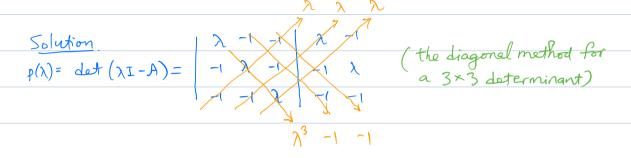
1. Find the eigenvalues of A, and for each eigenvalue find a corresponding eigenvector.

$$A = \left[ \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array} \right]$$

(Hints: To find the roots of a cubic (degree 3) polynomial, use the rational root test.)



$$= \lambda^3 - 1 - 1 - \lambda - \lambda - \lambda$$
$$= \lambda^3 - 3\lambda - 2$$

By the radional root test, the possible radional roots are  $\pm 1, \pm 2$ .  $\lambda = -1 \Rightarrow (-1)^3 - 3(-1) - 2 = -1 + 3 - 2 = 0 \Rightarrow \lambda = -1$  is a root.

Therefore, n+1 is a factor. By long division,

$$\lambda^{2} - \lambda - 2$$

$$\lambda + 1 \int \lambda^{3} -3\lambda - 2$$

$$-\lambda^{3} + \lambda^{2}$$

$$-\lambda^{2} - 3\lambda$$

$$-\lambda^{2} - \lambda$$

$$-2\lambda - 2$$

$$-2\lambda - 2$$

So  $P(\lambda) = \lambda^3 - 3\lambda - 2 = (\lambda + 1)(\lambda^2 - \lambda - 2)$ =  $(\lambda + 1)(\lambda + 1)(\lambda - 2) = (\lambda + 1)^2(\lambda - 2)$ .

The eigenvalues are -1, -1, 2.

For  $\lambda = -1$ ,  $(\lambda I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -( & -1) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 & -( & -1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ 

iff 
$$x + y + \xi = 0$$

iff 
$$3 = -x - y$$

iff 
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} = x \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$
.

The eigenvectors corresponding to  $\lambda = -1$  are generated by  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

Row reduction

$$\Rightarrow \begin{array}{c} x = 3 \\ y = 3 \\ 3 = 3 \end{array} \Rightarrow \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For  $\lambda=2$ , the eigenvectors are multiples of

2.	Given	a matrix	A and	an eiger	vector	of $A$	, find	l
				()			,	

- (a) the eigenvalue  $\lambda$  to which **v** corresponds.
- (b) the associated solution of  $D\mathbf{x} = A\mathbf{x}$ .

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$$
,  $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ .

Solution (a) 
$$A\vec{v} = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
.

The corresponding eigenvalue of [-2] is -3.

(b) The associated solution of 
$$D\vec{x} = A\vec{x}$$
 is

$$\vec{x} = e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

3. (a) Calculate the determinant of the **lower triangular matrix** 

$$A = \left[ \begin{array}{rrrr} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 7 & -2 & -5 & 0 \\ -3 & 1 & 5 & 6 \end{array} \right].$$

- (b) What can you say about the determinant of a lower triangular matrix?
- (c) Find the eigenvalues of the matrix *A* in part (a).
- (d) What can you say about the eigenvalues of a lower triangular matrix? (Note. The same conclusions hold for an upper triangular matrix.)

(a) 
$$\det A = 2 \cdot 3 \cdot (-5) \cdot 6 = -180$$

(b) The determinant of a lower triangular matrix is the product of the diagonal entries.

(c) 
$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 & 0 & 0 \\ -1 & \lambda - 3 & 0 & 0 \\ -7 & 2 & \lambda + 5 & 0 \end{vmatrix} = (\lambda - 2)(\lambda - 3)(\lambda + 5)(\lambda - 6)$$

$$\begin{vmatrix} -7 & 2 & \lambda + 5 & 0 \\ 3 & -1 & -5 & \lambda - 6 \end{vmatrix} = (2, 3, -5, 6)$$

(d) The eigenvalues of a lower triangular matrix are the diagonal entries.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix}
1 & 1 & -1 \\
4 & -1 & 2 \\
1 & 0 & 2 \\
3 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
R_2 \to R_2 - 4R_1 \\
0 & -5 & 6 \\
0 & -1 & 3 \\
0 & -2 & 4
\end{bmatrix}
\begin{bmatrix}
R_3 \to -R_3 \\
0 & 1 & -3 \\
0 & -5 & 6 \\
0 & -5 & 6 \\
0 & -2 & 4
\end{bmatrix}$$

Although this matrix is not reduced, it already shows that  $(3\sqrt{7}+c_2\sqrt{2}+c_3\sqrt{3}=0 \Rightarrow c_3=0 \Rightarrow c_2=0 \Rightarrow c_1=0$ . Hence, the three vectors are linearly independent.

5. Find the general solution of 
$$D\mathbf{x} = A\mathbf{x}$$
, where  $A$  is the matrix  $\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$ .

Solution.  

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 3 \\ 3 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$$

The eigenvalues are -2, 4,

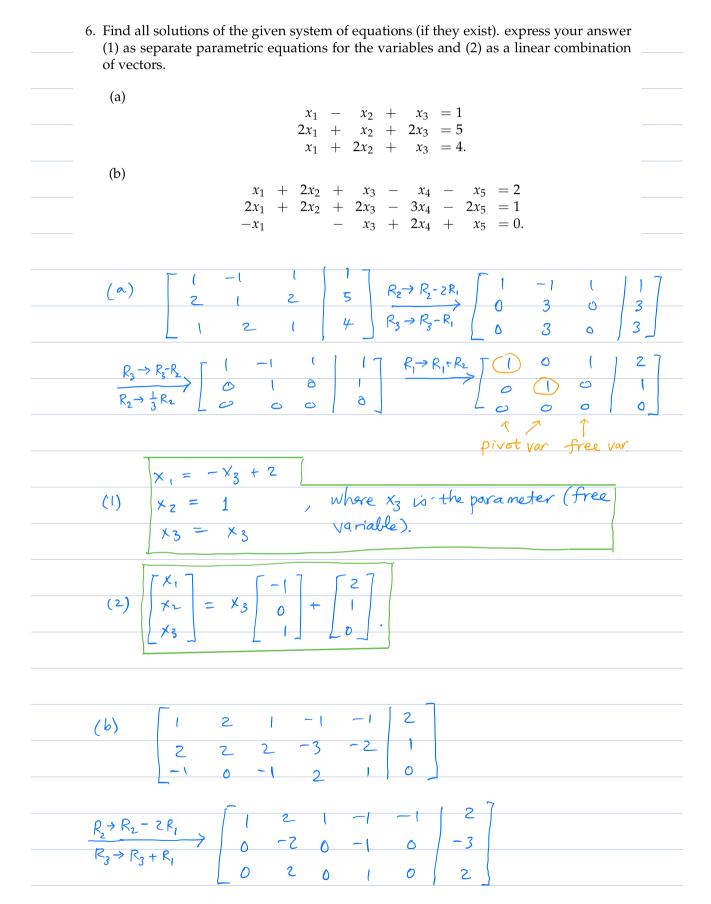
For 
$$\lambda = -2$$
,  $\begin{bmatrix} \lambda I - A & 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 

$$x_1 = x_2$$
  $\Rightarrow$  Eigenvector  $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 \Rightarrow \text{Solution } e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 

For 
$$\lambda = 4$$
,  $\begin{bmatrix} \lambda L - A & \sigma \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

$$X_1 = -X_L$$
  $\Rightarrow x^2 = \begin{bmatrix} -1 \\ X_2 = X_2 \end{bmatrix}$   $\Rightarrow$  Solution  $e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ 

The general solution is 
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
.



$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	
The last equation is $0 = -1$ , which is impossible.	
The last equation is $0 = -1$ , which is impossible. Hence, this system has no solutions	Д