Problem Set 4
Linear Independence; Constant Coefficient Linear ODES (real roots)

Math 51 Fall 2021

due 2022-02-14 at 11:00 PM

Reminders

- Midterm 1 is February 14 in the open block – 12:00-1:20 PM.

These problems cover (Nitecki and Guterman 1992, secs. 2.4, 2.5)

1. Decide whether the indicated functions are linearly independent on the interval $(-\infty, \infty)$. If the functions are linearly independent, show that this is the case using the definition, or using the Wronskian test. To show that the functions $f_1(t), f_2(t), \ldots, f_n(t)$ are linearly dependent, you need to give explicit values $c_1, c_2, \cdots, c_n$ for which at least one c_i is non-zero and such that $0 = c_1 h_1(t) + c_2 h_2(t) + \cdots + c_n h_n(t)$ for every t.

   a. $h_1(t) = 1, \quad h_2(t) = t - 2, \quad h_3(t) = (t-2)^2$.

   Solution:

   The functions are linearly independent. Indeed, their Wronskian is given by

   $$W(1, t-2, (t-2)^2) = \det \begin{bmatrix} 1 & t-2 & (t-2)^2 \\ 0 & 1 & 2(t-2) \\ 0 & 0 & 2 \end{bmatrix} = 2.$$

   So for any t_0 – for example, for $t_0 = 0$ – we have $W(1, t-2, (t-2)^2)(0) = 2 \neq 0$.

   Thus the Wronskian test shows the functions to be linearly indep.

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   b. $h_1(t) = t^5, \quad h_2(t) = |t^5|$.

   Solution:

   Let's argue using the definition that the functions are linearly independent. Suppose that there are constants c_1, c_2 for which

   $$0 = c_1 t^5 + c_2 |t^5|$$

   When $t = 1$, we then find that $0 = c_1 \cdot 1^5 + c_2 \cdot |1^5| = c_1 + c_2$ and when $t = -1$ we find that $0 = c_1 \cdot (-1)^5 + c_2 \cdot |(-1)^5| = -c_1 + c_2$.

   Thus

   $$\begin{cases} 0 & = c_1 + c_2 \\ 0 & = -c_1 + c_2 \end{cases}$$

   Adding the equations, find that $2c_2 = 0$ so that $c_2 = 0$ and then also $c_1 = 0$.

   Since we have shown that $c_1 = c_2 = 0$, we have confirmed that h_1 and h_2 are linearly independent.

c. $h_1(t) = \sin^2(t) + 1, \quad h_2(t) = 2\cos^2(t), \quad h_3(t) = 10$

Solution:

A familiar trig identity says that $\sin^2(t) + \cos^2(t) = 1$.

Consider constants a, b, c and consider the expression

$$a(\sin^2(t) + 1) + b \cdot \cos^2(t) + c10 = a\sin^2(t) + 2b\cos^2(t) + 10c + a \quad (\clubsuit)$$

If we take $b = a/2$ we find that

$$(\clubsuit) = a\sin^2(t) + 2\frac{a}{2}\cos^2(t) + 10c + \frac{a}{2} = a(\sin^2(t) + \cos^2(t)) + 10c + \frac{a}{2} = \frac{3a}{2} + 10c$$

If we take $a = 2$, $b = \dfrac{a}{2} = 1$, and $c = \dfrac{-3}{10}$ we see that

$(\clubsuit) = 0$. This shows that

$2h_1(t) + h_2(t) - \dfrac{3}{10}h_3(t) = 0$

so that h_1, h_2, h_3 are linearly dependent.

d. $h_1(t) = e^t, \quad h_2(t) = e^{t+1}, \quad h_3(t) = 1.$

Solution:

These functions are linearly dependent. Indeed, notice that $h_2(t) = e^{t+1} = e \cdot e^t = e \cdot h_1(t)$. Thus if we take $c_1 = e, c_2 = -1, c_3 = 0$, we find that

$$c_1h_1(t) + c_2h_2(t) + c_3h_3(t) = e \cdot h_1(t) - h_2(t) + 0 \cdot h_3(t) = e \cdot e^t - e \cdot e^t = 0;$$

since c_1, c_2, c_3 are not all zero, we have now showed the functions to be linearly dependent.

2. Suppose that $h_1(t), h_2(t), h_3(t)$ are linearly dependent. Show that $h_1'(t), h_2'(t), h_3'(t)$ are linearly dependent, as well.

   You will need to use the definition of linear dependence to make this argument.

   Solution:

   We suppose that h_1, h_2, h_3 are linearly dependent. Thus there are numbers c_1, c_2, c_3 with $c_i \neq 0$ for at least one i for which

   $$0 = c_1h_1(t) + c_2h_2(t) + c_3h_3(t).$$

   Now apply the differentiation operator $D = \dfrac{d}{dt}$ to each side of this equation to find that

   $$0 = D[0] = D[c_1h_1(t) + c_2h_2(t) + c_3h_3(t)] = c_1D[h_1(t)] + c_2D[h_2(t)] + c_3D[h_3(t)] = c_1h_1'(t) + c_2h_2'(t) + c_3h_3'(t)$$

   This last equation establishes the linear dependence of h_1', h_2', h_3'.

3. Find the general solution of each of the following ODEs:

   a. $(D^2-2)(D+4)^2x=0$

   Solution:

   The characteristic polynomial has roots $\pm\sqrt{2}$ each with multiplicity 1 and -4 with multiplicity 2. The ODE thus has solutions

$$e^{\sqrt{2}t}, \quad e^{-\sqrt{2}t}, \quad e^{-4t}, \quad te^{-4t}$$

   A Theorem in the book/lecture tells us that these solutions are linearly independent. Since the ODE is of order 4, they generate the general solution.

   In other words, the general solution is given by

$$x = c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} + c_3e^{-4t} + c_4te^{-4t}.$$

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   b. $D(D^2-4)^2x=0$.

   Solution:

   The characteristic polynomial $r(r^2-4)^2$ has roots 2 and -2 each with multiplicity 2 together with the root 0 of multiplicity 1.

   Thus we find the following five solutions to the 5th order ODE:

$$e^{2t}, \quad te^{2t}, \quad e^{-2t}, \quad te^{-2t}, \quad e^{0t}=1.$$

   Since we know these functions to be linearly independent, they generate the general solution, which is

$$x(t) = c_1e^{2t} + c_2te^{2t} + c_3e^{-2t} + c_4te^{-2t} + c_5.$$

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   c. $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} - 4x = 0.$

   Solution:

   Rewriting the equation as

$$(D^2-2D-4)x=0,$$

   we see that the characteristic polynomial is r^2-2r-4.

   Using the quadratic formula, we find that this polynomial has roots

$$\frac{2\pm\sqrt{4+16}}{2} = 1\pm\sqrt{5}.$$

   Thus the general solution is generated by

$$e^{(1+\sqrt{5})t} \quad \text{and} \quad e^{(1-\sqrt{5})t}.$$

   i.e. the general solution is

$$x(t) = Ae^{(1+\sqrt{5})t} + Be^{(1-\sqrt{5})t}.$$

4. Solve the initial value problem

$$(D+2)^2 Dx = 0; \quad x(0) = x'(0) = 1, \quad x''(0) = 0.$$

Solution:

The char. poly $(r+2)^2 r$ has root -2 with multiplicity 2 and root 0 with multiplicity one. Thus the general solution has the form

$$x(t) = Ae^{-2t} + Bte^{-2t} + C.$$

Note that

$$x'(t) = -2Ae^{-2t} + B(1-2t)e^{-2t}$$

and

$$x''(t) = 4Ae^{-2t} + B(-4+4t)e^{-2t}.$$

Now we need

$$\begin{cases} 1 & = x(0) & = A + C \\ 1 & = x'(0) & = -2A + B \\ 0 & = x''(0) & = 4A - 4B \end{cases}$$

The third equation tells us that $A = B$. Thus the second equation says that $1 = -2A + A = -A$ so that $A = B = -1$.

Finally, the first equation now shows that $1 = -1 + C$ and hence that $C = 2$.

We now conclude that the solution to the initial value problem is

$$x(t) = -e^{-2t} - te^{-2t} + 2.$$

5. Use the exponential shift formula (see the reminder below) to compute the function $Lf = L[f]$ in each case:

   a. $L = D^2 + D - 1, \quad f(t) = e^t \sin(t)$

   Solution:

   Using the exponential shift formula, we find that

$$\begin{aligned}(D^2 + D - 1)\left[e^t \sin(t)\right] &= e^t \left((D+1)^2 + (D+1) - 1\right)[\sin(t)] \\ &= e^t \left(D^2 + 2D + 1 + D + 1 - 1\right)[\sin(t)] \\ &= e^t \left(D^2 + 3D + 1\right)[\sin(t)] \\ &= e^t \left(-\sin(t) + 3\cos(t) + \sin(t)\right) \\ &= 3e^t \cos(t)\end{aligned}$$

   b. $L = (D-1)(D^2 + D + 1), \quad f(t) = te^{2t}$.

   Solution:

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\begin{aligned}
(D-1)(D^2+D+1)[te^{2t}] &= e^{2t}(D+2-1)((D+2)^2+(D+2)+1)[t] \\
&= e^{2t}(D+1)(D^2+4D+4+(D+2)+1)[t] \\
&= e^{2t}(D+1)(D^2+5D+7)[t] \\
&= e^{2t}(D+1)[5+7t] \\
&= e^{2t}[7+(5+7t)] = e^{2t}[12+7t]
\end{aligned}
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Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. Differential Equations: A First Course. Saunders.