Midterm Exam

Math 51 Spring 2021 – Tufts University

2022-02-14

There are **6 problems** on the exam.

You may not use *calculators*, *books* or *notes* during the exam. All electronic devices (including your phones) must be *silenced* and *put away* for the duration of the exam.

After finishing your exam, you will submit this exam booklet. We will scan your submission and upload it to *Gradescope* for marking (you do *not* need to take images of your exam). You should write your name at the top of each page, as indicated (*especially* if you remove the staples from your exam booklet).

For the partial credit problems, always *show your work*. Try to fit this work in the *available space* if possible. There is a blank page at the back of your exam for use as scratch paper. If you need more space for a solution, please write clearly *in the indicated space* that your solution continues later.

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Please print your name, and sign your exam. With your signature, you pledge that you have neither given nor received assistance on this exam.

Name (printed): Solutions

Signature: ____________________

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Name: ____________________________

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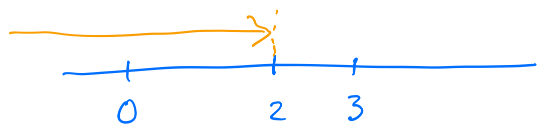
1. 20 points) Short-answer questions. Each question is worth two points.
No work needs to be shown, as only the answer will be graded. Write your answers in the indicated boxes.

(a) Consider the differential equation

$$(t-3)^2 x'' + \frac{1}{t-2} x' + 2x = 0.$$

Find the largest open interval containing $t = 0$ on which this o.d.e. is normal.



$\boxed{(-\infty, 2)}$

(b) The order of the differential equation $t^3(x'')^3 + 3x' + tx^4 = 0$ is

The order is the highest derivative in the equation, in this case x''.

$\boxed{2}$

(c) True or False. Let h_1, h_2, h_3 be solutions of a normal third-order linear differential equation on $(-\infty, \infty)$. If the Wronskian satisfies $W[h_1, h_2, h_3](t_0) = 0$ at one real number t_0 then it satisfies $W[h_1, h_2, h_3](t) = 0$ for *every* real number t.

$W[h_1, h_2, h_3](t_0) = 0$ at $t_0 \in \mathbb{R}$

$\Leftrightarrow$ h_1, h_2, h_3 do not generate the general solution of the associated homogeneous equation

$\Leftrightarrow$ $W[h_1, h_2, h_3](t) = 0$ at any $t \in \mathbb{R}$

$\boxed{\text{True}}$

(d) True or False. Let h_1, h_2, h_3 be solutions of a normal, **fourth**-order linear differentiable equation on $(-\infty, \infty)$. If the Wronskian satisfies $W[h_1, h_2, h_3](0) = 1$, then the general solution is given by

$$x(t) = c_1 h_1(t) + c_2 h_2(t) + c_3 h_3(t).$$

For a 4th-order linear o.d.e., four solutions are needed to generate.

$\boxed{\text{False}}$

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(e) True or False. The linear differential equation $t^3x'' + 3x' + x + t = 0$ is homogeneous.

In standard form,

$$t^3x'' + 3x' + x = -t.$$

False

(f) True or False. If $h_1(t), h_2(t)$ are solutions of the differential equation

$$x'' + x + 1 = 0$$

on an interval I, then the general solution is given by

$$x(t) = c_1h_1(t) + c_2h_2(t).$$

The equation $x'' + x = -1$ is nonhomogeneous.

False

(g) Answer Yes or No. Suppose $h_1(t)$ and $h_2(t)$ are any two solutions of the differential equation $t^3x'' + tx' + (t^2 - 1)x = 1$. Is $h_1(t) + h_2(t)$ a solution of this differential equation?

The o.d.e. is nonhomogeneous.

No

(h) Answer Yes or No. Suppose $h_1(t)$ and $h_2(t)$ are two any solutions of the differential equation $t^3x'' + tx' + (t^2 - 1)x = 0$. Is $h_1(t) + h_2(t)$ a solution of this differential equation?

The o.d.e. is homogeneous.

Yes

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(i) True or False. If $x = h(t)$ is a solution of the differential equation $(D-3)^2 x = 0$, then it is necessarily a solution of the differential equation $(D-3)^2(D+1)x = 0$.

True

(j) Multiple-Choice. Suppose $h_1(t), h_2(t), h_3(t)$ are solutions of $x''' - 3x' + x = 0$ on $(-\infty, \infty)$ and that the Wronskian satisfies $W[h_1, h_2, h_3](1) \neq 0$. Which of the following statements is true?

I. $h_1(t), h_2(t), h_3(t)$ generate the general solution of $L(x) = 0$ on $(-\infty, \infty)$.

II. $h_1(t), h_2(t), h_3(t)$ are linearly independent on $(-\infty, \infty)$.

Write one of A, B, C, D, or E in the box.

A. Only I is true.

B. Only II is true.

C. Both I and II are true.

D. Neither I nor II is true.

E. It is not possible to determine if the statements are true or false from the given information.

C

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2. (15 points)

(a) Calculate $\det \begin{bmatrix} 3 & 0 & 2 \\ 5 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$.

Method 1. $\begin{vmatrix} 3 & 0 & 2 \\ 5 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = 3\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix}$

(Expand along 1st row)

$= 3 + 2(-7) = \boxed{-11}$

Method 2. $\left|\begin{matrix} 3 & 0 & 2 \\ 5 & 1 & 0 \\ 2 & -1 & 1 \end{matrix}\right| \begin{matrix} 3 & 0 \\ 5 & 1 \\ 2 & -1 \end{matrix}$

$= 3\cdot 1\cdot 1 + 0\cdot 0\cdot 2 + 2\cdot 5\cdot(-1)$
$\quad - 2\cdot 1\cdot 2 - (-1)\cdot 0\cdot 3 - 1\cdot 5\cdot 0$

$= 3 - 10 - 4 = \boxed{-11}$

(b) The functions $f_1(t) = e^t$, $f_2(t) = te^t$, and $f_3(t) = 1$ are solutions to the differential equation

$$x''' - 2x'' + x' = (D-1)^2 Dx = 0.$$

Use the Wronskian test to confirm that these functions generate the general solution to this ODE.

$$W[e^t \ \ te^t \ \ 1] = \begin{vmatrix} e^t & te^t & 1 \\ e^t & e^t + te^t & 0 \\ e^t & 2e^t + te^t & 0 \end{vmatrix} = \begin{vmatrix} e^t & (1+t)e^t \\ e^t & (2+t)e^t \end{vmatrix}$$

$$= e^{2t}\begin{vmatrix} 1 & 1+t \\ 1 & 2+t \end{vmatrix} = e^t \neq 0$$

(c) Decide whether the functions $g_1(t) = \dfrac{t^2}{2}$, $g_2(t) = -t^2$ are *linearly independent*.

$$2\cdot \frac{t^2}{2} + (-t^2) = 0$$

$2g_1(t) + g_2(t) = 0$ is a nontrivial linear relation.

Therefore, $g_1(t)$ and $g_2(t)$ are not linearly independent.

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3. (15 points) Write the general solution for each differential equation below.

(a) $D(D^2 - 2D - 1)x = 0$.

$D(D-2)(D+1)\,x = 0$

Three solutions are 1, e^{2t}, e^{-t}.

The general solution is $\boxed{x(t) = c_1 + c_2 e^{2t} + c_3 e^{-t}.}$

(b) $(D^2 - 3)^2 x = 0$.

$(D+\sqrt{3})^2 (D-\sqrt{3})^2 x = 0$

Some solutions are

$$e^{-\sqrt{3}t},\quad te^{-\sqrt{3}t},\quad e^{\sqrt{3}t},\quad te^{\sqrt{3}t}.$$

These are linearly independant.

The general solution is

$$x(t) = \boxed{c_1 e^{-\sqrt{3}t} + c_2 t e^{-\sqrt{3}t} + c_3 e^{\sqrt{3}t} + c_4 t e^{\sqrt{3}t}.}$$

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4. (15 points) Use the method of variation of parameters to find the general solution of

$$\frac{dx}{dt} + 2x = e^{-2t}\sqrt{t^3}. \quad (N)$$

First solve the homogeneous equation

$$x' + 2x = 0 \quad (H).$$

The solution is $x = k e^{-2t}$.

Next vary the parameter k and plug $x = k(t) e^{-2t}$ into (N).

$$x' + 2x = k' e^{-2t} - \cancel{2k e^{-2t}} + \cancel{2k e^{-2t}}$$

$$= k' \cancel{e^{-2t}} = \cancel{e^{-2t}} \sqrt{t^3}.$$

$$k' = t^{3/2}$$

$$k = \int t^{3/2}\, dt = \frac{2}{5} t^{5/2} + C.$$

The general solution is

$$x(t) = \left(\frac{2}{5} t^{5/2} + C\right) e^{-2t}$$

$$= \boxed{\underbrace{\frac{2}{5} t^{5/2} e^{-2t}}_{\text{particular solution to (N)}} + \underbrace{C e^{-2t}}_{\text{general solution to (H)}}}.$$

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5. (15 points) A drug is absorbed from the bloodstream into the body at a rate proportional to the amount of the drug present in the bloodstream after t hours. If there are $x(t)$ mg of the drug present in the bloodstream at time t, assume that the drug is absorbed at a rate of $0.5x(t)$ mg/hour.

Also assume that the drug is administered intravenously into a patient's bloodstream at a constant rate of 3 mg/hour.

(a) $x(t)$ is a solution to a differential equation of the form $\frac{dx}{dt} = \lambda x + r$. Indicate the values of λ and r, and give the differential equation to which $x(t)$ is a solution.

$$\frac{dx}{dt} = -\frac{1}{2}x + 3$$

(absorbed by the body = leaving the blood stream. Hence, the coefficient $\frac{1}{2}$ comes with a minus sign.)

(b) Find the general solution to the differential equation you found in (a).

In standard form,

$$x' + \frac{1}{2}x = 3.$$

1) Integrating factor $\rho(t) = e^{\int \frac{1}{2}dt} = e^{\frac{1}{2}t}$

2) $e^{\frac{1}{2}t}x' + \frac{1}{2}e^{\frac{1}{2}t}x = 3e^{\frac{1}{2}t}$

3) $e^{\frac{1}{2}t}x = \int 3e^{\frac{1}{2}t}dt = 6e^{\frac{1}{2}t} + C$

$$\boxed{x = 6 + Ce^{-\frac{1}{2}t}}$$

(c) If the patient has 0 mg of the drug in their bloodstream at time $t = 0$, how much is present after 2 hours?

$x(0) = 6 + Ce^{0} = 6 + C = 0 \Rightarrow C = -6.$

$x = 6 - 6e^{-\frac{1}{2}t}$

$x(2) = 6 - 6e^{-1} = \boxed{6\left(1 - \frac{1}{e}\right)}.$

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6. (20 points)

(a) Solve the initial value problem

$$(D^2 - 16)x = 0, \quad x(0) = 0, \quad x'(0) = 4.$$

$(D+4)(D-4)x = 0$

The general solution is

$$x = c_1 e^{-4t} + c_2 e^{4t}.$$

Then $x' = -4c_1 e^{-4t} + 4c_2 e^{4t}.$

The initial conditions are

$$x(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 0 \qquad (1)$$

$$x'(0) = -4c_1 + 4c_2 = 4. \Rightarrow -c_1 + c_2 = 1. \qquad (2)$$

$(1)+(2)$ $\quad 2c_2 = 1 \Rightarrow c_2 = \frac{1}{2} \Rightarrow c_1 = -\frac{1}{2}.$

$$\boxed{x(t) = -\frac{1}{2} e^{-4t} + \frac{1}{2} e^{4t}}.$$

(b) Solve the initial value problem

$$t^2 \frac{dx}{dt} = x, \quad t > 0, \quad x(1) = 1.$$

$\frac{dx}{x} = \frac{dt}{t^2}$ since $x \neq 0$.

$$\int \frac{dx}{x} = \int \frac{dt}{t^2}$$

$$\ln|x| = -\frac{1}{t} + C$$

$$|x| = e^{\ln|x|} = e^{-\frac{1}{t} + C} = e^C e^{-\frac{1}{t}}$$

$$x = \pm e^C e^{-\frac{1}{t}} = k e^{-\frac{1}{t}}, \quad k \neq 0.$$

$$x(1) = k e^{-1} = 1 \Rightarrow k = e.$$

$$\boxed{x(t) = e\, e^{-\frac{1}{t}} = e^{(1-\frac{1}{t})}}.$$

End of exam