Problem Set 7
Linear Systems of ODES; independence

Math 51 Fall 2021

due Monday 2022-03-07 at 11:59 PM

These problems relate to material of (Nitecki and Guterman 1992, secs. 3.2, 3.3, 3.4).

Reading for the Week of 2022-02-28

- §3.2: Linear Systems, Matrices, and Vectors
- §3.3: Linear Systems of ODES: general properties
- §3.4: Linear independence of vectors

Problems

1. For each of the following systems of ODEs, decide whether it is linear. For each linear system, do also the following:

   - indicate whether it is homogeneous
   - find a matrix A and a vector E such that the system can be rewritten in the form

   $$D\mathrm{x} = A\mathrm{x} + \mathrm{E}$$

   where $\mathrm{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (or $\mathrm{x} = \begin{bmatrix} x \\ y \end{bmatrix}$).

   $$(a) \quad \begin{cases} x' = & ty - z \\ y' = & -\dfrac{x}{t} - z + 1 \\ z' = & -x - t^2 y + z + 2t \end{cases} \qquad (b) \quad \begin{cases} x' = & 2x - 3y \\ y' = & 3x^2 y + y + 1 \end{cases} \qquad (c) \quad \begin{cases} x' = & 7x + 11y \\ y' = & -2x + y \end{cases}$$

2. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and consider the non-homogeneous system

   $$(\clubsuit) \quad D\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix}.$$

   a. Show that $\mathrm{h}_1(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}, \quad \mathrm{h}_2(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$ are solutions to the corresponding homogeneous system $D\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$.

   b. Show that $\mathrm{p}(t) = \begin{bmatrix} 0 \\ -t \end{bmatrix}$ is a particular solution to the $(\clubsuit)$.

c. Show that the initial vectors $\mathrm{h}_1(0)$ and $\mathrm{h}_2(0)$ are linearly independent. Find the general solution to ($\clubsuit$).

3. Consider the linear ODE

$$\text{(N)} \quad (D-3)^2 x = e^{3t} \quad \text{i.e.} \quad (D^2 - 6x + 9)x = e^{3t}.$$

   a. Find the equivalent linear system ($\mathrm{S_N}$) of ODEs. Write this system in matrix form.

   b. Note that the general solution to the homogeneous equation (H) $(D-3)^2 x = 0$ is generated by $h_1(t) = e^{3t}$ and $h_2(t) = te^{3t}$. Find the corresponding vector solutions h_1 and h_2 to the homogeneous system ($\mathrm{S_H}$).

   c. Find a particular solution $p(t)$ to the equation $(D-3)^2 x = e^{3t}$, and find the corresponding vector solution $\mathrm{p}(t)$ to the system ($\mathrm{S_N}$).

   d. The general solution to (N) is given by $x(t) = p(t) + c_1 h_1(t) + c_2 h_2(t)$. What is the general solution to the system ($\mathrm{S_N}$)?

4. Consider the the following matrices A and lists of vector-valued functions $\mathrm{h_i}$. In each case, answer the following questions:

   - Which of the functions $\mathrm{h_i}$ are solutions to the homogeneous equation $D\mathrm{x} = A\mathrm{x}$? Be sure to indicate how you reach your conclusion.

   - Consider the functions that are solutions. Do they generate the general solution to $D\mathrm{x} = A\mathrm{x}$? Why or why not?

   a. $A = \begin{bmatrix} -3 & 8 \\ -3 & 7 \end{bmatrix}; \quad \mathrm{h}_1 = \begin{bmatrix} 2e^t \\ e^t \end{bmatrix}, \quad \mathrm{h}_2 = \begin{bmatrix} 2e^t - 4e^{3t} \\ e^t - 3e^{3t} \end{bmatrix}, \quad \mathrm{h}_3 = e^t \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$

   b. $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad \mathrm{h}_1 = e^t \begin{bmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \\ 0 \end{bmatrix}, \quad \mathrm{h}_2 = e^t \begin{bmatrix} 2\cos(t) + 2\sin(t) \\ 4\cos(t) \\ e^{-3t} \end{bmatrix}, \quad \mathrm{h}_3 = \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \end{bmatrix}.$

5. Let

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}, \quad \mathrm{E} = \begin{bmatrix} 0 \\ -5e^t \end{bmatrix}.$$

   The formulas

$$(\clubsuit) \quad \begin{cases} x_1 = & c_1 \cos(2t) + c_2 \sin(2t) + e^t \\ x_2 = & -2c_2 \cos(2t) + 2c_1 \sin(2t) - e^t \end{cases}$$

   describe a collection of solutions to the nonhomogeneous system $D\mathrm{x} = A\mathrm{x} + \mathrm{E}$.

   a. Write the collection ($\clubsuit$) of solutions in the form $\mathrm{x} = c_1\mathrm{h}_1 + c_2\mathrm{h}_2 + \mathrm{p}$ where h_1 and h_2 are solutions to the homogeneous system $D\mathrm{x} = A\mathrm{x}$.

   b. Decide whether the collection ($\clubsuit$) is complete.

6. Check the following list of vectors for linear independence:

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. Differential Equations: A First Course. Saunders.