Math 51 | Differential Equations Worksheet 5 | February 17–18, 2022

Complex Numbers

Definition. $i = \sqrt{-1}$.

Thus, i is a solution of $x^2 + 1 = 0$. A *complex number* is a number of the form $z = a + ib$, where a and b are real.

Definition. The *conjugate* of $z = a + ib$ is $\bar{z} = a - ib$.

Properties. $\overline{z + w} = \bar{z} + \bar{w}, \quad \overline{zw} = \bar{z}\bar{w}, \quad z\bar{z} = a^2 + b^2.$

1. Write down the power series (Maclaurin series).

   (a) $\cos v =$ ____

   (b) $\sin v =$ ____

   (c) $e^v =$ ____

   (d) $e^{iv} =$ ____

2. Use Problem 1 to prove Euler's formula $e^{iv} = \cos v + i \sin v$.

   Therefore, $e^{u+iv} = \boxed{}$

If we substitute $v = \pi$ into Euler's formula, we get

$$e^{i\pi} = -1, \quad \text{or} \quad \boxed{e^{i\pi} + 1 = 0}.$$

The formula, linking the five fundamental constants, 0, 1, e, π, i, is often voted in various polls "the most beautiful formula in mathematics."

Linear ODEs with Constant Coefficients: Complex Roots

1. Find generating solutions of the following homogeneous linear equations with constant coefficients.

   (a) $(D^2 - 2D + 4)x = 0.$

   (b) $(D^2 - 2D + 4)^3 x = 0.$

   (c) $(D - 1)^2(D^2 - 2D + 4)^2 x = 0.$

2. The exponential shift formulas states that

$$P(D)[e^{\lambda t} y] = e^{\lambda t} P(D + \lambda)[y].$$

   This formula say: to calculate $P(D)[e^{\lambda t} y]$,

   - take out the exponential $e^{\lambda t}$,
   - at the same time, replace D by $D + \lambda$.

   Apply the exponential shift formula to calculate

$$(D^2 - 2D + 4)[e^t \cos(\sqrt{3}t)].$$

Practicum Section: Name:

1. Consider the homogeneous linear differential equation $(D^2 - 4D + 5)x = 0$.

   (a) Find two linearly independent complex solutions.

   (b) Find two linearly independent real solutions.

2. Consider the homogeneous linear differential equation $(D^2 - 4D + 5)^2 x = 0$.

   (a) Find four linearly independent complex solutions.

   (b) Find four linearly independent real solutions.

(Worksheet continues on back)

3. Using only real solutions, find the general solution of the homogeneous linear differential equation
$$D^3(D^2+1)(D^2-4D+5)^3x = 0.$$

4. Apply the exponential shift formula to calculate $(D^2-4D+5)[e^t \sin t]$.