Computability - Exercise 4

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: May 7, 2020

Question 1

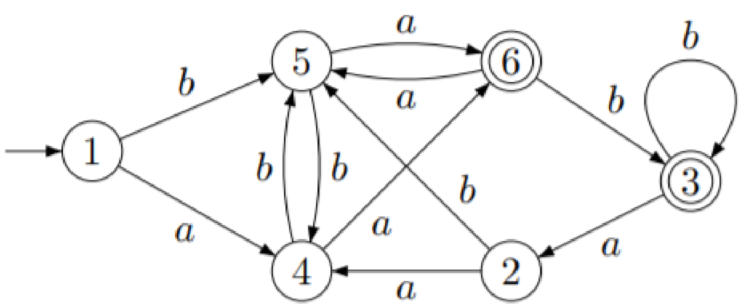
Recall that in the DFA minimization algorithm we saw in class, we calculated, given a DFA $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F\rangle$, an equivalence relation $\equiv \subseteq Q \times Q$ (the fixed-point of the $\equiv_i$ relations), and we proved that for every two states $q, s \in Q$, we have that $q \equiv s$ iff for every $z \in \Sigma^*$, it holds that $\delta^*(q, z) \in F$ iff $\delta^*(s, z) \in F$.

Let $Q/\equiv$ denote the set of equivalence classes of $\equiv$. That is, $Q/\equiv = \{[q] : q \in Q\}$. Consider the DFA $\mathcal{A}' = \langle Q/\equiv, \Sigma, \delta', [q_0], F'\rangle$, where $F' = \{[q] : q \in F\}$ and $\delta'([q], \sigma) = [\delta(q, \sigma)]$.

Assume all states in Q are reachable from q_0.

1. (a) Prove that $\equiv$ is indeed an equivalence relation.
   (b) Prove that δ' is well defined. That is, if $q \equiv s$, then $\delta(q, \sigma) \equiv \delta(s, \sigma)$ for every $\sigma \in \Sigma$.
   (c) Prove that $L(\mathcal{A}') = L(\mathcal{A})$.
   - Hint: prove that for every word $w \in \Sigma^*$, we have that $\delta'^*([q_0], w) = [\delta^*(q_0, w)]$, and show it implies that $L(\mathcal{A}) = L(\mathcal{A}')$.

   (d) Prove that $\mathcal{A}'$ has a minimal number of states, out of all the DFAs recognizing $L(\mathcal{A})$.
   - Hint: Assume by way of contradiction that there is a DFA $\mathcal{A}''$ with fewer states than $\mathcal{A}'$ such that $L(\mathcal{A}'') = L(\mathcal{A})$.
2. Minimize the following DFA:



Question 2

For the language $L \subseteq \Sigma^*$, we define a new equivalence relation $\approx_L \subseteq \Sigma^* \times \Sigma^*$, as follows:

$$x \approx_L y \iff \forall z, w \in \Sigma^*, \text{ it holds that } zxw \in L \iff zyw \in L$$

Show that L is regular $\iff \approx_L$ has a finite number of equivalence classes.

Hint: Recall that in the proof of the Myhill-Nerode theorem we saw in class, we showed that if L is regular, then it has a finite number of $\sim_L$ equivalence classes by defining an equivalence relation $\sim_{\mathcal{A}} \subseteq \Sigma^* \times \Sigma^*$ induced by a DFA for L: for $x, y \in \Sigma^*$, we have that $x \sim_{\mathcal{A}} y$ iff $\delta(q_0, x) = \delta(q_0, y)$. We then showed that for all $w, w' \in \Sigma^*$, if $w \sim_{\mathcal{A}} w'$ then $w \sim_L w'$. Make sure you understand why it proves that the number of equivalence classes of the Myhill-Nerode relation is finite.

A similar approach can be used here.

Question 3 (Not for submission, from exam)

For the language $L \subseteq \Sigma^*$, we define a new equivalence relation $\approx_L \subseteq \Sigma^* \times \Sigma^*$, as follows:

$$x \approx_L y \iff \forall z \in \Sigma^* \text{ such that } |z| \text{ is even, it holds that } xz \in L \iff yz \in L.$$

That is, $x \approx_L y$ iff there is no even-length separating suffix between x and y. For example, let $L = \{a^n : n \equiv 0 \mod 6\}$. Then it holds that:

- $a \approx_L a^3$, since for every $k \in \mathbb{N}$ it holds that $a^{2k+1} \notin L$ and $a^{2k+3} \notin L$.
- $a^2 \not\approx_L a^4$, because a^2 is a separating suffix of even length.

1. Let $L = (ab)^*$. How many equivalence classes does the relation $\approx_L$ induce on Σ^*?
2. Show that for all $x, y \in \Sigma^*$, it holds that $x \sim_L y \iff x \approx_L y$, and $x\sigma \approx_L y\sigma$ for all $\sigma \in \Sigma$.

Question 4

1. For each of the following languages, use the Myhill-Nerode theorem in order to decide whether they are regular or not. If they are regular, show that there is a finite number of equivalence classes. Otherwise, show that there are infinitely many equivalence classes.
   (a) $\left\{1^k : k \equiv 0 \pmod 3\right\}$ over $\Sigma = \{1\}$
   (b) $\left\{a^i b^j c^k : i + j = k\right\}$ over $\Sigma = \{a, b, c\}$
   (c) (**Not for submission**) $\left\{0^i 1^j : i > j\right\}$ over $\Sigma = \{0, 1\}$
2. Let $L = \{w \cdot z \in \{a, b\}^* : w \neq \epsilon \land \#_a(w) = \#_b(w)\}$. That is, the language of words that have a non empty prefix with the same number of a's and b's. Is $L \in REG$? Use the Myhill-Nerode theorem to prove your answer.

Question 5

1. For each of the following languages over $\Sigma = \{0, 1\}$, describe a CFG that generates the language. Briefly explain why your construction is correct.
   (a) $L_1 = \{w : w \neq w^{rev}\}$.
   (b) $L_2 = \{w \in \{0, 1\}^* : \#_0(w) = \#_1(w)\}$.
   (c) $L_3 = \{w \in \{0, 1\}^* :$ in every prefix of w there are at least as many 0's as there are 1's$\}$.
2. For each of the following grammars, what is the generated language? Briefly explain your answers.
   (a) $G_1 = \langle\{S, A\}, \{a, b, c\}, R_1, S\rangle$, with R_1 as follows.
   $$S \to aSc|A$$
   $$A \to aAb|\varepsilon$$
   (b) $G_2 = \langle\{S, A, B, C\}, \{0, 1\}, R_2, S\rangle$, with R_2 as follows.
   $$S \to CSC|A$$
   $$A \to 0B1|1B0$$
   $$B \to CB|\varepsilon$$
   $$C \to 1|0$$