GAME THEORY 1

Exercise 11

The exercise is due 27.6.2021 at 22:00.

1. A cooperative game (N,ν) is called additive if for every coalition $S \subseteq N$ we have $\nu(S) = \sum_{i\in S} \nu(\{i\})$. What is the Shapley value of player $i \in N$ in an additive game?

2. For every cooperative game (N,ν) we define its dual $(N,\nu^\star)$ by

$$v^*(S) = v(N) - v(N\backslash S)$$

   (a) Prove that the dual of the dual is the original game, namely $(\nu^\star)^\star = \nu$.

   (b) Prove that the Shapley value of player $i \in N$ in the game ν is the same as its Shapley value in $\nu^\star$. Namely, prove that $\psi_i(\nu) = \psi_i(\nu^\star)$.

3. **(Weighted majority games)** Compute the Shapley value of each player in the following games:

   (a) **(One big party and three small parties)** $\nu = [5; 3, 2, 2, 2]$**.**

   (b) **(Two big parties and three small parties)** $\nu = [5; 3, 3, 1, 1, 1]$**.**

   (c) **(Almost a dictator)** $n > 3$ and $\nu = [2n-2; 2n-3, 2, 2, 2...., 2]$. What happens when $n \to \infty$?

   (d) $n+1$ players, $\nu = [\frac{n}{2}; \frac{n}{3}, 1, 1..., 1]$. For simplicity assume that n is divisible by 3. What happens when $n \to \infty$?

4. **(The Knesset)** The Knesset consists of two camps. Camp R has a total of 56 mandats and camp L a total of 57, and one other party (which we denote by Y) that does not belong to either of the camps has 7 mandates. The parties in Camp R do not want to form a coalition with the parties in Camp L and vice versa. Therefore, they can only form a coalition if they cooperate with the third party. We formulate this situation as a coorperative game where $N = \{R, L, Y\}$ and

$$\begin{aligned} \nu(R) = \nu(L) = \nu(Y) = 0 \\ \nu(R,Y) = \nu(L,Y) = 1 \\ \nu(L,R) = 0,\ \nu(R,L,Y) = 1 \end{aligned}$$

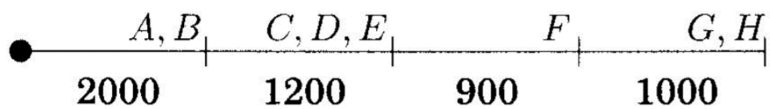
   Compute the Shapley value of each of the players.

5. Let N be a finite set and $S \subseteq N$. We define a cooperative game (N, u_S) by

$$u_S(T) = \begin{cases} 1 & T\cap S \neq \emptyset, \\ 0 & T \cap S = \emptyset. \end{cases}$$

   compute the Shapley value of each player.

6. (**Airport problem**) Many different airline companies are using the same airport. As a result they need to pay for its maintence together. However, smaller companies use smaller planes which requires a shorter landing strip. In this question we will use the Shapley value in order to determine how the cost is supposed to be splitted among the different airlines.
We define a cooperative game (N, ν) where N is the set of all airline companies. For every coalition S, $\nu(S)$ is the maintaince cost for the **shortest landing strip** that is used by the coalition.
In the following figure there is an example with eight airline companies A, B, C, D, E, F, G, H. The location of the company in the figure determines the part of the landing strip that their airplans require in order to land safely. For example, the air plans of company F needs the first three landing strips. The number below each segment is the maintaince cost of that section. For example $\nu(\{C, F, G\}) = 2000+1200+900+1000 = 5100$ and $\nu(\{A, D, E\}) = 2000+1200 = 3200$.



Prove that the Shapley value suggests that the maintaince cost of each section will be splitted equally by the airplans which uses that section. For example

$$\psi_A(\nu) = \frac{2000}{8} = 250$$

because all of the airplans are using the first landing strip. While

$$\psi_F(\nu) = \frac{2000}{8} + \frac{1200}{6} + \frac{900}{3}$$

because airline F pays for the first section together with everyone else, for the second section together with C, D, E, G, H and for the third section together with G, H. **Hint:** Write ν as a linear combination $\sum_i c_i u_{B_i}$ for $B_i \subseteq N$ where u_{B_i} are the games from Question 5.