Computability - Exercise 5

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: May 14, 2020

Question 1 - (Not for submission)

1. Give an implementation-level description of a TM for the language:
   $L = \{w \in \{a, b, c\}^* : w \text{ contains an equal number of } a\text{'s}, b\text{'s and } c\text{'s}\}$.
2. Prove that if $L \in R$ then $L^* \in R$.

Question 2

1. Let $A, B \in coRE$ be such that $A \cap B = \emptyset$. Prove that there exists a language $C \in R$ such that $A \subseteq C$ and $B \cap C = \emptyset$. We say that such a language separates A and B.
   - **Hint:** use the TMs associated with the given languages, in order to construct such a language.
2. Let M be a TM whose reading head can only move to the right. Prove that $L(M) \in REG$.
3. (a) Give a formal description of a TM that decides the language $L = \{a^n b^n c^n : n \in \mathbb{N}\}$. Then, give a high-level description of how your machine works.
   (b) Modify the TM from article (a) such that it recognizes the language, but does not decide it. Briefly explain the correctness of your modification.

Question 3

Prove the following claims:

1. If $L_1, L_2 \in coRE$ then $L_1 \cdot L_2 \in coRE$.
2. If $L \in RE$ then $L^* \in RE$. (Hint: Consider running TMs in parallel).
3. Let $L_1, L_2 \subseteq \Sigma^*$. Define $op(L_1, L_2) = \{xz \in \Sigma^* \mid \exists w \in L_1 \ s.t.\ xwz \in L_2\}$.
   Prove that if $L_1, L_2 \in RE$ then $op(L_1, L_2) \in RE$.

Question 4

Let M be a TM and let $t \in \mathbb{N}$. Consider the language L of all words $w \in \Sigma^*$ that are accepted by M within at most t steps. Show that $L \in REG$.

Question 5

Clumsy Smurf got drunk, and considered the following problem: given a TM M, we want to decide if $L(M) \neq \emptyset$. Clumsy Smurf suggested the following algorithm: construct from M a directed graph where the states are the vertices, and there is an edge (q, q') iff there is a transition $\delta(q, \sigma) = (q', \tau, D)$ for some $\sigma, \tau \in \Gamma$ and $D \in \{R, L\}$.
Then, run BFS from q_0, and see if q_{acc} is reachable. If it is, then $L(M) \neq \emptyset$, and otherwise $L(M) = \emptyset$.
Assume that there are no outgoing edges from M's halting states.
When Clumsy Smurf's hangover subsided, he told his suggestion to Brainy Smurf. Then Brainy Smurf explained to him that..
Prove or give a counterexample in order to complete the Smurfs episode!