# MathTools HW 12

1. Let $x_0, x_1, \dots, x_k \in \mathbb{R}^n$ be points. The set

$$V = \{x \in \mathbb{R}^n \,:\, \|x - x_0\|_2 \leq \|x - x_i\|_2 \text{ for all } 1 \leq i \leq k\}$$

is called the **Voronoi region** around $x_0$ with respect to $x_1, \dots, x_k$.

(a) Prove that $V$ is a polyhedron. That is, express it as $\{x \,:\, Ax \leq b\}$ for appropriate $A$ and $b$.

(b) Let $\mathcal{P} = \{Ax \leq b\}$ be any polyherdron with a non-empty interior.[1] Prove that one can find points $x_0, x_1, \dots, x_k$ (for some $k$) such that $\mathcal{P}$ is the Voronoi region of $x_0$ with respect to $x_1, \dots, x_k$.

*Hint:* Start with the case where $\mathcal{P}$ is just a halfspace, $\mathcal{P} = \{x \,:\, a^\top x \leq b\}$. Given $x_0 \in \mathcal{P}$ in the interior, meaning that $a^\top x_0 < b$, show how to find a point $x_1$ such that $\mathcal{P}$ is the Voronoi region of $x_0$ with respect to $x_1$. Drawing this might help...

2. Suppose you are given two sets of points in $\mathbb{R}^n$, $\{x_1, \dots, x_K\}$ and $\{y_1, \dots, y_L\}$. Your goal is to find a hyperplane that separates these sets: that is, a non-zero vector $0 \neq a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$a^\top x_i \geq b \text{ and } a^\top y_j \leq b \quad \text{for all} \quad 1 \leq i \leq K, 1 \leq j \leq L\,.$$

Assume that the rank of the $(L+K) \times (n+1)$ matrix

$$\begin{bmatrix} x_1^\top & 1 \\ \vdots & \\ x_L^\top & 1 \\ y_1^\top & 1 \\ \vdots & \\ y_L^\top & 1 \end{bmatrix}$$

is $n+1$. Formulate this as an LP feasibility problem.[2]

*Remark:* Be extra careful in formulating your LP, so to ensure that $a = 0, b = 0$ is not a solution!

*Hint:* Prove the following: under the assumption above, if there is a separating hyperplane, then there is also a separating hyperplane for which there exists a coordinate, either $1 \leq i \leq K$ or $1 \leq j \leq L$, for which $a^\top x_i \geq b + 1$ or $a^\top y_j \leq b - 1$. By adding more optimization variables, build an LP for which $(0, 0)$ is not a solution, but this particular $(a, b)$ is. To that end, consider the constraints of the form

$$a^\top x_i \geq b + t_i\,, \quad a^\top y_j \leq b - s_j,$$

[1] That is, there is a point $x_0$ that satisfies all the constraints with strict inequalities: $Ax_0 < b$.

[2] An LP feasibliy problem is the following: given a polytope $\mathcal{P}$ (described via inequalities $Ax \leq b$), either return some $x \in \mathcal{P}$ or declare that $\mathcal{P} = \emptyset$.

where $t_i, s_j$ are variables.

3. Consider the maximization problem $\max_{x \in \mathbb{R}^n} c^\top x$ subject to $\|x\|_\infty \le 1$, where $c = (c_1, \dots, c_n)$ is some given vector.
   (a) Show that this is an LP.
   (b) Write its dual. Show explicitly (without using strong duality) that the value of the dual is $\|c\|_1$ (as you should expect).

4. **Definition:** For a set of vectors $u_1, \dots, u_n$, their **conic hull** is the set

$$\mathrm{cone}(u_1, \dots, u_n) = \left\{ \sum_{i=1}^{n} \alpha_i u_i \;:\; \alpha_i \ge 0 \right\} ,$$

(this is similar to a convex hull, where we drop the requirement $\sum_{i=1}^{n} \alpha_i = 1$).

Let $\mathcal{P} = \{x \;:\; Ax \le b\}$ be a polyhedron, *assumed to be non-empty*. Denote by $a_1, \dots, a_m$ the rows of $A$. Show that $\mathcal{P}$ is bounded **if and only if** $\mathrm{cone}(a_1, \dots, a_m) = \mathbb{R}^n$.

*Hint:* Here is a possible approach: You can transform the question of whether $\mathcal{P}$ is bounded to the question whether a bunch of LPs has finite value (meaning, their maximum is attained). Use duality.

5. Let $A \in \mathbb{R}^{m \times n}$. Show that if there exists $y \in \mathbb{R}^m$ such that $A^\top y \ge 0$ and $b^\top y < 0$, then the set

$$\{x \;:\; Ax = b\,, x \ge 0\}$$

is empty.

6. (Complementary slackness in linear programming). Consider the following LP:

$$(P) \quad \text{maximize} \quad c^\top x \quad \text{subject to} \quad Ax \le b\,,$$

and its dual:

$$(D) \quad \text{minimize} \quad b^\top y \quad \text{subject to} \quad A^\top y = c\,, y \ge 0\,.$$

Suppose both problems are feasible, and attain their optima in the points $x, y$ respectively.

Prove that for *every* index $i$, *either* the primal constraint is tight, meaning $(Ax)_i = b_i$, *or* $y_i = 0$.