1. **First-order nonlinear equation**
   Find the particular solution that satisfies the initial condition:

$$x^2 \frac{dx}{dt} = t^2, \quad x(1) = 2.$$

Solution. First separate the variables:

$$x^2 dx = t^2 dt.$$

Integrate both sides to get

$$\int x^2 dx = \int t^2 dt,$$

$$\frac{x^3}{3} = \frac{t^3}{3} + C_1.$$

Now solve for x in terms of t:

$$x^3 = t^3 + 3C_1.$$

$$x = (t^3 + C)^{1/3},$$

where we relabel the constant $3C_1$ as C.

To determine C, plug in the initial condition $x(1) = 2$:

$$2 = (1 + C)^{1/3} \Rightarrow 8 = 1 + C \Rightarrow C = 7$$

The particular solution is

$$\boxed{x = (t^3 + 7)^{1/3}}.$$

Check.

$$\frac{dx}{dt} = \frac{1}{3}(t^3+7)^{-\frac{2}{3}} 3t^2 = t^2 (t^3+7)^{-\frac{2}{3}}$$

$$x^2 \frac{dx}{dt} = (t^3+7)^{\frac{2}{3}} t^2 (t^3+7)^{-\frac{2}{3}} = t^2. \quad ✓$$

$$x(1) = 8^{1/3} = 2. \quad ✓$$

$\square$

2. **First-order nonlinear equation**
Find the general solution of

$$\frac{dx}{dt} = x^2 - 1, \quad x \geq 1.$$

(*Hint*. Use partial fraction decomposition. Also remember that you cannot divide by 0.)

Solution: To separate the variables x and t, we first divide by x^2-1. However, this is possible only if $x^2-1 \neq 0$.

Case 1. $x^2-1=0$.

Then $x = \pm 1$ and the ode becomes $\frac{dx}{dt} = 0$.

Since $x \geq 1$, the solution is the constant function $x = 1$.

Case 2. $x^2-1 \neq 0$.

Then $\frac{dx}{x^2-1} = dt$.

To integrate $\frac{1}{x^2-1}$, we decompose it into partial fractions:

$$\frac{1}{x^2-1} = \frac{a}{x+1} + \frac{b}{x-1}.$$

Multiply both sides by x^2-1 to get

$$1 = a(x-1) + b(x+1) = (a+b)x + (-a+b).$$

When two polynomials are equal for all x, their corresponding coefficients must be equal. Hence,

$$\begin{cases} a+b = 0 \\ -a+b = 1 \end{cases} \Rightarrow b = a+1$$

So $a+b=0 \Rightarrow a+(a+1)=0 \Rightarrow 2a=-1 \Rightarrow a = -\frac{1}{2}$.

$$b = a+1 = -\tfrac{1}{2}+1 = \tfrac{1}{2}$$

Then $\frac{1}{x^2-1} = \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}$ and the ode becomes

$$\left(-\frac{1}{x+1} + \frac{1}{x-1}\right) dx = 2\,dt.$$

Integrating both sides gives

$$-\ln|x+1| + \ln|x-1| = \ln\left|\frac{x-1}{x+1}\right| = 2t + C.$$

Since $x \geq 1$, we can remove the absolute value

$$\frac{x-1}{x+1} = e^{2t+C}.$$

Now solve for x:

$$\frac{x-1}{x+1} = \frac{x+1-2}{x+1} = 1 - \frac{2}{x+1} = e^{2t+C} = e^{C} e^{2t}.$$

$$\frac{2}{x+1} = 1 - e^{C} e^{2t}$$

Take reciprocals: $$\frac{x+1}{2} = \frac{1}{1-e^{C}e^{2t}}$$

Set $k = e^{C}$: $$\boxed{x = \frac{2}{1-k e^{2t}} - 1 = \frac{1+ke^{2t}}{1-ke^{2t}}}.$$

<u>Check</u>. $$\frac{dx}{dt} = \frac{4ke^{2t}}{(1-ke^{2t})^2}$$

$$x^2+1 = \frac{(1+ke^{2t})^2}{(1-ke^{2t})^2} - \frac{(1-ke^{2t})^2}{(1-ke^{2t})^2} = \frac{4ke^{2t}}{(1-ke^{2t})^2}. \checkmark$$

3. **Compound interest**
When interest is compounded continuously, the rate of change of the principal is proportional to the principal. The constant of proportionality is called the *interest rate*.
(a) Set up a differential equation to model the principal $x = x(t)$ in an account accruing interest at 8% per year, compounded continuously.
(b) Solve the differential equation.
(c) Calculate the percentage increase in such a deposit over one year. (This is called the *effective annual rate*.)

Solution. (a) $\boxed{\frac{dx}{dt} = .08\,x, \qquad x > 0}$.

(b) By separation of variables,

$$\frac{dx}{x} = .08\,dt,$$

$$\int \frac{dx}{x} = \int .08\,dt,$$

$$\ln x = .08t + C$$

$$x = e^C e^{.08t} = k e^{.08t} \quad (\text{Setting } k = e^C).$$

Note that $x(0) = k e^0 = k$. Thus, k is the initial deposit and we can write $\boxed{x(t) = x(0)\, e^{.08t}}$.

(c) Increase in a year $= x(1) - x(0)$

$$= x(0)\, e^{.08} - x(0)$$

Percentage increase $= \dfrac{x(0)\, e^{.08} - x(0)}{x(0)}$

$$= e^{.08} - 1 \approx .0833 = \boxed{8.33\%}.$$

4. **Classification of ode's**
In this problem, no work needs to be shown. Only the answers will be graded.

(a) For each part, write down the order of the differential equation.

(i) $r^4\frac{d^3x}{dt^3} + t\frac{dx}{dt} - x = t^7$ 3

(ii) $\left(\frac{dx}{dt}\right)^5 + \frac{d^4x}{dt^4} - t^3x^7 + t^7 = 0$ 4

(iii) $x^8\frac{dx}{dt} + \frac{d^7x}{dt^7} = x + t^9$ 7

(iv) $(x')^2\, x''' = x^4x'' + t^5x'$ 3

(b) For each of the following o.d.e.'s, decide whether it is linear. If it is linear, is it homogeneous? In this problem, take x to be the dependent variable; $x' = dx/dt$. Set up three columns, the first column for the equation number (i), (ii), (iii), ..., the second column for "linear," and the third for "homogeneous." In each entry, answer the question by writing "Y" or "N" for "Yes" or "No." If the question does not apply to an equation, leave the entry blank.

	linear	homogeneous
(i) $x' + x + t = 0$	Y	N
(ii) $x' + xt = 0$	Y	Y
(iii) $x't + x = 0$	Y	Y
(iv) $x'x + t = 0$	N	—

	linear	homogeneous
(v) $x' + x + t^2 = 0$	Y	N
(vi) $(x')^2 + x + t = 0$	N	—
(vii) $x' + x^2 + t = 0$	N	—
(viii) $x't + xt^2 + 1 = 0$	Y	N

(c) For each of the following linear differential equations, decide whether it is normal on $0 < t < 2$. Answer "Yes" or "No."

(i) $(t-1)\frac{dx}{dt} - 5x = 3t$ No $(t-1=0$ at $t=1)$

(ii) $t\frac{dx}{dt} - e^tx = \sin t$ Yes $(t$ is never 0 on $(0,2))$. $\square$

5. **First-order equation.**
Find the general solution of $x' + 3x = e^{-3t}\cos 2t$ on $(-\infty, \infty)$.

Solution. First find the general solution of the homogeneous equation $x' = -3x$. It is

$$x = k e^{-3t}, \quad \text{where } k \text{ is a constant.}$$

Now let k be a function of t.

$$x' = k' e^{-3t} - 3k e^{-3t}$$

$$x' + 3x = k' e^{-3t} - \cancel{3k e^{-3t}} + \cancel{3k e^{-3t}}$$

$$= k' e^{-3t} = e^{-3t} \cos 2t.$$

This simplifies to $k' = \cos 2t$

$$k = \frac{1}{2}(\sin 2t) + C.$$

The general solution of $x' + 3x = e^{-3t}\cos 2t$ is

$$\boxed{x = \left(\frac{1}{2}(\sin 2t) + C\right) e^{-3t}, \quad t \in (-\infty, \infty)}$$

Using integrating factor

The integrating factor is $e^{\int r(t)\,dt} = e^{\int 3\,dt} = e^{3t}$.

Multiply the equation by the integrating factor

$$\underbrace{e^{3t} x' + 3e^{3t} x} = \cos 2t.$$

$$(e^{3t} x)' = \cos 2t$$

$$e^{3t} x = \frac{1}{2}(\sin 2t) + C$$

$$\boxed{x = \left(\frac{1}{2}(\sin 2t) + C\right) e^{-3t}, \quad t \in (-\infty, \infty)}.$$

6. **Variation of parameter.**

Try variation of parameters on the differential equation $x' + x^2 = t$ as follows.

(a) Find the general solution of $x' + x^2 = 0$ by separating variables. There will be a parameter in the solution representing a constant.

(b) Let the parameter vary and substitute the solution back into the original nonhomogeneous equation. What goes wrong? What can you conclude about the method of variation of parameter?

Solution. (a)

$$\frac{dx}{dt} = -x^2$$

$$-\frac{dx}{x^2} = dt \qquad (x \neq 0 \text{ since } x' + x^2 = t)$$

$$\int -\frac{dx}{x^2} = \int dt$$

$$\frac{1}{x} = t + k.$$

$$\boxed{x = \frac{1}{t+k}}.$$

(b) Let $k = k(t)$. Then

$$x' = -\frac{k'}{(t+k)^2}$$

$$x' + x^2 = -\frac{k'}{(t+k)^2} + \frac{1}{(t+k)^2}$$

$$= \boxed{\frac{1-k'}{(t+k)^2} = t}$$

This differential equation in k is worse than the original differential equation in x. For a linear first-order equation, we always end up with

$$k'(t) = \text{some function of } t,$$

which is easy to solve.

Conclusion. Variation of parameters does not work well for nonlinear nonhomogeneous first-order differential equations.