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The Laplace Transform
2. Initial Conditions

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The Laplace Transform of a Derivative

We begin by relating the Laplace transform of a function to the transform of its derivative: we can apply integration by parts, (with $u = e^{-st}$, $dv = x'(t)\,dt$) to the basic defining integral for the transform to get

$$\int e^{-st}x'(t)\,dt = \int u\,dv$$

$$= uv - \int v\,du = (e^{-st})(x(t)) - \int (x(t))(-se^{-st})\,dt$$

$$= e^{-st}x(t) + s\int e^{-st}x(t)\,dt$$

Evaluating both sides from $t = 0$ to $t = b$ and taking the limit as $b \to \infty$, we have

$$\int_0^\infty e^{-st}x'(t)\,dt = \left(\lim_{b\to\infty} e^{-sb}x(b)\right) - x(0) + s\int_0^\infty e^{-st}x(t)\,dt.$$

Note that in order for the integral defining the Laplace transform of $x(t)$ to converge, the integrand must go to zero as $t \to 0$; this forces the limit in our expression to be zero;
the two integrals are the Laplace transforms of, respectively,
$x'(0)$ and $x(t)$. This gives us the basic relationship

First Differentiation Formula (order 1):
For any function $x(t)$ which is differentiable and which has a Laplace transform,

$$\mathcal{L}[Dx] = s\mathcal{L}[x] - x(0)\,.$$

As we shall see shortly, this formula has many useful applications.

We can iterate the First Differentiation Formula to get a relationship between the transform of a function $x(t)$ and the transform of any (higher order) derivative $D^n x(t)$: for example, setting $y = Dx$, we have

$$\begin{aligned}\mathcal{L}\left[D^2x\right] &= \mathcal{L}[Dy] = s\mathcal{L}[y] - y(0) \\ &= s\mathcal{L}[Dx] - x'(0) = s\left(s\mathcal{L}[x] - x(0)\right) - x'(0) \\ &= s^2\mathcal{L}[x] - sx(0) - x'(0).\end{aligned}$$

Extrapolating from this (or using mathematical induction) we get the most general form of the basic formula for transforms of derivatives:

First Differentiation Formula (order n):

$$\begin{aligned}\mathcal{L}[D^n x] = s^n\mathcal{L}[x] - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots \\ \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).\end{aligned}$$

Example: Find the solution of the initial-value problem

$$\begin{aligned}(H) &\quad (D^2 + 4D + 3)x = 0 \\ (IC) &\quad x(0) = 1, \quad x'(0) = 1.\end{aligned}$$

We apply the Laplace transform to both sides of (H):

$$\mathcal{L}\left[D^2x\right] + 4\mathcal{L}[Dx] + 3\mathcal{L}[x] = 0$$

We use the First Differentiation Formula to calculate the first and second terms:

$$\begin{aligned}\mathcal{L}\left[D^2x\right] &= s^2\mathcal{L}[x] - sx(0) - x'(0) \\ &= s^2\mathcal{L}[x] - s - 1 \\ 4\mathcal{L}[Dx] &= 4s\mathcal{L}[x] - 4x(0) \\ &= 4s\mathcal{L}[x] - 4.\end{aligned}$$

Plugging these expressions back into the original equation, we have

$$s^2\mathcal{L}\left[x\right] - s - 1 + 4s\mathcal{L}\left[x\right] - 4 + 3\mathcal{L}\left[x\right] = 0$$

or

$$(s^2 + 4s + 3)\mathcal{L}\left[x\right] - s - 5 = 0$$

so

$$\mathcal{L}\left[x\right] = \frac{s+5}{s^2+4s+3}.$$

This expression does not appear among the functions listed in the table of (inverse) Laplace transforms. However, you may recall from Calc 2 that one of the tricks of formal integration is the use of partial fractions; if the denominator can be factored, then we can write the fraction as a sum of simpler fractions, each of which has as its denominator a power of one of the factors. In the present case, it is easy to see that

$$s^2 + 4s + 3 = (s+1)(s+3),$$

so we expect to be able to write

$$\frac{s+5}{s^2+4s+3} = \frac{A}{s+1} + \frac{B}{s+3}$$

for some constants A and B. If we combine the two fractions on the right by cross-multiplication, we see that we need A and B so that

$$\frac{s+5}{s^2+4s+3} = \frac{A(s+3)+B(s+1)}{(s+1)(s+3)}.$$

In order to have these match, we need the coefficient of s, as well as the constant term, to be the same in the two numerators:

$$\begin{aligned} 1 &= A + B \\ 5 &= 3A + B. \end{aligned}$$

We easily conclude that $A = 2$ and $B = -1$, so

$$\mathcal{L}[x] = \frac{s+5}{s^2+4s+3} = \frac{2}{s+1} - \frac{1}{s+3}.$$

We immediately recognize these two fractions as the transforms of exponential functions, and arrive at the solution of our initial-value problem

$$x = 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = 2e^{-t} - e^{-3t}.$$

Partial Fraction Decomposition

The Partial Fraction Decomposition is a purely algebraic tool which you have already used in the context of formal integration; it is also extremely useful in finding inverse Laplace transforms–which are the "endgame" in solving initial value problems using the Laplace transform. We briefly recall how it works.
A polynomial $p(x)$ with real coefficients is reducible if it can be factored into a product of polynomials $p(x) = q_1(x)\, q_2(x)$ with real coefficients of lower degree, and irreducible if no such factoring is possible.
A consequence of the Fundamental Theorem of Algebra is that *the only irreducible polynomials (up to multiplication by constants) are either degree one:* $p(x) = x - \alpha$
or irreducible quadratics: $p(x) = (x-\alpha)^2 + \beta^2$.

Partial Fraction Decomposition:
Every fraction $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with real coefficients, can be expressed uniquely in the form

$$\frac{P(x)}{Q(x)} = R(x) + \frac{p_1(x)}{(q_1(x))^{k_1}} + \cdots + \frac{p_n(x)}{(q_n(x))^{k_n}}$$

subject to the following rules:

- $R(x)$ is a polynomial, which is nonzero only if $\deg P(x) \geq \deg Q(x)$;
- Each $q_i(x)$ is an irreducible factor of $Q(x)$, and $\deg p_i(x) < \deg q_i(x)$;
- $(q_i(x))^{k_i}$ divides $Q(x)$; and
- for $i \neq j$, either $k_i \neq k_j$ or $q_i(x) \neq q_i(x)$.

Example: Find the solution of the initial value problem

$$\begin{aligned} (H) \quad & (D^2+4)x = e^t \\ (IC) \quad & x(0) = 0, \quad x'(0) = 0. \end{aligned}$$

We apply the Laplace transform to both sides of (H):

$$\mathcal{L}\left[D^2x\right] + 4\mathcal{L}\left[x\right] = \mathcal{L}\left[e^t\right]$$

Note that since both initial conditions are zero, the First Differentiation Formula is particularly easy to apply:

$$\mathcal{L}\left[D^2x\right] = s^2\mathcal{L}\left[x\right] - s\cdot 0 - 0 = s^2\mathcal{L}\left[x\right]$$

so we get

$$(s^2+4)\mathcal{L}\left[x\right] = \mathcal{L}\left[e^t\right] = \frac{1}{s-1}$$

or

$$\mathcal{L}\left[x\right] = \frac{1}{(s-1)(s^2+4)}.$$

Since s^2+4 is an irreducible quadratic, The form of the partial fraction on the right is

$$\frac{1}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4};$$

we combine the two fractions on the right by putting them over a common denominator, and compare to the desired value

$$\begin{aligned} \frac{A(s^2+4)+(Bs+C)(s-1)}{(s-1)(s^2+4)} \\ = \frac{As^2+4A+Bs^2-Bs+Cs-C}{(s-1)(s^2+4)} \\ = \frac{1}{(s-1)(s^2+4)}. \end{aligned}$$

Matching coefficients leads to the equations

$$\begin{array}{rrrcl} A & +B & & = & 0 \\ & -B & +C & = & 0 \\ 4A & & -C & = & 1 \end{array}$$

with solution

$$A = \frac{1}{5}, \quad B = C = -\frac{1}{5}$$

so

$$\mathcal{L}\left[x\right] = \frac{1}{5}\frac{1}{s-1} - \frac{1}{5}\left(\frac{s}{s^2+4} + \frac{1}{s^2+4}\right).$$

Using our inverse Laplace Transform table, we recognize that this means the solution to our initial value problem is

$$x(t) = \frac{e^t}{5} - \frac{1}{5}\left(\cos 2t + \frac{1}{2}\sin 2t\right).$$

It is instructive to compare this to what happens if we keep the same differential equation but change the initial conditions. To solve

$$\begin{array}{rl} (H) & (D^2+4)x = e^t \\ (IC) & x(0) = 2, \quad x'(0) = 1, \end{array}$$

we again apply the Laplace transform to both sides of (H). This time the calculation is

$$s^2\mathcal{L}\left[x\right] - 2s - 1 + 4\mathcal{L}\left[x\right] = \frac{1}{s-1}$$

or

$$\mathcal{L}\left[x\right] = \frac{2s+1}{s^2+4} + \frac{1}{(s-1)(s^2+4)}.$$

While if starting from scratch we would probably combine the two fractions on the right and work from there, since we have already found the partial fraction decomposition of the second fraction, and the first is already in partial fraction form, we can simply jump to

$$\begin{aligned}
&\mathcal{L}[x] = \\
&\quad \frac{2s+1}{s^2+4} + \frac{1}{5}\frac{1}{s-1} - \frac{1}{5}\left(\frac{s}{s^2+4} + \frac{1}{s^2+4}\right) \\
&= \frac{1}{5}\frac{1}{s-1} + \left(2 - \frac{1}{5}\right)\frac{s}{s^2+4} + \left(1 - \frac{1}{5}\right)\frac{1}{s^2+4} \\
&\qquad = \frac{1}{5}\mathcal{L}\left[e^t\right] + \frac{9}{5}\mathcal{L}[\cos 2t] + \frac{4}{5}\left(\frac{1}{2}\mathcal{L}[\sin 2t]\right)
\end{aligned}$$

leading to the solution

$$x = \frac{e^t}{5} + \frac{9}{5}\cos 2t + \frac{2}{5}\sin 2t.$$