GAME THEORY 1

Exercise 6

The exercise is due 23/5/2021 at 22:00.

- Let $A \in M_{m\times n}(\mathbb{R})$ be the payoff matrix of a zero-sum game. The utility of the first player in mixed strategies is given by
$$u_1(x,y) = u(x,y) = x^T \cdot A \cdot y$$
where $x \in \Delta_m$ and $y \in \Delta_n$.

- The value of the game in mixed strategies is given by,
$$\mathrm{Val}(A) = \underbrace{\max_{x\in\Delta_m} \min_{y\in\Delta_n} u(x,y)}_{(1)} = \max_{x\in\Delta_m} \min_{1\le i\le n} u(x,e_i) = \underbrace{\min_{y\in\Delta_n} \max_{x\in\Delta_m} u(x,y)}_{(2)} = \min_{y\in\Delta_m} \max_{1\le i\le m} u(e_i,y).$$

- An optimal mixed strategy $x^* \in \Delta_m$ for the first player is a strategy which maximizes (1). Namely, it is a strategy which satisfies
$$\min_{y\in\Delta_m} u(x^*,y) \ge \min_{y\in\Delta_m} u(x,y)$$
for every $x \in \Delta_m$.

- Similarly, an optimal mixed strategy $y^* \in \Delta_n$ is a strategy which minimizes (2).
$$\max_{x\in\Delta_m} u(x,y^*) \le \max_{x\in\Delta_m} u(x,y)$$
for every $y \in \Delta_n$.

1. Consider the following zero-sum game
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$

   (a) Fnd all the optimal strategies in pure strategies for both players. Does the game have a value in pure strategies?

   (b) Let $x = \begin{pmatrix} p \\ 1-p \end{pmatrix}$ and $y = \begin{pmatrix} q \\ 1-q \end{pmatrix}$. Compute $u(x,y)$ as a function of p and q. It is convenient to write $u(p,q)$ instead of $u\left(\begin{pmatrix} p \\ 1-p \end{pmatrix}, \begin{pmatrix} q \\ 1-q \end{pmatrix}\right)$.

   (c) Draw the functions $u(p,0)$ and $u(p,1)$ as a function of p. Use that to compute the value
$$\mathrm{Val}(A) = \max_{0\le p\le 1} \min\{u(p,0), u(p,1)\}$$
in mixed strategies.

(d) Similarly, draw $u(0,q)$ and $u(1,q)$ as functions of q and show that

$$\text{Val}(A) = \min_{0 \le q \le 1} \max\{u(0,q), u(1,q)\}.$$

(e) Find all the optimal strategies (in mixed strategies) for both players.

2. Consider the following game: Two players choose a number from $\{2,3\}$, simultanuously. Player 1 wins if the sum is odd and player 2 wins if the sum is even. The loser has to pay the winner the product of the chosen numbers.

   (a) Describe the game using a payoff matrix A.

   (b) Compute the optimal strategies in pure strategies for both players. Does the game have a value in pure strategies?

   (c) Compute the value and the optimal strategies in mixed strategies for both players (use the same method as in Question 1).

3. Let $A = (a_{ij}) \in M_{m \times n}(\mathbb{R})$ be the payoff matrix of a zero-sum game. Let $c > 0$ and $d \in \mathbb{R}$. We define a new game $B = (b_{ij})$ by

$$b_{ij} = c \cdot a_{ij} + d$$

   (a) Prove that $\text{Val}(B) = c \cdot \text{Val}(A) + d$ in mixed strategies.

   (b) How are the optimal strategies in A and B related?

4. Let $A \in M_{n \times n}(\mathbb{R})$ be an anti-symmetric matrix $A^T = -A$.

   (a) Prove that $\text{Val}(A) = 0$ and that every optimal strategy (in mixed strategies) of one player is an optimal strategy of the other player. (Why does it make sense?)
   Hint: use the fact that

$$\min_{x \in X}(-f(x)) = -\max_{x \in X} f(x), \ \max_{x \in X}(-f(x)) = -\min_{x \in X} f(x).$$

   (b) What is the value of the game Rock Paper Scissors from Tirgul 6?