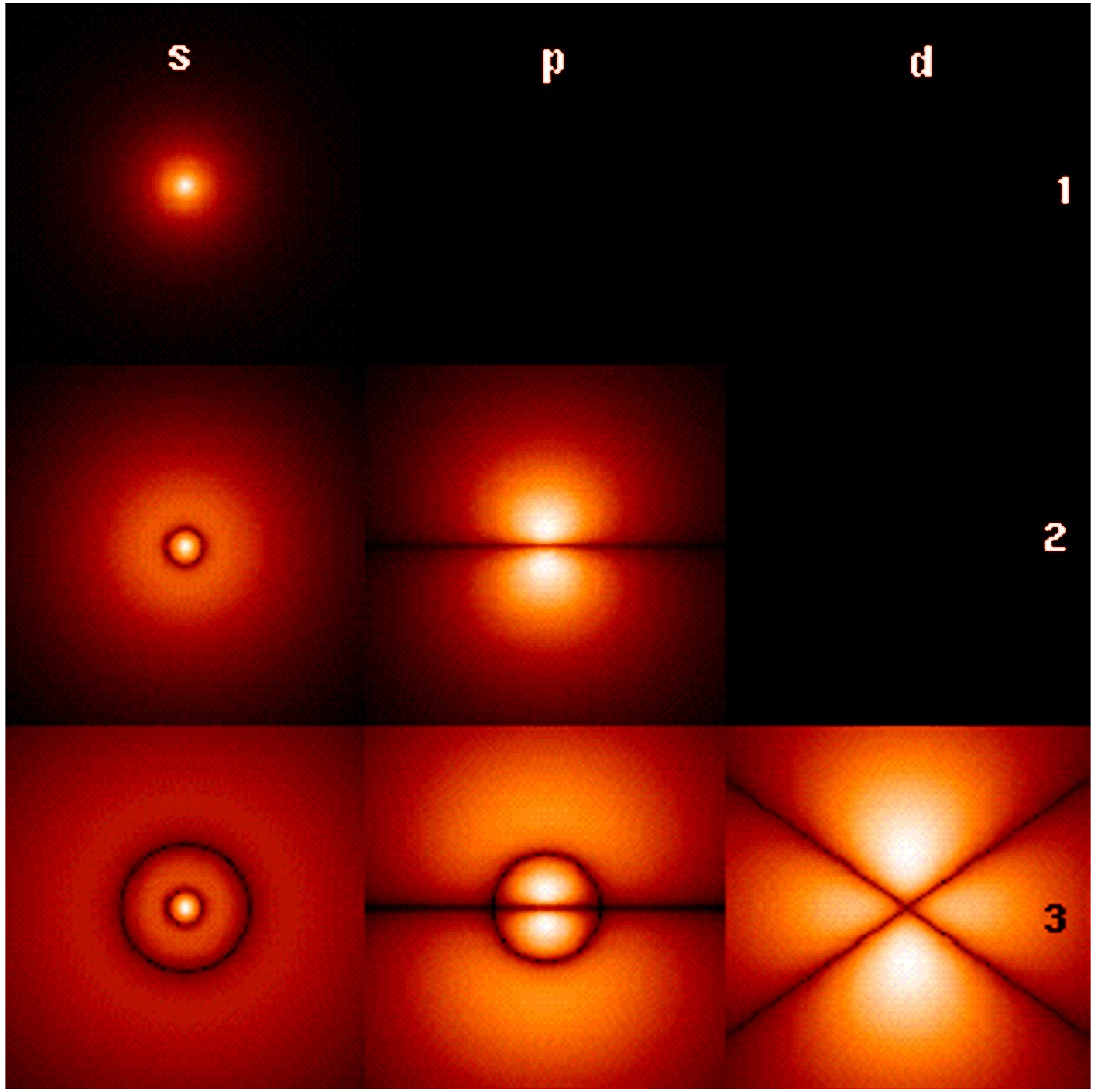
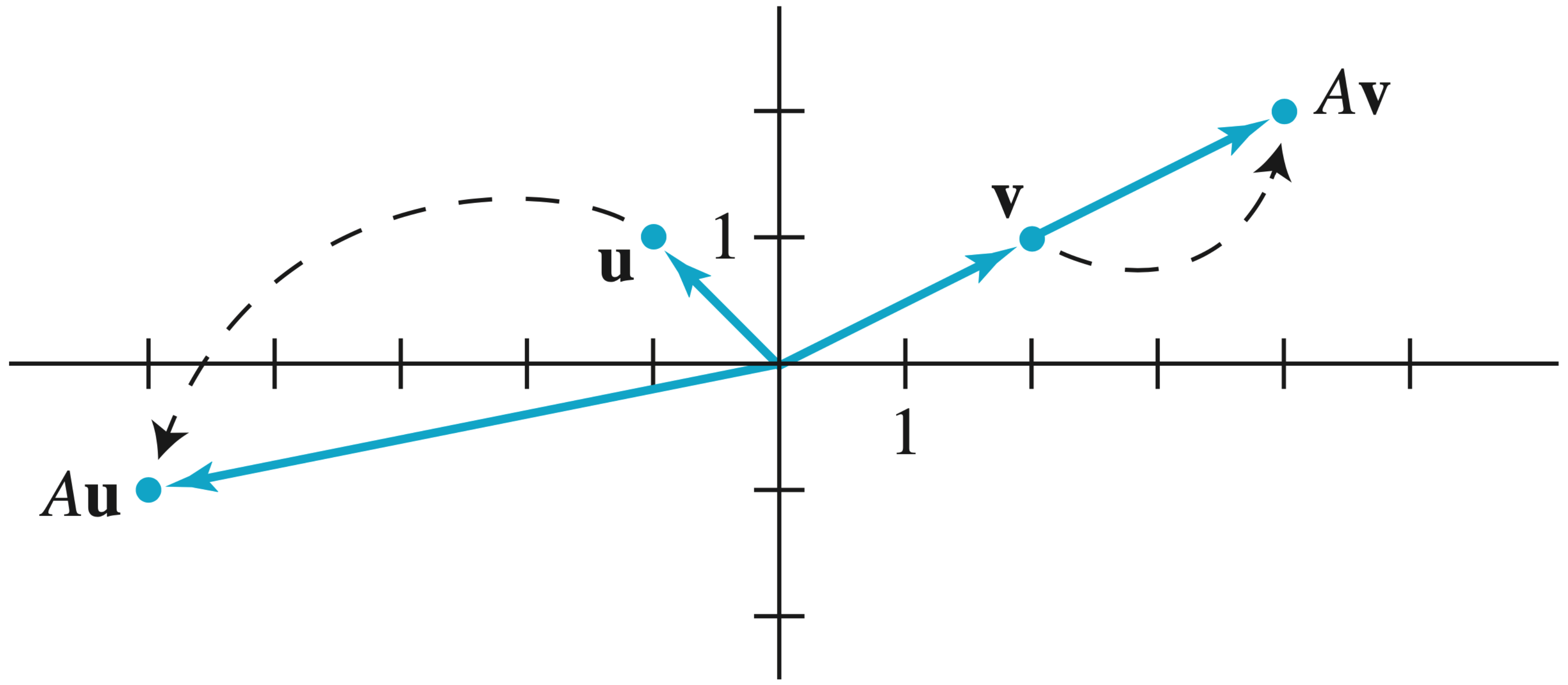


Differential Equations

Homogeneous Systems, Eigenvalues, and Eigenvectors

Eigenvalues and Eigenvectors



Def: Let A be an $n \times n$ matrix with constant entries. We say the number λ is an eigenvalue of A if there exists a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}. \qquad (\dagger)$$

Any nonzero vector satisfying the equality $(\dagger)$ is an eigenvector of A corresponding to λ.

A general description of eigenvectors corresponding to the same eigenvalue λ will be given by a linear combination of eigenvectors corresponding to λ.

Ex: Let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then

$$A\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\vec{v},$$

so $\lambda = 2$ is an eigenvalue of A and $\vec{v}$ is an eigenvector of A corresponding to the eigenvalue 2.

Def: The identity matrix I is the $n \times n$ matrix whose entries a_{ij} are zero if $i \neq j$ and 1 if $i = j$:

$$I := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Note: For any n-vector $\vec{v}$, $I\vec{v} = \vec{v}$, as

$$I\vec{v} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{v}.$$

Finding Eigenvalues

- The number λ is an eigenvalue of A.

$$\Updownarrow$$

- There exists some $\vec{v} \neq 0$ such that

$$A\vec{v} = \lambda\vec{v}.$$

$$\Updownarrow$$

- There exists some $\vec{v} \neq 0$ such that

$$\vec{0} = A\vec{v} - \lambda\vec{v} = A\vec{v} - \lambda I\vec{v} = (A - \lambda I)\vec{v}.$$

$$\text{(Cramer's } \Updownarrow \text{ determinant test)}$$

- $$\det(A - \lambda I) = 0$$

Thus,

Fact: The number λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Ex: Find the eigenvalues of

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -3/2 & 3/2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Sln: We expand $\det(A - \lambda I)$ along the first column:

$$\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & -1 & 0 \\ 1 & -3/2-\lambda & 3/2 \\ 0 & 1 & -1-\lambda \end{bmatrix}$$

$$= -(\lambda+1)\det \begin{bmatrix} -3/2-\lambda & 3/2 \\ 1 & -1-\lambda \end{bmatrix} - \det \begin{bmatrix} -1 & 0 \\ 1 & -1-\lambda \end{bmatrix}$$

$$= -(\lambda+1)\left(\lambda^2 + \frac{5}{2}\lambda + 1\right)$$

$$= -(\lambda+1)(\lambda+2)\left(\lambda + \frac{1}{2}\right).$$

The eigenvalues of A are roots of this polynomial.

$$\lambda = -1,\ \lambda = -2, \ \text{ and } \ \lambda = -\frac{1}{2}.$$

Ex: Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Sln: Here

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 0 & 4 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} \\ &= (3-\lambda) \det \begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix} \\ &= (3-\lambda)\left(\lambda^2 - 2\lambda - 3\right) \\ &= (3-\lambda)(\lambda-3)(\lambda+1), \end{aligned}$$

where the second equality follows from expansion along the second row. The eigenvalues of A are

$$\lambda = 3 \quad \text{and} \quad \lambda = -1.$$

Fact: If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial in λ of degree n.

Def: the characteristic polynomial of $A :=$

$$\det(A - \lambda I)$$

Note: Eigenvalues of A are the roots of the characteristic polynomial of A.

Fact: Let A be an n by n matrix and $\vec{s}$ an arbitrary n-vector. The equation $A\vec{v} = \vec{s}$ has a unique solution for $\vec{v}$ if and only if $\det(A) \neq 0$.

Pf: Write out the entries A, $\vec{v}$ and $\vec{s}$. Equating the vectors $A\vec{v}$ and $\vec{s}$ entrywise gives a system

$$\begin{aligned} v_1 a_{11} + v_2 a_{12} + \cdots + v_n a_{1n} &= s_1 \\ v_1 a_{21} + v_2 a_{22} + \cdots + v_n a_{2n} &= s_2 \\ &\vdots \\ v_1 a_{n1} + v_2 a_{n2} + \cdots + v_n a_{nn} &= s_n. \end{aligned}$$

Apply Cramer's determinant test. $\square$

Solutions of a Homogeneous System

Focus: homogeneous linear systems

$$D\vec{x} = A\vec{x} \tag{H}$$

with constant coefficients.

Special case:

If (H) is the equivalent system of a single equation $P(D)x = 0$, then the characteristic polynomial of A agrees with the $P(r)$ up to a sign.

If λ is a root of the polynomial $P(r)$, then $e^{\lambda t}$ is a solution of $P(D)x = 0$, and (H) has a solution

$$\vec{x} = \begin{bmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \end{bmatrix} = \begin{bmatrix} e^{\lambda t} \\ \lambda e^{\lambda t} \\ \vdots \\ \lambda^{n-1} e^{\lambda t} \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix} := e^{\lambda t}\vec{v}.$$

Since λ is also a root of the characteristic polynomial of A, λ is an eigenvalue. We can show that $\vec{v}$ is an eigenvector of A corresponding to λ.

Fact: $\vec{v}$ is an eigenvector of A corresponding to the eigenvalue λ if and only if

$$\vec{x} = e^{\lambda t}\vec{v}$$

is a solution of $D\vec{x} = A\vec{x}$, where $v \neq 0$.

General case:

· $\vec{v}$ is an eigenvector of A corresponding to the eigenvalue λ.

$$\Updownarrow$$

·

$$A\vec{v} = \lambda\vec{v}.$$

$$\Updownarrow$$

·

$$D(e^{\lambda t}\vec{v}) = D\begin{bmatrix} v_1 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix} = \begin{bmatrix} v_1 \lambda e^{\lambda t} \\ \vdots \\ v_n \lambda e^{\lambda t} \end{bmatrix}$$

$$= e^{\lambda t}\lambda\vec{v} = e^{\lambda t}A\vec{v} = A(e^{\lambda t}\vec{v})$$

$$\Updownarrow$$

· $\vec{x} = e^{\lambda t}\vec{v}$ is a solution of $D\vec{x} = A\vec{x}$.

Ex: In the first example, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 2$. The vector valued function

$$\vec{x} = e^{2t}\vec{v} = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$$

is a solution of $D\vec{x} = A\vec{x}$.

To find the general solution of a homogeneous linear system with constant coefficients, our strategy now is to find the eigenvalues and eigenvectors first, and hope we get n solutions with independent initial vectors.

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