Problem Set 7 – Solutions
Linear Systems of ODES; independence

Math 51 Fall 2021

due Monday 2022-03-07 at 11:59 PM

Problems

1. For each of the following systems of ODEs, decide whether it is linear. For each linear system, do also the following:

   - indicate whether it is homogeneous
   - find a matrix A and a vector E such that the system can be rewritten in the form

   $$D\mathrm{x} = A\mathrm{x} + \mathrm{E}$$

   where $\mathrm{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (or $\mathrm{x} = \begin{bmatrix} x \\ y \end{bmatrix}$).

   $$(a) \quad \begin{cases} x' = & ty - z \\ y' = & -\dfrac{x}{t} - z + 1 \\ z' = & -x - t^2 y + z + 2t \end{cases} \qquad (b) \quad \begin{cases} x' = & 2x - 3y \\ y' = & 3x^2 y + y + 1 \end{cases} \qquad (c) \quad \begin{cases} x' = & 7x + 11y \\ y' = & -2x + y \end{cases}$$

   Solution:

   (a) The system is linear, but is not homogeneous. It can be written

   $$D \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & t & -1 \\ \dfrac{-1}{t} & 0 & -1 \\ -1 & -t^2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2t \end{bmatrix}$$

   (b) The system is not linear, because the dependence $y' = 3x^2 y + y + 1$ is not linear (it involves the non-linear term $x^2 y$).

   (c) The system is linear and homogeneous. It can be written

   $$D \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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2. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and consider the non-homogeneous system

   $$(\clubsuit) \quad D \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix}.$$

a. Show that $\mathrm{h}_1(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}, \quad \mathrm{h}_2(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$ are solutions to the corresponding homogeneous system $D\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$.

Solution:

To check that h_1 is a solution, we compute

$$D\mathrm{h}_1 = \begin{bmatrix} \frac{d}{dt}\sin(t) \\ \frac{d}{dt}\cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

and

$$A\mathrm{h}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} = \begin{bmatrix} 0\sin(t) + 1\cos(t) \\ -1\sin(t) + 0\cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}.$$

Since these expressions agree, h_1 is a solution.

To check that h_2 is a solution, we compute

$$D\mathrm{h}_2 = \begin{bmatrix} \frac{d}{dt}\cos(t) \\ \frac{d}{dt}[-\sin(t)] \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}$$

and

$$A\mathrm{h}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} = \begin{bmatrix} 0\cos(t) + 1(-\sin(t)) \\ -1\cos(t) + 0(-\sin(t)) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}.$$

Since these expressions agree, h_2 is a solution.

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b. Show that $\mathrm{p}(t) = \begin{bmatrix} 0 \\ -t \end{bmatrix}$ is a particular solution to the (♣).

Solution:

Compute

$$D\mathrm{p} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and

$$A\mathrm{p} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -t \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Since these expressions are equation, p is a solution to (♣).

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c. Show that the initial vectors $\mathrm{h}_1(0)$ and $\mathrm{h}_2(0)$ are linearly independent. Find the general solution to (♣).

Solution:

Note that $\mathrm{h}_1(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathrm{h}_2(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; since $\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$, these vectors are linearly independent. Thus h_1 and h_2 generate the general solution to the homogeneous system $D\mathrm{x} = A\mathrm{x}$ and so the the general solution to the inhomogeneous system is given by

$$\begin{aligned} \mathrm{x}(t) &= \mathrm{p}(t) + c_1\mathrm{h}_1(t) + c_2\mathrm{h}_2(t) \\ &= \begin{bmatrix} 0 \\ -t \end{bmatrix} + c_1\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + c_2\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} \end{aligned}$$

3. Consider the linear ODE

$$\text{(N)} \quad (D-3)^2 x = e^{3t} \quad \text{i.e.} \quad (D^2 - 6x + 9)x = e^{3t}.$$

a. Find the equivalent linear system $(\mathrm{S_N})$ of ODEs. Write this system in matrix form.

Solution:

We set $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. For a solution x of (N) we set $x_1 = x$ and $x_2 = x'$.

We have that $x_1' = x_2$ and

$$x_2' = x'' = -9x + 6x' + e^{3t} = -9x_1 + 6x_2 + e^{3t}.$$

Thus we the system in matrix form

$$D \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$$

is equivalent to (N).

b. Note that the general solution to the homogeneous equation (H) $(D-3)^2 x = 0$ is generated by $h_1(t) = e^{3t}$ and $h_2(t) = te^{3t}$. Find the corresponding vector solutions h_1 and h_2 to the homogeneous system $(\mathrm{S_H})$.

Solution:

$$\mathrm{h}_1 = \begin{bmatrix} h_1(t) \\ h_1'(t) \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} \quad \text{and} \quad \mathrm{h}_2 = \begin{bmatrix} h_2(t) \\ h_2'(t) \end{bmatrix} = \begin{bmatrix} te^{3t} \\ (1+3t)e^{3t} \end{bmatrix}$$

c. Find a particular solution $p(t)$ to the equation $(D-3)^2 x = e^{3t}$, and find the corresponding vector solution $\mathrm{p}(t)$ to the system $(\mathrm{S_N})$.

Solution:

We can use the method of undetermined coefficients. We take $A(D) = D - 3$. The general solution to $A(D)(D-3)^2 = (D-3)^3$ is generated by $e^{3t}, te^{3t}, t^2e^{3t}$ but the first two functions are already solutions to the homogeneous equation (H) and may be eliminated.

Thus our simplified guess for a particular solution is kt^2e^{3t} and we must find the constant k.

For this, we apply the operator $(D-3)^2$ and use the exponential shift formula:

$$(D-3)^2[kt^2e^{3t}] = ke^{3t}(D+3-3)^2[t^2] = ke^{3t}D^2[t^2] = 2ke^{3t}.$$

We need $2ke^{3t} = e^{3t}$ so $k = 1/2$ and our particular solution is $p(t) = \frac{1}{2}t^2e^{3t}$.

In vector form we have

$$\mathrm{p} = \begin{bmatrix} p(t) \\ p'(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t^2e^{3t} \\ e^{3t}(t + \frac{3}{2}t^2) \end{bmatrix} = e^{3t} \begin{bmatrix} t^2/2 \\ (3t^2 + 2t)/2 \end{bmatrix}.$$

d. The general solution to (N) is given by $x(t) = p(t) + c_1h_1(t) + c_2h_2(t)$. What is the general solution to the system ($\mathrm{S_N}$)?

Solution:

$$\mathrm{x} = \mathrm{p} + c_1\mathrm{h}_1 + c_2\mathrm{h}_2 = e^{3t}\begin{bmatrix} t^2/2 \\ (3t^2+2t)/2 \end{bmatrix} + c_1\begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} + c_2\begin{bmatrix} te^{3t} \\ (1+3t)e^{3t} \end{bmatrix}$$

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4. Consider the the following matrices A and lists of vector-valued functions $\mathrm{h_i}$. In each case, answer the following questions:

- Which of the functions $\mathrm{h_i}$ are solutions to the homogeneous equation $D\mathrm{x} = A\mathrm{x}$? Be sure to indicate how you reach your conclusion.
- Consider the functions that are solutions. Do they generate the general solution to $D\mathrm{x} = A\mathrm{x}$? Why or why not?

a. $A = \begin{bmatrix} -3 & 8 \\ -3 & 7 \end{bmatrix}$; $\quad \mathrm{h}_1 = \begin{bmatrix} 2e^t \\ e^t \end{bmatrix}, \quad \mathrm{h}_2 = \begin{bmatrix} 2e^t - 4e^{3t} \\ e^t - 3e^{3t} \end{bmatrix}, \quad \mathrm{h}_3 = e^t\begin{bmatrix} 4 \\ 3 \end{bmatrix}.$

Solution:

h_1 and h_2 are solutions while h_3 is not a solution.

Indeed, let's check.

- $D\mathrm{h}_1 = \begin{bmatrix} 2e^t \\ e^t \end{bmatrix}$ and $A\mathrm{h}_1 = \begin{bmatrix} -3 & 8 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 2e^t \\ e^t \end{bmatrix} = \begin{bmatrix} -3\cdot 2e^t + 8\cdot e^t \\ -3\cdot 2e^t + 7\cdot e^t \end{bmatrix} = \begin{bmatrix} 2\cdot e^t \\ e^t \end{bmatrix}$
- $D\mathrm{h}_2 = D\begin{bmatrix} 2e^t - 4e^{3t} \\ e^t - 3e^{3t} \end{bmatrix} = \begin{bmatrix} 2e^t - 12e^{3t} \\ e^t - 9e^{3t} \end{bmatrix}$ and $A\mathrm{h}_2 = \begin{bmatrix} -3 & 8 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 2e^t - 4e^{3t} \\ e^t - 3e^{3t} \end{bmatrix} = \begin{bmatrix} -3\cdot(2e^t - 4e^{3t}) + 8\cdot(e^t - 3e^{3t}) \\ -3\cdot(2e^t - 4e^{3t}) + 7\cdot(e^t - 3e^{3t}) \end{bmatrix} = \begin{bmatrix} 2e^t - 12e^{3t} \\ e^t - 9e^{3t} \end{bmatrix}$

Thus $D\mathrm{h}_1 = A\mathrm{h}_1$ and $D\mathrm{h}_2 = A\mathrm{h}_2$ so that h_1 and h_2 are solutions.

- $D\mathrm{h}_3 = D\begin{bmatrix} 4e^t \\ 3e^t \end{bmatrix} = \begin{bmatrix} 4e^t \\ 3e^t \end{bmatrix} = e^t\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $A\mathrm{h}_3 = e^t\begin{bmatrix} -3 & 8 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = e^t\begin{bmatrix} -3\cdot 4 + 8\cdot 3 \\ -3\cdot 4 + 7\cdot 3 \end{bmatrix} = e^t\begin{bmatrix} 12 \\ -9 \end{bmatrix}$

Since $D\mathrm{h}_3 \neq A\mathrm{h}_3$, h_3 is not a solution.

Finally, we claim that h_1 and h_2 generate the general solution to $D\mathrm{x} = A\mathrm{x}$. For this, we use the Wronskian test. The Wronskian matrix W has columns h_1 and h_2; i.e.

$$W = \begin{bmatrix} 2e^t & 2e^t - 4e^{3t} \\ e^t & e^t - 3e^{3t} \end{bmatrix}.$$

Evaluating the matrix at $t = 0$ gives $W|_{t=0} = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}$

Now

$$\det W|_{t=0} = -4 - (-2) = -2.$$

Since this determinant is non-0, the general solution is given by $\mathrm{x} = c_1\mathrm{h}_1 + c_2\mathrm{h}_2$.

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b. $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$; $\quad \mathrm{h}_1 = e^t\begin{bmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \\ 0 \end{bmatrix}, \quad \mathrm{h}_2 = e^t\begin{bmatrix} 2\cos(t) + 2\sin(t) \\ 4\cos(t) \\ e^{-3t} \end{bmatrix}, \quad \mathrm{h}_3 = \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \end{bmatrix}.$

Solution:

A direct check confirms that $\mathrm{h_i}$ is a solution for $1 \leq i \leq 3$. To see that whether they generate the general solution, consider the vectors

$$\mathrm{v_i} = \mathrm{h_i}(0).$$

Thus

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathrm{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \quad \mathrm{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We now compute the determinant of the matrix whose columns are the vectors $\mathrm{v_i}$:

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0$$

This shows that $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ are linearly dependent, and thus $\mathrm{h}_1, \mathrm{h}_2$ and h_3 do not generate the general solution to $D\mathrm{x} = A\mathrm{x}$.

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5. Let

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}, \quad \mathrm{E} = \begin{bmatrix} 0 \\ -5e^t \end{bmatrix}.$$

The formulas

$$(\clubsuit) \quad \begin{cases} x_1 = & c_1 \cos(2t) + c_2 \sin(2t) + e^t \\ x_2 = & -2c_2 \cos(2t) + 2c_1 \sin(2t) - e^t \end{cases}$$

describe a collection of solutions to the nonhomogeneous system $D\mathrm{x} = A\mathrm{x} + \mathrm{E}$.

a. Write the collection ($\clubsuit$) of solutions in the form $\mathrm{x} = c_1\mathrm{h}_1 + c_2\mathrm{h}_2 + \mathrm{p}$ where h_1 and h_2 are solutions to the homogeneous system $D\mathrm{x} = A\mathrm{x}$.

Solution:

$$\mathrm{x} = c_1 \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ -2\cos(2t) \end{bmatrix} + \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$$

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b. Decide whether the collection ($\clubsuit$) is complete.

Solution:

The indicated solution will be complete provided that h_1 and h_2 generate the general solution to the homogeneous equation $D\mathrm{x} = A\mathrm{x}$.

To decide this, we consider the Wronskian matrix W whose columns are h_1 and h_2; thus

$$W = \begin{bmatrix} \cos(2t) & \sin(2t) \\ \sin(2t) & -2\cos(2t) \end{bmatrix}$$

We compute the determinant after evaluation at $t = 0$:

$$\det W|_{t=0} = \det \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = -2.$$

Since this determinant is non-zero, h_1 and h_2 generate the general solution to the homogeneous equation $D\mathrm{x} = A\mathrm{x}$, and this confirms that ($\clubsuit$) is complete (since p is a particular solution).

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6. Check the following list of vectors for linear independence:

$$\mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution:

Suppose that c_1, c_2, c_3 are scalars and that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 \\ c_3 \\ c_1 \\ c_1 + 3c_2 \end{bmatrix}$$

Examination of the 2nd and 3rd entries shows that $c_1 = c_3 = 0$. Then examination of the 1st (or 4th) entry shows that $c_2 = 0$.

Since all c_i must be zero, we have confirmed that the vectors are linearly independent.

Bibliography