Choosing the weights for the logarithmic pooling of probability distributions.

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Motivation

- Obtain important insights on consensus belief formation and group decision making (Genest and Zidek, 1986);
- Applications in a range of fields, from infectious disease modelling (Coelho and Codeço, 2009) and wildlife conservation (Poole and Raftery, 2000) to engineering (Savchuk and Martz, 1994);
- One important question remains, however: how to give each expert/information source a weight without being arbitrary?

Logarithmic pooling

Let $\mathbf{F}(\theta) = \{f_0(\theta), f_1(\theta), f_2(\theta), \ldots, f_K(\theta)\}$ be the set of prior distributions representing the opinions of $K+1$ experts and let $\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_K\}$ be the vector of weights, such that $\alpha_i > 0 \, \forall i$ and $\sum_{i=0}^{K} \alpha_i = 1$. Then the log-pooled prior is

$$\pi(\theta) = t(\boldsymbol{\alpha}) \prod_{i=0}^{K} f_i(\theta)^{\alpha_i} \tag{1}$$

with $t(\boldsymbol{\alpha}) = \int_{\boldsymbol{\Theta}} \prod_{i=0}^{K} f_i(\theta)^{\alpha_i} d\theta$.

- Enjoys rather desirable properties, such as *external Bayesianity* (Genest and Zidek, 1986);
- Poole and Raftery (2000) prove that $t(\boldsymbol{\alpha})$ is always finite for the case $K = 1$ (2 experts/priors), which we extend for any finite K. See Theorem 1 in `http://arxiv.org/pdf/1502.04206v1.pdf` for a simple proof.

Maximise entropy

- If there is no information about the reliabilities of the experts one might want to assign $\boldsymbol{\alpha}$ so as to maximise entropy of the resulting distribution:

$$H_\pi(\theta) = -\int_{\Theta} \pi(\theta) \ln \pi(\theta) d\theta$$

$$H_\pi(\theta; \boldsymbol{\alpha}) = \sum_{i=0}^{K} \alpha_i E_\pi[-\ln f_i(\theta)] - \ln t(\boldsymbol{\alpha})$$

- Formally, we want to find $\hat{\boldsymbol{\alpha}}$ such that

$$\hat{\boldsymbol{\alpha}} := \arg\max H_\pi(\theta; \boldsymbol{\alpha})$$

- Caveat: is not guaranteed to yield an unique solution.

Minimise Kullback-Leibler divergence

- What if we want to minimise conflict between the consensus and each individual opinion?
- Let $d_i = \text{KL}(f_i || \pi)$ and let $L(\boldsymbol{\alpha})$ be a loss function such that

$$L(\boldsymbol{\alpha}) = \sum_{i=0}^{K} d_i \tag{2}$$

$$= -K \ln t(\boldsymbol{\alpha}) + \sum_{i=0}^{K} \sum_{j \neq i}^{K} \alpha_j \text{KL}(f_i || f_j) \tag{3}$$

$$\hat{\boldsymbol{\alpha}} := \arg\min L(\boldsymbol{\alpha}) \tag{4}$$

- Contrary to the above, the loss function is convex, thus there is a unique solution (Rufo et al., 2012).

Place a prior on the weights

- An appealing alternative is to place a (hyper) prior on the weights ($\boldsymbol{\alpha}$);
- Two options:
  (a) Dirichlet prior:
  $$\pi(\boldsymbol{\alpha}) = \frac{1}{\mathcal{B}(\mathbf{X})} \prod_{i=0}^{K} \alpha_i^{x_i - 1}$$
  (b) logistic-normal:
  $$\alpha_i = \frac{e^{m_i}}{\sum_{i=0}^{K} e^{m_i}}, \; m_i \sim N(\mu_i, \sigma_i^2)$$
- Advantage: accomodates uncertainty in natural way, and is very flexible;
- Caveat(s): may yield inconsistent results and hardly ever allows for analytical solutions.

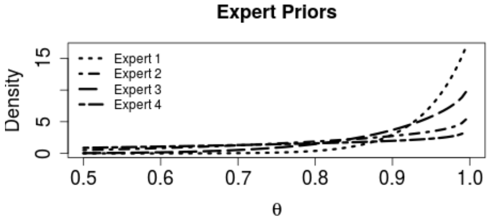
Application: binomial probabilities

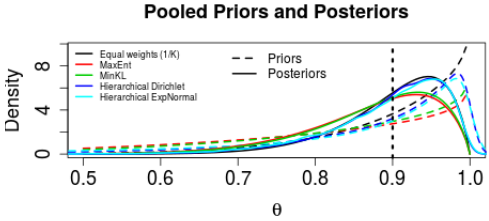
- $Y \sim \textit{Bernoulli}(\theta)$ and

$$f_i(\theta; a_i, b_i) = \frac{\Gamma(a_i + b_i)}{\Gamma(a_i b_i)} \theta^{a_i - 1}(1-\theta)^{b_i - 1}$$

- Allows for simple expressions for the entropy and KL divergence [$\pi(\theta; \boldsymbol{\alpha})$ is also Beta], and efficient sampling from the hyperpriors;
- Savchuk and Martz (1994) consider an example in which four experts are required supply prior information about the survival probability of a certain unit for which there have been $y = 9$ successes out of $n = 10$ trials;
- We propose to evaluate performance using integrated (marginal) likelihoods, a.k.a., prior evidence.

~~very~~ Preliminary results





~~very~~ Preliminary results II

Table : Weights obtained using the three methods for the proportion estimation problem. [1] – Kullback-Leibler [2] – Posterior mean for $\boldsymbol{\alpha}$.

Method	α_0	α_1	α_2	α_3
Maximum entropy	0.00	1.00	0.00	0.00
Minimum KL[1] divergence	0.04	0.96	0.00	0.00
Hierarchical prior[2]	0.26	0.24	0.26	0.23

Table : Integrated likelihoods for the priors of each expert as well as the combined priors. [1] Calculated using the posterior mean of $\boldsymbol{\alpha}$

Expert priors		Pooled priors	
Expert 0	0.237	Equal weights	0.254
Expert 1	0.211	Maximum entropy	0.211
Expert 2	0.256	Minimum KL	0.223
Expert 3	0.163	Hierarchical[1]	0.255

Conclusions and perspectives

- Our results are not yet decisive regarding which method is better;
- The Dirichlet approach seems the most natural from a Bayesian perspective, but prior sensitivity is currently unknown;
- Generalise the hyperprior approach to the exponential family;
- The relationship between the integrated likelihood of the combined (pooled) distribution in relation to that of each individual distribution.
- Applications to dynamic models (Poole and Raftery, 2000).

Thank you!

- Thank you very much for your attention!

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