

# Problem Set 7

## Variation of parameters & Linear systems of ODEs

Math 51 Fall 2021

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These problems concern (Nitecki and Guterman 1992, sec. 2.8 & §3.2).

1. Find the general solution to

$$x'' - 2x' + x = e^t \ln(t), \quad t > 0.$$

### Solution:

Since the characteristic polynomial is  $r^2 - 2r + 1 = (r - 1)^2$ , the general solution to the corresponding homogeneous equation is generated by

$$h_1(t) = e^t, \quad h_2(t) = te^t.$$

To solve the inhomogeneous equation, it isn't clear how to use the method of *undetermined coefficients* (what should  $A(D)$  be!?) So we use *variation of parameters*.

The Wronskian matrix is given by

$$W = W(h_1, h_2)(t) = \det \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix}.$$

And following the method, we need to solve the matrix equation

$$W \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix} \cdot \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \ln(t) \end{bmatrix}.$$

We now use *Cramer's Rule* to find  $c_i'$ . First note that  $\det W = (t+1)e^{2t} - te^{2t} = e^{2t}$ .

We now have

$$c_1' = \frac{\det \begin{bmatrix} 0 & te^t \\ e^t \ln(t) & (t+1)e^t \end{bmatrix}}{\det W} = \frac{-e^{2t} \ln(t)t}{e^{2t}} = -t \ln(t)$$

and

$$c_2' = \frac{\det \begin{bmatrix} e^t & 0 \\ e^t & e^t \ln(t) \end{bmatrix}}{\det W} - \frac{\ln(t)e^{2t}}{e^{2t}} = \ln(t).$$

We now find the functions  $c_1, c_2$  by anti-differentiation:

$$c_1(t) = \int c_1' dt = - \int t \ln(t) dt = \frac{-t^2 \ln(t)}{2} + \frac{t^2}{4} + C = \frac{t^2(1 - 2 \ln(t))}{4} + C$$

(integrate by parts with  $u = \ln(t)$  and  $dv = t$ ).

and

$$c_2(t) = \int c_2' dt = \int \ln(t) dt = t \ln(t) - t + D$$

Now, we only need a particular solution to the ODE, so we take  $C = D = 0$ , and we find the particular solution

$$\begin{aligned} p(t) &= c_1(t)h_1(t) + c_2(t)h_2(t) \\ &= \left( \frac{t^2(1 - 2\ln(t))}{4} \right) \cdot e^t + (t \ln(t) - t) \cdot te^t \\ &= \left( \frac{-2\ln(t) + 1}{4} \right) \cdot t^2 e^t + \frac{4\ln(t) - 4}{4} \cdot t^2 e^t \\ &= \frac{2\ln(t) - 3}{4} t^2 e^t \end{aligned}$$

Now the general solution is

$$x(t) = p(t) + k_1 e^t + k_2 t e^t = \frac{2\ln(t) - 3}{4} t^2 e^t + k_1 e^t + k_2 t e^t$$

for constants  $k_1, k_2$ .

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2. Verify that  $t$  and  $e^t$  are solutions of the homogeneous equation corresponding to

$$(N) \quad [(t-1)D^2 - tD + 1]x = (t-1)^2 e^t, \quad t > 1,$$

and find the general solution of the nonhomogeneous equation (N).

(Remember to put the ODE in *standard form* when using *variation of parameters*!)

**Solution:**

To verify that  $t$  is a homogeneous solution, we compute

$$((t-1)D^2 - tD + 1)[t] = (t-1)D^2[t] - tD[t] + t = 0 - t + t = 0$$

and to verify that  $e^t$  is a homog. solution, we compute

$$((t-1)D^2 - tD + 1)[e^t] = (t-1)D^2[e^t] - tD[e^t] + e^t = (t-1)e^t - te^t + e^t = 0.$$

We are going to describe a particular solution in the form  $p(t) = c_1(t)t + c_2(t)e^t$ ; we'll find the functions  $c_1$  and  $c_2$  using our *Wronskian method*. In order to use this method, we need to first put the ODE in *standard form*. i.e.

$$\left[ D^2 - \frac{t}{t-1}D + \frac{1}{t-1} \right] x = (t-1)e^t, \quad t > 1,$$

Now let  $W$  denote the Wronskian matrix  $W = \begin{bmatrix} t & e^t \\ 1 & e^t \end{bmatrix}$ . We must solve the matrix equation

$$W \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ (t-1)e^t \end{bmatrix}$$

(it was this step that requires standard form!!)

Notice that  $\det W = (t-1)e^t$ . Now use Cramer's Rule to find

$$c_1' = \frac{\det \begin{bmatrix} 0 & e^t \\ (t-1)e^t & e^t \end{bmatrix}}{(t-1)e^t} = \frac{-(t-1)e^{2t}}{(t-1)e^t} = -e^t$$

and

$$c_2' = \frac{\det \begin{bmatrix} t & 0 \\ 1 & (t-1)e^t \end{bmatrix}}{(t-1)e^t} = \frac{t(t-1)e^t}{(t-1)e^t} = t.$$

We find now that

$$c_1 = \int c_1' dt = - \int e^t dt = -e^t + C$$

and

$$c_2 = \int c_2' dt = \int t dt = \frac{t^2}{2} + D$$

Since we only need one particular solution, we may and will take  $C = D = 0$ . Now our particular solution is

$$p(t) = c_1 t + c_2 e^t = -te^t + \frac{t^2 e^t}{2} = \frac{(t^2 - 2t)e^t}{2}$$

and the general solution to the inhomog. ODE is

$$x(t) = p(t) + k_1 t + k_2 e^t = \frac{(t^2 - 2t)e^t}{2} + k_1 t + k_2 e^t.$$


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3. For each of the following systems of ODEs, decide whether it is linear. For each linear systems, do also the following:

- indicate whether it is homogeneous
- find a matrix  $A$  and a vector  $\mathbf{E}$  such that the system can be rewritten in the form

$$D\mathbf{x} = A\mathbf{x} + \mathbf{E}$$

where  $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  (or  $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ ).

a. 
$$\begin{cases} x' = ty - z \\ y' = -\frac{x}{t} - z + 1 \\ z' = -x - t^2 y + z + 2t \end{cases}$$

**Solution:**

The system is linear, but is not homogeneous. It can be written

$$D \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & t & -1 \\ -\frac{1}{t} & 0 & -1 \\ -1 & -t^2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2t \end{bmatrix}$$


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b. 
$$\begin{cases} x' = 2x - 3y \\ y' = 3x^2y + y + 1 \end{cases}$$

**Solution:**

The system is not linear, because the dependence  $y' = 3x^2y + y + 1$  is not linear (it involves the non-linear term  $x^2y$ ).

c. 
$$\begin{cases} x' = 7x + 11y \\ y' = -2x + y \end{cases}$$

**Solution:**

The system is linear and homogeneous. It can be written

$$D \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

4. Let  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .

a. Show that  $\mathbf{h}_1(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$ ,  $\mathbf{h}_2(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$  are solutions to the homogeneous system

$$D \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}.$$

**Solution:**

To check that  $\mathbf{h}_1$  is a solution, we compute

$$D\mathbf{h}_1 = \begin{bmatrix} \frac{d}{dt} \sin(t) \\ \frac{d}{dt} \cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

and

$$A\mathbf{h}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} = \begin{bmatrix} 0 \sin(t) + 1 \cos(t) \\ -1 \sin(t) + 0 \cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}.$$

Since these expressions agree,  $\mathbf{h}_1$  is a solution.

To check that  $\mathbf{h}_2$  is a solution, we compute

$$D\mathbf{h}_2 = \begin{bmatrix} \frac{d}{dt} \cos(t) \\ \frac{d}{dt} [-\sin(t)] \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}$$

and

$$A\mathbf{h}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} = \begin{bmatrix} 0 \cos(t) + 1(-\sin(t)) \\ -1 \cos(t) + 0(-\sin(t)) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}.$$

Since these expressions agree,  $\mathbf{h}_2$  is a solution.

b. Show that  $\mathbf{p}(t) = \begin{bmatrix} 0 \\ -t \end{bmatrix}$  is a *particular solution* to the inhomogeneous equation

$$(\clubsuit) \quad D \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix}.$$

**Solution:**

Compute

$$D\mathbf{p} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and

$$A\mathbf{p} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Since these expressions are equal,  $\mathbf{p}$  is a solution to  $(\clubsuit)$ .

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- c. Let  $\mathbf{x} = \mathbf{p}(t) + c \cdot \mathbf{h}_1(t)$  for a constant  $c$ . By computing  $D[\mathbf{x}]$  and  $A\mathbf{x} + \begin{bmatrix} t \\ -1 \end{bmatrix}$  and comparing the results, confirm that  $\mathbf{x}$  is a solution to  $(\clubsuit)$  for all  $c$ .

(**Note:** In fact,  $\mathbf{x} = \mathbf{p}(t) + c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$  is the *general solution* to  $(\clubsuit)$ ; we'll see how to confirm this later in class).

**Solution:**

Using **a.** and **b.** we find for any  $c$  that

$$D[\mathbf{p} + c\mathbf{h}_1] = D[\mathbf{p}] + cD[\mathbf{h}_1] = \left( A\mathbf{p} + \begin{bmatrix} t \\ -1 \end{bmatrix} \right) + cA\mathbf{h}_1 = A[\mathbf{p} + c\mathbf{h}_1] + \begin{bmatrix} t \\ -1 \end{bmatrix}$$

which confirms that  $\mathbf{x} = \mathbf{p} + c\mathbf{h}_1$  is indeed a solution to  $(\clubsuit)$ .

(as an aside, we know that  $\mathbf{p}(t) + c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$  is the *general solution* to  $(\clubsuit)$  provided that  $c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$  is the *general solution* to the homogeneous equation  $D\mathbf{x} = A\mathbf{x}$ . For this, we need to check the linear independence of the vectors  $\mathbf{h}_1(0)$  and  $\mathbf{h}_2(0)$ .)

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5. Consider the linear ODE

$$(\mathbf{N}) \quad (D - 3)^2x = e^{3t} \quad \text{i.e.} \quad (D^2 - 6D + 9)x = e^{3t}.$$

- a. Find the equivalent linear system  $(\mathbf{S}_\mathbf{N})$  of ODEs. Write this system in *matrix form*.

**Solution:**

We set  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ . For a solution  $x$  of  $(\mathbf{N})$  we set  $x_1 = x$  and  $x_2 = x'$ .

We have that  $x'_1 = x_2$  and

$$x'_2 = x'' = -9x + 6x' + e^{3t} = -9x_1 + 6x_2 + e^{3t}.$$

Thus we the system in matrix form

$$D \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$$

is equivalent to  $(\mathbf{N})$ .

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- b. Note that the general solution to the homogeneous equation  $(\mathbf{H}) \quad (D - 2)^2x = 0$  is generated by  $h_1(t) = e^{3t}$  and  $h_2(t) = te^{3t}$ . Find the corresponding vector solutions  $\mathbf{h}_1$  and  $\mathbf{h}_2$  to the homogeneous system  $(\mathbf{S}_\mathbf{H})$ .

**Solution:**

$$\mathbf{h}_1 = \begin{bmatrix} h_1(t) \\ h_1'(t) \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{bmatrix} h_2(t) \\ h_2'(t) \end{bmatrix} = \begin{bmatrix} te^{3t} \\ (1+3t)e^{3t} \end{bmatrix}$$


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- c. Find a particular solution  $p(t)$  to the equation  $(D-3)^2x = e^{3t}$ , and find the corresponding vector solution  $\mathbf{p}(t)$  to the system  $(\mathbf{S}_\mathbf{N})$ .

**Solution:**

We can use the method of *undetermined coefficients*. We take  $A(D) = D-3$ . The general solution to  $A(D)(D-3)^2 = (D-3)^3$  is generated by  $e^{3t}, te^{3t}, t^2e^{3t}$  but the first two functions are already solutions to the homogeneous equation  $(\mathbf{H})$  and may be eliminated.

Thus our simplified guess for a particular solution is  $kt^2e^{3t}$  and we must find the constant  $k$ .

For this, we apply the operator  $(D-3)^2$  and use the exponential shift formula:

$$(D-3)^2[kt^2e^{3t}] = ke^{3t}(D+3-3)^2[t^2] = ke^{3t}D^2[t^2] = 2ke^{3t}.$$

We need  $2ke^{3t} = e^{3t}$  so  $k = 1/2$  and our particular solution is  $p(t) = \frac{1}{2}t^2e^{3t}$ .

In vector form we have

$$\mathbf{p} = \begin{bmatrix} p(t) \\ p'(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t^2e^{3t} \\ e^{3t}(t + \frac{3}{2}t^2) \end{bmatrix} = e^{3t} \begin{bmatrix} t^2/2 \\ (3t^2 + 2t)/2 \end{bmatrix}.$$


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- d. The general solution to  $(\mathbf{N})$  is given by  $x(t) = p(t) + c_1h_1(t) + c_2h_2(t)$ . Indicate the the corresponding solutions  $\mathbf{x}$  to the system  $(\mathbf{S}_\mathbf{N})$ .

**Solution:**

$$\mathbf{x} = \mathbf{p} + c_1\mathbf{h}_1 + c_2\mathbf{h}_2 = e^{3t} \begin{bmatrix} t^2/2 \\ (3t^2 + 2t)/2 \end{bmatrix} + c_1 \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} te^{3t} \\ (1+3t)e^{3t} \end{bmatrix}$$


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## Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. *Differential Equations: A First Course*. Saunders.