Math 51 Fall 2020

The Laplace Transform
1. Basic Definitions

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The Laplace Transform

The Laplace transform presents an approach to solving initial-value problems for certain differential equations which is quite different from approaches you have seen before.
It is an operator–a "function of functions" which assigns to a function $f(t)$(of the variable t) a new function, $F(s) = \mathcal{L}[f(t)]$
(of a new variable, s),
according to the formula

$$\boxed{F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\ dt.}$$

This formula requires some unpacking.:

- The value of the new function $F(s)$ at a particular numerical value of the variable s is the integral on the right, a kind of weighted average of the values of the old function $f(t)$. In particular, during the integration on the right, s *acts as a constant.*
- The right side of the formula is an improper integral over an unbounded domain; this is defined to be the limit

$$\int_0^\infty e^{-st} f(t)\ dt = \lim_{b\to\infty} \int_0^b e^{-st} f(t)\ dt$$

The integral in this definition looks pretty intimidating, but in practice the actual formulas for the Laplace transforms of the functions we are working with will turn out to be surprisingly benign.

Why bother? An overview

Before proceeding with a study of the Laplace transform, let's understand the goal of this study. It turns out that there is a relation between the transform of a function $\mathcal{L}[f(t)]$ and the transform of its derivatives, $\mathcal{L}\left[D^k f(t)\right]$, involving the value(s) of f and its derivatives at $t = 0$, which allows us to turn an initial-value problem for a differential equation (N) $P(D)x = E(t)$ into an algebra problem involving the transform $F(s) = \mathcal{L}[f(t)]$ of the solution of (N), that is relatively easy to solve for $F(s)$.
Then we will learn how to go back from knowing the transform $F(s)$ of a solution of (N) to a formula for the solution $x = f(t)$ itself.
(This process is called the *inverse Laplace transform.*)

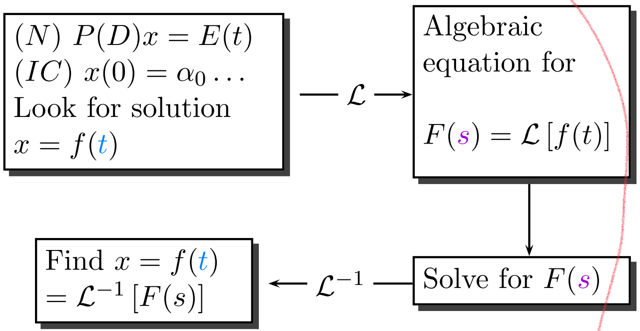

Figure 1: Strategy for using Laplace transform to solve initial-value problems

Basic Calculations

Example:

$$\mathcal{L}\left[e^{\lambda t}\right] = \int_0^\infty e^{-st} e^{\lambda t}\, dt = \lim_{b\to\infty} \int_0^b e^{-(s-\lambda)t}\, dt;$$

Note that if the constant $-(s-\lambda)$ multiplying t in the last expression is positive, the integrand diverges to ∞ as $t \to \infty$, so the (limit of the) integral diverges.

If it is zero, the integrand is b, and the integral still diverges.

So the definition of $F(s) = \mathcal{L}\left[e^{\lambda t}\right]$ only makes sense for $s > \lambda$, and in that case

$$\int_0^b e^{-(s-\lambda)t}\, dt = -\frac{e^{-(s-\lambda)b}}{s-\lambda} + \frac{1}{s-\lambda};$$

we see that the first fraction $\to 0$ as $b \to \infty$, so we get the formula $\boxed{\mathcal{L}\left[e^{\lambda t}\right] = \frac{1}{s-\lambda} \quad \text{for } s > \lambda.}$

Example:

$$\mathcal{L}[\cos bt] = \int_0^\infty e^{-st} \cos bt\, dt = \lim_{b\to\infty} \int_0^b e^{-st} \cos bt\, dt$$

A standard Calc 2 calculation (integration by parts, twice) leads to the indefinite integral

$$\int e^{-st} \cos bt\, dt = \frac{be^{-st}\sin bt - se^{-st}\cos bt}{s^2+b^2} + C;$$

Again, this diverges as $t \to \infty$ unless $s > 0$, in which case it goes to zero; so noting that $\sin 0 = 0$, this leaves us with

$$\boxed{\mathcal{L}[\cos bt] = \frac{s}{s^2+b^2} \quad \text{for } s > 0.}$$

A similar calculation gives us

$$\boxed{\mathcal{L}[\sin bt] = \frac{b}{s^2+b^2} \quad \text{for } s > 0.}$$

Example: By definition,

$$\mathcal{L}\left[t^n\right] = \lim_{b\to\infty} \int_0^b e^{-st} t^n \, dt.$$

Integration by parts gives the formula (for n a positive integer)

$$\int_0^b e^{-st} t^n \, dt = \frac{-b^n e^{-sb}}{s} + \frac{n}{s} \int_0^b e^{-st} t^{n-1} \, dt;$$

When $s > 0$, the first term on the right goes to zero as $b \to \infty$, and the two integrals that are left converge to $\mathcal{L}\left[t^n\right]$ (*resp.* $\mathcal{L}\left[t^n\right]$). So we have the reduction formula

$$\mathcal{L}\left[t^n\right] = \frac{n}{s} \mathcal{L}\left[t^{n-1}\right].$$

To make use of this, we have to have a starting point. But the case $n = 0$ is easy:

$$\mathcal{L}\left[t^0 = 1\right] = \lim_{b\to\infty} \int_0^b e^{-st} 1 \, dt = \lim_{b\to\infty} \frac{1}{s}(1 - e^{-sb}) = \frac{1}{s}.$$

So we get the recursive calculation

$$\mathcal{L}\left[t^0\right] = \frac{1}{s},$$

$$\mathcal{L}\left[t^1\right] = \frac{1}{s}\mathcal{L}\left[t^0\right] = \frac{1}{s^2}, \quad \mathcal{L}\left[t^2\right] = \frac{2}{s}\mathcal{L}\left[t^1\right] = \frac{2\cdot 1}{s^3},$$

$$\ldots, \mathcal{L}\left[t^n\right] = \frac{n}{s}\mathcal{L}\left[t^{n-1}\right] = \frac{n\cdot(n-1)!}{s^n} = \frac{n!}{s^{n+1}}.$$

For n a non-negative integer, $\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}$ for $s > 0$.

Example: **Linearity, baby!**

We know that integration is a linear process: $\int(\alpha f(t) + \beta g(t)) \, dt = \alpha \int f(t) \, dt + \beta \int g(t) \, dt$, and applying this to the integral defining the Laplace transform, we see that the Laplace transform is linear:

$$\boxed{\mathcal{L}\left[\alpha f(t) + \beta g(t)\right] = \alpha \mathcal{L}\left[f(t)\right] + \beta \mathcal{L}\left[g(t)\right].}$$

Using linearity, we can calculate many different Laplace transforms:

Example:

$$\begin{aligned}\mathcal{L}\left[2e^{3t} + 5t^3 - 6\sin 3t\right] &\\ &= 2\mathcal{L}\left[e^{3t}\right] + 5\mathcal{L}\left[t^3\right] - 6\mathcal{L}\left[\sin 3t\right] \\ &= \frac{2}{s-3} + \frac{30}{s^4} - \frac{18}{s^2+9}.\end{aligned}$$

Inverse Laplace Transforms

To find the inverse transform $f(t)$ of a function $F(s)$, we do something analogous to how we find indefinite integrals from tables of derivatives: we reverse-engineer the table of Laplace Transforms. As in the case of indefinite integrals, it is sometimes useful to rewrite our transform formulas in a form adapted to how we use them, The table on the next page shows what we can do with what we have already found.

$F(s) = \mathcal{L}\left[f(t)\right]$	$f(t) = \mathcal{L}^{-1}\left[F(s)\right]$
$\mathcal{L}\left[e^{\lambda t}\right] = \frac{1}{s-\lambda}$	$\mathcal{L}^{-1}\left[\frac{1}{s-\lambda}\right] = e^{\lambda t}$
$\mathcal{L}\left[\cos bt\right] = \frac{s}{s^2+b^2}$	$\mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt$
$\mathcal{L}\left[\sin bt\right] = \frac{b}{s^2+b^2}$	$\mathcal{L}^{-1}\left[\frac{1}{s^2+b^2}\right] = \frac{1}{b}\sin bt$
$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}$	$\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

We note also that the inverse transform is linear:

$$\mathcal{L}^{-1}\left[\alpha F(s) + \beta G(s)\right] = \alpha\mathcal{L}\left[F(s)\right] + \beta\mathcal{L}\left[G(s)\right].$$