(1) Which of the following ODEs are linear? Of the ones that are linear, rewrite them as linear operators, and say which ones are homogeneous.

(a) $\frac{d^2x}{dt^2} + t^2\frac{dx}{dt} - t\sin(t) = 0$
(b) $\frac{d^2x}{dt^2} + x\frac{dx}{dt} = 0$
(c) $\frac{d^2x}{dt^2} + \sin(t)\frac{dx}{dt} = t^2x$
(d) $\frac{d^3x}{dt^3} + e^t\frac{d^2x}{dt^2} + tx = te^t$

Solution: The only nonlinear equation is (b) thanks to the term $x\frac{dx}{dt}$. Of the remaining ones, rewriting them in the form $\mathcal{L}x = f(t)$, we have
(a) $(D^2 + t^2D)x = t\sin(t)$
(c) $(D^2 + \sin(t)D - t^2)x = 0$
(d) $(D^3 + e^tD^2 + t)x = te^t$
where only (c) is homogeneous since $f(t) = 0$.

(2) Calculate (and simplify) the determinant of the following matrix:

$$\begin{bmatrix} e^t & \sin t & \cos t \\ e^t & \cos t & -\sin t \\ e^t & -\sin t & -\cos t \end{bmatrix}$$

Solution: To compute the determinant, let's use the first column rather than the first row so that the determinant is

$$\begin{aligned}
\det M &= e^t\begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix} - e^t\begin{vmatrix} \sin t & \cos t \\ -\sin t & -\cos t \end{vmatrix} + e^t\begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} \\
&= e^t\left[-\cos^2 t - \sin^2 t + \cos t\sin t - \cos t\sin t - \sin^2 t - \cos^2 t\right] \\
&= -2e^t.
\end{aligned}$$

(3) Consider the differential equation $Lx = 0$ where $L = D^3 + D^2$. Using the Wronskian test, determine whether or not the given solutions generate the general solution:

$$h_1(t) = e^{-t}, \quad h_2(t) = t + 3e^{-t}, \quad h_3(t) = t.$$

Solution: Let's go ahead and compute the Wronskian as follows:

$$W[h_1, h_2, h_3] = \begin{vmatrix} e^{-t} & t + 3e^{-t} & t \\ -e^{-t} & 1 - 3e^{-t} & 1 \\ e^{-t} & 3e^{-t} & 0 \end{vmatrix}$$

Our life will be easiest if we use the final column to compute the determinant, since this way we'll make use of the zero and have one less 2×2 determinant to compute. Doing so yields:

$$\begin{aligned}
W[h_1, h_2, h_3] &= t\begin{vmatrix} -e^{-t} & 1 - 3e^{-t} \\ e^{-t} & 3e^{-t} \end{vmatrix} - \begin{vmatrix} e^{-t} & t + 3e^{-t} \\ e^{-t} & 3e^{-t} \end{vmatrix} \\
&= t\left(-3e^{-2t} - e^{-t} + 3e^{-2t}\right) - \left(3e^{-2t} - te^{-t} - 3e^{-2t}\right) \\
&= -te^{-t} + te^{-t} \\
&= 0
\end{aligned}$$

Unfortunately, the Wronskian test tells us that we do not have a complete set of solutions!

(4) For the differential equation $(D^3 + D^2 - D + 2)x = 0$,
(a) Find all solutions of the form $e^{\lambda t}$ or t^α.
(b) Determine whether the solutions found in (a) generate a complete collection of solutions.

Solution: (a) Let's start with part (a), testing out $e^{\lambda t}$. Plugging this in and factoring yields:

$$\begin{aligned}
(D^3 + D^2 - D + 2)e^{\lambda t} &= 0 \\
\Rightarrow (\lambda^3 + \lambda^2 - \lambda + 2)e^{\lambda t} &= 0 \\
\Rightarrow (\lambda + 2)(\lambda^2 - \lambda + 1)e^{\lambda t} &= 0
\end{aligned}$$

Since we need this equation to hold for all times t, we need to set one of the left factors to 0. Clearly $\lambda = -2$ works, but we don't yet have a way of dealing with the complex roots. For now, the only solution we have

$$x = e^{-2t}.$$

Now let's try t^α. Plugging in and computing derivatives gives us

$$\alpha(\alpha - 1)(\alpha - 2)t^{\alpha-3} + \alpha(\alpha - 1)t^{\alpha-2} - \alpha t^{\alpha-1} + 2t^\alpha = 0$$

Unfortunately, this isn't going to work! One way to see this is to note that we're trying to use constants to set different powers of t equal to each other, kind of like trying to find constants so that t^2 and t^3 cancel out. To prove this, note that no value of α can cancel all four of the terms on the left. The best we can do is choose $\alpha = 1$ to kill off three of the terms, but we would still be left with $2t = 0$ which is not true for all t.

(b) We have only found one candidate solution, $x = e^{-2t}$. Since the o.d.e. is third order, we would need three candidate solutions to generate the general solution, so no, the solutions found in (a) do not constitute a complete set of solutions.

(5) Use Cramer's determinant test to determine whether or not the following system has solutions for all values of the right side.

$$\begin{aligned}
x - y + 3z &= a \\
x + y - 3z &= b \\
3x - y + 3z &= c
\end{aligned}$$

Solution: Okay, let's use Cramer's determinant test. We take the determinant (using the top row) of the matrix of coefficients:

$$\det \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & -3 \\ 3 & -1 & 3 \end{bmatrix} = \begin{vmatrix} 1 & -3 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -3 \\ 3 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = 3 - 3 + 3 + 9 + 3(-1 - 3) = 0$$

and therefore Cramer's determinant test tells us that the system does not have solutions for all values of a, b, and c.

(6) Prove that linear combinations of solutions to linear *nonhomogeneous* o.d.e.'s need not be solutions themselves. To do this, suppose that $x = \phi(t)$ is a solution to the o.d.e. $Lx = f(t)$ where $f(t) \neq 0$. Show that even though x is a solution, $y(t) = x + x$ is not a solution. Is this surprising?

Solution: Suppse $y = x + x$ is a solution to the o.d.e. Then, $Ly = f$, while at the same time $Ly = L(x + x) = Lx + Lx = f + f = 2f$. This is a contradiction since $f = 2f$ would imply that $f = 0$ which assumed is not the case. Therefore, $y = x + x$ cannot be a solution.