One remark on distributors between groupoids and the comprehensive factorization.

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Abstract

In this short note we prove that distributors between groupoids form the bicategory of relations relative to the comprehensive factorization system.

1 Introduction

Distributors (also called profunctors, or bimodules) where introduced by Bénabou in [1] (see also [2], and [3, §7.8]), and they are often presented as a notion of *relation between categories*.

Actually, although the abstraction process leading from relations to distributors is far from being straightforward, in the case of groupoids this process is somehow clearer and it has a nice interpretation.

As recalled in section 2, in the set-theoretical case, relations can be introduced as being *relative* to a specified factorization system: the (surjective, injective). As a matter of fact, such a factorization system can be obtained from a *comprehension schema* (see [5]). Without entering too much into the details, for any set Y, one can consider the *comprehension adjunction*

$$\begin{tikzcd} \mathbf{Set}/Y \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{2}^Y \arrow[l, shift left] \end{tikzcd}, \tag{1}$$

where $\mathbf{2}^Y$ is the partially ordered set of the subsets of Y.

Then, for any function $X \xrightarrow{f} Y$, the (surjective) unit of the adjunction provides the factorization:

$$\begin{tikzcd} X \arrow[r, "\eta_f"] \arrow[dr, "f"'] & Im(f) \arrow[d, "m"] \\ & Y \end{tikzcd}$$

where m is a monomorphism.

As observed in [7], similar arguments can be used starting with the adjunction

$$\begin{tikzcd} \mathbf{Cat}/\mathbb{Y} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Set}^{\mathbb{Y}} \arrow[l, shift left] \end{tikzcd},$$

but climbing up one dimension produces two distinct factorizations of a functor: (initial, discrete opfibration) and (final, discrete fibration). The first was named *comprehensive factorization of a functor* in [7], as arising from a *comprehension scheme*.

Indeed, it is immediate to observe that these two factorization systems coincide if we consider the subcategory **Gpd** of groupoids.

The purpose of this note is to show that, when restricted to the category of groupoids, distributors form a bicategory of relations relative to the comprehensive factorization system.

2 Relations in Set

Classically, a *relation* between two sets A and B

$$A \overset{S}{\longrightarrow\!\!\!\!\!\!\!\!\mid\,\,\,\,} B$$

is just a subset S of the cartesian product $A \times B$. The set of relations between A and B

$$\mathbf{Rel}(A, B)$$

has an obvious category structure, where the arrows are inclusions. Moreover, a composition of relations is defined: given relations S and T

$$A \overset{S}{\longrightarrow\!\!\!\!\!\!\!\!\mid\,\,\,\,} B \overset{T}{\longrightarrow\!\!\!\!\!\!\!\!\mid\,\,\,\,} C$$

the pair $(a, c) \in A \times C$ is in $T \circ S$ if there exists $b \in B$ such that $(a, b) \in A \times B$ and $(b, c) \in B \times C$. The composition of relations is associative, with identities given by diagonals $\Delta_A \subset A \times A$, and these data form the bicategory **Rel**, in fact a mere 2-category.

We wish to point out the following elementary and very well known fact. Given sets A and B, there is a bijection between relations and functions:

$$\frac{S \hookrightarrow A \times B}{A \times B \xrightarrow[\chi_S]{} \mathbf{2}} \tag{2}$$

where the codomain $\mathbf{2}$ of the the characteristic function χ_S is the set of truth values. In this case one may think $\mathbf{2} = \{\notin, \in\}$.

Finally, we observe that the fact that relations organize themselves in a 2-category is a consequence of the choice of dealing with subsets, i.e. isomorphism classes of injective functions. More generally, one can consider mere injective functions, and end up with the weaker structure of a bicategory.

Now, let us turn our attention to *spans*. A span between two sets A and B is a pair of functions

$$\begin{tikzcd} & E \arrow[dl, "e_1"'] \arrow[dr, "e_2"] & \\ A & & B \end{tikzcd}$$

The set of spans between A and B form a category: a morphism between two spans (e_1, E, e_2) and (e'_1, E', e'_2) is just a function $E \xrightarrow{\gamma} E'$ satisfying $e_1 = e'_1 \cdot \gamma$ and $e_2 = e'_2 \cdot \gamma$

The composition of spans is defined by taking the pullback and then composing the projections, as in the diagram provided below:

$$\begin{tikzcd}
 & & F \diamond E \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] & & \\
 & E \arrow[dl] \arrow[dr] & & F \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}$$

These data, together with the obvious identity spans, form a bicategory denoted by **Span**.

For any pair of sets A and B, there is a (regular) epimorphic reflection

$$\begin{tikzcd}
\mathbf{Rel}(A,B) \arrow[r, shift right=1.5ex, "i_{A,B}"'] \arrow[r, phantom, "\perp"] & \mathbf{Span}(A,B) \arrow[l, shift right=1.5ex, "r_{A,B}"']
\end{tikzcd}$$

where $i_{A,B}$ embeds relations in spans by taking the composition with product projections, while the reflection is given by the (surjective, injective) factorization system available in the category of sets. More precisely, given a span (e_1, E, e_2) one obtains its associated relation by taking the image $r_{A,B}(E)$ of the function

$$E \xrightarrow{\langle e_1, e_2 \rangle} A \times B$$

As a matter of fact, the (epi, mono) factorization establishes the connection between the composition of relations and the composition of spans. Indeed, one can show that, given two relations S and T as above, their composition as relations is precisely the reflection of their composition as spans:

$$T \circ S = r_{A,C}(T \diamond S)\,, \tag{3}$$

and this is enough in order to extend the above reflection to a constant on object lax biadjunction

$$\begin{tikzcd}
\mathbf{Rel} \arrow[r, shift right=1.5ex, "i"'] \arrow[r, phantom, "\perp"] & \mathbf{Span} \arrow[l, shift right=1.5ex, "r"']
\end{tikzcd}$$

Actually, only the 2-functor i is truly lax, since r is in fact a pseudo 2-functor.

More generally, one can start with any finitely complete category $\mathcal{C}$, so that she can define internally the bicategory of spans. Then one adds the hypothesis that $\mathcal{C}$ is also endowed with a $(\mathcal{E}, \mathcal{M})$ (orthogonal) factorization system. This means that we have two classes of maps, $\mathcal{E}$ and $\mathcal{M}$ such that:

- they contain all the isomorphisms and they are closed under composition;

- every arrow of $\mathcal{C}$ can be factored as $f = m \cdot e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$;
- the factorization is functorial.

Then, given two objects A and B, one defines the categories $\mathbf{Rel}(A, B)$ together with the local reflections $r_{A,B} \dashv i_{A,B}$. However, element-wise composition of relations is missing, so that one may define the composition using the local reflections. In other words, one may take the formula (3) as the definition of the composition of relations. Indeed, such a composition needs not be associative. As a consequence, in general we do not obtain a bicategory. When we do get a bicategory $\mathbf{Rel}(\mathcal{C})$, then we call it:

> the bicategory of relations in $\mathcal{C}$ *relative to the factorization system* $(\mathcal{E}, \mathcal{M})$.

This happens, for instance, when $\mathcal{C}$ is regular, or more generally, when $(\mathcal{E}, \mathcal{M})$ is a proper factorization system with the class $\mathcal{E}$ stable under pullbacks, but these conditions are not strictly necessary. The literature on this subject is wide, and the interested reader shall consult the excellent [6] and the references therein.

3 Preliminaries on distributors

We mainly refer to [3] for definitions and notation.

Definition 3.1 (Bénabou, [1])**.** *A distributor S is a set valued functor*

$$\mathbb{B}^{\mathsf{op}} \times \mathbb{A} \xrightarrow{S} \mathbf{Set} \ .$$

where $\mathbb{B}^{\mathsf{op}}$ is the opposite category of $\mathbb{B}$.

As explained in [3], it is convenient to interpret the distributor

$$\mathbb{B} \xleftarrow{S} \!\!\!\!\!\!|\ \ \ \mathbb{A} \ ,$$

as giving rise to a categorical relation from the category $\mathbb{B}$ to the category $\mathbb{A}$. Then, keeping on the analogy with relations, the definition of distributor extends to categories the point of view expressed by the *denominator* of (2). This way, given two objects b and a of $\mathbb{B}$ and $\mathbb{A}$ respectively, one may think that there is *a set $S(b, a)$ of ways* in which b and a are related by S. An element $s \in S(b, a)$ will be represented by a dashed arrow connecting b with a:

$$a \overset{s}{\dashleftarrow} b$$

Functoriality of S then simply means that (i) the category $\mathbb{B}$ acts on from left on the disjoint union of the sets $S(b, a)$, (ii) the category $\mathbb{A}$ acts on the right, and (iii) that these two actions are compatible, i.e. the set $S(b, a)$ of ways in which b and a are related takes into account the categorical structure of $\mathbb{B}$ and of $\mathbb{A}$. This is made evident if we describe the two actions simply as an *external* composition: given α, β and s as in the diagram below,

$$a' \xleftarrow{\alpha} a \overset{s}{\leftarrow\!-\!-} b \xleftarrow{\beta} b'$$

the compatibility condition looks like a sort of associativity axiom

$$(\alpha \cdot s) \cdot \beta = \alpha \cdot (s \cdot \beta)\,.$$

Being functors $\mathbb{B}^{\mathsf{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}$, distributors from $\mathbb{A}$ to $\mathbb{B}$ naturally form a category denoted by $\mathbf{Dist}(\mathbb{A}, \mathbb{B})$.

Let us consider two distributors S and T as represented below:

$$\mathbb{C} \overset{T}{\leftarrow\!\!\!+\!\!\!-} \mathbb{B} \overset{S}{\leftarrow\!\!\!+\!\!\!-} \mathbb{A} \tag{4}$$

Their composition $T \otimes S$ is defined as follows: if c and a are objects of $\mathbb{C}$ and $\mathbb{A}$ respectively, $T \otimes S\,(c, a)$ is the quotient set of $\coprod_{b\in\mathbb{B}} T(c, b) \times S(b, a)$ determined by the equivalence relation generated by

$$(t, s) \sim (t', s') \qquad \text{if there exists } \beta \text{ such that } s = s' \cdot \beta,\ \beta \cdot t = t'\,. \tag{5}$$

We shall denote the equivalence class of (t, s) by $s \otimes t$.

All such sets $T \otimes S\,(c, a)$ are compatible with the actions of $\mathbb{C}$ and $\mathbb{A}$, therefore they can be arranged in a set valued functor, which is usually described as the coend

$$T \otimes S = \int^{b\in\mathbb{B}} T(-, b) \times S(b, -)\colon\quad \mathbb{C}^{\mathsf{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}\ . \tag{6}$$

Composition of distributor is not associative on the nose, but only up to natural isomorphisms. Distributors organize themselves in a bicategory denoted by **Dist**, identities being given by hom-functors $\mathrm{Hom}_{\mathbb{A}}(-, -)$.

To conclude our survey on distributors, we shall briefly return to relations and to the correspondence recalled in (2). In order to compare the bicategory **Dist** with the bicategory **Span**(**Cat**) (composition given by *strict* pullbacks), we have to expand our analogy with relations and find a solution to the following:

$$\frac{S \hookrightarrow A \times B}{A \times B \xrightarrow[\chi_S]{} \mathbf{2}} = \frac{???}{\mathbb{B}^{\mathsf{op}} \times \mathbb{A} \xrightarrow[S]{} \mathbf{Set}} \tag{7}$$

It is well-known that there are several possible candidate solutions. The most natural choice would be to consider the category of elements of the functor S or its opposite. This would lead us to consider a discrete opfibration over $\mathbb{B}^{\mathsf{op}} \times \mathbb{A}$ or a discrete fibration over its opposite $\mathbb{B} \times \mathbb{A}^{\mathsf{op}}$. In both cases, we would end up with a span over categories with opposite variance, situation which is not straightforwardly suitable for our intentions to relate distributor composition with the composition of the corresponding spans – this point of view surely deserves further investigations that the author is considering for a subsequent work.

Another possibility is offered by the more symmetrical notion of discrete two-sided fibration over $\mathbb{A} \times \mathbb{B}$. We shall describe explicitly the constructions that concern the case we are considering, but first let us give the formal definition.

A discrete two-sided fibration is the discrete version of the more general notion of two-sided fibration introduced by N. Yoneda in [8], and the formalized by J.W. Gray in [4] with the name of $(0,1)$-bifibration.

Definition 3.2. *A discrete two-sided fibration is a span of categories and functors*

$$\begin{tikzcd} & \mathbb{E} \arrow[dl, "Q"'] \arrow[dr, "P"] & \\ \mathbb{A} & & \mathbb{B} \end{tikzcd} \tag{8}$$

such that:

(i) each arrow $b \longrightarrow P(e')$ *in* $\mathbb{B}$ *has a unique* Q*-vertical lift at* e'*, i.e. such that it lies in the fiber over* $Q(e')$*.*

(ii) each arrow $Q(e) \longrightarrow a$ *in* $\mathbb{A}$ *has a unique* P*-vertical lift at* e*, i.e. such that it lies in the fiber over* $P(e)$*.*

(iii) for every arrow $e \xrightarrow{\epsilon} e'$ *in* $\mathbb{E}$*, the codomain of the* P*-vertical lift of* $Q(\epsilon)$ *at* e *is equal to the domain of the* Q*-vertical lift of* $P(\epsilon)$ *at* e'*, and the composite of the two lifts is* ϵ*.*

Now we can complete diagram (7) as follows.

Fact 3.3. *Giving a distributor* $\mathbb{B}^{\mathsf{op}} \times \mathbb{A} \xrightarrow{S} \mathbf{Set}$ *is equivalent to giving a discrete two-sided fibration*

$$\begin{tikzcd} & \mathbb{S} \arrow[dl, "S_1"'] \arrow[dr, "S_2"] & \\ \mathbb{A} & & \mathbb{B} \end{tikzcd}$$

which can be described as follows:

- $\mathbb{S}$ *is the category with objects*

$$\mathbb{S}_0 = \coprod_{a \in \mathbb{A}, b \in \mathbb{B}} S(b,a)$$

- *for* $s \in S(b,a)$ *and* $s' \in S(b',a')$*, an arrow* $s \longrightarrow s'$ *is a pair of arrows* (α, β) *with* $\alpha \in A(a,a')$ *and* $\beta \in B(b,b')$ *such that* $\alpha \cdot s = s' \cdot \beta$*. The* *reader may find it convenient to visualize arrows of $\mathbb{S}$ pretending they are commutative squares:*

$$\begin{tikzcd} a \arrow[d, "\alpha"'] & b \arrow[l, dashed, "s"'] \arrow[d, "\beta"] \\ a' & b' \arrow[l, dashed, "s'"] \end{tikzcd} \tag{9}$$

- *composition and identities are induced from those of $\mathbb{A}$ and $\mathbb{B}$.*
- *S_1 and S_2 are the obvious projections, i.e. $S_1(\alpha, \beta) = \alpha$ and $S_2(\alpha, \beta) = \beta$.*

4 Distributors between groupoids

Let us recall that a functor $\mathbb{A} \xrightarrow{F} \mathbb{B}$ is called *discrete fibration* if for every arrow $\xrightarrow{\beta} F(a')$, there exists a unique lift $\xrightarrow{\alpha} a'$ of β at a'. F is called *discrete opfibration* if F^{op} is a discrete fibration. We shall denote by $\mathcal{D}$ the class of discrete fibrations, and with $\mathcal{D}^{\mathsf{op}}$ the class of discrete opfibrations.

If we denote by $\mathcal{F}$ the class of functors that are left orthogonal to the class $\mathcal{D}$, we obtain a factorization system $(\mathcal{F}, \mathcal{D})$ for the category **Cat**. Not surprisingly, one can obtain another factorization system by taking opposites, namely $(\mathcal{F}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}})$. This is called *comprehensive factorization* in [7], where the authors introduce it from a categorical *comprehension schema*. Functors in $\mathcal{F}$ are called *final*, while functors in $\mathcal{F}^{\mathsf{op}}$ are called *initial*. We will refer to the $(\mathcal{F}, \mathcal{D})$ as to the (*final/discrete fibration*) factorization system.

The following characterization can be found in [*loc. cit.*]:

Proposition 4.1. *A functor $\mathbb{A} \xrightarrow{F} \mathbb{B}$ is final if, and only if, for every object b of $\mathbb{B}$, the comma category (b/F) is non-empty and connected.*

Some relevant facts occur when we consider groupoids instead of just categories.

Proposition 4.2. *Let $\mathbb{A} \xrightarrow{F} \mathbb{B}$ be a functor between groupoids. Then the following statements are equivalent:*

(i) F is final;

(ii) F is initial;

(iii) F is full and essentially surjective on objects.

Proof. Since clearly $\mathcal{D} = \mathcal{D}^{\mathsf{op}}$, one has $\mathcal{F} = \mathcal{F}^{\mathsf{op}}$ by the very definition of a factorization system. Hence it suffices to prove that (i) is equivalent to (iii).

Let F be final, and consider an arrow $F(a) \xrightarrow{\beta} F(a')$ in $\mathbb{B}$ and the following two objects of $(F(a)/F)$: $(id_{F(a)}, a)$ and (β, a'). Since $(F(a)/F)$ is connected by hypothesis, there is a path connecting the two objects above, but since $(F(a)/F)$ is itself a groupoid, such a path can be replaced with an arrow $(id_{F(a)}, a) \xrightarrow{\alpha} (\beta, a')$, and $F(\alpha) = \beta$. Moreover, for any object b, (b/F) is non-empty, i.e. there exists an object a and an arrow β such that $(\beta, a) \in (b/F)$. Since $\mathbb{B}$ is a groupoid, β is an isomorphism and this shows that F is essentially surjective on objects.

Conversely, let us suppose that F is full and essentially surjective on objects. We are to show that for any choice of b in $\mathbb{B}$, the comma category (b/F) is non-empty and connected. It is clearly non-empty since F is essentially surjective on objects, so let us prove it is connected. To this end, let us consider two objects (β, a) and (β', a') of (b/F). Since $\mathbb{B}$ is a groupoid, one can consider the composition $F(a) \xrightarrow{\beta' \cdot \beta^{-1}} F(a')$. Since F is full, one can find an arrow $a \xrightarrow{\alpha} a'$ in $\mathbb{A}$ such that $F(\alpha) = \beta' \cdot \beta^{-1}$, i.e. α underlies an arrow connecting (β, a) with (β', a') in (b/F), and this completes the proof. □

Another relevant fact concerns our representation of distributors in terms of a discrete two-sided fibration. It is given in the following proposition.

Proposition 4.3. *Let $\mathbb{A}$ and $\mathbb{E}$ be groupoids, and $\mathbb{B}$ a category. For a functor*

$$\mathbb{E} \xrightarrow{\langle Q,P \rangle} \mathbb{A} \times \mathbb{B}$$

the following statements are equivalent:

(i) $\langle Q, P \rangle$ yields a discrete two-sided fibration as in diagram (8);

(ii) $\langle Q, P \rangle$ is a discrete fibration.

Proof. (i)⇒(ii). We consider an arrow $(a, b) \xrightarrow{(\alpha,\beta)} (Q(e), P(e))$ in $\mathbb{A} \times \mathbb{B}$. By (ii) in Definition 3.2, we can lift $Q(e) \xrightarrow{\alpha^{-1}} a$ to a unique $e \xrightarrow{\hat{\alpha}^{-1}} e_a$ such that $P(\hat{\alpha}^{-1}) = id_{P(e)}$. By (i) in Definition 3.2, we can lift β to the unique $e_b \xrightarrow{\hat{\beta}} e_a$ such that $Q(\hat{\beta}) = id_{Q(e_a)=Q(a)}$. Hence $\hat{\alpha} \cdot \hat{\beta}$ is a lift at e of (α, β) along $\langle Q, P \rangle$: indeed, $Q(\hat{\alpha} \cdot \hat{\beta}) = Q(\hat{\alpha}) \cdot Q(\hat{\beta}) = \alpha \cdot id = \alpha$ and $P(\hat{\alpha} \cdot \hat{\beta}) = P(\hat{\alpha}) \cdot P(\hat{\beta}) = id \cdot \beta = \beta$. This lift is unique: suppose there is another one $e_b \xrightarrow{\epsilon} e$ of (α, β) along $\langle Q, P \rangle$. By (iii) in Definition 3.2, we get a factorization $\epsilon = \alpha' \cdot \beta'$, and one immediately sees that $\alpha' = \hat{\alpha}$ and $\beta' = \hat{\beta}$.

(ii)⇒(i). Conversely, first suppose we are given an arrow $b \xrightarrow{\beta} P(e')$. The (unique) lift required by point (i) in Definition 3.2 is given by the unique lift at e' of the arrow $(id_{Q(e')}, \beta)$ along the discrete fibration $\langle Q, P \rangle$. Then suppose we are given an arrow $Q(e) \xrightarrow{\alpha} a'$. The (unique) lift required by point (i) in Definition 3.2 is obtained by lifting at e the arrow $(\alpha^{-1}, id_{P(e)})$ along $\langle Q, P \rangle$, then taking its inverse. Finally, point (iii) in Definition 3.2 is obtained by observing that, for every arrow $e \xrightarrow{\epsilon} e'$, there is the factorization $(Q(\epsilon), P(\epsilon)) = (id_{Q(e')}, P(\epsilon)) \cdot (Q(\epsilon), id_{P(e)})$. □

The corollary below follows immediately.

Corollary 4.4. *In the category of groupoids, we consider a span*

$$\mathbb{E} \xrightarrow{\langle Q,P\rangle} \mathbb{A} \times \mathbb{B}$$

The following statements are equivalent:

(i) $\langle Q, P\rangle$ *is a discrete two-sided fibration;*

(ii) $\langle Q, P\rangle$ *is a discrete fibration;*

(iii) $\langle Q, P\rangle$ *is a discrete opfibration.*

Remark 4.5. In the case of the span representing the distributor $\mathbb{B}^{\mathsf{op}} \times \mathbb{A} \xrightarrow{S} \mathbf{Set}$, with reference to the description given in Fact 3.3, the discrete fibration can be described as follows. Suppose we are given two arrows $a \xrightarrow{\alpha} a'$ and $b \xrightarrow{\beta} b'$ together with an element $s' \in S(b, a)$. Then the unique lift of (α, β) at s' is the arrow:

$$(\alpha, \beta)\colon\ s \longrightarrow s'$$

where $s = \alpha^{-1} \cdot s' \cdot \beta$.

Since $(\mathcal{F}, \mathcal{D})$ is a factorization system, we need not prove the following statement.

Proposition 4.6. *Given two groupoids* $\mathbb{A}$ *and* $\mathbb{B}$*, the comprehensive factorization gives a reflection* R *to the inclusion of distributors into spans, i.e. there is an adjoint pair*

$$\begin{tikzcd} \mathbf{Dist}(\mathbb{A},\mathbb{B}) \arrow[r, shift right, "I_{\mathbb{A},\mathbb{B}}"'] \arrow[r, phantom, "\perp"] & \mathbf{Span}(\mathbb{A},\mathbb{B}) \arrow[l, shift right, "R_{\mathbb{A},\mathbb{B}}"'] \end{tikzcd}$$

with $R_{\mathbb{A},\mathbb{B}} \cdot I_{\mathbb{A},\mathbb{B}} \simeq id_{\mathbf{Dist}(\mathbb{A},\mathbb{B})}$.

Indeed, given a span as in diagram (8) the reflection is obtained by the factorization

$$\begin{tikzcd} \mathbb{E} \arrow[r, "R"] \arrow[dr, "\langle Q,P\rangle"'] & R(\mathbb{E}) \arrow[d, "\langle S_1,S_2\rangle"] \\ & \mathbb{A} \times \mathbb{B} \end{tikzcd}$$

where R is final, and $\langle S_1, S_2\rangle$ a discrete fibration.

Remark 4.7. Let us observe that by Yoneda embedding, Proposition 4.3 and the Corollary hold in any category with finite limits. As a consequence, Proposition 4.6 holds in every category where internal groupoids admits the comprehensive factorization system, as for instance in any exact category in the sense of Barr.

Theorem 4.8.

$$\mathbf{Dist}(\mathbf{Gpd}) = \mathbf{Rel}(\mathbf{Gpd}) \ \textit{w.r.t.}\ (\mathcal{F}, \mathcal{D}),$$

i.e. the bicategory of distributors between groupoids is the bicategory of relations in **Gpd** *relative to the (final/ discrete fibration) factorization system.*

The proof of the theorem follows immediately from the following lemma.

Lemma 4.9. *Distributor composition agrees with (the reflection of) span composition, i.e for given groupoids* $\mathbb{A}$*,* $\mathbb{B}$ *and* $\mathbb{C}$*, the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{Dist}(\mathbb{A},\mathbb{B}) \times \mathbf{Dist}(\mathbb{B},\mathbb{C}) \arrow[r, "\otimes"] \arrow[d, "I\times I"'] & \mathbf{Dist}(\mathbb{A},\mathbb{B}) \\
\mathbf{Span}(\mathbb{A},\mathbb{B}) \times \mathbf{Span}(\mathbb{B},\mathbb{C}) \arrow[r, "\diamond"'] & \mathbf{Span}(\mathbb{A},\mathbb{B}) \arrow[u, "R"']
\end{tikzcd}$$

where $\otimes$ *is the composition of distributors and* $\diamond$ *is the composition of spans, namely,* $T \diamond S$ *is given by the pullback of* S_2 *with* T_1*.*

Proof. The way the composition $R \cdot \diamond \cdot I \times I$ acts on a pair of distributors S and T is visualized in the following diagram

$$\begin{tikzcd}
& & T \diamond S \arrow[rr, "F"] \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & & R(T \diamond S) \arrow[ddllll, "R_1"'] \arrow[dd, "R_2"] \\
& \mathbb{S} \arrow[dl, "S_1"'] \arrow[dr, "S_2"'] & & \mathbb{T} \arrow[dl, "T_1"] \arrow[dr, "T_2"'] & \\
\mathbb{A} & & \mathbb{B} & & \mathbb{C}
\end{tikzcd}$$

As a matter of fact, we have two factorizations of the functor $\langle S_1 \cdot P_1, T_2 \cdot P_2 \rangle$:

$$\begin{tikzcd}
T \diamond S \arrow[r, "F"] \arrow[d, "Q"'] \arrow[dr, "\langle S_1 \cdot P_1, T_2 \cdot P_2 \rangle" description] & R(T \diamond S) \arrow[d, "\langle R_1, R_2 \rangle"] \\
T \otimes S \arrow[r, "\langle (T\otimes S)_1, (T\otimes S)_2 \rangle"'] & \mathbb{A} \times \mathbb{C}
\end{tikzcd}$$

The first is the comprehensive factorization given by the final functor F followed by the discrete fibration $\langle R_1, R_2 \rangle$. The second is given by the functor Q we shall describe below, followed by $\langle (T \otimes S)_1, (T \otimes S)_2 \rangle$.

Now, this last functor is a discrete fibration since the pair $((T \otimes S)_1, (T \otimes S)_2)$ is the span representing the distributor $T \otimes S$. Hence it will be sufficient to show that Q is final in order to conclude, by uniqueness of the factorization, that $R(T \diamond S) \simeq T \otimes S$ as desired.

The functor Q is readily described below:

$$T \diamond S \xrightarrow{\quad Q \quad} T \otimes S$$

$$\begin{tikzcd}
a \arrow[d, "\alpha"'] & b \arrow[l, dashed, "s"'] \arrow[d, "\beta"] & c \arrow[l, dashed, "t"'] \arrow[d, "\gamma"] & \mapsto & a \arrow[d, "\alpha"'] & c \arrow[l, dashed, "s\otimes t"'] \arrow[d, "\gamma"] \\
a' & b' \arrow[l, dashed, "s'"] & c' \arrow[l, dashed, "t'"] & & a' & c' \arrow[l, dashed, "s'\otimes t'"]
\end{tikzcd}$$

In fact Q is final. In order to prove it, we must prove that, for any object $\sigma\otimes\tau$ of $T \otimes S$, the comma category $(F\,/\sigma\otimes\tau)$ is nonempty and connected. Recall that $(F\,/\sigma\otimes\tau)$ has objects the pairs $(\phi,(s,t))$ where $\sigma\otimes\tau \xrightarrow{\phi} s\otimes t$, and arrows $(\alpha,\beta,\gamma)\colon\ (\phi,(s,t)) \longrightarrow (\phi',(s',t'))$, with $F(\alpha,\beta,\gamma)\cdot\phi = (\alpha,\gamma)\cdot\phi = \phi'$.

- $(F\,/\sigma\otimes\tau)$ is nonempty. Indeed, the object $(id_{\sigma\otimes\tau},(\sigma,\tau))$ is in $(F/\,\sigma\otimes\tau)$.
- $(F\,/\sigma\otimes\tau)$ is connected. Let us consider two objects $(\phi,(s,t))$ and $(\phi',(s',t'))$ of $(F\,/\sigma\otimes\tau)$. Then ϕ and ϕ' are the classes of some

$$\begin{aligned}
(\phi_A,\phi_B,\phi_C) \quad &: \quad (\sigma,\tau) \longrightarrow (s,t) \\
(\phi'_A,\phi'_B,\phi'_C) \quad &: \quad (\sigma,\tau) \longrightarrow (s',t')
\end{aligned}$$

  so that the arrow $(\phi'_A\cdot\phi_A^{-1},\phi'_B\cdot\phi_B^{-1},\phi'_C\cdot\phi_C^{-1},)$ connects $(\phi,(s,t))$ with $(\phi',(s',t'))$.

$\square$

5 Composition of distributors revisited

Let us consider two composable distributors S and T as in diagram (4). Their composition, obtained via the coend formula (6), can be interpreted as connected components $\pi_0(\mathbb{H})$ of a suitable category $\mathbb{H}$. This category has arrows $(s,t) \xrightarrow{\beta} (s't')$ as described below:

$$\begin{tikzcd}
& b \arrow[dl, dashed, "s"'] \arrow[dd, "\beta"] & \\
a & & c \arrow[ul, dashed, "t"'] \arrow[dl, dashed, "t'"] \\
& b' \arrow[ul, dashed, "s'"] &
\end{tikzcd}$$

i.e. with $s'\cdot\beta = s$ and $t' = \beta\cdot t$.

Let us notice that the class of objects of $\mathbb{H}$ coincides with the class of objects $(T\diamond S)_0$ of the pullback of T_1 with S_2. Concerning the class of arrows, they can be obtained as follows. First recall that a left (resp. right) categorical action amounts to a discrete fibration (resp. opfibration). For instance, for the distributor S, the action of $\mathbb{B}$ on $\mathbb{S}_0$ gives rise to the discrete fibration:

$$\mathbb{S}_R \longrightarrow \mathbb{B}$$

$$\begin{tikzcd} & b \arrow[dl, dashed, "s"'] \arrow[dd, "\beta"] \\ a & \\ & b' \arrow[ul, dashed, "s'"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} b \arrow[dd, "\beta"] \\ \\ b' \end{tikzcd}$$

where $s \xrightarrow{\beta} s'$ is an arrow of $\mathbb{S}_R$ if $s' \cdot \beta = s$. Similarly, for the distributor T, the action of $\mathbb{B}$ on $\mathbb{T}_0$ gives rise to the discrete opfibration:

$$\mathbb{T}_L \longrightarrow \mathbb{B}$$

$$\begin{tikzcd} b \arrow[dd, "\beta"'] & \\ & c \arrow[ul, dashed, "t"'] \arrow[dl, dashed, "t'"] \\ b' & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} b \arrow[dd, "\beta"'] \\ \\ b' \end{tikzcd}$$

where $t \xrightarrow{\beta} t'$ is an arrow of $\mathbb{T}_L$ if $t' = \beta \cdot t$.

Then one easily observe that $\mathbb{H}$ can be obtained as the pullback $\mathbb{S}_R \times_{\mathbb{B}} \mathbb{T}_L$ of the two functors above.

Indeed, not only $(T \otimes S)_0 = \pi_0(\mathbb{H})$, but the whole category $T \otimes S$ is a π_0 of a double category. We shell now describe this double category in general, even if we will limit ourselves to develop the theory for the case of groupoids.

A double cells of this double category is a diagram

$$\begin{tikzcd} (s_1, t_1) \arrow[r, "\beta"] \arrow[d, "(\alpha,\beta_1,\gamma)"'] & (s_2, t_2) \arrow[d, "(\alpha,\beta_2,\gamma)"] \\ (s_1', t_1') \arrow[r, "\beta'"'] & (s_2', t_2') \end{tikzcd}$$

where $\beta_2 \cdot \beta = \beta' \cdot \beta_1$. It may be helpful to visualize such double cells as follows:

$$
\begin{tikzcd}
a \arrow[dd, "\alpha"'] & & b_1 \arrow[ll, dashed, "s_1"'] \arrow[dl, "\beta"'] \arrow[dd, "\beta_1" near start] & \\
& b_1 \arrow[ul, dashed, "s_2"] \arrow[dd, "\beta_2" near end] & & c \arrow[ll, dashed, "t_2"'] \arrow[ul, dashed, "t_1"'] \arrow[dd, "\gamma"] \\
a' & & b_1' \arrow[ll, dashed, "s_1'"'] \arrow[dl, "\beta'"] & \\
& b_2' \arrow[ul, dashed, "s_2'"] & & c' \arrow[ll, dashed, "t_2'"] \arrow[ul, dashed, "t_1'"']
\end{tikzcd}
$$

If we denote by $(T \boxtimes S)_0$ the class of arrows of $\mathbb{H}$, and by $(T \boxtimes S)_1$ the class of double cells described just above, we can represent the double category as a double reflexive graph:

$$
\begin{tikzcd}
(T \boxtimes S)_1 \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & (T \diamond S)_1 \arrow[l, "e" description] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\
(T \boxtimes S)_0 \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] \arrow[u, "e" description] & (T \diamond S)_0 \arrow[l, "e" description] \arrow[u, "e" description]
\end{tikzcd}
\tag{10}
$$

In the case of groupoids, diagram (10) clearly underlies a double groupoid. Moreover, the factorization process that realizes the composition of S and T can be given the following interpretation.

If we see the double groupoid of diagram (10) as an (horizontal) groupoid in **Gpd**, then the π_0 (the groupoid of connected components) is given by the coequalizer in **Gpd** of domain $D = (d, d)$ and codomain $C = (c, c)$. Actually, from this point of view one concludes that the functor Q is final, and eventually computes the composition $T \otimes S$ as the induced factorization:

$$
\begin{tikzcd}
T \boxtimes S \arrow[r, shift left=2, "C"] \arrow[r, shift right=2, "D"'] & T \diamond S \arrow[l, "E" description] \arrow[rr, "Q"] \arrow[drr, "\langle S_1 \cdot P_1, T_2 \cdot P_2 \rangle"'] & & T \otimes S \arrow[d, "\langle T \otimes S_1, T \otimes S_2 \rangle"] \\
& & & \mathbb{A} \times \mathbb{C}
\end{tikzcd}
$$

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