Readings for the Week of January 31, 2022

§1.6, 2.2: Existence and Uniqueness; Linear ODEs
§2.3, App. A: Cramer's Rule and the Wronskian

Problem Set 3

(Due Monday, February 7, 2022, at 11 p.m.)

For a 10% penalty on your score, you may hand in the problem set late, until February 8, 2022, 11 p.m.

1. For each of the following ODEs, answer the following questions:
   - is the ODE linear?
   - if the ODE is linear, write a linear differential operator L and use it to rewrite the ODE in theform $Lx = E(t)$.
   - is the ODE homogeneous?

   (a) $\dfrac{d^5x}{dt^5} + t^2\dfrac{dx}{dt} = te^t$.

   (b) $\dfrac{d^2x}{dt^2} = x\dfrac{dx}{dt} + t$.

   (c) $\dfrac{d^3x}{dt^3}\sin(t)\dfrac{dx}{dt} = t^2x$.

   (d) $\dfrac{d^3x}{dt^3} + e^t\dfrac{d^2x}{dt^2} + tx = e^t$.

2. Calculate and simplify the determinant of the following matrix:

$$\begin{bmatrix} e^t & \sin t & \cos t \\ e^t & \cos t & -\sin t \\ e^t & -\sin t & -\cos t \end{bmatrix}$$

3. Consider the differential equation $Lx = 0$ where $L = D^3 - 4D$.

   (a) Check that each of $h_1(t) = 1$, $h_2(t) = e^{2t}$ and $h_3(t) = e^{-2t}$ are solutions to this ODE.

   (b) Use the Wronskian test to confirm that h_1, h_2, h_3 generate the general solution.

   (c) Indicate‘ the system of linear equations that you would need to solve in order to find c_1, c_2, c_3 such that

$$x(t) = c_1h_1(t) + c_2h_2(t) + c_3h_3(t)$$

   is a solution to the initial value problem $Lx = 0$, $x(0) = 1$, $x'(0) = 1$, $x''(0) = 1$. *Do not solve this system of equations.*

4. Find the solution to the initial value problem $D^2x = sin(2t)$, $x(\pi) = 1$, $x'(\pi) = 0.5$.

5. Let $L = D^2 - 3D + 2$.
   (a) Check that $h_1(t) = e^{2t}$ and $h_2(t) = e^t$ are both solutions to the Ordinary Differential Equation $Lx = 0$.
   (b) Use the Wronskian test to show that $x(t) = c_1h_1(t) + c_2h_2(t)$ is the general solution of $Lx = 0$.
   (c) Note that $L[t] = 2t - 3$. Thus $p(t) = t$ is a solution to the ODE

$$(\clubsuit) Lx = 2t - 3.$$

   Find the general solution to the ODE(♣).