GAME THEORY 1

Exercise 10

The exercise is due 20.6.2021 at 22:00.

1. Three players play the following game: The players choose (simultaneously) whether to show one or two fingers to the other players. If everyone chose the same then nothing happens. Otherwise, the player with the different choice gets 1$ from each one of the other players.

   (a) Write the payoff matrices.

   (b) Find all Nash Equilibrium.

2. The triangle ABC is divided into sub-simplices in a non-trivial way (there exists at least one point on each of the segments AB, AC, BC other than A, B and C). We label the vertices by numbers in $\{1, 2, 3\}$ according to the following rules:

   - A gets 2 or 3, B gets 1 or 3 and C gets 1 or 2.
   - Any vertex on AB gets 3.
   - Any vertex on AC gets 2.
   - Any vertex on BC gets 1.
   - Any other vertex gets 1, 2 or 3.

   Is it true that there must exists a triangle labeled by $1, 2, 3$? Prove or find a counterexample.

3. (**From a test**) We start with a rectangle whose vertices A, B, C, D are numbered by $1, 2, 3, 4$, respectively. We divide this rectangle to smaller rectangles by drawing lines that are in parallel with the segments. We associate to each of the new vertices a number $1, 2, 3, 4$ such that if a vertex is on a segment of the original rectangle, then it is labeled by one of the numbers of the corresponding vertices (for instance, any vertex on AB is labeled by 1 or by 2). Prove that there exists a smaller rectangle labeled by at least three different letters. Is it true that there exists a smaller rectangle labeled by $1, 2, 3$ and 4? (**Hint:** Use a double count argument as in Sperners Lemma).

4. Let $C \subseteq \mathbb{R}^d$ be a closed convex and non-empty set. In the proof of the separating hyperplane theorem we saw that for every $z \notin C$ there exists a point $a \in C$ such that $\|a - z\| \leq \|x - z\|$ for every $x \in C$. We also saw that

$$\langle a - z, x - a\rangle \geq 0$$

   for all $x \in C$.

   (a) Let $z \notin C$. Prove that the closes point in C to Z is unique. Namely, show that if $a, a' \in C$ satisfy that

$$\|a - z\|, \|a' - z\| \leq \|x - z\|$$

   for every $x \in C$ then $a = a'$. Give an example in the case where C is not convex. **Hint:** write

$$\|a - a'\|^2 = \langle a - a', a - a'\rangle = \langle a - z + z - a', a - a'\rangle$$

   and use the inequality $\langle a - z, x - a\rangle \geq 0$. Or, equivalently draw the picture and use high-school geometry.

(b) Let $\Psi\colon \mathbb{R}^d \to C$ be the map that sends z to its closest point in C. Prove that for every $y, z \in \mathbb{R}^d$ we have,

$$\|\Psi(y) - \Psi(z)\| \leq \|y - z\|$$

and conclude that Ψ is continuous. **Hint:** show that

$$\langle \Psi(z) - z, x - \Psi(z) \rangle \geq 0$$

for every $x \in C$ and $z \in \mathbb{R}^d$. Use this inequality and the Cauchy-Schwartz inequality to prove that

$$\|\Psi(y) - \Psi(z)\|^2 \leq \|y - z\| \cdot \|\Psi(y) - \Psi(z)\|$$

5. Let $\Delta_3 \subseteq \mathbb{R}^3$ the two dimensional simplex in $\mathbb{R}^3$. Let $f : \Delta_3 \to \mathbb{R}^3$ be the following function

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \frac{y+z}{2} \\ \frac{y}{2} \\ \frac{2x+z}{2} \end{pmatrix}$$

(a) Prove that $\mathrm{Im} f \subseteq \Delta_3$.

(b) Find all of the fixed points of f.