Computability - Exercise 12

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: July 22, 2020

Question 1

The language $ALL_{NFA} = \{\langle \mathcal{A} \rangle : \mathcal{A} \text{ is an NFA and } L(\mathcal{A}) = \Sigma^*\}$ is known to be PSPACE-complete. Use this fact in order to prove that the language

$$MIN_{NFA} = \{\langle \mathcal{A}, k \rangle : \text{there exists an NFA } \mathcal{B} \text{ with } L(\mathcal{B}) = L(\mathcal{A}) \text{ and } \mathcal{B} \text{ has at most } k \text{ states}\}$$

is also PSPACE-complete.

Note: We restrict ourselves to *total* NFAs, that is for every state q and letter σ, there is a σ-labeled transition going out from q.[1]

Question 2

Can there exists a solution to TQBF by polynomial-time nondeterministic TM? Describe the relationship between the classes P, PSPACE, coNP, and NP, in case such a solution exists.

Question 3

We define a log-space verifier to be a Turing machine with three tapes as described below, where the input is a pair $\langle x, y \rangle$ that is written on two of the tapes. The three tapes have the following roles:

- The input tape: A read-only tape, on which the word x is written.
- The witness tape: This is a read-only one-way tape, namely the machine cannot write on this tape, and the head is only allowed to move to the right on it. The word y is written on the witness tape.
- The work tape: this is the usual work tape for space bounded machines. Since we want the verifier to be log-space, the verifier machine is only allowed to use $O(\log |x|)$ space. Note that the space depends only on x, and not on y.

1. Prove that a language A is in NL if and only if these exists a log-space verifier V such that $A = \{x : \text{ there exists } y \text{ such that } \langle x, y \rangle \in L(V)\}$.
2. Suppose we remove the restriction on the witness tape, allowing the head to move both ways on it (but leaving it to remain read only). Show that every language in PSPACE has such a verifier.

[1] It is easy to see that every NFA can be converted to a total NFA by adding a sink and a transition to the sink for every q and σ that do not satisfy this condition.

Question 4

1. Show that the following languages are NL-complete:

   (a) $\mathsf{2\text{-}PATH} = \{\langle G, s, t\rangle : G$ is a directed graph, and there exist at least two different paths in G from s to $t\}$.

   (b) $E_{\mathsf{NFA}} = \{\langle \mathcal{A} \rangle : \mathcal{A}$ is an NFA and $L(\mathcal{A}) = \emptyset\}$.

2. We define the following language:

$$2SAT = \{\langle \varphi \rangle : \varphi \text{ is a satisfiable 2CNF formula}\}$$

   In the recitation we saw that $2SAT \in \text{NL}$. Prove that $2SAT$ is NL-hard to conclude that $2SAT$ is NL-complete.

Question 5

In the *iterated matrix multiplication problem*, we are given a sequence of $n \times n$ Boolean matrices and two indices i and j, and we have to decide whether the value of entry i, j in the product of the matrices is 1.

Reminder: we define the product C of two Boolean matrices A, B by $C_{ij} = \bigvee_{k=1}^{n} A_{ik} B_{kj}$.

Show that the iterated matrix multiplication problem is NL-complete.