CS-67720 Metric Embeddings Theory **Fall 2021/22**

Exercise 1

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1. (a) Given a weighted tree graph (T, E, w) let the distance between any two points be the length of the path between them in the tree. Any subset $X \subseteq T$ for some weighted tree graph (T, E, w), with the above distance function is called a *tree metric*. Prove that any 3-point metric embeds isometrically in a tree metric.

   (b) Prove that any 3-point metric embeds isometrically in l_1^2.

   (c) Let $G = (V, E)$ be the 4-point star graph $V = \{w, x, y, z\}, E = \{(w, x), (w, y), (w, z)\}$.

      i. Provide an embedding of G into the Euclidean plane with distortion $2/\sqrt{3}$.

      ii. Prove the following Euclidean Poincaré inequality: for every $x, y, z, w \in l_2$:

$$\|x-y\|_2^2 + \|y-z\|_2^2 + \|x-z\|_2^2 \leq 3\left[\|x-w\|_2^2 + \|y-w\|_2^2 + \|z-w\|_2^2\right].$$

      Hint: Consider each coordinate separately and give a characterization for w for which the inequality becomes tight.

      iii. Use the above inequality to give a tight lower bound on the distortion of any embedding of G into Euclidean space.

2. (a) Prove that the n-point equilateral space embeds isometrically in $l_\infty^{O(\log n)}$.

   (b) Prove that for any $0 < \epsilon \leq 1$ the n-point equilateral space embeds into l_p^k with distortion $1+\epsilon$, where $k = O_p(\frac{\log n}{\epsilon^2})$, for all $1 \leq p \leq \infty$. Note that the $O_p(\cdot)$ notation stands for a constant factor depending on p. How does the dimension behave as function of p ? *Hint:* Apply a random embedding into the k-dimensional hypercube, and apply Chernoff bounds.

3. (a) Denote the doubling dimension of metric space X by $\dim(X)$. Prove the following claim: Let (X, d_X) and (Y, d_Y) be any metric spaces. Let $f : X \to Y$ be an embedding with distortion $\alpha \geq 1$. It holds that $\dim(f(X)) = \dim(X) \cdot O(\log \alpha)$.

   (b) Give an example of a d dimensional normed space $(V, \|\|_V)$, and a subset $X \subset V$ such that the subspace spanning its vectors is of full vector space dimension d, and yet its doubling dimension is a constant, i.e. $\dim(X) = O(1)$.