Laplace transform formulas

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definition

- $F(s) = \mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt$

basic formulas

$$\begin{array}{lcl}
\mathscr{L}[e^{\lambda t}] = \dfrac{1}{s-\lambda} \quad \text{for } s > \lambda & , & \mathscr{L}^{-1}\left[\dfrac{1}{s-\lambda}\right] = e^{\lambda t} \\
\mathscr{L}[t^n] = \dfrac{n!}{s^{n+1}} \quad \text{for } s > 0 & , & \mathscr{L}^{-1}\left[\dfrac{1}{s^{n+1}}\right] = \dfrac{t^n}{n!} \\
\mathscr{L}[\sin(\beta t)] = \dfrac{\beta}{s^2+\beta^2} \quad \text{for } s > 0 & , & \mathscr{L}^{-1}\left[\dfrac{1}{s^2+\beta^2}\right] = \dfrac{1}{\beta}\sin(\beta t) \\
\mathscr{L}[\cos(\beta t)] = \dfrac{s}{s^2+\beta^2} \quad \text{for } s > 0 & , & \mathscr{L}^{-1}\left[\dfrac{s}{s^2+\beta^2}\right] = \cos(\beta t)
\end{array}$$

first differentiation formula:

- $\mathscr{L}[Dx] = s\mathscr{L}[x] - x(0)$,
- $\mathscr{L}[D^2x] = s^2\mathscr{L}[x] - sx(0) - x'(0)$,
- $\mathscr{L}[D^kx] = s^k\mathscr{L}[x(t)] - s^{k-1}x(0) - s^{k-2}x'(0) - \cdots - sx^{(k-2)}(0) - x^{(k-1)}(0)$ for $k \geq 1$.

first shift formula

- if $\mathscr{L}[f(t)] = F(s)$ then $\mathscr{L}[e^{\alpha t}f(t)] = F(s-\alpha)$.
- $\mathscr{L}^{-1}[F(s)] = e^{\alpha t}\mathscr{L}^{-1}[F(s+\alpha)]$

second differentiation formula

- $\mathscr{L}[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}\mathscr{L}[f(t)]$

second shift formula

- $\mathscr{L}[u_a(t)f(t)] = e^{-as}\mathscr{L}[f(t+a)]$.
- if $f(t) = \mathscr{L}^{-1}[F(s)]$ then $\mathscr{L}^{-1}[e^{-as}F(s)] = u_a(t)f(t-a)$.

convolution

- definition: $(f * g)(t) = \int_0^t f(t-u)g(u)du$.
- $\mathscr{L}[(f * g)(t)] = \mathscr{L}[f(t)]\mathscr{L}[g(t)]$.
- $\mathscr{L}^{-1}[F(s)G(s)] = \mathscr{L}^{-1}[F(s)] * \mathscr{L}^{-1}[G(s)]$