PS #9 Solutions

1. In reduced-row-echelon form, A becomes

$$\begin{bmatrix} 1 & 0 & 4/5 & 9/5 \\ 0 & 1 & 2/15 & -1/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From here, one can find that the general solution is of the form

$$X = \left(x_3 \begin{bmatrix} -4/5 \\ -2/15 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -9/5 \\ 1/5 \\ 0 \\ 1 \end{bmatrix} \right) \cdot C$$

for any $C \in \mathbb{R}$. where x_3 + x_4 are free variables.

2a As A is upper triangular, the eigenvalues are 3 + 1, with 1 having a multiplicity of 2.

b. Given the tools currently at our disposal, we cannot compute the general solution as existence + uniqueness tells us we should have 3 linearly independent solutions. We currently only have 2!

3a. The eigenvalues are $-2, -1, 0, + 1$.

b. for $\lambda = 0$, the corresponding eigenvector is parallel to $\vec{v} = \begin{pmatrix} 9 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

4. The eigenvalues of A are $-1, 1, + 2$ with corresponding eigenvectors

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, + \begin{pmatrix} 8 \\ 10 \\ 3 \end{pmatrix}.$$

Thus, the general solution is

$$\vec{X}_{gen} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 8 \\ 10 \\ 3 \end{pmatrix}$$

5. This matrix has eigenvalues of $\frac{5 + \sqrt{17}}{2}$ + $\frac{5 - \sqrt{17}}{2}$ with eigenvectors $\begin{pmatrix} -3 + \sqrt{17} \\ 4 \end{pmatrix}$ + $\begin{pmatrix} -3 - \sqrt{17} \\ 4 \end{pmatrix}$. The general solution is

$$\vec{X}_{gen} = c_1 e^{\frac{5+\sqrt{17}}{2}t} \begin{pmatrix} -3 + \sqrt{17} \\ 4 \end{pmatrix} + c_2 e^{\frac{5-\sqrt{17}}{2}t} \begin{pmatrix} -3 - \sqrt{17} \\ 4 \end{pmatrix}$$

To find the coefficients given the initial condition $X(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, solve

$$\begin{bmatrix} -3 + \sqrt{17} & -3 - \sqrt{17} \\ 4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{3}{\sqrt{17}} \\ \frac{1}{2} - \frac{3}{\sqrt{17}} \end{bmatrix}$$