1. Consider the following matrix $A$ and list of vector-valued functions.

$$A = \begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}; \quad \mathbf{h}_1(t) = \begin{bmatrix} 3e^{4t} + e^{-4t} \\ e^{4t} + 3e^{-4t} \end{bmatrix}, \quad \mathbf{h}_2(t) = \begin{bmatrix} 3e^{4t} - e^{-4t} \\ e^{4t} - 3e^{-4t} \end{bmatrix}.$$

(a) Are the functions $\mathbf{h}_1$ and $\mathbf{h}_2$ solutions to the equation $D\mathbf{x} = A\mathbf{x}$?
(b) Of the functions that are solutions, do they generate the general solution? Explain why or why not.

**Solution:**
(a) We check this by hand, by evaluating both $A\mathbf{h}_i$ and $D\mathbf{h}_i$, as follows.

$$A\mathbf{h}_1 = \begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 3e^{4t} + e^{-4t} \\ e^{4t} + 3e^{-4t} \end{bmatrix} = \begin{bmatrix} 5(3e^{4t} + e^{-4t}) - 3(e^{4t} + 3e^{-4t}) \\ 3(3e^{4t} + e^{-4t}) - 5(e^{4t} + 3e^{-4t}) \end{bmatrix} = \begin{bmatrix} 12e^{4t} - 4e^{-4t} \\ 4e^{4t} - 12e^{-4t} \end{bmatrix}$$

while

$$D\mathbf{h}_1 = D \begin{bmatrix} 3e^{4t} + e^{-4t} \\ e^{4t} + 3e^{-4t} \end{bmatrix} = \begin{bmatrix} 12e^{4t} - 4e^{-4t} \\ 4e^{4t} - 12e^{-4t} \end{bmatrix}$$

and therefore $\mathbf{h}_1$ is a solution. Similarly:

$$A\mathbf{h}_2 = \begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 3e^{4t} - e^{-4t} \\ e^{4t} - 3e^{-4t} \end{bmatrix} = \begin{bmatrix} 5(3e^{4t} - e^{-4t}) - 3(e^{4t} - 3e^{-4t}) \\ 3(3e^{4t} - e^{-4t}) - 5(e^{4t} - 3e^{-4t}) \end{bmatrix} = \begin{bmatrix} 12e^{4t} + 4e^{-4t} \\ 4e^{4t} + 12e^{-4t} \end{bmatrix}$$

while

$$D\mathbf{h}_2 = D \begin{bmatrix} 3e^{4t} - e^{-4t} \\ e^{4t} - 3e^{-4t} \end{bmatrix} = \begin{bmatrix} 12e^{4t} + 4e^{-4t} \\ 4e^{4t} + 12e^{-4t} \end{bmatrix}$$

so $\mathbf{h}_2$ is also a solution.

(b) We have two candidate solutions for a second order system, so we're in good shape. To see if they are a complete set we use the Wronskian test:

$$W[\mathbf{h}_1, \mathbf{h}_2] = \det \begin{bmatrix} 3e^{4t} + e^{-4t} & 3e^{4t} - e^{-4t} \\ e^{4t} + 3e^{-4t} & e^{4t} - 3e^{-4t} \end{bmatrix} = -16 \neq 0$$

and therefore $\mathbf{h}_1$ and $\mathbf{h}_2$ are a complete set of solutions.

2. Let

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}, \quad \mathbf{E}(t) = \begin{bmatrix} 0 \\ -5e^t \end{bmatrix}.$$

The formulas

$$\begin{cases} x_1 = c_1 \cos 2t + c_2 \sin 2t + e^t \\ x_2 = -2c_2 \cos 2t + 2c_1 \sin 2t - e^t \end{cases}$$

describe a collection of solutions of the nonhomogeneous system $D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$. Decide whether the collection is complete.

**Solution:** Since we know that the above constitute solutions, we do not need to verify the particular solution and can focus only on whether or not the homogeneous components generate the general solution to the related homogeneous equation. Taking careful note of the constants, our two candidate solutions are

$$\mathbf{h}_1 = \begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix}, \quad \mathbf{h}_2 = \begin{bmatrix} \sin 2t \\ -2\cos 2t \end{bmatrix}.$$

Using the Wronskian we have that

$$W(\mathbf{h}_1, \mathbf{h}_2) = \det \begin{bmatrix} \cos 2t & \sin 2t \\ 2\sin 2t & -2\cos 2t \end{bmatrix} = -2\cos^2 2t - 2\sin^2 2t = -2 \neq 0$$

and therefore the general solution to the nonhomogeneous equation is

$$\mathbf{x} = c_1\mathbf{h}_1 + c_2\mathbf{h} + \mathbf{p} = c_1 \begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t \\ -2\cos 2t \end{bmatrix} + \begin{bmatrix} e^t \\ -e^t \end{bmatrix}.$$

3. Check the following set of vectors for linear independence:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

**Solution:** We do this manually. We take an arbitrary linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ and set it equal to 0. The resulting system of equations is

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c3 &= 0 \\ c1 &= 0 \\ c1 + 3c_2 &= 0 \end{aligned}$$

The second and third equation do most of the work for us, and combining them with either the first or last equation gives us that $c_1 = c_2 = c_3 = 0$ and therefore the vectors are linearly independent.

4. Show that if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are independent, then the set of vectors obtained by deleting $\mathbf{v}_i$ is also independent, where $1 \leq i \leq n$.

**Solution:** Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a set of linearly independent vectors of the same length. Suppose now that the same set with one vector removed is linearly dependent. Without loss of generality, we may assume that we remove the vector $\mathbf{v}_n$. Since we have assumed $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ is linearly dependent, there is at least one nonzero $c_i$ such that $c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1} = 0$. Now, if we reinclude $\mathbf{v}_n$ in our linearly combination, with the constant $c_n = 0$, we have a nontrivial solution to $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$, and therefore this would mean that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent which contradicts our hypothesis.

5. In words (though pictures may be drawn to help illustrate), describe what it means for vectors to be linearly independent and linearly dependent in both 2 and 3 dimensions.

**Solution:**

Let $\mathbf{v}$ be any 2-vector. If the set $\mathbf{v}, \mathbf{w}$ is linearly dependent, then $c_1\mathbf{v} + c_2\mathbf{w} = 0$ where at least one of the two constants is nonzero. This means that $\mathbf{v} = k\mathbf{w}$ for some constant $k$ (including 0). Therefore, $\mathbf{v}$ and $\mathbf{w}$ must be multiples of each other, and either point in the exact same direction or in opposite directions (i.e. if $k < 0$).

Now let's think about 3D. If we have two vectors $\mathbf{u}, \mathbf{v}$ that are linearly dependent, then the scenario is similar to 2D where they must be scalar multiples of one another. If instead we have two vectors $\mathbf{u}, \mathbf{v}$ that are linearly independent, and we include a third vector $\mathbf{w}$, it gets slightly more interesting. If the set $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is linearly dependent, then $\mathbf{w}$ may be written as some combination $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. This means that $\mathbf{w}$ lies on the plane of all vectors that can be written as combinations of $\mathbf{u}$ and $\mathbf{v}$! If on the other hand $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a linearly independent set, then $\mathbf{w}$ will be point away from the $\mathbf{uv}$ plane.

6. (a) Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are $n$ linearly independent $n$-vectors. Show that every $n$-vector $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. (*Hint*: View this as a system of equations for the unknown coefficients in the linear combination. What can you say about the determinant of coefficients?)

   (b) Prove that any set $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ consisting of $n+1$ different $n$-vectors must be linearly dependent.

   (c) Does a set of $k \leq n$ different $n$-vectors have to be independent?

**Solution:**

(a) Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are $n$ linearly independent $n$-vectors. Let's see if we can write some other arbitrary vector $\mathbf{w}$ in terms of our set (we'll use $\mathbf{w}$ just so that it's easier to keep track of things, it doesn't really matter what we call our new vector). In other words, we want to find constants such that

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{w}.$$

Let's write this out with vectors and matrices:

$$c_1 \begin{bmatrix} v_{11} \\ \vdots \\ v_{1n} \end{bmatrix} + \cdots + c_n \begin{bmatrix} v_{n1} \\ \vdots \\ v_{nn} \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

or, viewing the constants as their own matrix,

$$\begin{bmatrix} v_{11} & \cdots & v_{n1} \\ \vdots & \ddots & \vdots \\ v_{1n} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}.$$

Since the vectors are linearly independent, the determinant of the matrix above is nonzero, and therefore the equation has a unique solution $c_1, \ldots, c_n$ for any choice of $w_1, \ldots, w_n$, i.e. any $\mathbf{w}$.

(b) This follows immediately from part (a): take $\mathbf{v}_{,} \ldots, \mathbf{v}_n$ and use $\mathbf{v}_{n+1}$ instead of $\mathbf{w}$.

(c) No, a counterexample is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$