Review material for Midterm 1 – Solutions

Math 51 Spring 2022

exam date 2022-02-14

1. (multiple-choice) Which of the following represents a linear ODE?
   a. $x \cdot x'' + x + 1 = \sin(t)$
   b. $t \cdot x'' + x^2 + 1 = \sin(t)$
   c. $t^2 \cdot x'' + (t+1) \cdot x + 1 = \sin(t)$
   d. $(D^2 + D + t)x^2 = \sin(t)$

   Solution: c.

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2. (multiple-choice) Consider the Wronskian $W(t) = W(f_1, f_2, f_3)(t)$ of the functions

   $$f_1(t) = 1, \qquad f_2(t) = 1 + t \qquad \text{and} \qquad f_3(t) = \ln(1+t).$$

   Which of the following statements is most correct?

   a. The Wronskian is given by $W(t) = -1/(1+t)^2$; since $W(1) = -1/4$ is non-zero, the functions are linearly independent on the interval $(-1, \infty)$.
   b. Since $W(1) = 0$, the functions are linearly dependent on $(-1, \infty)$.
   c. Since $W(t)$ is not defined on $(-\infty, \infty)$, the Wronskian test doesn't apply.
   d. None of the above.

   Solution: a.

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3. (multiple-choice) Let $P(D)$ be a differential operator of order 4, and suppose that $h_1(t), h_2(t), h_3(t), h_4(t)$ are solutions to the homogeneous equation

   $$(\heartsuit) \quad P(D)x = 0.$$

   Suppose that

   $$h_1(t) + h_2(t) + h_3(t) + h_4(t) = 0$$

   for every t, $-\infty < t < \infty$.

   Which of the following statements is most correct?

   a. The general solution to $(\heartsuit)$ is given by

   $$x(t) = c_1 h_1(t) + c_2 h_2(t) + c_3 h_3(t) + c_4 h_4(t).$$

   b. The functions $h_1(t), h_2(t), h_3(t), h_4(t)$ are linearly dependent.
   c. A particular solution to $(\heartsuit)$ has the form

   $$q(t) = \int h_1(t) dt.$$

d. For some values of k_1, k_2 and k_3, the expression $q(t) = k_1 h_1(t) + k_2 h_2(t) + k_3 h_3(t)$ provides a particular solution to the ODE

$$P(D)x = e^t.$$

Solution: b.

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4. Consider the ODE $\dfrac{dx}{dt} = x^2 \cos(t)$.

a. Find the general solution to this ODE.

Solution:

Separating variables, we find that

$$\begin{aligned} \int \frac{dx}{x^2} &= \int \cos(t)dt \\ \implies \frac{-1}{x} &= \sin(t) + C \\ \implies x &= \frac{-1}{\sin(t) + C} \end{aligned}$$

So we have the solutions $x = \dfrac{-1}{\sin(t) + C}$ for each constant C, and we also have the constant solution $x(t) = 0$.

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b. Find a solution x satisfying $x(0) = 1$.

Solution:

If $1 = x(0) = \dfrac{-1}{\sin(0) + C} = \dfrac{-1}{C}$ we find that $C = -1$ so that

$$x(t) = \frac{-1}{\sin(t) - 1}.$$

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c. What is the largest interval containing $t_0 = 0$ on which this solution is defined?

Solution:

This solution is defined on $\left(\dfrac{-3\pi}{2}, \dfrac{\pi}{2}\right)$ and no larger interval (because $\sin\left(\dfrac{\pi}{2}\right) = \sin\left(\dfrac{-3\pi}{2}\right) = 1$ so that $x(t)$ is not defined at the endpoints of this interval).

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5. Consider the differential equation

$$(\clubsuit) \quad (t+2)\frac{dx}{dt} + 2x = t + 1.$$

a. Find the largest interval containing $t_0 = 0$ on which this equation is normal.

Solution:

This ODE is normal on $(-2, \infty)$ since the leading coefficient $t + 2$ is never 0 for any t in $(-2, \infty)$.

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b. Find the general solution to (♣).

Solution:

We consider this ODE for $t > -2$.

We first solve the corresponding homogeneous equation

$$(t+2)\frac{dx}{dt} = -2x$$

by separating variables. We find that

$$\begin{aligned}
&\int \frac{dx}{x} = -2\int \frac{dt}{t+2} \\
\implies &\ln|x| = -2\ln|t+2| + C \\
\implies &|x| = e^C e^{-2\ln(t+2)} \\
\implies &x = k(t+2)^{-2}
\end{aligned}$$

for an arbitrary constant k.

The possibility that $k = 0$ already accounts for the constant solution $x(t) = 0$.

In fact, to solve (♣) we only need one homogeneous solution, so we take $k = 1$ and $h(t) = (t+2)^{-2} = \dfrac{1}{(t+2)^2}$.

Now we seek solutions to (♣) of the form $x(t) = k(t)h(t)$. In order to use our formula for $k(t)$, we need to put the equation in standard form!!

i.e. we consider the equation

$$\frac{dx}{dt} + \frac{2x}{t+2} = \frac{t+1}{t+2}$$

Now the function $k(t)$ satisfies the condition

$$k'(t) = \frac{1}{h(t)} \cdot \frac{t+1}{t+2} = (t+2)^2 \cdot \frac{t+1}{t+2} = (t+1)(t+2) = t^2 + 3t + 2.$$

Thus

$$k(t) = \int (t^2 + 3t + 2)dt = \frac{t^3}{3} + \frac{3t^2}{2} + 2t + C$$

so that the general solution to (♣) is given by

$$x(t) = h(t)k(t) = \frac{1}{(t+2)^2}\left(\frac{t^3}{3} + \frac{3t^2}{2} + 2t + C\right)$$

for an arbitrary constant C.

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6. Consider the ODE $\dfrac{dx}{dt} = \dfrac{x}{t} + 1$ for $t > 0$.

a. Find the general solution $x(t)$ to this ODE.

Solution:

We first solve he corresponding homogeneous equation $\dfrac{dx}{dt} = \dfrac{x}{t}$ using separation of variables. We find that

$$\int \frac{dx}{x} = \int \frac{dt}{t}$$
$$\implies \ln|x| = \ln|t| + C$$
$$\implies |x| = te^C$$
$$\implies x = kt$$

for an arbitrary constant k. Note that $k = 0$ accounts for the constant solution $x(t) = 0$. To solve the ODE, we need one non-zero homogeneous solution, so we take $k = 1$ and

$$h(t) = t.$$

This ODE is already in standard form. We seek a solution of the form $x(t) = k(t)h(t)$; the function $k(t)$ satisfies the equation

$$k'(t) = \frac{1}{h(t)} \cdot 1 = \frac{1}{t}.$$

Thus

$$k(t) = \int \frac{dt}{t} = \ln|t| + C = \ln(t) + C$$

since $t > 0$.

So the general solution is given by

$$x(t) = k(t)h(t) = (\ln(t) + C)t = t\ln(t) + Ct.$$

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b. find the particular solution of the ODE for which $x(1) = 0$.

Solution:

Using the general solution $x(t) = t\ln(t) + Ct$, we need $0 = x(1) = 1\ln(1) + C$ so that $C = 0$. Thus $x(t) = t\ln(t)$ is the solution to the initial value problem.

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7. For what value(s) of α is the determinant

$$\det \begin{bmatrix} 1 & \alpha & 1 \\ 1 & 1 & 0 \\ 0 & 1 & \alpha \end{bmatrix}$$

equal to 0?

Solution:

Write D for the indicated determinant. Using expansion on the 3rd row, we have

$$D = -\det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha \det \begin{bmatrix} 1 & \alpha \\ 1 & 1 \end{bmatrix} = -(-1) + \alpha(1-\alpha) = 1 + \alpha - \alpha^2.$$

Using the quadratic formula, we see that $D = 0$ precisely when $\alpha = \dfrac{1 \pm \sqrt{5}}{2}$.

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8. Consider the system of linear equations

$$\begin{aligned} u_1 + 2u_2 + 3u_3 &= -1, \\ 3u_1 + 2u_2 + 1u_3 &= -1, \\ 5u_1 - 2u_2 + 2u_3 &= -1. \end{aligned}$$

The coefficient matrix has

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 5 & -2 & 2 \end{bmatrix} = -48$$

Use Cramer's Rule to give a formula for u_3 in terms of determinants. Do not evaluate the determinants.

Solution:

Write $M = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 5 & -2 & 2 \end{bmatrix}$ for the indicated matrix.

Cramer's Rule shows that

$$u_3 = \frac{\det \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 5 & -2 & -1 \end{bmatrix}}{\det M} = \frac{\det \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 5 & -2 & -1 \end{bmatrix}}{-48}$$

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9. Consider a 3rd order linear ODE which is normal on $(-\infty, \infty)$ and suppose that x_1 and x_2 are solutions. Which of the following statements is most correct?

   a. If $x_1(0) = x_2(0)$ and $x_1(1) = x_2(1)$, then $x_1 = x_2$.

   b. If $x_1(0) = x_2(0)$, $x_1'(0) = x_2'(0)$ and $x_1''(0) = x_2''(0)$, then $x_1 = x_2$.

   c. If $x_1(0) > 0$ then it is also true that $x_2(0) > 0$.

   Solution: b.

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10. Indicate which of the following ODEs is normal on the interval $(0, 2\pi)$.

   a. $\dfrac{1}{t}\dfrac{d^3x}{dt^3} + \sin(t)\dfrac{dx}{dt} = \cos(t)$.

   b. $D^2x + \cos(t)Dx = \ln(t-1)$

   c. $(t+1)D^5x + x = \dfrac{1}{\cos(t/8)}$

   Solution: a. and c.

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11. Consider the functions

$$h_1(t) = -1 + 7t + 8t^2, \qquad h_2(t) = 1 + 2t + t^2, \qquad h_3(t) = -1 + t + 2t^2.$$

   a. Find constants a, b so that $h_1(t) = a \cdot h_2(t) + b \cdot h_3(t)$.

   Hint: Equate coefficients of powers of t.

   Solution:

Consider the equation

$$\begin{aligned}-1+7t+8t^2 &= ah_2(t)+bh_3(t)\\ &= a(1+2t+t^2)+b(-1+t+2t^2)\\ &= (a-b)+(2a+b)t+(a+2b)t^2\end{aligned}$$

Comparing coefficients we find the system of equations

$$(\heartsuit) \qquad \begin{cases}-1=a-b\\ 7=2a+b\\ 8=a+2b\end{cases}$$

Adding the first two equations leads to the equation

$$6=3a \qquad \Longrightarrow \qquad a=2.$$

And then the first equation leads to

$$-1=2-b \implies b=3$$

You can check that $\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$ is a solution to ($\heartsuit$) (by plugging in these values in all 3 equations).

Thus

$$h_1(t)=2\cdot h_2(t)+3\cdot h_3(t)$$

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b. Are the functions $h_1(t), h_2(t), h_3(t)$ linearly dependent? (What does your answer to (a) tell you about linear dependence?)

Solution:

According to part a., the functions $h_1(t), h_2(t), h_3(t)$ are linearly dependent (the equation $h_1(t) = 2\cdot h_2(t)+3\cdot h_3(t)$ confirms the linear dependence).

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12. A particular solution to the equation

$$(\clubsuit) \qquad (D^2-16)x=e^{4t}$$

is $p(t)=\frac{1}{8}\cdot t\cdot e^{4t}+e^{-4t}$. Find the general solution.

Solution:

The general solution to the homogeneous equation

$$(D^2-16)x=0$$

is given by $x=c_1e^{4t}+c_2e^{-4t}$.

Since $p(t)$ is a particular solution, the general solution to the ($\clubsuit$) is given by

$$x(t)=p(t)+c_1e^{4t}+c_2e^{-4t}=\frac{1}{8}\cdot t\cdot e^{4t}+e^{-4t}+c_1e^{4t}+c_2e^{-4t}.$$

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13. Use the exponential shift formula

$$P(D)[e^{\lambda t}y] = e^{\lambda t}P(D+\lambda)[y]$$

to compute the function $P(D)[f]$ in each of the following cases:

a. $P(D) = D^2 + D - 6$ and $f = t^2e^{2t}$.

Solution:

$$\begin{aligned}
(D^2 + D - 6)[t^2e^{2t}] &= e^{2t}((D+2)^2 + (D+2) - 6)[t^2] \\
&= e^{2t}(D^2 + 4D + 4 + D + 2 - 6)[t^2] \\
&= e^{2t}(D^2 + 5D)[t^2] \\
&= e^{2t}(2 + 10t)
\end{aligned}$$

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b. $P(D) = D^2 + 3$ and $f = e^t\cos(3t)$.

Solution:

$$\begin{aligned}
(D^2 + 3)[e^t\cos(3t) &= e^t((D+1)^2 + 3)[\cos(3t) \\
&= e^t(D^2 + 2D + 1 + 3)[\cos(3t)] \\
&= e^t(D^2 + 2D + 4)[\cos(3t)] \\
&= e^t(-9\cos(3t) - 6\sin(3t) + 4\cos(3t)) \\
&= e^t(-5\cos(3t) - 6\sin(3t))
\end{aligned}$$

(I originally used the wrong expression for $P(D)$; here is that calculation for comparison

$$\begin{aligned}
(D^2 + D)[e^t\cos(3t) &= e^t((D+1)^2 + (D+1))[\cos(3t) \\
&= e^t(D^2 + 2D + 1 + D + 1)[\cos(3t)] \\
&= e^t(D^2 + 3D + 2)[\cos(3t)] \\
&= e^t(-9\cos(3t) - 9\sin(3t) + 2\cos(3t)) \\
&= e^t(-7\cos(3t) - 9\sin(3t))
\end{aligned}$$

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c. $P(D) = (D+3)(D-1)^2$ and $f = t^2e^t$.

$$\begin{aligned}
(D+3)(D-1)^2[t^2e^t] &= e^t(D+1+3)(D+1-1)^2[t^2] \\
&= e^t(D+4)D^2[t^2] \\
&= e^t(D+4)[2] \\
&= 8e^t
\end{aligned}$$

14. Find the general solution to the differential equation

$$(t+1)x' = \frac{x}{t-1}, \quad t > 1.$$

Solution:

Separating variables, we are led to the integrals

$$(\clubsuit) \quad \int \frac{1}{x} dx = \int \frac{1}{(t+1)(t-1)} dt$$

To find the integral on the right, use the method of partial fractions. We must solve

$$\frac{1}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} = \frac{A(t-1)+B(t+1)}{(t+1)(t-1)}.$$

Thus, we require that

$$0t + 1 = 1 = A(t-1) + B(t+1) = (A+B)t + (B-A).$$

Comparing coefficients, we find that

$$\begin{aligned} 0 &= A + B \\ 1 &= -A + B \end{aligned}$$

Adding the two equations, we find that $2B = 1$ so that $B = 1/2$ and then $A = -1/2$. Thus,

$$\frac{1}{(t+1)(t-1)} = \frac{-1}{2(t+1)} + \frac{1}{2(t-1)} = \frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right).$$

Returning now to the integrals ($\clubsuit$), we find that

$$\begin{aligned} \ln|x| &= \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt \\ &= \frac{1}{2}\left(\ln(t-1) - \ln(t+1)\right) + C \\ &= \ln(\sqrt{t-1}) - \ln(\sqrt{t+1}) + C \end{aligned}$$

After taking the exponential of each side, we find that

$$x = k\sqrt{\frac{t-1}{t+1}}$$

for an arbitrary constant k.

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15. Solve the initial value problem

$$(\clubsuit) \quad 2\frac{dx}{dt} - x = t \cdot e^t; \quad x(0) = 1$$

Solution:

We rewrite the equation in standard form: $\dfrac{dx}{dt} - \dfrac{x}{2} = \dfrac{t \cdot e^t}{2}$; thus the right-hand-side is $q(t) = \dfrac{te^t}{2}$.

We first find a solution to the corresponding homogeneous equation $x' = \dfrac{x}{2}$; one solution is $h(t) = e^{t/2}$.

Now search for a solution to (♣) of the form $x(t) = k(t)h(t)$ for an unknown function $k(t)$.

For a first order linear ODE in standard form, we know that

$$k'(t) = \frac{1}{h(t)} \cdot q(t) = e^{-t/2}\frac{te^t}{2} = \frac{te^{t/2}}{2}.$$

We now find $k(t)$ by integrating:

$$k(t) = \frac{1}{2}\int te^{t/2}dt.$$

We use integration by parts with $u = t$ and $dv = e^{t/2}dt$. Thus $du = dt$ and $v = 2e^{t/2}$, so that

$$k(t) = \frac{1}{2}\left(2te^{t/2} - 2\int e^{t/2}dt\right) = te^{t/2} - 2e^{t/2} + C.$$

Thus we find that the general solution to (♣) is given by

$$x(t) = k(t)h(t) = te^t - 2e^t + Ce^{t/2}.$$

To solve the initial value problem, we require that $x(0) = 1$, and we find that

$$1 = x(0) = -2 + C \implies C = 3.$$

Thus the solution to the IVP is

$$x(t) = te^t - 2e^t + 3e^{t/2}.$$

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16. Solve the initial value problem

$$4x'' + 4x' - 3x = 0; \quad x(0) = 0, \quad x'(0) = 1.$$

Solution:

The characteristic polynomial factors as

$$4r^2 + 4r - 3 = (2r + 3)(2r - 1)$$

and thus has roots $\dfrac{-3}{2}$ and $\dfrac{1}{2}$.

It follows that the general solution to the indicated ODE is given by

$$x(t) = c_1e^{-3t/2} + c_2e^{t/2}$$

Note that

$$x'(t) = \frac{-3c_1}{2}e^{-3t/2} + \frac{c_2}{2}e^{t/2}.$$

We now require that

$$\begin{aligned} 0 &= x(0) = c_1 + c_2 \\ 1 &= c'(0) = \frac{-3c_1}{2} + \frac{c_2}{2} \end{aligned}$$

i.e.

$$\begin{aligned} 0 &= c_1 + c_2 \\ 2 &= -3c_1 + c_2 \end{aligned}$$

Subtracting the first equation from the second gives $2 = -4c_1$ so that $c_1 = -1/2$, and then the first equation shows that $c_2 = -c_1 = 1/2$.

Thus the solution to the IVP is

$$x(t) = \frac{-e^{-3t/2}}{2} + \frac{e^{t/2}}{2}$$
