gmd_heat_equation

November 27, 2017

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        import seaborn as sns

        sns.set_context('paper')
        sns.set_style('darkgrid')
```

Setting Initial Conditions for Wall Accelerating to U at *t*=0, C=0.1

```
In [2]: y_min = 0.0
        y_max = 1.0

        u_min = 0.0
        u_max = 1.0

        t_min = 0.0
        t_max = 2.0

        y_step = 0.1
        t_step = 0.001

        y_array = np.arange(y_min, y_max+y_step, y_step)
        t_array = np.arange(t_min, t_max+t_step, t_step)
        y_spacing = len(y_array)
        t_spacing = len(t_array)
        u_grid = np.zeros((y_spacing, t_spacing))

        #Setting Initial Conditions
        u_grid[0, :] = u_max

In [3]: def heat_slope_func(u_array, y_step, v=1.0):
            d2u_dy2_array = np.copy(u_array)
            for i in range(0, len(u_array)-1):
                d2u_dy2_array[i] = (u_array[i+1] + u_array[i-1] - 2.0*u_array[i])/(y_step**2.0
            d2u_dy2_array[-1] = 0.0
            d2u_dy2_array[0] = 0.0
            return v*d2u_dy2_array
```

Note that we solve the implicit Euler scheme with an iterative fixed-point method instead of a Newton-Rhapson solver.

Implicit Euler: $u_{j+1} = u_j + h \times \frac{du_{j+1}}{dt}$

For an initial guess of $\frac{du_{j+1}^{[0]}}{dt}$ we use $\frac{du_j}{dt}$. Then $u_{j+1}^{[i]} = u_j + h \times \frac{du_{j+1}^{[i-1]}}{dt}$ until $\frac{|u^{[i]} - u^{[i-1]}|}{u^{[i-1]}} < 10^{-16}$ or $i = 1000$.

For the Crank-Nicholson (CN) we combined the implicit backwards Euler (BE) with the forward Euler (FE): $\frac{du^{\text{CN}}}{dt} = \frac{1}{2} \times (\frac{du^{\text{BE}}}{dt} + \frac{du^{\text{FE}}}{dt})$

Where $\frac{du^{\text{FE}}}{dt} = \frac{du_j}{dt}$ and $\frac{du^{\text{BE}}}{dt} = \frac{du_{j+1}}{dt}$. We use an iterative method as above for the same tolerance conditions.

```
In [4]: def forward_euler_t(heat_slope_func, u_old, t_old, y_step, t_step=t_step, v=1.0):
            return u_old + t_step*heat_slope_func(u_old, y_step, v=v)

        def backward_euler_t(heat_slope_func, u_old, t_old, y_step, t_step=t_step, v=1.0, tol=
            t_new = t_old + t_step
            u_guess = forward_euler_t(heat_slope_func, u_old, t_old, y_step, t_step=t_step, v=
            error = 0.1
            counter = 0
            while error > tol:
                u_new = u_old + t_step*heat_slope_func(u_guess, y_step, v=v)
                error = np.nanmedian(np.abs(u_new - u_guess)/u_guess)
                u_guess = u_new
                counter += 1
                if counter > max_iter:
                    #print("Did not converge within ", max_iter, ' iterations')
                    error = tol/10.0
            return u_new

        def crank_nicholson_t(heat_slope_func, u_old, t_old, y_step, t_step=t_step, v=1.0, tol
            forward_slope = heat_slope_func(u_old, y_step, v=v)
            u_guess = u_old + t_step*forward_slope
            error = 0.1
            counter = 0
            while error > tol:
                test_slope = 0.5*(heat_slope_func(u_old, y_step, v=v) + heat_slope_func(u_gues
                u_new = u_old + t_step*test_slope
                error = np.nanmedian(np.abs(u_new - u_guess)/u_guess)
                u_guess = u_new
                counter += 1
                if counter > max_iter:
                    #print("Did not converge within ", max_iter, ' iterations')
                    error = tol/10.0
            return u_new

        def integrate_t(u_grid_old, method_func, heat_slope_func, t_array, y_step, t_step=t_ste
            u_grid_new = np.copy(u_grid_old)
            for i in range(0, len(t_array)-1):
```

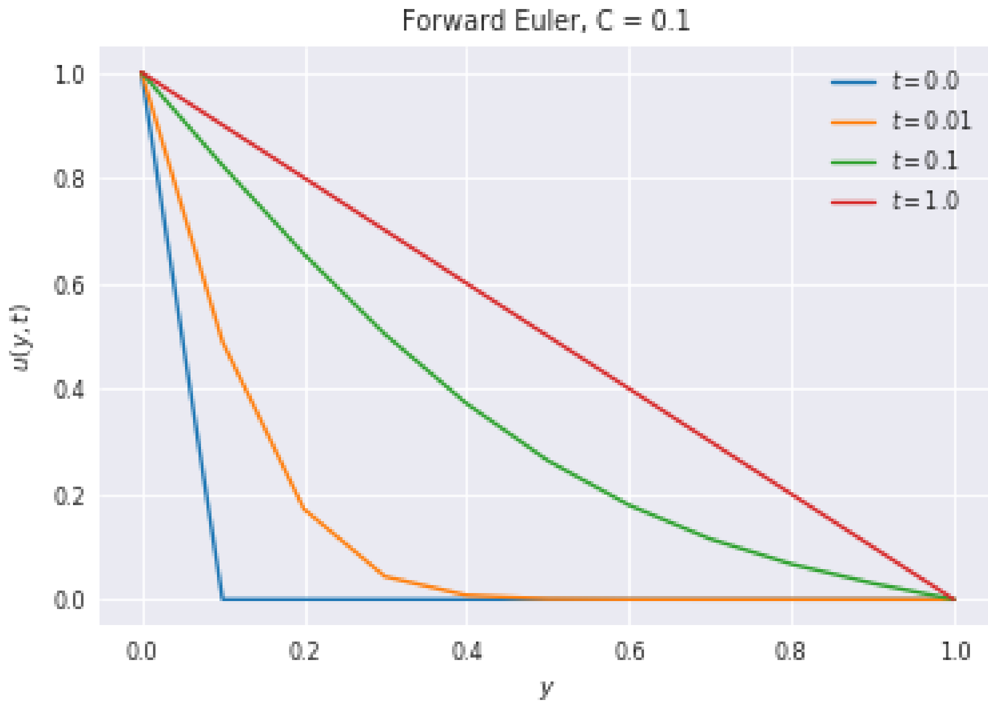
```
            u_old = u_grid_new[:, i]
            t_old = t_array[i]
            u_grid_new[:, i+1] = method_func(heat_slope_func, u_old, t_old, y_step, t_step
        return u_grid_new
```

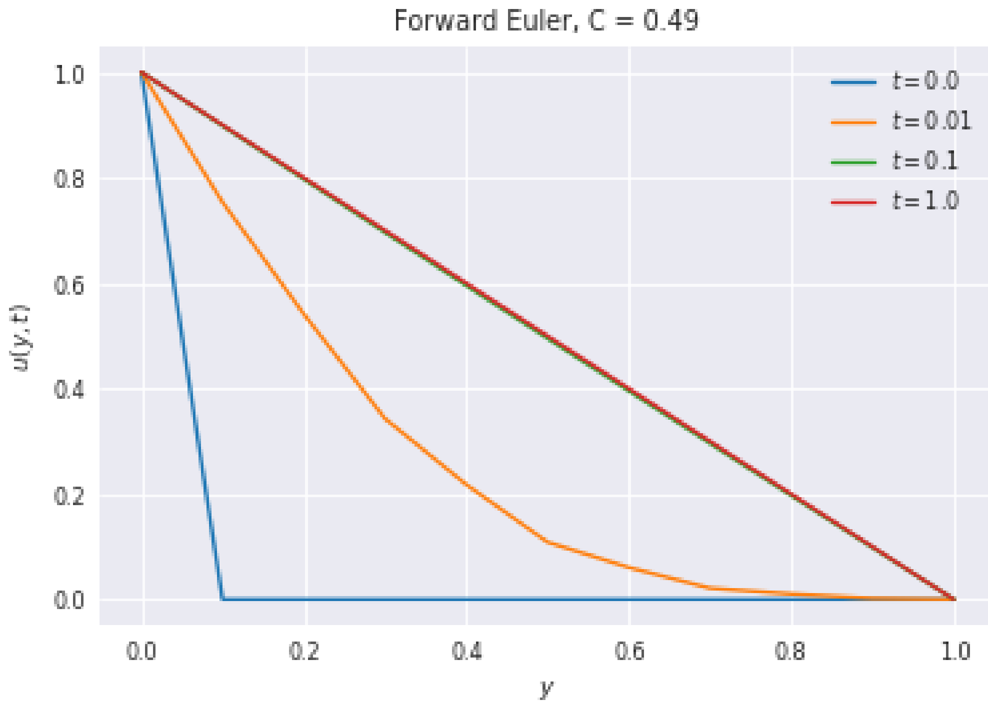
```
In [5]: u_grid_euler = integrate_t(u_grid, forward_euler_t, heat_slope_func, t_array, y_step)
        u_grid_crank = integrate_t(u_grid, crank_nicholson_t, heat_slope_func, t_array, y_step
        u_grid_euler_049 = integrate_t(u_grid, forward_euler_t, heat_slope_func, t_array, y_st
        u_grid_crank_049 = integrate_t(u_grid, crank_nicholson_t, heat_slope_func, t_array, y_
        u_grid_euler_051 = integrate_t(u_grid, forward_euler_t, heat_slope_func, t_array, y_st
        u_grid_crank_051 = integrate_t(u_grid, crank_nicholson_t, heat_slope_func, t_array, y_
        u_grid_euler_055 = integrate_t(u_grid, forward_euler_t, heat_slope_func, t_array, y_st
        u_grid_crank_055 = integrate_t(u_grid, crank_nicholson_t, heat_slope_func, t_array, y_
```

```
/home/gmduvvuri/miniconda3/envs/astroconda/lib/python3.5/site-packages/ipykernel_launcher.py:2
/home/gmduvvuri/miniconda3/envs/astroconda/lib/python3.5/site-packages/ipykernel_launcher.py:2
```

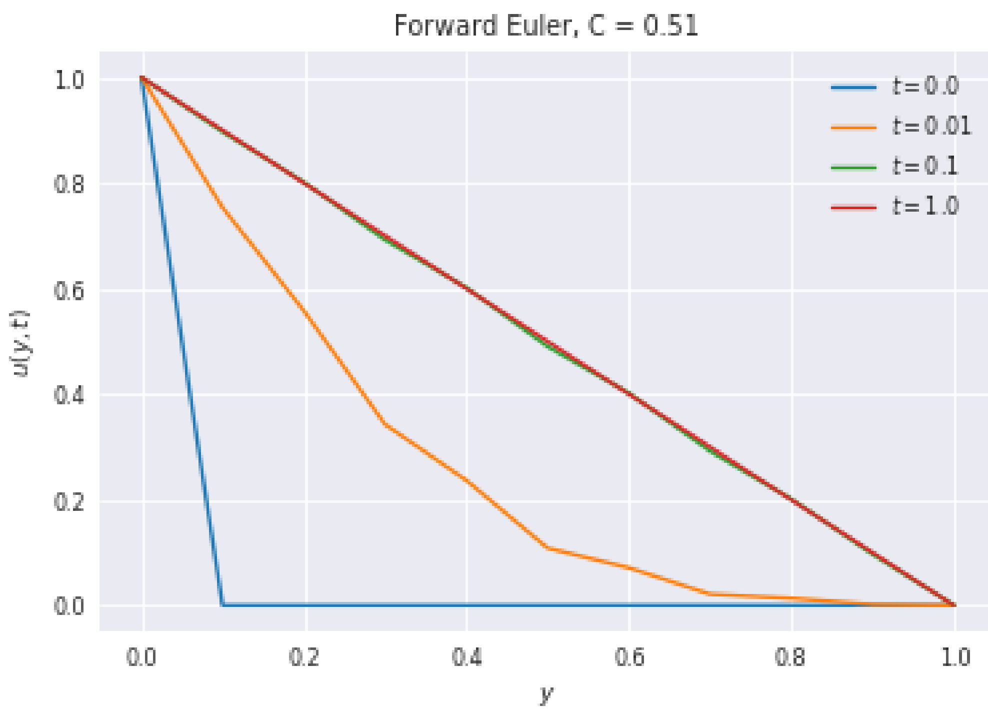
```
In [6]: plt.plot(y_array, u_grid_euler[:, 0], label=r'$t=$' + str(t_array[0]))
        plt.plot(y_array, u_grid_euler[:, 10], label=r'$t=$' + str(t_array[10]))
        plt.plot(y_array, u_grid_euler[:, 100], label=r'$t=$' + str(t_array[100]))
        plt.plot(y_array, u_grid_euler[:, 1000], label=r'$t=$' + str(t_array[1000]))
        plt.ylabel(r'$u(y, t)$')
        plt.xlabel(r'$y$')
        plt.title('Forward Euler, C = 0.1')
        plt.legend()
        plt.show()

        plt.plot(y_array, u_grid_euler_049[:, 0], label=r'$t=$' + str(t_array[0]))
        plt.plot(y_array, u_grid_euler_049[:, 10], label=r'$t=$' + str(t_array[10]))
        plt.plot(y_array, u_grid_euler_049[:, 100], label=r'$t=$' + str(t_array[100]))
        plt.plot(y_array, u_grid_euler_049[:, 1000], label=r'$t=$' + str(t_array[1000]))
        plt.ylabel(r'$u(y, t)$')
        plt.xlabel(r'$y$')
        plt.title('Forward Euler, C = 0.49')
        plt.legend()
        plt.show()

        plt.plot(y_array, u_grid_euler_051[:, 0], label=r'$t=$' + str(t_array[0]))
        plt.plot(y_array, u_grid_euler_051[:, 10], label=r'$t=$' + str(t_array[10]))
        plt.plot(y_array, u_grid_euler_051[:, 100], label=r'$t=$' + str(t_array[100]))
        plt.plot(y_array, u_grid_euler_051[:, 1000], label=r'$t=$' + str(t_array[1000]))
        plt.ylabel(r'$u(y, t)$')
        plt.xlabel(r'$y$')
        plt.title('Forward Euler, C = 0.51')
        plt.legend()
        plt.show()
```

```
plt.plot(y_array, u_grid_euler_055[:, 0], label=r'$t=$' + str(t_array[0]))
plt.plot(y_array, u_grid_euler_055[:, 10], label=r'$t=$' + str(t_array[10]))
plt.plot(y_array, u_grid_euler_055[:, 100], label=r'$t=$' + str(t_array[100]))
plt.plot(y_array, u_grid_euler_055[:, 1000], label=r'$t=$' + str(t_array[1000]))
plt.ylabel(r'$u(y, t)$')
plt.xlabel(r'$y$')
plt.title('Forward Euler, C = 0.55')
plt.legend()
plt.show()
```

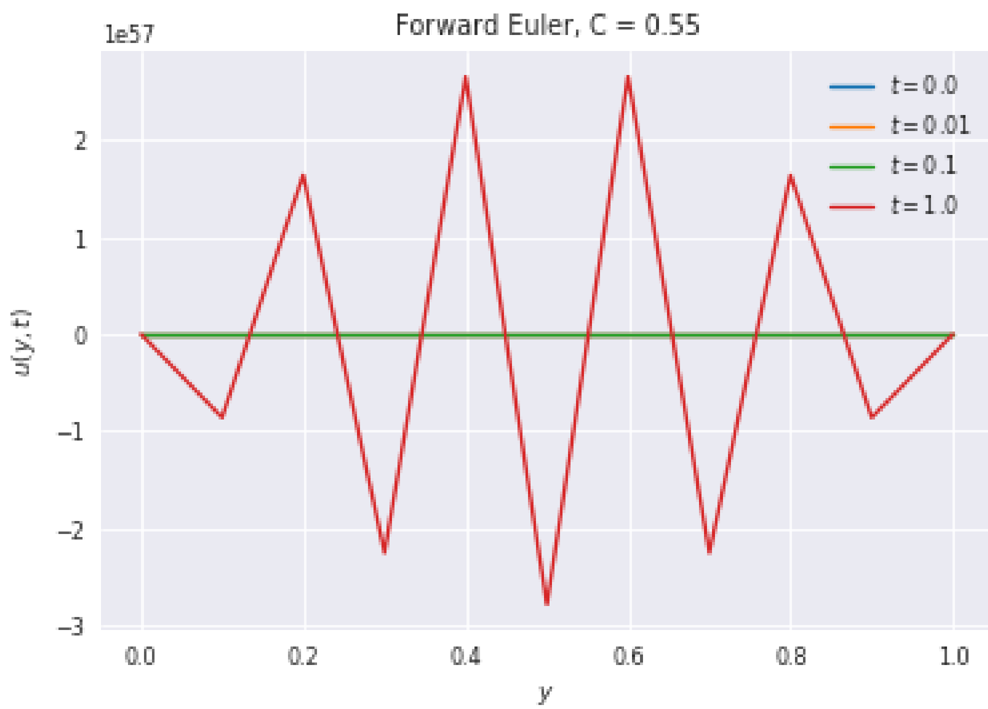




Forward Euler, C = 0.49
1.0
0.8
0.6
0.4
0.2
0.0
$u(y,t)$
0.0
0.2
0.4
0.6
0.8
1.0
y
$t = 0.0$
$t = 0.01$
$t = 0.1$
$t = 1.0$

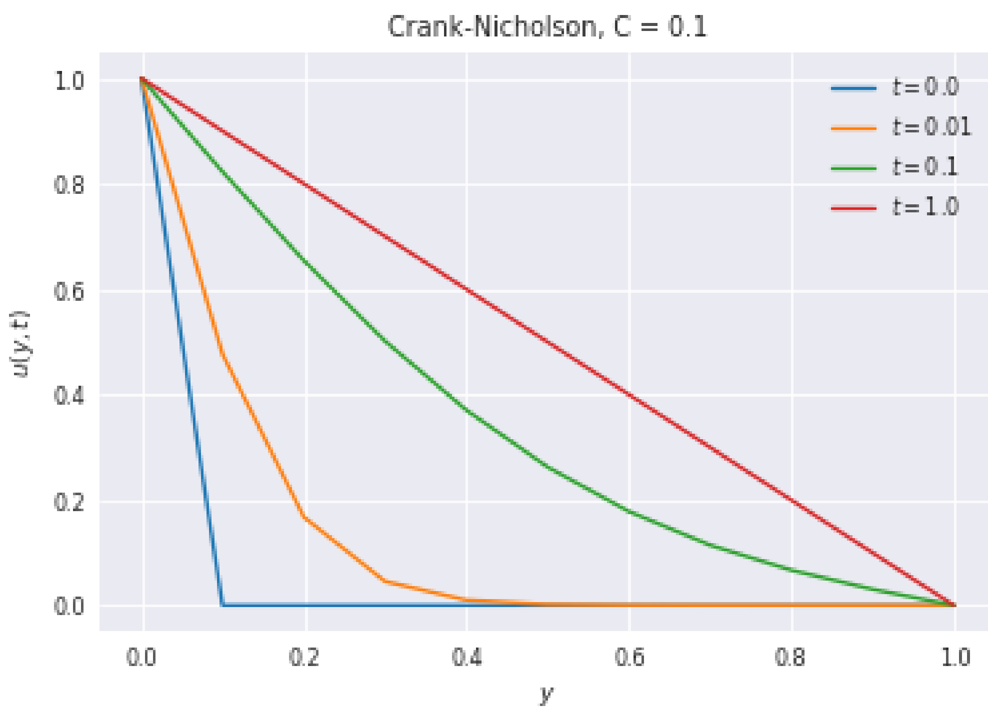


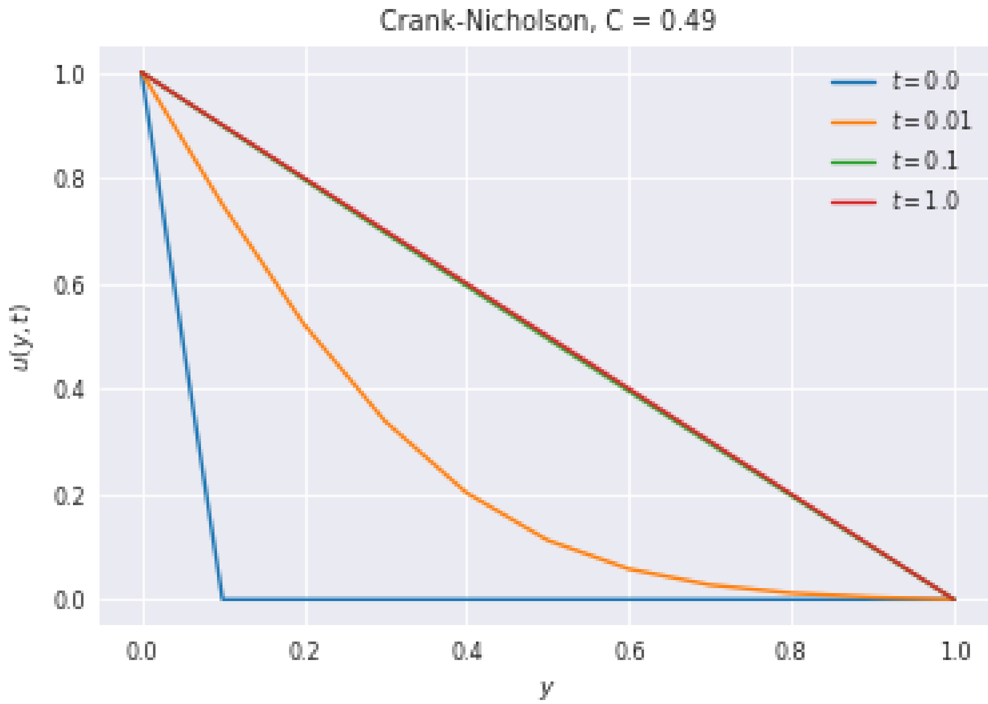
Forward Euler, C = 0.51
1.0
0.8
0.6
0.4
0.2
0.0
$u(y,t)$
0.0
0.2
0.4
0.6
0.8
1.0
y
$t = 0.0$
$t = 0.01$
$t = 0.1$
$t = 1.0$

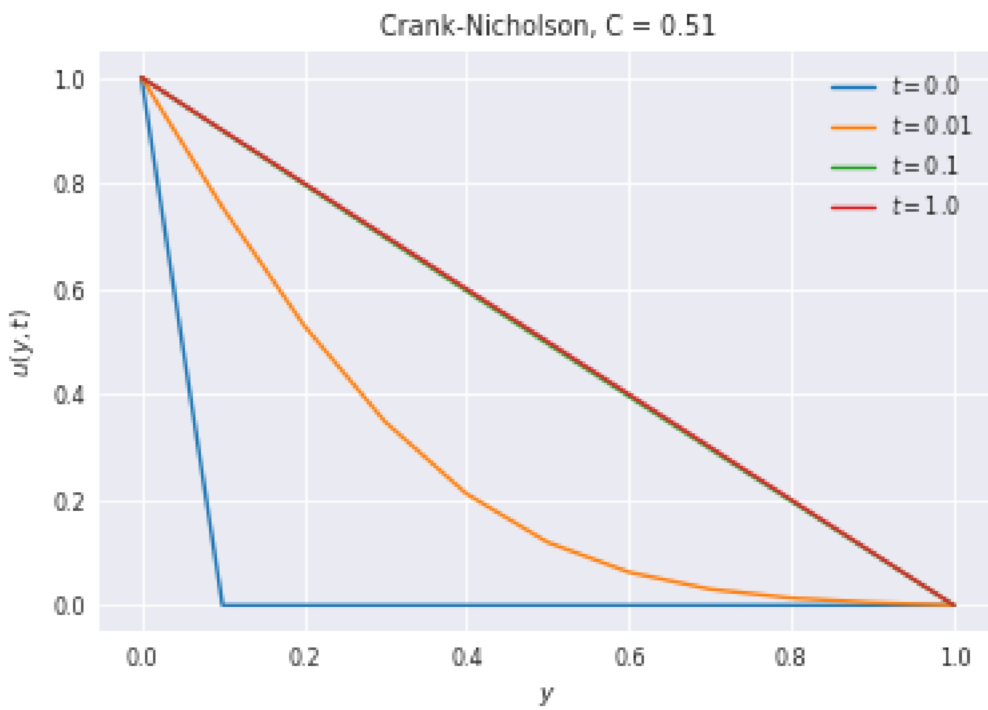


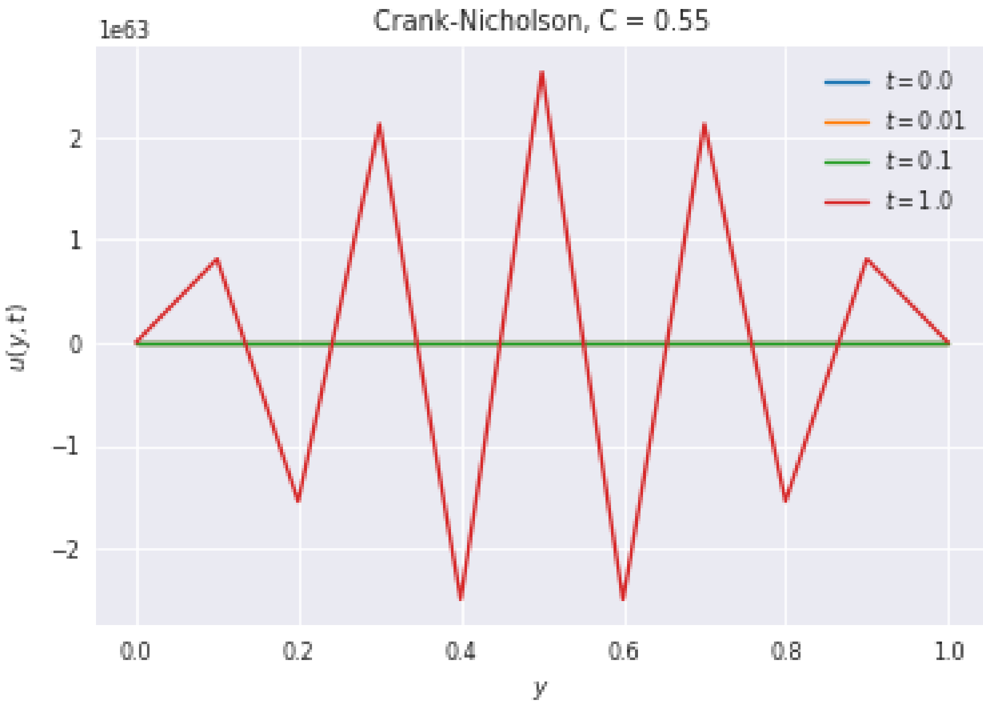

```
In [7]: plt.title('Crank-Nicholson, C = 0.1')
        plt.plot(y_array, u_grid_crank[:, 0], label=r'$t=$' + str(t_array[0]))
        plt.plot(y_array, u_grid_crank[:, 10], label=r'$t=$' + str(t_array[10]))
        plt.plot(y_array, u_grid_crank[:, 100], label=r'$t=$' + str(t_array[100]))
        plt.plot(y_array, u_grid_crank[:, 1000], label=r'$t=$' + str(t_array[1000]))
        plt.ylabel(r'$u(y, t)$')
        plt.xlabel(r'$y$')
        plt.legend()
        plt.show()

        plt.title('Crank-Nicholson, C = 0.49')
        plt.plot(y_array, u_grid_crank_049[:, 0], label=r'$t=$' + str(t_array[0]))
        plt.plot(y_array, u_grid_crank_049[:, 10], label=r'$t=$' + str(t_array[10]))
        plt.plot(y_array, u_grid_crank_049[:, 100], label=r'$t=$' + str(t_array[100]))
        plt.plot(y_array, u_grid_crank_049[:, 1000], label=r'$t=$' + str(t_array[1000]))
        plt.ylabel(r'$u(y, t)$')
        plt.xlabel(r'$y$')
        plt.legend()
        plt.show()

        plt.title('Crank-Nicholson, C = 0.51')
        plt.plot(y_array, u_grid_crank_051[:, 0], label=r'$t=$' + str(t_array[0]))
        plt.plot(y_array, u_grid_crank_051[:, 10], label=r'$t=$' + str(t_array[10]))
        plt.plot(y_array, u_grid_crank_051[:, 100], label=r'$t=$' + str(t_array[100]))
```

```
plt.plot(y_array, u_grid_crank_051[:, 1000], label=r'$t=$' + str(t_array[1000]))
plt.ylabel(r'$u(y, t)$')
plt.xlabel(r'$y$')
plt.legend()
plt.show()

plt.title('Crank-Nicholson, C = 0.55')
plt.plot(y_array, u_grid_crank_055[:, 0], label=r'$t=$' + str(t_array[0]))
plt.plot(y_array, u_grid_crank_055[:, 10], label=r'$t=$' + str(t_array[10]))
plt.plot(y_array, u_grid_crank_055[:, 100], label=r'$t=$' + str(t_array[100]))
plt.plot(y_array, u_grid_crank_055[:, 1000], label=r'$t=$' + str(t_array[1000]))
plt.ylabel(r'$u(y, t)$')
plt.xlabel(r'$y$')
plt.legend()
plt.show()
```




Crank-Nicholson, C = 0.49
$u(y,t)$
y
$t = 0.0$
$t = 0.01$
$t = 0.1$
$t = 1.0$


Crank-Nicholson, C = 0.51
$u(y,t)$
y
$t = 0.0$
$t = 0.01$
$t = 0.1$
$t = 1.0$



My implicit scheme is not stable for $C > 0.5$, indicating that my iterative method does not work properly. The explicit scheme blows up as expected. Perhaps the error introduced by the first forward Euler step for the initial guess is sufficient to render this scheme ineffectively implicit. A better version would be Gauss' elimination method to solve the matrix.

Implementing an Oscillating Boundary

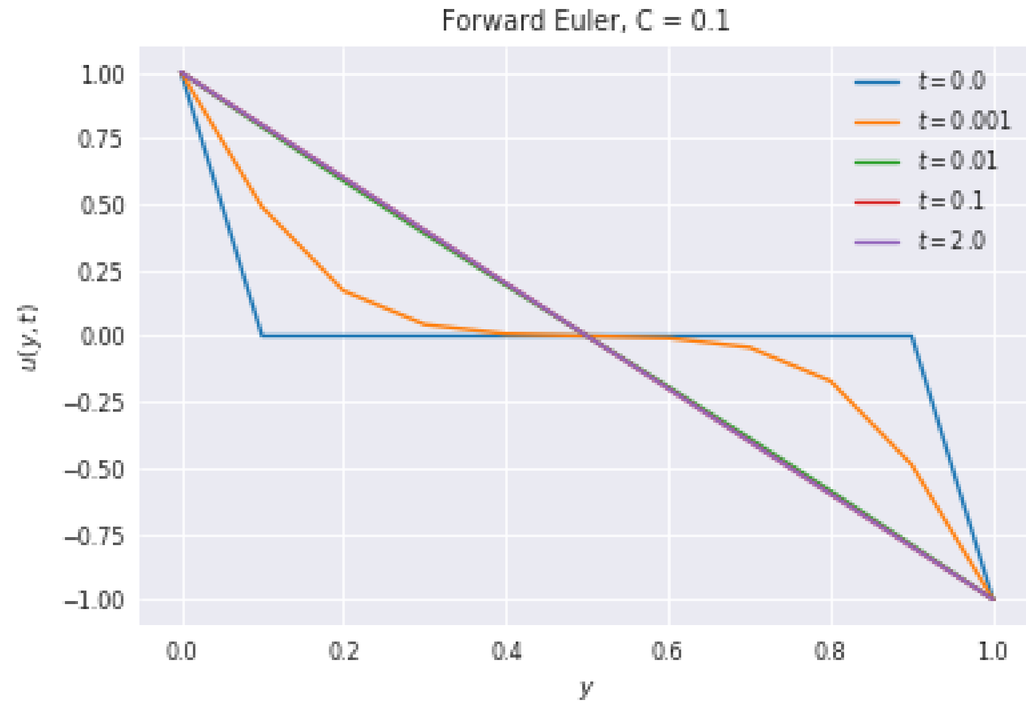
In [8]:
```
y_min = 0.0
y_max = 1.0

u_min = 0.0
u_max = 1.0

t_min = 0.0
t_max = 2.0

y_step = 0.1
t_step = 0.0001

y_osc = np.arange(y_min, y_max+y_step, y_step)
t_osc = np.arange(t_min, t_max+t_step, t_step)
y_osc_spacing = len(y_osc)
t_osc_spacing = len(t_osc)
u_grid_osc = np.zeros((y_osc_spacing, t_osc_spacing))

#Setting Initial Conditions
```

```
omega = 1.0*np.pi/2.0
u_grid_osc[0, :] = np.cos(omega*t_osc)
u_grid_osc[-1, :] = -1.0
```

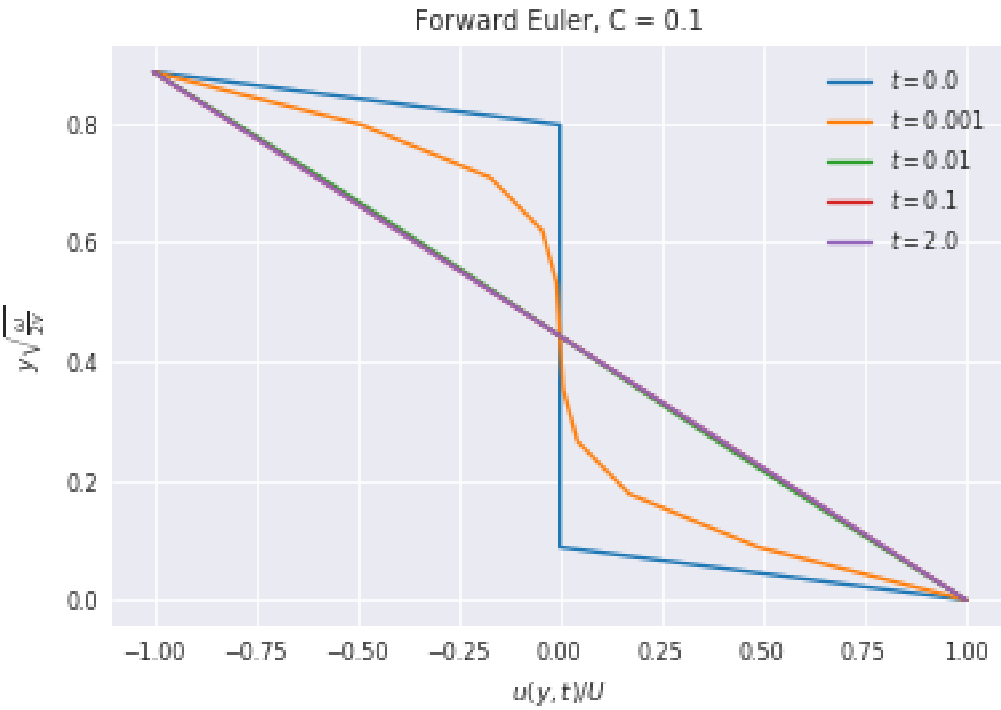
```
In [9]: u_grid_euler_osc = integrate_t(u_grid_osc, forward_euler_t, heat_slope_func, t_osc, y_s
        u_grid_crank_osc = integrate_t(u_grid_osc, crank_nicholson_t, heat_slope_func, t_osc, y
```

```
/home/gmduvvuri/miniconda3/envs/astroconda/lib/python3.5/site-packages/ipykernel_launcher.py:2
/home/gmduvvuri/miniconda3/envs/astroconda/lib/python3.5/site-packages/ipykernel_launcher.py:2
```

```
In [10]: plt.plot(y_osc, u_grid_euler_osc[:, 0], label=r'$t=$' + str(t_osc[0]))
         plt.plot(y_osc, u_grid_euler_osc[:, 10], label=r'$t=$' + str(t_osc[10]))
         plt.plot(y_osc, u_grid_euler_osc[:, 100], label=r'$t=$' + str(t_osc[100]))
         plt.plot(y_osc, u_grid_euler_osc[:, 1000], label=r'$t=$' + str(t_osc[1000]))
         plt.plot(y_osc, u_grid_euler_osc[:, -1], label=r'$t=$' + str(t_osc[-1]))
         plt.ylabel(r'$u(y, t)$')
         plt.xlabel(r'$y$')
         plt.title('Forward Euler, C = 0.1')
         plt.legend()
         plt.show()

         laminar_flow = y_osc*np.sqrt(omega/(2.0))
         plt.plot(u_grid_euler_osc[:, 0], laminar_flow,label=r'$t=$' + str(t_osc[0]))
         plt.plot(u_grid_euler_osc[:, 10], laminar_flow,label=r'$t=$' + str(t_osc[10]))
         plt.plot(u_grid_euler_osc[:, 100], laminar_flow,label=r'$t=$' + str(t_osc[100]))
         plt.plot(u_grid_euler_osc[:, 1000], laminar_flow,label=r'$t=$' + str(t_osc[1000]))
         plt.plot(u_grid_euler_osc[:, 1000], laminar_flow,label=r'$t=$' + str(t_osc[-1]))
         plt.xlabel(r'$u(y, t)/U$')
         plt.ylabel(r'$y\sqrt{\frac{\omega}{2v}}$')
         plt.title('Forward Euler, C = 0.1')
         plt.legend()
         plt.show()
```



Forward Euler, C = 0.1
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
$u(y,t)$
0.0
0.2
0.4
0.6
0.8
1.0
y
$t = 0.0$
$t = 0.001$
$t = 0.01$
$t = 0.1$
$t = 2.0$



Forward Euler, C = 0.1
0.8
0.6
0.4
0.2
0.0
$y\sqrt{\frac{\omega}{2\nu}}$
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
$u(y,t)/U$
$t = 0.0$
$t = 0.001$
$t = 0.01$
$t = 0.1$
$t = 2.0$