Homework Exercises:

1. Find the general solution to the system

$$\begin{aligned} x_1' &= \quad\quad -x_2 \\ x_2' &= 2x_1 + 3x_2. \end{aligned}$$

In this problem, write the answer not as a vector but in the form $x_1 = \cdots$, $x_2 = \cdots$.

Solution: If we rewrite this system in matrix form $D\mathbf{x} = A\mathbf{x}$ then we can compute the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$. The characteristic polynomial $p(\lambda)$ of A is

$$p(\lambda) = \det(A - \lambda I) = (-\lambda)(3-\lambda) + 2 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

and therefore A has eigenvalues 1 and 2. We compute the corresponding eigenvectors by solving the system $(A - \lambda I|\mathbf{0})$. Let's do this for $\lambda = 1$. We have

$$(A - 1 * I|\mathbf{0}) = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

which yields $v_1 = -v_2$ and therefore an eigenvector for $\lambda = 1$ is $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. By similar process we find the the eigenvector for $\lambda = 2$ and have the general solution

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

and finally, rewriting the solutions in the appropriate form we have

$$x_1 = c_1 e^t + c_2 e^{2t}, \quad x_2 = -c_1 e^t - 2c_2 e^{2t}$$

Note that we may not absorb minus signs into the constants in the expression for x_2, since that would alter the constants in x_1.

2. Find a generating set of real solutions to the system $D\mathbf{x} = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}.$$

Solution: As in the first problem, we compute eigenvalues and their eigenvectors and write solutions as $e^{\lambda t}\mathbf{v}$. The characteristic polynomial $p(\lambda)$ of A (computing the determinant using the first column) is

$$p(\lambda) = (1-\lambda)((1-\lambda)(4-\lambda) + 2) - 0 + (-1)(-2 - 2(1-\lambda))$$

Often, we want to see if there's a way to factor this without expanding it out entirely. Unfortunately in this case we can't quite do that, so we simplify, and compute the factors using the rational root test. We find that

$$p(\lambda) = -\lambda^3 + 6\lambda^2 - 13\lambda + 10 = -(\lambda - 2)(\lambda^2 - 4\lambda + 5)$$

which has roots $\lambda = 2$ and $\lambda = 2 \pm i$. To see an example of complex row reduction, see problem 4. For $\lambda = 2 - i$ the eigenvector found is $\begin{bmatrix} 2 \\ 1-i \\ 2 \end{bmatrix}$.

$$e^{(2-i)t}\begin{bmatrix} 2 \\ 1-i \\ 2 \end{bmatrix} = e^{2t}(\cos t - i\sin t)\begin{bmatrix} 2 \\ 1-i \\ 2 \end{bmatrix} = e^{2t}\begin{bmatrix} 2\cos t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix} + ie^{2t}\begin{bmatrix} -2\sin t \\ -\sin t - \cos t \\ -2\sin t \end{bmatrix}$$

In actuality, we find the two solutions corresponding the complex root of the characteristic polynomial by taking the real and imaginary components of the above expression. Together with the real eigenvalue $\lambda = 2$ and the corresponding eigenvector, we have the general solution

$$\mathbf{x}(t) = c_1e^{2t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2e^{2t}\begin{bmatrix} 2\cos t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix} + c_3e^{2t}\begin{bmatrix} -2\sin t \\ -\sin t - \cos t \\ -2\sin t \end{bmatrix}.$$

3. Consider the system $D\mathbf{x} = A\mathbf{x}$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

By computing eigenvalues and eigenvectors, construct as many solutions as possible. Do they constitute a complete set of solutions?

Solution: There characteristic polynomial of A is given by

$$p(\lambda) = -\lambda((-\lambda)(-1-\lambda)-1)+1 = -\lambda^3 - \lambda^2 + \lambda + 1 = -(\lambda-1)(\lambda+1)^2$$

and therefore we have eigenvalues $\lambda = 1$ and $\lambda = -1$ with multiplicity 2. As we can probably anticipate from the question, we are unfortunately only able to find a single eigenvector for $\lambda = -1$, namely $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and the best we can do for a general solution is

$$\mathbf{x}(t) = c_1e^{t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

which of course is not sufficient as we only have two linearly independent solutions for a third order system.

4. Solve the following system of equations for x, y, and z using row reduction on the related augmented matrix:

$$\begin{array}{rcrcrcr} x & + & iy & + & (-3+i)z & = & -1-i, \\ 2x & + & (1+3i)y & + & (4-2i)z & = & 2i, \\ 2ix & - & 2y & + & (-2-3i)z & = & -1+i. \end{array}$$

Note that the solutions may themselves be complex. Be sure to detail the row operations being performed.

Solution: Alright, we first identify the augmented matrix:

$$
\begin{aligned}
(A \mid \mathbf{b}) &= \left[\begin{array}{ccc|c} 1 & i & -3+i & -1-i \\ 2 & 1+3i & 4-2i & 2i \\ 2i & -2 & -2-3i & -1+i \end{array}\right] \\
&= \left[\begin{array}{ccc|c} 1 & i & -3+i & -1-i \\ 0 & 1+i & 10-4i & 2+4i \\ 0 & 0 & 3i & -3+3i \end{array}\right] \quad (R_2 = R_2 - 2R_1, R_3 = R_3 - 2iR_1) \\
&= \left[\begin{array}{ccc|c} 1 & i & -3+i & -1-i \\ 0 & 1 & 3-7i & 3+i \\ 0 & 0 & 1 & 1+i \end{array}\right] \quad (R_3 = R_3 \cdot (-i/3), R_2 = R_2 \cdot (1-i)/2) \\
&= \left[\begin{array}{ccc|c} 1 & i & 0 & 3+i \\ 0 & 1 & 0 & -7+5i \\ 0 & 0 & 1 & 1+i \end{array}\right] \quad (R_1 = R_1 - (-3+i)R_3, R_2 = R_2 - (3-7i)R_3)
\end{aligned}
$$

We could do the last step, but note that we certainly have enough here to solve the system now. It is clear that

$$y = -7+5i,\ z = 1+i, \quad \Rightarrow \quad x = 3+i-iy = 3+i-i(-7+5i) = 8+8i.$$

5. Solve the initial value problem $D\mathbf{x} = A\mathbf{x}$ where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Solution: Phew, back to an easier problem hopefully. The eigenvalues of A are easy enough to compute, given that the characteristic polynomial is

$$p(\lambda) = (-1-\lambda)(1-\lambda)+2 = \lambda^2+1 \quad \Rightarrow \quad \lambda = \pm i.$$

As before, we only need to work with one root. Using $\lambda = -i$, we have

$$
\begin{aligned}
(A - iI \mid \mathbf{0}) &= \left[\begin{array}{cc|c} -1+i & 2 & 0 \\ -1 & 1+i & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} 1 & -1-i & 0 \\ -1 & 1+i & 0 \end{array}\right] \quad (R_1 = R_1 \cdot \frac{-1-i}{2}) \\
&= \left[\begin{array}{cc|c} 1 & -1-i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad (R_2 = R_2 + R_1)
\end{aligned}
$$

and therefore

$$\mathbf{v} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

is a valid eigenvector. The two solutions then come from taking the real and imaginary parts of $e^{\lambda t}\mathbf{v}$ which gives us the general solution:

$$
\begin{aligned}
\mathbf{x}(t) &= c_1 \mathrm{Re}\left((\cos t - i \sin t)\begin{bmatrix} 1+i \\ 1 \end{bmatrix}\right) + c_2 \mathrm{Im}\left((\cos t - i \sin t)\begin{bmatrix} 1+i \\ 1 \end{bmatrix}\right) \\
&= c_1 \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t - \sin t \\ -\sin t \end{bmatrix}
\end{aligned}
$$

The last thing to do is apply the initial conditions. We have

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

which translates to

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

which implies that $c_1 = -1, c_2 = 1$ which yields the specific solution

$$\mathbf{x}(t) = -\begin{bmatrix} \cos t + \sin t \\ +\cos t \end{bmatrix} + \begin{bmatrix} \cos t - \sin t \\ -\sin t \end{bmatrix} = \begin{bmatrix} -2\sin t \\ -\cos t - \sin t \end{bmatrix}$$

6. Let $\lambda_1, \ldots, \lambda_k$ be k distinct eigenvalues of an $n \times n$ matrix, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ respectively. This exercise proves that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent for $k \geq 2$.

   Assume that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent, i.e., there is a nontrivial linear relation among them. We will try to get a contracdiction. Renumbering the eigenvectors if necessary, let

$$c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r = 0 \tag{1}$$

   be the shortest nontrivial linear relation among $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

   (a) Multiply equation (1) by A and simplify to get a new equation labelled as (2).
   (b) Multiply equation (1) by λ_1 to get a new equation labelled as (3).
   (c) Subtract equation (3) from equation (2) and explain why this results in a contradiction.

   The contradiction proves that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must be linearly independent.

Solution: Before starting, let's look at our linear combination and consider the superlatives given. Is it possible that any of the coefficients $c_1, \ldots, c_r$ are 0? The answer is no, because if any of them were 0, we could remove that component of the linear combination to have a shorter nontrivial linear relation among the original set of vectors. For example, if we have that $2\mathbf{v}_1 - 3\mathbf{v}_2 + 0\mathbf{v}_3 = 0$, then clearly the three vectors are linearly dependent, but a shorter nontrivial linear relation between them could be given by $2\mathbf{v}_1 - 3\mathbf{v}_2 = 0$. So, let's keep this in mind for the end of the proof: none of the constants in our linear combination are 0.

(a) Applying A to the right-hand side of A yields $\mathbf{0}$ so we focus on the left-hand side, using the fact that matrices act linearly on vectors:

$$A(c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r = c_1A\mathbf{v}_1 + \cdots + c_rA\mathbf{v}_r = c_1\lambda_1\mathbf{v}_1 + \cdots + c_r\lambda_r\mathbf{v}_r.$$

So our new equation (2) is

$$c_1\lambda_1\mathbf{v}_1 + \cdots + c_r\lambda_r\mathbf{v}_r = \mathbf{0} \tag{2}$$

(b) This is straightforward, we just multiply and distribute.

$$c_1\lambda_1\mathbf{v}_1 + \cdots + c_r\lambda_1\mathbf{v}_r = 0 \tag{3}$$

(c) Now we take the difference of equations (2) and (3). The λ_1 terms cancel, and we're left with

$$c_2(\lambda_2 - \lambda_1)\mathbf{v}_2 + \cdots + c_r(\lambda_r - \lambda_1)\mathbf{v}_r = 0$$

Note that none of the λ differences are 0, since we assumed that all eigenvalues are distinct (i.e. $\lambda_i - \lambda_j \neq 0$ if $i \neq j$). Also, at the beginning of the proof we noted that none of the constants $c_1, \ldots, c_r$ are 0. Therefore, we have constructed a *shorter* nontrivial relation among the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. This is a contradiction, and therefore the $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must be linearly independent.