Problem Set 1
ODEs via integration

Math 51 Fall 2021

due 2022-01-23 at 5:00 PM

Reminders on the rubric:

1. A correct answer does not guarantee credit.

   Your goal in answering a problem should be to convince the grader of comprehension. In short: Show your work!

2. Use your words!

   Many problems in this course can be answered purely in math terms. While that may at times be sufficient, it is often helpful to use words between steps and to use words to justify choices made in solving a problem.

3. Legibility of work is important.

   It is important that the graders can clearly read the work. If work cannot be read or easily followed, then the grader may not be convinced of complete understanding of the material and points may be lost.

Problems

These problems relate to material of (Nitecki and Guterman 1992, sec. 1.1). In these exercises, you may use that the general solution to an ODE of the form

$$x' = kx$$

– where k is a constant – is given by $x(t) = x_0 e^{kt}$ for a constant x_0.

1. Find all values of the constant k for which the given function $x = \phi(t)$ is a solution of the indicated differential equation:

   a. $x = t^k, t > 0; \quad t^2 \cdot x \cdot x'' - 2t \cdot x \cdot x' - 10x^2 = 0.$

   b. $x = kte^{3t}; \quad x'' - 3x' = e^{3t}.$

2. (adapted from (Nitecki and Guterman 1992, sec. 1.1 exercise 24 and 1.2 exercise 22)).

   A tanker carrying $200,000$ liters of oil runs aground off the coast of Alaska. Water pours in the tanker at one end at a rate of 1500 liters per hour while the polluted water-oil mixture pours out at the other end, also at a rate of 1500 liters per hour.

   We wish to describe the number of liters $x(t)$ of oil in the tanker at time t. Of course, $x(0) = 200000$.

   a. Explain why the percentage of oil in the tanker at time $t \geq 0$ is given by the fraction $\dfrac{x(t)}{200000}$.

b. Use the answer to a. to set up a differential equation to predict the amount $x = x(t)$ of oil in the tanker at time t. Explain how you arrived at your formulation.

c. Solve the differential equation found in part b. How much oil remains in the tanker after 5 days?

3. Radioactive decay The atoms of a radioactive substance tend to decompose into atoms of a more stable substance at a rate proportional to the number $x = x(t)$ of unstable atoms present. Suppose that time t is measured in seconds, and that the half-life of a certain compound is 10^4 seconds. In other words, if $x_0 = x(0)$ then $x(10^4) = 0.5x_0$. Find a differential equation for $x(t)$.

4. For each of the following ODES, find the general solution, and find the particular solution satisfying the given initial conditions.

   a. $\dfrac{d^2x}{dt^2} = t - 1; \quad x(0) = 1, x'(0) = 2.$

   b. $x'' = \dfrac{-1}{(t+1)^2}, t > -1; \quad x(0) = 2, x'(0) = 3.$

Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. Differential Equations: A First Course. Saunders.