Computability - Exercise 3

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: April 30, 2020

Question 1

Use the inductive proof that you saw in Recitation 3 in order to convert the regular expression $(a \cup b)^* \cdot a$ into an equivalent NFA. No need to prove your answer.

Question 2 (Not for submission)

In the recitation you saw an algorithm that converts NFAs to regular expressions. Run this algorithm on the following NFA, describe the intermediate GNFAs (generalized NFAs), and the resulting regular expression.

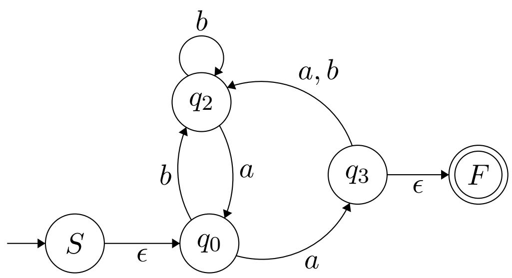

Figure 1: Translate me to a regular expression!

Question 3

Two years ago, Dana decided not to solve Exercise 3 and slept all day instead. In the exam, the following question appeared:

> Consider the following language over $\Sigma = \{a, b\}$:
>
> $$L = \{a^i \cdot b^j : 0 \leq i < j\}.$$
>
> Is $L \in \text{REG}$? Prove your answer.

Dana wrote the following solution to the question:

It holds that $L \notin \text{REG}$.

Proof. We assume by contradiction that $L \in \text{REG}$. Hence, according to the pumping lemma, there exists a constant $p > 0$, such that for every $w \in L$ such that $|w| > p$, we can decompose w into subwords $w = xyz$, such that:

1. $|y| > 0$,
2. $|xy| \leq p$, and
3. for all $i \in \mathbb{N} \cup \{0\}$, we have that $xy^i z \in L$.

Consider the word $w = a^{p-1}b^p$. Clearly, $|w| > p$ and $w \in L$. We partition w as follows: $x = a^{p-2}, y = a, z = b^p$. It holds that $|y| > 0$ and that $|xy| \leq p$, but when we examine the third condition and choose $i = 3$, we get $w' = xy^3z = a^{p-2}aaab^p$. Since $w' \notin L$, we get a contradiction. Hence, L does not satisfy the pumping lemma and so $L \notin \text{REG}$. $\square$

1. Find and explain the mistake in Dana's solution.
2. Write a correct solution to the question.
3. Prove that the language $L = \{aabbb\}$ satisfies the pumping lemma.

Question 4

For each of the following languages, determine if it is regular. If the language is not regular, prove it using the pumping lemma.

1. $\Sigma = \{a, b\}$, $L_1 = \{v \cdot u \cdot u : v, u \in \{a, b\}^*, u \neq \epsilon\}$
2. $\Sigma = \{a, b\}$, $L_2 = \{a \cdot w \cdot a : w \in \{a, b\}^*\}$
3. $\Sigma = \{1\}$, $L_3 = \{1^p : p \text{ is a prime number}\}$
4. $\Sigma = \{0, 1, \ldots, 9\}$, $L_4 = \{w \in \Sigma^* : w \text{ is divisible by 3 and } \epsilon \notin L_4\}$
5. $\Sigma = \{0, 1\}$, $L_5 = \{0^{i_1}10^{i_2}10^{i_3}10^{i_4}10^{i_5}10^{i_6}1 : i_1 > i_2 > i_3 > i_4 > i_5 > i_6 \text{ and } i_1 < 100\}$
6. $\Sigma = \{0, 1\}$, $L_6 = \{0^{i_1}10^{i_2}10^{i_3}10^{i_4}10^{i_5}10^{i_6}1 : i_1 > i_2 > i_3 > i_4 > i_5 > i_6 \text{ and } i_2 < 100\}$
7. $\Sigma = \{0, 1\}$, $L_7 = \{0^{i_1}10^{i_2}10^{i_3}10^{i_4}10^{i_5}10^{i_6}1 : i_1 > i_2 > i_3 > i_4 > i_5 > i_6 \text{ and } i_3 < 100\}$

Question 5

We saw that the pumping lemma holds for every regular language. In this question, we show that the converse does not hold: A language satisfying the pumping lemma isn't necessarily regular. Consider the language $L = \left\{a^i b^j c^k : \text{either } i \text{ is even or } j = k\right\}$.

1. Prove that L satisfies all the conditions of the pumping lemma with pumping constant $p = 2$.
2. Prove that L is not regular.