

# Math 51 Spring 2022 - Final Exam - some review problems

2022-04-27

- Differential equations via integration (§ 1.1)
  - Separation of variables (§ 1.2)
  - Linear Differential Equations (§ 1.3)
  - Existence & Uniqueness; Linear ODEs (§ 1.6, 2.2)
  - Cramer's Rule and the Wronskian (§ 2.3 & App. A)
  - Linear Independence (§ 2.4)
  - Const coeff linear ODEs (real roots) (§ 2.5)
  - Const coeff linear ODEs (complex roots) (§ 2.6)
  - Non-homog linear ODEs via undetermined coeffs (§ 2.7)
  - Non-homog linear ODEs via variation of parameters (§ 2.8)
  - Linear Systems (§ 3.2, 3.3)
  - Linear systems and independence (§ 3.4)
  - Eigenvalues, Eigenvectors (§ 3.5)
  - Row Reduction (§ 3.6)
  - Homogeneous linear systems (real roots) (§ 3.7)
  - Homogeneous linear systems (complex roots) (§ 3.8)
  - Homogeneous linear systems (double roots) (§ 3.9)
  - Homogeneous linear systems (higher multiplicity roots) (§ 3.10)
  - Non-homogeneous Systems (§ 3.11)
  - The Laplace transform  $\mathcal{L}$  and initial value problems (§ 5.2, 5.3)
  - Properties of  $\mathcal{L}$  and  $\mathcal{L}^{-1}$  (§ 5.4)
  - Piecewise functions (§ 5.5)
  - ~~Convolution (§ 5.6)~~
1. Indicate which of the following best represents a *simplified guess* for a particular solution  $p(t)$  to the non-homogeneous linear ODE:
- $$(D - 3)(D - 1)x = te^{3t} + \cos(2t)$$
- a.  $p(t) = k_1 te^{3t} + k_2 \cos(2t) + k_3 \sin(2t)$
  - b.  $p(t) = k_1 te^{3t} + k_2 \cos(2t)$
  - c.  $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t)$
  - d.  $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t) + k_4 \sin(2t)$
2. Indicate which of the following represents the general solution to the homogeneous linear ODE  $(D^2 - 2D + 2)^2 x = 0$ .
- a.  $h(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t) + c_3 t e^{-t} \cos(t) + c_4 t e^{-t} \sin(t)$

- b.  $h(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t)$   
 c.  $h(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t) + c_3 t e^t \cos(t) + c_4 t e^t \sin(t)$   
 d.  $h(t) = c_1 t e^t \cos(t) + c_2 t e^t \sin(t) + c_3 t^2 e^t \cos(t) + c_4 t^2 e^t \sin(t)$
3. The matrix  $A = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix}$  has characteristic polynomial  $\lambda^2 - 2\lambda + 2$  and thus its eigenvalues are  $\lambda = 1 + i$  and  $\lambda = 1 - i$ .

Which of the following is an eigenvector for  $A$ ?

- a.  $A$  has no eigenvectors.  
 b.  $\begin{bmatrix} 3 - i \\ 2 \end{bmatrix}$   
 c.  $\begin{bmatrix} 2 \\ -3 + i \end{bmatrix}$   
 d.  $\begin{bmatrix} 3 + i \\ 2 \end{bmatrix}$
4. Consider the linear system of ODEs

$$(\diamond) \quad D\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

A third order linear ODE is *equivalent* to this system if for each of its solutions  $x(t)$ , the vector-valued function  $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x'(t) \\ x''(t) \end{bmatrix}$  is a solution to  $(\diamond)$ . Which of the following linear ODEs is equivalent to  $(\diamond)$ ?

- a.  $(D^3 - 2D^2 - D - 5)x = e^t$   
 b.  $(D^3 - 5D^2 - D - 2)x = e^t$   
 c.  $(D^3 + 2D^2 + D + 5)x = -e^t$   
 d.  $(D^3 + 5D^2 + D + 2)x = -e^t$
5. Let  $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ .  $\lambda = 2$  is an eigenvalue of  $A$  with multiplicity two. The matrix  $A - 2\mathbf{I}_3$

satisfies  $(A - 2\mathbf{I}_3)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Thus the generalized eigenvectors of  $A$  for  $\lambda = 2$  are generated by  $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ .

Which of the following represents a solution  $\mathbf{h}(t)$  to the system  $D\mathbf{x} = A\mathbf{x}$  with the property that  $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ ?

a.  $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

b.  $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 + 12t \\ 2 \\ 6 \end{bmatrix}$

c.  $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 + t \\ 2 \\ 6 \end{bmatrix}$

d. No solution  $\mathbf{h}(t)$  has the property that  $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ .

6. Consider the homogeneous system  $(\diamond) \quad D\mathbf{x} = A\mathbf{x}$  where  $A$  is a  $3 \times 3$  matrix, and let  $\mathbf{h}_1(t), \mathbf{h}_2(t)$  be solutions to  $(\diamond)$ . Which of the following statements is correct?

a.  $\mathbf{h}_1(0)$  and  $\mathbf{h}_2(0)$  are *eigenvectors* for  $A$ .

b. The system  $(\diamond)$  has exactly two solutions.

c. If the vectors  $\mathbf{h}_1(0), \mathbf{h}_2(0)$  are linearly independent, then the general solution to  $(\diamond)$  is given by  $\mathbf{x}(t) = c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$ .

d. None of the above statements is correct.

7. The matrix  $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$  has characteristic polynomial  $\lambda(\lambda - 3)$  and hence has eigenvalues  $\lambda = 0$  and  $\lambda = 3$ . An eigenvector for  $\lambda = 0$  is given by  $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  and an eigenvector for  $\lambda = 3$  is given by  $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

Find a particular solution  $\mathbf{p}(t)$  for the system of linear ODEs

$$D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

8. Let  $A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$ .

The characteristic polynomial of  $A$  is  $r^2 - 4r + 5$  so the eigenvalues of  $A$  are  $\lambda = 2 \pm i$ .

Moreover,  $\mathbf{v} = \begin{bmatrix} 2 - i \\ 5 \end{bmatrix}$  is an eigenvector for  $\lambda = 2 + i$ .

a. Find the general solution to  $D\mathbf{x} = A\mathbf{x}$ .

b. Solve the initial value problem  $D\mathbf{x} = A\mathbf{x}$ ,  $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

9. Solve the initial value problem  $(4D^2 - 4D + 1)x = 0$ ,  $x(2) = x'(2) = e$ .

10. Consider the matrix  $B = \begin{bmatrix} 5 & -3 & -6 \\ 0 & 2 & 0 \\ 3 & -3 & -4 \end{bmatrix}$ .

- a. The vector  $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$  is an eigenvector for  $B$ . What is the corresponding eigenvalue?

**Hint:** Compute the vector  $B\mathbf{v}$  and compare with  $\mathbf{v}$ .

- b. Find an eigenvector for  $B$  for the eigenvalue  $\lambda = -1$ .

11. Laplace Transforms:

- a. Compute the inverse Laplace transform  $\mathcal{L}^{-1}[F(s)]$  of the function  $F(s) = \frac{3s^2 + s + 1}{(s+1)(s^2+2)}$ .

- b. If  $x$  is a solution to  $(D^2 + D + 1)x = 1$  with  $x(0) = 0$  and  $x'(0) = 1$ , find an expression for  $\mathcal{L}[x]$  as a function of  $s$ .

12. Let  $W = W(h_1(t), h_2(t))$  denote the *Wronskian matrix* of the functions  $h_1(t) = e^{2t}$  and  $h_2(t) = te^{2t}$ . Which of the following represents the *determinant* of  $W$ ?

- a.  $e^{4t}$   
 b.  $(1 + 4t)e^{4t}$   
 c.  $e^{2t}$   
 d.  $(1 + 4t)e^{2t}$

13. Consider the vectors  $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ , and  $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$  in  $\mathbf{R}^4$ , and let  $A =$

$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$  be the  $4 \times 3$  matrix whose columns are the  $\mathbf{v}_i$ . Which of the following statements is correct?

- a. The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are *linearly dependent*.

- b. Since  $A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ , the only solution to the equation  $A\mathbf{w} = \mathbf{0}$  is  $\mathbf{w} = \mathbf{0}$  so the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are *linearly independent*.

- c. The equation  $A\mathbf{w} = \mathbf{x}$  has a solution for every vector  $\mathbf{x}$  in  $\mathbf{R}^4$ .

- d. The determinant of  $A$  is  $\neq 0$ .

14. Let  $A$  be an  $n \times n$  matrix with constant coefficients  $a_{ij}$ , and let  $\mathbf{E}(t)$  be a vector with  $n$  components. If  $\mathbf{v}$  is any vector in  $\mathbf{R}^n$ , must there be a solution  $\mathbf{x}(t)$  to the system of equations  $D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$  for which  $\mathbf{x}(0) = \mathbf{v}$ ?

- a. No, this conclusion is only guaranteed when the system is *homogeneous*.  
 b. No, this conclusion is only guaranteed when the entries of the vector  $\mathbf{E}(t)$  are *constant* functions of  $t$ .

- c. Yes, this conclusion is the content of the *Existence and Uniqueness Theorem for Solutions of Linear Systems*.
- d. No, this conclusion is only guaranteed when  $\det A \neq 0$ .
15. Consider the homogeneous system  $(\diamond) \quad D\mathbf{x} = A\mathbf{x}$  where  $A$  is a  $3 \times 3$  matrix.
- a. If  $\mathbf{h}(t)$  is a solution, must  $\mathbf{h}(0)$  be an eigenvector for  $A$ ? Why or why not?
- b. Show that the vectors  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  are linearly independent ~~dependent~~.
- c. Let  $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$  be solutions to  $(\diamond)$ . Suppose that  $\mathbf{h}_1(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{h}_2(0) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ , and  $\mathbf{h}_3(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  are the vectors from b. Do the solutions  $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$  generate the general solution to  $(\diamond)$ ? Why or why not?
16. A drug is absorbed by the body at a rate proportional to the amount of the drug present in the bloodstream after  $t$  hours. If there are  $x(t)$  mg of the drug present in the bloodstream at time  $t$ , assume that the drug is absorbed at a rate of  $0.5x(t)$  /hour. If a patient receives the drug intravenously at a constant rate of 3 mg/hour, to which of the following ODEs is  $x(t)$  a solution?
- a.  $x'(t) = -0.5x(t) + 3$
- b.  $x'(t) = -0.5x(t); \quad x(0) = 3$
- c.  $x'(t) = 0.5x(0) + 3$
- d.  $x'(t) = .5x(t) - 3$

17. You are given that a particular solution to

$$(\heartsuit) \quad (D^2 - 2D + 1)x = e^t$$

is  $p(t) = \frac{t^2 e^t}{2}$ . Which of the following best represents the general solution to  $(\heartsuit)$ ?

- a.  $c_1 e^t + c_2 t e^t$ .
- b.  $\frac{t^2 e^t}{2} + c_1 e^t + c_2 t e^t$ .
- c.  $\frac{t^2 e^t}{2} + c e^t$ .
- d.  $\frac{t^2 e^t}{2} + c_1 e^t + c_2 e^{-t}$ .
18. Let  $x_1(t)$  and  $x_2(t)$  be solutions to the ODE  $(t+1)x'' + x' + x = 0$ . Suppose that  $x_1(0) = x_2(0)$  and that  $x_1'(0) = x_2'(0)$ . Which of the following statements is most correct?
- a.  $x_1(t) = x_2(t)$  for every  $t$ .

- b. Since the ODE is *normal* on the interval  $(-1, \infty)$ , we can conclude that  $x_1(t) = x_2(t)$  for  $-1 < t < \infty$ .
- c. No conclusion is possible because the existence and uniqueness theorem does not apply to this ODE.
- d. We can only conclude that  $x_1(t) = x_2(t)$  for all  $t$  if we also assume that  $x_1''(0) = x_2''(0)$ .

19. Show that the functions

$$f_1(t) = e^t \cos(t), \quad f_2(t) = e^t \sin(t), \quad f_3(t) = e^t$$

are linearly independent.

You have been told that functions like this are independent. However, here we want you to demonstrate it directly in this case. You may use the *Wronskian test* (with all details needed to justify using it) or other, direct arguments from the definition.

20. Find the Laplace transform of the function

$$f(t) = \begin{cases} 1 & \text{for } t < 1, \\ t - 1 & \text{for } 1 \leq t < 2, \\ 1 & \text{for } t \geq 2. \end{cases}$$

21. Suppose  $g(t)$  is the inverse Laplace transform of

$$F(s) = \frac{2se^{\pi s/2}}{(s^2 + 4)}.$$

Find  $g\left(\frac{\pi}{4}\right)$ .