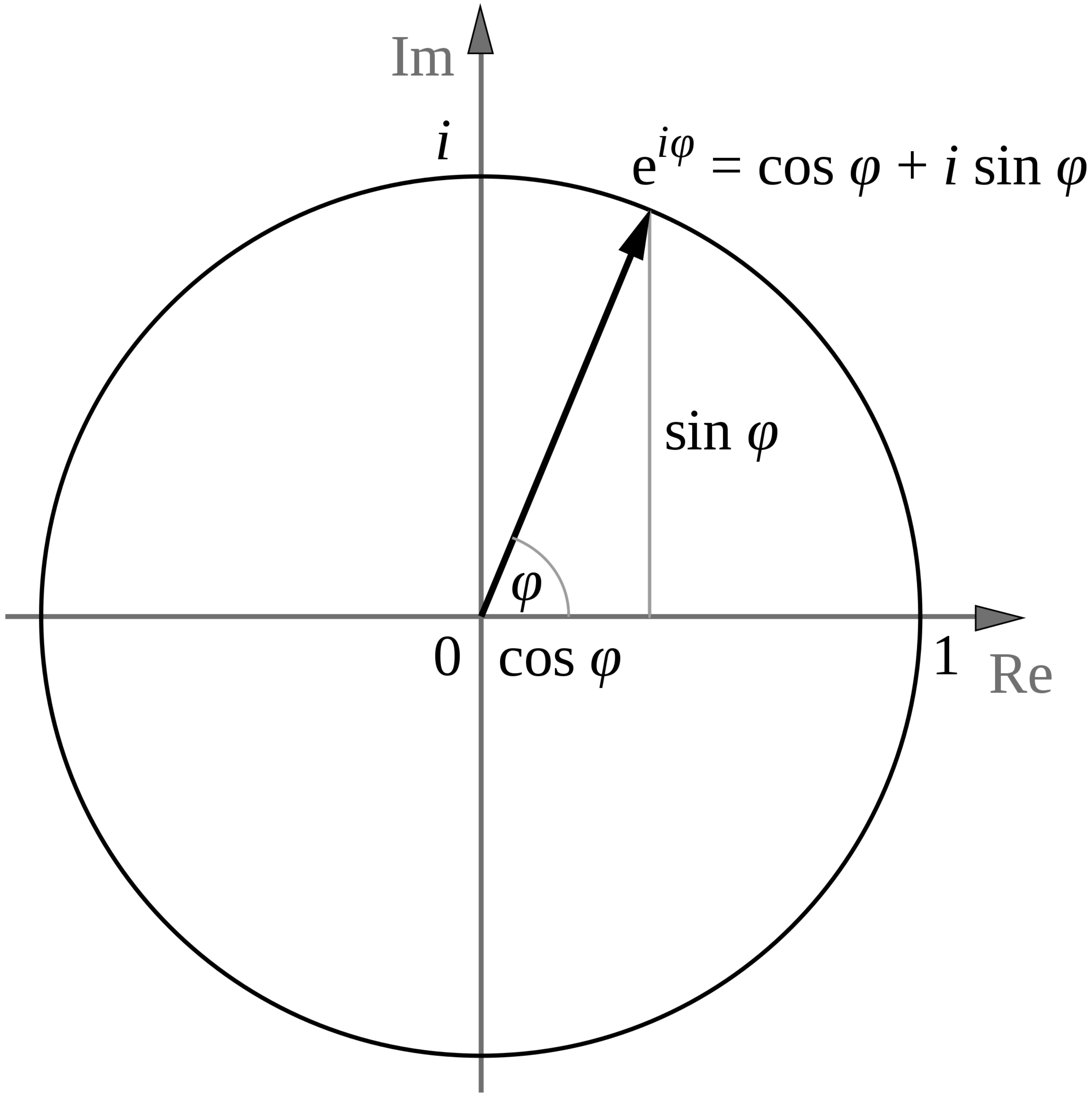

Differential Equations

Homogeneous Systems: Complex Roots

Complex Eigenvalues

Let A be a real $n \times n$ matrix.

Recall:
· (Section 3.5) $\det(A - \lambda I)$ is a polynomial in λ of degree n with real coefficients.
· (Section 2.6) Complex (non-real) roots of such polynomials come in conjugate pairs $\alpha \pm \beta i$, $\beta \neq 0$.

Fact: Complex eigenvalues of A come in conjugate pairs:

$$\begin{aligned} &\alpha + \beta i \text{ is a root of } \det(A - \lambda I) \\ \Leftrightarrow\ &\alpha - \beta i \text{ is a root of } \det(A - \lambda I) \end{aligned}$$

Strategy: Find linearly independent (complex) eigenvectors $\vec{v}$ corresponding to $\alpha + \beta i$, then the associated solutions of $D\vec{x} = A\vec{x}$ will be given by the real and imaginary parts of

$$e^{(\alpha+\beta i)t}\vec{v} = e^{\alpha}(\cos\beta t + i\sin\beta t)\vec{v}$$

by Euler's formula.

Ex: Solve $D\vec{x} = A\vec{x}$, where

$$A = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix}.$$

Sln: The characteristic polynomial of A is

$$\det(A - \lambda I) = (-1-\lambda)^2 - (-1) \times 4 = \lambda^2 + 2\lambda + 5,$$

so A has the pair of complex eigenvalues

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2} = -1 \pm 2i.$$

To find an eigenvector for $\lambda = -1 + 2i$, we solve $[A - (-1+2i)I]\vec{v} = \vec{0}$, or reduce the augmented matrix $[A - (-1+2i)I \mid \vec{0}\,]$:

$$\left[\begin{array}{cc|c} -2i & -1 & 0 \\ 4 & -2i & 0 \end{array}\right] \xrightarrow[R_2 \to R_2/4]{R_1 \to R_1 \cdot i/2} \left[\begin{array}{cc|c} 1 & -i/2 & 0 \\ 1 & -i/2 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_1]{} \left[\begin{array}{cc|c} 1 & -i/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We solve the equivalent system

$$\begin{aligned} v_1 - (i/2)v_2 &= 0 \\ 0 &= 0 \end{aligned}$$

for the pivot variable v_1 in terms of the free variable v_2:

$$v_1 = v_2 \cdot i/2.$$

Take the free variable v_2 to be 2, we get an eigenvector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} i \\ 2 \end{bmatrix}.$$

Ignoring the fact that λ is complex, we get a complex "solution" of $D\vec{x} = Ax$ associated to $\vec{v}$,

$$\vec{g}_1(t) = e^{(-1+2i)t}\vec{v}.$$

To rewrite this vector valued function in terms of its real and imaginary parts, we apply Euler's formula:

$$\vec{g}_1(t) = e^{-t}(\cos 2t + i \sin 2t) \begin{bmatrix} i \\ 2 \end{bmatrix} = \begin{bmatrix} e^{-t}(-\sin 2t + i \cos 2t) \\ e^{-t}(2\cos 2t + 2i \sin 2t) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t} \sin 2t \\ 2e^{-t} \cos 2t \end{bmatrix} + i \begin{bmatrix} e^{-t} \cos 2t \\ 2e^{-t} \sin 2t \end{bmatrix}.$$

To find an eigenvector for the other eigenvalue $\lambda = -1 - 2i$, we reduce $[A - (-1-2i)I \mid \vec{0}\,]$ to get

$$\left[\begin{array}{cc|c} 1 & i/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We get an eigenvector

$$\vec{w} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$$

of A corresponding to $\lambda = -1 - 2i$. Associated to $\vec{w}$ is another complex "solution"

$$\vec{g}_2(t) = e^{(-1-2i)t}\vec{w} = \begin{bmatrix} -e^{-t} \sin 2t \\ 2e^{-t} \cos 2t \end{bmatrix} - i \begin{bmatrix} e^{-t} \cos 2t \\ 2e^{-t} \sin 2t \end{bmatrix}.$$

The "general solution" of $D\vec{x} = A\vec{x}$ is

$$\begin{aligned} \vec{x} &= k_1\vec{g}_1(t) + k_2\vec{g}_2(t) \\ &= (k_1 + k_2) \begin{bmatrix} -e^{-t} \sin 2t \\ 2e^{-t} \cos 2t \end{bmatrix} + i\,(k_1 - k_2) \begin{bmatrix} e^{-t} \cos 2t \\ 2e^{-t} \sin 2t \end{bmatrix} \end{aligned}$$

$$= c_1 \begin{bmatrix} -e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\cos 2t \\ 2e^{-t}\sin 2t \end{bmatrix}.$$

Do the two real vector valued functions

$$\vec{h}_1(t) = \begin{bmatrix} -e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix}, \quad \vec{h}_2(t) = \begin{bmatrix} e^{-t}\cos 2t \\ 2e^{-t}\sin 2t \end{bmatrix}$$

generate the general solution of $D\vec{x} = A\vec{x}$?

Substitution will show that $\vec{h}_1(t)$ and $\vec{h}_2(t)$ are indeed solutions. Their initial vectors

$$\vec{h}_1(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \vec{h}_2(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are linearly independent, so the actual general solution of our second-order system is

$$\vec{x} = c_1\vec{h}_1(t) + c_2\vec{h}_2(t) = c_1 \begin{bmatrix} -e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\cos 2t \\ 2e^{-t}\sin 2t \end{bmatrix}.$$

Fact: Let $\alpha \pm \beta i$ be eigenvalues of the real $n \times n$ matrix A. If $\vec{v}$ is an eigenvector of A corresponding to $\alpha + \beta i$, then

$$\vec{h}_1(t) = \mathrm{Re}\left(e^{(\alpha+\beta i)t}\vec{v}\right), \quad \vec{h}_2(t) = \mathrm{Im}\left(e^{(\alpha+\beta i)t}\vec{v}\right)$$

are solutions of $D\vec{x} = A\vec{x}$ with linearly independent initial vectors.

Note: We only need to work with one of the two eigenvalues $\alpha \pm \beta i$ to find two solutions.

General Solution of $D\vec{x} = A\vec{x}$

Fact: Associate real vector valued functions to a real $n \times n$ matrix A as follows:

- For each real eigenvalue λ, find as many linearly independent eigenvectors corresponding to λ as possible.

  To each such eigenvector $\vec{v}$, associate $e^{\lambda t}\vec{v}$.

- For each pair of complex eigenvalues $\lambda = \alpha \pm \beta i$, find as many linearly independent eigenvectors corresponding to *one of the eigenvalues*, $\alpha + \beta i$, as possible.

  To each such eigenvector $\vec{v}$, associate the two functions $\mathrm{Re}\left(e^{(\alpha+\beta i)t}\vec{v}\right)$ and $\mathrm{Im}\left(e^{(\alpha+\beta i)t}\vec{v}\right)$.

These functions are solutions of $D\vec{x} = A\vec{x}$ with linearly independent initial vectors. In particular, *if we find n such functions $\vec{h}_1(t), \ldots, \vec{h}_n(t)$*, then the general solution of $D\vec{x} = A\vec{x}$ is $\vec{x} = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)$.

Ex: Solve $D\vec{x} = A\vec{x}$, where

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Sln: The characteristic polynomial is

$$\det(A - \lambda I) = -(1+\lambda)\left(\lambda^2 + 2\lambda + 2\right),$$

so the eigenvalues of A are

$$\lambda = -1, \quad \lambda = -1 \pm i.$$

To find eigenvectors for $\lambda = -1$, we reduce $[A - (-1)I \mid \vec{0}\,]$:

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The choice for the free variable $v_3 = -2$ leads to an eigenvector

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix},$$

and the associated solution of $D\vec{x} = A\vec{x}$

$$\vec{h}_1(t) = e^{-t}\vec{v} = \begin{bmatrix} e^{-t} \\ 0 \\ -2e^{-t} \end{bmatrix}.$$

For the complex eigenvalues $-1 \pm i$, we choose one of the pair, say $\lambda = -1+i$, and look for the corresponding eigenvectors by reducing $[A - (-1+i)I \mid \vec{0}\,]$:

$$\left[\begin{array}{ccc|c} -i & -1 & 0 & 0 \\ 2 & -i & 1 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \xrightarrow[R_2 \to R_2/2]{R_1 \to R_1 \cdot i} \left[\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ 1 & -i/2 & 1/2 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ 0 & i/2 & 1/2 & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \xrightarrow[R_2 \to -2i \cdot R_2]{} \left[\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 1 & -i & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 + i \cdot R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The complex vector

$$\begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}$$

is an eigenvector for $\lambda = -1 + i$. The associated complex solution of $D\vec{x} = A\vec{x}$ is

$$\begin{aligned} e^{(-1+i)t} \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} &= e^{-t}(\cos t + i \sin t) \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -e^{-t}\cos t - ie^{-t}\sin t \\ -e^{-t}\sin t + ie^{-t}\cos t \\ e^{-t}\cos t + ie^{-t}\sin t \end{bmatrix} \\ &= \begin{bmatrix} -e^{-t}\cos t \\ -e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix} + i \begin{bmatrix} -e^{-t}\sin t \\ e^{-t}\cos t \\ e^{-t}\sin t \end{bmatrix}. \end{aligned}$$

The real and imaginary parts of this complex solution,

$$\vec{h}_2(t) = \begin{bmatrix} -e^{-t}\cos t \\ -e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}, \quad \vec{h}_3(t) = \begin{bmatrix} -e^{-t}\sin t \\ e^{-t}\cos t \\ e^{-t}\sin t \end{bmatrix}$$

are solutions of $D\vec{x} = A\vec{x}$ with independent initial vectors.

The general solution of our third-order system is

$$\begin{aligned} \vec{x} &= c_1\vec{h}_1(t) + c_2\vec{h}_2(t) + c_3\vec{h}_3(t) \\ &= c_1 \begin{bmatrix} e^{-t} \\ 0 \\ -2e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t}\cos t \\ -e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix} + c_3 \begin{bmatrix} -e^{-t}\sin t \\ e^{-t}\cos t \\ e^{-t}\sin t \end{bmatrix}. \end{aligned}$$

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