# Review material for Midterm 2

Math 51 Spring 2022

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## Problems

1. Solve the initial value problem $(D^2 - 6D + 10)x = 0$, $x(0) = x'(0) = 1$.

   **Solution:**

   Note first that the polynomial $r^2 - 6r + 10$ has roots $\lambda = 3 \pm i$.

   Thus we see that $h_1(t) = e^{3t}\cos(t)$ and $h_2(t) = e^{3t}\sin(t)$ generate the general solution to the homogeneous ODE.

   We seek a solution of the form

   $$x(t) = c_1 h_1(t) + c_2 h_2(t) = c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t)$$

   for constants $c_1, c_2$.

   Note that $x'(t) = c_1 e^{3t}(-\sin(t) + 3\cos(t) + c_2 e^{3t}(\cos(t) + 3\sin(t))$.

   We need the conditions $1 = x(0) = c_1$ and $1 = x'(0) = 3c_1 + c_2$.

   Thus $c_1 = 1$ and $c_2 = -2$ so the required solution is

   $$x(t) = e^{3t}\cos(t) - 2e^{3t}\sin(t) = e^{3t}(\cos(t) - 2\sin(t))$$

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2. Consider the linear ODE $(D^2 - 4)x = e^{2t} + e^{3t}$.

   a. Find a *simplified guess* for a particular solution $p(t)$ for the ODE.

      **Solution:**

      We must first find an operator $A(D)$ for which $A(D)[e^{2t} = e^{3t}] = 0$. For this, we may take $A(D) = (D-2)(D-3)$.

      Now, any solution to the given ODE will be a solution to the homogeneous ODE $A(D) \cdot (D^2 - 4)x = 0$, i.e. to

      $$(D-2)^2(D-3)(D+2)x = 0.$$

      This homogeneous ODE has solutions

      $$e^{2t}, te^{2t}, e^{3t}, e^{-2t}.$$

      Of these,$e^{2t}$ and $e^{-2t}$ are solutions to the homogeneous equation $(D^2 - 4)x = 0$ corresponding to the original ODE, so we may eliminate these solutions.

      Thus our simplified guess is given by

$$p(t) = k_1 te^{2t} + k_2 e^{3t}$$

for constants $k_1, k_2$.

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b. Use the method of *undetermined coefficients* to find $p(t)$.

**Solution:**

We need $(D^2 - 4)[p(t)] = e^{2t} + e^{3t}$.

We first compute

$$(D^2 - 4)[te^{2t}] = e^{2t}((D+2)^2 - 4)[t] = e^{2t}(D^2 + 4D)[t] = 4e^{2t}$$

and

$$(D^2 - 4)[e^{3t}] = e^{3t}((D+3)^2 - 4)[1] = e^{3t}(D^2 + 6D + 5)[1] = 5e^{3t}$$

With these computations, we now find

$$(D^2 - 4)[k_1 te^{2t} + k_2 e^{3t}] = k_1 4e^{2t} + k_2 5e^{3t}.$$

Thus we *require* that

$$k_1 4e^{2t} + k_2 5e^{3t} = e^{2t} + e^{3t}.$$

Since the functions $te^{2t}$ and $e^{3t}$ are linearly independent we find that

$$4k_1 = 1 \quad \text{and} \quad 5k_2 = 1.$$

Thus $k_1 = 1/4$ and $k_2 = 1/5$ so that

$$p(t) = \frac{te^{2t}}{4} + \frac{e^{3t}}{5}.$$

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3. Use the method of *variation of parameters* to find the general solution to the linear ODE

$$(D^2 + 4)x = \frac{1}{\sin(2t)}.$$

**Solution:**

The general solution to the homog. equation $(D^2 + 4)x = 0$ is generated by $\cos(2t)$ and $\sin(2t)$.

The Wronskian matrix is given by

$$W = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}.$$

Note that $\det W = 2\cos^2(2t) + 2\sin^2(2t) = 2$.

Now, we seek a particular solution of the form

$$p(t) = c_1(t)\cos(2t) + c_2(t)\sin(2t)$$

where $\mathbf{c}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$ satisfies the matrix equation

$$W\mathbf{c}'(t) = \begin{bmatrix} 0 \\ 1/\sin(2t) \end{bmatrix}.$$

Now, Cramer's Rule shows that

$$c_1'(t) = \frac{\det \begin{bmatrix} 0 & \sin(2t) \\ 1/\sin(2t) & 2\cos(2t) \end{bmatrix}}{2} = 1/2$$

and

$$c_2'(t) = \frac{\det \begin{bmatrix} \cos(2t) & 0 \\ -2\sin(2t) & 1/\sin(2t) \end{bmatrix}}{2} = \frac{\cot(2t)}{2}$$

We now find $c_1(t) and $c_2(t)$ by anti-differentiation. Remember that we may choose any antiderivative. We find $c_1(t) = \frac{t}{2}$ and $c_2(t) = \frac{1}{4}\ln(|\sin(2t)|)$.

Thus our particular solution is given by

$$p(t) = \frac{t\cos(2t)}{2} + \frac{1}{4}\ln(|\sin(2t)|)\sin(2t)$$

and the general solution is then

$$x(t) = \frac{t\cos(2t)}{2} + \frac{1}{4}\ln(|\sin(2t)|)\sin(2t) + c_1\cos(2t) + c_2\sin(2t).$$

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4. Consider the linear ODE $(\diamondsuit) \qquad (D^3 + D)x = e^t.$

   a. Find a matrix $A$ for which

$$(\heartsuit) \qquad D\vec{x} = A\vec{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}$$

   is an *equivalent linear system* to $(\diamondsuit)$.

   **Solution:**

   Note that if $x$ is a solution to $(\diamondsuit)$, then $x''' = -x' + e^t$.

   Setting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ x' \\ x'' \end{bmatrix}$ we find that

$$D\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

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   b. If $u(t)$ is a solution to $(\diamondsuit)$, explain why $\vec{v}(t) = \begin{bmatrix} u(t) \\ u'(t) \\ u''(t) \end{bmatrix}$ is a solution to $(\heartsuit)$.

   **Solution:**

   If $\mathbf{v} = \begin{bmatrix} u(t) \\ u'(t) \\ u''(t) \end{bmatrix}$ then

$$D\mathbf{v} = \begin{bmatrix} u'(t) \\ u''(t) \\ u'''(t) \end{bmatrix} = \begin{bmatrix} u'(t) \\ u''(t) \\ -u'(t) + e^t \end{bmatrix} = \begin{bmatrix} u'(t) \\ u''(t) \\ -u'(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}$$

On the other hand

$$A\begin{bmatrix}u(t)\\u'(t)\\u''(t)\end{bmatrix}+\begin{bmatrix}0\\0\\e^t\end{bmatrix}=\begin{bmatrix}u'(t)\\u'''(t)\\-u'(t)\end{bmatrix}+\begin{bmatrix}0\\0\\e^t\end{bmatrix}$$

Since these two expressions coincide, $\mathbf{v}$ is indeed a solution.

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5. Find the *real* and *imaginary* parts of the vector $(\cos(t)+i\sin(t))\cdot\begin{bmatrix}2+i\\1\\i\end{bmatrix}$ for any value of $t$.

**Solution:**

First notice that

$$(\cos(t)+i\sin(t))(2+i)=2\cos(t)-\sin(t)+i(\cos(t)+2\sin(t))$$

Thus

$$(\cos(t)+i\sin(t))\cdot\begin{bmatrix}2+i\\1\\i\end{bmatrix}=\begin{bmatrix}2\cos(t)-\sin(t)+i(\cos(t)+2\sin(t))\\\cos(t)+i\sin(t)\\-\sin(t)+i\cos(t)\end{bmatrix}=\begin{bmatrix}2\cos(t)-\sin(t)\\\cos(t)\\-\sin(t)\end{bmatrix}+i\begin{bmatrix}\cos(t)+2\sin(t)\\\sin(t)\\\cos(t)\end{bmatrix}$$

So the real part is $\begin{bmatrix}2\cos(t)-\sin(t)\\\cos(t)\\-\sin(t)\end{bmatrix}$ and the imaginary part is $\begin{bmatrix}\cos(t)+2\sin(t)\\\sin(t)\\\cos(t)\end{bmatrix}$

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6. Let $A=\begin{bmatrix}2&2&1\\0&2&0\\0&0&-1\end{bmatrix}$.

a. Find the characteristic polynomial and show that eigenvalues of $A$ are $\lambda=2,-1$.

**Solution:**

Since $A$ is upper triangular, $A-\lambda\mathbf{I}_3$ is also upper triangular. Thus, the char poly – i.e. the determinant of $A-\lambda\mathbf{I}_3$ – is equal to $(2-\lambda)^2(1-\lambda)$

Thus the eigenvalues of $A$ are as indicated.

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b. Find two linearly independent generalized eigenvectors for $\lambda=2$. Note that there are not 2 linearly independent *eigenvectors* for $\lambda=2$.

**Solution:**

Examining the matrix

$$A-2\mathbf{I}_3=\begin{bmatrix}0&2&1\\0&0&0\\0&0&-3\end{bmatrix}\sim\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

we see that $\mathbf{e}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$ is the only eigenvector for $\lambda=2$.

On the other hand, consider

$$(A-2\mathbf{I}_2)^2=\begin{bmatrix}0&2&1\\0&0&0\\0&0&-3\end{bmatrix}^2=\begin{bmatrix}0&0&-3\\0&0&0\\0&0&9\end{bmatrix}\sim\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

Recall that the generalized eigenvectors are the solutions to $(A-2\mathbf{I}_2)^2\mathbf{v}=\mathbf{0}$. Thus, in this case the generalized eigenvectors are generated by the two linearly independent vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

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c. Find the general solution to the linear system $D\vec{x} = A\vec{x}$.

**Solution:**

Using the generalized eigenvectors from b. we find solutions

$$h_1(t) = e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

and

$$\begin{aligned} h_2(t) &= e^{2t}\left(\begin{bmatrix}0\\1\\0\end{bmatrix} + t(A-2\mathbf{I}_3)\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = e^{2t}\left(\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}0&2&1\\0&0&0\\0&0&-3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \\ &= e^{2t}\left(\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\0\\0\end{bmatrix}\right) \\ &= e^{2t}\begin{bmatrix}2t\\1\\0\end{bmatrix} \end{aligned}$$

It remains to find an eigenvector for $\lambda = -1$.

For this, we look at

$$A+\mathbf{I}_3 = \begin{bmatrix}3&2&1\\0&3&0\\0&0&0\end{bmatrix} \sim \begin{bmatrix}3&0&1\\0&1&0\\0&0&0\end{bmatrix}$$

and find the eigenvector $\mathbf{w} = \begin{bmatrix}-1\\0\\3\end{bmatrix}$.

Thus we find the solution

$$\mathbf{h}_3(t) = e^{-t}\mathbf{w} = e^{-t}\begin{bmatrix}-1\\0\\3\end{bmatrix}.$$

So the general solution is

$$\mathbf{x}(t) = c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t) + c_3\mathbf{h}_3(t) = c_1e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2e^{2t}\begin{bmatrix}2t\\1\\0\end{bmatrix} + c_3e^{-t}\begin{bmatrix}-1\\0\\3\end{bmatrix}$$

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7. Let $B = \begin{bmatrix}1&-1&0&1&1\\0&2&-1&2&0\\0&0&0&1&3\\0&0&0&0&0\end{bmatrix}$

a. Find all solutions to the matrix equation $B\vec{v} = \vec{0}$.

**Solution:**

The matrix is (essentially) already in echelon form, and we see that the variables $v_1$, $v_2$, and $v_4$ are *pivot variables* while $v_3$ and $v_5$ are *free variables.*

We find one solution by setting $v_3 = 1$ and $v_5 = 0$: we find in this case that $v_4 = 0$, $2v_2 = v_3$, and $v_1 = v_2$. Thus one solution is

$$\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We find another solution by setting $v_3 = 0$ and $v_5 = 1$. In this case we find that $v_4 = -3v_5 = -3$, $2v_2 = -2v_4 = 6$ and $v_1 = v_2 - v_4 - v_5 = 3 + 3 - 1$. Thus the other solution is

$$\begin{bmatrix} 5 \\ 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

Thus every solution of $B\mathbf{v} = \mathbf{0}$ has the form

$$v_3 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_5 \begin{bmatrix} 5 \\ 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

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b. Let $\vec{b}_1, \ldots, \vec{b}_5$ denote the *columns* of the matrix $B$. Thus the $\vec{b}_i$ are vectors in $\mathbf{R}^4$. Are these vectors linearly independent? Why or why not?

**Solution:**

Since $B \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$, we see that

$$(1/2)\mathbf{b}_1 + (1/2)\mathbf{b_2} + \mathbf{b}_3 = 0$$

which shows that the vectors are *linearly dependent* (i.e. *not* linearly independent).

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8. Find the solution $\vec{h}(t)$ to the homogeneous system

$$D\vec{x} = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

which satisfies $\vec{h}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. What is $\vec{h}(1)$?

**Solution:**

The char. poly is $-(\lambda - 2)^2(\lambda - 1)$.

An eigenvector for $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = 2$ are generated by

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

.

Thus the general solution is

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

In order that $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, we need

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

To solve this equation, we perform row operations on the augment matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

This shows that $c_1 = 2$, $c_2 = -1$ and $c_3 = 0$ so the required solution is

$$\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + -e^{2t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

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9. Suppose that the $3 \times 3$ matrix $A$ has eigenvalues 2 and $1 \pm 3i$, that $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is an eigenvector for $\lambda = 2$, and that $\mathbf{w} = \begin{bmatrix} u_1 + w_1 i \\ u_2 + w_2 i \\ u_3 + w_3 i \end{bmatrix}$ is an eigenvector for $\lambda = 1 + 3i$.

Describe three real solutions to the homogeneous system of linear ODES $D\mathbf{x} = A\mathbf{x}$ with linearly independent initial vectors.

**Solution:**

One real solution is determined by the eigenvector $\mathbf{v}$; it is given by

$$\mathbf{h}_1 = e^{2t} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

For the other two solutions, we use the eigenvector $\mathbf{w}$. We consider the "complex solution"

$$e^{(1+3i)t}\mathbf{w} = e^t(\cos(3t) + i\sin(3t))\mathbf{w} \quad (*)$$

We take for $\mathbf{h}_2(t)$ the *real part* of $(*)$, and we take $\mathbf{h}_3(t)$ to be the *imaginary part* of $(*)$.

This gives the required 3 solutions.

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## Bibliography