Problem Set 13 – Solutions
Properties of the Laplace transform and functions defined in pieces

Math 51 Fall 2021

due Monday 2022-05-02

1. Calculate the Laplace transform of the following function:

   (a) $t^n e^{mt}$

   Solution:

$$\mathscr{L}[t^n e^{mt}] = (-1)^n \frac{d^n}{ds^n}\mathscr{L}[e^{mt}] = (-1)^n \frac{d^n}{ds^n}\left[\frac{1}{s-m}\right] = (-1)^n(-1)^n \frac{n!}{(s-m)^{n+1}} = \frac{n!}{(s-m)^{n+1}}$$

   (b) $te^{2t}\sin(t)$

   Solution:

$$\mathscr{L}[te^{2t}\sin(t)] = -\frac{d}{ds}\mathscr{L}[e^{2t}\sin(t)] = -\frac{d}{ds}\left[\frac{1}{(s-2)^2+1}\right] = -\frac{d}{ds}\left[\frac{1}{s^2-4s+5}\right] = \frac{2s-4}{(s^2-4s+5)^2}$$

   (c) $(t^3+3)^2$

   Solution:

$$\mathscr{L}[(t^3+3)^2] = \mathscr{L}[t^6+6t^3+9] = \frac{6!}{s^7} + \frac{6\cdot 3!}{s^4} + \frac{9}{s}$$

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2. Compute the inverse transform of the following functions:

   (a) $\dfrac{1}{s^2+2s+5}$

   Solution:

   Completing the square, we find that

$$s^2+2s+5 = (s+1)^2 - 1 + 5 = (s+1)^2 + 4$$

   Thus using the first shift formula with $\alpha = -1$, we find that

$$\mathscr{L}^{-1}\left[\frac{1}{s^2+2s+5}\right] = \mathscr{L}^{-1}\left[\frac{1}{(s+1)^2+4}\right] = e^{-t}\mathscr{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{e^{-t}}{2}\sin(2t)$$

(b) $\dfrac{1}{3s+6}$

Solution:

$$\mathscr{L}^{-1}\left[\frac{1}{3s+6}\right] = \frac{1}{3}\mathscr{L}^{-1}\left[\frac{1}{s+2}\right] = \frac{e^{-2t}}{3}$$

(c) $\dfrac{s+3}{s^2+10s+25}$

Solution:

Use the first shift formula with $\alpha = -5$:

$$\mathscr{L}^{-1}\left[\frac{s+3}{s^2+10s+25}\right] = \mathscr{L}^{-1}\left[\frac{s+3}{(s+5)^2}\right] = e^{-5t}\mathscr{L}^{-1}\left[\frac{s-2}{s^2}\right] = e^{-5t}\mathscr{L}^{-1}\left[\frac{1}{s}-\frac{2}{s^2}\right] = e^{-5t}\,(1-2t)$$

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3. Using the Laplace Transform, solve the following initial value problems:

(a) $(D^2+2D+2)x = 0$; $x(0) = x'(0) = 0$

Solution:

Applying $\mathscr{L}$ to each side yields

$$0 = (s^2+2s+2)\mathscr{L}[x]$$

from which one deduces $\mathscr{L}[x] = 0$.

It follows that $x(t) = 0$ for all t – i.e. x is the constant solution 0.

(b) $(D^2+4)x = t$; $x(0) = -1$, $x'(0) = 0$

Solution:

Application of $\mathscr{L}$ yields

$$(s^2+4)\mathscr{L}[x] + s = \frac{1}{s^2}.$$

Thus

$$\mathscr{L}[x] = \frac{1}{s^2(s^2+4)} - \frac{s}{s^2+4} = \frac{1-s^3}{s^2(s^2+4)}$$

We now solve the partial fractions problem

$$\frac{1-s^3}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

We need

$$1-s^3 = As(s^2+4) + B(s^2+4) + (Cs+D)s^2 = (A+C)s^3 + (B+D)s^2 + 4As + 4B$$

Comparing coefficients, we find a system of linear equations corresponding to the following augmented matrix

$$\left[\begin{array}{cccc|c} 0 & 4 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1/4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/4 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1/4 \end{array}\right] \sim$$

This shows that $\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 1/4 \\ -1 \\ -1/4 \end{bmatrix}$ so that

$$\frac{1-s^3}{s^2(s^2+4)} = \frac{1}{4}\left(\frac{1}{s^2} + \frac{-4s-1}{s^2+4}\right) = \frac{1}{4}\left(\frac{1}{s^2} + \frac{-4s}{s^2+4} + \frac{-1}{s^2+4}\right)$$

Thus we find that

$$x = \mathscr{L}^{-1}\left[\frac{1}{4}\left(\frac{1}{s^2} + \frac{-4s}{s^2+4} + \frac{-1}{s^2+4}\right)\right] = \frac{1}{4}\left(t - 4\cos(2t) - \frac{1}{2}\sin(2t)\right)$$

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4. Write the function

$$g(t) = \begin{cases} t^2 & t < 3 \\ e^{-t} & t \geq 3 \end{cases}$$

in step-function notation, where $u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

Solution:

$$g(t) = t^2 + u_3(t) \cdot (-t^2 + e^{-t}) = t^2 - u_3(t) \cdot t^2 + u_3(t) \cdot e^{-t}$$

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5. Find the inverse transform of the following functions:

(a) $\dfrac{se^{-2s}}{s^2+2}$

Solution:

Use the second shift formula with $a = 2$. Since $f(t) = \mathscr{L}^{-1}\left[\dfrac{s}{s^2+2}\right] = \cos(\sqrt{2}t)$ we see that

$$\mathscr{L}^{-1}\left[\frac{se^{-2s}}{s^2+2}\right] = u_2(t)f(t-2) = u_2(t) \cdot \cos(\sqrt{2}(t-2))$$

(b) $\dfrac{e^{-s}}{s(s+3)}$.

Solution:

We first find $f(t) = \mathscr{L}^{-1}\left[\dfrac{1}{s(s+3)}\right]$. Partial fractions decomposition gives

$$\frac{1}{s(s+3)} = \frac{1}{3}\left(\frac{1}{s} + \frac{-1}{s+3}\right)$$

so that

$$f(t) = \frac{1}{3}\mathscr{L}^{-1}\left[\frac{1}{s} + \frac{-1}{s+3}\right] = \frac{1}{3}\left(1 - e^{-3t}\right)$$

Now use the second shift formula with $a = 1$ to find that

$$\mathscr{L}^{-1}\left[\frac{e^{-s}}{s(s+3)}\right] = u_1(t)f(t-1) = \frac{1}{3}u_1(t)(1 - e^{-3(t-1)}) = \frac{u_1(t)}{3}(1 - e^{-3t+3})$$

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6. Solve the following initial value problem:

$$(D^3 - D)x = \begin{cases} 4 & t < 4 \\ 0 & t \geq 4 \end{cases}; \quad x(0) = x'(0) = x''(0) = 0$$

Solution:

Applying $\mathscr{L}$ to the ODE and using the first differentiation formula, we find that

$$(s^3 - s)\mathscr{L}[x] = \mathscr{L}[4 - 4u_4(t)] = \frac{4}{s} - \frac{4e^{-4s}}{s}.$$

Thus

$$(\clubsuit) \quad \mathscr{L}[x] = \frac{4}{s^2(s^2-1)} - \frac{4e^{-4s}}{s^2(s^2-1)}.$$

We now solve the partial fractions problem:

$$\frac{1}{s^2(s^2-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$$

We need:

$$1 = As(s^2-1) + B(s^2-1) + Cs^2(s+1) + Ds^2(s-1) = (A+C+D)s^3 + (B+C-D)s^2 - As - B$$

Comparing coefficients immediately gives that $A = 0$ and $B = -1$. Now C and D must satisfy the equations $C + D = 0$ and $C - D = 1$, and we conclude that $\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1/2 \\ -1/2 \end{bmatrix}$ so that

$$\frac{1}{s^2(s^2+1)} = \frac{1}{2}\left(\frac{-2}{s^2} + \frac{1}{s-1} - \frac{1}{s+1}\right).$$

We now compute

$$f(t) = \mathscr{L}^{-1}\left[\frac{4}{s^2(s^2-1)}\right] = \mathscr{L}^{-1}\left[\frac{4}{2}\left(\frac{-2}{s^2} + \frac{1}{s-1} - \frac{1}{s+1}\right)\right] = 2\left(-2t + e^t - e^{-t}\right) = -4t + 2e^t - 2e^{-t}$$

Now use the second shift formula with $a = -4$ to find that

$$\mathscr{L}^{-1}\left[\frac{4e^{-4s}}{s^2(s^2-1)}\right] = u_4(t)f(t-4) = u_4(t) \cdot (-4(t-4) + 2e^{t-4} - 2e^{-t+4})$$

Thus ($\clubsuit$) together with the previous two calculations show that

$$x = -4t + 2e^t - 2e^{-t} - u_4(t) \cdot \left(-4(t-4) + 2e^{t-4} - 2e^{-t+4}\right).$$