Review material for Midterm 1

Math 51 Spring 2022

exam date 2022-02-14

The midterm will cover course material from the following sections of (Nitecki and Guterman 1992):

- Introduction to differential equations §1.1
- Separation of variables §1.2
- First–order linear ODEs §1.3
- Existence & Uniqueness Theorem §1.6
- Linear differential equations §2.2
- Homogeneous linear ODEs & the Wronskian §2.3
- Linear independence of functions §2.4
- const coeff linear ODES (Real Roots) §2.5

This is the material covered in the lectures; you can find the corresponding lecture notes on Canvas

You should review the problem sets PS1, PS2, PS3, PS4.

Here are some further review problems:

1. (multiple-choice) Which of the following represents a linear ODE?

   a. $x \cdot x'' + x + 1 = \sin(t)$

   b. $t \cdot x'' + x^2 + 1 = \sin(t)$

   c. $t^2 \cdot x'' + (t+1) \cdot x + 1 = \sin(t)$

   d. $(D^2 + D + t)x^2 = \sin(t)$

2. (multiple-choice) Consider the Wronskian $W(t) = W(f_1, f_2, f_3)(t)$ of the functions

   $$f_1(t) = 1, \qquad f_2(t) = 1 + t \qquad \text{and} \qquad f_3(t) = \ln(1+t).$$

   Which of the following statements is most correct?

   a. The Wronskian is given by $W(t) = -1/(1+t)^2$; since $W(1) = -1/4$ is non-zero, the functions are linearly independent on the interval $(-1, \infty)$.

   b. Since $W(1) = 0$, the functions are linearly dependent on $(-1, \infty)$.

   c. Since $W(t)$ is not defined on $(-\infty, \infty)$, the Wronskian test doesn't apply.

   d. None of the above.

3. (multiple-choice) Let $P(D)$ be a differential operator of order 4, and suppose that $h_1(t), h_2(t), h_3(t), h_4(t)$ are solutions to the homogeneous equation

   $$(\heartsuit) \quad P(D)x = 0.$$

   Suppose that

   $$h_1(t) + h_2(t) + h_3(t) + h_4(t) = 0$$

   for every t, $-\infty < t < \infty$.

Which of the following statements is most correct?

a. The general solution to ($\heartsuit$) is given by

$$x(t) = c_1 h_1(t) + c_2 h_2(t) + c_3 h_3(t) + c_4 h_4(t).$$

b. The functions $h_1(t), h_2(t), h_3(t), h_4(t)$ are linearly dependent.

c. A particular solution to ($\heartsuit$) has the form

$$q(t) = \int h_1(t)dt.$$

d. For some values of k_1, k_2 and k_3, the expression $q(t) = k_1 h_1(t) + k_2 h_2(t) + k_3 h_3(t)$ provides a particular solution to the ODE
$$P(D)x = e^t.$$

4. Consider the ODE $\dfrac{dx}{dt} = x^2 \cos(t)$.

   a. Find the general solution to this ODE.

   b. Find a solution x satisfying $x(0) = 1$.

   c. What is the largest interval containing $t_0 = 0$ on which this solution is defined?

5. Consider the differential equation

$$(\clubsuit) \quad (t+2)\frac{dx}{dt} + 2x = t + 1.$$

   a. Find the largest interval containing $t_0 = 0$ on which this equation is normal.

   b. Find the general solution to ($\clubsuit$).

6. Consider the ODE $\dfrac{dx}{dt} = \dfrac{x}{t} + 1$ for $t > 0$.

   a. Find the general solution $x(t)$ to this ODE.

   b. find the particular solution of the ODE for which $x(1) = 0$.

7. For what value(s) of α is the determinant

$$\det \begin{bmatrix} 1 & \alpha & 1 \\ 1 & 1 & 0 \\ 0 & 1 & \alpha \end{bmatrix}$$

   equal to 0?

8. Consider the system of linear equations

$$\begin{aligned} u_1 + 2u_2 + 3u_3 &= -1, \\ 3u_1 + 2u_2 + 1u_3 &= -1, \\ 5u_1 - 2u_2 + 2u_3 &= -1. \end{aligned}$$

   The coefficient matrix has
$$\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 5 & -2 & 2 \end{bmatrix} = -48$$

   Use Cramer's Rule to give a formula for u_3 in terms of determinants. Do not evaluate the determinants.

9. Consider a 3rd order linear ODE which is normal on $(-\infty, \infty)$ and suppose that x_1 and x_2 are solutions. Which of the following statements is most correct?
   a. If $x_1(0) = x_2(0)$ and $x_1(1) = x_2(1)$, then $x_1 = x_2$.
   b. If $x_1(0) = x_2(0)$, $x_1'(0) = x_2'(0)$ and $x_1''(0) = x_2''(0)$, then $x_1 = x_2$.
   c. If $x_1(0) > 0$ then it is also true that $x_2(0) > 0$.
10. Indicate which of the following ODEs is normal on the interval $(0, 2\pi)$.
    a. $\dfrac{1}{t}\dfrac{d^3x}{dt^3} + \sin(t)\dfrac{dx}{dt} = \cos(t)$.
    b. $D^2x + \cos(t)Dx = \ln(t-1)$
    c. $(t+1)D^5x + x = \dfrac{1}{\cos(t/8)}$
11. Consider the functions

$$h_1(t) = -1 + 7t + 8t^2, \qquad h_2(t) = 1 + 2t + t^2, \qquad h_3(t) = -1 + t + 2t^2.$$

    a. Find constants a, b so that $h_1(t) = a \cdot h_2(t) + b \cdot h_3(t)$.

       Hint: Equate coefficients of powers of t.
    b. Are the functions $h_1(t), h_2(t), h_3(t)$ linearly dependent? (What does your answer to (a) tell you about linear dependence?)
12. A particular solution to the equation

$$(\clubsuit) \qquad (D^2 - 16)x = e^{4t}$$

    is $p(t) = \dfrac{1}{8} \cdot t \cdot e^{4t} + e^{-4t}$. Find the general solution.
13. Use the exponential shift formula

$$P(D)[e^{\lambda t}y] = e^{\lambda t}P(D+\lambda)[y]$$

    to compute the function $P(D)[f]$ in each of the following cases:
    a. $P(D) = D^2 + D - 6$ and $f = t^2e^{2t}$.
    b. $P(D) = D^2 + 3$ and $f = e^t\cos(3t)$.
    c. $P(D) = (D+3)(D-1)^2$ and ft^2e^t.
14. Find the general solution to the differential equation

$$(t+1)x' = \frac{x}{t-1}, \quad t > 1.$$

15. Solve the initial value problem

$$(\clubsuit) \quad 2\frac{dx}{dt} - x = t \cdot e^t; \quad x(0) = 1$$

16. Solve the initial value problem

$$4x'' + 4x' - 3x = 0; \quad x(0) = 0, \quad x'(0) = 1.$$

Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. Differential Equations: A First Course. Saunders.