Math 51 Lecture 1

Integration, Separation of Variables

A differential equation is an equation involving derivatives.

1. Integration

Example. Suppose the acceleration of a particle is

$$a(t) = -\frac{2t}{(t^2+1)^2},$$

What is the position $x(t)$ at time t if $x(0)=0$ and $x'(0)=0$?

The d.e. is $x''(t) = -\frac{2t}{(t^2+1)^2}$.

If

$$x^{(n)}(t) = \boxed{\frac{d^n x}{dt^n} = f(t)},$$

then x is obtained by integrating $f(t)$ n times.

Solution. $x'(t) = \int -\frac{2t}{(t^2+1)^2}\,dt$ (u-substitution: Let $u = t^2+1$. Then $du = 2t\,dt$.)

$$= \int -\frac{du}{u^2}$$

$$= \frac{1}{u} + C_1$$

$$= \frac{1}{t^2+1} + C_1. \qquad (1)$$

To find $x(t)$, integrate its derivative:

$$x(t) = \int \frac{1}{t^2+1} + C_1\,dt$$

$$= \boxed{\arctan t + C_1 t + C_2}. \qquad (2)$$

(This is called the general solution of the o.d.e.

It describes all possible solutions and involves undetermined constants.)

To determine the constants C_1 and C_2, we need to use the two initial conditions.

Since $x'(0) = 0$, by (1),

$$x'(0) = \frac{1}{0^2+1} + C_1 = 1 + C_1 = 0.$$

Therefore,

$$C_1 = -1.$$

Since $x(0) = 0$, by (2),

$$x(0) = \arctan 0 + (-1)\cdot 0 + C_2 = 0$$

Although $0, \pm\pi, \pm 2\pi, \ldots$ all have tan equal to 0, by definition,

$$-\frac{\pi}{2} < \arctan t < \frac{\pi}{2},$$

Therefore, $\arctan 0 = 0$ and $C_2 = 0$.

Finally, $x(t) = \arctan t - t$ is the <u>specific solution</u> that satisfies the o.d.e. and the two initial conditions. (A specific solution does not involve any constant.)

2. <u>Separation of Variables</u>

<u>Example</u> Suppose the temperature $x(t)$ increases $.1\%$ a year.

Find the temperature $x(t)$ in the year t. (t is a real number.)

Reinterpretation:

The rate of change of x w.r.t. t is $.001$ of x, i.e.,

$$x'(t) = \boxed{\frac{dx}{dt} = .001\,x}.$$

<u>Sol</u>. Bring all x to one side & all t to the other side:

<u>Case</u> 1. $x \neq 0$. Can divide by x:

$$\frac{dx}{x} = .001\,dt.$$

Integrate both sides:

$$\int \frac{1}{x}\,dx = \int .001\,dt.$$

$$\ln|x| = .001t + C.$$

$$|x| = e^{\ln|x|} = e^{.001t + C} = e^{.001t}e^{C}$$

$$= k e^{.001t} \quad (\text{where } k = e^{C} > 0).$$

$$\boxed{x = \pm k e^{.001t} \quad (k > 0)}.$$

<u>Case</u> 2. $x = 0$

The d.e. becomes $\frac{dx}{dt} = 0$.

$\boxed{x = 0}$ is a solution of this d.e.

The sol of the orig. eq. is $\pm k e^{.001t}$ or 0, $k > 0$.

A better way of describing the sol is

$$\boxed{x(t) = k e^{.001t}, \text{ where } k \text{ is any real number.}}$$ $\square$

In general, if a first-order differential equation is of the form

$$\frac{dx}{dt} = f(x)\, g(t),$$

then there are two cases:

Case 1. $f(x) \neq 0$.

In this case one can divide by $f(x)$ and separate the variables:

$$\frac{dx}{f(x)} = g(t)\, dt.$$

Integrate both sides: $\displaystyle\int \frac{dx}{f(x)} = \int g(t)\, dt$

and solve for x.

Case 2. $f(x) = 0$.

In this case, the d.e. becomes

$$\frac{dx}{dt} = 0 \cdot g(t) = 0.$$

If the number r is a root of $f(x)$ so that $f(r) = 0$, then the constant function $x = r$ satisfies $\frac{dx}{dt} = 0$. Therefore, the roots of $f(x)$ are also solutions of the d.e.

In summary, a first-order d.e. of the form $dx/dt = f(x)\, g(t)$ has two sets of solutions, one set coming from separation of variables and the other set consists of constant functions coming from the roots of $f(x)$.

Why Does Separation of Variables Work?

$\frac{dx}{dt}$ is a notation for the derivative, not a fraction.

In $\int \sin x \, dx$, dx is a notation, not a real thing.

How does one justify $\frac{dx}{f(x)} = g(t)\, dt$?

Write the d.e. as

$$x'(t) = f(x)\, g(t)$$

Then

$$\frac{x'(t)}{f(x(t))} = g(t).$$

Since both sides are functions of t, we can integrate with respect to t:

$$\int \frac{1}{f(x(t))} x'(t)\, dt = \int g(t)\, dt \qquad (3)$$

By the change of variables formula, if $x = x(t)$, then

$$\int \frac{1}{f(x)}\, dx = \int \frac{1}{f(x(t))} x'(t)\, dt$$

$$= \int g(t)\, dt \quad \text{by (3).}$$

Hence,

$$\boxed{\int \frac{1}{f(x)}\, dx = \int g(t)\, dt}$$

This derivation did not treat $x'(t)$ as a fraction, but it arrives at the same result as separation of variables. For this reason, separation of variables is a legitimate technique.