Carefully PRINT your full name:

Math 51 Differential Equations Spring 2022

Alternate Exam 2 (90 pts.+ 10 bonus pts)

You may not use calculators, books or notes during the exam. All electronic devices (including your phones) must be silenced and put away for the duration of the exam.

After finishing your exam, you will submit this exam booklet. We will scan your submission and upload it to Gradescope for marking (you do not need to take images of your exam). You should write your name at the top of each page, as indicated (especially if you remove the staples from your exam booklet).

For the partial credit problems, always show your work. Try to fit this work in the available space if possible.

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1. (16 points) True-false and multiple choice. Circle the correct choice.

   (a) Let $\mathbf{h}_1, \ldots, \mathbf{h}_n$ be n solutions of an order-n linear system $D\vec{x} = A\vec{x}$ on an interval I. Is it possible that the Wronkisan $W[\mathbf{h}_1, \ldots, \mathbf{h}_n](t)$ is 0 at one point t_0 of I but nonzero at another point t_1 of I?

      A. Yes

      B. No

   (b) True or False. An $n \times n$ real matrix must have n linearly independent eigenvectors, some of which may be complex.

   (c) True or False. Let A be a matrix of real numbers. The linear system $D\vec{x} = A\vec{x}$ must have n linearly independent real solutions.

   (d) True or False. Let A be a matrix of real numbers. If $\vec{x}$ is a complex solution of $D\vec{x} = A\vec{x}$, then both $\operatorname{Re}\vec{x}$ and $\operatorname{Im}\vec{x}$ are real solutions of $D\vec{x} = A\vec{x}$.

   (e) True or False. Five vectors in $\mathbb{R}^5$ are linearly independent if and only if they generate (span) $\mathbb{R}^5$.

   (f) True or False. Let A be an $n \times n$ matrix with an eigenvalue λ of multiplicity 3. An eigenvector corresponding to λ is also a generalized eigenvector.

   (g) True or False. Let A be an $n \times n$ real matrix. The general solution of $D\vec{x} = A\vec{x}$ can be generated by fewer than n solutions.

   (h) True or False. For every eigenvalue λ of an $n \times n$ matrix, there must be a corresponding eigenvector.

2. (8 pts) Write down an annihilator of smallest possible order with real coefficients for the function

$$4te^{3t} + t^2 e^{2t} \sin t.$$

3. (10 points) Make a *simplified* guess for a particular solution of the differential equation

$$(D+2)^7(D^2+1)^6 x = te^{-2t} + \cos t.$$

Do not solve for the coefficients.

4. (14 points)

(a) Convert the differential equation

$$x''' - t^2 x'' + \pi x' + tx = \sin t$$

into a linear system of three equations in three unknowns x_1, x_2, x_3.

(b) Write the linear system in the form $D\vec{x} = A(t)\vec{x} + \vec{E}(t)$ for some matrix $A(t)$ and vector $\vec{E}(t)$.

5. (15 points)

(a) (10 pts) The matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ has eigenvalue $\lambda = 2$ of multiplicity 2. It has an eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and a generalized eigenvector $\vec{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Using these two vectors, write down two solutions of $D\vec{x} = A\vec{x}$ that generate the general solution.

(b) (5 pts) Suppose we have found three solutions of a linear system $D\vec{x} = A\vec{x}$:

$$e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} t^2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right), \quad e^t \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} - t^2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right),$$

$$e^t \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{2} t^2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right).$$

Explain why these three solutions generate the general solution of $D\vec{x} = A\vec{x}$.

6. (12 points) Suppose $i, -i$ are eigenvalues of the 2×2 matrix A with corresponding eigenvectors $\begin{bmatrix} 1 \\ i \end{bmatrix}$, $\begin{bmatrix} 1 \\ -i \end{bmatrix}$, respectively. Write down two linearly independent real solutions of $D\vec{\mathbf{x}} = A\vec{\mathbf{x}}$. Show your work and simplify your answers.

7. (15 points) Find the general solution of the differential equation

$$5x'' - 10x' + 5x = t^{1/5}e^t,$$

given that two independent solutions of the related homogeneous equation are e^t and te^t.

(Continuation of Question 7)

(End of Exam)