Computability - Exercise 9

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: June 11, 2020

Question 1

In this question, we consider closure properties of P and NP. Let $L_1, L_2 \in \text{P}$ and $L_3, L_4 \in \text{NP}$. Prove the following claims.

1. $\overline{L_1} \in \text{P}$.
2. $L_3 \cap L_4 \in \text{NP}$.
3. $L_1 \cdot L_2 \in \text{P}$.
4. $L_3 \cdot L_4 \in \text{NP}$.

Question 2

We define the class $\text{coNP} = \{\overline{L} : L \in \text{NP}\}$. It is unknown whether NP = coNP. Prove the following claims.

1. **(Not for submission)** If P = NP, then NP = coNP.
2. If NP $\subseteq$ coNP, then NP = coNP.
3. coNP $\subseteq$ EXPTIME.
4. If P = NP then EXPTIME = NEXPTIME.
   Hint: For a language $L \in \text{NEXPTIME}$, consider a language $L' = \{w\#1^{2^{|w|^d}} : w \in L\}$ (i.e., we pad w with the unary representation of $2^{|w|^d}$), where d is a wisely-chosen constant.

Question 3

Let p be some polynomial, and let $N = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\mathsf{acc}}, q_{\mathsf{rej}} \rangle$ be an NTM such that for every input $x \in L(N)$ **there exists** an accepting run of N on x of length at most $p(|x|)$.

Prove that $L(N) \in \text{NP}$.

Question 4 (Not for submission)

König's Lemma: Let T be a rooted tree with infinitely many nodes, such that each node has finitely many children. Then, T contains a *ray*, that is, there exists an infinite sequence $x_0, x_1, x_2, \dots$ of nodes, where x_0 is the root of T, and for each $i \in \mathbb{N}$, x_i is a child of x_{i-1}.

1. Prove König's lemma.
2. Aladdin consumed too much coffee and thought about the following (wrong) idea: Given an NTM N, we build an equivalent NTM N', such that that every run of N' on every input w halts:
   - Nondeterministically write an integer $n \in \mathbb{N}$.
   - Simulate N on w for n steps.
   - If N accepted within n steps, accept. Otherwise, reject.

   Explain Aladdin's mistake.
3. Recall that we defined the runtime function of an NTM N to be

   $$t_N(n) = \max_{w \in \Sigma^*, |w| \leq n} \{\text{length of the longest run of } N \text{ on } w\}$$

   Prove that the runtime of an NTM is well defined. That is, prove that for every deciding NTM N and $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $t_N(n) = k$.

Question 5

In the subset sum problem, the goal is to determine whether in a set $S \subseteq \mathbb{N}$ (given in binary) there is subset whose sum is $k \in \mathbb{N}$. Formally, the language is defined as follows.
$SUBSET - SUM = \{\langle S, k \rangle : S \text{ contains a subset whose sum is } k\}$.

1. Show that $SUBSET - SUM \in$ NP by describing an NTM N that decides $SUBSET - SUM$ in polynomial time (with respect to the input size).
2. Show that $SUBSET - SUM \in$ NP by describing a polynomial time verifier for this language.
3. The language $UNARY - SUBSET - SUM$ is defined similarly, but the numbers are given in unary. That is, $S \subseteq \{1^n : n \in \mathbb{N}\}$ and then

   $UNARY - SUBSET - SUM = \{\langle S, 1^k \rangle : S \text{ contains a subset whose sum of lengths is } k\}$.

   Show that $UNARY - SUBSET - SUM \in$ P.