Differential Equations

Row Reduction

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Systems of Algebraic Equations

A system of m algebraic equations in n unknowns

$$\begin{aligned} v_1 a_{11} &+ v_2 a_{12} + \cdots + v_n a_{1n} = s_1 \\ v_1 a_{21} &+ v_2 a_{22} + \cdots + v_n a_{2n} = s_2 \\ &\vdots \\ v_1 a_{m1} &+ v_2 a_{m2} + \cdots + v_n a_{mn} = s_m. \end{aligned}$$

can be written as the vector equation

$$A\vec{v} = \vec{s},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}.$$

Ex: The system

$$\begin{aligned} 2u_1 + u_2 + 4u_3 + 3u_4 &= 3 \\ 2u_2 + 2u_4 + 5u_5 &= 8 \\ -u_1 + 4u_2 - 2u_3 + 3u_4 + 6u_5 &= 6 \\ u_1 + 2u_3 + u_4 - 2u_5 &= -2 \end{aligned} \tag{S}$$

can be written as

$$\begin{bmatrix} 2 & 1 & 4 & 3 & 0 \\ 0 & 2 & 0 & 2 & 5 \\ -1 & 4 & -2 & 3 & 6 \\ 1 & 0 & 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 6 \\ -2 \end{bmatrix}.$$

Def: The augmented matrix of $A\vec{v} = \vec{s}$ is

$$[\, A \mid \vec{s}\,] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & s_1 \\ a_{21} & a_{22} & \dots & a_{2n} & s_2 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & s_m \end{array}\right].$$

Operations on the equations that do not change the solutions:

- Adding a multiple of one equation to another equation.
  $(\alpha) \to (\alpha) + c(\beta)$
- Multiplying an equation by a nonzero number.
  $(\alpha) \to c(\alpha)$
- Swapping two equations.
  $(\alpha) \leftrightarrow (\beta)$

Operations on the augmented matrix that do not change the solutions:

- Adding a multiple of one row to another row.
  $R_\alpha \to R_\alpha + cR_\beta$
- Multiplying a row by a nonzero number.
  $R_\alpha \to cR_\alpha$
- Swap two rows.
  $R_\alpha \leftrightarrow R_\beta$

Def: The three matrix operations are called row operations. Two matrices are row equivalent if we can get from one to the other by a sequence of row operations.

Fact: If $[A \mid \vec{s}\,]$ and $[B \mid \vec{r}\,]$ are row equivalent, then $A\vec{v} = \vec{s}$ and $B\vec{v} = \vec{r}$ have the same solutions.

Def: A matrix is reduced if

1. Any rows of all zeros are at the bottom.
2. The first nonzero entry of each nonzero row (a pivot) is 1.
3. Each pivot is to the right of the pivots of the preceding rows.
4. Each pivot is the only nonzero entries in its column.

Fact: Every matrix is row equivalent to exactly one reduced matrix.

The process of finding the reduced matrix is called reduction.

Ex: Solve (S).

Sln: We reduce the augmented matrix:

$$\left[\begin{array}{ccccc|c} 2 & 1 & 4 & 3 & 0 & 3 \\ 0 & 2 & 0 & 2 & 5 & 8 \\ -1 & 4 & -2 & 3 & 6 & 6 \\ 1 & 0 & 2 & 1 & -2 & -2 \end{array}\right] \xrightarrow[R_4 \leftrightarrow R_1]{} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -2 & -2 \\ 0 & 2 & 0 & 2 & 5 & 8 \\ -1 & 4 & -2 & 3 & 6 & 6 \\ 2 & 1 & 4 & 3 & 0 & 3 \end{array}\right] \xrightarrow[\substack{R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 2R_1}]{} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -2 & -2 \\ 0 & 2 & 0 & 2 & 5 & 8 \\ 0 & 4 & 0 & 4 & 4 & 4 \\ 0 & 1 & 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow[R_4 \leftrightarrow R_2]{}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -2 & -2 \\ 0 & 1 & 0 & 1 & 4 & 7 \\ 0 & 4 & 0 & 4 & 4 & 4 \\ 0 & 2 & 0 & 2 & 5 & 8 \end{array}\right] \xrightarrow[\substack{R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 2R_2}]{} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -2 & -2 \\ 0 & 1 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 & -12 & -24 \\ 0 & 0 & 0 & 0 & -3 & -6 \end{array}\right] \xrightarrow[\substack{R_3 \to R_3/(-12) \\ R_4 \to R_4/(-3)}]{}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -2 & -2 \\ 0 & 1 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_3]{} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -2 & -2 \\ 0 & 1 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 - 4R_3}]{} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We get an equivalent system

$$\begin{aligned} u_1 \quad + 2u_3 + u_4 \quad &= \quad 2 \\ u_2 \qquad\quad + u_4 \quad &= -1 \\ u_5 &= \quad 2 \\ 0 &= \quad 0. \end{aligned}$$

We solve for the pivot variables u_1, u_2, and u_5 in terms of the nonpivot (free) variables:

$$u_1 = 2 - 2u_3 - u_4, \quad u_2 = -1 - u_4, \quad u_5 = 2.$$

Any choice of the free variables u_3 and u_4, say $u_3 = a$ and $u_4 = b$, leads to a solution of (S)

$$u_1 = 2-2a-b, \;\; u_2 = -1-b, \;\; u_3 = a, \;\; u_4 = b, \;\; u_5 = 2,$$

or in vector form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 2-2a-b \\ -1-b \\ a \\ b \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + a \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Ex: The eigenvalues of

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -3/2 & 3/2 \\ 0 & 1 & -1 \end{bmatrix}$$

are $\lambda = -1, -2, 1/2$ (from Section 3.5). Find the eigenvectors of A corresponding to each eigenvalue.

Sln: To find eigenvectors of A corresponding to $\lambda = -1$, we solve $[A-(-1)I]\vec{v} = \vec{0}$, or reduce $[A+I \mid \vec{0}\,]$:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1/2 & 3/2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{} \begin{bmatrix} 1 & -1/2 & 3/2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_2 \to -R_2]{}$$

$$\begin{bmatrix} 1 & -1/2 & 3/2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 + R_2/2} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The equivalent system

$$\begin{aligned} v_1 \quad + (3/2)v_3 &= 0 \\ v_2 \qquad\qquad &= 0 \\ 0 &= 0. \end{aligned}$$

can be solved for the pivot variables v_1 and v_2 in terms of the free variable v_3:

$$v_1 = -(3/2)v_3, \quad v_2 = 0.$$

Any choice of v_3, say $v_3 = 2a$, leads to a solution. In vector form, the solutions are

$$\vec{v} = \begin{bmatrix} -3a \\ 0 \\ 2a \end{bmatrix} = a \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}.$$

The nonzero vectors of this form are the eigenvectors for $\lambda = -1$.

To find the eigenvectors for $\lambda = -2$, we solve $(A+2I)\vec{w} = \vec{0}$, or reduce $[A+2I \mid \vec{0}\,]$:

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1/2 & 3/2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3/2 & 3/2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 \to 2R_2/3]{}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The system corresponding to the last matrix is

$$\begin{aligned} w_1 \quad + w_3 &= 0 \\ w_2 + w_3 &= 0 \\ 0 &= 0. \end{aligned}$$

Any choice of the free variable w_3, say $w_3 = a$, leads to a solution

$$\vec{w} = \begin{bmatrix} -a \\ -a \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The eigenvectors for $\lambda = -2$ are the nonzero vectors of this form.

To find the eigenvectors for $\lambda = -1/2$, we solve $[A - (-1/2)I]\vec{u} = \vec{0}$ by reducing $[A + (1/2)I \mid \vec{0}\,]$:

$$\begin{bmatrix} -1/2 & -1 & 0 \\ 1 & -1 & 3/2 \\ 0 & 1 & -1/2 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{} \begin{bmatrix} 1 & -1 & 3/2 \\ -1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 + R_1/2]{} \begin{bmatrix} 1 & -1 & 3/2 \\ 0 & -3/2 & 3/4 \\ 0 & 1 & -1/2 \end{bmatrix}$$

$$\xrightarrow[R_3 \leftrightarrow R_2]{} \begin{bmatrix} 1 & -1 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & -3/2 & 3/4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_2/2]{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, we have

$$u_1 = -u_3, \quad u_2 = \frac{1}{2} u_3.$$

The eigenvectors of A with eigenvalues $\lambda = -1/2$ are the nonzero vectors of the form

$$\vec{u} = \begin{bmatrix} -2a \\ a \\ 2a \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Ex: The eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

are $\lambda = 3, -1$ (from Section 3.5). Find the eigenvectors of A corresponding to each eigenvalue.

Sln: To find eigenvectors of A corresponding to $\lambda = 3$, we solve $(A - 3I)\vec{w} = \vec{0}$, by reducing $[A - 3I \mid \vec{0}\,]$:

$$\begin{bmatrix} -2 & 0 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_1]{} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 2R_1]{} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, w_2 and w_3 are free, and we can solve for the pivot variable w_1 in terms of w_2 and w_3:

$$w_1 = 2w_3.$$

Any choice of w_2 and w_3, say $w_2 = a$ and $w_3 = b$, gives a solution of $(A - 3I)\vec{w} = \vec{0}$ in the form

$$\vec{w} = \begin{bmatrix} 2b \\ a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} 2b \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Any nonzero vector of this form is an eigenvector. In particular, the vectors

$$\vec{w_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (\text{take } a = 1 \text{ and } b = 0)$$

and

$$\vec{w_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad (\text{take } a = 0 \text{ and } b = 1)$$

are eigenvectors for $\lambda = 3$. Note that the two eigenvectors are linearly independent.

To find eigenvectors of A corresponding to $\lambda = -1$, we solve $[A - (-1)I]\vec{v} = \vec{0}$, or reduce $[A + I \mid \vec{0}\,]$:

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_1]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2/4 \\ R_3 \to R_3 - 2R_1}]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equivalent system

$$\begin{aligned} v_1 \quad + 2v_3 &= 0 \\ v_2 \qquad &= 0 \\ 0 &= 0. \end{aligned}$$

can be solved for the pivot variables v_1 and v_2 in terms of the free variable v_3:

$$v_1 = -2v_3, \quad v_2 = 0.$$

Any choice of v_3, say $v_3 = a$, leads to a solution in vector form

$$\vec{v} = \begin{bmatrix} -2a \\ 0 \\ a \end{bmatrix} = a \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

The nonzero vectors of this form are the eigenvectors of A corresponding to the eigenvalue $\lambda = -1$.

Fact: The solutions of a system $A\vec{v} = \vec{0}$ are described by an arbitrary linear combination

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

of k solutions $\vec{v}_1, \ldots, \vec{v}_k$, where k is the number of free variables in the reduced augmented matrix of the system. The k vectors are linearly independent.

Fact: If all eigenvectors $\vec{v}$ of A corresponding to the same eigenvalue λ are described by

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k,$$

where each c_i stands for an arbitrary choice of a free variable in the reduced augmented matrix of the system $(A - \lambda I)\vec{v} = \vec{0}$, then $\vec{v}_1, \ldots, \vec{v}_k$ are linearly indepedent eigenvectors of A corresponding to λ.

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