Practicum Session: PRINT your name:

Section Instructor: (Choose from Hasselblatt, McNinch, Smith, Tu)

Math 51 | Differential Equations | February 14, 2022
Exam 1 (100 points) | noon–1:20 p.m.

This is a closed-book exam. No books, notes, or calculators are permitted.

At the end of the exam you are required to sign a pledge that you have not cheated. Students found violating the pledge will be reported to the Dean of Student Affairs.

You must show your work in all the questions that require calculations, explanations, or proofs. Simplify all answers.

1. (21 points) Short-answer questions. Two points per question except in (a). No work needs to be shown, as only the answer will be graded.

   (a) (3 points) Consider the differential equation

   $$(t-3)^2 x'' + \frac{1}{t-2} x' + 2x = 0.$$

   Find the largest open interval containing $t = 0$ on which the o.d.e. is normal.

   (b) The order of the o.d.e. $t^3(x'')^3 + 3x' + tx^4 = 0$ is

   (c) True or False. Let h_1, h_2, h_3 be solutions of a normal third-order linear differential equation on an open interval I. Then the Wronskian $W[h_1, h_2, h_3](t_0) = 0$ at one point $t_0 \in I$ if and only if $W[h_1, h_2, h_3](t) = 0$ at every point $t \in I$.

   (d) True or False. Let h_1, h_2, h_3 be arbitrary infinitely differentiable functions on an open interval I. Then the Wronskian $W[h_1, h_2, h_3](t_0) = 0$ at one point $t_0 \in I$ if and only if h_1, h_2, h_3 are linearly dependent on I.

(e) True or False. The linear o.d.e $t^3x'' + 3x' + x + t = 0$ is homogeneous.

(f) True or False. If $h_1(t), h_2(t), h_3(t)$ are three solutions of the differential equation $x'' + x + 1 = 0$ on an interval I, then they cannot be linearly independent on I.

(g) True or False. Let $P(r), Q(r)$ be two polynomials and $F(r) = P(r)Q(r)$. If $x = h(t)$ is a solution of the differential equation $P(D)x = 0$, then it is necessarily a solution of the differential equation $F(D)x = 0$.

(h) Yes or No. Suppose $h_1(t)$ and $h_2(t)$ are any two solutions of the differential equation $t^3x'' + tx' + (t^2 - 1)x = 1$. Is $h_1(t) + h_2(t)$ necessarily a solution of this differential equation?

(i) Yes or No. Suppose $h_1(t)$ and $h_2(t)$ are two any solutions of the differential equation $t^3x'' + x' + x^2 = 0$. Is $h_1(t) + h_2(t)$ necessarily a solution of this differential equation?

(j) Multiple-Choice. Let $L(x) = x''' - 3x' + x = 0$. Suppose $h_1(t), h_2(t), h_3(t)$ are solutions of $L(x) = 0$ on $(-\infty, \infty)$ and the Wronskian $W[h_1, h_2, h_3](1) \neq 0$. Which of the following statements is true?

I. $h_1(t), h_2(t), h_3(t)$ generate the general solution of $L(x) = 0$ on $(-\infty, \infty)$.

II. $h_2(t), h_2(t), h_3(t)$ are linearly independent on $(-\infty, \infty)$.

Write one of A, B, C, D, or E in the box.

A. Only I is true.

B. Only II is true.

C. Both I and II are true.

D. Neither I nor II is true.

E. It is not possible to determine the truth or falsity of I or II from the given information.

2. (12 points) Determine whether each collection of functions is linearly independent or dependent on the interval $-\infty < t < \infty$. Justify your answer.

   (a) $f_1(t) = e^t, \quad f_2(t) = te^t, \quad f_3(t) = 1.$

   (b) $g_1(t) = t^2, \quad g_2(t) = -t^2.$

3. (20 points) Write the general solution for each differential equation below.

   (a) $D(D^2 - 9)(D^2 - 2D - 1)x = 0.$

   (b) $(D^2 - 2D - 15)^2 x = 0.$

4. (6 points)

   (a) (3 pts) Calculate

$$\det \begin{bmatrix} 3 & 0 & 2 \\ 5 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}.$$

   (b) (3 pts) Suppose

$$\begin{aligned} u_1 + 2u_2 + 3u_3 &= a, \\ 3u_1 + 2u_2 + 1u_3 &= b, \\ 5u_1 - 2u_2 + 2u_3 &= c, \end{aligned}$$

   where it is given that the determinant of the coefficient matrix is nonzero. Write down the formula for u_3 in terms of determinants. Do not evaluate the determinants.

5. (16 points) Use variation of parameters to find the general solution of

$$\frac{dx}{dt} + \frac{2}{t}x = \frac{e^{2t}}{t^2}, \qquad t > 0.$$

   (If you use a different method, you will get at most 10 points.)

6. (10 points) Using the exponential shift formula $P(D)[e^{\lambda t}y] = e^{\lambda t}[P(D+\lambda)y]$, evaluate $(D+5)^3\left[(t^3 - 4t^2 + 2t - 4)e^{-5t}\right]$.

7. (15 points) Consider the equation $t^2x'' - tx' + x = 0$ on the interval $(0, \infty)$.

   (a) Show that $h_1(t) = t$ and $h_2(t) = t\ln t$ are solutions on this interval.

   (b) Do these two solutions generate the general solution? Explain your reasoning.

PLEDGE: I pledge that during this exam I have neither given nor received assistance or cheated in any other way.

Signature:

(End of Exam)