Worksheet 2

1. The following example shows that in some physical situations, nonuniqueness is natural and obvious, not pathological.[1]

   Consider a water bucket with a hole in the bottom. If you see an empty bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be nonunique when integrated backwards in time.

   Here's a crude model of the situation. Let $h(t)$ =height of the water remaining in the bucket at time t; a =area of the hole; A =cross-sectional area of the bucket (assumed constant); $v(t)$ =velocity of the water passing through the hole.

   (a) Show that $av(t) = Ah(t)$. What physical law are you invoking?

   (b) To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount Δh and that the water has density ρ . Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation $v^2 = 2gh$.

   (c) Combining the previous two items, show that $\dot{h} = -C\sqrt{h}$, where $C = \sqrt{2g}\dfrac{a}{A}$ and $\dot{h} = \frac{d}{dt}h$.

   (d) Given $h(O) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is nonunique in backwards time, i.e., for $t < 0$.

[1] See Steven H. Strogatz: Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering

2. Consider the ODE
$$x'' + x' - 6x = 3t + 2.$$

(a) Find all linear solutions.

(b) Find all linear solutions of the associated homogeneous equation.

(c) Find all solutions of the associated homogeneous equation that are of the form $\sin at$ or $\cos bt$.

(d) Find all solutions of the associated homogeneous equation that are of the form e^{ct}.

(e) Use the Wronskian test to check whether the solutions from the previous three items generate the general solution.

(f) Find the general solution of $x'' + x' - 6x = 3t + 2$.