Lab 7

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Lab 7

1 Problem Descriptions

已知加速度，用 Romberg 数值积分计算速度和位移。

2 Analysis and Algorithms

现在要用数值方法求 $\int_a^b f(x)\,dx$，
设 $h=\frac{b-a}{n}$，已知：
复化梯形积分 $T_n(f)=h\left[\frac{1}{2}f(a)+\sum_{i=1}^{n-1}f(a+ih)+\frac{1}{2}f(b)\right]$、
复化 Simpson 积分 $S_n(f)=\frac{h}{3}\left[f(a)+4\sum_{i=0}^{m-1}f(x_{2i+1})+2\sum_{i=1}^{m-1}f(x_{2i})+f(b)\right]$.
将 $(T_n(f)-T_{2n}(f))$ 作为 $T_{2n}(f)$ 的修正值补充到 $I(f)$，即

$$I(f)\approx T_{2n}(f)+\frac{1}{3}\left(T_{2n}(f)-T_n(f)\right)=\frac{4}{3}T_{2n}-\frac{1}{3}T_n=S_n$$

其结果是将复化梯形求积公式组合成复化 Simpson 求积公式，截断误差由 $O(h^2)$ 提高到 $O(h^4)$. 这种手段称为外推算法，该算法在不增加计算量的前提下提高了误差的精度. 不妨对 $S_{2n}(f),S_n(f)$ 再作一次线性组合：

$$I(f)-S_n(f)=-\frac{f^{(4)}(\xi)}{180}h^4(b-a)\approx dh^4$$
$$I(f)-S_{2n}(f)=-\frac{f^{(4)}(\eta)}{180}\left(\frac{h}{2}\right)^4(b-a)\approx d\left(\frac{h}{2}\right)^4$$

$$I(f)\approx S_{2n}(f)+\frac{1}{15}\left(S_{2n}(f)-S_n(f)\right)=C_n(f)$$

复化 Simpson 公式组成复化 Cotes 公式，其截断误差是 $O(h^6)$. 同理对 Cotes 公式进行线性组合：

$$I(f)-C_{2n}(f)=e\left(\frac{h}{2}\right)^6 I(f)-C_n(f)=eh^6$$

得到具有 7 次代数精度和截断误差是 $O(h^8)$ 的 Romberg 公式：

$$R_n(f)=C_{2n}(f)+\frac{1}{63}\left(C_{2n}(f)-C_n(f)\right)$$

为了便于在计算机上实现 Romberg 算法，将 $T_n,S_n,C_n,R_n,\cdots$ 统一用 $R_{k,j}$ 表示，列标 $j=1,2,3,4$ 分别表示梯形、Simpson、Cotes 、Romberg 积分，行标 k 表示步长 $h_k=\frac{h}{2^{k-1}}$，得到 Romberg 计算公式：

$$R_{k,j}=R_{k,j-1}+\frac{R_{k,j-1}-R_{k-1,j-1}}{4^{j-1}-1},k=2,3,\cdots$$

对每一个 k,j 从 2 做到 k, 一直做到 $|R_k,k-R_{k-1,k-1}|$ 小于给定控制精度时停止计算.

3 Results

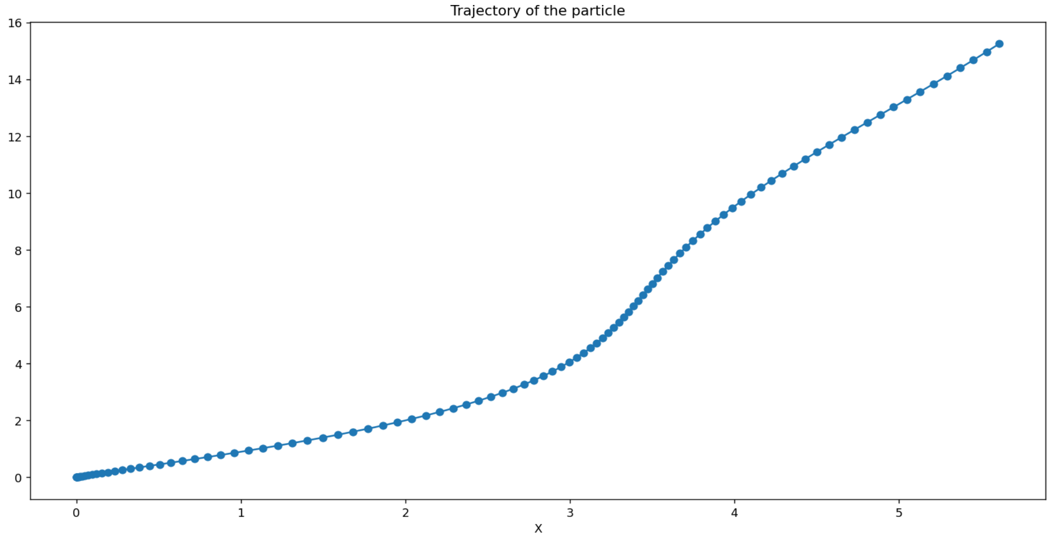
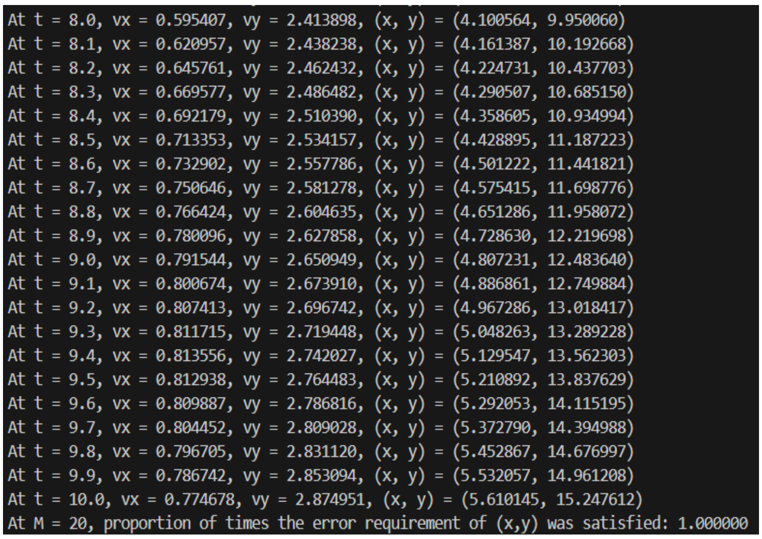


图 1: M=8 时的粒子轨迹



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At t = 8.0, vx = 0.595407, vy = 2.413898, (x, y) = (4.100564, 9.950060)
At t = 8.1, vx = 0.620957, vy = 2.438238, (x, y) = (4.161387, 10.192668)
At t = 8.2, vx = 0.645761, vy = 2.462432, (x, y) = (4.224731, 10.437703)
At t = 8.3, vx = 0.669577, vy = 2.486482, (x, y) = (4.290507, 10.685150)
At t = 8.4, vx = 0.692179, vy = 2.510390, (x, y) = (4.358605, 10.934994)
At t = 8.5, vx = 0.713353, vy = 2.534157, (x, y) = (4.428895, 11.187223)
At t = 8.6, vx = 0.732902, vy = 2.557786, (x, y) = (4.501222, 11.441821)
At t = 8.7, vx = 0.750646, vy = 2.581278, (x, y) = (4.575415, 11.698776)
At t = 8.8, vx = 0.766424, vy = 2.604635, (x, y) = (4.651286, 11.958072)
At t = 8.9, vx = 0.780096, vy = 2.627858, (x, y) = (4.728630, 12.219698)
At t = 9.0, vx = 0.791544, vy = 2.650949, (x, y) = (4.807231, 12.483640)
At t = 9.1, vx = 0.800674, vy = 2.673910, (x, y) = (4.886861, 12.749884)
At t = 9.2, vx = 0.807413, vy = 2.696742, (x, y) = (4.967286, 13.018417)
At t = 9.3, vx = 0.811715, vy = 2.719448, (x, y) = (5.048263, 13.289228)
At t = 9.4, vx = 0.813556, vy = 2.742027, (x, y) = (5.129547, 13.562303)
At t = 9.5, vx = 0.812938, vy = 2.764483, (x, y) = (5.210892, 13.837629)
At t = 9.6, vx = 0.809887, vy = 2.786816, (x, y) = (5.292053, 14.115195)
At t = 9.7, vx = 0.804452, vy = 2.809028, (x, y) = (5.372790, 14.394988)
At t = 9.8, vx = 0.796705, vy = 2.831120, (x, y) = (5.452867, 14.676997)
At t = 9.9, vx = 0.786742, vy = 2.853094, (x, y) = (5.532057, 14.961208)
At t = 10.0, vx = 0.774678, vy = 2.874951, (x, y) = (5.610145, 15.247612)
At M = 20, proportion of times the error requirement of (x,y) was satisfied: 1.000000
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图 2: 部分输出结果

4 Conclusion

M=4 时 Romberg 积分达到要求精度的比例为 0.05，M=8 时为 0.88，M=12,16,20 时均为 1，从结果可看出 M 越大，Romberg 积分达到要求精度的比例越高。但由于 M 越大，计算量越大，耗时越长，因此需要根据实际情况，平衡达到要求精度的比例和 M 的大小，例如在本实验中可以取 M=12。