Computability - Exercise 2

All questions should be answered *formally* and *rigorously* unless stated otherwise

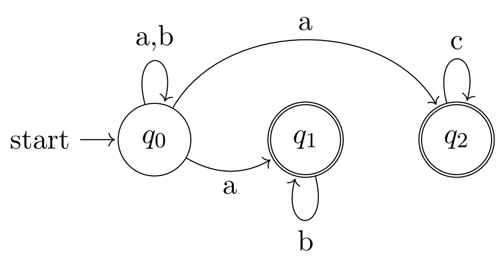
Due: April 23, 2020

Question 1

1. Let $L = \{w \in \{a,b\}^* | \text{ every } b \text{ in } w \text{ appears immediately after three successive } a\text{'s}\}$. Describe a DFA for the language L. Briefly explain your answer (no need to formally prove correctness).

2. Let $L = \{w_1 \cdot w_2 \in \{a,b\}^* | \ w_1 \text{ starts with an } a \text{ and ends with the letter } b, \text{ and } w_2 \text{ contains } abab\}$. Describe an NFA for L. Briefly explain your answer (no need to formally prove correctness).

3. Determinize the following NFA over $\Sigma = \{a,b,c\}$ using the subset construction. There is no need to draw states that are not reachable from the initial state.



Question 2 - Not For Submission

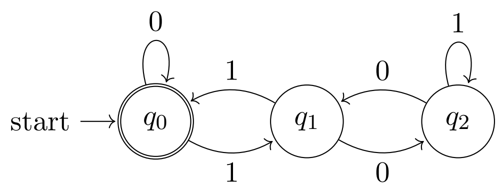
In the recitation, we saw that given an NFA $\mathcal{A} = \langle Q, \Sigma, \delta, Q_0, F\rangle$, one can efficiently construct an equivalent NFA with no ε-transitions. In this question, you will prove the correctness of the construction.

For every state $q \in Q$, let $E(q) = \{q' \in Q | \ q' \text{ is reachable from } q \text{ by } \varepsilon\text{-transitions}\}$. We already saw that the set $E(q)$ can be computed in time polynomial in $|Q| + |\Sigma|$. Consider the NFA $\mathcal{B} = \langle Q, \Sigma, \rho, \bigcup_{q \in Q_0} E(q), F\rangle$, where for every $q \in Q$ and $\sigma \in \Sigma$, we have that $\rho(q, \sigma) = \bigcup_{q' \in \delta(q,\sigma)} E(q')$. Prove that $L(\mathcal{A}) = L(\mathcal{B})$.

Question 3

1. Let $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F\rangle$ be a DFA, and let $q \in Q$ be a state of $\mathcal{A}$. Prove that for every two words $u, v \in \Sigma^*$, it holds that $\delta^*(q, u \cdot v) = \delta^*(\delta^*(q, u), v)$.

2. Describe the language L of the following DFA $\mathcal{A}$, and prove formally that $L(\mathcal{A}) = L$.



Question 4

For a language $L \subseteq \Sigma^*$, we define

- Pref(L) $= \{x \in \Sigma^* | \; \exists u \in \Sigma^*$ such that $x \cdot u \in L\}$.
- Suff(L) $= \{x \in \Sigma^* | \; \exists u \in \Sigma^*$ such that $u \cdot x \in L\}$.

Intuitively, Pref(L) is the language of all the words that are prefixes of words in L, and Suff(L) is the language of all the words that are suffixes of words in L. Prove the following claims:

1. Pref(Pref(L)) = Pref(L).
2. Pref(Suff(L)) = Suff(Pref(L)).
3. If L is regular, then Pref(L) is regular.

Question 5

Let L be a language over Σ. Prove the following claims:

1. (**Not For Submission**) L is regular **if and only if** L^{rev} is regular, where

$$L^{rev} = \{\sigma_n \cdot \sigma_{n-1} \cdots \sigma_1 | \; \sigma_1 \cdots \sigma_{n-1} \cdot \sigma_n \in L, \text{ and for all } 1 \leq i \leq n, \text{ we have that } \sigma_i \in \Sigma\}.$$

2. If L is regular, then so is $L_{\frac{1}{2}} = \{w \in \Sigma^* | \; w \cdot w \in L\}$.

   Hint: Let $\mathcal{A}$ be an automaton with $L(\mathcal{A}) = L$. To recognize $L_{\frac{1}{2}}$: for an input word $w \in \Sigma^*$, consider simulating runs of $\mathcal{A}$ on w in parallel. How are the simulated runs related?

3. The regular languages are closed under the "star" operation. That is, if L is regular, then so is $L^* = \{w \in \Sigma^* | \; w = w_1 \cdot w_2 \cdot \ldots \cdot w_n$, where $n \geq 0$ and for all $1 \leq i \leq n$, we have that $w_i \in L\}$. Recall that L^* contains ε as a concatenation of length 0.

   Hint: Given a DFA $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ for L, you can use the following construction for L^*.

   Define $\mathcal{A}' = \langle Q \cup \{q_{\mathsf{start}}\}, \Sigma, \delta', \{q_{\mathsf{start}}\}, \{q_{\mathsf{start}}\} \rangle$, where q_{start} is a new state, where for each $q \in Q \cup \{q_{\mathsf{start}}\}$ and $\sigma \in \Sigma$, we define

$$\delta'(q, \sigma) = \begin{cases} \{\delta(q, \sigma)\} & \text{if } q \in Q \\ \emptyset & \text{if } q = q_{\mathsf{start}} \end{cases}$$

   and

$$\delta'(q, \varepsilon) = \begin{cases} \emptyset & \text{if } q \in Q \setminus F \\ \{q_{\mathsf{start}}\} & \text{if } q \in F \\ \{q_0\} & \text{if } q = q_{\mathsf{start}} \end{cases}$$