Exercise 2

Lecturer: Yair Bartal

1. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Let $f : X \to Y$, and $g : Y \to Z$ be any embeddings. Let $n = |X|$. In what follows, if $|Y| > n$ we use the notation ℓ_q-*dist*(g) to mean ℓ_q-*dist*$(g|_{f(X)})$, where $g|_{f(X)}$ is the restriction of g to $f(X)$. Also, note that f and g may be assumed to be bijections, otherwise the values of ℓ_q-*dist* would be infinite.

   (a) Provide an example of such metric spaces on n-points ($n > 2$) such that $\ell_1\text{-}dist(g \circ f) > \ell_1\text{-}dist(f) \cdot \ell_1\text{-}dist(g)$. Moreover, provide an example where $\frac{\ell_1\text{-}dist(g\circ f)}{\ell_1\text{-}dist(f)\cdot \ell_1\text{-}dist(g)} \to \infty$ as $n \to \infty$.

   (b) Prove that for any $1 \le q < \infty$, for (p, s) s.t. $\frac{1}{p} + \frac{1}{s} = 1$, it holds that

   $$\ell_q\text{-}dist(g \circ f) \le \ell_{qp}\text{-}dist(f) \cdot \ell_{qs}\text{-}dist(g).$$

   (c) Consider the case where the target space Z is scalable (e.g. normed spaces):

   Definition 2.1 (Scalable Metric Space)**.** *A metric space (M, d) is called scalable if for all $A \subset M$ and $\alpha > 0$ we have that the dilation of A by factor α, i.e. the metric space $(A, \alpha d)$, embeds isometrically in M.*

   Assume Y is scalable. Prove that for any $1 \le q < \infty$, there exists an embedding $h : X \to Y$ such that $\ell_q\text{-}dist(h) = O\left(\sqrt{\ell_q\text{-}expans(f)} \cdot \sqrt{\ell_q\text{-}contr(f)}\right)$. Use this to prove that if Z is scalable, f is non-contractive, g is non-expansive, then for any $1 \le q < \infty$ there exists an embedding $\hat{h} : X \to Z$ such that

   $$\ell_q\text{-}dist(\hat{h}) = O\left(\sqrt{\ell_q\text{-}dist(f)} \cdot \sqrt{\ell_q\text{-}dist(g)}\right).$$

   (d) Let $f : X \to \ell_2^d$ be any embedding. Denote by $g : \ell_2^d \to \ell_2^k$ the JL transform (for some $d \ge k$) for which we have shown a bound on the expected ℓ_q distortion. Prove that for any $1 \le q < \infty$ it holds that:

   $$E\left[\ell_q\text{-}dist(g \circ f)\right] \le \ell_q\text{-}dist(f) \cdot \left(E\left[(\ell_q\text{-}dist(g))^q\right]\right)^{1/q}.$$

   (Note: We have used this in class to obtain an approximation the the optimal embedding via the bound proved for JL: $\left(E\left[(\ell_q\text{-}dist(g))^q\right]\right)^{1/q} = 1 + O(q/k) + O(1/\sqrt{k})$.)

2. Provide an embedding of K_n into $\mathbb{R}$ with $\ell_1\text{-}dist(f) = O(\sqrt{\log n})$. Hint: Use a claim stated in Question 1(c). (Note: You may use the claim even if you didn't solve this question).

3. The following shows that we can't get similar ℓ_q-distortion bounds to those of the JL analysis we've seen in class if the implementation of the JL transform uses only a fixed

set of values (independent of n). Let $E_d \subset \ell_2^d$ be the set of the standard basis vectors (of size $n = d$). Assume that the linear transformation $f : E_d \to \ell_2^k$ is given by $f(x) = T \cdot x$, for $x \in E_d$, where T is a $k \times d$ matrix. Assume that all the values of entries of the matrix T belong to the set U. Show that if $|U| < d^{1/k}$ then ℓ_q-$dist(f) = \infty$, and $REM_q(f) = \infty$, for any $1 \leq q \leq \infty$.

BONUS: Let (X, d) be any metric space. Given any real $0 < \gamma \leq 1$, denote d^γ the function defined by $d^\gamma(x, y) = (d(x, y))^\gamma$. Prove that (X, d^γ) is a metric space. Prove that any n-point set $X \subset \mathbb{R}^d$ with metric $(\ell_1)^{1/p}$ can be embedded into ℓ_p with distortion $1 + \epsilon$, for any given $\epsilon \leq 1/2$,and any $p \geq 1$.