In finding the eigenvalues of a square matrix, it is necessary to solve for the roots of the characteristic polynomial. In degree 2, one can always use the quadratic formula.

Roots and factors. A number a is a root of a polynomial $P(x)$ if and only if $x - a$ is a factor of $P(x)$.

Proof. By polynomial long division,

$$P(x) = (x - a)Q(x) + r,$$

for some constant r (the remainder). Then

$$\begin{aligned} & a \text{ is a root of } P(x) \\ \iff\ & 0 = P(a) = 0 \cdot Q(x) + r = r \\ \iff\ & P(x) = (x - a)Q(x) \\ \iff\ & x - a \text{ is a factor of } P(x). \end{aligned}$$

Integer root test. Let $P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be an integer polynomial with leading coefficient 1. If a is an integer root of $P(x)$, then a divides the constant c_0.

Proof. If a is a root of $P(x)$, then $P(x)$ factors:

$$\begin{aligned} P(x) &= x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \\ &= (x - a)(x^{n-1} + \cdots + b_1 x + b_0). \end{aligned}$$

Thus, $c_0 = -a \cdot b_0$, which shows that a divides c_0.

Rational root test. Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be an integer polynomial. If p/q is a rational root in lowest terms of $P(x)$, then p divides the constant c_0 and q divides the leading coefficient c_n.

1. Factor $x^3 - 2x^2 - 2x - 3$ into linear factors. (First find a linear factor. Then carry out polynomial long division.)

2. (a) Explain why an echelon matrix with fewer rows than columns $(n \times m, n < m)$ must have some columns without corners (pivots).

   (b) Use part (a) to explain why a homogeneous system of linear equations with more unknowns than equations must have infinitely many solutions.

3. Solve the following system of linear equations by first finding the echelon form of the augmented matrix:

$$\begin{array}{rcrcrl} x_1 & + & 3x_2 & - & x_3 & = 5 \\ & & x_2 & - & x_3 & = 2 \\ x_1 & + & x_2 & + & x_3 & = 1. \end{array}$$

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4. Factor $2x^3 + x^2 + x - 1$ into linear factors.

5. Use row reduction to decide whether the following three vectors are linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

6. Consider the linear system $D\mathbf{x} = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$.

   (a) Find the characteristic polynomial A.

   (b) Find the eigenvalues of A.

   (c) For each eigenvalue of A, find a corresponding eigenvector.

   (d) Write down the general solution of $D\mathbf{x} = A\mathbf{x}$.