# GAME THEORY 1

## Exercise 7

The exercise is due 30/5/2021 at 22:00.

1. Prove that the mixed strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is optimal for both players in Rock Paper Scissors.

2. Let $A \in M_{n\times n}(\mathbb{R})$ be a square payoff matrix of a zero-sum game. Suppose that there exists some $v \in \mathbb{R}$ and $x, y \in \Delta_n$ such that

$$A \cdot y = \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix}, \; x^T \cdot A = \begin{pmatrix} v, & \cdots & ,v \end{pmatrix}.$$

   Prove that $v$ is the value of the game, $x$ is optimal for player 1 and $y$ optimal for player 2. **Hint:** Look at the statement in Lecture notes 6 section 3.1.

3. Player 2 chooses a number $j \in \{1, 2, 3, 4\}$ and player 1 trying to guess the number. If player 1 guessed correctly he gets 1\$. If he guessed a greater number than the one player 2 had chosen, he gets 0.5\$. In any other case he gets nothing. Write the payoff matrix of the game and find the value and one optimal strategy for each player. **Hint:** Guess that there are optimal strategies as in Question 2. Let $(y_1, y_2, y_3, y_4)$ be a mixed strategy of player 2 and $v$ the value of the game. The matrix equation in Question 2 leads to a (simple) linear equations in 5 variables $y_1, y_2, y_3, y_4, v$, solve it. Then, remember that $(y_1, y_2, y_3, y_4) \in \Delta_4$ which gives you another equation. Use the same method for player 1.

4. Let $G_1, G_2$ be two zero-sum games with payoff matrices

$$A_1 \in M_{m_1\times n_1}(\mathbb{R}), \;\; A_2 \in M_{m_2\times n_2}(\mathbb{R})$$

   We define their sum $G_1 \oplus G_2$ by the matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

   where 0 indicates a block of zeros of the correct size. Let $v_1$ and $v_2$ denote the values of $G_1$ and $G_2$ respectively.

   (a) Suppose that $v_1, v_2 \geq 0$ and that $(v_1, v_2) \neq (0, 0)$. For every $i \in \{1, 2\}$, let $x_i^*, y_i^*$ be optimal strategies in $G_i$ for player 1 and 2 respectively. We define the following strategies in $G_1 \oplus G_2$:
   - The strategy $x^*$ of player 1 is to play $x_1^*$ in $G_1$ with probability $\frac{v_2}{v_1+v_2}$ and $x_2^*$ in $G_2$in probability $\frac{v_1}{v_1+v_2}$.
   - The strategy $y^*$ of player 2 is to play $y_1^*$ in $G_1$ with probability $\frac{v_2}{v_1+v_2}$ and $y_2^*$ in $G_2$in probability $\frac{v_1}{v_1+v_2}$.

   Prove that $x^*$ and $y^*$ are optimal in $G_1 \oplus G_2$ and the value is $\frac{v_1 v_2}{v_1+v_2}$.

   (b) Suppose now that $v_1, v_2 \leq 0$ and $(v_1, v_2) \neq 0$. What is the value of the game? Describe an optimal strategy for each player.

5. Let

$$A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \in M_{n\times n}(\mathbb{R})$$

be the payoff matrix of a zero-sum game. (A is a diagonal matrix).

(a) Suppose that $\lambda_1, ..., \lambda_n > 0$ and find the value and optimal strategies for the players. **Hint:** Use Question 4 (you can also use Question 2).

(b) Suppose that $\lambda_1, ..., \lambda_n < 0$, find the value and optimal strategies for the players.

(c) Suppose that there exists $i, j$ such that $\lambda_i > 0$ and $\lambda_j < 0$. What is the value of the game? Find optimal strategies for the players. (**Hint:** Dominant strategies)

(d) Player 2 chooses a number $j \in \{0, 1, 2, 3, 4\}$. Player 1 guesses the number. If player 1 succeeds he gets $2^j$ dollars (and player 2 loses that much). If he fails, he gets nothing. What is the value of the game?