Computability - Exercise 10

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: June 18, 2020

Question 1

(a) Let $L_1, L_2 \subseteq \Sigma^*$. Prove that if $L_1 \leq_p L_2$ and $L_2 \in NP$, then $L_1 \in NP$.

(b) Prove that polynomial-time reductions are transitive. That is, if $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$ then $L_1 \leq_p L_3$, where $L_1, L_2, L_3 \in 2^{\Sigma^*}$.

(c) Prove that A_{TM} is **NP-Hard**.

(d) Prove that every non-trivial language L is **P-Hard**, i.e., that $K \leq_p L$ for every $K \in P$.

Question 2

We define coNP $= \{\overline{L} \mid L \in \text{NP}\}$. Show that if coNP $\cap$ NP$-$Complete $\neq \emptyset$ then NP $=$ coNP.
(In fact, it is unknown whether P=NP and whether NP = coNP).

Question 3

The feasibility problem of *Integer Linear Programming* is defined as follows: given a set of m linear constraints:

$$\sum_{j=1}^{n} a_{i,j} \cdot x_j \leq b_i,$$

where $1 \leq i \leq m$, we have to decide whether there exists a 0/1 assignment to $x_1, \ldots, x_n$ such that all constraints are satisfied. In matrix notation, an instance of the feasibility problem is given by a pair of an $m \times n$ matrix over $\mathbb{Z}$, denoted A, and a column vector of size m over $\mathbb{Z}$, denoted b, and we have to decide if there is some column vector of size n over $\{0,1\}$, denoted x, such that $Ax \leq b$.
Let $ILP = \{\langle A, b\rangle : \ A \in \mathbb{Z}_{m\times n},\, b \in \mathbb{Z}^m$ and there exists $x \in \{0,1\}^n$ with $Ax \leq b\}$.

Also, define the languages

$$SUBSET-SUM = \left\{ \langle \{x_1, \ldots, x_m\}, t\rangle : \exists I \subseteq [m] \text{ such that } \sum_{i\in I} x_i = t \right\}$$

and

$$PARTITION = \left\{ \langle x_1, \ldots, x_m\rangle : \exists I \subseteq [m] \text{ such that } \sum_{i\in I} x_i = \sum_{i\in [m]\setminus I} x_i \right\}.$$

where $x_1, ..., x_m, t \in \mathbb{Z}$ are numbers encoded in binary. As you will see in class, SUBSET-SUM is NP-Complete.

Questions:

(a) Prove that $ILP \in NP$ by showing a polynomial-time verifier for it.

(b) Prove that $SUBSET - SUM \leq_p PARTITION$.

(c) Prove that $3SAT \leq_p ILP$.

Question 4

An *independent set* in an undirected graph $G = (V, E)$ is a set $D \subseteq V$ such that for every $u, v \in D$ it holds that $\{u, v\} \notin E$. Define the language $IS = \{\langle G, k \rangle \mid \text{The graph } G \text{ has an independent set of size } k\}$.

(a) Prove that $IS \in$ NP, by showing a polynomial-time verifier for it.

(b) Prove that $CLIQUE \leq_p IS$.

(c) **(Not for submission)** Prove that $VC \leq_p IS$. (Remark: we saw that $CLIQUE \leq_p VC$, so (b) follows from (c) using 1(b). However, we want you to show a direct reduction).