Computability - Exercise 7

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: May 31, 2020

Question 1 (not for submission)

(a) Show that reductions are reflexive. That is, show that for every language L over Σ, it holds that $L \leq_m L$.

(b) Show that reductions need not be anti-symmetric. That is, show that there exist two languages L_1 and L_2 over Σ such that $L_1 \neq L_2$ and still $L_1 \leq_m L_2$ and $L_2 \leq_m L_1$.

(c) Show that reductions are transitive. That is, show that for every three languages L_1, L_2 and L_3 over Σ, it holds that if $L_1 \leq_m L_2$ and $L_2 \leq_m L_3$, then $L_1 \leq_m L_3$.

Question 2

(a) We say that a language L is *RE-hard* if for every language $L' \in RE$, it holds that $L' \leq_m L$. Prove that A_{TM} is RE-hard.

(b) We say that a language L is *R-hard* if for every language $L' \in R$, it holds that $L' \leq_m L$. Prove that every nontrivial language is R-hard. That is, for every $L \notin \{\emptyset, \Sigma^*\}$, it holds that L is R-hard.

Question 3 (2,3,5,7 not for submission)

For each of the following languages, determine whether it is in R, $RE \setminus R$, $coRE \setminus R$ or $\overline{RE \cup coRE}$. Prove your answers formally, don't use Rice's Theorem in your proofs (in all questions, M is a TM).

1. $L = \{\langle M, w\rangle : M$ accepts w and uses at most $|w|$ tape cells during the run on $w\}$.
2. $L = \{\langle M\rangle :$ there exists $w \in \Sigma^*$ such that M accepts w after at least $|w|$ steps$\}$.
3. $REACH_{TM} = \{\langle M, q\rangle : q \neq q_{acc}$ and M reaches the state q on every input$\}$.
4. $L = \{\langle M\rangle : L(M) = A_{TM}\}$.
5. $RE_{TM} = \{\langle M\rangle : L(M) \in \text{RE}\}$.
6. $MIN_{TM} = \{\langle M, k\rangle :$ there exists a TM D such that $L(M) = L(D)$ and D has less than k states$\}$.
7. $L_7 = \{\langle M\rangle :$ there exists w such that M uses unboundedly many tape cells in its run on $w\}$
8. $NONTRIVIAL_{TM} = \{\langle M\rangle : L(M) \neq \emptyset$ and $L(M) \neq \Sigma^*\}$.
9. $L = \{\langle M\rangle :$ There does not exist $w \in \Sigma^*$ such that M rejects $w\}$

Question 4 (from a midterm exam)

1. Let $L_1, L_2 \subseteq \Sigma^*$ and let $f, g : \Sigma^* \to \Sigma^*$ be two computable functions such that for every $x \in \Sigma^*$, it holds that $x \in L_1 \iff f(x) \in L_2$ and $g(x) \notin L_2$. Prove / Disprove the following claims:
   (a) If $L_2 \in R$ then $L_1 \in R$.
   (b) If $L_2 \in RE$ then $L_1 \in RE$.
2. Prove that there exists $L_2 \notin RE$ such that $\overline{L_2} \leq_m L_2$.
   Guidance: Consider "tweaking" some well-known undecidable language.

Question 5 (from an exam)

A *zig-zag enumerator* that correpsonds to a language $L \subset \Sigma^\star$ is a an enumerator that prints a sequence of words, $w_1, w_2, \ldots$ such that

$$\begin{aligned} L &= \{w_1, w_3, w_5, \ldots\} \\ \bar{L} &= \{w_2, w_4, w_6, \ldots\} \end{aligned}$$

That is, this enumerator (eventually) prints all words in $\Sigma^\star$, alternating between words in L and in $\bar{L}$ (and possibly writing the same word more than once). Prove or refute the following statement: For every **non-trivial** language $L \subset \Sigma^\star$, we have that $L \in R$ iff there exists a zigzag enumerator corresponding to L.