Baer sums and the Butterfly Effect

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**From some joint works and projects with A. Cigoli, S. Mantovani, E. M. Vitale*

Introduction

Please do not read this slide!

Let C be a group, and $B = B_\xi$ a C-module, $\xi \colon C \times B \to B$.

$$\mathbf{H}^2(C, B_\xi) = \mathbf{Z}^2(C, B_\xi)/\mathbf{B}^2(C, B_\xi)$$

where

$$\mathbf{Z}^2(C, B_\xi) = \{\text{functions } t \colon C \times C \to B \text{ s.t. } \cdots$$

$$\cdots\ x \cdot t(y, z) + t(x, yz) = t(x, y) + t(xy, z) \text{ and } t(x, 1) = 0 = t(1, y)\}$$

$$\mathbf{B}^2(C, B_\xi) = \{t \in \mathbf{Z}^2(C, B_\xi) \text{ s.t. } t = \delta s\}$$

where $s \colon C \to B$ is a function with $s(1) = 0$ and $\delta s(x, y) = x \cdot s(y) - s(xy) + s(x)$.

Theorem (MacLane)

There is a bijection $\omega \colon \mathbf{OpExt}(C, B_\xi) \to \mathbf{H}^2(C, B_\xi)$

$$[\ B \xrightarrow{k} E \xrightarrow{f} C\ ] \mapsto [t(x, y)] \text{ with } t \text{ defined by } s(x) + s(y) = t(x, y) + s(xy)$$

where $s \colon C \to E$ *a normalized set-theoretical section of* f.

Baer sums *à la* Yoneda

MacLane shows that the abelian group structure on $\mathbf{OpExt}(C, B_\xi)$ can be computed with a Yoneda-like construction on (iso classes of) extensions with abelian kernels:

$$[E] \oplus [E'] = [\, \nabla_B \, (E \times E') \, \Delta_C \,]$$

This is *Exercise IV.4.7* in [ML63].

$$\begin{tikzcd}
B \arrow[r] & E \oplus E' \arrow[r] & C \\
B \times B \arrow[u, "\nabla"] \arrow[r] \arrow[d, equal] \arrow[ur, phantom, "\text{push forward}"] & (E \times E') \underset{C \times C}{\times} C \arrow[u] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & C \arrow[u, equal] \arrow[d, "\Delta"] \\
B \times B \arrow[r, "k \times k'"'] & E \times E' \arrow[r, "f \times f'"'] & C \times C
\end{tikzcd}$$

[ML63] S. MacLane. Homology (1963).

Baer sums via Butterflies

In [B10] Bourn shows that the abelian group structure on $\mathbf{OpExt}(C, B_\xi)$ can be described as the composition of a kind of internal profunctors.

In [CM16], we explicit this composition using (flippable) butterflies, a normalized version of such internal profunctors:

$$[E] \oplus [E'] := [\mathcal{B}_{E'} \circ \mathcal{B}_E]$$

$$\begin{tikzcd}
 & & E \oplus E' \arrow[ddddll, bend right=60, "\overline{f p_1}"'] \arrow[ddddrr, bend left=60, "\overline{f' p_2}"] & & \\
 & & E \times_C E' \arrow[u, "\mathrm{coker}\langle k,-k'\rangle"'] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "p_2"] & & \\
B \arrow[urr, bend left, "\langle -k,0\rangle"] \arrow[dr, "-k"] \arrow[dd, "0"'] & & B \arrow[u, "\langle k,-k'\rangle"'] \arrow[dl, "k"'] \arrow[dr, "-k'"] \arrow[dd, "0"] & & B \arrow[ull, bend right, "\langle 0,k'\rangle"'] \arrow[dl, "k'"'] \arrow[dd, "0"] \\
 & E \arrow[dl, "f"'] \arrow[dr, "f"] & & E' \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\
C & & C & & C
\end{tikzcd}$$

[B10] D. Bourn. Internal profunctors and commutator theory..., TAC 24 (2010).
[CM16] A.S. Cigoli, G.M. Extension theory and the calculus of butterflies, J. Alg. 458 (2016).

In this talk...

In this talk, I would like to explain why Baer sum manifests itself in two ways:

- as an operation between objects
- as a composition of arrows

and I will provide a framework where both these phenomena occur.

Moreover, in this talk I would like to celebrate Enrico Vitale's 60th birthday.

Indeed, what follows is largely inspired by ideas, discussions and chats with Enrico, since my visits in LLN during my PhD studies. This eventually evolved into new projects and collaborations with Sandra Mantovani and Alan Cigoli (in chronological order).

Disclaimer. This talk is not intended as a historically accurate reconstruction, but rather as a tale of my research in the subjects, with friends and colleagues, since my lucky encounter with Enrico. Therefore, the references provided here only refer to this journey... More accurate references to the literature can be found in the cited articles.

Butterflies

2-dimensional algebra

Focus: study of internal categories in categories (varieties of algebras).
E.g. congruences, and their normalized version, ideals.

This viewpoint allows to introduce categorical tools and notions in the study of algebraic structures.

Facts: 1. If **C** is Malt'sev, internal categories are groupoids.

2. If **C** is semi-abelian, groupoids can be normalized: crossed modules.

$$\mathbf{Gpd}(\mathbf{C}) \simeq \mathbf{XMod}(\mathbf{C})$$

$$\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[d, "m"] & & X_0 \flat X \arrow[d, "\alpha"] \\
X_1 \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] & \leftrightarrow & X \arrow[d, "\partial"] \\
X_0 \arrow[u, "e" description] & & X_0
\end{tikzcd}$$

$$\xrightarrow{\ \ker\ }$$

$$\xleftarrow{\ \rtimes\ }$$

Modules and crossed modules

Definition

Let $\mathbf{C}$ be (strongly) semi-abelian.

$\mathbf{Mod} = \{B_\xi \colon C\flat B \xrightarrow{\xi} B$ s.t. B abelian, ξ internal action$\}$

$$\mathbf{XMod} = \left\{ X \xrightarrow{\partial} X_0 \ \text{s.t.} \begin{tikzcd} X_0\flat X \arrow[r, "\alpha"] \arrow[d, "1\flat\partial"'] \arrow[dr, phantom, "\text{(PXM)}"] & X \arrow[d, "\partial"] \\ X_0\flat X_0 \arrow[r, "\text{conj.}"] & X_0 \end{tikzcd} \ \text{and} \ \begin{tikzcd} X\flat X \arrow[r, "\text{conj.}"] \arrow[d, "\partial\flat 1"'] \arrow[dr, phantom, "\text{(PFF)}"] & X \arrow[d, equal] \\ X_0\flat X \arrow[r, "\alpha"] & X \end{tikzcd} \right\}$$

$\mathbf{XEXT} = \{\, 0 \longrightarrow B \xrightarrow{j} E \xrightarrow{\partial} E_0 \xrightarrow{p} C \longrightarrow 0$ exact sequence s.t. ∂ is a X-mod$\}$

Fact: $\mathbf{XEXT} \to \mathbf{C} : (j, \partial, p) \mapsto C$ is a fibration.

Fix C and the fibre $\mathbf{XEXT}(C)$ over C.
Fact: $\mathbf{XEXT}(C) \to \mathbf{Mod}(C) : (j, \partial) \mapsto C\flat B \to B$ is an opfibration.

Weak maps internally

In general, morphisms of crossed module translate internal functors

$$
\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] & Y_1 \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] \\
X_0 \arrow[r, "f_0"] \arrow[u, "e"] & Y_0 \arrow[u, "e"]
\end{tikzcd}
\quad \leftrightarrow \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\partial"'] & Y \arrow[d, "\partial"] \\
X_0 \arrow[r, "f_0"] & Y_0
\end{tikzcd}
$$

When $\mathbf{C} = \mathbf{Gp}$, internal groupoids are strict monodial categories (in fact 2-groups), and internal functors are strict monoidal. However, there is a weaker notion of morphism between strict monodial categories: **(weak) monoidal funtors**.

$$
\begin{tikzcd}
X_1 \arrow[r, dashed, "f_1"] \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] & Y_1 \arrow[d, shift right=2, "c"'] \arrow[d, shift left=2, "d"] \\
X_0 \arrow[r, dashed, "f_0"] \arrow[u, "e"] & Y_0 \arrow[u, "e"]
\end{tikzcd}
\quad \leftrightarrow \quad ?
$$

Questions: 1. Can they be described internally?
2. Can they be translated in terms of crossed modules?

Weak maps internally

Both questions have a positive answer.

First in [AMMV13] we introduced weak maps between crossed modules, so-called **butterflies**, that generalize the group case developed by B. Noohi.

Then in [MMV13] we dealt with the grupoid case, where we singled out a class of internal profunctors: **fractors**, the internal version of Benabou's *locally representable distributors*.

[AMMV13] O. Abbad, S. Mantovani, G.M., E.M. Vitale. Butterflies in a semi-abelian context, Adv. Math., 238 (2013).

[MMV13] S. Mantovani, G.M., E.M. Vitale. Profunctors in Mal'cev categories and fractions of functors, J.P.A.A. 217 (2013).

Proposition

For a nice base category **C***, fractors (resp. butterflies) form the bicategory of fractions of internal groupoids (resp. crossed modules) w.r.t. weak equivalences.*

A key fact: the class of weak equivalences of internal groupoids in a regular protomodular category form a *bipullback congruence*, so that they admit a 2-dimensional calculus of fractions [V10].

[V10] E.M. Vitale. Bipullbacks and calculus of fractions, Cahiers de Top. et Géom. Diff. Cat. 51 (2010).

Butterflies

A weak map between internal groupoids

$$\mathbb{X} \xrightarrow{F} \mathbb{Y}$$

is described by a **butterfly** between the corresponding crossed modules, i.e. a diagram:

$$\begin{tikzcd}
X \arrow[dd, "\partial_X"'] \arrow[dr, "\kappa"] & & Y \arrow[dl, "\iota"'] \arrow[dd, "\partial_Y"] \\
& F \arrow[dl, "\sigma"'] \arrow[dr, "\rho"] & \\
X_0 & & Y_0
\end{tikzcd}$$

where

- the two triangles commute
- (ι, σ) is short exact
- (κ, ρ) is a complex, i.e. $\rho\kappa = 0$
- $$\begin{tikzcd}
F \flat X \arrow[r, "\sigma \flat 1"] \arrow[d, "1 \flat \kappa"'] & X_0 \flat X \arrow[r, "\alpha"] & X \arrow[d, "\kappa"] \\
F \flat F \arrow[rr, "\text{conj.}"] & & F
\end{tikzcd}$$
- $$\begin{tikzcd}
F \flat Y \arrow[r, "\rho \flat 1"] \arrow[d, "1 \flat \iota"'] & Y_0 \flat Y \arrow[r, "\alpha"] & Y \arrow[d, "\iota"] \\
F \flat F \arrow[rr, "\text{conj.}"] & & F
\end{tikzcd}$$

A 2-cell $\omega \colon (\kappa, \iota, F, \sigma, \rho) \Rightarrow (\kappa', \iota', F', \sigma', \rho')$ is a $\omega \colon F \to F'$ making all commute.

The bicategory **Bfly** of butterflies

Butterflies composition

$$\begin{tikzcd}
 & & G\circ F \arrow[ddddll, bend right, "\overline{\sigma p_1}"'] \arrow[ddddrr, bend left, "\overline{\rho' p_2}"] & & \\
 & & F\times_{Y_0} G \arrow[u, "\mathrm{coker}\langle\iota,\kappa'\rangle"'] \arrow[ddl, bend right, "p_1"] \arrow[ddr, bend left, "p_2"'] & & \\
X \arrow[urr, bend left, "\langle\kappa,0\rangle"] \arrow[dr, "\kappa"'] \arrow[dd, "\partial_X"'] & & Y \arrow[u, "\langle\iota,\kappa'\rangle"'] \arrow[dl, "\iota"] \arrow[dr, "\kappa'"'] \arrow[dd, "\partial_Y"] & & Z \arrow[ull, bend right, "\langle 0,\iota'\rangle"'] \arrow[dl, "\iota'"] \arrow[dd, "\partial_Z"] \\
 & F \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & & G \arrow[dl, "\sigma'"] \arrow[dr, "\rho'"'] & \\
X_0 & & Y_0 & & Z_0
\end{tikzcd}$$

Identity butterflies

$$\begin{tikzcd}
X \arrow[dr, "\ker(d)"] \arrow[dd, "\partial_X"'] & & X \arrow[dl, "\ker(c)"'] \arrow[dd, "\partial_X"] \\
 & X_1 \arrow[dl, "c"'] \arrow[dr, "d"] & \\
X_0 & & X_0
\end{tikzcd}$$

Flippable butterflies = equivalences

$$\begin{tikzcd}
X \arrow[dr, "\ker(\rho)"] \arrow[dd, "\partial_X"'] & & Y \arrow[dl, "\ker(\sigma)"'] \arrow[dd, "\partial_Y"] \\
 & E \arrow[dl, "\sigma"'] \arrow[dr, "\rho"] & \\
X_0 & & Y_0
\end{tikzcd}$$

A (bi)category of fractions

Theorem (AMMV13)

The 2-functor $\mathcal{B}$: **XMod** $\to$ **Bfly**

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\partial_X"'] & Y \arrow[d, "\partial_Y"] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} X \arrow[dr] \arrow[dd, "\partial_X"'] & & Y \arrow[dl] \arrow[dd, "\partial_Y"] \\ & X_0 \times_{Y_0} Y_1 \arrow[dl, shift left] \arrow[dr] & \\ X_0 \arrow[ur, shift left] & & Y_0 \end{tikzcd}$$

presents **Bfly** *as the bicategory of fractions of* **XMod** *w.r.t. weak equivalences, i.e. crossed modules morphisms inducing iso on kernels and cokernels.*

Butterflies with the s.e.s. being split are called **representable**.

Set $[\mathbf{Bfly}] := \mathcal{C}\ell(\mathbf{Bfly})$.

Proposition

The functor $\mathcal{B}$: **XMod** $\to$ [**Bfly**] *presents* [**Bfly**] *as the category of fractions of* **XMod** *w.r.t. weak equivalences.*

This is how fractions look like: the left leg

(p_1, σ)

is a weak equivalence.

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\kappa \# \iota"] & \\ X \arrow[dr, dotted, "\kappa"'] \arrow[dd, "\partial_X"'] & & Y \arrow[dl, dotted, "\iota"] \arrow[dd, "\partial_Y"] \\ & F \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\ X_0 & & Y_0 \end{tikzcd}$$

Butterflies and crossed extensions

Butterflies extend to crossed extensions...

Proposition

The functor $\mathcal{B} \colon \mathbf{XEXT} \to [\mathbf{BEXT}]$ *presents* $[\mathbf{BEXT}]$ *as the category of fractions of* $\mathbf{XEXT}$ *w.r.t. weak equivalences.*

...and a functor

$$\mathcal{P} \colon \mathbf{BEXT} \to \mathbf{Mod}$$

is defined as follows:

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[dd, "j"'] & B \arrow[r] \arrow[d] & B' \arrow[dd, "j'"] \\
& X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\kappa \# \iota"] & \\
X \arrow[dr, "\kappa"] \arrow[dd, "\partial_X"'] & & Y \arrow[dl, "\iota"'] \arrow[dd, "\partial_Y"] \\
& F \arrow[dl, "\sigma"'] \arrow[dr, "\rho"] \arrow[dd] & \\
X_0 \arrow[d, "p"'] & & Y_0 \arrow[d, "p'"] \\
C \arrow[r, equal] & C \arrow[r] & C'
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
C \flat B \arrow[r] \arrow[d, "\xi_X"'] & C' \flat B' \arrow[d, "\xi_Y"] \\
B \arrow[r] & B'
\end{tikzcd}$$

Notice: in order to make the diagrams fit the slides, we won't always draw the kernel and cokernels of crossed extensions.

Cohomology 2-groups

Baer sums don't exist (G. Janelidze)

Baer sums arise naturally as a consequence of two elementary observations:

(1) $\mathbf{C}$ with fin. prod. and $F\colon \mathbf{C} \to \mathbf{Set}$ preserves them $\Rightarrow$ F sends grps to grps.

(2) $\mathbf{C}$ additive $\Rightarrow$ $\mathbf{C} = \mathbf{Ab}(\mathbf{C})$

Then, (1) + (2) $\Rightarrow$ $F\colon \mathbf{C} \to \mathbf{Ab} \to \mathbf{Set}$.

Example

For C in $\mathbf{C}$, $\mathbf{Mod}(C) = \{B_\xi\colon\ C\flat B \xrightarrow{\xi} B\ \}$ category of C-modules

- $\mathbf{H}^2_{op}(C,-) \simeq \mathbf{OpExt}(C,-)\colon \mathbf{Mod}(C) \to \mathbf{Set}$

  where $\mathbf{OpExt}(C, B_\xi) = \{[\ 0 \longrightarrow B \xrightarrow{k} E \xrightarrow{f} C \longrightarrow 0\ ]\}$

- $\mathbf{H}^3_{op}(C,-) \simeq \mathbf{XExt}(C,-)\colon \mathbf{Mod}(C) \to \mathbf{Set}$

  where $\mathbf{XExt}(C, B_\xi) = \{[\ 0 \longrightarrow B \xrightarrow{k} E \xrightarrow{\partial} E_0 \xrightarrow{f} C \longrightarrow 0\ ]\}$

Question: what if we replace sets **OpExt** and **XExt** with cats **OPEXT** and **XEXT**?

Monoidal Baer sums don't exist either!

In [CMM23a] and [CMM23b] we analyzed the "don't exist" argument in dimension 2.

We introduced the symmetric 2-groups:

$$\mathbb{H}^2(C, B_\xi) := (\mathbf{OPEXT}(C, B_\xi), \oplus, O)$$

$$\mathbb{H}^3(C, B_\xi) := (\mathbf{XEXT}(C, B_\xi), \oplus, O)$$

[CMM23a] A.S. Cigoli, S. Mantovani, G.M. On pseudofunctors sending groups to 2-groups. Mediterr. J. Math. 20 (2023).

[CMM23b] A.S. Cigoli, S. Mantovani, G.M. The third cohomology 2-group, Milan J. Math. (2023).

Consider an opfibration $P\colon \mathbf{X} \to \mathbf{B}$ such that $\mathbf{X}$ and $\mathbf{B}$ have finite products.

Definition

P is **cartesian monoidal** if it preserves finite products and cocartesian maps are product stable.

Proposition

If P has groupoidal fibres, then P is cartesian monoidal iff it preserves finite products.

(In fact, every map in $\mathbf{X}$ is cocartesian.)

Lifting the monoidal structure

Theorem

If $\mathbf{B}$ *is additive (and therefore* $\mathbf{B} \cong \mathbf{CMon}(\mathbf{B})$*),*

- *the fibres of a cartesian monoidal opfibration* $P : \mathbf{X} \to \mathbf{B}$ *are symmetric monoidal*[a]
- *they are symmetric 2-groups iff they are groupoids*

[a] I first learned how the monoidal structure lifts from D. Bourn, attending his PhD lectures in Milan, 2007.

Definition

A **symmetric 2-group** $\mathbb{G}$ is a symmetric monoidal groupoid such that every object is weakly invertible w.r.t. the tensor product.

With every (symmetric) 2-group is 2-functorially associated a group module

$$\pi_0(\mathbb{G}) \times \pi_1(\mathbb{G}) \to \pi_1(\mathbb{G})$$

where $\pi_0(\mathbb{G})$ is the (abelian) group of the connected components of $\mathbb{G}$ and $\pi_1(\mathbb{G})$ the abelian group of the automorphisms of the monoidal unit.

The second cohomology 2-group

Fix C in $\mathbf{C}$.

$$P\colon \mathbf{OPEXT}(C) \to \mathbf{Mod}(C)$$

is a cartesian monoidal opfibration with groupoidal fibres.

The fibre $\mathbf{OPEXT}(C, B_\xi)$ is a symmetric 2-group $\mathbb{H}^2(C, B_\xi)$ with

$$\begin{tikzcd}
B \arrow[r] & E \oplus E' \arrow[r] & C \\
B \times B \arrow[u, "\nabla = +"] \arrow[r] \arrow[ur, phantom, "\text{p.f. } = \text{cocart. lift}"] \arrow[d, equal] & E \underset{C}{\times} E' \arrow[u] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b. } = \text{product}/C"] & C \arrow[u, equal] \arrow[d, "\Delta"] \\
B \times B \arrow[r, "k \times k'"'] & E \times E' \arrow[r, "f \times f'"'] & C \times C
\end{tikzcd}$$

and

$$O_\xi := \; B \longrightarrow B \rtimes_\xi C \longrightarrow C$$

Proposition

(i) $\pi_0(\mathbb{H}^2(C, B_\xi)) \cong \mathbf{H}^2(C, B_\xi)$

(ii) $\pi_1(\mathbb{H}^2(C, B_\xi)) \cong \mathbf{Z}^1(C, B_\xi)$

proof:

(i) clear from the construction of Baer sums, when you mod out iso classes

(ii) automorphisms of split extensions that fix ker and coker correspond to 1-cocycles

The third cohomology 2-group

Fix C in $\mathbf{C}$.

$$P\colon \mathbf{XEXT}(C) \to \mathbf{Mod}(C)$$

is a cartesian monoidal opfibration, but fibres are not groupoids. Take fractions:

$$\overline{P}\colon [\mathbf{BEXT}(C)] \to \mathbf{Mod}(C)$$

The fibre $[\mathbf{BEXT}(C)](C, B_\xi)$ is a symmetric 2-group $\mathbb{H}^3(C, B_\xi)$ with

$$\begin{tikzcd}
B \arrow[r] & E \oplus E' \arrow[r, "\partial\oplus\partial'"] & E_0 \times_C E'_0 \arrow[r] & C \\
B \times B \arrow[r] \arrow[u, "\nabla = +"] \arrow[ur, phantom, "\text{p.f. } = \text{cocart. lift}"] & E \times E' \arrow[r] \arrow[u] & E_0 \times_C E'_0 \arrow[r] \arrow[u, equal] \arrow[d] \arrow[dr, phantom, "\text{p.b. } = \text{product}/C"] & C \arrow[u, equal] \arrow[d, "\Delta"] \\
B \times B \arrow[r, "j\times j'"'] \arrow[u, equal] & E \times E' \arrow[r, "\partial\times\partial'"'] \arrow[u, equal] & E_0 \times E'_0 \arrow[r, "p\times p'"'] & C \times C
\end{tikzcd}$$

and

$$O'_\xi := \; B = B \xrightarrow{0} C = C$$

Proposition

(i) $\pi_0(\mathbb{H}^3(C, B_\xi)) \cong \mathbf{H}^3(C, B_\xi)$

(ii) $\pi_1(\mathbb{H}^3(C, B_\xi)) \cong \mathbf{H}^2(C, B_\xi)$

proof:

(i) clear from the construction of Baer sum, when you mod out connected components $=$ iso classes of fractions

(ii) what do automorphisms of O'_ξ look like?

Please wait for the next slide...

Automorphisms of O'_ξ

Here they are:

$$\begin{tikzcd} B \arrow[dd, "0"'] \arrow[dr, "-k"] & & B \arrow[dd, "0"] \arrow[dl, "k"'] \\ & E \arrow[dl, "f"] \arrow[dr, "f"'] & \\ C & & C \end{tikzcd}$$

...and if you compute their composition as (iso classes of) butterflies, it comes out that their composite precisely corresponds to Baer sums.

Therefore we obtain

Corollary

$$\pi_0(\mathbb{H}^2(C, B_\xi)) \cong \mathbf{H}^2(C, B_\xi) \cong \pi_1(\mathbb{H}^3(C, B_\xi))$$

Is this the end?

The isomorphism $\pi_0(\mathbb{H}^2(C, B_\xi)) \cong \pi_1(\mathbb{H}^3(C, B_\xi))$ has a 2-dimensional explanation.

Proposition (CMM23b)

The assignment

$$\Phi: \quad B \xrightarrow{k} E \xrightarrow{f} C \qquad \mapsto \qquad \begin{tikzcd} B \arrow[dd, "0"'] \arrow[dr, "-k"] & & B \arrow[dd, "0"] \arrow[dl, "k"'] \\ & E \arrow[dl, "f"] \arrow[dr, "f"'] & \\ C & & C \end{tikzcd}$$

establishes a monoidal equivalence $\mathbb{H}^2(C, B_\xi) \simeq \mathbf{Eq}(O'_\xi) = \mathbf{XEXT}_{id}(O'_\xi, O'_\xi)$

The two natures of 2-dimensional Baer Sums demystified

The proof in [CMM23b] is rather technical, however one can consider more formal argument. In fact Φ extends to an isomorphism of pseudofunctors

$$\begin{tikzcd} \mathbf{Mod}(C) \arrow[r, bend left, "\mathbf{OPEXT}(C,-)"] \arrow[r, bend right, "\mathbf{Eq}(O'_{\_})"'] \arrow[r, phantom, "\Downarrow \Phi"] & \mathbf{Gpd} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{OPEXT}(C,\xi) \arrow[r, "\mathbf{OPEXT}(C,h)"] \arrow[d, "\Phi_\xi"'] & \mathbf{OPEXT}(C,\psi) \arrow[d, "\Phi_\psi"] \\ \mathbf{Eq}(O'_\xi) \arrow[r, "\mathbf{Eq}(O'_h)"'] & \mathbf{Eq}(O'_\psi) \end{tikzcd}$$

that turns into an **isomorphism of cartesian monoidal opfibrations**
$\Rightarrow$ the monoidal structures lifted from $\mathbf{Mod}(C)$ induces $\mathbb{H}^2(C, B_\xi) \simeq \mathbf{Eq}(O'_\xi)$

- *Such equivalences urge us to find a context where the 2-group $\mathbf{Eq}(O'_\xi)$ is a cohomological gadget*
- *But this context cannot be $\mathbb{H}^3(C, B_\xi)$, since its 1-cells are (already) isomorphism classes!*

Cohomology 3-groups

(work in progress)

The cohomology 3-group $\mathbb{H}(C, B_\xi)$

Fix a C-module B_ξ in $\mathbf{C}$ and consider a structure $\mathbb{H}(C, B_\xi)$ made of the following data:

- **objects:** ∂_X, ∂_Y are crossed extensions over B_ξ
- 1-**cells:** $E, E' : \partial_X \to \partial_Y$ are (necessarily flippable) butterflies over B_ξ
- 2-**cells:** $\alpha : E \to E'$ are (necessarily iso) morphisms of such butterflies

$$\begin{tikzcd}
X \arrow[ddd, "\partial_X"'] \arrow[dr, dashed] \arrow[ddr] & & Y \arrow[ddd, "\partial_Y"] \arrow[dl, dashed] \arrow[ddl] \\
& E \arrow[d, "\alpha"] \arrow[ddl, dashed] \arrow[ddr, dashed] & \\
& E' \arrow[dl] \arrow[dr] & \\
X_0 & & Y_0
\end{tikzcd}$$

Facts:

- $\mathbb{H}(C, B_\xi)$ is a sub-bicategory of $\mathbf{Bfly}(\mathbf{C})$
- $\mathbb{H}(C, B_\xi)$ is a bigroupoid: 1-cells are equivalences, and 2-cells are isomorphism
- $\mathbb{H}(C, B_\xi)$ is a monoidal bigroupoid, and all objects are weakly invertible w.r.t. $\oplus$.

2-monoidal structure of $\mathbb{H}(C, B_\xi)$

Rough idea: how to extend $\oplus$ to butterflies:

$$\begin{tikzcd}
 & \mathbb{X} \oplus \mathbb{Y} \arrow[rr, dashed, "E \oplus F"] & & \mathbb{X}' \oplus \mathbb{Y}' \\
\mathbb{X} \times \mathbb{Y} \arrow[ur, "\widehat{+}"] \arrow[rr, "E \times F"'] & & \mathbb{X}' \times \mathbb{Y}' \arrow[ur, "\widehat{+}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\mathbf{BEXT}(C)]} \arrow[d, "\overline{P}"] \\
\mathbf{Mod}(C)
\end{tikzcd}$$

$$\begin{tikzcd}
 & B \arrow[rr, equal] & & B \\
B \times B \arrow[ur, "+"] \arrow[rr, equal] & & B \times B \arrow[ur, "+"'] &
\end{tikzcd}$$

uses that $\overline{P}$ is an opfibration on isomorpism classes of butterflies.

A speculative answer

Conjecture

$\mathbb{H}(C, B_\xi)$ is a symmetric 3-group, i.e. a monoidal bigroupoid where all object are invertible w.r.t. the tensor product up to coherent equivalences.

Proposition

There are canonical natural monoidal equivalences

$$\pi_0(\mathbb{H}(C, B_\xi)) \simeq \mathbb{H}^3(C, B_\xi) \qquad \pi_1(\mathbb{H}(C, B_\xi)) \simeq \mathbb{H}^2(C, B_\xi)$$

that determine canonical natural homomorphisms of abelian groups

$$\pi_0\pi_0(\mathbb{H}(C, B_\xi)) \cong \mathbf{H}^3(C, B_\xi) \qquad \pi_1\pi_1(\mathbb{H}(C, B_\xi)) \cong \mathbf{Z}^1(C, B_\xi)$$

$$\pi_0\pi_1(\mathbb{H}(C, B_\xi)) \cong \mathbf{H}^2(C, B_\xi) \cong \pi_1\pi_0(\mathbb{H}(C, B_\xi))$$

proof: All statements relies on previous results. In particular

$$\pi_1(\mathbb{H}(C, B_\xi)) := \mathbf{Eq}(O'_\xi) \simeq \mathbb{H}^2(C, B_\xi)$$

A jump in the past...

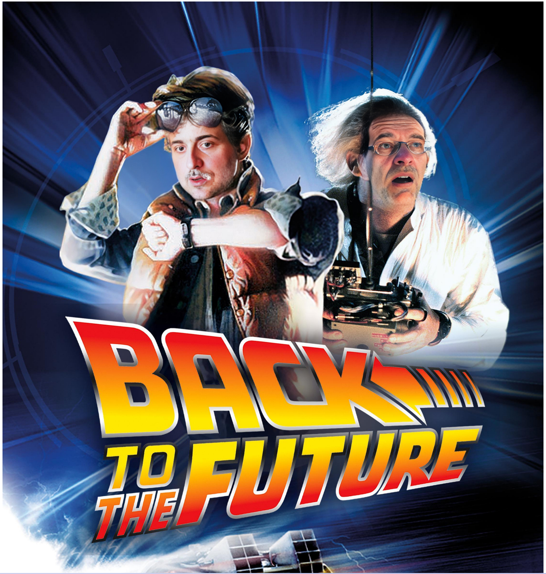
Although a general theory of 3-groups and their cohomological invariants is still missing (afaik), a semi-strict version of previous proposition comes from a semi-strict version of the general theory developed in my PhD Thesis (2008), under the guidance of Enrico Vitale.

We studied **exact sequences of pointed n-groupoid** (strict n-categories with weak inverses), and starting with one such, we obtained a ziqqurat of exact sequences...

Lemma 9.1 *(of* [KMV11]*). The following diagram commutes*

$$\begin{tikzcd}
n\text{-}\mathbf{Gpd}_\star \arrow[r, "\pi_0^{(n)}"] \arrow[d, "\pi_1^{(n)}"] & (n-1)\text{-}\mathbf{Gpd}_\star \arrow[d, "\pi_1^{(n-1)}"] \\
(n-1)\text{-}\mathbf{Gpd}_\star \arrow[r, "\pi_0^{(n-1)}"] & (n-2)\text{-}\mathbf{Gpd}_\star
\end{tikzcd}$$

The semi-strict version of the theory emerges from the consideration that, as 2-groups "are" with bigroupoids with one object, 3-groups "are" trigroupoids with one object.


BACK TO THE FUTURE

References

AMMV13 O. Abbad, S. Mantovani, G.M., E.M. Vitale. Butterflies in a semi-abelian context, Adv. Math., 238 (2013).

Bo2010 D. Bourn, Internal profunctors and commutator theory; applications to extensions classification and categorical Galois Theory, T.A.C. 24 (2010).

CMM23a A.S. Cigoli, S. Mantovani, G.M. On pseudofunctors sending groups to 2-groups. Mediterr. J. Math. 20 (2023).

CMM23b A.S. Cigoli, S. Mantovani, G.M. The third cohomology 2-group, Milan J. Math. (2023).

CM2016 A.S. Cigoli, G. Metere, Extension theory and the calculus of butterflies, J. Alg. 458 (2016).

KMV11 S. Kasangian, G.M., E.M. Vitale. The ziqqurath of exact sequences of n-groupoids, C. Top. et Géom. Diff. Cat. 52 (2011).

MMV13 S. Mantovani, G.M., E.M. Vitale. Profunctors in Mal'cev categories and fractions of functors, J.P.A.A. 217 (2013).

V10 E.M. Vitale. Bipullbacks and calculus of fractions, C. Top. et Géom. Diff. Cat. 51 (2010).

THANK YOU FOR YOUR ATTENTION!