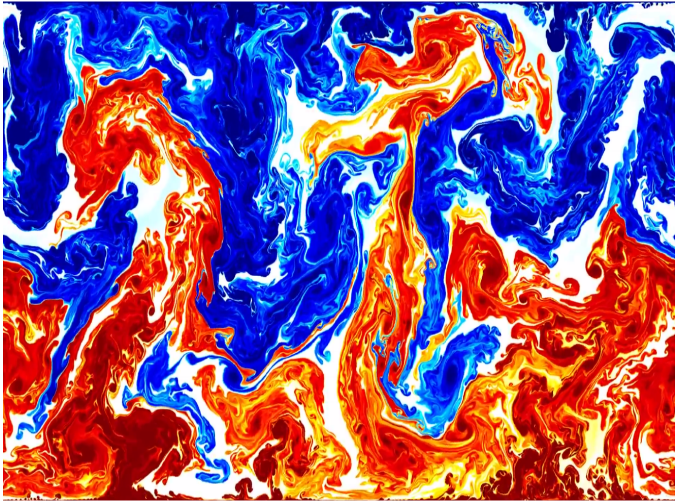

Differential Equations

Nonhomogeneous Systems

Variation of Parameter

§2.8: n-th order linear o.d.e. in standard form

$$Lx = q(t) \tag{N}$$

Step 1. Find the general solution

$$H(t) = c_1 h_1(t) + \cdots + c_n h_n(t)$$

of the related homogeneous equation

$$Lx = 0. \tag{H}$$

Step 2. Look for a particular solution of (N) of the form

$$p(t) = c_1(t)h_1(t) + \cdots + c_n(t)h_n(t).$$

Substituting this guess into (N) yields

$$\begin{aligned}
c_1'(t)h_1(t) \quad &+ \cdots + c_n'(t)h_n(t) \quad &= 0,\\
c_1'(t)h_1'(t) \quad &+ \cdots + c_n'(t)h_n'(t) \quad &= 0,\\
&\vdots\\
c_1'(t)h_1^{(n-1)}(t) &+ \cdots + c_n'(t)h_n^{(n-1)}(t) &= q(t).
\end{aligned}$$

Alg: (Variation of Parameters)

n-th order linear system of o.d.e.'s

$$D\vec{x} = A\vec{x} + \vec{E}(t) \tag{S}$$

Step 1. Find the general solution

$$\vec{H}(t) = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)$$

of the related homogeneous system

$$D\vec{x} = A\vec{x}. \tag{H}$$

Step 2. Look for a particular solution of (S) of the form

$$\vec{p}(t) = c_1(t)\vec{h}_1(t) + \cdots + c_n(t)\vec{h}_n(t).$$

Substituting this guess into (S) yields

$$c_1'(t)\vec{h}_1(t) + \cdots + c_n'(t)\vec{h}_n(t) = \vec{E}(t), \tag{$*$}$$

which we solve for $c_1'(t), \ldots, c_n'(t)$.

Step 3. Integrate to find $c_1(t), \ldots, c_n(t)$, taking the constants of integration to be 0, and obtain a particular solution of (N)

$$p(t) = c_1(t)h_1(t) + \cdots + c_n(t)h_n(t).$$

The general solution of (N) is

$$x(t) = H(t) + p(t).$$

which we solve for $c_1'(t), \ldots, c_n'(t)$.

Step 3. Integrate to find $c_1(t), \ldots, c_n(t)$, taking the constants of integration to be 0, and obtain a particular solution of (S)

$$\vec{p}(t) = c_1(t)\vec{h}_1(t) + \cdots + c_n(t)\vec{h}_n(t).$$

The general solution of (S) is

$$\vec{x}(t) = \vec{H}(t) + \vec{p}(t).$$

Claim: Substituting

$$\vec{p}(t) = c_1(t)\vec{h}_1(t) + \cdots + c_n(t)\vec{h}_n(t)$$

into $D\vec{x} = A\vec{x} + \vec{E}(t)$ gives

$$c_1'(t)\vec{h}_1(t) + \cdots + c_n'(t)\vec{h}_n(t) = \vec{E}(t). \qquad (*)$$

Pf: By linearity of D and the product rule, the left side is

$$\begin{aligned} D\vec{p}(t) &= D\left[c_1(t)\vec{h}_1(t)\right] + \cdots + D\left[c_n(t)\vec{h}_n(t)\right] \\ &= \left[c_1'(t)\vec{h}_1 + c_1(t)\vec{h}_1'\right] + \cdots + \left[c_n'(t)\vec{h}_n + c_n(t)\vec{h}_n'\right]. \end{aligned}$$

By linearity of matrix-vector products, the right side is

$$\begin{aligned} A\vec{p}(t) + \vec{E}(t) &= c_1(t)A\vec{h}_1(t) + \cdots + c_n(t)A\vec{h}_n(t) + \vec{E}(t) \\ &= c_1(t)D\vec{h}_1(t) + \cdots + c_n(t)D\vec{h}_n(t) + \vec{E}(t) \\ &= c_1(t)\vec{h}_1' + \cdots + c_n(t)\vec{h}_n' + \vec{E}(t). \end{aligned}$$

Equating $D\vec{p}(t)$ with $A\vec{p}(t) + \vec{E}(t)$, and canceling

$$c_1(t)\vec{h}_1' + \cdots + c_n(t)\vec{h}_n'$$

on both sides, we are left with

$$c_1'(t)\vec{h}_1(t) + \cdots + c_n'(t)\vec{h}_n(t) = \vec{E}(t). \qquad (*)$$

$\square$

Note: The methods of variation of parameters for an o.d.e. $Lx = E(t)$ and for an equivalent system of o.d.e.'s $D\vec{x} = A\vec{x} + \vec{E}(t)$ agree.

(Page 319, Note 2)

Ex: Solve

$$D\vec{x} = A\vec{x} + \vec{E}(t), \tag{S}$$

where

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \vec{E}(t) = \begin{bmatrix} -2e^{t} \\ 9t \\ e^{t} \end{bmatrix}.$$

Sln: We found the general solution of the related homogeneous system $D\vec{x} = A\vec{x}$ in Section 3.7:

$$\vec{H}(t) = c_1 \begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}.$$

We look for a particular solution of (S) of the form $\vec{x} = \vec{p}(t)$, where

$$\vec{p}(t) = c_1(t) \begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_2(t) \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix} + c_3(t) \begin{bmatrix} 2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}.$$

To find functions $c_1(t), c_2(t), c_3(t)$ that work, we solve

$$c_1'(t) \begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_2'(t) \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix} + c_3'(t) \begin{bmatrix} 2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} = \begin{bmatrix} -2e^{t} \\ 9t \\ e^{t} \end{bmatrix} \tag{$*$}$$

for the unknowns $c_1'(t), \ldots, c_n'(t)$, by reducing the augmented matrix $\left[\vec{h}_1(t) \quad \vec{h}_2(t) \quad \vec{h}_3(t) \mid E(t)\right]$:

$$\left[\begin{array}{ccc|c} -2e^{-t} & 0 & 2e^{3t} & -2e^{t} \\ 0 & e^{3t} & 0 & 9t \\ e^{-t} & 0 & e^{3t} & e^{t} \end{array}\right] \xrightarrow[R_3 \to R_3 + (1/2)R_1]{R_2 \to e^{-3t} R_2}$$

$$\left[\begin{array}{ccc|c} -2e^{-t} & 0 & 2e^{3t} & -2e^{t} \\ 0 & 1 & 0 & 9te^{-3t} \\ 0 & 0 & 2e^{3t} & 0 \end{array}\right] \xrightarrow[R_3 \to (e^{-3t}/2) R_3]{R_1 \to (-e^{t}/2) R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -e^{4t} & e^{2t} \\ 0 & 1 & 0 & 9te^{-3t} \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + e^{4t} R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & e^{2t} \\ 0 & 1 & 0 & 9te^{-3t} \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The solution of the algebraic system corresponding to the reduced augmented matrix is

$$c_1'(t) = e^{2t}, \quad c_2'(t) = 9te^{-3t}, \quad c_3'(t) = 0.$$

We integrate these, taking the integration constants to be zero, and get

$$c_1(t) = \frac{e^{2t}}{2}, \quad c_2(t) = -3te^{-3t} - e^{-3t}, \quad c_3(t) = 0.$$

A particular solution of (S) is

$$\vec{p}(t) = \frac{e^{2t}}{2} \begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} - \left(3te^{-3t} + e^{-3t}\right) \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} -e^{t} \\ -3t - 1 \\ e^{t}/2 \end{bmatrix}.$$

The general solution of (S) is

$$\vec{x} = \vec{H}(t) + \vec{p}(t)$$
$$= c_1 \begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + \begin{bmatrix} -e^{t} \\ -3t - 1 \\ e^{t}/2 \end{bmatrix}.$$

Thm: (Cramer's Rule)
If $B = \begin{bmatrix} \vec{b}_1 & \dots & \vec{b}_n \end{bmatrix}$ is an $n \times n$ matrix with nonzero determinant, then the unique solution to the system

$$B\vec{v} = \vec{r}$$

is given by

$$v_1 = \frac{\Delta_1}{\det B}, \quad v_2 = \frac{\Delta_2}{\det B}, \quad \dots, \quad v_n = \frac{\Delta_n}{\det B},$$

where

$$\Delta_i = \det \begin{bmatrix} \vec{b}_1 & \dots & \vec{b}_{i-1} & \vec{r} & \vec{b}_{i+1} & \dots & \vec{b}_n \end{bmatrix}.$$

for $i = 1, 2, \dots, n$.

Note: We can solve the system $(*)$ by row reduction or Cramer's rule.

Ex: Solve

$$D\vec{x} = A\vec{x} + \vec{E}(t), \tag{S}$$

where

$$A = \begin{bmatrix} -1 & -1 \\ 4 & -1 \end{bmatrix} \quad \text{and} \quad \vec{E}(t) = \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix}.$$

Sln: We solved the related homogeneous system of (S) in Section 3.8. The general solution was $\vec{x} = \vec{H}(t)$, where

$$\vec{H}(t) = c_1 \begin{bmatrix} -e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\cos 2t \\ 2e^{-t}\sin 2t \end{bmatrix}.$$

We look for a particular solution of (S) in the form

$$\vec{p}(t) = c_1(t) \begin{bmatrix} -e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix} + c_2(t) \begin{bmatrix} e^{-t}\cos 2t \\ 2e^{-t}\sin 2t \end{bmatrix}.$$

To find $c_1(t), c_2(t)$, we solve

$$c_1'(t) \begin{bmatrix} -e^{-t}\sin 2t \\ 2e^{-t}\cos 2t \end{bmatrix} + c_2'(t) \begin{bmatrix} e^{-t}\cos 2t \\ 2e^{-t}\sin 2t \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix} \tag{$*$}$$

or

$$\begin{bmatrix} -e^{-t}\sin 2t & e^{-t}\cos 2t \\ 2e^{-t}\cos 2t & 2e^{-t}\sin 2t \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix}$$

for $c_1'(t), c_2'(t)$. By Cramer's rule, the unique solution of the system $(*)$ is

$$c_1'(t) = \frac{\det \begin{bmatrix} 2e^{-t} & e^{-t}\cos 2t \\ 0 & 2e^{-t}\sin 2t \end{bmatrix}}{\det \begin{bmatrix} -e^{-t}\sin 2t & e^{-t}\cos 2t \\ 2e^{-t}\cos 2t & 2e^{-t}\sin 2t \end{bmatrix}} = -2\sin 2t,$$

$$c_2'(t) = \frac{\det \begin{bmatrix} -e^{-t}\sin 2t & 2e^{-t} \\ 2e^{-t}\cos 2t & 0 \end{bmatrix}}{\det \begin{bmatrix} -e^{-t}\sin 2t & e^{-t}\cos 2t \\ 2e^{-t}\cos 2t & 2e^{-t}\sin 2t \end{bmatrix}} = 2\cos 2t.$$

We can take

$$c_1(t) = \cos 2t \quad \text{and} \quad c_2(t) = \sin 2t$$

to obtain the particular solution

$$\vec{p}(t) = \cos 2t \begin{bmatrix} -e^{-t} \sin 2t \\ 2e^{-t} \cos 2t \end{bmatrix} + \sin 2t \begin{bmatrix} e^{-t} \cos 2t \\ 2e^{-t} \sin 2t \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{-t} \end{bmatrix}.$$

The general solution of (S) is

$$\begin{aligned} \vec{x} &= \vec{H}(t) + \vec{p}(t) \\ &= c_1 \begin{bmatrix} -e^{-t} \sin 2t \\ 2e^{-t} \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \cos 2t \\ 2e^{-t} \sin 2t \end{bmatrix} + \begin{bmatrix} 0 \\ 2e^{-t} \end{bmatrix}. \end{aligned}$$