Math 51 **Homework 5 Solutions** **Spring 2021**

1. Find the general solution to

$$(D^4 + 2D^2 + 1)x = 0.$$

Solution: To find the general solution we start by factoring the polynomial corresponding to our operator: $P(x) = x^4 + 2x^2 + 1 = (x^2 + 1)^2$. This has single pair of complex roots $0 \pm i$, of multiplicity 2. The general solution to our o.d.e. is therefore

$$x(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t.$$

2. Solve the initial value problem

$$(D^3 - 2D^2 + 2D - 4)x = 0, \quad x(0) = 0,\ x'(0) = 4,\ x''(0) = 12.$$

Solution: As in problem 1, we want to begin by factoring our polynomial. In this case, we'll want to use the Rational Root Theorem. Since the leading order coefficient is 1, the only possible rational roots of our polynomial $P(x) = x^3 - 2x^2 + 2-$ are the divisors of 4, namely $\pm 1, \pm 2, \pm 4$. As usual, let's start with the easy ones. Testing ± 1 fails, however $P(2) = 0$ so we've found a root! One could proceed and check the remainder of the possible rational roots, but as most polynomials have complex roots this is usually a waste of time. The usual way to proceed is by division, which leads to

$$P(x) = (x - 2)(x^2 + 2)$$

which has roots 2 and $0 \pm \sqrt{(}2)i$ and therefore general solution

$$x(t) = c_1 e^{2t} + c_2 \cos \sqrt{2}t + c_3 \sin \sqrt{2}t.$$

Now, we plug in the three conditions for x, x' and x'' giving us a system of three equations and three unknowns:

$$\begin{aligned} 0 &= x(0) = c_1 + c_2 + 0 \\ 4 &= x'(0) = 2c_1 + 0 + \sqrt{2}c_3 \\ 12 &= x''(0) = 4c_1 - 2c_2 + 0 \end{aligned}$$

The first and third equations give us that $c_1 = -c_2$ and then $6c_1 = 12$ so that $c_1 = 2, c_2 = -2$. Then the second equation implies that $c_3 = 0$ so our specific solution is

$$x(t) = 2e^{2t} - 2\cos \sqrt{2}t.$$

3. Consider a nonhomogeneous linear differential equation, $Lx = E(t)$. In a few sentences, explain the idea of the annihilator method for solving this o.d.e. Some vocabulary that might be useful to include: homogeneous equation, nonhomogeneous equation, homogeneous solution, particular solution.

Solution: The annihilator method for solving ODEs turns nonhomogeneous equations into homogeneous equations that we know how to solve. First, solve the homogeneous problem for the homogeneous solution $x_h(t)$ by solving $Lx_h = 0$. Then, determine the annihilator operator $A(D)$ that sends $E(t)$ to zero, and solve the corresponding homogeneous problem $A(D)Lx = 0$. We do this by finding the characteristic polynomial of the operator $A(D)L$, finding the form of the general solution x, and the particular solution x_p are the terms in x that are not already in x_h. We finish by plugging x_p into the original ODE $Lx_p = E(t)$ to find the coefficients of x_p. The general solution is given by $x(t) = x_h(t) + x_p(t)$.

4. Find an annihilator of smallest possible order for:
   (a) $e^t + \sin 2t - 3$
   (b) $t^2 + e^t \sin 3t$

   Solution: To find the annihilator, we want to try to identify each function given with roots of a differential polynomial operator. So for (a), we associate e^t with $(D-1)$, $\sin 2t$ with (D^2+4), and any constant is just killed off by D. For (b), t^2 must be differentiated three times to go to 0, and $e^t \sin 3t$ we associate with the root $1 \pm 3t$, which means we'll need $(D^2 - 2D + 10)$. Thus, the annihilators are
   (a) $(D-1)(D^2+4)D$
   (b) $D^3(D^2-2D+10)$

5. Make a *simplified guess* for a particular solution to the following equation (note: you do not need to solve for the coefficients),

$$(D-1)^2(D^2+1)^3(D+2)x = t^2e^{3t} + e^t + e^{-t}\sin 3t + t^4.$$

   Solution: This problem entails finding an annihilator $A(D)$ for $E(t)$ in the nonhomogeneous equation above $Lx = E$, then finding the simplified guess for the homogeneous equation $A(D)Lx = 0$. To begin, we identify the necessary operators to annihilate the righthand-side. For each one:

$$\begin{aligned} t^2e^{3t} &\Leftrightarrow (D-3)^3 \\ e^t &\Leftrightarrow (D-1) \\ e^{-t}\sin 3t &\Leftrightarrow (D+1+3i)(D+1-3i) = (D^2+2D+10) \\ t^4 &\Leftrightarrow D^5 \end{aligned}$$

   Therefore

$$A(D) = (D-3)^3(D-1)(D^2+2D+10)D^5.$$

   Now we want to combine $A(D)$ and L in annoyingly large homogeneous o.d.e.

$$A(D)Lx = (D-3)^3(D^2+2D+10)D^5(D-1)^3(D^2+1)^3(D+2)x = 0$$

   Note that $A(D)$ and L only had one mutual factor, namely $(D-1)$ which L had two copies of. Therefore, in the simplified guess that we want to make, we need to make sure we have copies of all solutions to $A(D)Lx = 0$ *that are not covered in the general solution to* $Lx = 0$. This means we will need a term t^2e^t (because of $(D-1)^3$) but *not* e^t or te^t since these are solutions to $Lx = 0$ and therefore cannot be matched to the forcing term $E(t)$. Thus, our guess $g(t)$ is

$$g(t) = c_1 e^{3t} + c_2 t e^{3t} + c_3 t^2 e^{3t} + c_4 t^2 e^t + c_5 e^{-t}\cos 3t + c_6 e^{-t}\sin 3t + c_7 + c_8 t + c_9 t^2 + c_{10} t^3 + c_{11} t^4$$

It may not appear to be simple, but note that we do not have any terms present that L would annihilate.

6. Let us define the constant coefficient differential operator $L = D^3 - 2D^2 + D$, the function $f(t) = 1 + e^{-2t}$, and let x be a function of t.
   (a) Solve the o.d.e. $Lx = 0$ given the initial values
   $$x(0) = 0,\ \ x'(0) = 0,\ \ x''(0) = 1.$$
   (b) Find an annihilator of f, i.e. find a polynomial $A(D)$ such that $A(D)f = 0$.
   (c) Find the general solution to the o.d.e. $Lx = f$.

   Solution:
   (a) Factor L as $D(D-1)^2$ so the general form is $x(t) = c_1 + c_2 e^t + c_3 t e^t$, and we then can setup the system of equations
   $$\begin{aligned} x(t) &= \quad c_1 + c_2 e^t + c_3 t e^t \\ x'(t) &= \quad c_2 e^t + c_3 e^t + c_3 t e^t \\ x''(t) &= \quad c_2 e^t + 2c_3 e^t + c_3 t e^t \end{aligned}$$
   So at $t = 0$ we have the system of equations
   $$\begin{aligned} 0 &= \quad c_1 + c_2 \\ 0 &= \quad c_2 + c_3 \\ 1 &= \quad c_2 + 2c_3 \end{aligned}$$
   From which we find $c_1 = 1, c_2 = -1, c_3 = 1$.
   (b) Noting that f is solved by a general ODE with roots 0 and -2, we know that $D(D+2)$ will annihilate f.
   (c) Solving $A(D)Lx = D^2(D+2)(D-1)^2 x = 0$ we obtain the general solution $c_1 + c_2 t + c_3 e^{-2t} + c_4 e^t + c_5 t e^t$. Considering only the terms that don't solve $Lx = 0$ and differentiating them by L. (relabeling constants)
   $$(D^3 - 2D^2 + D)c_1 t + c_2 e^{-2t} = f$$
   This becomes
   $$-18c_2 e^{-2t} + c_1 = 1 + e^{-2t}$$
   So $c_1 = 1$ and $c_2 = -\frac{1}{18}$ so we get the general solution
   $$c_1 + t - \frac{1}{18}e^{-2t} + c_2 e^t + c_3 t e^t$$