Lab 1

圆形镜面成像问题

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1 Algorithm Description

设 $T(cos\theta, sin\theta)$, 则有

$$PT + QT = \sqrt{(P_x - \cos\theta)^2 + \sin^2\theta} + \sqrt{(Q_x - \cos\theta)^2 + (Q_y - \sin\theta)^2}$$
$$= \sqrt{P_x^2 - 2P_x\cos\theta + 1} + \sqrt{Q_x^2 + Q_y^2 + 1 - 2Q_x\cos\theta - 2Q_y\sin\theta}$$

由费马原理,光线沿 $PT + QT$ 最短的路径传播,因此只需对上式求导求极小值点。关于 θ 求导得

$$\frac{P_x sin\theta}{\sqrt{P_x^2 - 2P_x cos\theta + 1}} + \frac{Q_x sin\theta - Q_y cos\theta}{\sqrt{Q_x^2 + Q_y^2 + 1 - 2Q_x cos\theta - 2Q_y sin\theta}}$$

故只需用二分法解非线性方程

$$P_x sin\theta\sqrt{Q_x^2 + Q_y^2 + 1 - 2Q_x cos\theta - 2Q_y sin\theta} + (Q_x sin\theta - Q_y cos\theta)\sqrt{P_x^2 - 2P_x cos\theta + 1} = 0$$

$$T_x = cos\theta$$

$$T_y = sin\theta$$

由对称性易知

$$R_x = \frac{2Q_y - Q_x tan\theta - kQ_x + k(2T_x - Q_x) - 2T_y}{k - tan\theta}$$
$$R_y = Q_y - (Q_x - R_x)\theta$$

2 Results

$$P = (-2, 0), Q = (-1, 1) : T = (-0.885670, 0.464316), R = (-0.380057, 0.674993)$$

$$P = (-10, 0), Q = (-2, 1) : T = (-0.959312, 0.282350), R = (0.304214, 0.321811)$$

$$P = (-1.000001, 0), Q = (-2, 2) : T = (-1.000000, 0.000002), R = (0.000007, 1.999996)$$

$$P = (-2, 0), Q = (-1, 0.000001) : T = (-1.000000, 0.000001), R = (-1.000000, 0.000001)$$

$$P = (-2.33, 0), Q = (-3, 1) : T = (-0.989279, 0.146038), R = (1.182424, 0.382590)$$

$$P = (-3, 0), Q = (-1, 0.5) : T = (-0.922615, 0.385721), R = (-0.786920, 0.410917)$$

$$P = (-3, 0), Q = (-2, 10) : T = (-0.827028, 0.562160), R = (8.380296, 2.944148)$$

$$P = (-3, 0), Q = (-3, 1) : T = (-0.987408, 0.158192), R = (1.187435, 0.329136)$$

$$P = (-10, 0), Q = (-2, 1) : T = (-0.959312, 0.282350), R = (0.304214, 0.321811)$$

$$P = (-1024, 0), Q = (-8, 4) : T = (-0.970066, 0.242842), R = (7.000894, 0.244735)$$

3 Conclusion

本实验提高精度的主要措施：

1. 使用 long double 类型；

2. 用较多位数来表示 π；

3. 将含有除法的方程交叉相乘，通过乘法代替除法，以减少在求商时的误差；

4. 在用二分法解非线性方程时，限制条件是结果的绝对值 $\leq 10^{-7}$。

但由于本实验的方程较为复杂，含有三角函数、根式、平方等易造成误差放大的因素，暂时还未找到其他较好的减少误差的办法，但还尝试了另一种思路，叙述如下：

设 OT 的延长线交 PQ 于 R，则由角平分线定理，有

$$\frac{QT}{PT} = \frac{QR}{RP} = \frac{Q_y}{R_y} - 1$$

设 $T(x, y), k = \frac{Q_y}{Q_x - P_x}$，带入上述方程化简得：

$$\frac{Q_y^2(kx - y)^2 + k^2 P_x^2 y^2 - 2kQ_y P_x y(kx - y)}{k^2 P_x^2 y^2} = \frac{Q_y^2 + Q_x^2 + 1 - 2yQ_y - 2xQ_x}{1 + P_x^2 - 2xP_x}$$

故只需用 for 循环遍历或二分法解非线性方程

$$(Q_y^2(kx-y)^2+k^2P_x^2y^2-2kQ_yP_xy(kx-y))(1+P_x^2-2xP_x)-(k^2P_x^2y^2)(Q_y^2+Q_x^2+1-2yQ_y-2xQ_x) = 0$$

$$y = \sqrt{1 - x^2}$$

由对称性易知

$$R_x = \frac{\frac{Q_x T_y}{T_x} + \frac{T_x(2T_x - Q_x)}{T_y} + 2T_y - 2Q_y}{\frac{T_x}{T_y} + \frac{T_y}{T_x}}$$

$$R_y = Q_y - \frac{T_y}{T_x(Q_x - R_x)}$$

但验证结果时发现，上述方法在遇到一些极端情况如 $P = (-1.000001, 0), Q = (-2, 2)$ 时，中间的运算过程会出现极小的浮点数导致无法继续运算，所以这种思路在细节上仍有待改进。