Metric Embedding Theory and its Algorithms (67720) – Exercise 2

Mike Greenbaum 211747639

November 2021

1. (a) We will define $Y, Z = \mathbb{R}^n, X = \{e_1, \ldots, e_n\}$, $d_X = d_Y = d_Z = l_\infty$. define the following linear function: $\forall i \in [n], f(e_i) = g(e_i) = \begin{cases} n \cdot e_i & i = 1 \\ e_i & else \end{cases}$, notice that it is enough to define the entire function, f and g.

Notice that $f, g, g \circ f$ are non contracting and therefore we can replace $dist(u,v)$ with $expans(u,v)$ in the calculations later.
Notice that $\forall u, v \in X \setminus \{e_1\}$, $\frac{d_Y(f(u),f(v))}{d_X(u,v)} = \frac{d_Y(u,v)}{d_X(u,v)} = \frac{1}{1} = 1$ and if $u = e_1$, then $\frac{d_Y(f(u),f(v))}{d_X(u,v)} = \frac{d_Y(n \cdot u,v)}{d_X(u,v)} = \frac{n}{1} = n$, therefore:

$$l_1 - dist(f) \overset{\text{def}}{=} \left(\frac{\sum_{u \neq v \in X} dist_f(u,v)^1}{\binom{n}{2}} \right)^{\frac{1}{1}} = \frac{\sum_{u \neq v \in X} expans_f(u,v)}{\binom{n}{2}} = \frac{\sum_{u \neq v \in X} \frac{d_Y(f(u),f(v))}{d_X(u,v)}}{\binom{n}{2}}$$
$$= \frac{\sum_{u \neq v \in X \setminus \{e_1\}} \frac{d_Y(f(u),f(v))}{d_X(u,v)} + \sum_{(u=e_1 \vee v=e_1) \wedge u \neq v} \frac{d_Y(f(u),f(v))}{d_X(u,v)}}{\binom{n}{2}}$$
$$= \frac{\sum_{u \neq v \in X \setminus \{e_1\}} 1 + \sum_{(u=e_1 \vee v=e_1) \wedge u \neq v} n}{\binom{n}{2}} = \frac{\binom{n-1}{2} + n \cdot 2 \cdot (n-1)}{\binom{n}{2}} = \Theta(1)$$

Notice that $\forall u, v \in f(X) \setminus \{n \cdot e_1\}$, $\frac{d_Z(g(u),g(v))}{d_Y(u,v)} = \frac{d_Z(u,v)}{d_Y(u,v)} = \frac{1}{1} = 1$ and if $u = n \cdot e_1$, then $\frac{d_Z(g(u),g(v))}{d_Y(u,v)} = \frac{d_Z(n \cdot u,v)}{d_Y(u,v)} = \frac{n}{1} = n$, therefore:

$$l_1 - dist(g) \overset{\text{def}}{=} \left(\frac{\sum_{u \neq v \in f(X)} dist_g(u,v)^1}{\binom{n}{2}} \right)^{\frac{1}{1}} = \frac{\sum_{u \neq v \in f(X)} expans_g(u,v)}{\binom{n}{2}} = \frac{\sum_{u \neq v \in f(X)} \frac{d_Z(g(u),g(v))}{d_Y(u,v)}}{\binom{n}{2}}$$
$$= \frac{\sum_{u \neq v \in f(X) \setminus \{n \cdot e_1\}} \frac{d_Z(g(u),g(v))}{d_Y(u,v)} + \sum_{(u=n \cdot e_1 \vee v=n \cdot e_1) \wedge u \neq v} \frac{d_Z(g(u),g(v))}{d_Y(u,v)}}{\binom{n}{2}}$$
$$= \frac{\sum_{u \neq v \in f(X) \setminus \{n \cdot e_1\}} 1 + \sum_{(u=n \cdot e_1 \vee v=n \cdot e_1) \wedge u \neq v} n}{\binom{n}{2}} = \frac{\binom{n-1}{2} + n \cdot 2 \cdot (n-1)}{\binom{n}{2}} = \Theta(1)$$

Notice that $\forall u, v \in X \setminus \{e_1\}$, $\frac{d_Z(g(f(u)),g(f(v)))}{d_X(u,v)} = \frac{d_Z(g(u),g(v))}{d_X(u,v)} = \frac{d_Z(u,v)}{d_X(u,v)} = \frac{1}{1} = 1$ and if $u = e_1$, then $\frac{d_Z(g(f(u)),g(f(v)))}{d_X(u,v)} = \frac{d_Z(g(n \cdot u),g(v))}{d_X(u,v)} = \frac{d_Z(n^2 \cdot u,v)}{d_X(u,v)} = \frac{n^2}{1} = n^2$, therefore:

$$l_1 - dist(g \circ f) \overset{\text{def}}{=} \left(\frac{\sum_{u \neq v \in X} dist_{(g \circ f}(u,v)^1}{\binom{n}{2}} \right)^{\frac{1}{1}} = \frac{\sum_{u \neq v \in X} expans_{(g \circ f}(u,v)}{\binom{n}{2}} = \frac{\sum_{u \neq v \in X} \frac{d_Z(g(u),g(v))}{d_X(u,v)}}{\binom{n}{2}}$$
$$= \frac{\sum_{u \neq v \in X \setminus \{e_1\}} \frac{d_Z(g(u),g(v))}{d_X(u,v)} + \sum_{(u=e_1 \vee v=e_1) \wedge u \neq v} \frac{d_Z(g(u),g(v))}{d_X(u,v)}}{\binom{n}{2}}$$
$$= \frac{\sum_{u \neq v \in X \setminus \{e_1\}} 1 + \sum_{(u=e_1 \vee v=e_1) \wedge u \neq v} n^2}{\binom{n}{2}} = \frac{\binom{n-1}{2} + n^2 \cdot 2 \cdot (n-1)}{\binom{n}{2}} = \Theta(n)$$

Therefore we showed that $l_1 - dist(g \circ f) = \Theta(n)$, and $l_1 - dist(f) = l_1 - dist(g) = \Theta(1)$, therefore $l_1 - dist(f) \cdot l_1 - dist(g) = \Theta(1)$. We got that $l_1 - dist(g \circ f) = \Theta(n)$ and $l_1 - dist(f) \cdot l_1 - dist(g) = \Theta(1)$ and therefore due to the definition of the Θ notation, $\lim_{n \to \infty} \frac{l_1 - dist(g \circ f)}{l_1 - dist(f) \cdot l_1 - dist(g)} = \infty$. In class it was stated that it is enough to show that the inequality $l_1 - dist(g \circ f) > l_1 - dist(f) \cdot l_1 - dist(g)$ is true from some N and not for all $n \geq 2$ and it follows immediately from the limit.

(b) denote $|X| = n$

Define $\forall u \neq v \in X, y_{u,v} \overset{\text{def}}{=} dist_f(u,v)^q, x_{u,v} \overset{\text{def}}{=} dist_g(f(u),f(v))^q$. Notice that:

$$\begin{aligned}
\|x\|_s^{\frac{1}{q}} &= \left(\left(\sum_{u\neq v\in X} (dist_g(f(u),f(v))^q)^s\right)^{\frac{1}{s}}\right)^{\frac{1}{q}} = \left(\sum_{u\neq v\in X} dist_g(f(u),f(v))^{qs}\right)^{\frac{1}{qs}} \\
&= \left(\frac{\sum_{u\neq v\in X} dist_g(f(u),f(v))^{qs}}{\binom{n}{2}} \cdot \binom{n}{2}\right)^{\frac{1}{qs}} = \left(\frac{\sum_{u\neq v\in X} dist_g(f(u),f(v))^{qs}}{\binom{n}{2}}\right)^{\frac{1}{qs}} \cdot \binom{n}{2}^{\frac{1}{qs}} \\
&= \left(\frac{\sum_{u\neq v\in f(X)} dist_g(u,v)^{qs}}{\binom{n}{2}}\right)^{\frac{1}{qs}} \cdot \binom{n}{2}^{\frac{1}{qs}} \overset{\text{def}}{=} l_{qs} - dist(g) \cdot \binom{n}{2}^{\frac{1}{qs}}
\end{aligned}$$

Furthermore, notice that:

$$\begin{aligned}
\|y\|_s^{\frac{1}{q}} &= \left(\left(\sum_{u\neq v\in X} (dist_f(u,v)^q)^p\right)^{\frac{1}{p}}\right)^{\frac{1}{q}} = \left(\sum_{u\neq v\in X} dist_f(u,v)^{qp}\right)^{\frac{1}{qp}} \\
&= \left(\frac{\sum_{u\neq v\in X} dist_f(u,v)^{qp}}{\binom{n}{2}} \cdot \binom{n}{2}\right)^{\frac{1}{qp}} = \left(\frac{\sum_{u\neq v\in X} dist_f(u,v)^{qp}}{\binom{n}{2}}\right)^{\frac{1}{qp}} \cdot \binom{n}{2}^{\frac{1}{qp}} \\
&\overset{\text{def}}{=} l_{qp} - dist(f) \cdot \binom{n}{2}^{\frac{1}{qp}}
\end{aligned}$$

Then

$$\begin{aligned}
l_q - dist(g\circ f) &\overset{\text{def}}{=} \left(\frac{\sum_{u\neq v\in X} dist_{g\circ f}(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = \left(\frac{\sum_{u\neq v\in X} \max\left\{expans_{g\circ f}(u,v), contr_{g\circ f}(u,v)\right\}^q}{\binom{n}{2}}\right)^{\frac{1}{q}} \\
&= \left(\frac{\sum_{u\neq v\in X} \max\left\{\frac{d_Z(g(f(u)),g(f(v)))}{d_X(u,v)}, \frac{d_X(u,v)}{d_Z(g(f(u)),g(f(v)))}\right\}^q}{\binom{n}{2}}\right)^{\frac{1}{q}} \\
&= \left(\frac{\sum_{u\neq v\in X} \max\left\{\frac{d_Z(g(f(u)),g(f(v)))}{d_Y(f(u),f(v))}\cdot\frac{d_Y(f(u),f(v))}{d_X(u,v)}, \frac{d_X(u,v)}{d_Y(f(u),f(v))}\cdot\frac{d_Y(f(u),f(v))}{d_Z(g(f(u)),g(f(v)))}\right\}^q}{\binom{n}{2}}\right)^{\frac{1}{q}} \\
&\leq \left(\frac{\sum_{u\neq v\in X} \left(\max\left\{\frac{d_Y(f(u),f(v))}{d_X(u,v)}, \frac{d_X(u,v)}{d_Y(f(u),f(v))}\right\}\cdot\max\left\{\frac{d_Z(g(f(u)),g(f(v)))}{d_Y(f(u),f(v))}, \frac{d_Y(f(u),f(v))}{d_Z(g(f(u)),g(f(v)))}\right\}\right)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} \\
&= \left(\frac{\sum_{u\neq v\in X} dist_f(u,v)^q \cdot dist_g(f(u),f(v))^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = \left(\frac{\langle x,y\rangle}{\binom{n}{2}}\right)^{\frac{1}{q}} \overset{\text{Holder-inequality}}{\leq} \left(\frac{\|x\|_s\cdot\|y\|_p}{\binom{n}{2}}\right)^{\frac{1}{q}} \\
&= \|x\|_s^{\frac{1}{q}}\cdot\|y\|_p^{\frac{1}{q}}\cdot\left(\frac{1}{\binom{n}{2}}\right)^{\frac{1}{q}} = l_{qp} - dist(f)\cdot\binom{n}{2}^{\frac{1}{qp}}\cdot l_{qs} - dist(g)\cdot\binom{n}{2}^{\frac{1}{qs}}\cdot\left(\frac{1}{\binom{n}{2}}\right)^{\frac{1}{q}} \\
&= l_{qp} - dist(f)\cdot l_{qs} - dist(g)\cdot\binom{n}{2}^{\frac{1}{q}\cdot\left(\frac{1}{p}+\frac{1}{s}\right)}\cdot\left(\frac{1}{\binom{n}{2}}\right)^{\frac{1}{q}} \\
&= l_{qp} - dist(f)\cdot l_{qs} - dist(g)\cdot\binom{n}{2}^{\frac{1}{q}}\cdot\left(\frac{1}{\binom{n}{2}}\right)^{\frac{1}{q}} = l_{qp} - dist(f)\cdot l_{qs} - dist(g)
\end{aligned}$$

We proved that $l_q - dist(g\circ f) \leq l_{qp} - dist(f)\cdot l_{qs} - dist(g)$ as required.

(c) Denote $\alpha = \sqrt{\frac{l_q - contr(f)}{l_q - expans(f)}}$. We will look at $h(x) = f(x)$, where $h : (X, d_X) \to (f(X), d_{\alpha f(X)})$. Notice that

$$l_q - expans(h) \overset{\text{def}}{=} \left(\frac{\sum_{u\neq v\in X} expans_h(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = \left(\frac{\sum_{u\neq v\in X}\left(\frac{d_{\alpha f(X)}(h(u),h(v))}{d_X(u,v)}\right)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = \left(\frac{\sum_{u\neq v\in X}\left(\frac{d_{\alpha f(X)}(u,v)}{d_X(u,v)}\right)^q}{\binom{n}{2}}\right)^{\frac{1}{q}}$$

$$\overset{\text{f(X)−scalable−metric}}{=} \left(\frac{\sum_{u\neq v\in X}\left(\frac{\alpha\cdot d_Y(u,v)}{d_X(u,v)}\right)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = \alpha\cdot\left(\frac{\sum_{u\neq v\in X} expans_f(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}}$$

$$= \alpha\cdot l_q - expans(f) = \sqrt{\frac{l_q - contr(f)}{l_q - expans(f)}}\cdot l_q - expans(f) = \sqrt{l_q - contr(f)}\cdot\sqrt{l_q - expans(f)}$$

Now notice that

$$l_q - contr(h) \overset{\text{def}}{=} \left(\frac{\sum_{u\neq v\in X} contr_h(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = \left(\frac{\sum_{u\neq v\in X}\left(\frac{d_X(u,v)}{d_{\alpha f(X)}(h(u),h(v))}\right)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = \left(\frac{\sum_{u\neq v\in X}\left(\frac{d_X(u,v)}{d_{\alpha f(X)}(u,v)}\right)^q}{\binom{n}{2}}\right)^{\frac{1}{q}}$$

$$\overset{\text{f(X)−scalable−metric}}{=} \left(\frac{\sum_{u\neq v\in X}\left(\frac{d_X(u,v)}{\alpha\cdot d_Y(u,v)}\right)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = \alpha^{-1}\cdot\left(\frac{\sum_{u\neq v\in X} contr_f(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}}$$

$$= \alpha^{-1}\cdot l_q - contr(f) = \sqrt{\frac{l_q - expans(f)}{l_q - contr(f)}}\cdot l_q - contr(f) = \sqrt{l_q - contr(f)}\cdot\sqrt{l_q - expans(f)}$$

define $x_{u,v} = \frac{contr_h(u,v)}{\binom{n}{2}^{\frac{1}{q}}}, y_{u,v} = \frac{expans_h(u,v)}{\binom{n}{2}^{\frac{1}{q}}}, \forall u\neq v\in X$.

Finally notice that

$$l_q - dist(h) \overset{\text{def}}{=} \left(\frac{\sum_{u\neq v\in X} dist_h(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = \left(\frac{\sum_{u\neq v\in X}\max\{contr_h(u,v), expans_h(u,v)\}^q}{\binom{n}{2}}\right)^{\frac{1}{q}}$$

$$= \left(\frac{\sum_{u\neq v\in X}\max\{contr_h(u,v)^q, expans_h(u,v)^q\}}{\binom{n}{2}}\right)^{\frac{1}{q}} \leq \left(\frac{\sum_{u\neq v\in X}(contr_h(u,v) + expans_h(u,v))^q}{\binom{n}{2}}\right)^{\frac{1}{q}}$$

$$= \left(\sum_{u\neq v\in X}(x_{u,v}+y_{u,v})^q\right)^{\frac{1}{q}} = \|x+y\|_q \overset{\text{Minkowski−inequality}}{\leq} \|x\|_q + \|y\|_q$$

$$= \left(\frac{\sum_{u\neq v\in X} contr_h(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} + \left(\frac{\sum_{u\neq v\in X} expans_h(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}} = l_q - expans(h) + l_q - contr(h)$$

$$= 2\cdot\sqrt{l_q - contr(f)}\cdot\sqrt{l_q - expans(f)} = O\left(\sqrt{l_q - contr(f)}\cdot\sqrt{l_q - expans(f)}\right)$$

Now, we know that due to Y being a scaling space, there exists h_1 that embeds isometrically from $(\alpha f(X), d_{\alpha f(X)}) \to (f(X), d_{f(X)})$. Look at $\hat{h}(x) = h_1(h(x))$.

Notice that $\hat{h} : X \to f(X)$ and that $l_q - dist(\hat{h}) = l_q - dist(h) = O\left(\sqrt{l_q - contr(f)}\cdot\sqrt{l_q - expans(f)}\right)$ because $h1$ is embeds isometrically and therefore doesn't change the distances.

Therefore, we showed there exists a function $\hat{h} : X \to Y$ with distortion $O\left(\sqrt{l_q - contr(f)}\cdot\sqrt{l_q - expans(f)}\right)$.

Now assume that f is non-contractive and g is non-expansive, notice that Y, Z are scaling spaces. Notice that f is non-contractive and therefore $l_q - expans(f) = l_q - dist(f)$ and g is non-expansive and therefore $l_q - contr(g) = l_q - dist(g)$.

Notice that $l_q - expans(g\circ f) \leq l_q - expans(f) = l_q - expans(f) = l_q - dist(f)$ because g is non-expansive, and $l_q - contr(g\circ f) \leq l_q - contr(g) = l_q - contr(g) = l_q - dist(g)$ because f is non-contractive. Notice that $g\circ f : X \to Z$ and Z is scaling space.

Therefore due to what we proved there exists $\hat{h} : X \to Z$ with $l_q - dist(\hat{h}) = O\left(\sqrt{l_q - expans(g\circ f)}\cdot\sqrt{l_q - contr(g\circ f)}\right) = O\left(\sqrt{l_q - dist(f)}\cdot\sqrt{l_q - dist(g)}\right)$, as required.

(d) First notice that we saw in class that $\forall x$, s.t. $\|x\|_2 = 1$ it holds that $expans_g(x)^q = \left(\frac{Z}{k}\right)^{\frac{q}{2}}, contr_g(x)^q = \left(\frac{k}{Z}\right)^{\frac{q}{2}}$ when $Z \sim \chi_k^2$.

Notice that $expans_g(x), contr_g(x)$ are not functions of x, therefore $\max\{expans_g(x), contr_g(x)\}$ is also not a function of x.

Therefore, define $\forall x$ s.t. $\|x\|_2 = 1, \mathbb{E}\left[dist_g(x)\right] = E\left[\max\{expans_g(x), contr_g(x)\}\right] = c.$

Therefore, due to g being a linear function, $\forall u \neq v \in X, \mathbb{E}\left[dist_g(u,v)\right] = c$

Notice that

$$\mathbb{E}\left[l_q - dist(g)^q\right] \overset{\text{def}}{=} \mathbb{E}\left[\frac{\sum_{u\neq v\in X} dist_g(u,v)}{\binom{n}{2}}\right] = \frac{\sum_{u\neq v\in X}\mathbb{E}\left[dist_g(u,v)\right]}{\binom{n}{2}} = \frac{\sum_{u\neq v\in X} c}{\binom{n}{2}} = \frac{\binom{n}{2}\cdot c}{\binom{n}{2}} = c$$

Therefore, we got that $\forall u \neq v \in X, \mathbb{E}\left[dist_g(u,v)\right] = \mathbb{E}\left[l_q - dist(g)^q\right]$.

Notice that $f(x) = x^{\frac{1}{q}}$ is a concave function and therefore due to Jensen's inequaliy $\mathbb{E}[f(X)] \leq f(\mathbb{E}[X])$.

Therefore:

$$\begin{aligned}
\mathbb{E}[l_q - dist(g\circ f)] &\overset{\text{def}}{=} \mathbb{E}\left[\left(\frac{\sum_{u\neq v\in X} dist_{g\circ f}(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}}\right] = \mathbb{E}\left[\left(\frac{\sum_{u\neq v\in X}\max\left\{expans_{g\circ f}(u,v), contr_{g\circ f}(u,v)\right\}^q}{\binom{n}{2}}\right)^{\frac{1}{q}}\right] \\
&= \mathbb{E}\left[\left(\frac{\sum_{u\neq v\in X}\max\left\{expans_{g\circ f}(u,v), contr_{g\circ f}(u,v)\right\}^q}{\binom{n}{2}}\right)^{\frac{1}{q}}\right] \\
&= \mathbb{E}\left[\left(\frac{\sum_{u\neq v\in X}\max\left\{\frac{d_Z(g(f(u)),g(f(v)))}{d_X(u,v)}, \frac{d_X(u,v)}{d_Z(g(f(u)),g(f(v)))}\right\}^q}{\binom{n}{2}}\right)^{\frac{1}{q}}\right] \\
&= \mathbb{E}\left[\left(\frac{\sum_{u\neq v\in X}\max\left\{\frac{d_Z(g(f(u)),g(f(v)))}{d_Y(f(u),f(v))}\cdot\frac{d_Y(f(u),f(v))}{d_X(u,v)}, \frac{d_X(u,v)}{d_Y(f(u),f(v))}\cdot\frac{d_Y(f(u),f(v))}{d_Z(g(f(u)),g(f(v)))}\right\}^q}{\binom{n}{2}}\right)^{\frac{1}{q}}\right] \\
&\leq \mathbb{E}\left[\left(\frac{\sum_{u\neq v\in X}\max\left\{\frac{d_Y(f(u),f(v))}{d_X(u,v)}, \frac{d_X(u,v)}{d_Y(f(u),f(v))}\right\}^q\cdot\max\left\{\frac{d_Z(g(f(u)),g(f(v)))}{d_Y(f(u),f(v))}, \frac{d_Y(f(u),f(v))}{d_Z(g(f(u)),g(f(v)))}\right\}^q}{\binom{n}{2}}\right)^{\frac{1}{q}}\right] \\
&= \mathbb{E}\left[\left(\frac{\sum_{u\neq v\in X} dist_f(u,v)^q\cdot dist_g(f(u),f(v))^q}{\binom{n}{2}}\right)^{\frac{1}{q}}\right] \\
&\overset{\text{Jensen's-inequality}}{\leq} \left(\mathbb{E}\left[\frac{\sum_{u\neq v\in X} dist_f(u,v)^q\cdot dist_g(f(u),f(v))^q}{\binom{n}{2}}\right]\right)^{\frac{1}{q}} \\
&\overset{\text{f-is-not-random}}{=} \left(\frac{\sum_{u\neq v\in X} dist_f(u,v)^q\cdot\mathbb{E}\left[dist_g(f(u),f(v))^q\right]}{\binom{n}{2}}\right)^{\frac{1}{q}} \\
&= \left(\frac{\sum_{u\neq v\in X} dist_f(u,v)^q\cdot\mathbb{E}\left[l_q - dist(g)^q\right]}{\binom{n}{2}}\right)^{\frac{1}{q}} \\
&= \left(\frac{\sum_{u\neq v\in X} dist_f(u,v)^q}{\binom{n}{2}}\right)^{\frac{1}{q}}\cdot\mathbb{E}\left[l_q - dist(g)^q\right]^{\frac{1}{q}} \overset{\text{def}}{=} l_q - dist(f)\cdot\mathbb{E}\left[l_q - dist(g)^q\right]^{\frac{1}{q}}
\end{aligned}$$

Therefore, we proved that $\mathbb{E}[l_q - dist(g\circ f)] \leq l_q - dist(f)\cdot\mathbb{E}\left[l_q - dist(g)^q\right]^{\frac{1}{q}}$ as required.

2. Denote the vertexes of K_n to be $v_1,\ldots,v_n$. Define $c = d_X(v_1,v_2)$, notice that due to $X = K_n$, $\forall u \neq v \in X, d_X(u,v) = c$

Define $f : K_n \to \mathbb{R}$ in the following way: $\forall i \in [n], f(v_i) = \frac{c \cdot \log(n)}{n} \cdot i$. Define $X = K_n, Y = \mathbb{R}, d_Y = l_1$ Notice that:

$$l_1 - expans(f) \overset{\text{def}}{=} \left(\frac{\sum_{u \neq v \in X} expans_f(u,v)^1}{\binom{n}{2}} \right)^{\frac{1}{1}} = \left(\frac{\sum_{u \neq v \in X} \left(\frac{d_Y(f(u),f(v))}{d_X(u,v)} \right)}{\binom{n}{2}} \right) = \left(\frac{\sum_{v_i \neq v_j \in X} \left(\frac{d_Y(f(v_i),f(v_j))}{d_X(v_i,v_j)} \right)}{\binom{n}{2}} \right)$$

$$= \left(\frac{\sum_{v_i \neq v_j \in X} \left(\frac{d_Y(\frac{c \cdot \log(n)}{n} \cdot i, \frac{c \cdot \log(n)}{n} \cdot j)}{d_X(v_i,v_j)} \right)}{\binom{n}{2}} \right) = \left(\frac{\sum_{v_i \neq v_j \in X} \left(\frac{c \cdot \frac{\log(n)}{n} \cdot |i-j|}{c} \right)}{\binom{n}{2}} \right)$$

$$\leq \left(\frac{\sum_{v_i \neq v_j \in X} \left(\frac{\frac{\log(n)}{n} \cdot n}{1} \right)}{\binom{n}{2}} \right) = \left(\frac{\sum_{v_i \neq v_j \in X} (\log(n))}{\binom{n}{2}} \right) = \left(\frac{\binom{n}{2} \cdot \log(n)}{\binom{n}{2}} \right) = \log(n)$$

We saw in Infy and Dast that the following holds $\forall i < n$: $\sum_{j=i+1}^{n} \frac{1}{|i-j|} = \sum_{k=1}^{n-i} \frac{1}{k} = O(\log(n))$, Therefore:

$$l_1 - contr(f) \overset{\text{def}}{=} \left(\frac{\sum_{u \neq v \in X} contr_f(u,v)^1}{\binom{n}{2}} \right)^{\frac{1}{1}} = \left(\frac{\sum_{u \neq v \in X} \left(\frac{d_X(u,v)}{d_Y(f(u),f(v))} \right)}{\binom{n}{2}} \right) = \left(\frac{\sum_{v_i \neq v_j \in X} \left(\frac{d_X(v_i,v_j)}{d_Y(f(v_i),f(v_j))} \right)}{\binom{n}{2}} \right)$$

$$= \left(\frac{\sum_{v_i \neq v_j \in X} \left(\frac{d_X(v_i,v_j)}{d_Y(\frac{c \cdot \log(n)}{n} \cdot i, \frac{c \cdot \log(n)}{n} \cdot j)} \right)}{\binom{n}{2}} \right) = \left(\frac{\sum_{v_i \neq v_j \in X} \left(\frac{c}{\frac{c \cdot \log(n)}{n} \cdot |i-j|} \right)}{\binom{n}{2}} \right)$$

$$= \frac{n}{\log(n)} \cdot \left(\frac{\sum_{v_i \neq v_j \in X} \left(\frac{1}{|i-j|} \right)}{\binom{n}{2}} \right) = \frac{n}{\log(n)} \cdot \left(\frac{2 \cdot \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{1}{|i-j|}}{\binom{n}{2}} \right)$$

$$\leq \frac{n}{\log(n)} \cdot \left(\frac{2 \cdot \sum_{i=1}^{n} O(\log(n))}{\binom{n}{2}} \right) \leq \frac{n}{\log(n)} \cdot \left(\frac{O(n \cdot \log(n))}{\binom{n}{2}} \right) = O\left(\frac{n \cdot n \cdot \log(n)}{\log(n) \cdot n^2} \right) = O(1)$$

Therefore, we found a function f s.t. $l_1 - expans(f) = \log(n), l_1 - contr(f) = O(1)$. Notice that $\mathbb{R}$ is a norm space and therefore it is scalable metric space. Using the argument from the first part of 1c,there exists $h : K_n \to \mathbb{R}$ s.t. $l_1 - dist(h) = O\left(\sqrt{l_1 - expans(f)} \cdot \sqrt{l_1 - contr(f)}\right) = O\left(\sqrt{\log(n)} \cdot \sqrt{O(1)}\right) = O\left(\sqrt{\log(n)}\right)$, as required.

3. Note that $\forall i \in [n]$, every element of $f(e_i)$ is in U. Therefore $\forall i \in [n], f(e_i) \in U^k$.

   Notice that due to the assumption that $|U| < d^{\frac{1}{k}}$, we get that $|U^k| < \left(d^{\frac{1}{k}}\right)^k = d = n$.

   Notice that we have that $f(e_1), \ldots, f(e_n) \in U^k$ and $|U^k| < n$, therefore due to the pigeon principle $\exists i \neq j \in [n]$ s.t. $f(e_i) = f(e_j)$. Therefore, $d_Y(e_i, e_j) = 0$ and due to the definitions of $l_q - dist(f), REM_q(f) = \infty$ (In $l_q - dist(f)$ we will sum the contr which will be infinity and in $REM_q(f)$ due to dividing by 0).

   Therefore, We got that $l_q - dist(f) = \infty, REM_q(f) = \infty$, as required.

Bonus We will first prove that $d_X^\gamma(u,v)$ is a metric.

1. reflexivity: $\forall u, v \in X$, $d_X^\gamma(u,v) = 0 \iff (d_X(u,v))^\gamma = 0 \iff d_X(u,v) = 0 \overset{d_X-\text{is}-\text{a}-\text{metric}}{\iff} u = v$

2. symmetry: $\forall u, v \in X$, $d_X^\gamma(u,v) = (d_X(u,v))^\gamma \overset{d_X-\text{is}-\text{a}-\text{metric}}{=} (d_X(v,u))^\gamma = d_X^\gamma(v,u)$

3. triangle inequality: Define $f(x,y) = (x+y)^\gamma - x^\gamma - y^\gamma$. Notice that $\frac{\partial f}{\partial x} = \gamma \cdot (x+y)^{\gamma-1} - x^{\gamma-1}$. Due to $\gamma - 1 \leq 0$ and $y \geq 0$ we get that $\forall y \geq 0, \frac{\partial f}{\partial x} \leq 0$. Due to symmetry we get that $\forall x, y \geq 0, \frac{\partial f}{\partial x} \leq 0 \wedge \frac{\partial f}{\partial y} \leq 0$. Notice that $f(0,0) = 0$ therefore, $\forall x, y \geq 0, f(x,y) \leq 0$, therefore $(x+y)^\gamma \leq x^\gamma + y^\gamma$.

   Therefore $\forall u, v, w \in X$, due to $d_X(u,v), d_X(v,w) \geq 0$, we get that:

$$d_X^\gamma(u,w) = (d_X(u,w))^\gamma \overset{d_X-\text{is}-\text{a}-\text{metric}}{\leq} (d_X(u,v) + d_X(v,w))^\gamma \leq (d_X(u,v))^\gamma + (d_X(v,w))^\gamma = d_X^\gamma(u,v) + d_X^\gamma(v,w)$$

Therefore, we proved that $d_X^\gamma(u,v)$ is a metric.