MATH 51: HOMEWORK 6 SOLUTION

Homework exercise solution:

1. The general solution of the related homogeneous equation is $x = H(t)$, with
$$H(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$$
We look for a particular solution in the form $x = p(t)$ with
$$p(t) = c_1(t) e^{\frac{1}{2}t} + c_2(t) t e^{\frac{1}{2}t}$$
In this case the system of equations reads
$$c_1'(t) e^{\frac{1}{2}t} + c_2'(t) t e^{\frac{1}{2}t} = 0$$
$$\frac{1}{2} c_1'(t) e^{\frac{1}{2}t} + c_2'(t) e^{\frac{1}{2}t} + \frac{1}{2} c_2'(t) t e^{\frac{1}{2}t} = \frac{2}{t^2} e^{\frac{1}{2}t}$$
Multiply the second formula by 2 and minus the first formula from it, we have
$$2c_2'(t) e^{\frac{1}{2}t} = \frac{4}{t^2} e^{\frac{1}{2}t}; \quad c_2'(t) = \frac{2}{t^2}; \quad c_2(t) = -\frac{2}{t}$$
Insert $c_2' = \frac{2}{t^2}$, we have
$$c_1'(t) e^{\frac{1}{2}t} + \frac{2}{t} e^{\frac{1}{2}t} = 0; \quad c_1'(t) = -\frac{2}{t}; \quad c_1(t) = -2\ln|t|$$
Our particular solution is $x = p(t)$ with
$$p(t) = -2\ln|t| e^{\frac{1}{2}t} - 2e^{\frac{1}{2}t}$$
The general solution is
$$x = H(t) + p(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t} - 2\ln|t| e^{\frac{1}{2}t} - 2e^{\frac{1}{2}t}$$

2. The general solution of the related homogeneous equation is $x = H(t)$, with
$$H(t) = c_1 \cos t + c_2 \sin t$$
We look for a particular solution in the form $x = p(t)$ with
$$p(t) = c_1(t)\cos t + c_2(t)\sin t$$
In this case the system of equations reads
$$c_1'(t)\cos t + c_2' \sin(t) = 0$$
$$-c_1'(t)\sin t + c_2' \cos(t) = \tan t$$
Cramer's rule provides formulas for $c_1'(t)$ and $c_2'(t)$
$$c_1'(t) = \frac{-\sin t \tan t}{\cos^2 t + \sin^2 t} = \frac{\sin^2 t}{\cos t} = \frac{1 - \cos^2 t}{\cos t} = \sec t - \cos t$$
$$c_2'(t) = \frac{\cos t \tan t}{\cos^2 t + \sin^2 t} = \sin t$$
We integrate these formulas to get
$$c_1 = \ln|\sec t + \tan t| - \sin t$$
$$c_2 = -\cos t$$

Our particular solution is $x = p(t)$ with

$$p(t) = \ln|\sec t + \tan t| \cos t$$

The general solution is

$$x = H(t) + p(t) = c_1 \cos t + c_2 \sin t + \ln|\sec t + \tan t| \cos t$$

3. Assume $x = t$, then we have $Dx = 1$ and $D^2x = 0$, therefore

$$((t-1)D^2 - tD + 1)x = -t + t = 0$$

hence $x = t$ is a solution of the homogeneous equation.

Assume $x = e^t$, then we have $Dx = e^t$ and $D^2x = e^t$, therefore

$$((t-1)D^2 - tD + 1)x = (t-1)e^t - te^t + e^t = 0$$

hence $x = e^t$ is a solution of the homogeneous equation. The general solution of the related homogeneous equation is $x = H(t)$, with

$$H(t) = c_1 t + c_2 e^t$$

We look for a particular solution in the form $x = p(t)$ with

$$H(t) = c_1(t)t + c_2(t)e^t$$

In this case the system of equations read

$$c_1'(t)t + c_2'(t)e^t = 0$$

$$c_1'(t) + c_2'(t)e^t = (t-1)e^t$$

Minus the second equation from the first, we have

$$c_1'(t) = -e^t; \quad c_1(t) = -e^t$$

Insert $c_1'(t) = -e^t$, we have

$$c_2'(t)e^t = te^t; \quad c_2'(t) = t; \quad c_2(t) = \frac{1}{2}t^2$$

Our particular solution is $x = p(t)$ with

$$p(t) = -te^t + \frac{1}{2}t^2e^t$$

The general solution is

$$x = H(t) + p(t) = c_1 t + c_2 e^t - te^t + \frac{1}{2}t^2 e^t$$

4. (a).

$$D\mathbf{x}_1(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad D\mathbf{x}_2(t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(b).

$$A\mathbf{x}_1(t) + \mathbf{E} = \begin{bmatrix} \cos t + t \\ -\sin t - 1 \end{bmatrix} \quad A\mathbf{x}_2(t) + \mathbf{E} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(c). $\mathbf{x}_2$ is a solution and $\mathbf{x}_1$ is not a solution.

5. (a). Linear third order nonhomogeneous system.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & -t & -1 \\ -\frac{1}{t} & 0 & -1 \\ -1 & -t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix}$$

(b). Nonlinear system.

(c). Linear second order homogeneous system.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

6. (a).

$$(D-1)^2(D+1)x = (D^3 - D^2 - D + 1)x = t$$

Suppose $x = x(t)$ is a solution and set

$$x_1 = x; \quad x_2 = x'; \quad x_3 = x''$$

Then we have

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -x_1 + x_2 + x_3 + t \end{aligned}$$

(b). Using the method of undetermined coefficients to solve the general solution. The homogeneous equation has root 1 with multiplicity of 2 and root -1 with multiplicity of 1. Hence we have

$$H(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t}$$

Consider the annihilator of t has root 0 with multiplicity of 2. Hence we have

$$D^2(D-1)^2(D+1)x = 0$$

with particular solution

$$p(t) = k_1 + k_2 t$$

then we have

$$(D^3 - D^2 - D + 1)p(t) = t$$

with $D^3 p(t) = 0$, $D^2 p(t) = 0$, $Dp(t) = k_2$

$$-k_2 + k_1 + k_1 t = t$$

Hence we have $k_1 = 1$ and $k_2 = 1$. The general solution of (N) is

$$x(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t} + t + 1$$

The general solution of (S_N) is

$$\begin{aligned} x_1 &= c_1 e^t + c_2 t e^t + c_3 e^{-t} + t + 1 \\ x_2 &= c_1 e^t + c_2 e^t + c_2 t e^t - c_3 e^{-t} + 1 \\ x_3 &= c_1 e^t + 2c_2 e^t + c_2 t e^t + c_3 e^{-t} \end{aligned}$$

(c).

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

(d).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} te^t \\ e^t + te^t \\ 2e^t + te^t \end{bmatrix} + c_3 \begin{bmatrix} e^{-t} \\ -e^{-t} \\ e^{-t} \end{bmatrix} + \begin{bmatrix} t+1 \\ 1 \\ 0 \end{bmatrix}$$