Advanced Algorithms (67824) – Exercise 1

Idan Orzech 322362849, Mike Greenbaum 211747639

November 2021

1. The definition of a v being a vertex is that $c^T \cdot x \leq b_0$ is valid for $\mathcal{P}$ and $1 = |F_{c,b_0}| = \{x \in \mathbb{R}^n \mid c^T \cdot x = b_0\}$. Therefore, an equivalent definition is to say $c^T \cdot v = b_0$ and $\forall v \neq y \in \mathcal{P} c^T \cdot y < b_0$. The definition of a vertex that we will use in this segment is that v is a vertex iff $\exists a \in \mathbb{R}^n, b \in \mathbb{R}\ s.t.\ \ a^T x = b,\ \ \forall x \neq y \in \mathcal{P},\ a^T y = \langle a, y\rangle < b$

   (a) ($\Rightarrow$): If $x \in \text{Vert}(\mathcal{P})$, then by definition $\exists a \in \mathbb{R}^n, b \in \mathbb{R}\ s.t.\ \ a^T x = b,\ \ \forall x \neq y \in \mathcal{P},\ a^T y = \langle a, y\rangle < b$.

   Assume for contradiction that x isn't an extremum point of $\mathcal{P}$, namely $\exists 0 \neq u \in \mathbb{R}^n$ s.t. $x + u, x - u \in \mathcal{P}$.

   i. If $\langle a, u\rangle \geq 0$ then $\langle a, x + u\rangle = \langle a, x\rangle + \langle a, u\rangle \geq \langle a, x\rangle = b$, in contradiction to the fact that x is the only point in $\mathcal{P}$ that satisfies $\langle a, y\rangle \geq b$.

   ii. Otherwise $\langle a, u\rangle < 0$, and then $\langle a, x - u\rangle = \langle a, x\rangle - \langle a, u\rangle > \langle a, x\rangle = b$, in contradiction to the fact that every point $y \in \mathcal{P}$ satisfies $\langle a, y\rangle \leq b$.

   We reached a contradiction in both cases, hence if $x \in \text{Vert}(\mathcal{P})$ then indeed x is an extremum point of $\mathcal{P}$.

   ($\Leftarrow$): Suppose x an extremum point of $\mathcal{P}$. Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^n$ the matrices such that $\mathcal{P} = \{x \mid Ax \geq b\}$ by definition. Denote the rows of A as $a_1, \ldots, a_m$ and $I(y) := \{i \in [m] \mid a_i^T y = b_i\}$. First we prove that $span(\{a_i \mid i \in I(x)\}) = \mathbb{R}^n$.

   Assume for contradiction that span$(\{a_i \mid i \in I(x)\}) \neq \mathbb{R}^n$, therefore $\exists 0 \neq u \in \mathbb{R}^n$ s.t. $\langle a_i, u\rangle = 0, \forall i \in I(x)$. Note that $\langle a_i, x\rangle > b_i, \forall i \notin I(x)$, therefore we can find sufficiently small $\varepsilon > 0$ s.t. $\langle a_i, x \pm \varepsilon \cdot u\rangle > b_i, \forall i \notin I(x)$. Note that $\langle a_i, x \pm \varepsilon \cdot u\rangle = \langle a_i, x\rangle \pm \langle a_i, \varepsilon \cdot u\rangle = \langle a_i, x\rangle + 0 = b_i, \forall i \in I(x)$. Therefore, $\langle a_i, x \pm \varepsilon \cdot u\rangle \geq b_i, \forall i \in [m]$. Therefore, $x \pm \varepsilon \cdot u \in \mathcal{P}$, therefore x not an extremum point of $\mathcal{P}$ by definition – contradiction.

   So, now we know that $span(\{a_i \mid i \in I(x)\}) = \mathbb{R}^n$. Define $a := \sum_{i\in I(x)} a_i, b := \sum_{i\in I(x)} b_i$. Note that due to $span(\{a_i \mid i \in I(x)\}) = \mathbb{R}^n$, there exists a unique solution to $a_i^T \cdot y = b_i, \forall i \in I(x)$. Note that x is a solution to the equations and therefore $a^T \cdot x = (\sum_{i\in I(x)} a_i)^T \cdot x = \sum_{i\in I(x)} a_i^T \cdot x = \sum_{i\in I(x)} b_i = b$.

   Note that $\forall x \neq y \in \mathcal{P}, \exists i \in I(x)$ s.t. $a_i^T \cdot y > b_i$, due to $a_i^T \cdot y \geq b_i, \forall i$ (since $y \in \mathcal{P}$) and the uniqueness of the solution to the equations 4 rows above.

   Therefore, $\forall x \neq y \in \mathcal{P}, a^T \cdot y = (\sum_{i\in I(x)} a_i)^T \cdot y = \sum_{i\in I(x)} a_i^T \cdot y > \sum_{i\in I(x)} b_i = b$.

   We showed that there exists $a \in \mathbb{R}^n, b \in \mathbb{R}$ s.t. $a^T \cdot x = b$ and $a^T \cdot y < b, \forall x \neq y \in \mathcal{P} \wedge a^T \cdot x = b$ and therefore $x \in \text{Vert}(\mathcal{P})$ by definition. This proves the iff.

   (b) Assume for contradiction $\exists 0 \neq v \in \text{Vert}(cone(Y))$. Therefore by definition $\exists a \in \mathbb{R}^n, b \in \mathbb{R}$ s.t. $a^T v = b,\ \ \forall v \neq y \in cone(Y)\ a^T y < b$.

   Denote $Y = \{y_1, \ldots, y_k\}$. By definition, $\exists \lambda_1, \ldots, \lambda_k \geq 0, v = \sum_{i=1}^k \lambda_i \cdot y_i$. Due to $v \neq 0$, $\exists i \in [k]$ s.t. $y_i \neq 0 \wedge \lambda_i \neq 0$.

   i. If $\langle a, y_i\rangle \geq 0$ then $\langle a, v + y_i\rangle = \langle a, v\rangle + \langle a, y_i\rangle \geq b + 0 = b$, and note that $v + y_i = \sum_{j\neq i} \lambda_j y_j + (1+\lambda_i) \cdot y_i \in cone(Y)$, therefore we found a vector $v \neq v + y_i$ s.t. $\langle a, v + y_i\rangle \geq b$ in contradiction to the fact that v is the only vector in the cone to hold that inequality – contradiction.

ii. Otherwise, $\langle a, y_i \rangle < 0$ and then $\langle a, v - \lambda_i \cdot y_i \rangle = \langle a, v \rangle + \langle a - \lambda_i \cdot y_i \rangle \geq b - 0 = b$, and note that $v - \lambda_i \cdot y_i = \sum_{j \neq i} \lambda_j y_j \in cone(Y)$, therefore we found a vector $v \neq v - \lambda_i \cdot y_i$ s.t. $\langle a, v - \lambda_i \cdot y_i \rangle \geq b$ in contradiction to the fact that v is the only vector in the cone to hold that inequality – contradiction.

Hence, we conclude that $\text{Vert}(cone(Y)) \subseteq \{0\}$.

2. (a) Consider the linear program in canonical form in $\mathbb{R}$

$$\begin{aligned} &\text{minimize } \mathbf{0} \cdot x \\ &\text{subject to } \mathbf{0}x \geq \mathbf{0} \end{aligned}$$

Note that the program is bounded (the target function is always 0), and the set of feasible solutions is $\mathbb{R}$. Since $\mathbb{R}$ has no vertices as a polyhedron in $\mathbb{R}$ (by definition, for every $x \in \mathbb{R}$ is holds that $x \pm 1 \in \mathbb{R}$ so x is not a vertex by 1a), the minimum of the target function is not achieved in any vertex of $\mathbb{R}$.

(b) We first convert the program to standard form. Define 2 variables $x^+, x^- \geq 0$ such that $x = x^+ - x^-$. Define also a slack variable s. The program is now

$$\begin{aligned} &\text{minimize } \mathbf{0} \cdot \begin{pmatrix} x^+ \\ -x^- \\ s \end{pmatrix} \\ &\text{subject to } \begin{pmatrix} 0 & -0 & -1 \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = \mathbf{0}, \\ &\begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} \geq \mathbf{0} \end{aligned}$$

in standard form. 0 is a vertex of the induced polyhedron P: for any $0 \neq v = (v_1, v_2, v_3) \in \mathbb{R}^3$, $v_i \neq 0$ for some $i \in \{1, 2, 3\}$, and wlog $v_i > 0$. Then $\mathbf{0} - v \notin P$ since its i'th coordinate is negative, and all the coordinates need to be nonnegative. Thus by the equivalent definition of vertices (question 1a), $\mathbf{0} \in \text{Vert}(P)$. In addition, $\mathbf{0}$ achieves the minimum of $\mathbf{0} \cdot \begin{pmatrix} x^+ \\ -x^- \\ s \end{pmatrix}$ since it is a feasible solution and the target function is always 0, as required.

3. In the proof of Farkas' lemma, we used the fact that for every point $x \notin P$ there a separating hyperplane between x and P. The claim is correct only if P is convex. We show that the claim is incorrect for nonconvex P.

Let $P = \{y \in \mathbb{R}^n \mid 0.5 \leq \|y\|_2 \leq 1\}$ and note that P is indeed compact. In addition $x := 0$ is not in P.

Assume for contradiction that there exists a separating hyperplane between x and P, namely $\exists 0 \neq a \in \mathbb{R}^n, b \in \mathbb{R}$ s.t. $a^T x \leq b, \quad \forall y \in P, \; a^T y \geq b$.

Using the fact that $x = 0$, we get that $0 = a^T 0 = a^T x \leq b$.

Choose $y := \frac{-a}{\|a\|_2}$, and note that $y \in P$ because $\|y\|_2 = 1$. We get $-\|a\|_2 = a^T \cdot \frac{-a}{\|a\|_2} = a^T y \geq b$.

Combining the results we get that $0 \leq b \leq -\|a\|_2$, therefore $a = 0$ and we reach a contradiction to the assumption that $a \neq 0$. Hence there is no separating hyperplane, and this refutes the Farkas lemma for this case, as required.

Did you know? In addition to his discovery of the greenhouse effect, Joseph Fourier was also an enthusiastic explorer of ancient Egypt. On 1798, he accompanied Napoleon Bonaparte on an expedition to Egypt.