the back of your exam for use as scratch paper. If you need more space for a solution, please write clearly in the indicated space that your solution continues later.

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Please sign the pledge below. With your signature, you pledge that you have neither given nor received assistance on this exam.

Signature:

1. (25 points) Short-Answer Questions.

   (a) (2 pts.) True or False. Any set of vectors that include the zero vector $\mathbf{0}$ is linearly dependent.

   (b) (2 pts.) True or False. If $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$ are solutions of the nonhomogeneous system $D\vec{\mathbf{x}} = A\vec{\mathbf{x}} + \vec{\mathbf{E}}(t)$, then $\vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_2$ is a solution of the related homogeneous system.

   (c) (2 pts.) True or False. Assume that all the functions in this question are differentiable. Let $\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_n$ be solutions of the *nonhomogeneous* linear system $D\vec{\mathbf{x}} = A\vec{\mathbf{x}} + \vec{\mathbf{E}}(t)$ on an interval I and let t_0 be a point in I. Then $\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_n$ generate the general solution of the given system if and only if the Wronskian $W[\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_n](t_0) \neq 0$.

   (d) (2 pts.) True or False. Two vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are linearly dependent if and only if one of them is a constant multiple of the other.

   (e) (2 pts.) True or False. Three vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$ are linearly dependent if and only if each vectors is a constant multiple of another vector.

   (f) (2 pts.) Which of the following formulas for $\vec{\mathbf{x}}$ gives the general solution to the linear homogeneous ode $(D^2+1)^3\vec{\mathbf{x}} = \vec{\mathbf{0}}$?

      A. $c_1 \cos t + c_2 \sin t$
      B. $c_1 t^2 \cos t + c_2 t^2 \sin t$
      C. $c_1 t^2 \cos t + c_2 t^2 \sin t + c_3 t \cos t + c_4 t \sin t + c_5 \cos t + c_6 \sin t$
      D. None of the above.

   (g) (2 pts.) For which of the following expressions for $E(t)$ does the method of undetermined coefficients **not** apply when solving the linear nonhomogeneous ode $Lx = E(t)$?

   $$3t^4, \qquad \sin t, \qquad 2t^3 e^{-4t} \cos 5t, \qquad \ln t.$$

      A. Only $3t^4$.
      B. Only $\sin t$.
      C. Only $2t^3 e^{-4t} \cos 5t$.
      D. Only $\ln t$.
      E. The method of undetermined coefficients does not apply for at least two of the four functions.
      F. The method of undetermined coefficients applies for all four functions.

   (h) (3 pts.) The matrix

   $$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

   has a triple eigenvalue 1. Find three linearly independent generalized eigenvectors (you do not have to verify that they are linearly independent). (*Hint*: Approached correctly, this problem does not require any computation.)

(i) (5 pts.) Write down an annihilator of smallest possible order with real coefficients for the function $3e^t + 2te^{-t} + \sin t$.

(j) (3 pts.) Suppose

$$\begin{array}{rcrcrcl} 2u_1 & + & 5u_2 & + & 3u_3 & = & a \\ u_1 & - & 2u_2 & + & u_3 & = & b \\ u_1 & + & 4u_2 & - & u_3 & = & c, \end{array}$$

where it is given that the determinant of the coefficient matrix is nonzero. Write down the formula for u_3 in terms of determinants. Do not evaluate the determinants.

2. (5 points) Given that $\vec{\mathbf{h}}_2 = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ and $\vec{\mathbf{h}}_2 = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ are solutions of $D\vec{\mathbf{x}} = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, determine whether or not the general solution is $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{h}}_1(t) + c_2\vec{\mathbf{h}}_2(t)$

3. (10 points) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

and for each eigenvalue, find as many linearly independent eigenvectors as possible.

4. (10 points)

(a) Convert the differential equation

$$x''' - e^t x'' - 4tx + x = e^{2t}$$

into a linear system of three equations in three unknowns x_1, x_2, x_3.

(b) Write the linear system in the form $D\vec{\mathbf{x}} = A(t)\vec{\mathbf{x}} + \vec{\mathbf{E}}(t)$ for some matrix $A(t)$ and vector $\vec{\mathbf{E}}(t)$.

5. (10 points) Suppose $3 + 2i, 3 - 2i$ are eigenvalues of the 2×2 matrix A with corresponding eigenvectors $\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}$, respectively. Write down two linearly independent real solutions of $D\vec{\mathbf{x}} = A\vec{\mathbf{x}}$. Show your work and simplify your answers.

6. (10 points) Find the general solution of $4x'' - 4x' + x = \frac{8}{t^2}e^{t/2}$ for $t > 0$.

7. (10 points) The matrix A below has an eigenvalue λ and a generalized eigenvector $\vec{\mathbf{v}}$ as follows:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad \lambda = 1, \qquad \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find the general solution of $D\vec{\mathbf{x}} = A\vec{\mathbf{x}}$.

8. (10 points) Make a *simplified* guess for a particular solution of the differential equation

$$(D+2)^7(D^2+1)^6 = te^{-2t} + \sin t.$$

Do not solve for the coefficients.