GAME THEORY 1

Exercise 8

The exercise is due 6/6/2021 at 22:00.

1. Use dominating strategies to reduce the following payoff matrix as much as you can.

1	0	0	2	0	1
4	3	7	-5	1	2
3	2	0	2	1	1
4	3	1	3	2	2
3	3	4	-1	2	1
4	3	3	-2	2	2

2. Let $A \in M_{m\times n}(\mathbb{R})$ be the payoff matrix of a zero-sum game. Prove the following claims.

   (a) If a pure strategy $i \in \{1, ..., m\}$ of player 1 is strongly dominated by a pure strategy $j \neq i$. Then in **any** optimal strategy $x^* \in \Delta_m$ of player 1, $x_i^* = 0$.

   (b) Prove that the same claim as in (a) holds if i is strongly dominated by a mixed strategy $x' \in \Delta_m$.

   (c) Use (a) and (b) to find **all** of the optimal strategies in

2	3	3	1
1	1	-1	0
3	0	1	4
2.5	1	1	2

   (d) Give an example of a zero-sum game and pure strategies $i \neq j$ of player 1 such that j weakly domiantes i, but there exists an optimal strategy x^* for player 1 with $x_i^* > 0$.

   (e) Give an example of a zero-sum game and a pure strategy i of player 1 such that **every** optimal strategy x^* of player 1 satisfies $x_i = 0$, but i is not weakly (or strongly) dominated by any other strategy.

   Hint: Look at 2×2 matrices of the form $\begin{pmatrix} a & a \\ b & c \end{pmatrix}$.

3. We consider the following zero-sum game with two players:

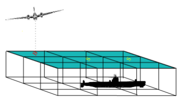
   - Every player chooses an integer in $\{1, 2, ..., 10\}$. We denote by d the absolute value of the differences between the chosen numbers.
   - If $d \leq 1$, nothing happens.
   - If $d = 2$, then the player with the greater value earns 2\$ (and the other one loses the same amount).
   - If $d \geq 3$, then the player with the smaller value earns d\$ (and the other one loses the same amount).

   (a) Write the payoff matrix of the game.

   (b) Find the dominated pure strategies and write the reduced matrix (After removing all dominated strategies).

(c) What is the value of the game? Find optimal strategies for both players. **Hint:** You can solve a system of inequalities as in Tirgul 8, or use Question 2 from Exercies 7.

4. (**Submarine game**): This game illustrate war between two nations. Player 1 plays the role of a fighter aircraft which tries to bomb a submarine (player 2) that is located deep inside the water. Player 1 flies above the water and can not see the submarine. He needs to guess where to drop the bomb. Formally, “the water” is a 3×3 table (as in the following figure).



Player 2 chooses where to located the submarine in this table. The size of the submarine is of 2 squares (namely 2×1 or 1×2 - it can not be sitting diagonally). Player 1 then drops a bomb on one of the squares on the table. If he hits part of the submarine he wins (gets a point), otherwise nothing happens. You can see the payoff matrix of this game in a figure at the end of this page This 9×12 payoff matrix seems to be complicated to solve. However, using symmetry we can reduce this matrix and solve the game.
First, observe that the game is invariant to rotations and reflections of the board. Player 1 has three **kinds** of moves: bomb the middle square, bomb one of the corners or bomb one of the midsides. If we rotate or reflect the board the middle always remain in its place, corners go to other corners and midsides to other midsides edges. Player 2 has two kinds of moves - place the submarine in the center or in the off-center. We can therefore assume that there exists optimal strategies where strategies of the same kind are chosen in the same probability. In other word, we can assume that player 1 has only 3 moves and player 2 has 2 moves.

(a) Write this payoff matrix (and remember that the utility of player 1 is the probability that the bomb will hit the submarine. For instance if he chooses a midside and player 2 chooses the center, the probability to win is 1/4). What is the value of the game? Find optimal strategies for player 1 and player 2 in the original game.

(b) Now consider the same game, but with a submarine of size 3×1 (or 1×3). Use symmetries to find optimal strategies and the value of the game.

(c) What about the case where the submarine is of size 1×1?

	off-center								center			
corner	1	1	0	0	0	0	0	0	0	0	0	0
	0	0	1	1	0	0	0	0	0	0	0	0
	0	0	0	0	1	1	0	0	0	0	0	0
	0	0	0	0	0	0	1	1	0	0	0	0
mid-side	1	0	0	0	0	0	0	1	1	0	0	0
	0	1	1	0	0	0	0	0	0	1	0	0
	0	0	0	1	1	0	0	0	0	0	1	0
	0	0	0	0	0	1	1	0	0	0	0	1
middle	0	0	0	0	0	0	0	0	1	1	1	1