SOLUTIONS TO MATH 51 FINAL F21

II.2(a) Given the complex eigenvector for A corresponding to the complex eigenvalue $\lambda = 2+i$:

$$\vec{v} = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$$

find the general solution of (H) $DX = AX$

The complex solution to (H) is

$$e^{2t}(\cos t + i \sin t) \begin{bmatrix} 2-i \\ 5 \end{bmatrix} = e^{2t} \begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + ie^{2t} \begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix}$$

so the real and imaginary parts of this generate the general solution

$$X(t) = C_1 e^{2t} \begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix}.$$

(b) Find the solution to (H) satisfying the initial condition

$$X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The value at $t = 0$ of the general solution given above is

$$X(0) = C_1 e^0 \begin{bmatrix} 2\cos 0 + \sin 0 \\ 5\cos 0 \end{bmatrix} + C_2 e^0 \begin{bmatrix} -\cos 0 + 2\sin 0 \\ 5\sin 0 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix};$$

setting this equal to the desired initial condition yields the system of equations

$$2C_1 - C_2 = 1$$
$$5C_1 + 0C_2 = 1$$

which can be solved by either reducing the augmented matrix

$$\begin{bmatrix} 2 & -1 & |1 \\ 5 & 0 & |1 \end{bmatrix}$$

or Cramer's Rule, or simply by noting that the second equation says $C_1 = \frac{1}{5}$, and substituting into the first equation yields $\frac{2}{5} - C_2 = 1$ or $C_2 = -\frac{3}{5}$.. Thus the desired solution of (H) is

$$X(t) = \frac{1}{5} e^{2t} \begin{bmatrix} 2\cos t + \sin t \\ 5\cos t \end{bmatrix} - \frac{3}{4} e^{2t} \begin{bmatrix} -\cos t + 2\sin t \\ 5\sin t \end{bmatrix} = e^{2t} \begin{bmatrix} \cos t - \sin t \\ \cos t - 3\sin t \end{bmatrix}.$$

I graded this on the basis of 10 for (a) (5 each for the two vector functions) and 5 for (b).

II.5 (a) Find the inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{3s^2+s+1}{(s+1)(s^2+2)}\right].$$

The partial fraction decomposition has the form

$$\frac{3s^2+s+1}{(s+1)(s^2+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2};$$

combining over a common denominator and matching coefficients leads to

$$\begin{array}{lcccl} s^2 \text{ terms}: & A & +B & & = 3 \\ s \text{ terms}: & & B & +C & = 1 \\ \text{constant terms}: & 2A & & +C & = 1 \end{array}$$

We can solve the first (respectively, second) equation for A (respectively, C) in terms of B:

$$\begin{aligned} A &= 3 - B \\ C &= 1 - B \end{aligned}$$

and substituting into the third equation yields

$$\begin{aligned} (6-2B) + (1-B) &= 1 \\ -3B &= -6 \\ B &= 2 \\ A &= 1 \\ C &= -1 \end{aligned}$$

so

$$\frac{3s^2+s+1}{(s+1)(s^2+2)} = \frac{1}{s+1} + \frac{2s-1}{s^2+2}.$$

Then the inverse transform is

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{3s^2+s+1}{(s+1)(s^2+2)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{2s}{s^2+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+2}\right] \\ &= e^{-t} + 2\cos t\sqrt{2} - \frac{1}{\sqrt{2}}\sin t\sqrt{2}. \end{aligned}$$

(b) Express the Laplace Transform of the solution of the o.d.e. $(D^2+D+1)x = 1$ satisfying the initial conditions $x(0) = 0$ and $x'(0) = 1$ as a function of s:

By the first differentiation formula, applying the Laplace Transform to both sides of the problem yields

$$\begin{aligned}
\mathcal{L}\left[D^2x\right] + \mathcal{L}\left[Dx\right] + \mathcal{L}\left[x\right] &= \mathcal{L}\left[1\right] \\
\{s^2\mathcal{L}\left[x\right] - sx(0) - x'(0)\} + \{s\mathcal{L}\left[x\right] - x(0)\} + \mathcal{L}\left[x\right] &= \mathcal{L}\left[1\right] \\
s^2\mathcal{L}\left[x\right] - 1 + s\mathcal{L}\left[x\right] + \mathcal{L}\left[x\right] &= \frac{1}{s} \\
(s^2+s+1)\mathcal{L}\left[x\right] &= 1 + \frac{1}{s} = \frac{1+s}{s} \\
\mathcal{L}\left[x\right] &= \frac{1+s}{s(s^2+2+1)}
\end{aligned}$$

I graded this as 10 for (a) (5 for partial fractions, 5 for inverse transform) and 5 for (b).