MathTools HW 8

1. **The ergodic theorem for Markov chains.** Consider an ergodic Markov chain over a finite state space $\mathcal{S}$, with the stationary distribution $\boldsymbol{\pi}$. Recall that for any $f : S \to \mathbb{R}$, the averages $\frac{1}{T}\sum_{t=1}^{T} f(X_i)$ converge in probability,[1] as $T \to \infty$, to the expectation $\mathbb{E}_{X\sim\boldsymbol{\pi}}[f(X)]$.

   (a) For a state $s \in \mathcal{S}$, let T_s be the amount of time the chain spends in s along the first T time steps, in other words,
   $$T_s = |\{1 \le t \le T \,:\, X_t = s\}| \,.$$
   T_s is, of course, a random variable. Show that[2] the proportion of time spent in s, $T_s/T \stackrel{T\to\infty}{\longrightarrow} \pi_s$.

   (b) Extend the ergodic theorem, in the following manner. Prove that for any $f : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$,
   $$\frac{1}{T}\sum_{t=1}^{T} f(X_t, X_{t+1}) \stackrel{T\to\infty}{\longrightarrow} \mathbb{E}[f(Y_1, Y_2)]\,,$$
   where Y_1, Y_2 are random variables that have the joint distribution $\Pr(Y_1 = a, Y_2 = b) = \pi_a P_{ab}$ (that is, first sample Y_1 according to $\boldsymbol{\pi}$, and then sample Y_2 according to the conditional distribution of the chain).

   Hint: Construct an ergodic Markov chain whose state space, $\mathcal{T}$, is a subset of $\mathcal{S} \times \mathcal{S}$. Apply the ergodic theorem for that chain.

   (c) For $a, b \in \mathcal{S}$, let $T_{a,b}$ be the amount of time the chain spends on the edge (a, b), that is,
   $$T_{a,b} = |\{1 \le t \le T \,:\, X_t = a \text{ and } X_{t+1} = b\}| \,.$$
   Show that $T_{a,b}/T \stackrel{T\to\infty}{\longrightarrow} \pi_a P_{ab}$.

2. **Gambler's ruin.** Consider the following 2-player game. Player 1 starts with k coins, and player 2 starts with $n - k$ coins. At each round, a fair coin is flipped; if the coin lands head, player 1 gains a coin and player 2 loses one; otherwise, player 1 loses a coin and player 2 gains one. The game ends once one of the players loses all his coins (and the other gains all n coins).

   (a) Let X_t be the number of coins of player 1 at round t. Explain why the process $X_1, X_2, \ldots$ constitutes a Markov chain (describe its transition matrix). Show that this chain is *not* irreducible.

   Henceforth, denote the transition matrix of the chain by P.

   (b) Let $\boldsymbol{q}$ be any starting distribution. Prove that
   $$\lim_{t\to\infty} (\boldsymbol{q}^\top P^t)_k = 0 \quad \text{ for all } 1 \le k \le n-1\,.$$

[1] We say that a sequence of random variables, $Z_1, Z_2, \ldots$ converges in probability to a number $a \in \mathbb{R}$ if for *every* $\varepsilon > 0$, the probability $\Pr(|Z_n - a| \ge \varepsilon) \to 0$ as $n \to \infty$.

[2] Convergence here = convergence in probability.

(c) Denote by $\mathbf{e}_k$ the probability take puts all its mass on k. Show that the stationary distributions of P are exactly the convex combinations of $\mathbf{e}_0$ and $\mathbf{e}_b$, that is, all the distributions of the form

$$\boldsymbol{\pi} = \alpha \mathbf{e}_0 + (1-\alpha)\mathbf{e}_n \,, \quad \text{where } \alpha \in [0,1]\,.$$

Remark: $\boldsymbol{\pi}$ is a stationary distribution of P whenever $\boldsymbol{\pi}^\top P = \boldsymbol{\pi}^\top$.

(d) Compute $\lim_{t\to\infty} \mathbf{e}_k^\top P^t$ for all k.

Hint: Let p_k be the probability that, assuming player 1 starts with k coins, he eventually loses them all. In other words, it is the probability, given that the initial state is $X_1 = k$, that the chain eventually hits the state 0. Find a recursive formula for p_k; guess a solution (or solve the equation systematically) - it should be very easy.

3. **Expander mixing lemma.** Let G be a d-regular graph on n vertices. Let A_G be the adjacency matrix of G, and denote its eigenvalues by $\lambda_1 \geq \ldots, \geq \lambda_n$. Recall that $\lambda_1 = d$, and denote $\lambda^* = \max\{|\lambda_2|, |\lambda_n|\}$. For sets $S, T \subset [n]$, denote by $e(S,T)$ the number of edges the cross between S and T, where edges that start and end in $S \cap T$ are counted twice; that is,

$$e(S,T) = |\{(s,t) \,:\, s \in S, t \in T\}|$$

(the number of *ordered* pairs). Prove that

$$\left| e(S,T) - \frac{d}{n}|S||T| \right| \leq \lambda^* \sqrt{|S||T|}\,.$$

Remark: Note that in a random graph $\mathcal{G}(n, d/n)$, with average degree d, one has $\mathbb{E}[e(S,T)] = \frac{d}{n}|S||T|$ (say when S, T are disjoint). This result says that expanders (in the sense of small λ^*) behave, in some sense, like random graphs.

4. **Max-Cut.** Let G be a graph on n vertices. A *cut* is a partition of its vertices in two parts (S, S^c) (where $S^c = [n] \setminus S$). The *size* of the cut is $e(S, S^c)$, the number of edges that cross the cut (see previous question). The maximum cut problem (Max-Cut) asks for the size of a maximum cut in G, denoted here by

$$c(G) = \max_{S \subset [n]} e(S, S^c)\,.$$

As many of you probably know, the maximum cut problem is in general NP-complete.

Assume that G is d-regular. Prove the following bounds on the maximum cut of G:

(a) $c(G) \leq \frac{n(d-\lambda_n)}{4}$, where λ_n is the *smallest* eigenvalue of the adjacency matrix.

Hint: The bound of Q(3) is unfortunately a bit too loose to prove this. Letting $\mathbf{1}_S$ be the indicator for the set S, analyze $\mathbf{1}_S^\top A_G \mathbf{1}_{S^c}$, where also note that $\mathbf{1}_{S^c} = \mathbf{1} - \mathbf{1}_S$.

(b) $c(G) \geq \frac{nd}{4}$.

Hint: Use the *probabilistic method*. Choose S randomly, and compute $\mathbb{E}[e(S, S^c)]$.