GAME THEORY 1

Exercise 4

The exercise is due 9/5/2021 at 22 : 00.

1. In this question you are **not** allowed to use the Sprgue-Grundy Theorem. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two progressively bounded impartial games and $G = G_1 \oplus G_2 = (V, E)$. Let $(x_1, x_2) \in V$.

   (a) Prove that if $\mathrm{sg}_{G_1}(x_1) \oplus \mathrm{sg}_{G_2}(x_2) \neq 0$ then there exists a legal move from (x_1, x_2) to a position (y_1, y_2) with
   $$\mathrm{sg}_{G_1}(y_1) \oplus \mathrm{sg}_{G_2}(y_2) = 0.$$

   (b) Prove that if $\mathrm{sg}_{G_1}(x_1) \oplus \mathrm{sg}_{G_2}(x_2) = 0$ then every legal move from (x_1, x_2) end in a position (y_1, y_2) with
   $$\mathrm{sg}_{G_1}(y_1) \oplus \mathrm{sg}_{G_2}(y_2) \neq 0.$$

   (c) Conclude that $(x_1, x_2) \in P$ if and only if $\mathrm{sg}_{G_1}(x_1) \oplus \mathrm{sg}_{G_2}(x_2) = 0$.

2. Recall the substruction game from Tirgul 1: Starting with a pile of m coins. Bob and Alice alternate removing $1, 2, 3$ or 4 coins from the pile. The player who takes the final coin wins.

   (a) Compute the sg values of the positions $x = 0, 1, 2, 3, 4, 5$.

   (b) Prove by induction that $\mathrm{sg}(x) = x \mod 5$ for every $x \in \mathbb{N} \cup \{0\}$.

   (c) Suppose now that there are two piles (m, n). Bob and Alice alternate choosing a pile and removing $1, 2, 3$ or 4 coins from that pile. Compute P and N .

   (d) Generalize the above to k piles $(n_1, n_2, ..., n_k)$.

3. **The game Hackenbush:** In Hackenbush we begin with a finite number of connected undirected graphs. Each graph is “attached” to the floor (The floor is labeled by a dotted line - see figure 1). Bob and Alice alternate cutting a chosen edge. If as a result a part of the graph is no longer attached to the floor, it falls and removed from the game (see figure 2 for example). The player who cuts the last edge wins. Note that if G is an Hackenbush game with k graphs then $G = G_1 \oplus G_2 \oplus ... \oplus G_k$ where G_i is the Hackenbush game with only the i-th graph.

   (a) When the starting position is k sticks (see figure 2), we obtain a version for a game of Nim with k piles. We refer to these graphs as Nim sticks. Computing the Sprague-Grundy value of such graphs is easy (it is the Nim sum of the lengths). Observe, that if we take these graphs and “merge them together” (as in figure 3), we get an equivalent game with the same Sprague-Grundy value. More generally, given k graphs $G_1, G_2, ..., G_k$ where G_i is only attached to the floor in one vertex x_i. We can merge them together to one graph G where $x_1, ..., x_k$ are glued together to one vertex x and nothing else changes (see figure 3). Explain why
   $$\mathrm{sg}(G) = \mathrm{sg}(G_1) \oplus \cdots \oplus \mathrm{sg}(G_k)$$

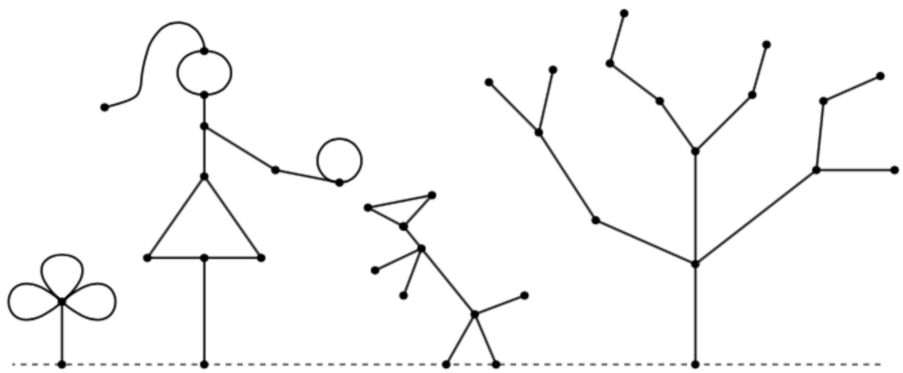


Figure 0.1: An example of a starting position in Hackenbush with four graphs. Taken from Fair Game by Richard Guy.

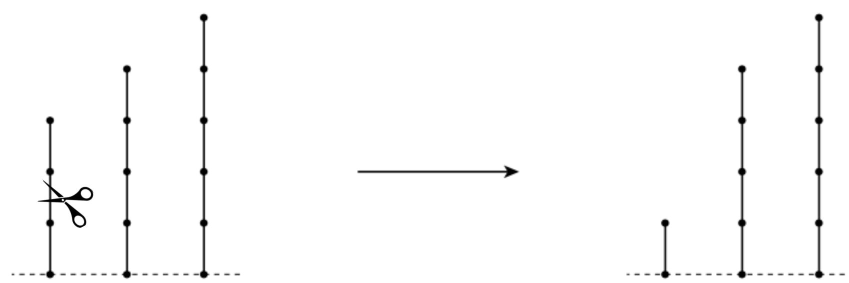


Figure 0.2: A position in Hackenbush which represents a game of Nim with 3 piles of sizes $(3, 4, 5)$ and a legal move.

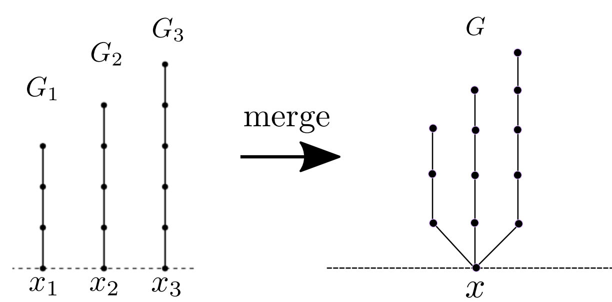
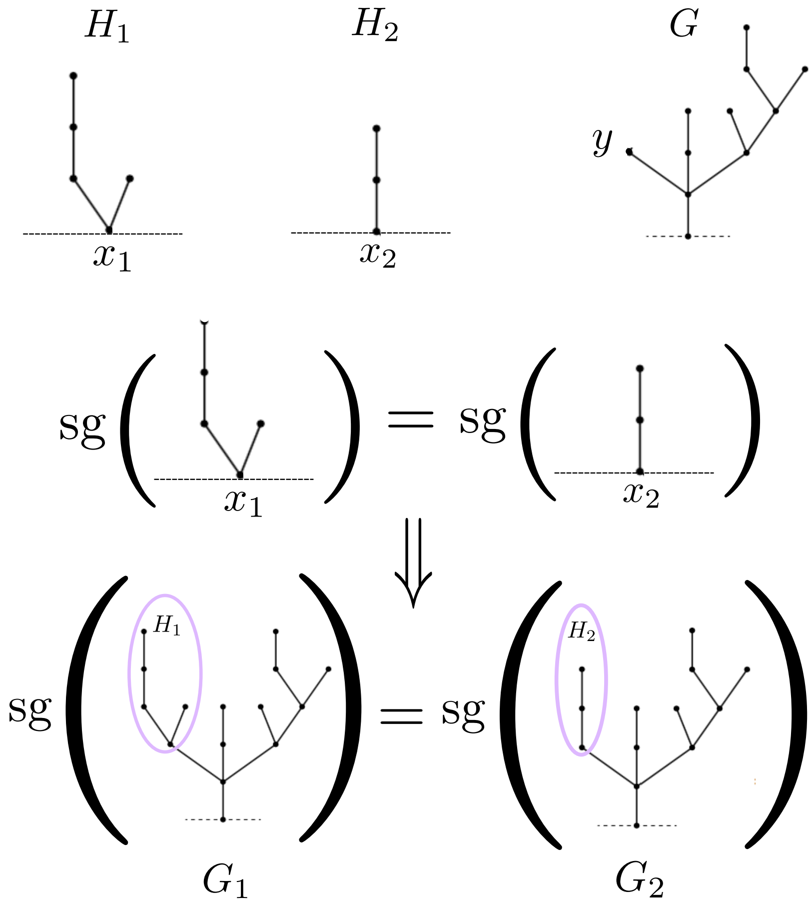


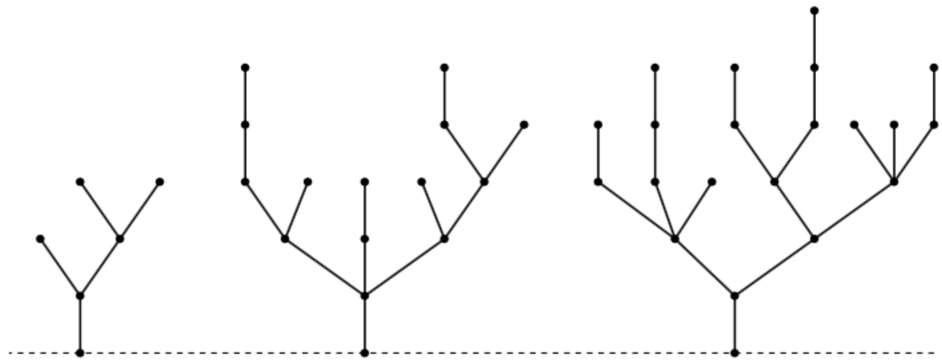
Figure 0.3: Three graphs being merged together in x.



איור 0.4: Merging H_1 and H_2 to a vertex y in G.

(b) Suppose we have two graphs H_1, H_2, where H_1 is attached to the floor in x_1 and H_2 in x_2. Let G be another graph. Given a vertex $y \in G$ we can merge G with H_1 by gluing x_1 to y and get a new graph G_1. Similarly, we can merge G with H_2 by gluing x_2 to y and get a new graph G_2 (see figure 4). If $\mathrm{sg}(H_1) = \mathrm{sg}(H_2)$, prove that $\mathrm{sg}(G_1) = \mathrm{sg}(G_2)$. (Hint: Look at the position (G_1, G_2) in hackenbush with two graphs $(\mathcal{G} \oplus \mathcal{G})$. Show that this position is in P by describing a winning strategy for the second player. Then use that fact that the Nim sum of two integers is zero if and only if they are equal).

(c) Using (a) and (b) we see how to compute the sg number of a tree: In each turn choose a vertex and replace the sub-tree which begins in that vertex with a Nim stick. As in (b) we see that the size of that Nim stick will be the Sprague-Grundy value of that sub-tree. For example, in figure 4, we replaced H_1 with a nim stick of size $2 = 3 \oplus 1 = \mathrm{sg}(H_1)$. We can continue this way until we get one Nim stick, the size of that Nim stick is the Sprague-Grundy value of the original graph. Compute the Sprague-Grundy value of the trees G_1, G_2



איור 0.5: Starting position with three trees.

and G_3 in figure 5. Then, show that the position (G_1, G_2, G_3) in Hackenbush game with three graphs is a position in N.