Problem Set 7
Variation of parameters & Linear systems of ODEs

Math 51 Fall 2021

due 2021-10-31 at 11:59 PM

These problems concern (Nitecki and Guterman 1992, sec. 2.8 & §3.2).

1. Find the general solution to

$$x'' - 2x' + x = e^t \ln(t), \quad t > 0.$$

Solution:

Since the characteristic polyomial is $r^2 - 2r + 1 = (r-1)^2$, the general solution to the corresponding homogeneous equation is generated by

$$h_1(t) = e^t, \quad h_2(t) = te^t.$$

To solve the inhomogeneous equation, it isn't clear how to use the method of *undetermined coefficients* (what should $A(D)$ be?!?) So we use *variation of parameters.*

The Wronskian matrix is given by

$$W = W(h_1, h_2)(t) = \det \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix}.$$

And following the method, we need to solve the matrix equation

$$W \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix} \cdot \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \ln(t) \end{bmatrix}.$$

We now use *Cramer's Rule* to find c_i'. First note that $\det W = (t+1)e^{2t} - te^{2t} = e^{2t}$.

We now have

$$c_1' = \frac{\det \begin{bmatrix} 0 & te^t \\ e^t \ln(t) & (t+1)e^t \end{bmatrix}}{\det W} = \frac{-e^{2t}\ln(t)t}{e^{2t}} = -t\ln(t)$$

and

$$c_2' = \frac{\det \begin{bmatrix} e^t & 0 \\ e^t & e^t \ln(t) \end{bmatrix}}{\det W} - \frac{\ln(t)e^{2t}}{e^{2t}} = \ln(t).$$

We now find the functions c_1, c_2 by anti-differentiation:

$$c_1(t) = \int c_1' dt = -\int t\ln(t) dt = \frac{-t^2\ln(t)}{2} + \frac{t^2}{4} + C = \frac{t^2(1 - 2\ln(t))}{4} + C$$

(*integrate by parts* with $u = \ln(t)$ and $dv = t$).

and

$$c_2(t) = \int c_2' dt = \int \ln(t) dt = t\ln(t) - t + D$$

Now, we only need a particular solution to the ODE, se we take $C = D = 0$, and we find the particular solution

$$\begin{aligned} p(t) &= c_1(t)h_1(t) + c_2(t)h_2(t) \\ &= \left(\frac{t^2(1-2\ln(t))}{4}\right) \cdot e^t + (t\ln(t) - t) \cdot te^t \\ &= \left(\frac{-2\ln(t)+1}{4}\right) \cdot t^2e^t + \frac{4\ln(t)-4}{4} \cdot t^2e^t \\ &= \frac{2\ln(t)-3}{4} t^2e^t \end{aligned}$$

Now the general solution is

$$x(t) = p(t) + k_1e^t + k_2te^t = \frac{2\ln(t)-3}{4}t^2e^t + k_1e^t + k_2te^t$$

for constants k_1, k_2.

---

2. Verify that t and e^t are solutions of the homogeneous equation corresponding to

$$(N) \quad [(t-1)D^2 - tD + 1]x = (t-1)^2e^t, \quad t > 1,$$

and find the general solution of the nonhomogeneous equation (N).

(Remember to put the ODE in *standard form* when using *variation of parameters*!)

Solution:

To verify that t is a homogeneous solution, we compute

$$((t-1)D^2 - tD + 1)[t] = (t-1)D^2[t] - tD[t] + t = 0 - t + t = 0$$

and to verify that e^t is a homog. solution, we compute

$$((t-1)D^2 - tD + 1)[e^t] = (t-1)D^2[e^t] - tD[e^t] + e^t = (t-1)e^t - te^t + e^t = 0.$$

We are going to describe a particular solution in the form $p(t) = c_1(t)t + c_2(t)e^t$; we'll find the functions c_1 and c_2 using our *Wronskian method*. In order to use this method, we need to first put the ODE in *standard form*. i.e.

$$[D^2 - \frac{t}{t-1}D + \frac{1}{t-1}]x = (t-1)e^t, \quad t > 1,$$

Now let W denote the Wronskian matrix $W = \begin{bmatrix} t & e^t \\ 1 & e^t \end{bmatrix}$. We must solve the matrix equation

$$W \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ (t-1)e^t \end{bmatrix}$$

(*it was this step that requires standard form!!*)

Notice that $\det W = (t-1)e^t$. Now use Cramer's Rule to find

$$c_1' = \frac{\det \begin{bmatrix} 0 & e^t \\ (t-1)e^t & e^t \end{bmatrix}}{(t-1)e^t} = \frac{-(t-1)e^{2t}}{(t-1)e^t} = -e^t$$

and

$$c_2' = \frac{\det \begin{bmatrix} t & 0 \\ 1 & (t-1)e^t \end{bmatrix}}{(t-1)e^t} = \frac{t(t-1)e^t}{(t-1)e^t} = t.$$

We find now that

$$c_1 = \int c_1' dt = -\int e^t dt = -e^t + C$$

and

$$c_2 = \int c_2' dt = \int t dt = \frac{t^2}{2} + D$$

Since we only need one particular solution, we may and will take $C = D = 0$. Now our particular solution is

$$p(t) = c_1 t + c_2 e^t = -te^t + \frac{t^2 e^t}{2} = \frac{(t^2-2t)e^t}{2}$$

and the general solution to the inhomog. ODE is

$$x(t) = p(t) + k_1 t + k_2 e^t = \frac{(t^2-2t)e^t}{2} + k_1 t + k_2 e^t.$$

---

3. For each of the following systems of ODEs, decide whether it is linear. For each linear systems, do also the following:

   - indicate whether it is homogeneous
   - find a matrix A and a vector $\mathbf{E}$ such that the system can be rewritten in the form

   $$D\mathbf{x} = A\mathbf{x} + \mathbf{E}$$

   where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (or $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$).

   a. $\begin{cases} x' = & ty - z \\ y' = & -\dfrac{x}{t} - z + 1 \\ z' = & -x - t^2 y + z + 2t \end{cases}$

   Solution:

   The system is linear, but is not homogeneous. It can be written

   $$D\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & t & -1 \\ \frac{-1}{t} & 0 & -1 \\ -1 & -t^2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2t \end{bmatrix}$$

---

b. $\begin{cases} x' = & 2x - 3y \\ y' = & 3x^2y + y + 1 \end{cases}$

Solution:

The system is not linear, because the dependence $y' = 3x^2y + y + 1$ is not linear (it involves the non-linear term x^2y).

---

c. $\begin{cases} x' = & 7x + 11y \\ y' = & -2x + y \end{cases}$

Solution:

The system is linear and homogeneous. It can be written

$$D\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

---

4. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

a. Show that $\mathbf{h_1}(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$, $\quad \mathbf{h_2}(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$ are solutions to the homogeneous system

$$D\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}.$$

Solution:

To check that $\mathbf{h_1}$ is a solution, we compute

$$D\mathbf{h_1} = \begin{bmatrix} \frac{d}{dt}\sin(t) \\ \frac{d}{dt}\cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

and

$$A\mathbf{h_1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} = \begin{bmatrix} 0\sin(t) + 1\cos(t) \\ -1\sin(t) + 0\cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}.$$

Since these expressions agree, $\mathbf{h_1}$ is a solution.

To check that $\mathbf{h_2}$ is a solution, we compute

$$D\mathbf{h_2} = \begin{bmatrix} \frac{d}{dt}\cos(t) \\ \frac{d}{dt}[-\sin(t)] \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}$$

and

$$A\mathbf{h_1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} = \begin{bmatrix} 0\cos(t) + 1(-\sin(t)) \\ -1\cos(t) + 0(-\sin(t)) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}.$$

Since these expressions agree, $\mathbf{h_2}$ is a solution.

---

b. Show that $\mathbf{p}(t) = \begin{bmatrix} 0 \\ -t \end{bmatrix}$ is a *particular solution* to the inhomogeneous equation

$$(\clubsuit) \quad D\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix}.$$

Solution:

Compute

$$D\mathbf{p} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and

$$A\mathbf{p} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -t \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Since these expressions are equation, $\mathbf{p}$ is a solution to ($\clubsuit$).

---

c. Let $\mathbf{x} = \mathbf{p}(t) + c \cdot \mathbf{h_1}(t)$ for a constant c. By computing $D[\mathbf{x}]$ and $A\mathbf{x} + \begin{bmatrix} t \\ -1 \end{bmatrix}$ and comparing the results, confirm that $\mathbf{x}$ is a solution to ($\clubsuit$) for all c.

(**Note:** In fact, $\mathbf{x} = \mathbf{p}(t) + c_1\mathbf{h_1}(t) + c_2\mathbf{h_2}(t)$ is the *general solution* to ($\clubsuit$); we'll see how to confirm this later in class).

Solution:

Using **a.** and **b.** we find for any c that

$$D[\mathbf{p} + c\mathbf{h_1}] = D[\mathbf{p}] + cD[\mathbf{h_1}] = \left(A\mathbf{p} + \begin{bmatrix} t \\ -1 \end{bmatrix} \right) + cA\mathbf{h_1} = A[\mathbf{p} + c\mathbf{h_1}] + \begin{bmatrix} t \\ -1 \end{bmatrix}$$

which confirms that $\mathbf{x} = \mathbf{p} + c\mathbf{h_1}$ is indeed a solution to ($\clubsuit$).

(*as an aside, we know that* $\mathbf{p}(t) + c_1\mathbf{h_1}(t) + c_2\mathbf{h_2}(t)$ *is the general solution to* ($\clubsuit$) *provided that* $c_1\mathbf{h_1}(t) + c_2\mathbf{h_2}(t)$ *is the general solution to the homogeneous equation* $D\mathbf{x} = A\mathbf{x}$*. For this, we need to check the linear independence of the vectors* $\mathbf{h_1}(0)$ *and* $\mathbf{h_2}(0)$*.*)

---

5. Consider the linear ODE

$$(\mathbf{N}) \quad (D-3)^2 x = e^{3t} \quad \text{i.e.} \quad (D^2 - 6D + 9)x = e^{3t}.$$

a. Find the equivalent linear system ($\mathbf{S_N}$) of ODEs. Write this system in *matrix form.*

Solution:

We set $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. For a solution x of ($\mathbf{N}$) we set $x_1 = x$ and $x_2 = x'$.

We have that $x_1' = x_2$ and

$$x_2' = x'' = -9x + 6x' + e^{3t} = -9x_1 + 6x_2 + e^{3t}.$$

Thus we the system in matrix form

$$D \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$$

is equivalent to ($\mathbf{N}$).

---

b. Note that the general solution to the homogeneous equation ($\mathbf{H}$) $(D-2)^2 x = 0$ is generated by $h_1(t) = e^{3t}$ and $h_2(t) = te^{3t}$. Find the corresponding vector solutions $\mathbf{h}_1$ and $\mathbf{h}_2$ to the homogeneous system ($\mathbf{S_H}$).

Solution:

$$\mathbf{h_1} = \begin{bmatrix} h_1(t) \\ h_1'(t) \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{h_2} = \begin{bmatrix} h_2(t) \\ h_2'(t) \end{bmatrix} = \begin{bmatrix} te^{3t} \\ (1+3t)e^{3t} \end{bmatrix}$$

---

c. Find a particular solution $p(t)$ to the equation $(D-3)^2 x = e^{3t}$, and find the corresponding vector solution $\mathbf{p}(t)$ to the system $(\mathbf{S_N})$.

Solution:

We can use the method of *undetermined coefficients*. We take $A(D) = D-3$. The general solution to $A(D)(D-3)^2 = (D-3)^3$ is generated by $e^{3t}, te^{3t}, t^2 e^{3t}$ but the first two functions are already solutions to the homogeneous equation $(\mathbf{H})$ and may be eliminated.

Thus our simplified guess for a particular solution is $kt^2 e^{3t}$ and we must find the constant k.

For this, we apply the operator $(D-3)^2$ and use the exponential shift formula:

$$(D-3)^2[kt^2 e^{3t}] = ke^{3t}(D+3-3)^2[t^2] = ke^{3t} D^2[t^2] = 2ke^{3t}.$$

We need $2ke^{3t} = e^{3t}$ so $k = 1/2$ and our particular solution is $p(t) = \dfrac{1}{2} t^2 e^{3t}$.

In vector form we have

$$\mathbf{p} = \begin{bmatrix} p(t) \\ p'(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} t^2 e^{3t} \\ e^{3t}(t + \dfrac{3}{2} t^2) \end{bmatrix} = e^{3t} \begin{bmatrix} t^2/2 \\ (3t^2+2t)/2 \end{bmatrix}.$$

---

d. The general solution to $(\mathbf{N})$ is given by $x(t) = p(t) + c_1 h_1(t) + c_2 h_2(t)$. Indicate the the corresponding solutions $\mathbf{x}$ to the system $(\mathbf{S_N})$.

Solution:

$$\mathbf{x} = \mathbf{p} + c_1 \mathbf{h_1} + c_2 \mathbf{h_2} = e^{3t} \begin{bmatrix} t^2/2 \\ (3t^2+2t)/2 \end{bmatrix} + c_1 \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} te^{3t} \\ (1+3t)e^{3t} \end{bmatrix}$$

Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. *Differential Equations: A First Course*. Saunders.