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Math 51 | Differential Equations | Spring 2022

Alternate Exam 2 (90 pts.+ 10 bonus pts)

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For the partial credit problems, always show your work. Try to fit this work in the available space if possible.

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1. (16 points) True-false and multiple choice. Circle the correct choice.

   (a) Let $\mathbf{h}_1, \ldots, \mathbf{h}_n$ be n solutions of an order-n linear system $D\vec{x} = A\vec{x}$ on an interval I. Is it possible that the Wronkisan $W[\mathbf{h}_1, \ldots, \mathbf{h}_n](t)$ is 0 at one point t_0 of I but nonzero at another point t_1 of I?

   A. Yes
   B. No

   (b) True or False. An $n \times n$ real matrix must have n linearly independent eigenvectors, some of which may be complex.

   (c) True or False. Let A be a matrix of real numbers. The linear system $D\vec{x} = A\vec{x}$ must have n linearly independent real solutions.

   (d) True or False. Let A be a matrix of real numbers. If $\vec{x}$ is a complex solution of $D\vec{x} = A\vec{x}$, then both $\operatorname{Re}\vec{x}$ and $\operatorname{Im}\vec{x}$ are real solutions of $D\vec{x} = A\vec{x}$.

   (e) True or False. Five vectors in $\mathbb{R}^5$ are linearly independent if and only if they generate (span) $\mathbb{R}^5$.

   (f) True or False. Let A be an $n \times n$ matrix with an eigenvalue λ of multiplicity 3. An eigenvector corresponding to λ is also a generalized eigenvector.

   (g) True or False. Let A be an $n \times n$ real matrix. The general solution of $D\vec{x} = A\vec{x}$ can be generated by fewer than n solutions.

   (h) True or False. For every eigenvalue λ of an $n \times n$ matrix, there must be a corresponding eigenvector.

2. (8 pts) Write down an annihilator of smallest possible order with real coefficients for the function

$$4te^{3t} + t^2e^{2t}\sin t.$$

function	annihilator
e^{3t}	$D-3$
te^{3t}	$(D-3)^2$
$e^{2t}\sin t$	$(D-(2+i))(D-(2-i)) = D^2-4D+5$
$t^2e^{2t}\sin t$	$(D^2-4D+5)^3$
$4te^{3t}+t^2e^{2t}\sin t$	$\boxed{(D-3)^2(D^2-4D+5)^3}$

3. (10 points) Make a *simplified* guess for a particular solution of the differential equation

$$(D+2)^7(D^2+1)^6x = te^{-2t} + \cos t.$$

Do not solve for the coefficients.

The solutions to the homogeneous equation

$$(D+2)^7(D^2+1)^6x = 0$$

are $e^{-2t}, te^{-2t}, \dots, t^6e^{-2t}, \cos t, \sin t, t\cos t, t\sin t, \dots,$
$t^5\cos t, t^5\sin t$.

An annihilator of te^{-2t} is $(D+2)^2$.

An annihilator of $\cos t$ is $(D+i)(D-i) = D^2+1$.

Therefore, an annihilator of $te^{-2t}+\cos t$ is

$$(D+2)^2(D^2+1).$$

If x is a particular solution of (N), then

$$(D+2)^2(D^2+1)(D+2)^7(D^2+1)^6x = (D+2)^2(D^2+1)(te^{-2t}+\cos t),$$

$$(D+2)^9(D^2+1)^7x = 0.$$

Therefore,

$x = e^{-2t}, te^{-2t}, \dots, t^8e^{-2t}, \cos t, \sin t, \dots, t^6\cos t, t^6\sin t.$

Crossing out the homogeneous solutions, a simplified guess for a particular solution is

$$\boxed{c_1t^7e^{-2t} + c_2t^8e^{-2t} + c_3t^6\cos t + c_4t^6\sin t}.$$

4. (14 points)

(a) Convert the differential equation

$$x''' - t^2x'' + \pi x' + tx = \sin t$$

into a linear system of three equations in three unknowns x_1, x_2, x_3.

(b) Write the linear system in the form $D\vec{x} = A(t)\vec{x} + \vec{E}(t)$ for some matrix $A(t)$ and vector $\vec{E}(t)$.

(a) Set

$$x_1 = x$$

$$x_2 = x' = x_1'$$

$$x_3 = x'' = x_2'$$

$$x_3' = x''' = t^2x'' - \pi x' - tx + \sin t$$

$$= t^2 x_3 - \pi x_2 - t x_1 + \sin t.$$

Bring the derivatives to the left:

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -tx_1 - \pi x_2 + t^2 x_3 + \sin t \end{aligned}}.$$

(b)

$$\boxed{D\vec{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -t & -\pi & t^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}}.$$

5. (15 points)

(a) (10 pts) The matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ has eigenvalue $\lambda = 2$ of multiplicity 2. It has an eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and a generalized eigenvector $\vec{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Using these two vectors, write down two solutions of $D\vec{x} = A\vec{x}$ that generate the general solution.

$$(A - \lambda I)\vec{u} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\boxed{e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}, \quad e^{2t}\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \boxed{e^{2t}\begin{bmatrix} t \\ -1-t \end{bmatrix}}.$$

(b) (5 pts) Suppose we have found three solutions of a linear system $D\vec{x} = A\vec{x}$:

$$e^t\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}t^2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right), \quad e^t\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} - t^2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right),$$
$$e^t\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{2}t^2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right).$$

Explain why these three solutions generate the general solution of $D\vec{x} = A\vec{x}$.

Their initial vectors at time $t=0$,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent.

6. (12 points) Suppose $i, -i$ are eigenvalues of the 2×2 matrix A with corresponding eigenvectors $\begin{bmatrix} 1 \\ i \end{bmatrix}$, $\begin{bmatrix} 1 \\ -i \end{bmatrix}$, respectively. Write down two linearly independent real solutions of $D\vec{\mathbf{x}} = A\vec{\mathbf{x}}$. Show your work and simplify your answers.

A complex solution is

$$e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= (\cos t + i \sin t)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= \left((\cos t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + i\left((\cos t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

Two linearly independent real solutions are

$$\boxed{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.}$$

7. (15 points) Find the general solution of the differential equation

$$5x'' - 10x' + 5x = t^{1/5}e^t,$$

given that two independent solutions of the related homogeneous equation are e^t and te^t.

Solution. First put in standard form: $(D^2 - 2D + 1)x = \frac{1}{5} t^{1/5} e^t$.

1) Solutions to the homogeneous equation:

$$(D^2 - 2D + 1) = (D-1)^2 = 0$$

General solution: $\vec{h} = c_1 e^t + c_2 te^t$.

2) Vary c:

Solve
$$c_1' e^t + c_2' te^t = 0$$
$$c_1' e^t + c_2'(1+t)e^t = \frac{1}{5} t^{1/5} e^t.$$

Divide by e^t:

Need to solve $\left[\begin{array}{cc|c} 1 & t & 0 \\ 1 & 1+t & \frac{1}{5}t^{1/5} \end{array}\right]$.

The determinant of the coefficient matrix is

$$\Delta = \begin{vmatrix} 1 & t \\ 1 & 1+t \end{vmatrix} = 1.$$

By Cramer's rule,

$$c_1' = \begin{vmatrix} 0 & t \\ \frac{1}{5}t^{1/5} & 1+t \end{vmatrix} = -\frac{1}{5} t^{6/5} \Rightarrow c_1 = -\frac{1}{11} t^{11/5},$$

$$c_2' = \begin{vmatrix} 1 & 0 \\ 1 & \frac{1}{5}t^{1/5} \end{vmatrix} = \frac{1}{5} t^{1/5} \Rightarrow c_2 = \frac{1}{6} t^{6/5}.$$

A particular solution is

$$c_1 e^t + c_2 te^t = -\frac{1}{11} t^{11/5} e^t + \frac{1}{6} t^{11/5} e^t$$

$$= \frac{5}{66} t^{11/5} e^t.$$

General solution:

$$\boxed{c_1 e^t + c_2 te^t + \frac{5}{66} t^{11/5} e^t}.$$

(Continuation of Question 7)

(End of Exam)