Math 51 **Homework 11** **Spring 2021**

Homework Exercises:

1. Let $f(t) = te^{3t}$. Calculate the Laplace transform $F(s) = \mathcal{L}[f(t)]$ directly from the definition and indicate the values of s for which the integral defining $F(s)$ converges.

   Solution: From the definition,

$$\begin{aligned}
F(s) &= \lim_{h\to\infty} \int_0^h te^{3t}e^{-st}dt \\
&= \lim_{h\to\infty} \int_0^h te^{(3-s)t}dt \\
&= \lim_{h\to\infty} \left[-\int_0^h \frac{e^{(3-s)t}}{3-s}dt + \frac{te^{(3-s)t}}{3-s}\bigg|_{t=0}^{t=h} \right] \\
&= \lim_{h\to\infty} \left[-\frac{e^{(3-s)t}}{(3-s)^2}\bigg|_{t=0}^{t=h} + \frac{he^{(3-s)h}}{3-s} \right] \\
&= \lim_{h\to\infty} \left[-\frac{e^{(3-s)h}}{(3-s)^2} + \frac{1}{(3-s)^2} + \frac{he^{(3-s)h}}{3-s} \right]
\end{aligned}$$

   Let's consider each term. The second term does not depend on the limit and is reasonable for $s \neq 3$. The other two terms will converge as long as the argument of the exponential is negative, which means we need for $s > 3$. In each case, for $s > 3$, the exponential goes to 0 and therefore

$$F(s) = \frac{1}{(3-s)^2}, \quad s > 3.$$

2. For each of the following functions, calculate its Laplace transform $F(s) = \mathcal{L}[f(t)]$ using the linearity of $\mathcal{L}$ together with the basic formulae summarized at the end of $\S5.2$.

   Solution:

   (a)

$$\begin{aligned}
\mathcal{L}[f(t)] &= \mathcal{L}[-3t + e^{-3t} - 5\sin 6t] \\
&= -3\mathcal{L}[t] + \mathcal{L}[e^{-3t}] - 5\mathcal{L}[\sin 6t] \\
&= -\frac{3}{s^2} + \frac{1}{s+3} - \frac{30}{s^2+36}, \quad s > 0
\end{aligned}$$

   (b)

$$\begin{aligned}
\mathcal{L}[f(t)] &= \mathcal{L}[e^{2t+3}] \\
&= e^3\mathcal{L}[e^{2t}] \\
&= \frac{e^3}{s-2}, \quad s > 2
\end{aligned}$$

(c) For this one, we use the trig identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$:

$$\begin{aligned}
\mathcal{L}[f(t)] &= \mathcal{L}[\sin\left(t + \tfrac{\pi}{6}\right)] \\
&= \mathcal{L}[\sin t \cos(\tfrac{\pi}{6}) + \cos t \sin(\tfrac{\pi}{6})] \\
&= \mathcal{L}[\tfrac{\sqrt{3}}{2}\sin t + \tfrac{1}{2}\cos t] \\
&= \tfrac{\sqrt{3}}{2}\mathcal{L}[\sin t] + \tfrac{1}{2}\mathcal{L}[\cos t] \\
&= \frac{\sqrt{3}}{2}\frac{1}{s^2+1} + \frac{1}{2}\frac{s}{s^2+1} \\
&= \frac{\sqrt{3}+s}{2(s^2+1)}, \quad s > 0
\end{aligned}$$

3. For each of the following functions, calculate its inverse transform $f(t) = \mathcal{L}^{-1}[F(s)]$ using the linearity of $\mathcal{L}^{-1}$ together with the basic formulae summarized at the end of §5.2.
   (a) $F(s) = \frac{1}{3s+1}$
   (b) $F(s) = \frac{3}{s^2+1} - \frac{20}{s^4} + \frac{3}{s}$

   Solution:
   (a) $\mathcal{L}^{-1}\left[\frac{1}{3s+1}\right] = \mathcal{L}^{-1}\left[\frac{1}{3(s+1/3)}\right] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1/3}\right] = \frac{1}{3}e^{-t/3}$
   (b) By linearity,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{3}{s^2+1} - \frac{20}{s^4} + \frac{3}{s}\right] &= 3\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - 20\mathcal{L}^{-1}\left[\frac{1}{s^4}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{s}\right] \\
&= 3\sin t - 20\frac{t^{4-1}}{(4-1)!} + 3 \\
&= 3\sin t - \frac{10t^3}{3} + 3.
\end{aligned}$$

4. Use the first differentiation formula to find an expression for the Laplace transform $\mathcal{L}[x]$ where x is the solution of the given initial-value problem.
   (a) $(D-1)x = e^{3t}, \quad x(0) = 3$
   (b) $(D^2-1)x = e^{2t}, \quad x(0) = 0, \ x'(0) = 1$
   (c) $(D^2+1)x = \cos 3t, \quad x(0) = 0, \ x'(0) = 3$

   Solution:
   (a) Taking the Laplace transform of the left-hand side yields

$$\mathcal{L}[(D-1)x] = \mathcal{L}[Dx] - \mathcal{L}[x] = s\mathcal{L}[x] - x(0) - \mathcal{L}[x] = (s-1)\mathcal{L}[x] - 3$$

   while taking the transform of the right-hand side gives

$$\mathcal{L}[e^{3t}] = \frac{1}{s-3}$$

   and therefore we have that

$$(s-1)\mathcal{L}[x] - 3 = \frac{1}{s-3} \quad \Rightarrow \quad \mathcal{L}[x] = \frac{1}{(s-1)(s-3)} + \frac{3}{s-1}.$$

(b) Again, using linearity and the first differentiation formula, taking the Laplace transform of both sides yields

$$s^2\mathcal{L}[x] - sx(0) - x'(0) - \mathcal{L}[x] = \frac{1}{s-2}$$

and after grouping like terms and evaluating the boundary conditions we have

$$(s^2-1)\mathcal{L}[x] - 1 = \frac{1}{s-2}$$

which finally yields

$$\mathcal{L}[x] = \frac{1}{(s^2-1)(s-2)} + \frac{1}{s^2-1}.$$

(c) As before applying the Laplace transform yields

$$s^2\mathcal{L}[x] - sx(0) - x'(0) + \mathcal{L}[x] = \frac{s}{s^2+9}$$

which, after some rearranging, becomes

$$\mathcal{L}[x] = \frac{s}{(s^2+9)(s^2+1)} + \frac{3}{s^2+1}.$$

5. Find the inverse transform of $F(s) = \dfrac{s+4}{s^2+4s+3}$.

Solution: We'll need to use partial fraction decomposition here. The denominator factors as $s^2+4s+3 = (s+3)(s+1)$ and therefore we aim to find some a and b such that

$$F(s) = \frac{s+4}{(s+3)(s+1)} = \frac{a}{s+3} + \frac{b}{s+1}.$$

Let's multiply through both sides by the factors in the denominators, and then choose convenient values of s to determine a and b. We have

$$s+4 = a(s+1) + b(s+3)$$

and if we plug in $s=-3$ then we find that $a=-1/2$, whereas if we plug in $s=-1$ we get that $b=3/2$. Therefore

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[-\frac{1}{2(s+3)} + \frac{3}{2(s+1)}\right] = -\frac{e^{-3t}}{2} + \frac{3e^{-t}}{2}$$

6. Use the Laplace transform to solve the initial-value problem:

$$(D^2+4)x = t, \quad x(0) = -1, x'(0) = 0.$$

Solution: We start by applying the Laplace transform to both sides of the equation and using the first differentiation formula to get that

$$s^2\mathcal{L}[x] - sx(0) + x'(0) + 4\mathcal{L}[x] = \frac{1}{s^2}$$

which simplifies to

$$\mathcal{L}[x] = \frac{1}{s^2(s^2+4)} - \frac{s}{s^2+4}$$

The first component must be decomposed to move further. For some a_1, a_2, b_1, and b_2 we have

$$\frac{1}{s^2(s^2+4)} = \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{b_1 s + b_2}{s^2+4}.$$

To determine the constants, let's multiply both sides by the denominator on the left. This gives us

$$1 = a_1 s(s^2+4) + a_2(s^2+4) + b_1 s^3 + b_2 s^2.$$

Let's match terms by order. We have the following

$$\begin{aligned} 1 &= 4a_2 \\ 0 &= 4a_1 s \\ 0 &= a_2 s^2 + b_2 s^2 \\ 0 &= a_1 s^3 + b_1^3 \end{aligned}$$

This gives us that $a_2 = -b_2 = 1/4$ while $a_1 = b_1 = 0$. Therefore

$$\mathcal{L}[x] = \frac{1}{4s^2} - \frac{1}{4(s^2+4)} - \frac{s}{s^2+4}$$

which, upon inverting, yields the specific solution

$$x(t) = \frac{t}{4} - \frac{1}{8}\sin 2t - \cos 2t.$$