Math 51 **Problem Set 3 Solutions**

1. (a) $\frac{d^5x}{dt^5} + t^2\frac{dx}{dt} = te^t$ is linear but not homogeneous. This equation can be rewritten as $Lx = te^t$ where $L = D^5 + t^2 D$

   (b) $\frac{d^2x}{dt^2} = x\frac{dx}{dt} + t$ is neither linear nor homogeneous.

   (c) $\frac{d^3x}{dt^3}\sin(t)\frac{dx}{dt} = t^2x$ is neither linear nor homogeneous.

   (d) $\frac{d^3x}{dt^3} + e^t\frac{d^2x}{dt^2} + tx = e^t$ is linear but not homogeneous. This equation can be rewritten as $Lx = e^t$ where $L = D^3 + e^tD^2 + t$.

2. To calculate the determinant of the given matrix, denoted M, we will expand along the first column. Thus,

$$\begin{aligned}\det(M) &= e^t(-\cos^2(t) - \sin^2(t)) - e^t(-\sin(t)\cos(t) + \sin(t)\cos(t)) + e^t(-\sin^2(t) - \cos^2(t)) \\ &= -2e^t \neq 0\end{aligned}$$

3. (a) Notice that $D^3(1) = 0,\ 4D(1) = 0$, so $0 - 0 = 0$ so h_1 is a solution.
   Next, notice that $D^3(e^{2t}) = 8e^{2t},\ 4D(e^{2t}) = 8e^{2t}$, so $8e^{2t} - 8e^{2t} = 0$, so h_2 is a solution.
   Finally, notice that $D^3(e^{-2t}) = -8e^{-2t},\ 4D(e^{-2t}) = -8e^{2t}$, so $-8e^{2t} + 8e^{2t} = 0$, so h_3 is a solution.

   (b) The Wronskian matrix W is

$$W = \begin{bmatrix} 1 & e^{2t} & e^{-2t} \\ 0 & 2e^{2t} & -2e^{-2t} \\ 0 & 4e^{2t} & 4e^{-2t} \end{bmatrix}$$

   so $\det(W) = 8e^{2t}e^{-2t} + 8e^{2t}e^{-2t} = 16 \neq 0$, so h_1, h_2, h_3 can in fact generate the general solution.

   (c) The system of equations in matrix form is given by

$$\begin{bmatrix} 1 & e^{2t} & e^{-2t} \\ 0 & 2e^{2t} & -2e^{-2t} \\ 0 & 4e^{2t} & 4e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4. Given the equation

$$\frac{d^2x}{dt^2} = \sin(2t)$$

with initial conditions $x(\pi) = 1, x'(\pi) = \frac{1}{2}$, we can uniquely specify a solution. Integrating the above equation once will give us

$$\frac{dx}{dt} = \frac{-\cos(2t)}{2} + C_1$$

We can solve for C_1 by plugging in the initial condition, which gives $C_1 = 1$. Integrating again, we get that

$$x(t) = \frac{-\sin(2t)}{4} + t + C_2$$

and using the initial condition again we find that $C_2 = 1 - \pi$, so our final solution is

$$x(t) = \frac{-\sin(2t)}{4} + t + 1 - \pi$$

5. (a) Note that $Le^{2t} = 4e^{2t} - 6e^{2t} + 2e^{2t} = 0$ and $Le^t = e^t - 3e^t + 2e^t = 0$ so h_1 and h_2 are solutions.

   (b) The Wronskian determinant is $\det(W) = e^{2t}e^t - 2e^{2t}e^t = -e^{3t} \neq 0$ so these solutions form the general solution for the homegeneous differential equation.

   (c) If $p(t) = t$ is the particular solution, denoted x_p, then the general solution is given by $x_g(t) = x_p(t) + x_h(t) = t + c_1e^{2t} + c_2e^t$.