Math 51 Lecture 10

Multiple Roots, Nonhomogeneous Systems

Double Roots

Sometimes an eigenvalue λ comes with multiplicity, but there are not enough independent eigenvectors.

Example. $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

Eigenvalues: $\det[\lambda I - A] = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda-2 \end{vmatrix} = \lambda^2 - 2\lambda + 1$

$= (\lambda - 1)^{2}$ ← multiplicity

$\lambda = 1, 1$

Eigenvectors: $\begin{bmatrix} \lambda & -1 \\ 1 & \lambda-2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Hence, $\begin{array}{l} v_1 = v_2 \\ v_2 = v_2 \end{array} \Rightarrow \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This gives a sol. of DE: $\vec{x} = e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, but we don't have enough eigen vectors to get 2 independent solutions.

Def. If λ is a double root of $\det(A - \lambda I) = 0$, then a <u>generalized eigenvector</u> of λ is a vector $\vec{v} \neq \vec{0}$ s.t.

$$(A - \lambda I)^2 \vec{v} = \vec{0}.$$

- Every eigenvector $\vec{v}$ is a generalized eigenvector, because

$\vec{v}$ eigenvector $\Rightarrow (A - \lambda I)\vec{v} = \vec{0}$

$\Rightarrow (A - \lambda I)^2 \vec{v} = (A - \lambda I)\underbrace{(A - \lambda I)\vec{v}}_{\vec{0}} = \vec{0}$.

Method for a double root

- Find 2 independent generalized eigenvectors.
- To each generalized vector $\vec{v}$, there is associated a solution

$$e^{\lambda t}(\vec{v} + t(A-\lambda I)\vec{v}).$$

Example. $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$, eigenvalues $\lambda = 1, 1$.

Generalized eigenvectors:

$$(A-I)^2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$(A-I)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad (A-I)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Indep. sol.:

$$e^t\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = e^t\begin{bmatrix} 1-t \\ -t \end{bmatrix},$$

$$e^t\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = e^t\begin{bmatrix} t \\ 1+t \end{bmatrix}.$$

Gen. sol.:

$$c_1 e^t\begin{bmatrix} 1-t \\ -t \end{bmatrix} + c_2 e^t\begin{bmatrix} t \\ 1+t \end{bmatrix}.$$

Triple roots

A gen. eigenvector of a triple root λ is $\vec{v} \neq 0$ s.t.

$$(A-\lambda I)^3\vec{v} = \vec{0}.$$

To each generalized eigenvector $\vec{v}$, associate the solution

$$e^{\lambda t}\left(\vec{v} + t(A-\lambda I)\vec{v} + \frac{1}{2}t^2(A-\lambda I)^2\vec{v}\right).$$

Quadruple roots: A solution is

$$e^{\lambda t}\left(\vec{v} + t(A-\lambda I)\vec{v} + \cdots + \frac{1}{3!}t^3(A-\lambda I)^3\vec{v}\right).$$

Nonhomogeneous Linear Systems $D\vec{x} = A\vec{x} + \vec{E}(t)$

Step 1. Solve homo system $D\vec{x} = A\vec{x}$.

$$\vec{h}(t) = \sum c_i \vec{h}_i = c_1\vec{h}_1 + \cdots + c_n\vec{h}_n, \qquad c_i \in \mathbb{R},$$

where $D\vec{h}_i = \vec{h}_i' = A\vec{h}_i$.

Step 2. Let the parameters c_i vary as functions of t.

$$\vec{x} = \sum c_i(t)\vec{h}_i(t)$$

Plug into $D\vec{x} = A\vec{x} + \vec{E}(t)$:

$$D\vec{x} = \vec{x}' = \sum c_i'\vec{h}_i + \sum c_i\vec{h}_i' = A\vec{x} + \vec{E}(t)$$

$$\sum c_i'\vec{h}_i + \cancel{\sum c_i A\vec{h}_i} = A\sum c_i\vec{h}_i + \vec{E}(t)$$

(because $\vec{h}_i' = A\vec{h}_i$)

$$= \cancel{\sum c_i A\vec{h}_i} + \vec{E}(t)$$

(by linearity of multiplication by A)

Therefore,

$$\boxed{\sum c_i'\vec{h}_i = \vec{E}(t)}$$

Solve for $c_1', \dots, c_n'$.

Step 3. Integrate $c_1', \dots, c_n'$ to get $c_1, \dots, c_n$.

This gives a particular sol $\vec{p}(t) = \sum c_i(t)\, h_i(t)$.

Step 4. The general solution is $\vec{x} = \vec{h}(t) + \vec{p}(t)$.

Example. Solve $D\vec{x} = A\vec{x} + \vec{E}(t)$, $A = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}$, $\vec{E}(t) = \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix}$.

Step 1. Homo. system $D\vec{x} = A\vec{x}$.

Eigenvalues: $\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 1 \\ -4 & \lambda+1 \end{vmatrix} = \lambda^2 + 2\lambda + 1 + 4$

$$= \lambda^2 + 2\lambda + 5$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm i2$$

Eigenvectors for $\lambda = -1+2i$: (some work) $\vec{v} = \begin{bmatrix} i \\ 2 \end{bmatrix}$.

Homo sol : $e^{(-1+i2)t}\begin{bmatrix} i \\ 2 \end{bmatrix} = e^{-t}(\cos 2t + i \sin 2t)\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$$= e^{-t}\left(\cos 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$+ i e^{-t}\left(\cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

Gen. sol : $\vec{x} = c_1 e^{-t}\begin{bmatrix} -\sin 2t \\ 2\cos 2t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix}$

Step 2. $c_1' \begin{bmatrix} -e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix} + c_2' \begin{bmatrix} e^{-t}\cos 2t \\ 2e^{-t}\sin 2t \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} -\sin 2t & \cos 2t & 2 \\ 2\cos 2t & 2\sin 2t & 0 \end{array}\right]$, $\Delta = -2\sin^2 2t - 2\cos^2 2t$
$= -2$

By Cramer's rule,

$$c_1' = \frac{\begin{vmatrix} 2 & \cos 2t \\ 0 & 2\sin 2t \end{vmatrix}}{-2} = -2\sin 2t \Rightarrow c_1 = \cos 2t$$

$$c_2' = \frac{\begin{vmatrix} -\sin 2t & 2 \\ 2\cos 2t & 0 \end{vmatrix}}{-2} = 2\cos 2t \Rightarrow c_2 = \sin 2t$$

Step 3. Particular sol.

$$\vec{p}(t) = c_1(t)\vec{h}_1(t) + c_2(t)\vec{h}_2(t)$$

$$= (\cos 2t)e^{-t}\begin{bmatrix} -\sin 2t \\ 2\cos 2t \end{bmatrix} + (\sin 2t)e^{-t}\begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{-t} \end{bmatrix}.$$

Step 4. General sol. $\boxed{\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} -\sin 2t \\ 2\cos 2t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix} + \begin{bmatrix} 0 \\ 2e^{-t} \end{bmatrix}}$.