Problem Set 6
Non-homogeneous equations

Math 51 Fall 2021

due Monday 2022-02-28 at 11:59 PM

1. Find the constant-coefficient linear operator $A(D)$ of smallest order for which

$$A(D)[te^{-t}\cos(2t) + 2e^{3t}] = 0.$$

Solution:

The term $te^{t}\cos(t)$ is made 0 by an operator whose characteristic polynomial has $1+i$ as a root with multiplicity two. That characteristic polynomial must be $[(r-(1+i))(r-(1-i))]^2 = (r^2-2r+2)^2$; thus the operator is $(D^2-2D+2)^2$.

The term $2e^{3t}$ is made zero by the operator $D-3$

Thus $A(D) = (D^2-2D+2)^2 \cdot (D-3)$.

2. Make a simplified guess for a particular solution to the ODE

$$(D-1)^2(D^2+1)x = te^{3t} + e^t - \sin(t).$$

Note: In this problem, you aren't asked to solve for the coefficients.

Solution:

Write $P(D) = (D-1)^2(D^2+1)$.

The smallest order operator $A(D)$ which annihilates the right-hand-side of the ODE is

$$A(D) = (D-3)^2(D-1)(D^2+1).$$

To find an the (unsimplified) guess, we must find solutions to the auxiliary equation

$$A(D)P(D)x = 0$$

i.e. to

$$(D-3)^2(D-1)^3(D^2+1)^2x = 0.$$

The solution to this auxiliary equation is generated by

$$e^{3t}, \quad te^{3t}, \quad e^t, \quad te^t, \quad t^2e^t, \quad \sin(t), \quad \cos(t), \quad t\sin(t), \quad t\cos(t)$$

To simplify this guess, we remove solutions to the homogeneous equation corresponding to the original equation

$$P(D)x = 0 \quad \text{i.e.} \quad (D-1)^2(D^2+1)x = 0.$$

The solutions to this corresponding homogeneous equation are generated by

$$e^t, \quad te^t, \quad \sin(t), \quad \cos(t).$$

Thus a simplified guess for a solution to the give equation is given by

$$p(t) = k_1e^{3t} + k_2te^{3t} + k_3t^2e^t + k_4t\sin(t) + k_5t\cos(t)$$

3. Find the general solution to the ODE

$$(D^3 - D^2 - 2D)x = 1 + e^{2t}.$$

Solution:

Observe that

$$(D^3 - D^2 - 2D) = D(D^2 - D - 2) = D(D-2)(D+1).$$

Now, a polynomial differential operator $A(D)$ for which $A(D)[1+e^{2t}]$ is

$$A(D) = D(D-2).$$

Thus, we find a guess for a particular solution $q(t)$ to (♣) $\quad (D^3 - D^2 - 2D)x = 1 + e^{2t}$ from among the solutions to the auxiliary homogeneous equation

$$0 = A(D)\cdot(D^3 - D^2 - 2D)x = D(D-2)(D^3 - D^2 - 2D) = D^2(D-2)^2(D+1)$$

The solution to this auxiliary homog equation is generated by

$$1, \quad t, \quad e^{2t}, \quad te^{2t}, \quad e^{-t}.$$

Of these functions, 1, e^{2t} and e^{-t} are already solutions to the homogeneous equation $(D^3 - D^2 - 2D)x = 0$ and hence we may and will discard them.

Thus our simplified guess for the particular solution is

$$q(t) = k_1t + k_2te^{2t}.$$

We must now solve for k_1, k_2. For this, we must compute

$$(D^3 - D^2 - 2D)[k_1t + k_2te^{2t}].$$

We have $(D^3 - D^2 - 2D)[t] = -2$ and using the exponential shift formula we find that

$$\begin{aligned}(D^3 - D^2 - 2D)[te^{2t}] &= e^{2t}((D+2)^3 - (D+2)^2 - 2(D+2))[t] \\ &= e^{2t}(D^3 + 6D^2 + 12D + 8 - D^2 - 4D - 4 - 2D - 4))[t] \\ &= e^{2t}(D^3 + 5D^2 + 6D))[t] \\ &= 6e^{2t}.\end{aligned}$$

This shows that

$$(D^3 - D^2 - 2D)[k_1t + k_2te^{2t}] = -2k_1 + 6k_2e^{2t}.$$

So in order to arrange that $q(t) = k_1t + k_2te^{2t}$ is a particular solution to (♣), we need that

$$-2k_1 + +6k_2e^{2t} = 1 + e^{2t}.$$

Comparing coefficients, we find that

$$-2k_1 = 1 \quad \text{and} \quad 6k_2 = 1.$$

Thus $k_1 = -1/2$ and $k_2 = 1/6$ so the particular solution is

$$q(t) = \frac{te^{2t} - 3t}{6}.$$

Now, the general solution to (♣) has the form

$$x(t) = q(t) + H(t)$$

where $H(t)$ is the general solution to the corresponding homogeneous equation. Thus

$$x(t) = \frac{te^{2t} - 3t}{6} + c_1 + c_2 e^{2t} + c_3 e^{-t}$$

is the general solution to (♣).

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4. Solve the initial value problem:

$$(D^2 - 9)x = 9 + 12e^{-3t}; \quad x(0) = x'(0) = 0.$$

Solution:

We first note that the operator

$$A(D) = D(D+3)$$

satisfies

$$A(D)[9 + 12e^{-3t}] = 0.$$

Thus we may look for a particular solution $q(t)$ to

$$(\diamondsuit) \quad (D^2 - 9)x = 9 + 12e^{-3t}$$

among the general solutions to the auxiliary homogeneous equation

$$0 = A(D)(D^2 - 9)x = D \cdot (D+3)^2 \cdot (D-3)x.$$

The general solution to this auxiliary equation is generated by

$$1, \quad e^{-3t}, \quad te^{-3t}, \quad e^{3t}.$$

Among these solutions, e^{3t} and e^{-3t} are already solutions to the homogeneous equation $(D^2 + 9)x = 0$ corresponding to $(\diamondsuit)$. Thus we may and will omit them from our simplified guess.

Our guess for $q(t)$ is now:

$$q(t) = k_1 + k_2 te^{-3t}.$$

We must find k_1, k_2 for which $(D^2 - 9)[q(t)] = 9 + 12e^{-3t}$.

We compute, using the exponential shift formula when neeeded:

$$(D^2 - 9)[1] = -9$$

$$\begin{aligned}
(D^2 - 9)[te^{-3t}] &= e^{-3t}((D-3)^2 - 9)[t] \\
&= e^{-3t}(D^2 - 6D + 9 - 9)[t] \\
&= e^{-3t}(D^2 - 6D)[t] \\
&= -6e^{-3t}
\end{aligned}$$

We conclude that

$$(D^2-9)[k_1+k_2te^{-3t}] = -9k_1 - 6k_2e^{-3t}$$

Thus, in order to arrange that $q(t)$ is a particular solution to $(\diamondsuit)$, we need

$$-9k_1 - 6k_2e^{-3t} = 9 + 12e^{-3t}$$

Comparing coefficients, we find that

$$k_1 = -1, \quad k_2 = -2.$$

Thus

$$q(t) = -1 - 2te^{-3t}.$$

And we conclude the general solution to $(\diamondsuit)$ is given by

$$x(t) = q(t) + H(t) = -1 - 2te^{-3t} + c_1e^{3t} + c_2e^{-3t}.$$

Finally, we need

$$0 = x(0) = -1 + c_1 + c_2 = -3 + c_1 + c_2$$

and

$$0 = x'(0) = (-2e^{-3t}(1-3t) + 3c_1e^{3t} - 3c_2e^{-3t})\Big|_{t=0} = -2 + 3c_1 - 3c_2$$

So we must solve the system of equations

$$\begin{cases} 1 & = & c_1 & + & c_2 \\ 2 & = & 3c_1 & + & -3c_2 \end{cases}$$

One finds that $c_1 = \dfrac{5}{6}$ and $c_2 = \dfrac{1}{6}$, so the solution to the initial value problem is

$$\begin{aligned} x(t) &= -1 - 2te^{-3t} + \frac{5e^{3t}}{6} + \frac{e^{-3t}}{6} \\ &= -1 + \frac{5e^{3t}}{6} + \frac{(1-12t)e^{-3t}}{6} \end{aligned}$$

5. Find the general solution to

$$x'' - 2x' + x = e^t \ln(t), \quad t > 0.$$

(You should use variation of parameters)

Solution:

Since the characteristic polyomial is $r^2 - 2r + 1 = (r-1)^2$, the general solution to the corresponding homogeneous equation is generated by

$$h_1(t) = e^t, \quad h_2(t) = te^t.$$

To solve the inhomogeneous equation, it isn't clear how to use the method of undetermined coefficients (what should $A(D)$ be?!?) So we use variation of parameters.

The Wronskian matrix is given by

$$W = W(h_1, h_2)(t) = \det \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix}.$$

And following the method, we need to solve the matrix equation

$$W \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix} \cdot \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \ln(t) \end{bmatrix}.$$

We now use Cramer's Rule to find c_i'. First note that $\det W = (t+1)e^{2t} - te^{2t} = e^{2t}$.

We now have

$$c_1' = \frac{\det \begin{bmatrix} 0 & te^t \\ e^t \ln(t) & (t+1)e^t \end{bmatrix}}{\det W} = \frac{-e^{2t}\ln(t)t}{e^{2t}} = -t\ln(t)$$

and

$$c_2' = \frac{\det \begin{bmatrix} e^t & 0 \\ e^t & e^t \ln(t) \end{bmatrix}}{\det W} - \frac{\ln(t)e^{2t}}{e^{2t}} = \ln(t).$$

We now find the functions c_1, c_2 by anti-differentiation:

$$c_1(t) = \int c_1' dt = -\int t\ln(t) dt = \frac{-t^2\ln(t)}{2} + \frac{t^2}{4} + C = \frac{t^2(1 - 2\ln(t))}{4} + C$$

(integrate by parts with $u = \ln(t)$ and $dv = t$).

and

$$c_2(t) = \int c_2' dt = \int \ln(t) dt = t\ln(t) - t + D$$

Now, we only need a particular solution to the ODE, se we take $C = D = 0$, and we find the particular solution

$$\begin{aligned} p(t) &= c_1(t)h_1(t) + c_2(t)h_2(t) \\ &= \left(\frac{t^2(1-2\ln(t))}{4}\right) \cdot e^t + (t\ln(t) - t) \cdot te^t \\ &= \left(\frac{-2\ln(t)+1}{4}\right) \cdot t^2 e^t + \frac{4\ln(t) - 4}{4} \cdot t^2 e^t \\ &= \frac{2\ln(t) - 3}{4} t^2 e^t \end{aligned}$$

Now the general solution is

$$x(t) = p(t) + k_1 e^t + k_2 te^t = \frac{2\ln(t) - 3}{4} t^2 e^t + k_1 e^t + k_2 te^t$$

for constants k_1, k_2.