Problem Set 11
Non-homogeneous systems

Math 51 Spring 2022

2022-04-11 – because of Midterm 2, this assignment won't be collected

These problems concern (Nitecki and Guterman 1992, sec. 3.11).

1. The matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$ has eigenvalues $\lambda = 1 + \sqrt{3}$, $\mu = 1 - \sqrt{3}$, and the general solution to $D\mathbf{x} = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 e^{\lambda t} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + c_2 e^{\mu t} \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}.$$

   a. Find the general solution to the inhomogeneous equation $D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   Solution:

   We must find a *particular solution*

$$\mathbf{p}(t) = c_1(t)\mathbf{h}_1(t) + c_2(t)\mathbf{h}_2.$$

   We know that $\mathbf{c}' = \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix}$ is a solution to (♣) $\quad W\mathbf{c}' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where W is the *Wronskian matrix*

$$W = \begin{bmatrix} e^{\lambda t} & -e^{\mu t} \\ \sqrt{3}e^{\lambda t} & \sqrt{3}e^{\mu t} \end{bmatrix}$$

   We solve (♣) using *Cramer's Rule*. First note that

$$\det W = 2\sqrt{3}e^{(\lambda+\mu)t} = 2\sqrt{3}e^{2t}$$

   since $\lambda + \mu = 2$.

   Thus Cramer's Rule gives

$$c_1'(t) = \frac{\det \begin{bmatrix} 1 & -e^{\mu t} \\ 1 & \sqrt{3}e^{\mu t} \end{bmatrix}}{2\sqrt{3}e^{2t}} = \frac{(1+\sqrt{3})e^{\mu t}}{2\sqrt{3}e^{2t}} = \frac{\lambda}{2\sqrt{3}}e^{(\mu-2)t} = \frac{\lambda}{2\sqrt{3}}e^{-\lambda t}$$

   since $\mu - 2 = -\lambda$.

   Similarly,

$$c_2'(t) = \frac{\det \begin{bmatrix} e^{\lambda t} & 1 \\ \sqrt{3}e^{\lambda t} & 1 \end{bmatrix}}{2\sqrt{3}e^{2t}} = \frac{(1-\sqrt{3})e^{\lambda t}}{2\sqrt{3}e^{2t}} = \frac{\mu}{2\sqrt{3}}e^{(\lambda-2)t} = \frac{\mu}{2\sqrt{3}}e^{-\mu t}$$

   Integration now gives

$$c_1(t) = \int c_1'(t)dt = \frac{\lambda}{2\sqrt{3}} \int e^{-\lambda t} dt = \frac{-1}{2\sqrt{3}} e^{-\lambda t} + C_1$$

and

$$c_2(t) = \int c_2'(t)dt = \frac{\mu}{2\sqrt{3}} \int e^{-\mu t} dt = \frac{-1}{2\sqrt{3}} e^{-\mu t} + C_2$$

To find our particular solution we may take $C_1 = C_2 = 0$ and we find

$$\begin{aligned}
\mathbf{p}(t) &= c_1(t)\mathbf{h_1}(t) + c_1(t)\mathbf{h_1}(t) \\
&= c_1(t)e^{\lambda t}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + c_2(t)e^{\mu t}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \\
&= \frac{-1}{2\sqrt{3}} e^{-\lambda t}e^{\lambda t}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \frac{-1}{2\sqrt{3}} e^{-\mu t}e^{\mu t}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \\
&= \frac{-1}{2\sqrt{3}} \left(\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \right) \\
&= \frac{-1}{2\sqrt{3}} \begin{bmatrix} 0 \\ 2\sqrt{3} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\end{aligned}$$

Now we see that the general solution to $D\mathbf{x} = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = \mathbf{p}(t) + c_1\mathbf{h_1}(t) + c_1\mathbf{h_1}(t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_1e^{\lambda t}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + c_2e^{\mu t}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$$

---

b. Solve the initial value problem $D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution:

Using the solution found in (a), we must find c_1, c_2 for which

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_1e^{\lambda t}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + c_2e^{\mu t}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \right)\Bigg|_{t=0} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_1\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + c_2\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$$

Thus we need

$$\begin{bmatrix} 1 & -1 \\ \sqrt{3} & \sqrt{3} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To solve this equation, we perform row operations on the augmented matrix:

$$\begin{aligned}
\left[\begin{array}{cc|c} 1 & -1 & 1 \\ \sqrt{3} & \sqrt{3} & 1 \end{array}\right] &\sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 3 & 3 & \sqrt{3} \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 6 & -3+\sqrt{3} \end{array}\right] \\
&\sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & (-3+\sqrt{3})/6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & (3+\sqrt{3})/6 \\ 0 & 1 & (-3+\sqrt{3})/6 \end{array}\right]
\end{aligned}$$

Thus $c_1 = (3+\sqrt{3})/6$ and $c_2 = (-3+\sqrt{3})/6$ and the solution to the initial value problem is

$$\mathbf{x}(t) = \mathbf{p}(t) + \frac{(3+\sqrt{3})}{6}\mathbf{h}_1(t) + \frac{(-3+\sqrt{3})}{6}\mathbf{h}_2(t)$$

---

2. The matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has characteristic polynomial $\lambda^2 - 2\lambda$, and thus has eigenvalues $\lambda = 0, 2$.

   The general solution to $(H) \quad D\mathbf{x} = A\mathbf{x}$ is given by

$$\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   Find the general solution to the inhomogeneous equation $D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}$.

   Solution:

   We must find a particular solution

$$\mathbf{p}(t) = c_1(t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2(t) e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   If $\mathbf{c} = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$ then we know that $\mathbf{c}'$ is a solution to $W\mathbf{c}' = \begin{bmatrix} 0 \\ t \end{bmatrix}$ where W is the *Wronskian matrix*

$$W = \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix}$$

   We find $\mathbf{c}'$ using *Cramer's Rule*. First note that $\det W = -2e^{2t}$.

   Next, *Cramer's Rule* gives

$$c_1'(t) = \frac{\det \begin{bmatrix} 0 & e^{2t} \\ t & e^{2t} \end{bmatrix}}{-2e^{2t}} = \frac{-te^{2t}}{-2e^{2t}} = \frac{t}{2}$$

$$c_2'(t) = \frac{\det \begin{bmatrix} -1 & 0 \\ 1 & t \end{bmatrix}}{-2e^{2t}} = \frac{-t}{-2e^{2t}} = \frac{te^{-2t}}{2}$$

   Now integration gives

$$c_1(t) = \int c_1'(t)dt = \int \frac{t}{2} dt = \frac{t^2}{4} + C_1$$

   and integrating by parts we get

$$c_2(t) = \int c_2'(t)dt = \int \frac{te^{-2t}}{2} dt = \frac{-1}{8} e^{-2t}(2t+1) + C_2$$

   In forming the particular solution we may take $C_1 = C_2 = 0$, and we find

$$\mathbf{p}(t) = \frac{t^2}{4} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{-1}{8} e^{-2t}(2t+1)e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{t^2}{4} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{-1}{8}(2t+1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   The general solution is now given by

$$\mathbf{x}(t) = \mathbf{p}(t) + c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. *Differential Equations: A First Course*. Saunders.