

# Math 51 Spring 2022 - Final Exam - some review problems

## Solutions

2022-05-01

1. Indicate which of the following best represents a *simplified guess* for a particular solution  $p(t)$  to the non-homogeneous linear ODE:

$$(D - 3)(D - 1)x = te^{3t} + \cos(2t)$$

- a.  $p(t) = k_1 te^{3t} + k_2 \cos(2t) + k_3 \sin(2t)$
- b.  $p(t) = k_1 te^{3t} + k_2 \cos(2t)$
- c.  $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t)$
- d.  $p(t) = k_1 te^{3t} + k_2 t^2 e^{3t} + k_3 \cos(2t) + k_4 \sin(2t)$

**Solution:**

correct response was d.

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2. Indicate which of the following represents the general solution to the homogeneous linear ODE  $(D^2 - 2D + 2)^2 x = 0$ .

- a.  $h(t) = c_1 e^{-t} \cos(t) + c_2 e^{-t} \sin(t) + c_3 t e^{-t} \cos(t) + c_4 t e^{-t} \sin(t)$
- b.  $h(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t)$
- c.  $h(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t) + c_3 t e^t \cos(t) + c_4 t e^t \sin(t)$
- d.  $h(t) = c_1 t e^t \cos(t) + c_2 t e^t \sin(t) + c_3 t^2 e^t \cos(t) + c_4 t^2 e^t \sin(t)$

**Solution:**

correct response was ~~a~~ c.

(The roots of the char poly are  $\lambda = 1 \pm i$ , which leads to solutions of the form  $t^j \sin(t)$  and  $t^j \cos(t)$ .)

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3. The matrix  $A = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix}$  has characteristic polynomial  $\lambda^2 - 2\lambda + 2$  and thus its eigenvalues are  $\lambda = 1 + i$  and  $\lambda = 1 - i$ .

Which of the following is an eigenvector for  $A$ ?

- a.  $A$  has no eigenvectors.

- b.  $\begin{bmatrix} 3-i \\ 2 \end{bmatrix}$
- c.  $\begin{bmatrix} 2 \\ -3+i \end{bmatrix}$
- d.  $\begin{bmatrix} 3+i \\ 2 \end{bmatrix}$

**Solution:**

correct response was d.

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4. Consider the linear system of ODEs

$$(\diamond) \quad D\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

A third order linear ODE is *equivalent* to this system if for each of its solutions  $x(t)$ , the vector-valued function  $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x'(t) \\ x''(t) \end{bmatrix}$  is a solution to  $(\diamond)$ . Which of the following linear ODEs is equivalent to  $(\diamond)$ ?

- a.  $(D^3 - 2D^2 - D - 5)x = e^t$
- b.  $(D^3 - 5D^2 - D - 2)x = e^t$
- c.  $(D^3 + 2D^2 + D + 5)x = -e^t$
- d.  $(D^3 + 5D^2 + D + 2)x = -e^t$

**Solution:**

correct response was b.

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5. Let  $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ .  $\lambda = 2$  is an eigenvalue of  $A$  with multiplicity two. The matrix  $A - 2\mathbf{I}_3$

satisfies  $(A - 2\mathbf{I}_3)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Thus the generalized

eigenvectors of  $A$  for  $\lambda = 2$  are generated by  $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ .

Which of the following represents a solution  $\mathbf{h}(t)$  to the system  $D\mathbf{x} = A\mathbf{x}$  with the property

that  $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ ?

a.  $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

b.  $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 + 12t \\ 2 \\ 6 \end{bmatrix}$

c.  $\mathbf{h}(t) = e^{2t} \begin{bmatrix} 1 + t \\ 2 \\ 6 \end{bmatrix}$

d. No solution  $\mathbf{h}(t)$  has the property that  $\mathbf{h}(0) = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ .

**Solution:**

correct response was b.

6. Consider the homogeneous system  $(\diamond) \quad D\mathbf{x} = A\mathbf{x}$  where  $A$  is a  $3 \times 3$  matrix, and let  $\mathbf{h}_1(t), \mathbf{h}_2(t)$  be solutions to  $(\diamond)$ . Which of the following statements is correct?
- a.  $\mathbf{h}_1(0)$  and  $\mathbf{h}_2(0)$  are *eigenvectors* for  $A$ .
  - b. The system  $(\diamond)$  has exactly two solutions.
  - c. If the vectors  $\mathbf{h}_1(0), \mathbf{h}_2(0)$  are linearly independent, then the general solution to  $(\diamond)$  is given by  $\mathbf{x}(t) = c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$ .
  - d. None of the above statements is correct.

**Solution:**

correct response was d.

To see that **a.** is incorrect, consider solutions  $e^{\lambda t}\mathbf{v}$  and  $e^{\mu t}\mathbf{w}$  arising from eigenvectors  $\mathbf{v}$  and  $\mathbf{w}$ .

Then there is a solution  $h(t) = e^{\lambda t}\mathbf{v} + e^{\mu t}\mathbf{w}$  but  $h(0) = \mathbf{v} + \mathbf{w}$  which is not an eigenvector if  $\lambda \neq \mu$ .

**b.** is incorrect. Indeed, all linear combinations  $c_1\mathbf{h}_1(t) + c_2\mathbf{h}_2(t)$  are solutions, so there are always *infinitely many solutions*.

Finally, **c.** is incorrect because for a  $3 \times 3$  system the general solution is generated by three solutions with linearly independent initial vectors; two solutions are not enough.

7. The matrix  $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$  has characteristic polynomial  $\lambda(\lambda - 3)$  and hence has eigenvalues  $\lambda = 0$  and  $\lambda = 3$ . An eigenvector for  $\lambda = 0$  is given by  $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  and an eigenvector for  $\lambda = 3$  is given by  $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

Find a particular solution  $\mathbf{p}(t)$  for the system of linear ODEs

$$D\mathbf{x} = A\mathbf{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

**Solution:**

The general solution is generated by the solutions obtained from eigenvectors:

$$\mathbf{h}_1(t) = e^{0t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{h}_2(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

To find a particular solution, form the Wronskian matrix

$$W = \begin{bmatrix} -1 & e^{3t} \\ 1 & 2e^{3t} \end{bmatrix}$$

and notice that  $\det W = -3e^{3t}$ .

A particular solution has the form  $\mathbf{p}(t) = c_1(t)\mathbf{h}_1(t) + c_2(t)\mathbf{h}_2(t)$ , where the vector  $\mathbf{c} = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$  satisfies the matrix equations

$$W\mathbf{c}' = \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

Using Cramer's Rule, we find that

$$c_1'(t) = \frac{\det \begin{bmatrix} 0 & e^{3t} \\ t & 2e^{3t} \end{bmatrix}}{-3e^{3t}} = \frac{-te^{3t}}{-3e^{3t}} = \frac{t}{3}.$$

$$c_2'(t) = \frac{\det \begin{bmatrix} -1 & 0 \\ 1 & t \end{bmatrix}}{-3e^{3t}} = \frac{-t}{-3e^{3t}} = \frac{te^{-3t}}{3}$$

Now we integrate to find  $c_1(t)$  and  $c_2(t)$ :

$$c_1(t) = \int c_1'(t)dt = \frac{1}{3} \int tdt = \frac{t^2}{6} + A.$$

For  $c_2$  we integrate by parts with  $u = t, dv = e^{-3t}dt$ :

$$c_2(t) = \int c_2'(t)dt = \frac{1}{3} \int te^{-3t}dt = \frac{1}{3} \left( \frac{-t}{3}e^{-3t} + \frac{1}{3} \int e^{-3t}dt \right) = \frac{-1}{9}e^{-3t} \left( t + \frac{1}{3} \right) + B$$

We may take  $A = B = 0$  since we only seek a particular solution. This gives

$$\begin{aligned} \mathbf{p}(t) &= \frac{t^2}{6} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{-1}{9}e^{-3t} \left( t + \frac{1}{3} \right) e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{t^2}{6} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{-1}{9} \left( t + \frac{1}{3} \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

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8. Let  $A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$ .

The characteristic polynomial of  $A$  is  $r^2 - 4r + 5$  so the eigenvalues of  $A$  are  $\lambda = 2 \pm i$ .

Moreover,  $\mathbf{v} = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$  is an eigenvector for  $\lambda = 2 + i$ .

a. Find the general solution to  $D\mathbf{x} = A\mathbf{x}$ .

**Solution:**

The complex solution to (H) is

$$e^{2t}(\cos t + i \sin t) \begin{bmatrix} 2-i \\ 5 \end{bmatrix} = e^{2t} \begin{bmatrix} 2 \cos t + \sin t \\ 5 \cos t \end{bmatrix} + ie^{2t} \begin{bmatrix} -\cos t + 2 \sin t \\ 5 \sin t \end{bmatrix}$$

so the real and imaginary parts of this expression generate the general solution

$$x(t) = C_1 e^{2t} \begin{bmatrix} 2 \cos t + \sin t \\ 5 \cos t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -\cos t + 2 \sin t \\ 5 \sin t \end{bmatrix}.$$


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b. Solve the initial value problem  $D\mathbf{x} = A\mathbf{x}$ ,  $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**Solution:**

The value at  $t = 0$  of the general solution given above is

$$X(0) = C_1 e^0 \begin{bmatrix} 2 \cos 0 + \sin 0 \\ 5 \cos 0 \end{bmatrix} + C_2 e^0 \begin{bmatrix} -\cos 0 + 2 \sin 0 \\ 5 \sin 0 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix};$$

setting this equal to the desired initial condition yields the system of equations

$$\begin{aligned} 2C_1 - C_2 &= 1 \\ 5C_1 + 0C_2 &= 1 \end{aligned}$$

which can be solved by performing row operations on the augmented matrix

$$\left[ \begin{array}{cc|c} 2 & -1 & 1 \\ 5 & 0 & 1 \end{array} \right],$$

or by using Cramer's Rule, or simply by noting that the second equation says  $C_1 = \frac{1}{5}$ , and substituting into the first equation yields  $\frac{2}{5} - C_2 = 1$  or  $C_2 = -\frac{3}{5}$ .

Thus the desired solution of (H) is

$$X(t) = \frac{1}{5} e^{2t} \begin{bmatrix} 2 \cos t + \sin t \\ 5 \cos t \end{bmatrix} - \frac{3}{5} e^{2t} \begin{bmatrix} -\cos t + 2 \sin t \\ 5 \sin t \end{bmatrix} = e^{2t} \begin{bmatrix} \cos t - \sin t \\ \cos t - 3 \sin t \end{bmatrix}.$$


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9. Solve the initial value problem  $(4D^2 - 4D + 1)x = 0$ ,  $x(2) = x'(2) = e$ .

**Solution:**

The polynomial  $4r^2 - 4r + 1$  has root  $r = 1/2$  with multiplicity 2. Thus the general solution is given by

$$x(t) = c_1 e^{t/2} + c_2 t e^{t/2}.$$

Note that

$$\begin{aligned} x'(t) &= D[x(t)] = \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} D[t] \\ &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} (D + 1/2)[t] \\ &= \frac{c_1}{2} e^{t/2} + c_2 e^{t/2} (1 + t/2) \end{aligned}$$

Now, we need

$$e = x(2) = c_1 e + 2c_2 e$$

and

$$e = x'(2) = \frac{1}{2} e c_1 + 2e c_2$$

Thus we must solve the matrix equation

$$\begin{bmatrix} e & 2e \\ e/2 & 2e \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e \\ e \end{bmatrix}$$

This can be solved in several ways – e.g. by row operations on the augmented matrix, as follows:

$$\left[ \begin{array}{cc|c} e & 2e & e \\ e/2 & 2e & e \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 4 & 2 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2 & 1 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 1 \end{array} \right]$$

Thus  $c_1 = 0$  and  $c_2 = 1/2$  so that the solution to the initial value problem is given by

$$x(t) = \frac{t e^{t/2}}{2}.$$

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10. Consider the matrix  $B = \begin{bmatrix} 5 & -3 & -6 \\ 0 & 2 & 0 \\ 3 & -3 & -4 \end{bmatrix}$ .

- a. The vector  $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$  is an eigenvector for  $B$ . What is the corresponding eigenvalue?

**Hint:** Compute the vector  $B\mathbf{v}$  and compare with  $\mathbf{v}$ .

**Solution:**

The product  $B\mathbf{v}$  is equal to

$$B\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2\mathbf{v}$$

so the eigenvalue is  $\lambda = 2$ .

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b. Find an eigenvector for  $B$  for the eigenvalue  $\lambda = -1$ .

**Solution:** Perform row operations on the matrix  $B - (-1)\mathbf{I}_3 = B + \mathbf{I}_3$ :

$$\begin{bmatrix} 6 & -3 & -6 \\ 0 & 3 & 0 \\ 3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Considering this echelon matrix, we see that an eigenvector for  $\lambda = -1$  is given by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

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11. Laplace Transforms:

a. Compute the inverse Laplace transform  $\mathcal{L}^{-1}[F(s)]$  of the function  $F(s) = \frac{3s^2 + s + 1}{(s+1)(s^2+2)}$ .

**Solution:**

The partial fraction decomposition has the form

$$\frac{3s^2 + s + 1}{(s+1)(s^2+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2};$$

combining over a common denominator and matching coefficients leads to

$$\begin{array}{lcl} s^2 \text{ terms :} & A + B & = 3 \\ s \text{ terms :} & B + C & = 1 \\ \text{constant terms :} & 2A + C & = 1 \end{array}$$

We can solve the first (respectively, second) equation for  $A$  (respectively,  $C$ ) in terms of  $B$ :

$$\begin{aligned} A &= 3 - B \\ C &= 1 - B \end{aligned}$$

and substituting into the third equation yields

$$\begin{aligned}(6 - 2B) + (1 - B) &= 1 \\ -3B &= -6 \\ B &= 2 \\ A &= 1 \\ C &= -1\end{aligned}$$

so

$$\frac{3s^2 + s + 1}{(s + 1)(s^2 + 2)} = \frac{1}{s + 1} + \frac{2s - 1}{s^2 + 2}.$$

Then the inverse transform is

$$\begin{aligned}\mathcal{L}^{-1} \left[ \frac{3s^2 + s + 1}{(s + 1)(s^2 + 2)} \right] &= \mathcal{L}^{-1} \left[ \frac{1}{s + 1} \right] + \mathcal{L}^{-1} \left[ \frac{2s}{s^2 + 2} \right] - \mathcal{L}^{-1} \left[ \frac{1}{s^2 + 2} \right] \\ &= e^{-t} + 2 \cos t\sqrt{2} - \frac{1}{\sqrt{2}} \sin t\sqrt{2}.\end{aligned}$$

- b. If  $x$  is a solution to  $(D^2 + D + 1)x = 1$  with  $x(0) = 0$  and  $x'(0) = 1$ , find an expression for  $\mathcal{L}[x]$  as a function of  $s$ .

**Solution:**

By the first differentiation formula, applying the Laplace Transform to both sides of the problem yields

$$\begin{aligned}\mathcal{L}D^2x + \mathcal{L}Dx + \mathcal{L}x &= \mathcal{L}1 \\ \{s^2\mathcal{L}x - sx(0) - x'(0)\} + \{s\mathcal{L}x - x(0)\} + \mathcal{L}x &= \mathcal{L}1 \\ s^2\mathcal{L}x - 1 + s\mathcal{L}x + \mathcal{L}x &= \frac{1}{s} \\ (s^2 + s + 1)\mathcal{L}x &= 1 + \frac{1}{s} = \frac{1 + s}{s} \\ \mathcal{L}x &= \frac{1 + s}{s(s^2 + 2 + 1)}\end{aligned}$$

12. Let  $W = W(h_1(t), h_2(t))$  denote the *Wronskian matrix* of the functions  $h_1(t) = e^{2t}$  and  $h_2(t) = te^{2t}$ . Which of the following represents the *determinant* of  $W$ ?
- a.  $e^{4t}$
  - b.  $(1 + 4t)e^{4t}$
  - c.  $e^{2t}$
  - d.  $(1 + 4t)e^{2t}$

**Solution:**

~~correct answer is c.~~ correct answer is a - i.e.  $e^{4t}$ .



13. Consider the vectors  $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ , and  $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$  in  $\mathbf{R}^4$ , and let  $A =$

$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$  be the  $4 \times 3$  matrix whose columns are the  $\mathbf{v}_i$ . Which of the following statements is correct?

a. The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are *linearly dependent*.

b. Since  $A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ , the only solution to the equation  $A\mathbf{w} = \mathbf{0}$  is  $\mathbf{w} = \mathbf{0}$  so the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are *linearly independent*.

c. The equation  $A\mathbf{w} = \mathbf{x}$  has a solution for every vector  $\mathbf{x}$  in  $\mathbf{R}^4$ .

d. The determinant of  $A$  is  $\neq 0$ .

**Solution:**

The correct response is b.

Answer a. is incorrect because the vectors are independent.

Answer c. is incorrect because the given equation has no solution when  $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ .

Answer d. is incorrect because the determinant of a  $4 \times 3$  (non-square!) matrix is not defined.

14. Let  $A$  be an  $n \times n$  matrix with constant coefficients  $a_{ij}$ , and let  $\mathbf{E}(t)$  be a vector with  $n$  components. If  $\mathbf{v}$  is any vector in  $\mathbf{R}^n$ , must there be a solution  $\mathbf{x}(t)$  to the system of equations  $D\mathbf{x} = A\mathbf{x} + \mathbf{E}(t)$  for which  $\mathbf{x}(0) = \mathbf{v}$ ?

a. No, this conclusion is only guaranteed when the system is *homogeneous*.

b. No, this conclusion is only guaranteed when the entries of the vector  $\mathbf{E}(t)$  are *constant* functions of  $t$ .

c. Yes, this conclusion is the content of the *Existence and Uniqueness Theorem for Solutions of Linear Systems*.

d. No, this conclusion is only guaranteed when  $\det A \neq 0$ .

**Solution:**

correct response is c.

The existence and uniqueness theorem applies for non-homogeneous systems (so a is incorrect) and applies so long as the entries of  $A$  and of  $\mathbf{E}$  are *continuous* functions of  $t$  (so

b. is incorrect). Finally, the existence and uniqueness theorem is valid even when  $A$  has determinant 0 (so d. is incorrect).

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15. Consider the homogeneous system  $(\diamond) \quad D\mathbf{x} = A\mathbf{x}$  where  $A$  is a  $3 \times 3$  matrix.

a. If  $\mathbf{h}(t)$  is a solution, must  $\mathbf{h}(0)$  be an eigenvector for  $A$ ? Why or why not?

**Solution:**

No,  $\mathbf{h}(0)$  need not be an eigenvector. Suppose for example that  $\mathbf{v}$  and  $\mathbf{w}$  are eigenvectors for  $A$  with eigenvalues  $\lambda, \mu$ , and suppose that  $\lambda \neq \mu$ . Then  $\mathbf{v} + \mathbf{w}$  is not an eigenvector.

Indeed, since  $\lambda \neq \mu$  we know that  $\mathbf{v}$  and  $\mathbf{w}$  are *linearly independent*. Now, for any number  $\beta$ ,

$$(A - \beta\mathbf{I})(\mathbf{v} + \mathbf{w}) = (\lambda - \beta)\mathbf{v} + (\mu - \beta)\mathbf{w}$$

Since  $\lambda \neq \mu$ , at least one of  $\lambda - \beta$  or  $\mu - \beta$  is non-zero, so the linear independence of  $\mathbf{v}$  and  $\mathbf{w}$  shows that  $(A - \beta\mathbf{I})(\mathbf{v} + \mathbf{w})$  is non-zero. This shows that  $\mathbf{v} + \mathbf{w}$  is not an eigenvector (for *any* eigenvalue  $\beta$ ).

Now, the function

$$h(t) = e^{\lambda t}\mathbf{v} + e^{\mu t}\mathbf{w}$$

is a solution to  $(\diamond)$ , and  $h(0) = \mathbf{v} + \mathbf{w}$ .

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b. Show that the vectors  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  are linearly independent ~~dependent~~.

**Solution:**

We perform row operations on the matrix whose columns are given by these vectors:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the resulting echelon matrix has 3 pivots and no free variables, the only solution  $\mathbf{c}$  to the matrix equation

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is  $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$ . (Alternatively, you could have obtained this conclusion by noting that the

*determinant* of the matrix  $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$  is equal to  $-4$ ).

Thus the only coefficients satisfying the following equation

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \mathbf{0}$$

are  $c_1 = c_2 = c_3 = 0$ ; this shows that the vectors are linearly independent.

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- c. Let  $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$  be solutions to  $(\diamond)$ . Suppose that  $\mathbf{h}_1(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{h}_2(0) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ , and  $\mathbf{h}_3(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$  are the vectors from b. Do the solutions  $\mathbf{h}_1(t), \mathbf{h}_2(t), \mathbf{h}_3(t)$  generate the general solution to  $(\diamond)$ ? Why or why not?

**Solution:**

Yes. Since  $A$  is a  $3 \times 3$  matrix, one knows that three solutions  $\mathbf{h}_1(t)$ ,  $\mathbf{h}_2(t)$  and  $\mathbf{h}_3(t)$  generate the general solution provided that the “initial vectors”  $\mathbf{h}_1(0), \mathbf{h}_2(0), \mathbf{h}_3(0)$  are linearly independent; thus the result in part b. shows that the  $\mathbf{h}_i(t)$  generate the general solution.

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16. A drug is absorbed by the body at a rate proportional to the amount of the drug present in the bloodstream after  $t$  hours. If there are  $x(t)$  mg of the drug present in the bloodstream at time  $t$ , assume that the drug is absorbed at a rate of  $0.5x(t)$  /hour. If a patient receives the drug intravenously at a constant rate of 3 mg/hour, to which of the following ODEs is  $x(t)$  a solution?
- a.  $x'(t) = -0.5x(t) + 3$
  - b.  $x'(t) = -0.5x(t); \quad x(0) = 3$
  - c.  $x'(t) = 0.5x(0) + 3$
  - d.  $x'(t) = .5x(t) - 3$

**Solution:**

correct response is a.

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17. You are given that a particular solution to

$$(\heartsuit) \quad (D^2 - 2D + 1)x = e^t$$

is  $p(t) = \frac{t^2 e^t}{2}$ . Which of the following best represents the general solution to  $(\heartsuit)$ ?

- a.  $c_1 e^t + c_2 t e^t$ .
- b.  $\frac{t^2 e^t}{2} + c_1 e^t + c_2 t e^t$ .
- c.  $\frac{t^2 e^t}{2} + c e^t$ .
- d.  $\frac{t^2 e^t}{2} + c_1 e^t + c_2 e^{-t}$ .

**Solution:**

correct response was b.

- 
18. Let  $x_1(t)$  and  $x_2(t)$  be solutions to the ODE  $(t+1)x'' + x' + x = 0$ . Suppose that  $x_1(0) = x_2(0)$  and that  $x_1'(0) = x_2'(0)$ . Which of the following statements must be correct?
- $x_1(t) = x_2(t)$  for every  $t$ .
  - Since the ODE is *normal* on the interval  $(-1, \infty)$ , we can conclude that  $x_1(t) = x_2(t)$  for  $-1 < t < \infty$ .
  - No conclusion is possible because the existence and uniqueness theorem does not apply to this ODE.
  - We can only conclude that  $x_1(t) = x_2(t)$  for all  $t$  if we also assume that  $x_1''(0) = x_2''(0)$ .

**Solution:**

correct response was b. Indeed since the equation is of 2nd order and since it is normal on the interval  $(-1, \infty)$ , the existence and uniqueness theorem guarantees for any  $\alpha, \beta$  that there is only one solution  $x$  which  $x(0) = \alpha$  and  $x'(0) = \beta$ .

Assertion a. need not be true since the ODE is not normal on  $(-\infty, \infty)$ .

And assertion d. is incorrect – the existence and uniqueness theorem doesn't require a condition on the second derivative in this case.

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19. Show that the functions

$$f_1(t) = e^t \cos(t), \quad f_2(t) = e^t \sin(t), \quad f_3(t) = e^t$$

are linearly independent.

You have been told that functions like this are independent. However, here we want you to demonstrate it directly in this case. You may use the *Wronskian test* (with all details needed to justify using it) or other, direct arguments from the definition.

**Solution:**

There are several possible strategies for solving this problem; here we list a few of them.

First, you can use the Wronskian test. This requires computation of the first and second derivatives of the  $f_i$ , which is perhaps most easily done using the *exponential shift formula*.

One finds:

$$D[e^t \cos(t)] = e^t(D+1)[\cos(t)] = e^t(\cos(t) - \sin(t))$$

$$D[e^t \sin(t)] = e^t(D+1)[\sin(t)] = e^t(\cos(t) + \sin(t))$$

$$D^2[e^t \cos(t)] = D[e^t(\cos(t) - \sin(t))] = e^t(D+1)[\cos(t) - \sin(t)] = -2e^t \sin(t).$$

$$D^2[e^t \sin(t)] = D[e^t(\cos(t) + \sin(t))] = e^t(D+1)[\cos(t) + \sin(t)] = 2e^t \cos(t).$$

Thus the Wronskian matrix is given by

$$W = W(f_1, f_2, f_3) = \begin{bmatrix} e^t \cos(t) & e^t \sin(t) & e^t \\ e^t(\cos(t) - \sin(t)) & e^t(\cos(t) + \sin(t)) & e^t \\ -2e^t \sin(t) & 2e^t \cos(t) & e^t \end{bmatrix}$$

Now, according to the Wronskian test, the functions will be linearly independent (on the interval  $(-\infty, \infty)$ ) provided that  $\det W(t_0)$  is non-zero for some  $t_0$ . If we take  $t_0 = 0$ , we find that

$$\det W \Big|_{t=0} = \det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = -1 + 2 = 1$$

Since this determinant is non-zero, the Wronskian test confirms the linear independence of  $f_1, f_2, f_3$ .

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A second method of solving this problem just uses the *definition of linear independence*.

Suppose that  $c_1, c_2, c_3$  are constants and that

$$c_1 e^t \cos(t) + c_2 e^t \sin(t) + c_3 e^t = 0.$$

To show that the functions are linearly independent, we must *argue* that  $c_1 = c_2 = c_3 = 0$ .

Factoring out the quantity  $e^t$ , our assumption shows that

$$e^t(c_1 \cos(t) + c_2 \sin(t) + c_3) = 0.$$

Since  $e^t \neq 0$  for all  $t$ , we find that

$$c_1 \cos(t) + c_2 \sin(t) + c_3 = 0.$$

Now, since this equation holds for all times  $t$ , we may choose some particular values of  $t$  to find equations for the constants  $c_i$ .

When  $t = 0$ , we find that

$$0 = c_1 \cos(0) + c_2 \sin(0) + c_3 = c_1 + c_3.$$

When  $t = \pi/2$ , we find that

$$0 = c_1 \cos(\pi/2) + c_2 \sin(\pi/2) + c_3 = c_2 + c_3.$$

When  $t = \pi$ , we find that

$$0 = c_1 \cos(\pi) + c_2 \sin(\pi) + c_3 = -c_1 + c_3.$$

Now, we solve the system of equations

$$\begin{aligned} 0 &= c_1 + c_3 \\ 0 &= c_2 + c_3 \\ 0 &= -c_1 + c_3 \end{aligned}$$

Adding the first and third equation gives  $0 = 2c_3$  so that  $c_3 = 0$ . Now the first equation shows that  $c_1 = 0$  and the second shows that  $c_2 = 0$ .

Since we have argued that  $c_1 = c_2 = c_3 = 0$ , we conclude from the definition that  $f_1, f_2, f_3$  are linearly independent.

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20. Find the Laplace transform of the function

$$f(t) = \begin{cases} 1 & \text{for } t < 1, \\ t - 1 & \text{for } 1 \leq t < 2, \\ 1 & \text{for } t \geq 2. \end{cases}$$

**Solution:**

In order to be able to compute the Laplace transform, We first rewrite the function  $f(t)$  using the *unit step functions*.

We have

$$\begin{aligned} f(t) &= 1 + u_1(t) \cdot (-1 + (t - 1)) + u_2(t) \cdot ((-(t - 1) + 1) \\ &= 1 + u_1(t) \cdot (t - 2) + u_2(t) \cdot (-t + 2). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}[f(t)] &= \mathcal{L}[1 + u_1(t) \cdot (t - 2) + u_2(t) \cdot (-t + 2)] \\ &= \mathcal{L}[1] + \mathcal{L}[u_1(t) \cdot (t - 2)] + \mathcal{L}[u_2(t) \cdot (-t + 2)] \\ &= \mathcal{L}[1] + e^{-s} \mathcal{L}[(t + 1) - 2] + e^{-2s} \mathcal{L}[-(t + 2) + 2] \\ &= \mathcal{L}[1] + e^{-s} \mathcal{L}[t - 1] + e^{-2s} \mathcal{L}[-t] \\ &= (1 - e^{-s}) \mathcal{L}[1] + e^{-s} \mathcal{L}[t] - e^{-2s} \mathcal{L}[t] \\ &= (1 - e^{-s}) \mathcal{L}[1] + (e^{-s} - e^{-2s}) \mathcal{L}[t] \\ &= \frac{1 - e^{-s}}{s} + \frac{e^{-s} - e^{-2s}}{s^2} \end{aligned}$$

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21. Suppose  $g(t)$  is the inverse Laplace transform of

$$F(s) = \frac{2se^{\pi s/2}}{(s^2 + 4)}.$$

Find  $g\left(\frac{\pi}{4}\right)$ .

**Solution:**

We use the *second shift formula* to find  $g(t)$ . Notice that if we set

$$f(t) = \mathcal{L}^{-1} \left[ \frac{s}{s^2 + 4} \right] = \cos(2t)$$

then the second shift formula yields

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}[F(s)] = 2\mathcal{L}^{-1} \left[ e^{(\pi/2)s} \frac{s}{s^2 + 4} \right] \\ &= 2u_{\pi/2}(t) f(t - \pi/2) \end{aligned}$$

Thus  $u_{\pi/2}(\pi/4) = 0$  so that  $g(\pi/4) = 2u_{\pi/2}(\pi/4) \cdot f(\pi/4 - \pi/2) = 0$ .

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