Final Exam Solutions

Math 51 Fall 2021 – Tufts University – Z. Nitecki and G. McNinch

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I. Multiple Choice Problems (24 points)

1. (4pts) Indicate which of the following best represents a simplified guess for a particular solution $p(t)$ to the non-homogeneous linear ODE:

$$(D-3)(D-1)x = te^{3t} + \cos(2t)$$

   a. $p(t) = k_1 te^{3t} + k_2 \cos(2t) + k_3 \sin(2t)$
   b. $p(t) = k_1 te^{3t} + k_2 \cos(2t)$
   c. $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t)$
   d. $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t) + k_4 \sin(2t)$

Solution:

correct response was d.

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2. (4pts) Indicate which of the following represents the general solution to the homogeneous linear ODE $(D^2 - 2D + 2)^2 x = 0$.

   a. $h(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t) + c_3 te^t \cos(t) + c_4 te^t \sin(t)$
   b. $h(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t) + c_3 te^{-t} \cos(t) + c_4 te^{-t} \sin(t)$
   c. $h(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t)$
   d. $h(t) = c_1 te^t \cos(t) + c_2 te^t \sin(t) + c_3 t^2 e^t \cos(t) + c_4 t^2 e^t \sin(t)$

Solution:

correct response was a.

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3. (4pts) The matrix $A = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix}$ has characteristic polynomial $\lambda^2 - 2\lambda + 2$ and thus its eigenvalues are $\lambda = 1 + i$ and $\lambda = 1 - i$.

   Which of the following is an eigenvector for A?

   a. A has no eigenvectors.

   b. $\begin{bmatrix} 3 - i \\ 2 \end{bmatrix}$

   c. $\begin{bmatrix} 2 \\ -3 + i \end{bmatrix}$

   d. $\begin{bmatrix} -3 + i \\ 2 \end{bmatrix}$

   Solution:

   Typo; sorry. Should have been $\begin{bmatrix} 3 + i \\ 2 \end{bmatrix}$

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4. (4pts) Consider the linear system of ODEs

$$(\diamondsuit) \quad D\mathrm{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 5 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

   A third order linear ODE is equivalent to this system if for each of its solutions $x(t)$, the vector-valued function $\mathrm{x}(t) = \begin{bmatrix} x(t) \\ x'(t) \\ x''(t) \end{bmatrix}$ is a solution to $(\diamondsuit)$. Which of the following linear ODEs is equivalent to $(\diamondsuit)$?

   a. $(D^3 - 2D^2 - D - 5)x = e^t$

   b. $(D^3 - 5D^2 - D - 2)x = e^t$

   c. $(D^3 + 2D^2 + D + 5)x = -e^t$

   d. $(D^3 + 5D^2 + D + 2)x = -e^t$

   Solution:

   correct response was b.

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5. (4pts) Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. $\lambda = 2$ is an eigenvalue of A with multiplicity two. The matrix $A - 2\mathrm{I}_3$ satisfies $(A - 2\mathrm{I}_3)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the generalized eigenvectors of A for $\lambda = 2$ are generated by $\mathrm{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathrm{w} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

Which of the following represents a solution $\mathrm{h}(t)$ to the system $D\mathrm{x} = A\mathrm{x}$ with the property that $\mathrm{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$?

a. $\mathrm{h}(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

b. $\mathrm{h}(t) = e^{2t} \begin{bmatrix} 1 + 6t \\ 2 \\ 6 \end{bmatrix}$

c. $\mathrm{h}(t) = e^{2t} \begin{bmatrix} 1 + t \\ 2 \\ 6 \end{bmatrix}$

d. No solution $\mathrm{h}(t)$ has the property that $\mathrm{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$.

Solution:

credit awarded. another mistake. Correct response should have been $e^{2t} \begin{bmatrix} 1 + 12t \\ 2 \\ 6 \end{bmatrix}$

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6. (4pts) Consider the homogeneous system $(\diamondsuit)\quad D\mathrm{x} = A\mathrm{x}$ where A is a 3×3 matrix, and let $\mathrm{h}_1(t), \mathrm{h}_2(t)$ be solutions to $(\diamondsuit)$. Which of the following statements is correct?

   a. $\mathrm{h}_1(0)$ and $\mathrm{h}_2(0)$ are eigenvectors for A.

   b. The system $(\diamondsuit)$ has exactly two solutions.

   c. If the vectors $\mathrm{h}_1(0), \mathrm{h}_2(0)$ are linearly independent, then the general solution to $(\diamondsuit)$ is given by $\mathrm{x}(t) = c_1\mathrm{h}_1(t) + c_2\mathrm{h}_2(t)$.

   d. None of the above statements is correct.

Solution:

correct response was d.

To see that a. is incorrect, consider solutions $e^{\lambda t}\mathrm{v}$ and $e^{\mu t}\mathrm{w}$ arising from eigenvectors v and w.

Then there is a solution $h(t) = e^{\lambda t}\mathrm{v} + e^{\mu t}\mathrm{w}$ but $h(0) = \mathrm{v} + \mathrm{w}$ which is not an eigenvector if $\lambda \neq \mu$.

b. in incorrect since all linear combinations $c_1\mathrm{h}_1(t) + c_2\mathrm{h}_2(t)$ are solutions, so there are always infinitely many solutions.

Finally, c. is incorrect because for a 3×3 system the general solution is generated by three solutions with linearly independent initial vectors; two solutions is not enough.

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II. Partial Credit problems (75 points)

1. (15pts) The matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ has characteristic polynomial $\lambda(\lambda - 3)$ and hence has eigenvalues $\lambda = 0$ and $\lambda = 3$. An eigenvector for $\lambda = 0$ is given by $\mathrm{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and an eigenvector for $\lambda = 3$ is given by $\mathrm{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   Find a particular solution $\mathrm{p}(t)$ for the system of linear ODEs

$$D\mathrm{x} = A\mathrm{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

Solution:

The general solution is generated by the solutions obtained from eigenvectors:

$$\mathrm{h}_1(t) = e^{0t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathrm{h}_2(t) = e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

To find a particular solution, form the Wronskian matrix

$$W = \begin{bmatrix} -1 & e^{3t} \\ 1 & 2e^{3t} \end{bmatrix}$$

and notice that $\det W = -3e^{3t}$.

A particular solution has the form $\mathrm{p}(t) = c_1(t)\mathrm{h}_1(t) + c_2(t)\mathrm{h}_2(t)$, where the vector $\mathrm{c} = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$ satisfies the matrix equations

$$W\mathrm{c}' = \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

Using Cramer's Rule, we find that

$$c_1'(t) = \frac{\det \begin{bmatrix} 0 & e^{3t} \\ t & 2e^{3t} \end{bmatrix}}{-3e^{3t}} = \frac{-te^{3t}}{-3e^{3t}} = \frac{t}{3}.$$

$$c_2'(t) = \frac{\det \begin{bmatrix} -1 & 0 \\ 1 & t \end{bmatrix}}{-3e^{3t}} = \frac{-t}{-3e^{3t}} = \frac{te^{-3t}}{3}$$

Now we integrate to find $c_1(t)$ and $c_2(t)$:

$$c_1(t) = \int c_1'(t)dt = \frac{1}{3}\int t\,dt = \frac{t^2}{6} + A.$$

For c_2 we integrate by parts with $u = t, dv = e^{-3t}dt$:

$$c_2(t) = \int c_2'(t)dt = \frac{1}{3}\int te^{-3t}dt = \frac{1}{3}\left(\frac{-t}{3}e^{-3t} + \frac{1}{3}\int e^{-3t}dt\right) = \frac{-1}{9}e^{-3t}\left(t + \frac{1}{3}\right) + B$$

We may take $A = B = 0$ since we only seek a particular solution. This gives

$$\begin{aligned}\mathrm{p}(t) =& \frac{t^2}{6}\begin{bmatrix}-1\\1\end{bmatrix} + \frac{-1}{9}e^{-3t}\left(t + \frac{1}{3}\right)e^{3t}\begin{bmatrix}1\\2\end{bmatrix}\\ =& \frac{t^2}{6}\begin{bmatrix}-1\\1\end{bmatrix} + \frac{-1}{9}\left(t + \frac{1}{3}\right)\begin{bmatrix}1\\2\end{bmatrix}\end{aligned}$$

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2. (15pts) Let $A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$.

The characteristic polynomial of A is $r^2 - 4r + 5$ so the eigenvalues of A are $\lambda = 2 \pm i$.

Moreover, $\mathrm{v} = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$ is an eigenvector for $\lambda = 2 + i$.

a. Find the general solution to $D\mathrm{x} = A\mathrm{x}$.

Solution:

The complex solution to (H) is

$$e^{2t}(\cos t + i \sin t)\begin{bmatrix} 2-i \\ 5 \end{bmatrix} = e^{2t}\begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + ie^{2t}\begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix}$$

so the real and imaginary parts of this generate the general solution

$$X(t) = C_1 e^{2t}\begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix}.$$

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b. Solve the initial value problem $D\mathrm{x} = A\mathrm{x}, \quad \mathrm{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution:

The value at $t = 0$ of the general solution given above is

$$X(0) = C_1 e^0 \begin{bmatrix} 2\cos 0 + \sin 0 \\ 5\cos 0 \end{bmatrix} + C_2 e^0 \begin{bmatrix} -\cos 0 + 2\sin 0 \\ 5\sin 0 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix};$$

setting this equal to the desired initial condition yields the system of equations

$$\begin{aligned} 2C_1 - C_2 &= 1 \\ 5C_1 + 0C_2 &= 1 \end{aligned}$$

which can be solved by either reducing the augmented matrix

$$\begin{bmatrix} 2 & -1 & |1 \\ 5 & 0 & |1 \end{bmatrix}$$

or Cramer's Rule, or simply by noting that the second equation says $C_1 = \frac{1}{5}$, and substituting into the first equation yields $\frac{2}{5} - C_2 = 1$ or $C_2 = -\frac{3}{5}$.

Thus the desired solution of (H) is

$$X(t) = \frac{1}{5}e^{2t}\begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} - \frac{3}{4}e^{2t}\begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix} = e^{2t}\begin{bmatrix} \cos t - \sin t \\ \cos t - 3\sin t \end{bmatrix}.$$

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3. (15pts) Solve the initial value problem $(4D^2 - 4D + 1)x = 0, \quad x(2) = x'(2) = e$.

Solution:

The polynomial $4r^2 - 4r + 1$ has root $r = 1/2$ with multiplicity 2. Thus the general solution is given by

$$x(t) = c_1 e^{t/2} + c_2 t e^{t/2}.$$

Note that

$$\begin{aligned} x'(t) = D[x(t)] &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} D[t] \\ &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} (D + 1/2)[t] \\ &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} (1 + t/2) \end{aligned}$$

Now, we need

$$e = x(2) = c_1 e + 2c_2 e$$

and

$$e = x'(2) = \frac{1}{2} e c_1 + 2 e c_2$$

Thus we must solve the matrix equation

$$\begin{bmatrix} e & 2e \\ e/2 & 2e \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e \\ e \end{bmatrix}$$

This can be solved in several ways – e.g. by row operations on the augmented matrix, as follows:

$$\left[\begin{array}{cc|c} e & 2e & e \\ e/2 & 2e & e \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 4 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 1 \end{array}\right]$$

THus $c_1 = 0$ and $c_2 = 1/2$ so that

the solution to the initial value problem is given by

$$x(t) = \frac{t e^{t/2}}{2}.$$

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4. (15pts) Consider the matrix $B = \begin{bmatrix} 5 & -3 & -6 \\ 0 & 2 & 0 \\ 3 & -3 & -4 \end{bmatrix}$.

a. The vector $\mathrm{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector for B. What is the corresponding eigenvalue?

Hint: Compute the vector $B\mathrm{v}$ and compare with v.

Solution:

The product $B\mathrm{v}$ is equal to

$$B\mathrm{v} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2\mathrm{v}$$

so the eigenvalue is $\lambda = 2$.

b. Find an eigenvector for B for the eigenvalue $\lambda = -1$.

Solution: Perform row operations on the matrix $B - (-1)\mathrm{I}_3 = B + \mathrm{I}_3$:

$$\begin{bmatrix} 6 & -3 & -6 \\ 0 & 3 & 0 \\ 3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Considering this matrix, we see that an eigenvector for $\lambda = -1$ is given by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

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5. (15pts) Laplace Transforms:

a. Compute the inverse Laplace tranform $\mathcal{L}^{-1}[F(s)]$ of the function $F(s) = \dfrac{3s^2+s+1}{(s+1)(s^2+2)}$.

Solution:

The partial fraction decomposition has the form

$$\frac{3s^2+s+1}{(s+1)(s^2+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2};$$

combining over a common denominator and matching coefficients leads to

$$\begin{array}{lcccl} s^2 \text{ terms}: & A & +B & & = & 3 \\ s \text{ terms}: & & B & +C & = & 1 \\ \text{constant terms}: & 2A & & +C & = & 1 \end{array}$$

We can solve the first (respectively, second) equation for A (respectively, C) in terms of B:

$$\begin{aligned} A &= 3 - B \\ C &= 1 - B \end{aligned}$$

and substituting into the third equation yields

$$\begin{aligned} (6-2B) + (1-B) &= 1 \\ -3B &= -6 \\ B &= 2 \\ A &= 1 \\ C &= -1 \end{aligned}$$

so

$$\frac{3s^2+s+1}{(s+1)(s^2+2)} = \frac{1}{s+1} + \frac{2s-1}{s^2+2}.$$

Then the inverse transform is

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{3s^2+s+1}{(s+1)(s^2+2)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{2s}{s^2+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+2}\right] \\ &= e^{-t} + 2\cos t\sqrt{2} - \frac{1}{\sqrt{2}}\sin t\sqrt{2}. \end{aligned}$$

b. If x is a solution to $(D^2 + D + 1)x = 1$ with $x(0) = 0$ and $x'(0) = 1$, find an expression for $\mathcal{L}[x]$ as a function of s.

Solution:

By the first differentiation formula, applying the Laplace Transform to both sides of the problem yields

$$\begin{aligned}
\mathcal{L}D^2x + \mathcal{L}Dx + \mathcal{L}x &= \mathcal{L}1 \\
\{s^2\mathcal{L}x - sx(0) - x'(0)\} + \{s\mathcal{L}x - x(0)\} + \mathcal{L}x &= \mathcal{L}1 \\
s^2\mathcal{L}x - 1 + s\mathcal{L}x + \mathcal{L}x &= \frac{1}{s} \\
(s^2 + s + 1)\mathcal{L}x &= 1 + \frac{1}{s} = \frac{1+s}{s} \\
\mathcal{L}x &= \frac{1+s}{s(s^2+2+1)}
\end{aligned}$$

End of Exam