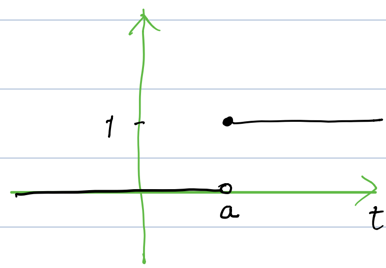
Math 51 Lecture 12

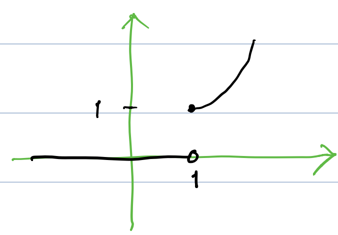
Step Functions, Diff + shift Formulas

Unit Step Function



$$u_a(t) = \begin{cases} 0 & \text{for } t < a, \\ 1 & \text{for } t \geq a. \end{cases}$$

Ex. $g(t) = \begin{cases} 0 & \text{for } t < 1 \\ t^2 & \text{for } t \geq 1 \end{cases}$, $= t^2 u_1(t)$.



Ex. Write in step notation:

$$f(t) = \begin{cases} t^2 & \text{for } t < 1, \\ 3t & \text{for } 1 \leq t < 4, \\ 1-t & \text{for } t \geq 4. \end{cases}$$

Start out w/ t^2.

At $t=1$, subtract $t^2 u_1(t)$,

add $3t\, u_1(t)$.

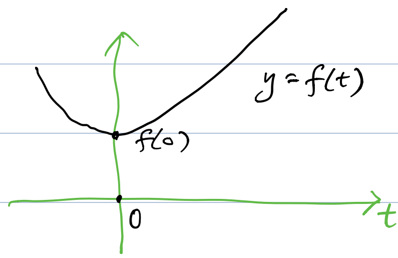
At $t=4$, subtract $3t\, u_4(t)$ + add $(1-t)\, u_4(t)$.

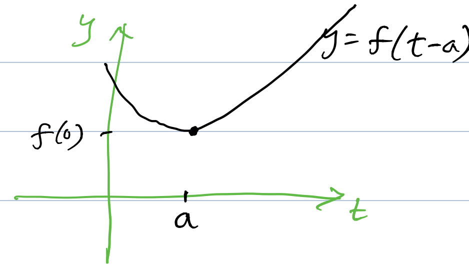
So $f(t) = t^2 + (-t^2+3t)\, u_1(t) + (-3t + 1 - t)\, u_4(t)$

$= \boxed{t^2 + (-t^2+3t)\, u_1(t) + (1-4t)\, u_4(t)}$.

Shifting a Function

Assume $a > 0$.





$y = f(t-a)$ will take on the value $f(0)$ when $t = a$.
Therefore, its graph is the graph of $y = f(t)$ shifted by a to the right.

Important Formulas for Laplace Transf.

Write $\mathcal{L}[f(t)] = F(s)$.

- $\mathcal{L}[D^2 x] = s^2 \mathcal{L}[x] - s\,x(0) - x'(0)$

  (1st diff formula)

- $\mathcal{L}[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n} F(s)$

  (2nd diff formula)

- $\mathcal{L}[e^{at} f(t)] = F(s-a)$

  (1st shift formula)

- $\mathcal{L}[u_a(t) f(t)] = e^{-as} \mathcal{L}[f(t+a)]$

  (2nd shift formula)

Some Consequences

In the 2nd shift formula, let $f(t) = 1$.

$$\boxed{\mathcal{L}[u_a(t)]} = e^{-as} \mathcal{L}[f(t+a)] = e^{-as} \mathcal{L}[1]$$
$$= \boxed{\frac{e^{-as}}{s}}.$$

Inverse Transf. of a shifted Function

From the 1st shift formula,

$$\boxed{\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t)}$$

Ex. $\mathcal{L}^{-1}\left[\frac{1}{(s-2)^6}\right]$ Let $F(s) = \frac{1}{s^6}$

$= \mathcal{L}^{-1}[F(s-2)]$

$= e^{2t}\mathcal{L}^{-1}\left[\frac{1}{s^6}\right]$ $\mathcal{L}[t^5] = \frac{5!}{s^6} \Rightarrow \mathcal{L}\left[\frac{1}{5!}t^5\right] = \frac{1}{s^6}$

$= e^{2t}\frac{1}{5!}t^5 = \boxed{\frac{1}{120}t^5 e^{2t}}$.

Inverse Transform of $e^{-as}F(s)$

The 2nd shift formula is

$$\mathcal{L}[u_a(t)\underbrace{f(t)}_{h(t-a)}] = e^{-as}\underbrace{\mathcal{L}[\underbrace{f(t+a)}_{h(t)}]}_{H(s)}$$

- Let $h(t) = f(t+a)$, Then $h(t-a) = f((t-a)+a)$
  $= f(t)$.
- The 2nd formula becomes

$$\mathcal{L}[u_a(t)\, h(t-a)] = e^{-as}H(s).$$

- The inverse is

$$\mathcal{L}^{-1}[e^{-as}H(s)] = u_a(t)\, h(t-a).$$

- Relabel h as f and H as F:

$$\boxed{\mathcal{L}^{-1}[e^{-as}F(s)] = u_a(t)\, f(t-a)}.$$

Ex. Find $\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s+5}\right]$

Let $F(s) = \frac{1}{s+5}$.

Then $f(s) = e^{-5t}$

By the inverse 2nd shift formula,

$$\mathcal{L}^{-1}\left[e^{-4s}F(s)\right] = u_4(t)\, f(t-4) \qquad a=4$$

$$= u_4(t)\, e^{-5(t-4)}$$

$$= \boxed{u_4(t)\, e^{-5t+20}}.$$

Ex. $\mathcal{L}^{-1}\left[\frac{1}{s^2+4s+5}\right]$

First complete the square:

$$s^2+4s+5 = (s^2+4s+4)+1$$

$$= (s+2)^2+1.$$

$\frac{1}{(s+2)^2+1}$ is a shift of $\frac{1}{s^2+1}$ by replacing

s by $s+2 = s-(-2)$.

Let $F(s) = \frac{1}{s^2+1}$. Then $f(t) = \sin t$.

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4s+5}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= \mathcal{L}^{-1}\left[F(s+2)\right]$$

$$= \mathcal{L}^{-1}\left[F(s-(-2))\right] \qquad a=-2$$

$$= e^{-2t} f(t) \qquad \text{(1st shift formula)}$$

$$= \boxed{e^{-2t}\sin t}.$$

Ex. $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^4}\left(1-e^{-3(s+1)}\right)\right]$.

Let $F(s) = \frac{1}{s^4}\left(1-e^{-3s}\right)$.

$= \mathcal{L}^{-1}[F(s+1)] = \mathcal{L}^{-1}[F(s-(-1))]$ $\quad a=-1$

$= e^{-t} f(t)$.

$= e^{-t} \mathcal{L}^{-1}\left[\frac{1}{s^4}\left(1-e^{-3s}\right)\right]$

$= e^{-t} \mathcal{L}^{-1}\left[\frac{1}{s^4}\right] - e^{-t}\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^4}\right]$

$= e^{-t} \frac{1}{6} t^3 - e^{-t}\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^4}\right]$

scratch work

$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$\mathcal{L}[t^3] = \frac{3!}{s^4}$

$\mathcal{L}\left[\frac{1}{6}t^3\right] = \frac{1}{s^4}$

$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^4}\right] = u_3(t)\, f(t-3)$,

where $F(s) = \frac{1}{s^4}$.

Then $f(t) = \frac{1}{6} t^3$.

$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^4}\right] = u_3(t)\,\frac{1}{6}(t-3)^3$.

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^4}\left(1-e^{-3(s+1)}\right)\right] = \boxed{\frac{1}{6}t^3 e^{-t} - e^{-t}\frac{1}{6}u_3(t)(t-3)^3}.$$