

DOCUMENTATION FOR REAL-HF CODE

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1. COMPUTER COMPUTATIONS

The Euler characteristic for $\widehat{HFR}(\Sigma_2(K))$ for a knot in S^3 can be computed combinatorially according to [GM25, Proposition 7.1], and we have written some computer code which will do so.

A real Heegaard diagram for $\Sigma_2(K)$ can be obtained from a Seifert surface F for K which has the property that $S^3 \setminus \nu(F)$ is a handlebody. The boundary of $\nu(F) \cong F \times I$ has an obvious involution with fixed set K . Therefore, we can obtain a real Heegaard diagram for $\Sigma_2(K)$ by choosing alpha curves for $S^3 \setminus \nu(F)$ and reflecting these across K to obtain the beta curves. The genus of the resulting Heegaard splitting is twice that of the Seifert surface F , so $|\mathbb{T}_\alpha \cap M^R|$ grows rather quickly.

Fix a real Heegaard diagram for $\Sigma_2(K)$ with orientable quotient. Number the alpha and beta arcs, and label every intersection point between them. Label the intersection points on the fixed set first: start at the basepoint and follow along the knot labeling intersection points as they appear according to the knot's orientation. We define a class “Real_Heegaard_Diagram”, which has attributes (alphas, betas, tau, alpha_int_beta, alpha_int_C), where:

- (1) “alphas” is a dictionary whose keys are the indices of the ordered alpha curves whose values are list consisting of the intersection points between this alpha curve and the beta curves, listed in the order they appear along the alpha curve (following the orientation); “betas” is defined analogously.
- (2) The input “tau” is a dictionary encoding the action of τ on the intersection points. Finally, “alpha_int_beta” is a dictionary which associated to the i th intersection point its sign as an intersection point between the alphas and betas.
- (3) The input “alpha_int_C” is a dictionary assigning to the i th intersection point its sign as an intersection point between the alpha curves and C (if this point is not on the fixed set, we just assign it the sign of its intersection as a point in $\alpha \cap \beta$).

From this data, one can compute the Euler characteristic of $\widehat{HFR}(\Sigma_2(K))$. The program generates a list of points in $\mathbb{T}_\alpha \cap M^R$, sorts them into real Spin^c -structures, and computes the associated signed count.

For example, consider the real Heegaard diagram for $\Sigma_2(K)$, where $K = 4_1$ shown in Figure 1.1. The intersection points have been labeled and the alpha and beta curves have been ordered and oriented. Note that the first intersection points are those on the fixed set which are ordered by the location of the basepoint and the orientation of the knot. This data specifies a “Real_Heegaard_Diagram” class.

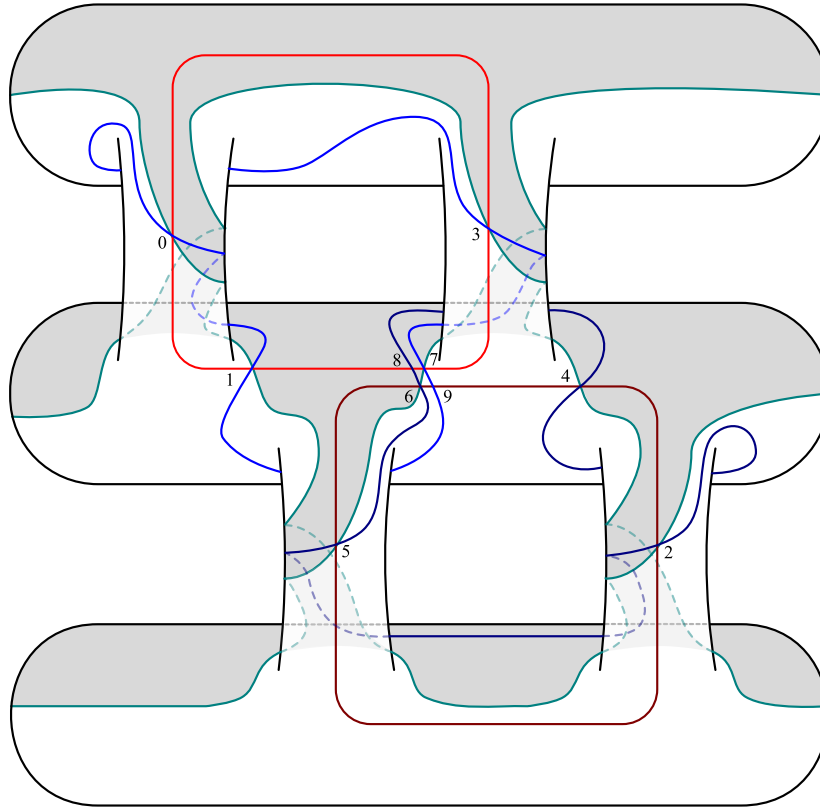
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: alphas = {0: [0,1,8,7,3], 1: [5,2,4,9,6]}
  betas = {0: [0,1,9,7,3], 1: [5,2,4,8,6]}
  tau = {0:0,1:1,2:2,3:3,4:4,5:5,6:6,7:7,8:9,9:8}
  alpha_int_beta = {0:1,1:-1,2:-1,3:1,4:-1,5:-1,6:1,7:1,8:-1,9:-1}
  alpha_int_C = {0:1,1:-1,2:-1,3:-1,4:1,5:1,6:-1,7:1,8:-1,9:-1}

H = Real_Heegaard_Diagram(alphas, betas, tau, alpha_int_beta, alpha_int_C)

```

These classes have a few useful attributes. As a sanity check, it can compute the first homology of the branched double cover (it returns its invariant factors), which should be the determinant of the knot.

FIGURE 1.1. A labeled real Heegaard diagram for $\Sigma_2(4_1)$.

```

alphas = {0: [0,1,8,7,3], 1: [5,2,4,9,6]}
betas = {0: [0,1,9,7,3], 1: [5,2,4,8,6]}
tau = {0:0,1:1,2:2,3:3,4:4,5:5,6:6,7:7,8:8,9:9}
alpha_int_beta = {0:1,1:-1,2:-1,3:1,4:-1,5:-1,6:1,7:1,8:-1,9:-1}
alpha_int_C = {0:1,1:-1,2:-1,3:-1,4:1,5:1,6:-1,7:1,8:-1,9:-1}

H = Real_Heegaard_Diagram(alphas, betas, tau, alpha_int_beta, alpha_int_C)

H.homology()

[5]

```

It will also, of course, return the Euler characteristics. The program fixes some arbitrary identification of $\text{Spin}^c(Y)$ with $H_1(Y)$ and returns the Euler characteristics preserving the cyclic ordering.

```

alphas = {0: [0,1,8,7,3], 1: [5,2,4,9,6]}
betas = {0: [0,1,9,7,3], 1: [5,2,4,8,6]}
tau = {0:0,1:1,2:2,3:3,4:4,5:5,6:6,7:7,8:8,9:9}
alpha_int_beta = {0:1,1:-1,2:-1,3:1,4:-1,5:-1,6:1,7:1,8:-1,9:-1}
alpha_int_C = {0:1,1:-1,2:-1,3:-1,4:1,5:1,6:-1,7:1,8:-1,9:-1}

H = Real_Heegaard_Diagram(alphas, betas, tau, alpha_int_beta, alpha_int_C)

H.real_euler_characteristics()

Starting computation...
Invariant beta points obtained...
Time to prepare for Euler characteristic computation: 0.45 seconds.
Computing Euler characteristics...
Total runtime: 0.45 seconds.
{(0,): -1, (1,): -1, (2,): 1, (3,): -1, (4,): -1}

```

Our program will also generate the data needed to compute the Euler characteristic of $\widehat{HFR}(\Sigma_2(K))$. It currently takes as input a braid word whose closure is the knot K . Applying Seifert's algorithm to such diagrams yields surfaces whose complements have obvious compressing disks, so, we can extract all the input data just from the braid word. Returning to our example, we can represent $K = 4_1$ as the closure of the braid $[1, -2, 1, -2]$. The function "real_HD" returns a "Real_Heegaard_Diagram" class for K .

```
# 4_1 is the braid closure of the braid [1,-2,1,-2]
H = real_HD([1,-2,1,-2])
print(H.alphas, H.betas, H.tau, H.alpha_int_beta, H.alpha_int_C)

{2: [3, 8, 7], 3: [2, 4, 9]} {2: [3, 9, 7], 3: [2, 4, 8]} {2: 2, 3: 3, 4: 4, 7: 7, 8: 9, 9: 8} {2: -1, 3: 1, 4: -1, 7: 1, 8: -1, 9: -1}
{2: -1, 3: -1, 4: 1, 7: 1, 8: -1, 9: -1}
```

This data comes from the Heegaard diagram shown in Figure 1.1; it has removed a few canceling intersection points. Representing K as a braid closure has the advantage that it is easy to choose a collection of compressing disks for the complement of the associated Seifert surface; the disadvantage is that the resulting surface may have needlessly large genus. For particular examples, it is sometimes possible to carefully choose a small free Seifert surface (for instance, the surface for 9_{46} in ?? is genus 1, while applying Seifert's algorithm to this diagram produces a larger genus surface; similarly, representing K as a braid closure also yields a surface of large genus).

REFERENCES

[GM25] Gary Guth and Ciprian Manolescu, *Real Heegaard Floer Homology*, Preprint, ADD (2025).

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