Review material for the Final Exam

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2021-12-17

Problems

1. Solve the initial value problem $(D^2 - 6D + 10)x = 0$, $x(0) = x'(0) = 1$.
2. Consider the linear ODE $(D^2 - 4)x = e^{2t} + e^{3t}$.
   a. Find a simplified guess for a particular solution $p(t)$ for the ODE.
   b. Use the method of undetermined coefficients to find $p(t)$.
3. Use the method of variation of parameters to find the general solution to the linear ODE
$$(D^2 + 4)x = \frac{1}{\sin(2t)}.$$
4. Consider the linear ODE $(\diamondsuit)$ $\qquad (D^3 + D)x = e^t$.
   a. Find a matrix A for which
$$(\heartsuit) \qquad D\vec{x} = A\vec{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}$$
   is an equivalent linear system to $(\diamondsuit)$.
   b. If $u(t)$ is a solution to $(\diamondsuit)$, explain why $\vec{v}(t) = \begin{bmatrix} u(t) \\ u'(t) \\ u''(t) \end{bmatrix}$ is a solution to $(\heartsuit)$.
5. Find the real and imaginary parts of the vector $(\cos(t) + i\sin(t)) \cdot \begin{bmatrix} 2+i \\ 1 \\ i \end{bmatrix}$ for any value of t.
6. Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$
   a. Find solutions $\vec{h}_1(t)$ and $\vec{h}_2(t)$ that generate the general solution to the linear system $D\vec{x} = A\vec{x}$.
   b. Suppose that $\vec{E}(t)$ is a vector valued function, and that we wish to find a particular solution to the inhomogeneous system
$$D\vec{x} = A\vec{x} + \vec{E}(t)$$
   of the form $\vec{p}(t) = c_1(t)\vec{h}_1(t) + c_2(t)\vec{h}_2(t)$ where the $\vec{h}_i(t)$ are the solutions you found in (c), and $c_1(t)$ and $c_2(t)$ are "unknown functions". What matrix equation must you solve in order to find the derivatives $c_1'(t)$ and $c_2'(t)$?
7. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

a. Find the characteristic polynomial and show that eigenvalues of A are $\lambda = 2, -1$.

b. Find two linearly independent generalized eigenvectors for $\lambda = 2$. Note that there are not 2 linearly independent eigenvectors for $\lambda = 2$.

c. Find the general solution to the linear system $D\vec{x} = A\vec{x}$.

8. Let $B = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   a. Find all solutions to the matrix equation $B\vec{v} = \vec{0}$.

   b. Let $\vec{b}_1, \ldots, \vec{b}_5$ denote the columns of the matrix B. Thus the $\vec{b}_i$ are vectors in R^4. Are these vectors linearly independent? Why or why not?

9. Find the solution $\vec{h}(t)$ to the homogeneous system

$$D\vec{x} = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

   which satisfies $\vec{h}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. What is $\vec{h}(1)$?

10. Suppose that the 3×3 matrix A has eigenvalues 2 and $1 \pm 3i$, that $\mathrm{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is an eigenvector for $\lambda = 2$, and that $\mathrm{w} = \begin{bmatrix} u_1 + w_1 i \\ u_2 + w_2 i \\ u_3 + w_3 i \end{bmatrix}$ is an eigenvector for $\lambda = 1 + 3i$.

    Describe three real solutions to the homogeneous system of linear ODES $D\mathrm{x} = A\mathrm{x}$ with linearly independent initial vectors.

11. Find the inverse Laplace transform $\mathcal{L}^{-1}\left[\dfrac{1+2s}{(s^2+9)s^2}\right]$.

12. Consider the initial value problem $(D^2 + D)x = e^{3t}; \quad x(0) = 0, x'(0) = 0$.

    a. Find an expression for the Laplace transform $\mathcal{L}[x]$ as a function of the variable s.

    b. Use your answer to (a) to find the solution $x = x(t)$ for the given initial value problem.

Bibliography