natural cubic splines

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1 Definition

(from Tibsirani, ESL)

A natural cubic spline with K knots is represented by K basis functions. One can start from a basis for cubic splines, and derive the reduced basis by imposing the boundary constraints. For example, starting from the truncated power series basis described in Section 5.2, we arrive at (Exercise 5.4):

$$N_1(X) = 1, \quad N_2(X) = X, \quad N_{k+2}(X) = d_k(X) - d_{K-1}(X), \qquad (5.4)$$

where

$$d_k(X) = \frac{(X - \xi_k)_+^3 - (X - \xi_K)_+^3}{\xi_K - \xi_k}. \qquad (5.5)$$

Each of these basis functions can be seen to have zero second and third derivative for $X \geq \xi_K$.

2 3 knot splines

We have N_1 and N_2 from the definition, but we need N_3 for a 3 knot spline.

$$N_3 = N_{1+2}(X) = d_1(X) - d_{3-1}(X)$$

where ξ_k is the location of the k^{th} knot and

$$d_1(X) = \frac{(X - \xi_1)_+^3 - (X - \xi_3)_+^3}{\xi_3 - \xi_1}$$

and

$$d_2(X) = \frac{(X - \xi_2)_+^3 - (X - \xi_3)_+^3}{\xi_3 - \xi_2}$$

so

$$N_3 = N_{1+2}(X) = \frac{(X-\xi_1)_+^3 - (X-\xi_3)_+^3}{\xi_3 - \xi_1} - \frac{(X-\xi_2)_+^3 - (X-\xi_3)_+^3}{\xi_3 - \xi_2}$$

and we simplify that.

2.1 Simplify

1. Multiply terms by 1

$$N_3 = \frac{\xi_3 - \xi_2}{\xi_3 - \xi_2} * \frac{(X-\xi_1)_+^3 - (X-\xi_3)_+^3}{\xi_3 - \xi_1} - \frac{\xi_3 - \xi_1}{\xi_3 - \xi_1} * \frac{(X-\xi_2)_+^3 - (X-\xi_3)_+^3}{\xi_3 - \xi_2}$$

2. Distribute

$$= \frac{[(\xi_3 - \xi_2) * ((X-\xi_1)_+^3 - (X-\xi_3)_+^3)] - [(\xi_3 - \xi_1) * ((X-\xi_2)_+^3 - (X-\xi_3)_+^3)]}{(\xi_3 - \xi_1) * (\xi_3 - \xi_2)}$$

$$= \frac{[(\xi_3 - \xi_2) * (X-\xi_1)_+^3 - (\xi_3 - \xi_2) * (X-\xi_3)_+^3] - [(\xi_3 - \xi_1) * (X-\xi_2)_+^3 - (\xi_3 - \xi_1) * (X-\xi_3)_+^3]}{(\xi_3 - \xi_1) * (\xi_3 - \xi_2)}$$

3. Cancel and simplify

$$= \frac{(\xi_3 - \xi_2) * (X-\xi_1)_+^3 - (\xi_3 - \xi_2) * (X-\xi_3)_+^3 - (\xi_3 - \xi_1) * (X-\xi_2)_+^3 + (\xi_3 - \xi_1) * (X-\xi_3)_+^3}{(\xi_3 - \xi_1) * (\xi_3 - \xi_2)}$$

$$= \frac{(\xi_3 - \xi_2) * (X-\xi_1)_+^3 - (\xi_3 - \xi_1) * (X-\xi_2)_+^3 \, [-(\xi_3 - \xi_2) + (\xi_3 - \xi_1)] * (X-\xi_3)_+^3}{(\xi_3 - \xi_1) * (\xi_3 - \xi_2)}$$

$$= \frac{(\xi_3 - \xi_2) * (X-\xi_1)_+^3 - (\xi_3 - \xi_1) * (X-\xi_2)_+^3 + (\xi_2 - \xi_1) * (X-\xi_3)_+^3}{(\xi_3 - \xi_1) * (\xi_3 - \xi_2)}$$

4. Divide by 1

$$= \frac{(\xi_3 - \xi_2) * (X-\xi_1)_+^3 - (\xi_3 - \xi_1) * (X-\xi_2)_+^3 + (\xi_2 - \xi_1) * (X-\xi_3)_+^3}{(\xi_3 - \xi_1) * (\xi_3 - \xi_2)} \div \frac{(\xi_3 - \xi_2)}{(\xi_3 - \xi_2)}$$

$$= \frac{(X-\xi_1)_+^3 - \left[\frac{(\xi_3 - \xi_1)}{(\xi_3 - \xi_2)} * (X-\xi_2)_+^3\right] + \left[\frac{(\xi_2 - \xi_1)}{(\xi_3 - \xi_2)} * (X-\xi_3)_+^3\right]}{(\xi_3 - \xi_1)}$$