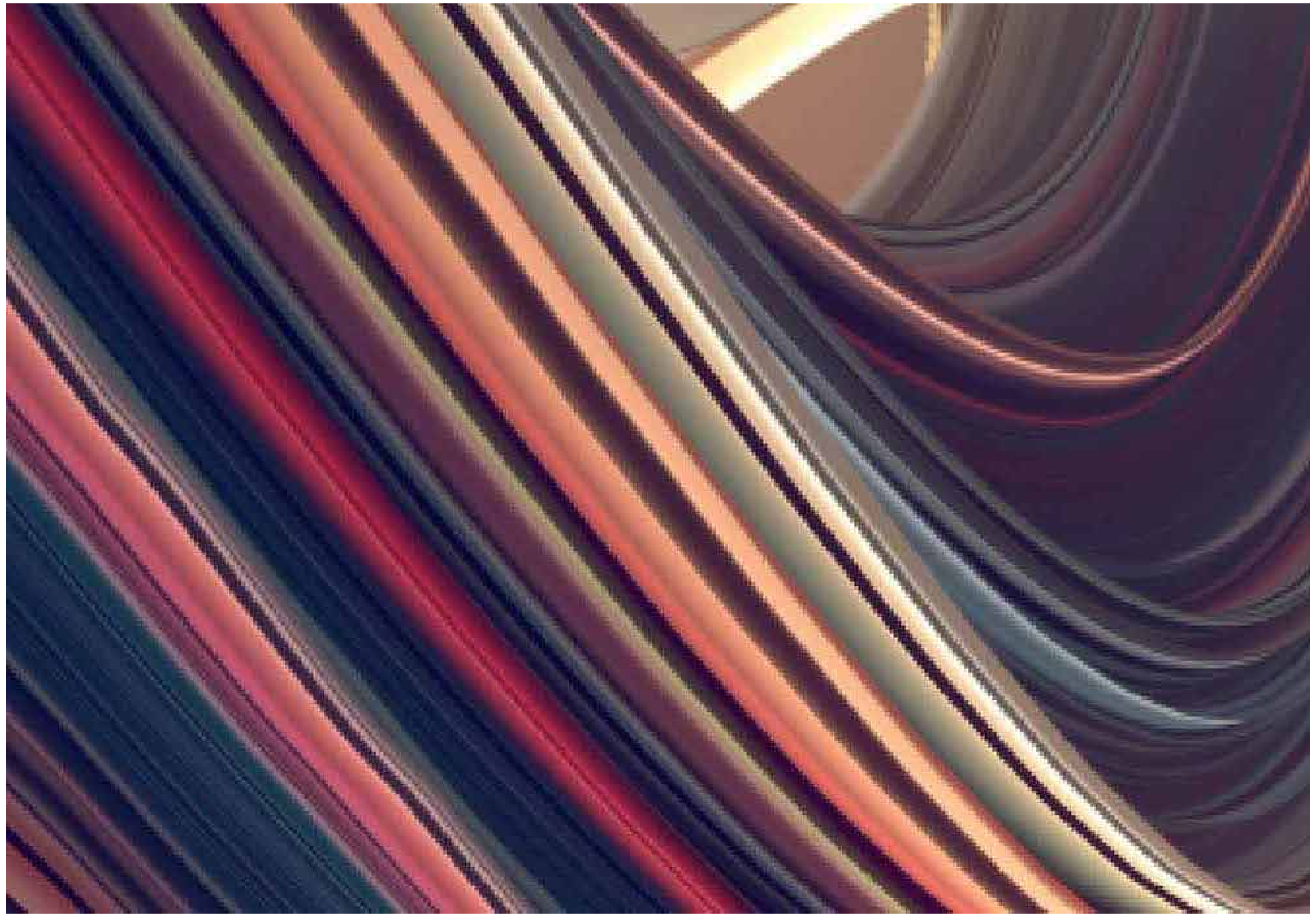

Differential Equations

Homogeneous Systems: Real Roots

Homogeneous Linear Systems

Recall:

- (Section 3.4) The general solution of an n-th order homogeneous system $D\vec{x} = A\vec{x}$ is generated by n solutions with linearly independent initial vectors .
- (Section 3.5) $\vec{v}$ is an eigenvector corresponding to the eigenvalue λ of A if and only if

$$\vec{h}(t) = e^{\lambda t}\vec{v}$$

is a solution of $D\vec{x} = A\vec{x}$ with initial vector

$$\vec{h}(0) = \vec{v} \neq 0.$$

Fact: For $D\vec{x} = A\vec{x}$ of order n, if we find n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively, then the associated solutions

$$\vec{h}_1(t) = e^{\lambda_1 t}\vec{v}_1, \ \ldots, \ \vec{h}_n(t) = e^{\lambda_n t}\vec{v}_n$$

generate the general solution of $D\vec{x} = A\vec{x}$

$$\vec{x} = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t).$$

Note: We allow the eigenvalues λ_i to repeat due to multiplicity. It is possible that $\lambda_i = \lambda_j$ with $i \neq j$.

Ex: Solve $D\vec{x} = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Sln: We found the eigenvalues and the corresponding eigenvectors of A in §3.5 and §3.6. Corresponding to the eigenvalue -1, we have an eigenvector

$$\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

We get a solution of $D\vec{x} = A\vec{x}$ in

$$\vec{h}_1(t) = e^{-t}\vec{v} = \begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}.$$

For the eigenvalue 3, we have two linearly independent eigenvectors

$$\vec{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

and two more solutions associated to $\vec{w}_1$ and $\vec{w}_2$

$$\vec{h}_2(t) = e^{3t}\vec{w}_1 = \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix}, \quad \vec{h}_3(t) = e^{3t}\vec{w}_2 = \begin{bmatrix} 2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}.$$

The three functions $\vec{h}_1(t), \vec{h}_2(t), \vec{h}_3(t)$ generate the general solution of the third order system if and only if the initial vectors $\vec{v}, \vec{w}_1, \vec{w}_2$ are linearly independent. To check for independence, we calculate the determinant of the matrix whose columns are these three vectors. We get

$$\det \begin{bmatrix} -2 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} = -4 \neq 0.$$

Thus, $\vec{v}, \vec{w}_1, \vec{w}_2$ are linearly independent and the general solution of $D\vec{x} = A\vec{x}$ is

$$\begin{aligned} \vec{x} &= c_1\vec{h}_1(t) + c_2\vec{h}_2(t) + c_3\vec{h}_3(t) \\ &= c_1 \begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}. \end{aligned}$$

Fact: If the eigenvectors corresponding to the same eigenvalue are independent, then a list of eigenvectors of A corresponding to distinct eigenvalues will also be linearly independent.

Fact: For each eigenvalue λ of A, an $n \times n$ matrix with constant entries, find as many independent eigenvectors corresponding to λ as possible, and associate to each independent eigenvector the solution $e^{\lambda t}\vec{v}$. These vector valued solutions of $D\vec{x} = A\vec{x}$ have independent initial vectors. In particular, if we find n such solutions $\vec{h}_1(t), \ldots, \vec{h}_n(t)$, then the general solution of $D\vec{x} = A\vec{x}$ is

$$\vec{x} = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)$$

Ex: Solve $D\vec{x} = A\vec{x}$, where

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -3/2 & 3/2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Sln: By what we found in §3.5 and §3.6, the vectors

$$\vec{v} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

are eigenvectors corresponding to the eigenvalues $-1, -2$ and $-1/2$, respectively. We get three solutions associated to these eigenvectors

$$\vec{h}_1(t) = e^{-t}\vec{v}, \quad \vec{h}_2(t) = e^{-2t}\vec{w}, \quad \vec{h}_3(t) = e^{-t/2}\vec{u}$$

with linearly independent initial vectors. Thus, the general solution of the third order system $D\vec{x} = A\vec{x}$ is

$$\begin{aligned} \vec{x} &= c_1\vec{h}_1(t) + c_2\vec{h}_2(t) + c_3\vec{h}_3(t) \\ &= c_1 \begin{bmatrix} -3e^{-t} \\ 0 \\ 2e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-2t} \\ -e^{-2t} \\ e^{-2t} \end{bmatrix} + c_3 \begin{bmatrix} -2e^{-t/2} \\ e^{-t/2} \\ 2e^{-t/2} \end{bmatrix}. \end{aligned}$$

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