Categorical Algebra, Gargnano September 27 - October 1, 2011

**Homological algebra, a projective approach**

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**1**. Homological algebra can be described as the study of exact sequences and of their preservation properties by functors. Its classical domain, after the initial categories of modules, is abelian categories; the extension achieves formal advantages (like duality) and a concrete enlargement of the domain of the theory (to sheaves, Serre's quotients, etc.).

Many homological procedures can be further extended to a *p-exact* category, i.e. an exact category in the sense of Puppe and Mitchell (see the texts by Mitchell, Adámek - Herrlich - Strecker and Freyd - Scedrov). This is a pointed category where every map factorises normal epi - normal mono. The new setting is also selfdual; a p-exact category is abelian iff it has finite products, iff it has finite sums.

This extension includes new categories, like the categories of projective spaces on a field, of cyclic groups, of vector fields of dimension lower than a fixed integer, of sets and partial bijections, etc. But our main motivation is of a structural kind: to develop a 'projective theory' of homological algebra, that includes the following points, for every p-exact category $\mathbf{E}$ (and is generally inconsistent with the assumption of the existence of products).

(a) A *transfer functor* of subobjects, $\mathrm{Sub}_{\mathbf{E}}\colon \mathbf{E} \to \mathbf{Mlc}$. Even if we deal with an abelian category, this functor takes values in the p-exact category of *modular lattices and modular connections*, that lacks products. This functor is exact, i.e. preserves kernels and cokernels (and also reflects them). The universal solution of 'making it faithful' yields the *projective* category $\mathrm{Pr}\mathbf{E}$ associated to $\mathbf{E}$, which is p-exact and not abelian, in general.

(b) *Duality between subobjects and quotients*. The kernel-cokernel duality of abelian categories is preserved in the extension.

(c) A *coherence theorem for homological algebra*. It proves that canonical isomorphisms between subquotients of $\mathbf{E}$ are closed under composition (and that all their diagrams commute) if and only if $\mathbf{E}$ is *distributive*, i.e. has distributive lattices of subobjects. This fact cannot be effectively expressed in the abelian framework, where distributivity is excluded, but *can be applied to the distributive expansion* $\mathrm{Dst}\mathbf{E}$ *of any p-exact* (possibly abelian) *category*; this expansion - again - is p-exact and not abelian. Distributivity has a counterpart in the property of *orthodoxy* of the semigroups of endorelations.

(d) *Universal models of spectral sequences*. Such models, a theoretical formulation of Zeeman diagrams, yield a graphic, evident way of investigating spectral sequences. In fact, we give universal models for various homological systems, including the usual sources of spectral sequences: filtered complexes, double complexes, Eilenberg's exact systems and Massey's exact couples.

We get thus a form of 'weakly non-abelian' homological algebra, which does *not yet* include all the situations of Algebraic Topology in which exact sequences are considered; the main exceptions are, likely, the homotopy sequences of a pair of pointed spaces or a (co)fibration, that are not even confined to the category of groups but degenerate in low degree into pointed sets and actions of groups. Therefore, their 'exactness' is usually described and studied step by step, even in complicated situations as the homotopy spectral sequence of a tower of fibrations (see the texts by Bousfield - Kan and Baues).

**2**. To take this into account, we introduced the setting of 'homological categories' (Como 1990).

A 'semiexact category', our basic notion for this generalisation, is a category equipped with a suitable ideal of 'null' morphisms, and provided with kernels and cokernels with respect to this ideal; this simple structure allows one to introduce exact sequences, exact functors, connected sequences of functors, homology theories and satellites. The kernel-cokernel duality now links *normal subobjects* and *normal quotients*. The transfer functor (of normal subobjects) takes values in the pointed semiexact category **Ltc** of lattices and adjunctions, an extension of **Mlc**. A stronger notion of 'homological category' allows one to deal with the homology of complexes and spectral sequences.

**Ltc** itself is pointed homological (i.e. homological with respect to zero maps), and not p-exact.

As a typical non-pointed example, let us mention the category $\mathbf{Top}_2$ of 'pairs' of topological spaces (X, A), in the usual sense of algebraic topology (X is a space and A is a subspace of X). This category is homological with regard to a natural ideal of null morphisms: the mappings of pairs $f: (X, A) \to (Y, B)$ such that $f(X) \subset B$. A short exact sequence in $\mathbf{Top}_2$ is always of the following type, up to isomorphism:

$$(A, B) \rightarrowtail (X, B) \twoheadrightarrow (X, A),$$

for a triple $X \supset A \supset B$ of spaces. If B is empty, and we identify as usual the pair $(X, \emptyset)$ with X, the short exact sequence above reduces to $A \rightarrowtail X \twoheadrightarrow (X, A)$, and accounts for the current way of reading the pair (X, A) as 'X modulo A': *it is indeed the quotient* X/A, *in our homological category.*

Analogous facts hold for the categories $\mathbf{Set}_2$ of 'pairs' of sets, or $\mathbf{Gp}_2$ of 'pairs' of groups.

Thus, the first four axioms of Eilenberg - Steenrod for a (relative) homology theory $H = ((H_n), (\partial_n))$, defined over all topological pairs, amount to saying that H is an exact connected sequence of functors from the homological category $\mathbf{Top}_2$ to some category of modules. Relative (co)homology of groups can be dealt with in a similar way, using the homological category $\mathbf{Gp}_2$.

Another example relevant for our applications is the category **Act** of actions of groups on pointed sets and consistent pairs of mappings: this category, homological with respect to a natural ideal, is the framework where we study the exact sequences and spectral sequences of unstable homotopy mentioned above. A quotient and category of fractions of **Act** (introduced at Cape Town, TC2009) is even more adequate for the study of the spectral sequence of a Massey's exact pair.

It should also be noted that there are important additive or semiadditive *homological* categories, that are not abelian. For instance, the category **Ltc** is semiadditive homological, with an idempotent sum of maps, while the category of Banach spaces and continuous linear maps is additive homological. Chain complexes over a semiadditive homological category and their homotopies are thus of interest.

Finally, let us remark that the present 'projective' hierarchy of

*semiexact*, *homological* and *p-exact* categories

is quite distinct from the settings that have been developed in the 'affine approach', based on finite limits: *Barr-exact*, *protomodular* (by Bourn), *Borceux-Bourn homological* and *semiabelian* category (by Janelidze, Márki and Tholen).