MathTools HW 5

1. **Complex matrices.** Recall that the standard inner product on $\mathbb{C}^n$ (over the complex numbers) is $\langle x, y\rangle = \sum_{i=1}^n x_i \overline{y}_i$, where $\overline{y}_i$ is the complex conjugate of y_i. For $A \in M_{n\times n}(\mathbb{C})$, denote by A^* the hermitian conjugate: $(A^*)_{ij} = \overline{A_{ji}}$ (in words: take both transpose and complex conjugates).

   (a) Prove that for any x, y, $\langle Ax, y\rangle = \langle x, A^* y\rangle$.

   (b) Let A be hermitian, meaning that $A = A^*$. Prove that

      i. If λ is an eigenvalue of A, then it is real ($\lambda \in \mathbb{R}$).

      ii. If u_1, u_2 are eigenvectors corresponding to different eigenvectors $\lambda_1 \neq \lambda_2$, then $\langle u_1, u_2\rangle = 0$.

2. **The power method.** Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix. As usual, denote its eigenvalues in decreasing order: $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$. Let $u_1, \dots, u_n$ be an orthonormal basis of corresponding eigenvectors.

   Suppose that $\lambda_1 > \max_{i=2,\dots,n} |\lambda_i|$ (notice the absolute value).

   (a) (Warm up). Let $x \in \mathbb{R}^n$ have $|\langle x, u_1\rangle| > 0$. Prove that

   $$\lim_{t\to\infty} \frac{x^\top A^{2t+1} x}{\|A^t x\|_2^2} = \lambda_1 \,.$$

   (b) **Power iterations.** Let $x^{(0)} \in S^{n-1}$ with $|\langle x^{(0)}, u_1\rangle| > 0.$[1] be a *unit vector* Consider the sequence $x^{(t)}$ defined by

   $$x^{(t+1)} = \frac{Ax^{(t)}}{\|Ax^{(t)}\|_2} \,.$$

   Prove that $\lim_{t\to\infty} |\langle x^{(t)}, u_1\rangle| = 1$ and $\lim_{t\to\infty} \|Ax^{(t)}\|_2 = \lambda_1$.

   Hint: Consider the sequence $b_t = 1 - \langle x^{(t)}, u_1\rangle^2$; it is always $b_t \geq 0$, and you need to show that $b_t \to 0$. Find an upper bound on b_{t+1} in terms of b_t. Remember that $\|x^{(t)}\|_2^2 = 1$ throughout the entire dynamic; it might also be useful to expand this in the basis $u_1, \dots, u_n$.

   (c) **Random initialization.** Let k be a parameter. Sample $x^{(0)} \in [-1, 1]^n$ to have i.i.d. coordinates, so that each $x_i^{(0)}$ is uniform in $\{-1, -1 + \frac{1}{k}, \dots, -\frac{1}{k}, 0, \frac{1}{k}, \dots, 1 - \frac{1}{k}, 1\}$.

   Show that $\Pr(\langle x^{(0)}, u_1\rangle = 0) = O(1/k)$. In particular, assuming k is large enough, $x^{(0)}$ is with high probability a good initialization for the power method.

[1] Notation: S^{n-1} stands for the Euclidean unit sphere in $\mathbb{R}^n$, that is, $\|x^{(0)}\|_2 = 1$.

3. **The spectrum of a graph.** Let G be a d-regular graph and denote by A_G its adjacency matrix.
   (a) Prove that $|\lambda_i| \leq d$ for all eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of A_G.

   Hint: For an eigenvector u, consider the vertex i such that $|u_i|$ is maximal.

   Recall we've shown in class d is an eigenvalue; thus, $\lambda_1 = d$.

   (b) Suppose that G has ℓ connected components. Show that the multiplicity of the eigenvalue d is exactly ℓ.

   In particular, if G is connected, we get $\lambda_i < d$ for all $i = 2, \ldots, n$.

   (c) Prove that $-d$ is an eigenvalue of G if and only if G has a **bipartite** connected component.

   Hint: Let u be an eigenvalue with eigenvector $-d$. Use the signs of its entries to define a partition of the vertices in two parts.

   In particular, this shows that if G is connected and not bipartite, then $|\lambda_i| < d$ for all $i = 2, \ldots, d$, as claimed in class.

   (d) Suppose that G is bipartite. Prove that if λ is an eigenvalue, then so is $-\lambda$.

4. **The spectrum of a graph - examples.** Find the spectrum (eigenvalues) of the followings graphs:
   (a) K_n: the complete graph on n vertices.

   (b) $K_{n,n}$: a bipartite graph on $2n$ vertices, each of whose sides has size n, and there is an edge between every vertex v on the right and w on the left.

   (c) $K_{n,m}$: a complete bipartite graph with *uneven* sides, one with size n and the other m.