Math 51

Convolution: Definition

Loring Tu

Problem. Find $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$.

We know $\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$, but we don't know how to calculate $\mathcal{L}^{-1}$ of the product of $\frac{1}{s^2+1}$ and $\frac{1}{s^2+1}$.

The convolution is the continuous analogue of power series multiplication.

Power series multiplication

$$(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$
$$= a_0 b_0 + (a_1 b_0 + a_0 b_1) x + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + \cdots.$$

In general, the coefficient of x^n is

$$c_n := a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n = \boxed{\sum_{u=0}^{n} a_{n-u} b_u}.$$

Continuous analogue: $n \rightsquigarrow t \in \mathbb{R}$

$$\sum_{u=0}^{n} \rightsquigarrow \int_0^t (\quad)\,du.$$

$$\sum_{u=0}^{n} a_{n-u} b_u \rightsquigarrow \boxed{\int_0^t a(t-u)\, b(u)\, du}.$$

This is the definition of the convolution.

Def. $(f*g)(t) = \int_0^t f(t-u)\, g(u)\, du.$

Example. $e^{at} * e^{bt}$

$$= \int_0^t e^{a(t-u)} e^{bu}\, du$$

$$= e^{at} \int_0^t e^{(b-a)u}\, du = e^{at} \frac{1}{b-a} \left[e^{(b-a)u} \right]_0^t$$

$$= \frac{1}{b-a} e^{at} \left[e^{(b-a)t} - 1 \right] = \boxed{\frac{1}{b-a}\left(e^{bt} - e^{at}\right)}$$

Note that $e^{at} * e^{bt} = e^{bt} * e^{at}$.