MATH 51: HOMEWORK 2 SOLUTION

Homework exercise solution:

1. Solve the initial-value problem:
$$\frac{dx}{dt} + 3x = 8e^t, \quad x(0) = 0.$$
Consider the homogeneous equation:
$$\frac{dx}{dt} = -3x$$
$$\int \frac{1}{x} dx = \int -3dt$$
$$\ln|x| = -3t + C, \quad x = Ce^{-3t}$$
Using variation of parameters, Assume $C = C(t)$
$$C'(t)e^{-3t} - 3Ce^{-3t} + 3C(t)e^{-3t} = 8e^t$$
$$C'(t)e^{-3t} = 8e^t$$
$$C'(t) = 8e^{4t}$$
$$C(t) = 2e^{4t} + k$$
$$x = 2e^t + ke^{-3t}, \quad x(0) = 0, \quad k = -2$$
Therefore $x = 2e^t - 2e^{-3t}$.

2. Solve the initial-value problem:
$$tx' - x = t^3, \quad x(1) = 0.$$
Consider the homogeneous equation:
$$\frac{dx}{dt} = \frac{x}{t}$$
$$\int \frac{1}{x} dx = \int \frac{1}{t} dt$$
$$\ln|x| = \ln|t| + c$$
$$x = Ct$$
Using variation of parameters, assume $C = C(t)$.
$$C'(t)t^2 + C(t)t - C(t)t = t^3$$
$$C'(t) = t, \quad C(t) = \frac{1}{2}t^2 + k$$
$$x = \frac{1}{2}t^3 + kt, \quad x(1) = 0, \quad k = -\frac{1}{2}$$
Therefore $x = \frac{1}{2}t^3 - \frac{1}{2}t$.

3. §1.3, exercise 30: Try variation of parameter on the nonlinear equation $x' + x^2 = t$ as follows: (a) Find the general solution of $x' + x^2 = 0$ by separating variables; (b) then replace the parameter with a variable and substitute back. What goes wrong?

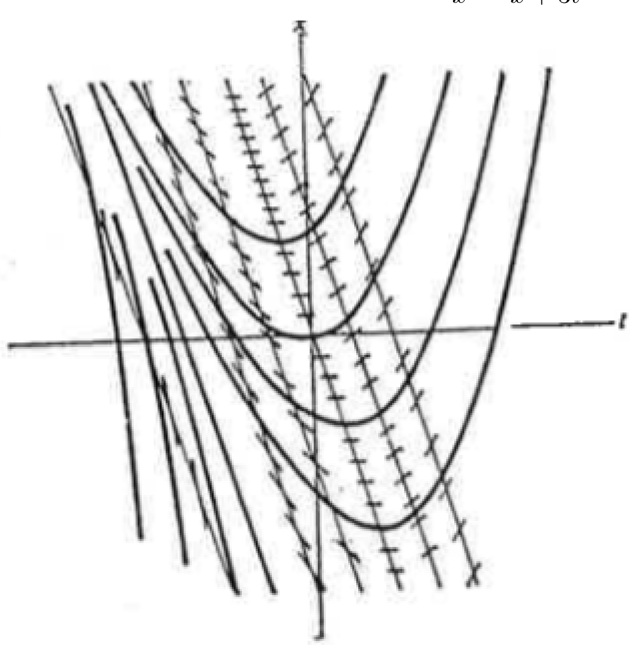
   (a). Consider the homogeneous equation:
$$x' = -x^2$$
$$\int \frac{1}{x^2}dx = \int -1dt$$
$$-\frac{1}{x} = -t + c$$
$$x = \frac{1}{t+c}$$
   (b). If we replace the parameter with a variable and substitute back, we would have $c(t)$ terms remain because the original differential equation is nonlinear.

4. §1.6, exercise 24: Show that for any nonnegative number a and b, the function
$$x(t) = \begin{cases} (t+a)^5, & t < -a \\ 0, & -a \le t \le b \\ (t-b)^5, & t > b \end{cases}$$
   is a solution of $dx/dt = 5x^{4/5}$ and satisfies $x(0) = 0$. Why doesn't this contradict with the uniqueness part of the theorem?

   For $t < -a$, $dx/dt = 5(t+a)^4 = 5x^{4/5}$, for $t > b$, $dx/dt = 5(t-b)^4 = 5x^{4/5}$ and for $x = 0, dx/dt = 0 = 5x^{4/5}$ and satisfy $x(0) = 0$, hence the given $x(t)$ is a solution.

   Consider $f(t,x) = 5x^{4/5}$. To have locally uniqueness at $x = 0$ the uniqueness requires $\frac{\partial f}{\partial x}$ continues at $x = 0$. However, $\frac{\partial f}{\partial x} = 4x^{-1/5}$ is undefined at $x = 0$, hence doesn't satisfy the existence and uniqueness theorem, therefore not a contradiction.

5. Plot graph of solutions of the following o.d.e. in the region $-2 \le x \le 2$ and $-2 \le t \le 2$. Use at least five isoclines and sketch at least three different solutions
$$x' = x + 3t$$



6. In this problem you are given a nonhomogeneous equation, the general solution $x = H(t)$ of the related homogeneous equation, and a function $p(t)$ involving two unknown constants A and B:

$$(D^2+1)x = e^t + 1, \quad H(t) = c_1 \sin 2t + c_2 \cos 2t, \quad p(t) = Ae^t + B.$$

(a) Find the values of A and B for which $x = p(t)$ is a particular solution of the nonhomogeneous equation.

$$D^2 p(t) = Ae^t$$
$$(D^2+1)p(t) = Ae^t + Ae^t + B = e^t + 1$$
$$A = \frac{1}{2}, \quad B = 1$$

(b) Find the general solution of the nonhomogeneous equation.

$$x(t) = H(t) + p(t) = c_1 \sin 2t + c_2 \cos 2t + \frac{1}{2}e^t + 1$$

(End of Homework 2)