GAME THEORY 1

Exercise 5

The exercise is due 16/5/2021 at 22:00.

A set $A \subseteq \mathbb{R}^n$ is said to be **convex** if for every $p, q \in A$ and $0 \leq \lambda \leq 1$, $\lambda p + (1-\lambda)q \in A$. (In other words, if p and q are in A then the segment which connects p and q is also in A.)

1. Let $\{A_\alpha\}_{\alpha \in I}$ be a family of convex sets in $\mathbb{R}^n$. Prove that $\bigcap_{\alpha \in I} A_\alpha$ is convex.

2. Let $x_1, ..., x_m \in \mathbb{R}^n$ and $\lambda_1, ..., \lambda_m \in \mathbb{R}$ such that $\lambda_i \geq 0$ for every $1 \leq i \leq m$ and $\sum_{i=1}^m \lambda_i = 1$. A linear combination of the form

   $$\lambda_1 x_1 + \cdots + \lambda_m x_m$$

   is called a **convex combination** of $x_1, ..., x_m$. Note that the convex combinations of x_1 and x_2 is are exactly the points on the segment which connects x_1 and x_2. Similarly, the convex combinations of x_1, x_2, x_3 are the points on the surface of the triangle whose vertices are x_1, x_2, x_3 and so on. Let $A \subseteq \mathbb{R}^n$ be a convex set and $x_1, ..., x_m \in A$. Prove that any convex combination of $x_1, ..., x_m$ is in A.
   Hint: Use the fact that

   $$\lambda_1 x_1 + \cdots + \lambda_m x_m = \lambda_1 x_1 + (1-\lambda_1)\left(\frac{\lambda_2}{1-\lambda_1}x_2 + \cdots + \frac{\lambda_m}{1-\lambda_1}x_m\right)$$

   whenever $\lambda_1 \neq 1$ and induction on m.

3. Let $A \subseteq \mathbb{R}^n$ (not necessarily convex). Prove that the following definitions for the **convex closure** $\operatorname{conv}(A)$ of A are equivalent:

   (a) $\operatorname{conv}_1(A)$ is the intersection of all convex sets which contains A. That is, $\operatorname{conv}_1(A) = \bigcap_{A \subseteq B,\, B \text{ is convex}} B$.

   (b) $\operatorname{conv}_2(A)$ is the minimal convex set which contains A. That is, for every $B \subseteq \mathbb{R}^n$, if B is convex and $A \subseteq B$, then $\operatorname{conv}_2(A) \subseteq B$.

   (c) $\operatorname{conv}_3(A) := \{\sum_{i=1}^m \lambda_i x_i \,|\, m \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1, x_i \in A\}$. Namely, $\operatorname{conv}_3(A)$ is the set of all convex combinations of points in A.

4. Let $n \in \mathbb{N}$ and $e_1, ..., e_n \in \mathbb{R}^n$ be the standard basis. Let $\Delta_n = \operatorname{conv}(e_1, ..., e_n) \subseteq \mathbb{R}^n$. Prove that Δ_n is compact.

5. Prove that the convex of any finite set of points in $\mathbb{R}^n$ is compact. **Hint:** use the previous question and the fact that the image of a compact set under a continuous function is compact.