1. Find all values of the constant k for which the given function $x = \phi(t)$ is a solution of the indicated differential equation:

   a. $x = t^k, t > 0$; $t^2 \cdot x \cdot x'' - 2t \cdot x \cdot x' - 10x^2 = 0$.

   b. $x = kte^{3t}$; $x'' - 3x' = e^{3t}$.

Solutions.

a. If $x = t^k$, then $x' = k t^{k-1}$ and $x'' = k(k-1)t^{k-2}$.

Plugging into the differential equation gives

$$t^2 \cdot t^k \cdot k(k-1)t^{k-2} - 2t \cdot t^k \cdot k t^{k-1} - 10 t^{2k} = 0,$$

$$k(k-1)t^{2k} - 2k t^{2k} - 10 t^{2k} = 0,$$

$$(k^2 - k - 2k - 10) t^{2k} = 0.$$

Since this equation is true for all $t > 0$, we can set $t = 1$ and get

$$k^2 - 3k - 10 = 0,$$

$$(k-5)(k+2) = 0.$$

Thus, $\boxed{k = 5 \text{ or } -2}$.

b. If $x = kte^{3t}$, then by the product rule,

$$x' = k e^{3t} + kt \cdot 3e^{3t} = k(1+3t)e^{3t},$$

$$x'' = k3e^{3t} + k(1+3t)3e^{3t}$$

$$= k3(2+3t)e^{3t}.$$

Plugging x' and x'' into the differential equation gives

$$k3(2+3t)\cancel{e^{3t}} - 3k(1+3t)\cancel{e^{3t}} = \cancel{e^{3t}}$$

$$3k(2+3t-1-3t) = 1 \Rightarrow 3k = 1 \Rightarrow \boxed{k = 1/3}. \quad \square$$

2. (adapted from (Nitecki and Guterman 1992, sec. 1.1 exercise 24 and 1.2 exercise 22)).

   A tanker carrying 200,000 liters of oil runs aground off the coast of Alaska. Water pours in the tanker at one end at a rate of 1500 liters per hour while the polluted water-oil mixture pours out at the other end, also at a rate of 1500 liters per hour.

   We wish to describe the number of liters $x(t)$ of oil in the tanker at time t. Of course, $x(0) = 200000$.

   a. Explain why the percentage of oil in the tanker at time $t \geq 0$ is given by the fraction $\dfrac{x(t)}{200000}$.

   b. Use the answer to a. to set up a differential equation to predict the amount $x = x(t)$ of oil in the tanker at time t. Explain how you arrived at your formulation.

   c. Solve the differential equation found in part b. How much oil remains in the tanker after 5 days?

Solution. Let $x(t)$ = liters of oil at time t (in hours).
$w(t)$ = liters of water at time t

Since water pours in at the same rate as oil pouring out, the water-oil mixture is always 200,000 liters. Therefore,

$$w(t) = 200{,}000 - x(t).$$

(a) Percentage of oil in tanker $= \dfrac{\text{amt of oil}}{\text{amt of water and oil}}$

$$= \frac{x(t)}{200\,000}.$$

(b) The mixture is pouring out at the rate of 1500 l/hr. Of this mixture, the percentage of oil is $\dfrac{x(t)}{200\,000}$. Therefore, oil is pouring out at the rate of

$$1500 \cdot \frac{x(t)}{200\,000} = \frac{3}{400}x(t).$$

This means $\boxed{\dfrac{dx}{dt} = -\dfrac{3}{400}x, \quad x(0) = 200\,000}$.

(c) By the fact stated in the forward to the problem set, the solution is

$$x(t) = x(0)\, e^{-\frac{3}{400}t}$$

$$= \boxed{200000\, e^{-\frac{3}{400}t}}.$$

After 5 days (120 hours), $x(t) = 200000\, e^{-\frac{360}{400}} \approx \boxed{81314 \text{ (liters)}}$ ▯

3. Radioactive decay The atoms of a radioactive substance tend to decompose into atoms of a more stable substance at a rate proportional to the number $x = x(t)$ of unstable atoms present. Suppose that time t is measured in seconds, and that the half-life of a certain compound is 10^4 seconds. In other words, if $x_0 = x(0)$ then $x(10^4) = 0.5x_0$. Find a differential equation for $x(t)$.

Solution.

"rate proportional to x" means

$$\frac{dx}{dt} = kx \quad \text{for some constant } k.$$

We need to determine k. The solution of this differential equation is

$$x = x_0 e^{kt}.$$

When $t = 10^4$,

$$x(10^4) = x_0 e^{k \cdot 10^4} = \frac{1}{2} x_0 .$$

Therefore,

$$e^{k \cdot 10^4} = \frac{1}{2}.$$

Taking ln of both sides gives $\ln\left(e^{k \cdot 10^4}\right) = \ln\left(\frac{1}{2}\right)$

or

$$10^4 k = -\ln 2.$$

Therefore,

$$k = -\frac{\ln 2}{10^4}.$$

The differential equation is

$$\boxed{\frac{dx}{dt} = -\frac{\ln 2}{10^4} x}.$$

□

4. For each of the following ODES, find the general solution, and find the particular solution satisfying the given initial conditions.

a. $\frac{d^2x}{dt^2} = t - 1; \quad x(0) = 1, x'(0) = 2.$

b. $x'' = \frac{-1}{(t+1)^2}, t > -1; \quad x(0) = 2, x'(0) = 3.$

Solutions.

a. Integrating both sides of $x'' = t-1$ gives

$$x' = \int t-1 \, dt = \frac{t^2}{2} - t + c_1. \qquad (1)$$

Integrating again gives the general solution

$$\boxed{x = \frac{t^3}{6} - \frac{t^2}{2} + c_1 t + c_2}. \qquad (2)$$

Plugging $x(0) = 1$ into (2) gives

$$1 = 0 - 0 + 0 + c_2 \Rightarrow c_2 = 1.$$

Plugging $x'(0) = 2$ into (1) gives

$$2 = 0 - 0 + c_1 \Rightarrow c_1 = 2.$$

The particular solution is

$$\boxed{x = \frac{t^3}{6} - \frac{t^2}{2} + t + 2}.$$

b. $x'' = -\frac{1}{(t+1)^2}, \quad t > -1.$

Integrating twice, we get

$$x' = \frac{1}{t+1} + c_1, \qquad (1)$$

$$\boxed{x = (\ln(t+1)) + c_1 t + c_2}, \quad t > -1. \qquad (2)$$

Plugging $x(0) = 2$ into (2) gives

$$2 = (\ln 1) + 0 + c_2 \Rightarrow c_2 = 2.$$

Plugging $x'(0) = 3$ into (1) gives

$$3 = 1 + c_1 \Rightarrow c_1 = 2.$$

The particular solution is

$$\boxed{x = (\ln(t+1)) + 2t + 2}. \qquad \square$$