Dipartimento di Matematica

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Relative ideals in homological categories

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An elementary example

R unital ring, $I \subseteq R$ (bilateral) ideal.

Fact. If $I \leqslant R$ in **Ring**, then $I = R$.

Can we deal with ideals of unital rings categorically?

Observe: $I \leqslant R$ in **Rng**, and **Ring** is a subcategory of **Rng**.

More precisely, ideals in **Ring** are kernels in the semi-abelian category **Rng**

Idea. Investigate the inclusion functor

$$U\colon \mathbf{Ring} \to \mathbf{Rng}$$

+ determine nice behavior of ideals that can be deduced from properties of U.

Observe:

- U is faithful, but not full.
- U is conservative, i.e. it reflects isomorphisms.
- U is a right adjoint, and its left adjoint F freely adds the unit element.

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Some glossary

A category **B** with finite limits is:

- **pointed:** $0 \to 1$ is an isomorphism.
- **regular:** p.b. stable regular epis + coequalizers of effective equiv. relations
- **Barr-exact:** regular + all equiv. relations are effective
- **protomodular:** $f^* \colon \mathbf{Pt}_{\mathbf{B}}(B) \to \mathbf{Pt}_{\mathbf{B}}(E)$ is conservative $\forall f \colon E \to B$.
- **with semidirect products:** $f^* \colon \mathbf{Pt}_{\mathbf{B}}(B) \to \mathbf{Pt}_{\mathbf{B}}(E)$ is monadic $\forall f \colon E \to B$.
- **homological:** regular + pointed + protomodular
- **semi-abelian:** Barr-exact + pointed + protomodular + finite coproducts

Basic setting and relative ideals

Definition (Lapenta, M., Spada)

A *basic setting* for relative U-ideals is an adjunction $\begin{tikzcd} \mathbf{B} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathbf{A} \arrow[l, bend right, "F"'] \end{tikzcd}$ where $\mathbf{A}$ is homological and U is conservative and faithful.

Observe:

- $\mathbf{A}$, $\mathbf{B}$ with finite limits (+ U preserves them), U conservative $\Rightarrow$ U faithful
- since U fin. limit. pres. + conservative, $\mathbf{A}$ protomodular $\Rightarrow$ $\mathbf{B}$ protomodular

Definition (Lapenta, M., Spada)

$k\colon A \to U(B)$ is a *relative U-ideal* of an object B in $\mathbf{B}$ if

there exists a morphism $f\colon B \to B'$ of $\mathbf{B}$ that makes the square diagram on the right a pullback in $\mathbf{A}$

$$\begin{tikzcd} B \arrow[d, "\exists\, f"'] & & A \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U(B) \arrow[d, "U(f)"] \\ B' & \text{s.t.} & 0 \arrow[r] & U(B') \end{tikzcd}$$

Relative ideals: examples

Unital rings:
$$\begin{tikzcd} \mathbf{Ring} \arrow[r, "U"'] \arrow[r, phantom, bend left=30, "\perp"] & \mathbf{Rng} \arrow[l, bend right=50, "F"'] \end{tikzcd}$$

$$F(R) = R \rtimes \mathbb{Z} \quad (r,n)(r',n') = (rn' + nr' + rr', nn')$$

Unital (associative) R-algebras:
$$\begin{tikzcd} \mathbf{UAlg}_R \arrow[r, "U"'] \arrow[r, phantom, bend left=30, "\perp"] & \mathbf{Alg}_R \arrow[l, bend right=50, "F"'] \end{tikzcd}$$

$$F(A) = A \rtimes R \quad (a,r)(a',r') = (r'a + ra' + aa', rr') \quad r(a,r') = (ra, rr')$$

Unital C^*-algebras:
$$\begin{tikzcd} \mathbf{UCStar} \arrow[r, "U"'] \arrow[r, phantom, bend left=30, "\perp"] & \mathbf{CStar} \arrow[l, bend right=50, "F"'] \end{tikzcd}$$

$F(A) = A \oplus \mathbb{C}$ with multiplication as above, and $(a,z)^* = (a^*, \bar{z})$

Algebraic varieties...

Relative ideals: examples

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Algebraic varieties...

The varietal case: basic setting for varieties

Recall from [BJ03] that a variety **V** is protomodular iff there exist $n \in \mathbb{N}$, 0-ary terms $e_1, \ldots, e_n$, binary terms $\alpha_1, \ldots, \alpha_n$, and $(n+1)$-ary term θ such that:

$$\theta(\alpha_1(x,y), \ldots, \alpha_n(x,y), y) = x, \quad \alpha_i(x,x) = e_i \quad \text{for } i = 1, \ldots, n$$

Fact. If **V** is semi-abelian, then $e_1 = \cdots = e_n = 0$.

Vice-versa, variety is called *classically ideally determined* (*BIT-speciale* in [U72]) if equations above hold for a specified constant $0 = e_1 = \cdots = e_n$.

Definition (Lapenta, M., Spada)

Let $\mathbf{A} = (\mathsf{A}, \Sigma_\mathsf{A}, Z_\mathsf{A})$ and $\mathbf{B} = (\mathsf{B}, \Sigma_\mathsf{B}, Z_\mathsf{B})$ be algebraic varieties, s.t.

- **A** homological, hence semi-abelian
- signatures $\Sigma_\mathsf{A} \subseteq \Sigma_\mathsf{B}$ and equations $Z_\mathsf{A} \subseteq Z_\mathsf{B}$

The forgetful functor $U\colon \mathbf{B} \to \mathbf{A}$ determines a special kind of basic setting that we call *basic setting for varieties.*

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0-ideals vs. U-ideals

Let $\mathbf{V}$ be a variety with a constant $0 \in \Sigma_{\mathbf{V}}$, $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_n)$.

- $t(\mathbf{x}, \mathbf{y})$ is a 0-ideal term in $\mathbf{y}$ if $t(\mathbf{x}, \mathbf{0}) = 0$ in $\mathbf{V}$, $\mathbf{0} = (0, \ldots, 0)$.
- $\emptyset \neq H \subseteq A$ is a 0-ideal of the algebra $A \in \mathbf{V}$, for every ideal term $t(\mathbf{x}, \mathbf{y})$

$$t(\mathbf{a}, \mathbf{h}) \in H, \qquad \mathbf{a} \in A^m, \ \mathbf{h} \in H^n.$$

Fact. In (classically) ideal determined varieties, {congruences} $\leftrightarrow$ {0-ideals}.

Proposition (Lapenta, M., Spada)

If $U\colon \mathbf{B} \to \mathbf{A}$ is a basic setting for varieties, $\mathbf{B}$ is classically ideally determined

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Let $U\colon \mathbf{B} \to \mathbf{A}$ be a basic setting for varieties. A subset H of an algebra $B \in \mathbf{B}$ is a 0-ideal iff $H \subseteq U(B)$ is a U-ideal of B with respect to $U\colon \mathbf{B} \to \mathbf{A}$.

A number of examples arise from the varietal case.
Moreover, one can consider topological models of the corresponding theories and develop other examples (if $\mathbf{Set}^{\mathbb{T}}$ is semi-abelian, $\mathbf{Top}^{\mathbb{T}}$ is homological).

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Augmentation ideals

Back to the *basic setting* $\mathbf{B} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{A}$ ($F \dashv U$), let $A \in \mathbf{A}$:

$$\begin{tikzcd} F(A) \arrow[d, "\exists! p_A"'] \\ I \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "0"'] & UF(A) \arrow[d, "U(p_A)"] \\ & U(I) \end{tikzcd}$$

Definition (Lapenta, M., Spada)

The unit η_A is an *augmentation U-ideal* if it is the kernel of $U(p_A)$.

Condition $(*)$

For every A in $\mathbf{A}$, η_A is an augmentation U-ideal

Proposition (Lapenta, M., Spada)

Condition $(*)$ *holds iff the unit η is cartesian*

Idea of the proof:

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "\eta_{A'}"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[r] \arrow[d, "\eta_A"] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d, "\eta_0"] \\ UF(A') \arrow[r, "UF(f)"'] & UF(A) \arrow[r, "UF(!_A)"'] & UF(0) \end{tikzcd}$$

$F(0) = I$

$F(!_A) = p_A$

Theorem (Lapenta, M., Spada)

Given a basic setting s.t. Condition $(*)$ *holds, with I initial in* $\mathbf{B}$, *the kernel functor*

$$K\colon \mathbf{B}/I \to \mathbf{A} \qquad f \mapsto Ker(U(f))$$

establishes an equivalence of categories.

Proof. (Janelidze's take)

Step 1. Given $\begin{tikzcd} \mathcal{B} \arrow[r, "\mathcal{U}"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{A} \arrow[l, bend right, "\mathcal{F}"'] \end{tikzcd}$, $\eta = id$ and $\mathcal{U}$ conservative $\Rightarrow$ $(\mathcal{F},\mathcal{U})$ equivalence.

Step 2. Given $\begin{tikzcd} \mathbf{B} \arrow[r, "U"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathbf{A} \arrow[l, bend right, "F"'] \end{tikzcd}$, and $X \in \mathbf{A}$, define the induced adjunction

$$\begin{tikzcd} \mathbf{B}/F(X) \arrow[r, "U^X"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathbf{A}/X \arrow[l, bend right, "F^X"'] \end{tikzcd} \qquad F^X(\alpha) = F(\alpha), \quad U^X(\beta) = (\eta_X)^*(U(\beta))$$

Step 3. Specialize to $\mathbf{B}/I = \mathbf{B}/F(0) \begin{tikzcd} {} \arrow[r, "U^0 = ker"'] \arrow[r, phantom, shift left=2, "\perp"] & {} \arrow[l, bend right, "F^0"'] \end{tikzcd} \mathbf{A}/0 = \mathbf{A}$ and apply Step 1.

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Step 1. Given $\mathcal{B} \underset{\mathcal{U}}{\overset{\mathcal{F}}{\underset{\perp}{\rightleftarrows}}} \mathcal{A}$, $\eta = id$ and $\mathcal{U}$ conservative $\Rightarrow (\mathcal{F},\mathcal{U})$ equivalence.

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Remarks

When the functor $K\colon \mathbf{B}/I \to \mathbf{A}$ is an equivalence,

- All objects of $\mathbf{A}$ can be seen as (augmentation) U-ideals of objects of $\mathbf{B}$, so that, in a sense, $\mathbf{A}$ sits inside $\mathbf{B}$.
- Since I is initial in $\mathbf{B}$, $\mathbf{B}/I = \mathbf{Pt}_{\mathbf{B}}(I)$.

  This means that one is motivated to describe the pseudoinverse H of K by a semidirect product in $\mathbf{A}$ (whenever $\mathbf{A}$ has semidirect products with I):

$$H\colon\ X \ \mapsto \quad \begin{array}{c} X \rtimes I \\ \uparrow\!\downarrow \\ I \end{array}$$

- In the examples considered, **Ring**, **UAlg**, **UCStar**, Condition $(*)$ holds.

Comparison with Ideally Exact Categories

At CT2023, G. Janelidze presented the novel notion of *ideally exact category*, as a first step towards "a development of a new non-pointed counterpart of semi-abelian categorical algebra" ([Ja23]).

In fact this notion shows a consistent connection with our basic setting for relative U-ideal. Let us clarify by starting again from the case of unital rings.

$$\begin{tikzcd}
\mathbf{Ring} \arrow[rr, "U"'] \arrow[dd, equal] \arrow[rr, phantom, shift left=2, "\perp"] & & \mathbf{Rng} \arrow[ll, bend right, "F"'] \arrow[dd, bend right, "H"'] \arrow[dd, phantom, "\dashv"] \\
& & \\
\mathbf{Ring}/\{0\} \arrow[rr, "!^*"'] \arrow[rr, phantom, shift left=2, "\perp"] & & \mathbf{Ring}/\mathbb{Z} \arrow[ll, bend right, "! \circ -"'] \arrow[uu, "K"']
\end{tikzcd}$$

- (F, U) basic setting
- (H, K) adjoint equivalence

  $\Rightarrow (F, U) \simeq (! \circ -, !^*)$

Idea: study properties of **Ring** that descend from properties of **Rng** via the monadic change of base functor along $! \colon \mathbb{Z} \to \{0\}$.

It turns out that the corresponding monad is **essentially nullary**.

Definition (Janelidze)

A monad $T = (T, \eta, \mu)$ on a cat. $\mathbf{X}$ with fin. coprod. is *essentially nullary* if, for every X in $\mathbf{X}$ the morphism $[T(!_X), \eta_X]\colon T(0) + X \to T(X)$ is a strong epi.

Examples.

- If $\mathbf{X}$ is a variety, any monad on $\mathbf{X}$ that adds constants + equations.
- If $\mathbf{X}$ is protomodular with finite coproducts, and T is a monad with cartesian units.

Definition (Janelidze)

A category $\mathbf{B}$ is ideally exact if it satisfies any of the following conditions:

(i) $\mathbf{B}$ Barr-exact protomodular with finite coprod. and $0 \to 1$ regular epi

(ii) $\mathbf{B}$ Barr-exact with finite coprod. and

$$\exists\ \mathbf{B} \to \mathbf{A} \text{ monadic, with } \mathbf{A} \text{ semi-abelian}$$

Notice that one can ask the monad in (ii) to be cartesian or essentially nullary.

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Ideally exact varieties

A **non-trivial** algebraic variety $\mathbf{V}$ is ideally exact iff it is protomodular.
If θ, $\alpha_1, \ldots, \alpha_n$ and $e_1, \ldots, e_n$ are terms that witness protomodularity, relevant examples are:

- $n = 2$, Heyting algebras, MV-algebras (we will discuss these later...)
- $n = 1$, groups (loops) with operations, unital R-algebras
- $n = 0$, in this case the characterization reduces to the existence of a unary term t satisfying the equation $t(x) = y$. There are two such varieties:

$$\emptyset \in \mathbf{V}_0 \qquad \text{and} \qquad \emptyset \notin \mathbf{V}_1$$

  Both are protomodular, but only $\mathbf{V}_1$ is ideally exact.

Basic setting for relative U-ideals VS Ideally Exact Categories

A relevant point is that the notion of Ideally Exact category is intrinsic:

(i) $\mathbf{B}$ Barr-exact protomodular with finite coprod. and $0 \to 1$ regular epi

However, concerning

(ii) $\mathbf{B}$ Barr-exact + fin. coprod. + $\exists\ U : \mathbf{B} \to \mathbf{A}$ monadic, with $\mathbf{A}$ semi-abelian

our *basic setting* has weaker assumptions, in that U is just (faithful and) conservative, and $\mathbf{A}$ is only homological.

In fact, it is possible to give up to Barr-exactness, while keeping monadicity.

Janelidze has shown that U comes from the change of base along $0 \to 1$ iff the unit of the adjunction is cartesian, which is the same as our Condition $(*)$ on augmentation ideals. Then, one could replace Barr-exactness with the requirement that $0 \to 1$ be effective descent.

Or, as it has been suggested by Bourn, one could consider an efficiently regular protomodular $\mathbf{B}$ with regular epi $0 \to 1$, so that $\mathbf{A} = \mathbf{B}/0$ is homological and $\mathbf{B} \to \mathbf{B}/0$ is monadic.

The category **Hoop** of hoops.

A *hoop* is an algebra $(A; \cdot, \rightarrow, 1)$ such that

(H0) $(A; \cdot, 1)$ is a commutative monoid

and the following equations hold:

(H1) $x \rightarrow x = 1$

(H2) $x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)$

(H3) $(x \cdot y) \rightarrow z = x \rightarrow (y \rightarrow z)$

Facts:

- Hoops are $\wedge$-semilattices, with $x \wedge y := x \cdot (x \rightarrow y)$.
- Hoops are partially ordered, with $x \leqslant y$ iff $x \rightarrow y = 1$ iff $\exists\, u$ s.t. $x = u \cdot y$.
- Hoops are residuated structures, with $x \cdot y \leqslant z$ iff $y \leqslant x \rightarrow z$.

A *bounded hoop* is an algebra $(A; \cdot, \rightarrow, 1, 0)$ such that $(A; \cdot, \rightarrow, 1)$ is a hoop, and the following equation holds:

(B) $0 \rightarrow x = 1$

Hoop is semi-abelian

Theorem (Lapenta, M., Spada)

Hoop *is semi-abelian.*

Proof. Since it is a pointed variety of algebras, it suffices to prove it's protomodular. Define terms:

$$e_1 := 1\,, \qquad e_2 := 1\,, \qquad \alpha_1(x,y) := x \to y\,,$$

$$\alpha_2(x,y) := ((x \to y) \to y) \to x\,, \qquad \theta(x,y,z) := (x \to z) \cdot y\,.$$

and apply the characterization in [BJ03].

Remark

Hoops satisfying $x \cdot x = x$ are called **idempotent**.
Idempotent hoops are (term equivalent to) Heyting $\wedge$-semilattices.

HSLat *is semi-abelian (* **HAlg** *is protomodular), proved by Johnstone in* [Jo04].

Here we use essentially the same terms as Johnstone's: same e_i and same α_i, while his $\beta(x,y,z) := (x \to z) \wedge y$ coincide with our θ under idempotency, but does not work verbatim for hoops.

Varieties of hoops

$(x \to y) \vee (y \to x) = 1$	*Basic hoops* (**BHoop**)	(P)
$(x \to y) \to y = (y \to x) \to x$	*Wajsberg hoops* (**WHoop**)	(W)
$x \cdot x = x$	*Idempotent hoops* (**IHoop**)	(I)
(P) + (I)	*Gödel hoops* (**GHoop**)	
(P) + $(x \to z) \vee ((y \to x \cdot y) \to x) = 1$	*Product hoops* (**PHoop**)	

where $x \vee y := ((x \to y) \to y) \wedge ((y \to x) \to x)$.

If the corresponding theories are expanded by adding a constant 0 and the axiom $0 \to x = 1$, one obtains the "equations"

$$\mathbf{WHoop} + 0 = \mathbf{WAlg} \ (\simeq \mathbf{MVAlg})$$

$$\mathbf{BHoop} + 0 = \mathbf{BAlg}$$

$$\mathbf{IHoop} + 0 \simeq \mathbf{HSLat} + 0 = \mathbf{HAlg}$$

$$\mathbf{GHoop} + 0 = \mathbf{GAlg}$$

$$\mathbf{PHoop} + 0 = \mathbf{PAlg}$$

Basic setting for varieties of hoops

Corollary (Lapenta, M., Spada)

The varieties of Basic, Wajsberg, Gödel and Product hoops are semi-abelian.

Proposition (Lapenta, M., Spada)

The forgetful functors $U\colon X\mathbf{Alg} \to X\mathbf{Hoop}$, *for* $X = \mathbf{B}, \mathbf{W}, \mathbf{G}, \mathbf{P}$ *determine basic settings for varieties:*

$$\begin{tikzcd} X\mathbf{Alg} \arrow[r, "U"'] \arrow[r, phantom, shift left=3, "\perp"] & X\mathbf{Hoop} \arrow[l, bend right, "F"'] \end{tikzcd}$$

Corollary (Lapenta, M., Spada)

The varieties of Basic, Wajsberg, Gödel and Product algebras are protomodular.

Corollary (See [Jo04])

The variety of Heyting semilattices is semi-abelian, while the variety of Heyting algebras is protomodular.

Case study: MV-algebras

An *MV-algebra* is an algebra $(A; \oplus, \neg, 0)$ such that $(A; \oplus, 0)$ is a commutative monoid and the the following equations hold

$$\neg\neg x = x\,, \qquad x \oplus \neg 0 = \neg 0\,, \quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x\,.$$

Other (derived) operations are be defined as follows:

$$\begin{aligned} &1 := \neg 0\,, \qquad x \odot y := \neg(\neg x \oplus \neg y)\,, \qquad x \to y := \neg x \oplus y \\ &x \vee y := \neg(\neg x \oplus y) \oplus y \quad x \wedge y := x \odot (\neg x \oplus y), \end{aligned}$$

Fact. $(A; \odot, \to, 1, 0)$ is a 0-bounded Wajsberg hoop (aka Wajsberg algebra). MV-algebras and W-algebras are the same, modulo term equivalence.

The functor $U\colon$ **MVAlg** $\to$ **WHoop** that forgets the 0 determines a *basic setting for varieties*.

[ACD10] describes the left adjoint, which is called *MV-closure*. We revisit the construction in the present context.

Observation. For MV-algebras the notions of *kernels* and *filters* are interchangeable. Here we focus on filters, described as relative U-ideals.

MV-closure $MV : \mathbf{WHoop} \to \mathbf{MVAlg}$

Given a Wajsberg hoop $W = (W; \odot, \to, 1)$, recall that a binary operation $\oplus_W$ can be canonically defined by letting $w \oplus_W w' := (w \to (w \odot w')) \to w'$.

Define the MV-closure of W

$$MV(W) := (W \times \mathbf{2}; \oplus, \neg, 0)$$

where

- $\mathbf{2} = \{0, 1\}$ is the initial MV-algebra,
- $\neg(w, i) := (w, 1 - i)$, $0 := (1, 0)$,
- the operation $\oplus$ is defined by letting

$$(w, 1) \oplus (w', 1) := (w \oplus_W w', 1)\,, \qquad (w, 0) \oplus (w', 0) := (w \odot w', 0)\,,$$
$$(w, 0) \oplus (w', 1) = (w', 1) \oplus (w, 0) := (w \to w', 1)\,.$$

then we obtain the split extension

$$\begin{tikzcd} W \arrow[r, "\eta_W"] & U\,MV(W) \arrow[r, shift left, "U(p_W)"] & U(\mathbf{2}) \arrow[l, shift left, "U(\sigma_W)"] \end{tikzcd}$$

with $\eta_W(w) := (w, 1)$, $p_W(w, i) := i$ and $\sigma_W(i) := (1, i)$. In particular, η_W is an augmentation ideal and the unit η is cartesian.

Adding 0 to other varieties of hoops (work in progress with F. Piazza)

- The case of product algebras vs product hoops has been analized in [GU23], where they show that the forgetful functor $U\colon \mathbf{PAlg} \to \mathbf{PHoop}$ has a left adjoint P defined as follows for a product hoop S:

  $$P(S) = B(S) \otimes_{\vee_S} C(S)$$

  where

  - $C(S) = \{x \to x^2 : x \in S\}$ (the cancellative elements)
  - $B(S) = MV(G(S))$, where MV is the MV-closure and $G(S) = \{(x \to x^2) \to x : x \in S\}$ (the boolean elements),
  - $\vee_S\colon B(S) \times C(S) \to C(S) := b \vee_S c = 1$ if $b \in G(S)$, $b \vee_S c = c$ otherwise,
  - the "tensor product" is defined as a suitable quotient of $B(S) \times C(S)$.

  The relevant fact to us is that S is a (maximal) U-ideal of $P(S)$, with $P(S)/S \cong \mathbf{2}$, and the canonical inclusion $S \to P(S)$ is the unit of the adjunction, that, therefore, satisfies Condition $(*)$.

- The other cases are under investigation...

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THANK YOU FOR YOUR ATTENTION!