MATH 51: HOMEWORK 8 SOLUTION

Homework exercise solution:

1. Consider the polynomial $\det(A - \lambda I)$, we have
$$p(\lambda) = -\lambda^3 + 3\lambda + 2$$
Using rational root test, we have $p(2) = 0$, divide $p(\lambda)$ by $\lambda - 2$, we have
$$p(\lambda) = (\lambda - 2)(\lambda^2 + 2\lambda + 1)$$
Hence we conclude the matrix has eigenvalue -1 with multiplicity of 2 and eigenvalue 2. For $\lambda = 2$, we have
$$A - 2\lambda = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$
Therefore the corresponding eigenvector is
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
For $\lambda = 2$, we have
$$A - 2\lambda = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
Therefore the corresponding eigenvectors are
$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

2. (a). Consider
$$A - \lambda I = \begin{bmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$
we have $1 - \lambda + 3 = 0$, hence $\lambda = 4$.
(b). The vector valued function
$$\mathbf{x} = e^{4t}\mathbf{v} = \begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix}$$
it the associated solution.

3. (a).
$$\det(A) = 1 \cdot \det \begin{bmatrix} 2 & -2 & 1 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{bmatrix} = 1 \cdot 2 \det \begin{bmatrix} -3 & 5 \\ 0 & 4 \end{bmatrix} = 1 \cdot 2 \cdot (-3) \cdot 4 = -24$$
(b). The determinant of the upper triangular matrix is the product of all diagonal terms.

(c).
$$\det(A - \lambda I) = (1-\lambda) \cdot \det \begin{bmatrix} 2-\lambda & -2 & 1 \\ 0 & -3-\lambda & 5 \\ 0 & 0 & 4-\lambda \end{bmatrix} = (1-\lambda)\cdot(2-\lambda)\cdot(-3-\lambda)\cdot(4-\lambda) = 0$$

Hence the eigenvalues are $1, 2, -3, 4$.

(d). The eigenvalues of upper triangular matrix are the diagonal terms.

4. Denote the ith row by R_i, first, consider $R_2 + R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & -1 & 1 & 1 \\ 2 & -1 & 3 & -1 \end{bmatrix}$$

Consider $-1 \times R_2$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 2 & -1 & 3 & -1 \end{bmatrix}$$

Consider $R_3 + R_2$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 2 & 0 & 2 & -2 \end{bmatrix}$$

Consider $R_3/2$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

Consider $R3 - R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -2 \end{bmatrix}$$

Consider $R_3 - 2R_2$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider $R_1 + 2R_2$, we have the reduced form:

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. (a). Label each equation by R_i, consider $R_2 - 3R_1$, $R_3 - 4R_1$ and $R_4 - R_1$, we have

$$\begin{bmatrix} 1 & 2 & 3 & -2 & 0 \\ 0 & -13 & -13 & 13 & 0 \\ 0 & -11 & -11 & 11 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

Consider $R_2/13$ and $R_3/11$, then $R_4 - R_2$ and $R_3 - R_2$, we have

$$\begin{bmatrix} 1 & 2 & 3 & -2 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider $R_1 + 2R_2$, we have

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which indicates that

$$x_1 + x_3 = 0$$
$$-x_2 - x_3 + x_4 = 0$$

Write in separate parametric equations, consider $x_3 = a$, we have $x_1 = -a$. Consider $x_2 = b$, we have $x_4 = a + b$. (There are many other choices of a and b). Write in vectors we have

$$\mathbf{x} = a \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(b). Label each equation by R_i, Consider $R_2 - 2R_1$ and $R_3 + R_1$, we have

$$\begin{bmatrix} 1 & 2 & 1 & -1 & -1 & 2 \\ 0 & -2 & 0 & -1 & 0 & -3 \\ 0 & 2 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Consider $R_3 + R_2$, we have

$$\begin{bmatrix} 1 & 2 & 1 & -1 & -1 & 2 \\ 0 & -2 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

R_3 indicates $0 = -1$, which means there's no solution.

6. (a). By definition, the number of pivots can't exceed the number of rows because for each row there exists at most 1 pivot. Since the number of rows is always less than the number of columns in this problem, we automatically have the number of pivots is less that the number of columns.

   (b). Since there are columns without pivots, there are more than 1 free variables, hence lead to infinitely many solutions.