Computability - Exercise 1

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: April 2nd, 2020

Question 1

Let Σ be some (non-empty and finite) alphabet, and let $L_1, L_2, L_3 \subseteq \Sigma^*$ be some languages.
Prove/disprove:

1. $L_1 \cdot L_2 = L_2 \cdot L_1$ (I.e., the operation of concatenation is commutative).
2. If L_1 is not empty and $L_1 \cdot L_2 = L_1 \cdot L_3$, then $L_2 = L_3$.
3. If $\Sigma = \{1\}$ and L_1 is an infinite language (i.e., $|L_1| = \aleph_0$) with $\epsilon \in L_1$, then $L_1 = L_1^*$.

Question 2

1. Describe a DFA (deterministic finite automaton) for each of the following languages. A drawing would suffice, but only if is exact and contains all the information needed to create from it a formal description of the automaton. No need to formally prove the correctness of your constructions. All languages are over the alphabet $\Sigma = \{0, 1\}$.
   (a) $L_a = \{w : w \text{ contains both a 0 and 1 }\}$
   (b) $L_b = \{w : \text{if } w \text{ contains a 1, then } w \text{ contains the sequence } 00\}$
2. Let $\Sigma = \{a, b\}$. What is the language of the DFA described in the figure below? Justify your answer shortly (For example, you can describe the role of each state). You don't need to prove your answer.

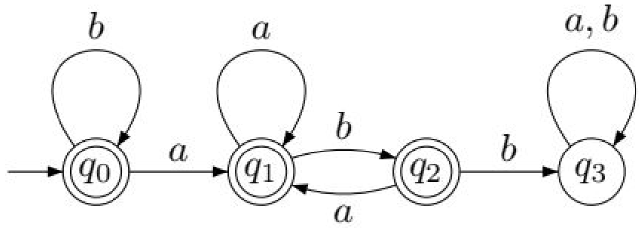

Figure 1: What's my language?

3. Give a formal description of the automaton above. That is, define it in terms of $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$.

Question 3

Let Σ be a finite alphabet.

1. Prove that the language $\emptyset$ is regular.
2. Prove that for every word $w \in \Sigma^*$, the language $\{w\}$ is regular.
3. Conclude that every finite language is regular.

Question 4

Let L_1 and L_2 be regular languages over the same alphabet. In class we saw that $L_1 \cup L_2$ is also regular. That is, the regular languages are closed under union.

1. Prove that regular languages are closed under complementation (show that if L is regular, then $\bar{L}$ is regular as well).

2. Conclude that regular languages are also closed under intersection.

Question 5 (Not for submission)

Let Σ be a finite alphabet. Let $\mathcal{F} = \{L \subseteq \Sigma^* : L \text{ is finite}\}$.

1. What is $\bigcap_{L\in\mathcal{F}} L$?

2. Is $\bigcup_{L\in\mathcal{F}} L$ finite?

3. What is $\bigcap_{L\notin\mathcal{F}} L$?

4. Decide true/false: $\overline{\mathcal{F}} = \{\overline{L} : L \in \mathcal{F}\}$.

Question 6 (Not for submission)

In this question we will show that an attempt to define a DFA by means of its generalized transition function can turn out to be erroneous and inconsistent. This justifies the given definition which relies on its transition function.
Let Q and Σ be two fixed finite nonempty sets, such that $q_0 \in Q$ and $|Q| > 1$.

1. Show using a counting argument, similar to what we did in class, that there exists a function $\alpha : Q \times \Sigma^* \to Q$ with $\alpha(q, \epsilon) = q$ for all $q \in Q$, for which there does not exist any function δ such that $\langle Q, \Sigma, \delta, q_0, Q\rangle$ is a DFA with $\delta^* = \alpha$.

2. Provide a specific function α maintaining this property.