A Complex Adaptive Harmonic IIR Notch Filter

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***Abstract*—A complex adaptive Harmonic IIR notch filter is proposed for estimating and tracking the frequency of periodic complex signals in a noisy harmonic environment. The transfer function of the developed notch filter consists of cascaded first-order complex transfer functions whose notch frequencies are constrained to the fundamental and harmonic frequencies. The least mean squares (LMS) algorithm is developed and a formula to determine the stability bound for the algorithm is derived. In addition, an improved simple scheme is devised to prevent the adaptive algorithm from converging to its local minima of the mean square error (MSE) function when the tracked signal fundamental frequency changes. Computer simulations validate the performance of the developed algorithm.**

***Keywords*—Adaptive filters, frequency tracking, and harmonic IIR notch filters**

I. INTRODUCTION

Estimation of unknown frequencies of periodic components in presence of noise using adaptive IIR notch filters has been attracted much research attention [1]–[12] and found wide applications for suppressing periodical noise in many signal processing systems. In the past years, many notch filter design methods such as the pole-zero constrained and all-pass decomposition approaches have been developed.

Recently, an adaptive harmonic IIR notch filter [13]–[15] has been developed for estimating and tracking of an unknown signal frequency when its harmonic frequency components exist, which may be produced due to nonlinear effects. The proposed harmonic IIR notch filter consists of the cascaded second-order IIR notch filters each with a single adaptive parameter. It can effectively be employed to estimate the signal fundamental frequency and harmonic frequencies using least mean squares (LMS) algorithm. As shown in [13], the MSE function has local minima. Thus the LMS algorithm may converge to a local minimum and reach to a wrong frequency value. To prevent the local minima convergence, a simple scheme [13] is devised to monitor the global minimum of the mean square error (MSE) function. Whenever a possible local minimum is detected [13], [15], the algorithm will reset the adaptive parameter using its new optimal value which is coarsely estimated via a block of data samples.

Although the most notch filter designs are based on real coefficients of the filter, complex notch filters [16]–[17] find many applications, especially in communication systems using quadrature modulation. Recently, a complex adaptive notch filter using the first-order complex all-pass transfer function is developed in [17]. The proposed design has a similar performance and features in comparison with an earlier real-coefficient prototype and can efficiently be used in frequency tracking in the complex domain. For the similar consideration, a complex adaptive harmonic IIR notch filter can also be developed.

In Section II, the complex adaptive harmonic IIR notch filter is developed and its associate LMS algorithm is derived. Section III depicts a simple scheme to monitor the global minimum of the MSE function. Next, a formula to determine the stability bound is derived. Section IV shows the computer simulations to validate the developed algorithm. Section V finally presents the conclusions.

II. COMPLEX ADAPTIVE HARMONIC IIR NOTCH FILTER AND PROPOSED ALGORITHM

A. Complex Adaptive Harmonic IIR Notch Filter

The proposed complex adaptive harmonic IIR notch filter has the following form

$$H(z) = H_1(z)H_2(z)\cdots H_M(z) = \prod_{m=1}^{M} H_m(z) \tag{1}$$

where the m-th first-order complex IIR sub-filter, $H_m(z)$, is defined as [16]–[17]:

$$H_m(z) = \frac{1-e^{jm\theta}z^{-1}}{1-re^{jm\theta}z^{-1}} \tag{2}$$

Let $y_m(n)$ be the m-th sub-filter output. It can be expressed as

$$y_m(n) = y_{m-1}(n) - e^{jm\theta}y_{m-1}(n-1) + re^{jm\theta}y_m(n-1) \tag{3}$$

$$m = 1, 2, \cdots, M$$

Note that $y_0(n) = x(n)$. From (1) and (2), it can be seen that the proposed notch filter has a single parameter θ; there are M zeros residing on the unit circle, which offer infinite-depth notches; and parameter r can be chosen to control the notch bandwidth (requiring $0 << r < 1$ for narrowband notches). If r is selected to be close to 1, the 3-dB notch bandwidth can be determined by $BW \approx 2(1-r)$ [18]. The MSE function is given by

$$E[|y_M(n)|^2] = E[|e(n)|^2] \tag{4}$$

Note that at the final stage, $e(n) = y_M(n)$. After adaptive parameter θ converges to the signal fundamental frequency, each $m\theta$ ($m = 2,3,\cdots,M$) will correspondingly be adapted to the harmonic frequencies. The mean square error (MSE) of the filter output in (4) is then minimized.

In order to drive the coefficient update equation, taking derivative of the instantaneous MSE error in (4) leads to the following:

$$\frac{\partial |y_M(n)|^2}{\partial \theta(n)} = \frac{\partial y_M^*(n) y_M(n)}{\partial \theta(n)} = \frac{\partial y_M^*(n)}{\partial \theta(n)} y_M(n) + \frac{\partial y_M(n)}{\partial \theta(n)} y_M^*(n) \tag{5}$$

Define a gradient function as

$$\beta_m(n) = \frac{\partial y_m(n)}{\partial \theta(n)} \tag{6}$$

Taking derivative of (3) achieves a recursion for the gradient function, that is,

$$\begin{aligned} \beta_m(n) &= \beta_{m-1}(n) - e^{jm\theta}\beta_{m-1}(n-1) - jme^{jm\theta} y_{m-1}(n-1) \\ &\quad + re^{jm\theta}\beta_m(n-1) + jrme^{jm\theta} y_m(n-1) \\ &\qquad m = 1,2,\cdots,M \end{aligned} \tag{7}$$

where

$$\beta_0(n) = \frac{\partial y_0(n)}{\partial \theta(n)} = \frac{\partial x(n)}{\partial \theta(n)} = 0, \ \beta_0(n-1) = 0$$

Substituting (7) with $m = M$ in (5) results in

$$\begin{aligned} \frac{\partial |y_M(n)|^2}{\partial \theta(n)} &= \beta_M^*(n) y_M(n) + \beta_M(n) y_M^*(n) \\ &= 2re\left\{ y_M(n)\beta_M^*(n) \right\} \end{aligned} \tag{8}$$

Similar to [13], the LMS algorithm using (6) is derived as

$$\theta(n+1) = \theta(n) - 2\mu Re\left\{ y_M(n)\beta_M^*(n) \right\} \tag{9}$$

where μ is a step size for controlling speed of adaptation. To avoid local convergence [13], the algorithm begins with an optimal initial value θ_0, which can coarsely be searched via the following frequency range:

$\theta = -180\pi/(180M),\cdots,-\pi/(180M),0,\pi/(180M),\cdots,179\pi/(180M)$,

That is,

$$\theta_0 = \underset{-\pi/M \le \theta < \pi/M}{\arg} (\min E[|e(n,\theta)|^2]) \tag{10}$$

Note that $E[|e(n,\theta)|^2])$ is the estimated MSE function. It can be computed by employing a block of N data samples:

$$E[|e(n,\theta)|^2]) \approx \frac{1}{N}\sum_{i=0}^{N-1} |y_M(n-i,\theta)|^2 \tag{11}$$

B. Global Minimum Monitoring

When a signal fundamental frequency changes, the algorithm may suffer from the local minimum convergence with a wrong frequency value or a slow convergence rate [15].

First, assuming the signal fundamental frequency changes with a small step, the global minimum can be monitored by examining the deviation of the fundamental frequency:

$$\Delta f = | f(n) - f_0 | \tag{12}$$

by comparing with a maximum allowable frequency deviation (25% of 3-dB notch bandwidth) given below:

$$\Delta f_{\max} = 0.5 \times (0.5BW) \tag{13}$$

where $f_0 = 0.5 f_s \theta_0 / \pi$ in Hz is the fundamental frequency, which is initially estimated using (10) and (11) while f_s denotes the sampling rate in Hz. The 3-dB bandwidth of the harmonic notch filter (BW) in Hz can be determined by $BW = (1-r) f_s / \pi$. We assume that if $\Delta f > \Delta f_{\max}$, the LMS algorithm may converge to the local minima.

Secondly, when a fundamental frequency switches at a large step, especially when a signal fundamental frequency changes from a negative value to a positive value, and vice versa, the MSE function is significantly changed [15]. At the estimated location of $\theta(n)$, the new MSE function may exhibit a flat surface, causing the algorithm stop adapting. This condition can be detected by checking if the gradient change within the notch filter bandwidth is less than the predefined threshold (10% of 3-dB notch bandwidth), that is

$$\Delta\beta_{1M}(n) = \left| \beta_M(n) - \beta_M(n)\big|_{\theta=\theta^* - BW/2} \right| < 0.1BW \tag{14}$$

$$\Delta\beta_{2M}(n) = \left| \beta_M(n)\big|_{\theta=\theta^* + BW/2} - \beta_M(n) \right| < 0.1BW \tag{15}$$

Therefore, either the condition for (12)-(13) or the condition for (14)-(15) is detected, the algorithm will reset adaptive parameter $\theta(n)$ to an estimated optimal value θ_0 (the neighborhood of the global minimum) utilizing (10) and (11). After resetting, the algorithm will resume the tracking process. We summarize the algorithm in Table I.

TABLE I COMPLEX ADAPTIVE HARMONIC NOTCH FILTER ALGORITHM

1: Estimate θ_0 by adopting (8) and (9): Search for $\theta_0 = \underset{0<\theta<\pi/M}{\arg} (\min E[\|e(n,\theta)\|^2])$ for the range: $\theta = -180\pi/(180M),\cdots,0\pi/(180M),\pi/(180M),\cdots,179\pi/(180M)$ Set $\theta(0) = \theta_0$ and $f_0 = 0.5 f_s \theta_0 / \pi$ Hz.
2: Perform the LMS algorithm using (3), (7), and (9) to obtain $\theta(n)$.
3: Convert $\theta(n)$ in radians to the desired estimated fundamental frequency in Hz: $f(n) = 0.5 f_s \theta(n) / \pi$.
4: Monitor the global minimum: If $\| f(n) - f_0 \| > \Delta f_{\max}$, go to Step 1 Or $\Delta\beta_{1M}(n) < 0.1BW$ and $\Delta\beta_{2M}(n) < 0.1BW$ go to Step 1 Otherwise go to Step 2.

C. Stability Bound

In this section, the upper bound for stability is derived. In order to achieve the simplified results, the second and higher order terms are ignored when performing the Taylor series expansion of the filter transfer function defined in (1), that is,

$$H(e^{j\Omega}, m\theta) \approx H_{\Omega}(m\theta)(\Omega - m\theta) \tag{16}$$

Note that the frequency response for the m-th filter section is given by

$$H_m(e^{j\Omega}) = \frac{(e^{j\Omega} - e^{jm\theta})}{(e^{j\Omega} - re^{jm\theta})} \tag{17}$$

Then taking derivative of (17) over Ω yields the following

$$\left.\frac{dH_m(e^{j\Omega})}{d\Omega}\right|_{\Omega=m\theta} = \left.\frac{je^{j\Omega}e^{jm\theta}(1-r)}{(e^{j\Omega} - re^{jm\theta})^2}\right|_{\Omega=m\theta} = \frac{j}{1-r} \tag{18}$$

Since

$$H_{\Omega}(\Omega) = \frac{dH(e^{j\Omega})}{d\Omega} = \sum_{i=1}^{M} \prod_{k=1,k\neq i}^{M} H_k(e^{j\Omega}) \frac{dH_i(e^{j\Omega})}{d\Omega} \tag{19}$$

Substituting (17) to (18) leads to

$$\begin{aligned} H_{\Omega}(m\theta) &= \sum_{i=1}^{M} \prod_{k=1,k\neq i}^{M} H_k(e^{jm\theta}) \frac{dH_i(e^{jm\theta})}{d\Omega} \\ &= \prod_{k=1,k\neq m}^{M} H_k(e^{jm\theta}) \frac{dH_m(e^{jm\theta})}{d\Omega} \end{aligned} \tag{20}$$

Thus the frequency responses at the harmonic frequencies are given by

$$\begin{aligned} H_{\Omega}(m\theta) &= \frac{j}{(1-r)} \prod_{k=1,k\neq m}^{M} H_k(e^{jm\theta}) \\ &= B(m\theta)\angle\phi_m \end{aligned} \tag{21}$$

The magnitude and phase of $H_{\Omega}(m\theta)$ in (21) are defined below:

$$B(m\theta) = |H_{\Omega}(m\theta)| \tag{22}$$

$$\phi_m = \angle H_{\Omega}(m\theta) \tag{23}$$

Now, consider the input signal $x(n)$ with harmonic amplitude A_m and phase α_m as

$$x(n) = \sum_{m=1}^{M} A_m e^{j[(m\theta)n + \alpha_m]} + v(n) \tag{24}$$

where $v(n)$ denotes the complex white Gaussian noise [17]. Similar to [8]–[9], the complex harmonic IIR notch filter output can be approximated as

$$y_M(n) = \sum_{m=1}^{M} mA_m B(m\theta) e^{j(m\theta)n + \alpha_m + \phi_m)} \delta_{\theta}(n) + v_1(n) \tag{25}$$

where $v_1(n)$ is the noise from the notch filter. Considering

$$\Omega - m\theta = m[\theta(n) - \theta] = m\delta_{\theta}(n) \tag{26}$$

with a gradient filter transfer function defined as $S_M(z) = \overline{\beta}_M(z) / X(z)$, the following recursion similar to [8]–[9] is achieved:

$$\begin{aligned} \beta_m(n) &= \beta_{m-1}(n) - e^{jm\theta}\beta_{m-1}(n-1) - jme^{jm\theta}y_{m-1}(n-1) \\ &\quad + re^{jm\theta}\beta_m(n-1) + jrme^{jm\theta}y_m(n-1) \end{aligned} \tag{27}$$

The z transform of (27) is given below:

$$\begin{aligned} \overline{\beta}_m(z) &= \overline{\beta}_{m-1}(z) - e^{jm\theta}\overline{\beta}_{m-1}(z)z^{-1} - jme^{jm\theta}Y_{m-1}(z)z^{-1} \\ &\quad + re^{jm\theta}\overline{\beta}_m(z)z^{-1} + jrme^{jm\theta}Y_m(z)z^{-1} \end{aligned} \tag{28}$$

The gradient filter transfer function at the output of the m-th filter section can expressed as

$$\begin{aligned} S_m(z) &= \frac{\overline{\beta}_m(z)}{X(z)} = \frac{\overline{\beta}_{m-1}(z)}{X(z)} \frac{1 - e^{jm\theta}z^{-1}}{1 - re^{jm\theta}z^{-1}} \\ &\quad - \frac{jme^{jm\theta}z^{-1}[Y_{m-1}(z) - rY_m(z)] / X(z)}{1 - re^{jm\theta}z^{-1}} \end{aligned} \tag{29}$$

Re-organizing (29) leads to the following recursion:

$$\begin{aligned} S_m(z) &= H_m(z)S_{m-1}(z) \\ &\quad - \frac{jme^{jm\theta}z^{-1}[1 - rH_m(z)]}{1 - re^{jm\theta}z^{-1}} H_1(z)\cdots H_{m-1}(z) \end{aligned} \tag{30}$$

with $S_0(z) = 0$. Again, expanding (30) gives

$$\begin{aligned} S_M(z) &= \sum_{m=1}^{M} \left[\prod_{k=1,k\neq m}^{M} H_k(z) \right] \frac{-jme^{jm\theta}z^{-1}}{1 - re^{jm\theta}z^{-1}} \\ &\quad + H(z) \sum_{m=1}^{M} \frac{jr \times me^{jm\theta}z^{-1}}{1 - re^{jm\theta}z^{-1}} \end{aligned} \tag{31}$$

At the point, $\Omega = m\theta$, the first summation term in (31) contains only one non-zero term corresponding to the term without the frequency response component $H_m(e^{jm\theta}) = 0$. This non-zero term is approximately constant and it can easily be verified that the point ($\Omega = m\theta$) is essentially the center of band-pass filter [9]. The second summation term in (31) is zero due to $H(e^{jm\theta}) = 0$. Therefore, the gradient filter frequency response at $\Omega = m\theta$ can be expressed as

$$\begin{aligned} S_M(e^{jm\theta}) &= -\left[\prod_{k=1,k\neq m}^{M} H_k(e^{jm\theta}) \right] \frac{jm}{1-r} \\ &= mB(m\theta)\angle(\phi_m + \pi) \end{aligned} \tag{32}$$

Using (32), we can derive the gradient filter output as

$$\beta_M(n) = \sum_{m=1}^{M} mB(m\theta)A_m e^{j[(m\theta)n + \alpha_m + \phi_m + \pi]} + v_2(n) \tag{33}$$

where $v_2(n)$ is the noise process in the gradient filter output.

Substituting (25) and (33) in (9), we achieve

$$E[\delta_{\theta}(n+1)] = E[\delta_{\theta}(n)] - E\{2\mu \operatorname{Re}[y_M(n)\beta_M^*(n)]\} \tag{34}$$

After simplification, (34) becomes

$$\begin{aligned} E[\delta_{\theta}(n+1)] &= E[\delta_{\theta}(n)] \\ &\quad - \mu \sum_{m=1}^{M} m^2 A_m^2 B^2(m\theta) e^{j(\phi_m - \phi_m - \pi)} E[\delta_{\theta}(n)] \\ &\quad - 2\mu E[v_1(n)v_2(n)] \end{aligned} \tag{35}$$

Thus, the stability bound in mean convergence is achieved as

$$\mu(\theta) < \frac{1}{\sum_{m=1}^{M} m^2 A_m^2 B^2(m\theta)} \tag{36}$$

As shown in (36), all the harmonic amplitudes are required to compute the stability bound. Assuming that each frequency component has the same amplitude, a simplified formula is yielded as

$$\mu(\theta) < \frac{1}{\sigma_x^2 \left[\frac{1}{M} \sum_{m=1}^{M} 2m^2 B^2(m\theta) \right]} \tag{37}$$

where σ_x^2 is the input signal power. Furthermore, for given M and the required frequency range, the upper bound $\mu_{\max}$ can numerically be searched, that is,

$$\mu_{\max} = \min_{-\pi/M \le \theta < \pi/M} [\arg(\mu(\theta))] \tag{38}$$

Fig. 1 shows the upper bounds estimated using (37) and (38) versus M with $\sigma_x^2 = 1$ for $r = 0.85$, $r = 0.9$, and $r = 0.96$, respectively. It can be seen that when r is close to 1, a small upper bound is required; and the upper bound decreases while the number of harmonics M increases.

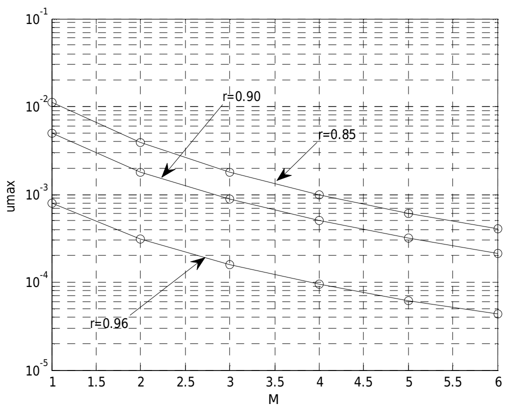

Fig. 1 Upper bound $\mu_{\max}$ in (38) versus M using $\sigma_x^2 = 1$.

III. Computer Simulations

In our simulations, an input signal having up to third harmonics ($M = 3$) with a fundamental frequency of f_a is adopted, that is,

$$\begin{aligned} x(n) = e^{j(2\pi \times f_a \times n / f_s)} + 0.5 e^{j(2\pi \times 2 f_a \times n / f_s + \pi/2)} \\ - 0.25 e^{j(2\pi \times 3 f_a \times n / f_s + \pi/2)} + v(n) \end{aligned} \tag{39}$$

where the sampling frequency is set to $f_s = 8000$ Hz. $v(n)$ is the zero-mean complex Gaussian noise process which is defined as

$$v(n) = v_R(n) + j v_I(n) \tag{40}$$

with the power given by

$$\sigma_v^2 = E[v_R^2(n)] + E[v_I^2(n)] \tag{41}$$

The fundamental frequency changed for every 2000 samples. For estimation of θ during the initial process and algorithm resetting, the block size of $N = 200$ samples was used. The upper bound $\mu_{\max} = 5.77 \times 10^{-4}$ was numerically searched using (36) and (38) for $r = 0.96$. $\mu = \mu_{\max}/20$ was chosen for our simulations.

Fig. 2 depicts characteristics of the developed algorithm. As shown in Fig. 2, the fundamental frequency begins with -875 Hz and changes for every 2000 samples with the following frequency values: 1225 Hz, 1100 Hz, -1125 Hz, and 1000 Hz, respectively. The signal to noise ratio was set to 12.15 dB for the simulations.

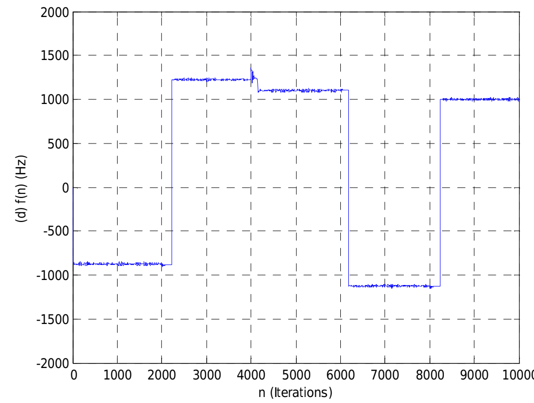

Fig. 2 Frequency tracking behavior comparisons: $SNR = 12.15$ dB, $\mu_{\max} = 5.77 \times 10^{-4}$, $\mu = \mu_{\max}/20$, and $r = 0.96$.

It is observed that the developed algorithm can track and estimate the unknown frequency of the complex signal along with its harmonic components very well. The developed scheme for global minimum monitoring can also effectively prevent the local minimum convergence.

IV. Conclusions

In this paper, a new complex adaptive harmonic IIR notch filter has been developed for estimating and tracking the unknown frequency of periodic complex signals in a noisy harmonic frequency environment. The developed algorithm is able to prevent the adaptive algorithm from its local minimum convergence due to signal fundamental frequency changes in the tracking process. In addition, a formula for determining the stability bound is derived.

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