CS-67720 Metric Embeddings Theory **Fall 2021/22**

Exercise 4

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1. (a) Show that any *weighted* cycle (C_n, w) on n points has a (1,2)-*spanning* tree cover (that is a tree cover composed of 2 spanning trees with distortion 1).

   (b) Show that for any probabilistic embedding of the *unweighted* cycle C_n into tree metrics, such that the distribution is over (a support of) at most k tree metrics, the expected distortion is at least $\Omega(n/k)$.

   (c) Show that for any $1 \le k \le n$, any *weighted* cycle (C_n, w) on n points probabilistically embeds into tree metrics, such that the distribution is over (a support of) k *spanning* trees, with expected distortion $O(n/k)$. **Directions:** Prove this first for the case of the unweighted cycle on n points, then extend the result.

   (d) Prove that for any Δ, the *weighted* cycle has a Δ-bounded probabilistic partition with padding parameter $O(1)$. **Hint:** First construct a simple Δ-bounded (deterministic) partition, then try to add randomization in the construction to make sure that the padding parameter condition is satisfied. It may help to think of the cycle as being continuous. Also, you can start by providing a construction for the line metric or even for an interval (e.g. $[0, 1]$). The same ideas should holds for the cycle. **An alternative direction:** You can modify the proof of Theorem 12.8 to prove the claimed bound, using the fact the the doubling dimension of the cycle is 1. More generally, this is a special case of a more general statement phrased in BONUS (i) below (so if you solve the bonus it implies in particular this special case).

BONUS (i): Prove that any metric space X has a Δ-bounded probabilistic partition with padding parameter $O(\dim(X))$, where $\dim(X)$ denotes the doubling dimension of X. Hint: Carefully modify the proof of Theorem 12.8.

2. Let G be a complete weighted undirected graph over n vertices, with weights obeying the triangle inequality.

   (a) Prove that G probabilistically embeds into its *spanning trees* with expected distortion $O(\log n)$. **Hint:** use the theorem on probabilistic embedding into ultrametrics as a black-box result. Directions: Prove that any k-HST (choose some constant $k \ge 2$) has a non-contractive $O(1)$-distortion embedding into a tree T (not necessarily a spanning tree) over n vertices (with no Steiner points). Explain how this implies the required claim.

   (b) Bound the expected weight of the obtained spanning tree in terms of the weight of the MST of G.

BONUS (ii): Prove that every finite ultrametric can be probabilistically embedded into a k-HST with distortion strictly less than k, for every $k > 1$. More challenging: Provide such a probabilistic embedding with distortion $O(\frac{k}{\log k})$.

BONUS (iii): Let X be the metric of an unweighted graph. Prove that X probabilistically embeds into *ultrametrics* with distortion $O(\log n)$, such that the distribution is over (a support of) poly(n) ultrametrics. **Hint:** Use the theorem on probabilistic embedding into ultrametrics, then apply Hoeffding/Bernstein's inequalities. Extra challenging: What is the best bound on the support's size achievable with each of these inequalities ? Can this method be applied to any metric space X ?

BONUS (iv): Given a finite metric space (X, d) prove that the snowflake metric (X, d^α), for $\alpha < 1$, embeds in ℓ_p with distortion $O\left(\frac{ddim(X)}{1-\alpha}\right)$, for any $1 \le p \le \infty$. **Hint:** Update the proof of the coarse scaling embedding into ℓ_p – modify each component i by multiplying it by $\Delta_i^{\alpha-1}$, and use the partition lemma stated in BONUS (i) above (you may use it even if you did not prove it).