Recitation worksheet
ODEs via integration

Math 51 Spring 2022

2022-01-27

Integrating factors: An alternative to "variation of parameters" for first-order linear ODEs.

This alternative is outlined briefly on (Nitecki and Guterman 1992, sec. 1.3, p. 28).

Consider the first order linear ODE

$$(\clubsuit) \quad \frac{dx}{dt} + r(t)x = q(t)$$

Put $\rho(t) = e^{R(t)}$ where $R(t) = \displaystyle\int r(t)dt$. If $x = x(t)$ is a solution to ($\clubsuit$), consider the function $u(t) = \rho(t)x(t)$.

The product rule shows that $\dfrac{du}{dt} = \dfrac{d}{dt}\left[\rho(t)x(t)\right] = \rho(t)x' + r(t)\rho(t)x$. Multiplying ($\clubsuit$) by $\rho(t)$ yields the equality

$$\rho(t)x' + r(t)\rho(t)x = \rho(t)q(t),$$

and we conclude that $u(t) = \rho(t)x(t)$ is a solution to the new ODE

$$(\diamondsuit) \quad \frac{du}{dt} = \rho(t)q(t).$$

We can now find all solutions $u(t)$ to ($\diamondsuit$) simply by integration:

$$u(t) = \int \rho(t)q(t)dt$$

and then we may recover the solutions $x(t)$ to ($\clubsuit$) simply by multiplying the solutions $u(t)$ by $\rho(t)^{-1} = 1/\rho(t)$.

The function $\rho(t)$ is called an *integrating factor*.

Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. *Differential Equations: A First Course*. Saunders.

1. Consider $x' + x/t = t^2$.

   a. Solve this equation using variation of parameters, and

   b. Solve this equation using integrating factors.

1. A pail of water at 20 degrees C is placed outside where the temperature is 10 degrees C.

   The water loses heat according to Newton's law of cooling, with constant of proportionality $\gamma = 1/10$ when time t is measure in hours. Suppose the temperature outdoors is decreasing at a constant rate of 3 degrees C per hour.

   a. Give a formula for the temperature outdoors after t hours.

   b. Write $x(t)$ of the temperature of the water in the bucket at time t. Use Newton's Law of Cooling to give a differential equation for which $x(t)$ is a solution.

   c. What is the temperature of the water at time t?

   d. What is the temperature of the water after 5 hours?