EXERCISE 1

The exercise is due 4/4/2021 at 22:00.

(1) Recall that in every finite combinatorial game with two players (Alice and Bob) which ends with “Bob wins”, “Alice wins” or a “draw” either one of the players has a winning strategy or both players can force a draw. Suppose now that we label each of the final positions by a pair of the form $(t, -t)$ where $t \in \mathbb{R}$ which indicates that Bob has to give t dollars to Alice (t can be negative!). Prove by induction that in this case exactly one of the following possibilities hold

  (a) Alice has a strategy which is guaranteed to result in a profit.
  (b) Bob has a strategy which is guaranteed to result in a profit.
  (c) Both players has a strategy which is guaranteed to result in $(0, 0)$.

(2) Starting with a pile of m coins. Two players take turns, in each turn each player can remove any number from 1 to d coins. The player who takes the last coin wins.

  (a) Draw the graph of the game (as in the Tirgul) for $m = 5$ and $d = 2$. For each position state whether it is in N or in P and describe the winning strategy of the relevant player.
  (b) Compute P and N with respect to any given d and m.
  (c) If instead of two players we had three players. Does any of the players have a winning strategy?

(3) The game SOS is played on a table of size $1 \times n$. In each turn a player chooses a square of size 1×1 and draws an S or an O. The first player to write an SOS wins and the game ends. Otherwise, after n turns the game ends in a draw.

  (a) If $n = 3$ prove that both players have a strategy which guaranteed to result in at least a draw.
  (b) If $n = 4$ and the game begins with $\begin{array}{|c|c|c|c|}\hline \text{S} & \; & \; & \; \\ \hline\end{array}$, show that the first player has a winning strategy.
  (c) If $n = 7$ show that the first player has a winning strategy (**hint:** use the middle square and the previous cases).