Final Exam

Math 51 Spring 2021 – Tufts University

2022-05-09

You may not use *calculators*, *books* or *notes* during the exam. All electronic devices (including your phones) must be *silenced* and *put away* for the duration of the exam.

After finishing your exam, you will submit this exam booklet. We will scan your submission and upload it to *Gradescope* for marking (you do *not* need to take/upload images of your exam with a phone). You should write your name at the top of each page, as indicated (*especially* if you remove the staples from your exam booklet).

For the partial credit problems, always *show your work*. Try to fit this work in the *available space* if possible. There is a blank page at the back of your exam for use as scratch paper. If you need more space for a solution, please write clearly *in the indicated space* that your solution continues later.

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Please print your name, and sign your exam. With your signature, you pledge that you have neither given nor received assistance on this exam.

Name (printed): ________________________________

Signature: ________________________________

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Name: ____________________________

1. (8 points) Indicate your response to the following. Each question is worth two (2) points.

   (a) Consider the system of linear ODEs

   $$(\clubsuit) \quad D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$$

   where A is a 3×3 matrix with constant entries and where the entries of $\mathbf{E}(t)$ are continuous functions of t on the interval $(0, \infty)$. If $\mathbf{h}(t)$ and $\mathbf{k}(t)$ are *solutions* to $(\clubsuit)$ and if $\mathbf{h}(1) = \mathbf{k}(1)$, must it be true that $\mathbf{h}(t) = \mathbf{k}(t)$ for all t? *Circle your answer.*

   Yes **No**

   (b) Let A be an $n \times n$ matrix with an eigenvalue of $\lambda = 2$ with *multiplicity 3*. Suppose that the vector $\mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^n$ is a solution to the equation $(A - 2\mathbf{I}_n)^3 \mathbf{x} = \mathbf{0}$. Give a formula for a solution $\mathbf{h}(t)$ to the homogeneous system of linear ODEs $D\mathbf{x} = A\mathbf{x}$ which satisfies $\mathbf{h}(0) = \mathbf{v}$.

   (c) Indicate whether the following statement is true or false: If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\mathbf{v} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ are vectors in $\mathbb{R}^3$ and if

   $$\det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} = 0$$

   then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are *linearly dependent*.

   True **False**

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(d) Indicate whether the following statement is true or false: If $h_1(t)$ and $h_2(t)$ are solutions to $P(D)x = e^t$, then $h_1(t) + h_2(t)$ is a solution to $P(D)x = 2e^t$.

True **False**

2. (20 points) Short-answer questions. Each question is worth five (5) points.

(a) Compute $\mathscr{L}^{-1}\left[\frac{e^{-s}}{s} + \frac{1}{s(s^2+4)}\right]$

(b) Let

$$f(t) = \begin{cases} e^{2t} & t < -1 \\ 0 & -1 \leq t \end{cases}$$

Compute $\mathscr{L}[f(t)]$.

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(c) Let $B = \begin{bmatrix} 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Find all solutions $\mathbf{x}$ to the matrix equation $B\mathbf{x} = \mathbf{0}$.

(d) Let $f_1(t) = e^t \cos(t)$, $f_2(t) = e^t \sin(t)$ and $f_3(t) = e^t$. Decide whether the functions $f_1(t), f_2(t), f_3(t)$ are *linearly dependent*. You should either use the *definition* of linear dependence, or the Wronskian - indicate your choice and show your work.

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3. (12 pts)

(a) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Find all generalized eigenvectors for A with eigenvalue $\lambda = 2$.

(b) Let $B = \begin{bmatrix} 0 & 1 \\ 1 & -5/2 \end{bmatrix}$. Find the general solution to the homogeneous system of ODEs $D\mathbf{x} = B\mathbf{x}$.

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4. (12 pts) Use the Laplace transform to solve the initial value problem

$$(D^2 - 9)x = u_1(t)e^t, \qquad x(0) = x'(0) = 0.$$

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5. (12 pts) Consider he ordinary differential equation

$$(\diamondsuit) \qquad \frac{dx}{dt} + tx = e^{t^2/2}$$

(a) Find the *general solution* $x(t)$ to $(\diamondsuit)$.

(b) Find the solution $x(t)$ to $(\diamondsuit)$ for which $x(0) = 1$.

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6. (12 pts) Consider the ordinary differential equation

$$(\heartsuit) \quad (D^2 - 4)x = e^{2t} + e^{-2t}.$$

a. An annihilator of $e^{2t} + e^{-2t}$ is the operator $A(D) = (D-2)(D+2) = D^2 - 4$. A solution to $(\heartsuit)$ must be a solution to the homogeneous equation $A(D) \cdot (D^2 - 4)x = (D-2)^2(D+2)^2 x = 0$. Briefly explain why a *simplified guess* for a solution $p(t)$ to $(\heartsuit)$ is given by

$$p(t) = k_1 \cdot te^{2t} + k_2 \cdot te^{-2t}$$

b. Use the *exponential shift* formula to compute $(D^2 - 4)[p(t)] = (D^2 - 4)[k_1 \cdot te^{2t} + k_2 \cdot te^{-2t}]$.

c. Use your answer to b.) to find a particular solution $p(t)$ to $(\heartsuit)$.

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7. (12 pts) The matrix $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ has eigenvalues ± 2. An eigenvector for $\lambda = 2$ is given by $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and an eigenvector for $\mu = -2$ is given by $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

   Find the general solution to the system of ODEs

$$D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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8. (12 pts) Solve the initial value problem

$$D(D^2 - 1)x = 0, \qquad x(0) = 0, x'(0) = 1, x''(0) = 0.$$

End of exam