Math 51 Lecture 13

Convolution, Integrating Factors, Review

Convolution.

Problem. Find $\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]$.

We know $\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$ and $\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$.

The convolution allows us to find $\mathcal{L}^{-1}$ of the product $\frac{s}{s^2+1} \cdot \frac{1}{s^2+1}$.

Convolution

not ∞

$$(f * g)(t) = \int_0^{t} f(t-u)\, g(u)\, du$$

constant variable

$$\mathcal{L}[(f*g)(t)] = F(s)G(s)$$

$$\mathcal{L}^{-1}[F(s)G(s)] = (f*g)(t) = \mathcal{L}^{-1}[F(s)] * \mathcal{L}^{-1}[G(s)].$$

Useful Formulas (not to be learned, in textbook, §5.6, Ex. 8, p.462)

(1) $(\sin \alpha t) * (\cos \alpha t) = \frac{t}{2} \sin \alpha t$

(2) $(\sin \alpha t) * (\sin \alpha t) = \frac{1}{2\alpha} \sin \alpha t - \frac{t}{2} \cos \alpha t$

(3) $(\cos \alpha t) * (\cos \alpha t) = \frac{1}{2\alpha} \sin \alpha t + \frac{t}{2} \cos \alpha t$.

Ex. $\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]$

$$= \mathcal{L}^{-1}\left[\frac{s}{s^2+1} \cdot \frac{1}{s^2+1}\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \cos t * \sin t = \frac{t}{2} \sin t.$$

Integrating Factors

Ex. Solve $3x' + 2x = 1$.

$$x' + \frac{2}{3}x = \frac{1}{3} \qquad (N)$$

Solve $x' + \frac{2}{3}x = 0$. $\quad (H)$

Earlier method: Solve (H) by separation of variables.

Then solve (N) by VoP.

Integrating factor for $\underset{\uparrow}{x}' + r(t)\,x = q(t) \qquad (*)$

leading coef 1

$$\rho(t) = e^{\int r(t)\,dt}.$$

- only for lin. 1st-order DE in standard form $(*)$

Ex. Solve $3x' + 2x = 1$.

(1) Divide by 3: $x' + \frac{2}{3}x = \frac{1}{3} \qquad (N)$

(2) Integrating factor: $\rho(t) = e^{\int \frac{2}{3}dt} = e^{\frac{2}{3}t}$

(3) Multiply (N) by $\rho(t)$: $\underbrace{e^{\frac{2}{3}t}x' + \frac{2}{3}e^{\frac{2}{3}t}x}_{(e^{\frac{2}{3}t}x)'} = \frac{1}{3}e^{\frac{2}{3}t}$ by prod. rule.

(4) Integrate: $e^{\frac{2}{3}t}x = \int \frac{1}{3}e^{\frac{2}{3}t} = \frac{1}{2}e^{\frac{2}{3}t} + C$

(5) Multiply by $e^{-\frac{2}{3}t}$: $\boxed{x = \frac{1}{2} + Ce^{-\frac{2}{3}t}}$

part. sol to (N) $\qquad$ gen. sol to (H)

This method always works for lin. 1st-order DE.

Review: Single Equation

First-Order DE

1) General: $A(x,t)\frac{dx}{dt} = B(x,t)$.

- $\frac{dx}{dt} = f(t) \Rightarrow$ integrate
- Separation of variables if possible

2) Linear: (H) $\frac{dx}{dt} + r(t)x = 0$. $\left(\Leftrightarrow \frac{dx}{dt} = -r(t)x\right)$

- separation of variables
- integrating factor
- formula: $x = Ce^{-\int r(t)\,dt}$

(N) $\frac{dx}{dt} + r(t)x = g(t)$

- Book: Solve (H) by separation of variables
  Then variation of parameters (not recommended)
- Better: $\boxed{\text{Integrating factor}}$

Higher-Order DE (Linear Constant Coefficients)

(H) homogeneous $L(D)x = 0$.

$L(D)$	lin indep solutions
D^3	$1, t, t^2$
$D-\lambda$	$e^{\lambda t}$
$(D-\lambda)^3$	$e^{\lambda t}, te^{\lambda t}, t^2e^{\lambda t}$
$D-(a\pm ib)$	$e^{at}\cos bt, e^{at}\sin bt$
$(D-(a\pm ib))^2$	$e^{at}\cos bt, e^{at}\sin bt, te^{at}\cos bt, te^{at}\sin bt$

(N) nonhomogeneous $(D^3 + b_1 D^2 + b_2 D + b_3)\, x = g(t)$.

- Annihilator (= undetermined coef)

  only works for $g(t)$ a linear combination of

  $$t^k e^{at} \cos bt, \quad t^k e^{at} \sin bt$$

- Variation of parameters

  1) Write down general solution to (H):

  $$h(t) = c_1 h_1 + c_2 h_2 + c_3 h_3$$

  2) Solve

  $$c_1' h_1 + c_2' h_2 + c_3' h_3 = 0$$

  $$c_1' h_1' + c_2' h_2' + c_3' h_3' = 0$$

  $$c_1' h_1'' + c_2' h_2'' + c_3' h_3'' = g(t).$$

- Laplace transform : only for IVP (initial-value problems)

<u>Review: Linear Systems</u> $D\vec{x} = A\vec{x} + \vec{E}(t)$

(H) homogeneous $D\vec{x} = A\vec{x}$

Real eigenvalue λ
w/ eigenvector $\vec{v}$ $\Rightarrow$ solution $e^{\lambda t}\vec{v}$

Complex eigenvalues λ
w/ complex eigenvectors $\vec{v}$ $\Rightarrow$ solutions: $\mathrm{Re}(e^{\lambda t}\vec{v})$, $\mathrm{Im}(e^{\lambda t}\vec{v})$

(Ignore the conjugate)

<u>Repeated eigenvalues</u>

- Find as many lin indep eigenvectors as you can
- Otherwise, find indep. generalized eigenvectors :

  λ multiplicity m $\Rightarrow$ solve $(A-\lambda I)^m \vec{v} = \vec{0}$.

$$\text{sol} = e^{\lambda t}\left[\vec{v} + t(A-\lambda I)\vec{v} + \frac{1}{2}t^2(A-\lambda I)^2\vec{v} + \cdots + \frac{1}{(m-1)!}t^{m-1}(A-\lambda I)^{m-1}\vec{v}\right].$$

(N) nonhomogeneous $D\vec{x} = A\vec{x} + \vec{E}(t)$

VoP : 1) Find general sol

$$\vec{h}(t) = c_1\vec{h}_1(t) + \cdots + c_n\vec{h}_n(t)$$

2) Solve

$$c_1'\vec{h}_1(t) + \cdots + c_n'\vec{h}_n(t) = 0.$$

3) Integrate $c_i'(t)$.

4) Then $\vec{p}(t) = \sum c_i(t)\vec{h}_i(t)$ is a part. sol of (N).