MATH 51: HOMEWORK 1 SOLUTIONS

(1) Find all values of the constant k for which the given functions $x = \phi(t)$ are solutions of the given equations:

(a) $x = t^k, \quad t > 0; \qquad 16t^2xx'' + 3x^2 = 0.$

Solution: Our differential equation is $16t^2xx'' + 3x^2 = 0$, and we need to check for which values of k the solution $4x = t^k 4$ is valid (for $t > 0$). So, we plug in our test solution to the differential equation! Doing this yields

$$16t^2(t^k)(t^k)'' + 3(t^k)^2 = 0$$

Differentiating and combining like terms we can rewrite this as

$$(16k^2 - 16k + 3)t^{2k} = 0$$

This will only hold for all time t if $16k^2 - 16k + 3 = 0$. Solving this quadratic equation yields $k = 1/4$ and $k = 3/4$. Therefore, $x = t^{1/4}$ and $x = t^{3/4}$ are both solutions to the differential equation valid for $t > 0$.

(b) $x = kte^{3t}; \qquad x'' - 3x' = e^{3t}.$

Solution: The process here is the same as for part (a). Plugging $x = kte^{3t}$ into the o.d.e. $x'' - 3x' = e^{3t}$ gives us

$$(kte^{3t})'' - 3(kte^{3t})' = e^{3t}$$

which simplifies after differentiating to the following:

$$3ke^{3t} = e^{3t}$$

Once again, this will only be true for all t when $3k = 1$, and therefore $k = 1/3$, so the only viable solution of the given form is $x = te^{3t}/3$.

(2) §1.1, exercise 24: A tanker carrying 100,000 gallons of oil runs aground off Nantucket. Water pours in the tanker at one end at 1000 gallons per hour while the polluted water-oil mixture pours out at the other end, also at 1000 gallons per hours. Set up a differential equation to predict the amount $x = x(t)$ of oil in the tanker. Explain how you arrived at your answer.

Solution: There are two factors at play in modeling the oil in the tanker: the outward flow of the contents of the tanker and the percentage of the tanker volume that is oil, in other words the concentration of oil. The outward flow is 1000 gallons per hour of the total mixture, so we should expect that the change in oil will be -1000 times the concentration – it is negative because the quantity of oil in the tanker is decreasig in time. Now, at any instant, if x is quantity of oil in the tanker then $x/100000$ is the percentage of oil in the tanker. Therefore, as the tanker 1000 gallons per hour, it will leak $1000 * x/100000 = x/100$ per hour. Thus, the o.d.e. is

$$\frac{dx}{dt} = -\frac{x}{100}.$$

(3) §1.2, exercise 22: Solve the differential equation found in exercise 24, §1.1. How much oil is left in the tanker after 10 days?

Solution: We solve this by separation of variables. Dividing by x, multiplying by dt and integrating yields

$$\ln|x| = -\frac{t}{100} + c$$

and after exponentiating we get

$$|x| = e^c e^{-t/100}$$

Now, since all terms on the right hand side are positive, if we define a new variable $k = \pm e^c$, this may be rewritten as

$$x = ke^{-t/100},$$

(one way to think of this is that if we say $|x| = a$ then it is the same thing as saying $x = \pm a$). The initial condition is of course $x(0) = 100,000$ since the tanker started full, so we may solve for k as follows

$$100000 = x(0) = ke^{-0/100} \quad \Rightarrow \quad k = 100000$$

Now, note that the flow rates are given per hour. Checking the oil quantity after 10 days is of course the same as after 10*24 = 240 hours, and therefore the amount of oil left in the tanker after 10 days is

$$x(240) = 100000e^{-240/100} \sim 9072 \text{ gallons}$$

Note: when solving the equation, we had to divide by 0. For the general solution, we should check whether or not the constant function $x(t) = 0$ is also a solution, but in this case we know that we have a nonzero initial condition so it was not of as much concern.

(4) Find the general solution of the following o.d.e.

$$\frac{dx}{dt} = (x \sin t)^2$$

Solution: This is a linear, homogeneous equation and is separable, so we rearrange terms to yield

$$\frac{dx}{x^2} = (\sin t)^2 dt$$

(assuming that $x \neq 0$) and after integrating both sides (using the trig identity $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$)

$$-\frac{1}{x} = \frac{t}{2} - \frac{\sin 2t}{4} + c$$

We now rearrange terms to find that

$$x = \frac{1}{\frac{\sin 2t}{4} - \frac{t}{2} - c}$$

Lastly, we need to check the constant solution $x(t) = 0$ that we ignored previously. This does indeed satisfy the equation which may be verified by plugging it in and seeing that both sides are 0 and therefore equal.

We can make this a little easier on the eyes by multiplying both sides by 4/4 and defining $k = -4c$ so that we have

$$x = \frac{4}{\sin 2t - 2t + k}$$

(5) Find the specific solution of the following o.d.e. Explicitly verify your answer by plugging it into the equation.

$$t^2 x' = x^2 + 1, \qquad x(1) = 0.$$

Solution: As before, this equation is separable. Rearranging terms we have

$$\frac{dx}{x^2+1} = \frac{dt}{t^2}$$

Integrating both sides yields

$$\arctan(x) = -\frac{1}{t} + c$$

and therefore we have that

$$x = \tan\left(c - \frac{1}{t}\right)$$

To find the specific solution, we apply the initial condition:

$$0 = x(1) = \tan\left(c - \frac{1}{1}\right)$$

This of course works for $c = 1$, and therefore we have

$$x = \tan\left(1 - \frac{1}{t}\right)$$

(6) Find the specific solution of the following o.d.e.

$$\frac{d^2x}{dt^2} = t\cos t, \qquad x(0) = x(\pi) = 0.$$

Solution: This does not require separation of variables, but rather direct integration twice. Integrating both sides once with respect to t (using integration by parts to deal with $t\cos t$) gives us

$$\frac{dx}{dt} = t\sin t + \cos t + c_1$$

and integrating a second time yields

$$x = 2\sin t - t\cos t + c_1 t + c_2.$$

Now, we have two initial conditions which we can use to specify c_1 and c_2. Starting with the first one, we have

$$0 = x(0) = 2\sin(0) - 0\cos(0) + c_1 \cdot 0 + c_2 \quad \Rightarrow \quad c_2 = 0$$

and now we can apply the second one,

$$0 = x(\pi) = 2\sin(\pi) - \pi\cos(\pi) + c_1 \cdot \pi = 0 + \pi + c_1\pi \quad \Rightarrow \quad c_1 = -1$$

and thus our specific solution is

$$x = 2\sin t - t\cos t - t.$$