# MathTools HW 3

1. **Concentration of the degree in a random graph.** Consider a random graph $G \sim \mathcal{G}(n,p)$ with $p = n^{-1/2}$. Show that one can find a constant $C > 0$ such that the following holds with high probability:

$$d_i \leq C \cdot (n \log n)^{1/2} \quad \text{holds simulatenously for all } i \in [n]\,,$$

   where $d_i$ is the degree of vertex $i$.

   *Remark:* A previous version of the problem used $C = 2.01$ explicitly.

2. **Threshold for the appearance of a clique.** Fix an integer $k \geq 3$. Show that the threshold for the appearance of a $k$-clique (copy of $K_k$) in $\mathcal{G}(n,p)$ is $p = n^{-\frac{2}{k-1}}$.

   *Hint:* Basically repeat the same argument you did in the lecture, for $k = 4$ (for the easier direction, you basically proved it in HW2 Q6 - make sure you see why). It might be convenient to use estimates like $\binom{n}{k} = \Theta(n^k)$ for $k$ constant, etc.

3. **Arithmetic progressions in random sets.** Let $S \subset \{1, \ldots, n\}$ be a uniformly chosen random subset of size $|S| = k$.[1] We say that $S$ contains an $\ell$**-arithmetic progression** if it contains $\ell$ numbers $a_1, \ldots, a_\ell$ such that

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_\ell - a_{\ell-1}\,.$$

   Notice that in what follows, $\ell$ is fixed and $k$ grows with $n$.

   (a) (Warm-up). Let $T \subset [n]$ with $|T| = t$, $t$ being constant (e.g, $t = 7$). Prove that $\Pr(T \subset S) = \Theta((k/n)^t)$.

   (b) Suppose that $k = \omega(n^{1/3})$. Prove that w.h.p, $S$ contains $\Omega(k^3/n)$ 3-arithmetic progressions.

   (c) Suppose that $k = o(\sqrt{n})$. Prove that w.h.p $S$ does not contain any 4-arithmetic progressions.

[1] Notice there are $\binom{n}{k}$ possible such subsets. We simply take one in random.