Math 51 Lecture 9

Homogeneous Linear Systems: Real and Complex Roots

Prob. Find the general solution of $D\vec{x} = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{bmatrix}.$$

Last time: If A has eigenvalue λ with eigenvector $\vec{v}$, then $\vec{x} = e^{\lambda t}\vec{v}$ is a sol with initial vector $\vec{v}$.

Recall. $\vec{h}_1(t), \vec{h}_2(t), \vec{h}_3(t)$ generate the general solution of a 3×3 system

iff $W[\vec{h}_1, \vec{h}_2, \vec{h}_3](t_0) \neq 0$ for any t_0.

iff $\det[\vec{h}_1(t_0)\ \vec{h}_2(t_0)\ \vec{h}_3(t_0)] \neq 0$ "

iff $\vec{h}_1(t_0), \vec{h}_2(t_0), \vec{h}_3(t_0)$ are linearly independent for any t_0.

Fact. The eigenvectors corresponding to distinct eigenvalues are lin. indep.

Fact. Suppose α, β are distinct eigenvalues,

$\vec{v}_1, \vec{v}_2$ are lin indep eigenvectors for α,

$\vec{v}_3, \vec{v}_4, \vec{v}_5$ " " " β.

Then $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$ are lin. indep.

Ex. The eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{bmatrix}$ are $1, 2, -3$

with eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -3 \\ 2 \\ 10 \end{bmatrix},$$

They are lin indep because they correspond to distinct eigenvalues.

General solution in vector form:

$$\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -3 \\ 2 \\ 10 \end{bmatrix}.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 e^{t} + c_2 e^{2t} - 3c_3 e^{-3t} \\ c_2 e^{2t} + 2c_3 e^{-3t} \\ 10 c_3 e^{-3t} \end{bmatrix}$$

General solution in terms of x_1, x_2, x_3:

$$\begin{aligned} x_1 &= c_1 e^{t} + c_2 e^{2t} - 3c_3 e^{-3t}, \\ x_2 &= c_2 e^{2t} + 2c_3 e^{-3t}, \\ x_3 &= 10 c_3 e^{-3t}. \end{aligned}$$

Goal: For each eigenvalue, find as many lin. indep. eigenvectors as you can.

Complex Eigenvalues of a Real Matrix

- The complex roots of a real polynomial come in conjugate pairs.
- The complex eigenvalues of a real matrix A come in conjugate pair.
- $\overline{zw} = \bar{z}\,\bar{w}$

1) If λ is a complex eigenvalue w/ eigenvector $\mathbf{v}$, then $\bar{\lambda}$ is an eigenvalue w/ e.vector $\bar{\mathbf{v}}$.

Reason: $A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}}$

$\Rightarrow \bar{A}\,\bar{\mathbf{v}} = \bar{\lambda}\,\bar{\mathbf{v}}$

$\Rightarrow A\bar{\mathbf{v}} = \bar{\lambda}\,\bar{\mathbf{v}}$ (A is real)

So $\bar{\mathbf{v}}$ is an eigenvector corresp. to $\bar{\lambda}$.

2) If $\mathbf{x}$ is a sol of $D\mathbf{x} = A\mathbf{x}$, then so $\bar{\mathbf{x}}$.

Reason. Let $\mathbf{x} = \mathbf{a} + i\mathbf{b}$.

$\overline{D\mathbf{x}} = \overline{(\mathbf{a}+i\mathbf{b})'} = \overline{\mathbf{a}' + i\mathbf{b}'}$

$= \mathbf{a}' - i\mathbf{b}' = (\mathbf{a}-i\mathbf{b})' = D\bar{\mathbf{x}}$.

$\overline{D\mathbf{x}} = \overline{A\mathbf{x}} \Rightarrow D\bar{\mathbf{x}} = \bar{A}\bar{\mathbf{x}} = A\bar{\mathbf{x}}$ (A is real)

$\Rightarrow \bar{\mathbf{x}}$ is a sol of $D\mathbf{x} = A\mathbf{x}$

3) $\boxed{\text{If } \mathbf{x} \text{ is a complex sol, then } \operatorname{Re}\mathbf{x} \text{ and } \operatorname{Im}\mathbf{x} \text{ are real sol.}}$

$\boxed{\text{Def. } \quad \operatorname{Re}(a+ib) = a, \quad \operatorname{Im}(a+ib) = b.}$

$$x = a + ib$$
$$\bar{x} = a - ib$$

Add: $x + \bar{x} = 2a$ $\qquad$ Subtract: $x - \bar{x} = 2ib$

So $a = \dfrac{\mathbf{x} + \bar{\mathbf{x}}}{2}$ $\qquad$ So $b = \dfrac{\mathbf{x} - \bar{\mathbf{x}}}{2i}$.

Since $\operatorname{Re}\bar{\mathbf{x}} = a$ and $\operatorname{Im}\mathbf{x} = b$ are lin comb of sol, they are also sol.

4) If $\mathbf{x} = e^{\lambda t}\mathbf{v}$ and λ not real, then $\mathbf{v}$ and $\bar{\mathbf{v}}$ are lin. indep. (because they are eigenvectors corresponding to distinct eigenvalues λ and $\bar{\lambda}$.)

5) If $\mathbf{x}$ is complex solutions, then $\operatorname{Re}\mathbf{x}$ and $\operatorname{Im}\mathbf{x}$ are lin. indep. real solutions.

(next chapter)

From Complex Solutions to Real Solutions $\qquad D\mathbf{x} = A\mathbf{x}$, $\quad A$ real

Suppose solutions are

$$\underbrace{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3}_{\text{real}}, \underbrace{\mathbf{x}_4, \bar{\mathbf{x}}_4, \mathbf{x}_5, \bar{\mathbf{x}}_5}_{\text{complex}}.$$

Then the real solutions are

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \operatorname{Re}\mathbf{x}_4, \operatorname{Im}\mathbf{x}_4, \operatorname{Re}\mathbf{x}_5, \operatorname{Im}\mathbf{x}_5.$$

Example. Solve $D\mathbf{x} = A\mathbf{x}$, $\quad A = \begin{bmatrix} -1 & -1 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Eigenvalues : $-1, \ -1 \pm i$

Eigenvectors : $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \ \underbrace{\begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}, \ \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}}_{\text{Conjugates}}$

Basis of complex solutions:

(A basis is a linear indep. generating set)

$$\underbrace{e^{-t}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}_{\text{real}}, \quad \underbrace{e^{(-1+i)t}\begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}}_{\text{Expand to find Re + Im parts}}, \quad \underbrace{e^{-(1-i)t}\begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}}_{\text{Do not expand this one.}}.$$

Do not expand this one. because it gives the same Re + Im parts as its conjugate.

Basis of real solutions

$$e^{(-1+i)t}\begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} = e^{-t}(\cos t + i \sin t)\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= e^{-t}\left(\cos t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$+ i e^{-t}\left(\begin{bmatrix} 0 \\ \cos t \\ 0 \end{bmatrix} + \begin{bmatrix} -\sin t \\ 0 \\ \sin t \end{bmatrix}\right)$$

$$= \underbrace{e^{-t}\begin{bmatrix} -\cos t \\ -\sin t \\ \cos t \end{bmatrix}}_{\text{Re}} + i \underbrace{e^{-t}\begin{bmatrix} -\sin t \\ \cos t \\ \sin t \end{bmatrix}}_{\text{Im}}$$

Real basis : $\quad e^{-t}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad e^{-t}\begin{bmatrix} -\cos t \\ -\sin t \\ \cos t \end{bmatrix}, \quad e^{-t}\begin{bmatrix} -\sin t \\ \cos t \\ \sin t \end{bmatrix}$