# Math 51 Spring 2022 - Final Exam - some review problems
# **Solutions**

2022-05-01

1. Indicate which of the following best represents a *simplified guess* for a particular solution $p(t)$ to the non-homogeneous linear ODE:

$$(D-3)(D-1)x = te^{3t} + \cos(2t)$$

   a. $p(t) = k_1 te^{3t} + k_2 \cos(2t) + k_3 \sin(2t)$
   b. $p(t) = k_1 te^{3t} + k_2 \cos(2t)$
   c. $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t)$
   d. $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t) + k_4 \sin(2t)$

   **Solution:**

   correct response was d.

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2. Indicate which of the following represents the general solution to the homogeneous linear ODE $(D^2 - 2D + 2)^2 x = 0$.

   a. $h(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t) + c_3 te^{-t} \cos(t) + c_4 te^{-t} \sin(t)$
   b. $h(t) = c_1 e^{t} \cos(t) + c_2 e^{t} \sin(t)$
   c. $h(t) = c_1 e^{t} \cos(t) + c_2 e^{t} \sin(t) + c_3 te^{t} \cos(t) + c_4 te^{t} \sin(t)$
   d. $h(t) = c_1 te^{t} \cos(t) + c_2 te^{t} \sin(t) + c_3 t^2 e^{t} \cos(t) + c_4 t^2 e^{t} \sin(t)$

   **Solution:**

   correct response was ~~a.~~ c.

   (The roots of the char poly are $\lambda = 1 \pm i$, which leads to solutions of the form $t^j \sin(t)$ and $t^j \cos(t)$.)

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3. The matrix $A = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix}$ has characteristic polynomial $\lambda^2 - 2\lambda + 2$ and thus its eigenvalues are $\lambda = 1 + i$ and $\lambda = 1 - i$.

   Which of the following is an eigenvector for $A$?

   a. $A$ has no eigenvectors.

b. $\begin{bmatrix} 3-i \\ 2 \end{bmatrix}$

c. $\begin{bmatrix} 2 \\ -3+i \end{bmatrix}$

d. $\begin{bmatrix} 3+i \\ 2 \end{bmatrix}$

**Solution:**

both b and d give eigenvectors. You can check that $\begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \pm i \\ 2 \end{bmatrix} = (1 \pm i) \begin{bmatrix} 3 \pm i \\ 2 \end{bmatrix}$

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4. Consider the linear system of ODEs

$$(\diamondsuit) \quad D\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

A third order linear ODE is *equivalent* to this system if for each of its solutions $x(t)$, the vector-valued function $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x'(t) \\ x''(t) \end{bmatrix}$ is a solution to $(\diamondsuit)$. Which of the following linear ODEs is equivalent to $(\diamondsuit)$?

a. $(D^3 - 2D^2 - D - 5)x = e^t$

b. $(D^3 - 5D^2 - D - 2)x = e^t$

c. $(D^3 + 2D^2 + D + 5)x = -e^t$

d. $(D^3 + 5D^2 + D + 2)x = -e^t$

**Solution:**

correct response was b.

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5. Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. $\lambda = 2$ is an eigenvalue of $A$ with multiplicity two. The matrix $A - 2\mathbf{I}_3$ satisfies $(A - 2\mathbf{I}_3)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the generalized eigenvectors of $A$ for $\lambda = 2$ are generated by $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

Which of the following represents a solution $\mathbf{h}(t)$ to the system $D\mathbf{x} = A\mathbf{x}$ with the property that $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$?

a. $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

b. $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 + 12t \\ 2 \\ 6 \end{bmatrix}$

c. $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 + t \\ 2 \\ 6 \end{bmatrix}$

d. No solution $\mathbf{h}(t)$ has the property that $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$.

**Solution:**

correct response was b.

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6. Consider the homogeneous system $(\diamondsuit) \quad D\mathbf{x} = A\mathbf{x}$ where $A$ is a $3 \times 3$ matrix, and let $\mathbf{h}_1(t), \mathbf{h}_2(t)$ be solutions to $(\diamondsuit)$. Which of the following statements is correct?

   a. $\mathbf{h}_1(0)$ and $\mathbf{h}_2(0)$ are *eigenvectors* for $A$.

   b. The system $(\diamondsuit)$ has exactly two solutions.

   c. If the vectors $\mathbf{h}_1(0), \mathbf{h}_2(0)$ are linearly independent, then the general solution to $(\diamondsuit)$ is given by $\mathbf{x}(t) = c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$.

   d. None of the above statements is correct.

**Solution:**

correct response was d.

To see that **a.** is incorrect, consider solutions $e^{\lambda t}\mathbf{v}$ and $e^{\mu t}\mathbf{w}$ arising from eigenvectors $\mathbf{v}$ and $\mathbf{w}$.

Then there is a solution $h(t) = e^{\lambda t}\mathbf{v} + e^{\mu t}\mathbf{w}$ but $h(0) = \mathbf{v} + \mathbf{w}$ which is not an eigenvector if $\lambda \neq \mu$.

**b.** in incorrect. Indeed, all linear combinations $c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$ are solutions, so there are always *infinitely many solutions.*

Finally, **c.** is incorrect because for a $3 \times 3$ system the general solution is generated by three solutions with linearly independent initial vectors; two solutions are not enough.

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7. The matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ has characteristic polynomial $\lambda(\lambda - 3)$ and hence has eigenvalues $\lambda = 0$ and $\lambda = 3$. An eigenvector for $\lambda = 0$ is given by $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and an eigenvector for $\lambda = 3$ is given by $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Find a particular solution $\mathbf{p}(t)$ for the system of linear ODEs

$$D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

**Solution:**

The general solution is generated by the solutions obtained from eigenvectors:

$$\mathbf{h}_1(t) = e^{0t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{h}_2(t) = e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

To find a particular solution, form the Wronskian matrix

$$W = \begin{bmatrix} -1 & e^{3t} \\ 1 & 2e^{3t} \end{bmatrix}$$

and notice that $\det W = -3e^{3t}$.

A particular solution has the form $\mathbf{p}(t) = c_1(t)\mathbf{h}_1(t)+c_2(t)\mathbf{h}_2(t)$, where the vector $\mathbf{c} = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$ satisfies the matrix equations

$$W\mathbf{c}' = \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

Using Cramer's Rule, we find that

$$c_1'(t) = \frac{\det \begin{bmatrix} 0 & e^{3t} \\ t & 2e^{3t} \end{bmatrix}}{-3e^{3t}} = \frac{-te^{3t}}{-3e^{3t}} = \frac{t}{3}.$$

$$c_2'(t) = \frac{\det \begin{bmatrix} -1 & 0 \\ 1 & t \end{bmatrix}}{-3e^{3t}} = \frac{-t}{-3e^{3t}} = \frac{te^{-3t}}{3}$$

Now we integrate to find $c_1(t)$ and $c_2(t)$:

$$c_1(t) = \int c_1'(t)dt = \frac{1}{3}\int tdt = \frac{t^2}{6} + A.$$

For $c_2$ we integrate by parts with $u = t, dv = e^{-3t}dt$:

$$c_2(t) = \int c_2'(t)dt = \frac{1}{3}\int te^{-3t}dt = \frac{1}{3}\left(\frac{-t}{3}e^{-3t} + \frac{1}{3}\int e^{-3t}dt\right) = \frac{-1}{9}e^{-3t}\left(t+\frac{1}{3}\right) + B$$

We may take $A = B = 0$ since we only seek a particular solution. This gives

$$\begin{aligned} \mathbf{p}(t) =& \frac{t^2}{6}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{-1}{9}e^{-3t}\left(t+\frac{1}{3}\right)e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ =& \frac{t^2}{6}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{-1}{9}\left(t+\frac{1}{3}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

8. Let $A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$.

The characteristic polynomial of $A$ is $r^2 - 4r + 5$ so the eigenvalues of $A$ are $\lambda = 2 \pm i$.

Moreover, $\mathbf{v} = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$ is an eigenvector for $\lambda = 2 + i$.

a. Find the general solution to $D\mathbf{x} = A\mathbf{x}$.

**Solution:**

The complex solution to (H) is

$$e^{2t}(\cos t + i \sin t) \begin{bmatrix} 2-i \\ 5 \end{bmatrix} = e^{2t} \begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + ie^{2t} \begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix}$$

so the real and imaginary parts of this expression generate the general solution

$$x(t) = C_1 e^{2t} \begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix}.$$

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b. Solve the initial value problem $D\mathbf{x} = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

**Solution:**

The value at $t = 0$ of the general solution given above is

$$X(0) = C_1 e^0 \begin{bmatrix} 2\cos 0 + \sin 0 \\ 5\cos 0 \end{bmatrix} + C_2 e^0 \begin{bmatrix} -\cos 0 + 2\sin 0 \\ 5\sin 0 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix};$$

setting this equal to the desired initial condition yields the system of equations

$$\begin{aligned} 2C_1 - C_2 &= 1 \\ 5C_1 + 0C_2 &= 1 \end{aligned}$$

which can be solved by performing row operations on the augmented matrix

$$\begin{bmatrix} 2 & -1 & |1 \\ 5 & 0 & |1 \end{bmatrix},$$

or by using Cramer's Rule, or simply by noting that the second equation says $C_1 = \frac{1}{5}$, and substituting into the first equation yields $\frac{2}{5} - C_2 = 1$ or $C_2 = -\frac{3}{5}$.

Thus the desired solution of (H) is

$$X(t) = \frac{1}{5} e^{2t} \begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} - \frac{3}{4} e^{2t} \begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix} = e^{2t} \begin{bmatrix} \cos t - \sin t \\ \cos t - 3\sin t \end{bmatrix}.$$

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9. Solve the initial value problem $(4D^2 - 4D + 1)x = 0, \quad x(2) = x'(2) = e.$

**Solution:**

The polynomial $4r^2 - 4r + 1$ has root $r = 1/2$ with multiplicity 2. Thus the general solution is given by

$$x(t) = c_1 e^{t/2} + c_2 t e^{t/2}.$$

Note that

$$\begin{aligned} x'(t) = D[x(t)] &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} D[t] \\ &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2}(D + 1/2)[t] \\ &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2}(1 + t/2) \end{aligned}$$

Now, we need

$$e = x(2) = c_1 e + 2c_2 e$$

and

$$e = x'(2) = \frac{1}{2} e c_1 + 2ec_2$$

Thus we must solve the matrix equation

$$\begin{bmatrix} e & 2e \\ e/2 & 2e \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e \\ e \end{bmatrix}$$

This can be solved in several ways – e.g. by row operations on the augmented matrix, as follows:

$$\left[\begin{array}{cc|c} e & 2e & e \\ e/2 & 2e & e \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 4 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 1 \end{array}\right]$$

Thus $c_1 = 0$ and $c_2 = 1/2$ so that the solution to the initial value problem is given by

$$x(t) = \frac{te^{t/2}}{2}.$$

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10. Consider the matrix $B = \begin{bmatrix} 5 & -3 & -6 \\ 0 & 2 & 0 \\ 3 & -3 & -4 \end{bmatrix}$.

   a. The vector $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector for $B$. What is the corresponding eigenvalue?

   **Hint:** Compute the vector $B\mathbf{v}$ and compare with $\mathbf{v}$.

**Solution:**

The product $B\mathbf{v}$ is equal to

$$B\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2\mathbf{v}$$

so the eigenvalue is $\lambda = 2$.

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b. Find an eigenvector for $B$ for the eigenvalue $\lambda = -1$.

**Solution:** Perform row operations on the matrix $B - (-1)\mathbf{I}_3 = B + \mathbf{I_3}$:

$$\begin{bmatrix} 6 & -3 & -6 \\ 0 & 3 & 0 \\ 3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Considering this echelon matrix, we see that an eigenvector for $\lambda = -1$ is given by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

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11. Laplace Transforms:

a. Compute the inverse Laplace tranform $\mathscr{L}^{-1}[F(s)]$ of the function $F(s) = \dfrac{3s^2 + s + 1}{(s+1)(s^2+2)}$.

**Solution:**

The partial fraction decomposition has the form

$$\frac{3s^2 + s + 1}{(s+1)(s^2+2)} = \frac{A}{s+1} + \frac{Bs + C}{s^2+2};$$

combining over a common denominator and matching coefficients leads to

$$\begin{array}{lccccc} s^2 \text{ terms}: & A & +B & & = & 3 \\ s \text{ terms}: & & B & +C & = & 1 \\ \text{constant terms}: & 2A & & +C & = & 1 \end{array}$$

We can solve the first (respectively, second) equation for $A$ (respectively, $C$) in terms of $B$:

$$A = 3 - B$$
$$C = 1 - B$$

and substituting into the third equation yields

$$\begin{aligned}(6-2B)+(1-B)&=1\\ -3B&=-6\\ B&=2\\ A&=1\\ C&=-1\end{aligned}$$

so

$$\frac{3s^2+s+1}{(s+1)(s^2+2)}=\frac{1}{s+1}+\frac{2s-1}{s^2+2}.$$

Then the inverse transform is

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{3s^2+s+1}{(s+1)(s^2+2)}\right]&=\mathscr{L}^{-1}\left[\frac{1}{s+1}\right]+\mathscr{L}^{-1}\left[\frac{2s}{s^2+2}\right]-\mathscr{L}^{-1}\left[\frac{1}{s^2+2}\right]\\ &=e^{-t}+2\cos t\sqrt{2}-\frac{1}{\sqrt{2}}\sin t\sqrt{2}.\end{aligned}$$

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b. If $x$ is a solution to $(D^2+D+1)x=1$ with $x(0)=0$ and $x'(0)=1$, find an expression for $\mathscr{L}[x]$ as a function of $s$.

**Solution:**

By the first differentiation formula, applying the Laplace Transform to both sides of the problem yields

$$\begin{aligned}\mathscr{L}D^2x+\mathscr{L}Dx+\mathscr{L}x&=\mathscr{L}1\\ \{s^2\mathscr{L}x-sx(0)-x'(0)\}+\{s\mathscr{L}x-x(0)\}+\mathscr{L}x&=\mathscr{L}1\\ s^2\mathscr{L}x-1+s\mathscr{L}x+\mathscr{L}x&=\frac{1}{s}\\ (s^2+s+1)\mathscr{L}x&=1+\frac{1}{s}=\frac{1+s}{s}\\ \mathscr{L}x&=\frac{1+s}{s(s^2+2+1)}\end{aligned}$$

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12. Let $W=W(h_1(t),h_2(t))$ denote the *Wronskian matrix* of the functions $h_1(t)=e^{2t}$ and $h_2(t)=te^{2t}$. Which of the following represents the *determinant* of $W$?

a. $e^{4t}$

b. $(1+4t)e^{4t}$

c. $e^{2t}$

d. $(1+4t)e^{2t}$

**Solution:**

~~correct answer is c~~ correct answer is a.

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13. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ in $\mathbf{R}^4$, and let $A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ be the $4 \times 3$ matrix whose columns are the $\mathbf{v}_i$. Which of the following statements is correct?

a. The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are *linearly dependent.*

b. Since $A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, the only solution to the equation $A\mathbf{w} = \mathbf{0}$ is $\mathbf{w} = \mathbf{0}$ so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are *linearly independent.*

c. The equation $A\mathbf{w} = \mathbf{x}$ has a solution for every vector $\mathbf{x}$ in $\mathbf{R}^4$.

d. The determinant of $A$ is $\neq 0$.

**Solution:**

The correct response is b.

Answer a. is incorrect because the vectors are independent.

Answer c. is incorrect because the given equation has no solution when $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Answer d. is incorrect because the determinant of a $4 \times 3$ (non-square!) matrix is not defined.

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14. Let $A$ be an $n \times n$ matrix with constant coefficients $a_{ij}$, and let $\mathbf{E}(t)$ be a vector with $n$ components. If $\mathbf{v}$ is any vector in $\mathbf{R}^n$, must there be a solution $\mathbf{x}(t)$ to the system of equations $D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$ for which $\mathbf{x}(0) = \mathbf{v}$?

a. No, this conclusion is only guaranteed when the system is *homogeneous.*

b. No, this conclusion is only guaranteed when the entries of the vector $\mathbf{E}(t)$ are *constant* functions of $t$.

c. Yes, this conclusion is the content of the *Existence and Uniqueness Theorem for Solutions of Linear Systems.*

d. No, this conclusion is only guaranteed when $\det A \neq 0$.

**Solution:**

correct response is c.

The existence and uniqueness theorem applies for non-homogeneous systems (so a is incorrect) and applies so long as the entries of $A$ and of $\mathbf{E}$ are *continuous* functions of $t$ (so

b. is incorrect). Finally, the existence and uniqueness theorem is valid even when $A$ has determinant 0 (so d. is incorrect).

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15. Consider the homogeneous system $(\diamondsuit) \quad D\mathbf{x} = A\mathbf{x}$ where $A$ is a $3 \times 3$ matrix.

 a. If $\mathbf{h}(t)$ is a solution, must $\mathbf{h}(0)$ be an eigenvector for $A$? Why or why not?

**Solution:**

No, $\mathbf{h}(0)$ need not be an eigenvector. Suppose for example that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for $A$ with eigenvalues $\lambda, \mu$, and suppose that $\lambda \neq \mu$. Then $\mathbf{v} + \mathbf{w}$ is not an eigenvector.

Indeed, since $\lambda \neq \mu$ we know that $\mathbf{v}$ and $\mathbf{w}$ are *linearly independent*. Now, for any number $\beta$,

$$(A - \beta\mathbf{I})(\mathbf{v} + \mathbf{w}) = (\lambda - \beta)\mathbf{v} + (\mu - \beta)\mathbf{w}$$

Since $\lambda \neq \mu$, at least one of $\lambda - \beta$ or $\mu - \beta$ is non-zero, so the linear independence of $\mathbf{v}$ and $\mathbf{w}$ shows that $(A - \beta\mathbf{I})(\mathbf{v} + \mathbf{w})$ is non-zero. This shows that $\mathbf{v} + \mathbf{w}$ is not an eigenvector (for *any* eigenvalue $\beta$).

Now, the function

$$h(t) = e^{\lambda t}\mathbf{v} + e^{\mu t}\mathbf{w}$$

is a solution to $(\diamondsuit)$, and $h(0) = \mathbf{v} + \mathbf{w}$.

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 b. Show that the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are linearly dependent.

**Solution:**

We perform row operations on the matrix whose columns are given by these vectors:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting echelon matrix has 3 pivots and no free variables, the only solution $\mathbf{c}$ to the matrix equation

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$. (Alternatively, you could have obtained this conclusion by noting that the *determinant* of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ is equal to 0).

Thus the only coefficients satisfying the following equation

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \mathbf{0}$$

are $c_1 = c_2 = c_3 = 0$; this shows that the vectors are linearly independent.

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c. Let $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$ be solutions to $(\diamondsuit)$. Suppose that $\mathbf{h}_1(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{h}_2(0) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{h}_3(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are the vectors from b. Do the solutions $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$ generate the general solution to $(\diamondsuit)$? Why or why not?

**Solution:**

Yes. Since $A$ is a $3 \times 3$ matrix, one knows that three solutions $\mathbf{h}_1(t)$, $\mathbf{h}_2(t)$ and $\mathbf{h}_3(t)$ generate the general solution provided that the "initial vectors" $\mathbf{h}_1(0), \mathbf{h}_2(0), \mathbf{h}_3(0)$ are linearly independent; thus the result in part b. shows that the $\mathbf{h}_i(t)$ generate the general solution.

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16. A drug is absorbed by the body at a rate proportional to the amount of the drug present in the bloodstream after $t$ hours. If there are $x(t)$ mg of the drug present in the bloodstream at time $t$, assume that the drug is absorbed at a rate of $0.5x(t)$ /hour. If a patient receives the drug intravenously at a constant rate of 3 mg/hour, to which of the following ODEs is $x(t)$ a solution?

    a. $x'(t) = -0.5x(t) + 3$

    b. $x'(t) = -0.5x(t); \quad x(0) = 3$

    c. $x'(t) = 0.5x(0) + 3$

    d. $x'(t) = .5x(t) - 3$

    **Solution:**

    correct response is a.

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17. You are given that a particular solution to

$$(\heartsuit) \quad (D^2 - 2D + 1)x = e^t$$

is $p(t) = \dfrac{t^2 e^t}{2}$. Which of the following best represents the general solution to $(\heartsuit)$?

    a. $c_1 e^t + c_2 t e^t$.

    b. $\dfrac{t^2 e^t}{2} + c_1 e^t + c_2 t e^t$.

    c. $\dfrac{t^2 e^t}{2} + c e^t$.

    d. $\dfrac{t^2 e^t}{2} + c_1 e^t + c_2 e^{-t}$.

    **Solution:**

    correct response was b.

18. Let $x_1(t)$ and $x_2(t)$ be solutions to the ODE $(t+1)x''+x'+x = 0$. Suppose that $x_1(0) = x_2(0)$ and that $x_1'(0) = x_2'(0)$. Which of the following statements must be correct?

    a. $x_1(t) = x_2(t)$ for every $t$.

    b. Since the ODE is *normal* on the interval $(-1, \infty)$, we can conclude that $x_1(t) = x_2(t)$ for $-1 < t < \infty$.

    c. No conclusion is possible because the existence and uniqueness theorem does not apply to this ODE.

    d. We can only conclude that $x_1(t) = x_2(t)$ for all $t$ if we also assume that $x_1''(0) = x_2''(0)$.

    **Solution:**

    correct response was b. Indeed since the equation is of 2nd order and since it is normal on the interval $(-1, \infty)$, the existence and uniqueness theorem guarantees for any $\alpha, \beta$ that there is only one solution $x$ which $x(0) = \alpha$ and $x'(0) = \beta$.

    Assertion a. need not be true since the ODE is not normal on $(-\infty, \infty)$.

    And assertion d. is incorrect – the existence and uniqueness theorem doesn't require a condition on the second derivative in this case.

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19. Show that the functions

$$f_1(t) = e^t \cos(t), \quad f_2(t) = e^t \sin(t), \quad f_3(t) = e^t$$

are linearly independent.

You have been told that functions like this are independent. However, here we want you to demonstrate it directly in this case. You may use the *Wronskian test* (with all details needed to justify using it) or other, direct arguments from the definition.

**Solution:**

There are several possible strategies for solving this problem; here we list a few of them.

First, you can use the Wronskian test. This requires computation of the first and second derivatives of the $f_i$, which is perhaps most easily done using the *exponential shift formula.*

One finds:

$$D[e^t \cos(t)] = e^t(D+1)[\cos(t)] = e^t(\cos(t) - \sin(t))$$

$$D[e^t \sin(t)] = e^t(D+1)[\sin(t)] = e^t(\cos(t) + \sin(t))$$

$$D^2[e^t \cos(t)] = D[e^t(\cos(t) - \sin(t))] = e^t(D+1)[\cos(t) - \sin(t)] = -2e^t \sin(t).$$

$$D^2[e^t \sin(t)] = D[e^t(\cos(t) + \sin(t))] = e^t(D+1)[\cos(t) + \sin(t)] = 2e^t \cos(t).$$

Thus the Wronskian matrix is given by

$$W = W(f_1, f_2, f_3) = \begin{bmatrix} e^t \cos(t) & e^t \sin(t) & e^t \\ e^t(\cos(t) - \sin(t)) & e^t(\cos(t) + \sin(t)) & e^t \\ -2e^t \sin(t) & 2e^t \cos(t) & e^t \end{bmatrix}$$

Now, according to the Wronskian test, the functions will be linearly independent (on the interval $(-\infty, \infty)$) provided that $\det W(t_0)$ is non-zero for some $t_0$. If we take $t_0 = 0$, we find that

$$\det W\Big|_{t=0} = \det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = -1 + 2 = 1$$

Since this determinant is non-zero, the Wronskian test confirms the linear independence of $f_1, f_2, f_3$.

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A second method of solving this problem just uses the *definition of linear independence.*

Suppose that $c_1, c_2, c_3$ are constants and that

$$c_1 e^t \cos(t) + c_2 e^t \sin(t) + c_2 e^t = 0.$$

To show that the functions are linearly independent, we must *argue* that $c_1 = c_2 = c_3 = 0$.

Factoring out the quantity $e^t$, our assumption shows that

$$e^t(c_1 \cos(t) + c_2 \sin(t) + c_3) = 0.$$

Since $e^t \neq 0$ for all $t$, we find that

$$c_1 \cos(t) + c_2 \sin(t) + c_3 = 0.$$

Now, since this equation holds for all times $t$, we may choose some particular values of $t$ to find equations for the constants $c_i$.

When $t = 0$, we find that

$$0 = c_1 \cos(0) + c_2 \sin(0) + c_3 = c_1 + c_3.$$

When $t = \pi/2$, we find that

$$0 = c_1 \cos(\pi/2) + c_2 \sin(\pi/2) + c_3 = c_2 + c_3.$$

When $t = \pi$, we find that

$$0 = c_1 \cos(\pi) + c_2 \sin(\pi) + c_3 = -c_1 + c_3.$$

Now, we solve the system of equations

$$\begin{aligned} 0 &= c_1 + c_3 \\ 0 &= c_2 + c_3 \\ 0 &= -c_1 + c_3 \end{aligned}$$

Adding the first and third equation gives $0 = 2c_3$ so that $c_3 = 0$. Now the first equation shows that $c_1 = 0$ and the second shows that $c_2 = 0$.

Since we have argued that $c_1 = c_2 = c_3 = 0$, we conclude from the definition that $f_1, f_2, f_3$ are linearly independent.

20. Find the Laplace transform of the function

$$f(t) = \begin{cases} 1 & \text{for } t < 1, \\ t - 1 & \text{for } 1 \le t < 2, \\ 1 & \text{for } t \ge 2. \end{cases}$$

**Solution:**

In order to be able to compute the Laplace transform, We first rewrite the function $f(t)$ using the *unit step functions.*

We have

$$\begin{aligned} f(t) &= 1 + u_1(t) \cdot (-1 + (t-1)) + u_2(t) \cdot ((-(t-1) + 1) \\ &= 1 + u_1(t) \cdot (t-2) + u_2(t) \cdot (-t+2). \end{aligned}$$

Thus

$$\begin{aligned} \mathscr{L}[f(t)] &= \mathscr{L}[1 + u_1(t) \cdot (t-2) + u_2(t) \cdot (-t+2)] \\ &= \mathscr{L}[1] + \mathscr{L}[u_1(t) \cdot (t-2)] + \mathscr{L}[u_2(t) \cdot (-t+2)] \\ &= \mathscr{L}[1] + e^{-s}\mathscr{L}[(t+1) - 2] + e^{-2s}\mathscr{L}[-(t+2) + 2] \\ &= \mathscr{L}[1] + e^{-s}\mathscr{L}[t-1] + e^{-2s}\mathscr{L}[-t] \\ &= (1 - e^{-s})\mathscr{L}[1] + e^{-s}\mathscr{L}[t] - e^{-2s}\mathscr{L}[t] \\ &= (1 - e^{-s})\mathscr{L}[1] + (e^{-s} - e^{-2s})\mathscr{L}[t] \\ &= \frac{1 - e^{-s}}{s} + \frac{e^{-s} - e^{-2s}}{s^2} \end{aligned}$$

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21. Suppose $g(t)$ is the inverse Laplace transform of

$$F(s) = \frac{2se^{\pi s/2}}{(s^2+4)}.$$

Find $g\left(\frac{\pi}{4}\right)$.

**Solution:**

We use the *second shift formula* to find $g(t)$. Notice that if we set

$$f(t) = \mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos(2t)$$

then the second shift formula yields

$$\begin{aligned} g(t) = \mathscr{L}^{-1}[F(s)] &= 2\mathscr{L}^{-1}\left[e^{(\pi/2)s}\frac{s}{s^2+4}\right] \\ &= 2u_{\pi/2}(t)f(t - \pi/2) \end{aligned}$$

Thus $u_{\pi/2}(\pi/4) = 0$ so that $g(\pi/4) = 2u_{\pi/2}(\pi/4) \cdot f(\pi/4 - \pi/2) = 0$.

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