# Review material for the Final Exam

Math 51 Fall 2021 – Z. Nitecki and G. McNinch

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## Overview

The final exam in the course is not cumulative, though of course the latter material probably relies in part on material learned in the first half of the course.

The Final will cover course material from the following sections of (Nitecki and Guterman 1992):

- §2.6 constant coeff linear ODES: Complex Roots
- §2.7 inhomogeneous linear ODEs: undetermined coefficients
- §2.8 inhomogeneous linear ODEs: Variation of Parameters,
- §3.2 linear systems and matrices
- §3.3 linear systems of ODEs
- §3.4 Linear Independence
- §3.5 Eigenvalues, Eigenvectors
- §3.6 Row Reduction
- §3.7 Homogeneous Linear Systems: Real eigenvalues
- §3.8 Homogeneous Linear Systems: Complex eigenvalues
- §3.9 Homogeneous Linear Systems: Double Roots
- §3.10 In-homogeneous Linear Systems (variation of parameters)
- §5.2, §5.3 The Laplace Transform

## Problems

1. Solve the initial value problem $(D^2 - 6D + 10)x = 0$, $x(0) = x'(0) = 1$.

   Solution:

   Note first that the polynomial $r^2 - 6r + 10$ has roots $\lambda = 3 \pm i$.

   Thus we see that $h_1(t) = e^{3t}\cos(t)$ and $h_2(t) = e^{3t}\sin(t)$ generate the general solution to the homogeneous ODE.

   We seek a solution of the form

   $$x(t) = c_1 h_1(t) + c_2 h_2(t) = c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t)$$

   for constants $c_1, c_2$.

   Note that $x'(t) = c_1 e^{3t}(-\sin(t) + 3\cos(t) + c_2 e^{3t}(\cos(t) + 3\sin(t))$.

   We need the conditions $1 = x(0) = c_1$ and $1 = x'(0) = 3c_1 + c_2$.

   Thus $c_1 = 1$ and $c_2 = -2$ so the required solution is

   $$x(t) = e^{3t}\cos(t) - 2e^{3t}\sin(t) = e^{3t}(\cos(t) - 2\sin(t))$$

2. Consider the linear ODE $(D^2-4)x = e^{2t} + e^{3t}$.

   a. Find a simplified guess for a particular solution $p(t)$ for the ODE.

   Solution:

   We must first find an operator $A(D)$ for which $A(D)[e^{2t} = e^{3t}] = 0$. For this, we may take $A(D) = (D-2)(D-3)$.

   Now, any solution to the given ODE will be a solution to the homogeneous ODE $A(D)\cdot(D^2-4)x = 0$, i.e. to

$$(D-2)^2(D-3)(D+2)x = 0.$$

   This homogeneous ODE has solutions

$$e^{2t}, te^{2t}, e^{3t}, e^{-2t}.$$

   Of these,$e^{2t}$ and $e^{-2t}$ are solutions to the homogeneous equation $(D^2-4)x = 0$ corresponding to the original ODE, so we may eliminate these solutions.

   Thus our simplified guess is given by

$$p(t) = k_1 te^{2t} + k_2 e^{3t}$$

   for constants $k_1, k_2$.

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   b. Use the method of undetermined coefficients to find $p(t)$.

   Solution:

   We need $(D^2-4)[p(t)] = e^{2t} + e^{3t}$.

   We first compute

$$(D^2-4)[te^{2t}] = e^{2t}((D+2)^2-4)[t] = e^{2t}(D^2+4D)[t] = 4e^{2t}$$

   and

$$(D^2-4)[e^{3t}] = e^{3t}((D+3)^2-4)[1] = e^{3t}(D^2+6D+5)[1] = 5e^{3t}$$

   With these computations, we now find

$$(D^2-4)[k_1te^{2t} + k_2e^{3t}] = k_1 4e^{2t} + k_2 5e^{3t}.$$

   Thus we require that

$$k_1 4e^{2t} + k_2 5e^{3t} = e^{2t} + e^{3t}.$$

   Since the functions $te^{2t}$ and $e^{3t}$ are linearly independent we find that

$$4k_1 = 1 \quad \text{and} \quad 5k_2 = 1.$$

   Thus $k_1 = 1/4$ and $k_2 = 1/5$ so that

$$p(t) = \frac{te^{2t}}{4} + \frac{e^{3t}}{5}.$$

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3. Use the method of variation of parameters to find the general solution to the linear ODE

$$(D^2+4)x = \frac{1}{\sin(2t)}.$$

Solution:

The general solution to the homog. equation $(D^2+4)x = 0$ is generated by $\cos(2t)$ and $\sin(2t)$.

The Wronskian matrix is given by

$$W = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}.$$

Note that $\det W = 2\cos^2(2t) + 2\sin^2(2t) = 2$.

Now, we seek a particular solution of the form

$$p(t) = c_1(t)\cos(2t) + c_2(t)\sin(2t)$$

where $\mathrm{c}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$ satisfies the matrix equation

$$W\mathrm{c}'(t) = \begin{bmatrix} 0 \\ 1/\sin(2t) \end{bmatrix}.$$

Now, Cramer's Rule shows that

$$c_1'(t) = \frac{\det \begin{bmatrix} 0 & \sin(2t) \\ 1/\sin(2t) & 2\cos(2t) \end{bmatrix}}{2} = 1/2$$

and

$$c_2'(t) = \frac{\det \begin{bmatrix} \cos(2t) & 0 \\ -2\sin(2t) & 1/\sin(2t) \end{bmatrix}}{2} = \frac{\cot(2t)}{2}$$

We now find $c_1(t) and $c_2(t)$ by anti-differentiation. Remember that we may choose any antiderivative.

We find $c_1(t) = \frac{t}{2}$ and $c_2(t) = \frac{1}{4}\ln(|\sin(2t)|)$.

Thus our particular solution is given by

$$p(t) = \frac{t\cos(2t)}{2} + \frac{1}{4}\ln(|\sin(2t)|)\sin(2t)$$

and the general solution is then

$$x(t) = \frac{t\cos(2t)}{2} + \frac{1}{4}\ln(|\sin(2t)|)\sin(2t) + c_1\cos(2t) + c_2\sin(2t).$$

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4. Consider the linear ODE $(\diamondsuit) \qquad (D^3+D)x = e^t$.

   a. Find a matrix $A$ for which

   $$(\heartsuit) \qquad D\vec{x} = A\vec{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}$$

   is an equivalent linear system to $(\diamondsuit)$.

Solution:

Note that if $x$ is a solution to ($\diamondsuit$), then $x''' = -x' + e^t$.

Setting $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ x' \\ x'' \end{bmatrix}$ we find that

$$D\mathrm{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

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b. If $u(t)$ is a solution to ($\diamondsuit$), explain why $\vec{v}(t) = \begin{bmatrix} u(t) \\ u'(t) \\ u''(t) \end{bmatrix}$ is a solution to ($\heartsuit$).

Solution:

If $\mathrm{v} = \begin{bmatrix} u(t) \\ u'(t) \\ u''(t) \end{bmatrix}$ then

$$D\mathrm{v} = \begin{bmatrix} u'(t) \\ u''(t) \\ u'''(t) \end{bmatrix} = \begin{bmatrix} u'(t) \\ u''(t) \\ -u'(t) + e^t \end{bmatrix} = \begin{bmatrix} u'(t) \\ u''(t) \\ -u'(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}$$

On the other hand

$$A \begin{bmatrix} u(t) \\ u'(t) \\ u''(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix} = \begin{bmatrix} u'(t) \\ u'''(t) \\ -u'(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}$$

Since these two expressions coincide, v is indeed a solution.

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5. Find the real and imaginary parts of the vector $(\cos(t) + i\sin(t)) \cdot \begin{bmatrix} 2+i \\ 1 \\ i \end{bmatrix}$ for any value of $t$.

Solution:

First notice that

$$(\cos(t) + i\sin(t))(2+i) = 2\cos(t) - \sin(t) + i(\cos(t) + 2\sin(t))$$

Thus

$$(\cos(t) + i\sin(t)) \cdot \begin{bmatrix} 2+i \\ 1 \\ i \end{bmatrix} = \begin{bmatrix} 2\cos(t) - \sin(t) + i(\cos(t) + 2\sin(t)) \\ \cos(t) + i\sin(t) \\ -\sin(t) + i\cos(t) \end{bmatrix} = \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix} + i \begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}$$

So the real part is $\begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}$ and the imaginary part is $\begin{bmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}$

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6. Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$

a. Find solutions $\vec{h}_1(t)$ and $\vec{h}_2(t)$ that generate the general solution to the linear system $D\vec{x} = A\vec{x}$.

Solution:

The char. poly is $\lambda^2 - 9$ so the eigenvalues are $\pm 3$. To find an eigenvector for $\lambda = 3$, we consider

$$A - 3\mathrm{I}_2 = \begin{bmatrix} -1 & 5 \\ 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$$

and find the eigenvector $\mathrm{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

To find an eigenvector for $\lambda = -3$, we consider

$$A + 3\mathrm{I}_2 = \begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and find the eigenvector $\mathrm{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Thus the general solution is given by

$$x(t) = c_1 e^{3t}\mathrm{v} + c_2 e^{-3t}\mathrm{w} = c_1 e^{3t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

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b. Suppose that $\mathrm{E}(t)$ is a vector valued function, and that we wish to find a particular solution to the non-homogeneous system

$$D\mathrm{x} = A\mathrm{x} + \mathrm{E}(t)$$

of the form $\mathrm{p}(t) = c_1(t)\mathrm{h}_1(t) + c_2(t)\mathrm{h}_2(t)$ where the $\mathrm{h}_i(t)$ are the solutions you found in (c), and $c_1(t)$ and $c_2(t)$ are "unknown functions". What matrix equation must you solve in order to find the derivatives $c_1'(t)$ and $c_2'(t)$?

Solution:

Let $W$ be the matrix whose columns are the solutions $\mathrm{h}_i$; thus

$$W = \begin{bmatrix} 5e^{3t} & -e^{-3t} \\ e^{3t} & e^{-3t} \end{bmatrix}.$$

Form the vector $\mathrm{c}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$ where $c_i(t)$ are the unknown coefficient functions. To find the $c_i(t)$, we must solve the matrix equation

$$W\mathrm{c}'(t) = \mathrm{E}(t)$$

i.e.

$$W \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \mathrm{E}(t).$$

To then find the $c_i(t)$, we must compute antiderivatives:

$$c_i(t) = \int c_i'(t)dt.$$

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7. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

a. Find the characteristic polynomial and show that eigenvalues of $A$ are $\lambda = 2, -1$.

Solution:

Since $A$ is upper triangular, $A - \lambda \mathrm{I}_3$ is also upper triangular. Thus, the char poly – i.e. the determinant of $A - \lambda \mathrm{I}_3$ – is equal to $(2-\lambda)^2(1-\lambda)$

Thus the eigenvalues of $A$ are as indicated.

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b. Find two linearly independent generalized eigenvectors for $\lambda = 2$. Note that there are not 2 linearly independent eigenvectors for $\lambda = 2$.

Solution:

Examining the matrix

$$A - 2\mathrm{I}_3 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that $\mathrm{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the only eigenvector for $\lambda = 2$.

On the other hand, consider

$$(A - 2\mathrm{I}_2)^2 = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Recall that the generalized eigenvectors are the solutions to $(A - 2\mathrm{I}_2)^2\mathrm{v} = 0$. Thus, in this case the generalized eigenvectors are generated by the two linearly independent vectors $\mathrm{e}_1$ and $\mathrm{e}_2$.

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c. Find the general solution to the linear system $D\vec{x} = A\vec{x}$.

Solution:

Using the generalized eigenvectors from b. we find solutions

$$h_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{aligned} h_2(t) &= e^{2t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t(A - 2\mathrm{I}_3)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = e^{2t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \\ &= e^{2t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) \\ &= e^{2t}\begin{bmatrix} 2t \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

It remains to find an eigenvector for $\lambda = -1$.

For this, we look at

$$A + \mathrm{I}_3 = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and find the eigenvector $\mathrm{w} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$.

Thus we find the solution

$$\mathrm{h}_3(t) = e^{-t}\mathrm{w} = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}.$$

So the general solution is

$$\mathrm{x}(t) = c_1 \mathrm{h}_1(t) + c_2 \mathrm{h}_2(t) + c_3 \mathrm{h}_3(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

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8. Let $B = \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   a. Find all solutions to the matrix equation $B\vec{v} = \vec{0}$.

   Solution:

   The matrix is (essentially) already in echelon form, and we see that the variables $v_1$, $v_2$, and $v_4$ are pivot variables while $v_3$ and $v_5$ are free variables.

   We find one solution by setting $v_3 = 1$ and $v_5 = 0$: we find in this case that $v_4 = 0$, $2v_2 = v_3$, and $v_1 = v_2$. Thus one solution is

$$\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

   We find another solution by setting $v_3 = 0$ and $v_5 = 1$. In this case we find that $v_4 = -3v_5 = -3$, $2v_2 = -2v_4 = 6$ and $v_1 = v_2 - v_4 - v_5 = 3 + 3 - 1$. Thus the other solution is

$$\begin{bmatrix} 5 \\ 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

   Thus every solution of $B\mathrm{v} = 0$ has the form

$$v_3 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v_5 \begin{bmatrix} 5 \\ 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

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b. Let $\vec{b}_1, \dots, \vec{b}_5$ denote the columns of the matrix $B$. Thus the $\vec{b}_i$ are vectors in $\mathrm{R}^4$. Are these vectors linearly independent? Why or why not?

Solution:

Since $B \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0$, we see that

$$(1/2)\mathrm{b}_1 + (1/2)\mathrm{b}_2 + \mathrm{b}_3 = 0$$

which shows that the vectors are linearly dependent (i.e. not linearly independent).

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9. Find the solution $\vec{h}(t)$ to the homogeneous system

$$D\vec{x} = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

which satisfies $\vec{h}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. What is $\vec{h}(1)$?

Solution:

The char. poly is $-(\lambda - 2)^2(\lambda - 1)$.

An eigenvector for $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = 2$ are generated by

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

.

Thus the general solution is

$$\mathrm{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

In order that $\mathrm{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, we need

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

To solve this equation, we perform row operations on the augment matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

This shows that $c_1 = 2$, $c_2 = -1$ and $c_3 = 0$ so the required solution is

$$\mathrm{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + -e^{2t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

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10. Suppose that the $3 \times 3$ matrix $A$ has eigenvalues 2 and $1 \pm 3i$, that $\mathrm{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is an eigenvector for $\lambda = 2$, and that $\mathrm{w} = \begin{bmatrix} u_1 + w_1 i \\ u_2 + w_2 i \\ u_3 + w_3 i \end{bmatrix}$ is an eigenvector for $\lambda = 1 + 3i$.

    Describe three real solutions to the homogeneous system of linear ODES $D\mathrm{x} = A\mathrm{x}$ with linearly independent initial vectors.

    Solution:

    One real solution is determined by the eigenvector v; it is given by

$$\mathrm{h}_1 = e^{2t} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

    For the other two solutions, we use the eigenvector w. We consider the "complex solution"

$$e^{(1+3i)t}\mathrm{w} = e^t(\cos(3t) + i\sin(3t))\mathrm{w} \quad (*)$$

    We take for $\mathrm{h}_2(t)$ the real part of $(*)$, and we take $\mathrm{h}_3(t)$ to be the imaginary part of $(*)$.

    This gives the required 3 solutions.

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11. Find the inverse Laplace transform $\mathcal{L}^{-1}\left[\dfrac{1+2s}{(s^2+9)s^2}\right]$.

    Solution:

    We first use partial fractions to rewrite the argument:

$$\frac{1+2s}{(s^2+9)s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9}.$$

    Getting a common denominator, we find that

$$1 + 2s = A \cdot (s^2+9) \cdot s + B(s^2+9) + (Cs+D)s^2.$$

    We now need four equations; we will find them by plugging in the values $s = 0, 1, -1, 2$.

    $s = 0$ leads to $1 = 9B$ or $B = 1/9$.

    $s = 1$ leads to $3 = 10A + 10B + C + D$.

    $s = -1$ leads to $-1 = -10A + 10B - C + D$.

    $s = 2$ leads to $5 = 26A + 13B + 8C + 4D$.

    Thus we need to solve the system with augmented matrix

$$\left[\begin{array}{cccc|c} 0 & 9 & 0 & 0 & 1 \\ 10 & 10 & 1 & 1 & 3 \\ -10 & 10 & -1 & 1 & -1 \\ 26 & 13 & 8 & 4 & 26 \end{array}\right]$$

The solution is c $= \begin{bmatrix} -1/6 \\ 1/9 \\ 11/3 \\ -1/9 \end{bmatrix}$, so

$$\frac{1+2s}{(s^2+9)s^2} = \frac{-1/6}{s} + \frac{1/9}{s^2} + \frac{11s/3 - 1/9}{s^2+9} = \frac{-1/6}{s} + \frac{1/9}{s^2} + \frac{11}{3}\frac{s}{s^2+9} - \frac{1}{9}\frac{1}{s^2+9}$$

So we find that

$$\mathcal{L}^{-1}[\frac{1+2s}{(s^2+9)s^2}] = \frac{-1}{6} + \frac{1}{9}t + \frac{11}{3}\cos(3t) - \frac{1}{9}\sin(3t)$$

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12. Consider the initial value problem $(D^2+D)x = e^{3t}; \quad x(0) = 0, x'(0) = 0$.

   a. Find an expression for the Laplace transform $\mathcal{L}[x]$ as a function of the variable $s$.

   Solution:

   Using linearity of $\mathcal{L}$, we find that

$$\mathcal{L}[D^2x] + \mathcal{L}[Dx] = \frac{1}{s-3}$$

   Now using the differentiation formula, and the fact that $x(0) = x'(0) = 0$ we find that

$$\frac{1}{s-3} = s^2\mathcal{L}[x] + s\mathcal{L}[x] = (s^2+s)\mathcal{L}[x]$$

   Solving for $\mathcal{L}[x]$ we find that

$$\mathcal{L}[x] = \frac{1}{(s-3)s(s+1)}.$$

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   b. Use your answer to (a) to find the solution $x = x(t)$ for the given initial value problem.

   Solution:

   In view of a., we have

$$x = \mathcal{L}^{-1}[\frac{1}{(s-3)s(s+1)}]$$

   We first solve the partial fractions problem:

$$\frac{1}{(s-3)s(s+1)} = \frac{A}{s-3} + \frac{B}{s} + \frac{C}{s+1}$$

   leads to

$$1 = A(s+1)s + B(s-3)(s+1) + C(s-3)s$$

   We now plug in $s = 0, -1, 3$ and we find

$$1 = -3B, \quad 1 = 4C, \quad 1 = 12A$$

Thus

$$\frac{1}{(s-3)s(s+1)} = \frac{1}{12}\frac{1}{s-3} - \frac{1}{3}\frac{1}{s} + \frac{1}{4}\frac{1}{s+1}$$

So

$$x = \mathcal{L}^{-1}\left[\frac{1}{(s-3)s(s+1)}\right] = \frac{1}{12}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$
$$= \frac{1}{12}e^{3t} - \frac{1}{3} + \frac{1}{4}e^{-t}$$

# Bibliography

Nitecki, Zbigniew, and Martin Guterman. 1992. Differential Equations: A First Course. Saunders.