Computability - Exercise 6

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: May 21, 2020

Remark (Universal Machines): For the purpose of this exercise (and henceforth) you may use the universal machine and its variants freely. Specifically, you may assume the existence of a TM U such that given an encoding $\langle M, w, t\rangle$ of a TM M, a word w, and a number t, U simulates the run of M on w for t steps, and halts with the configuration that M reached on its tape.

Question 1

In this question we will see another equivalent model to TMs, which allows the head to skip two cells in one step.

A *Jumping TM* (JTM) is $M = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}\rangle$ where all the components are similar to a TM, except the "type" of δ, which is $\delta : Q \times \Gamma \to Q \times \Gamma \times \{R, L, RR, LL\}$.

1. We start by defining the semantics of a JTM. Consider a configuration $uabqcdev$ (where $u, v \in \Gamma^*$ and $a, b, c, d, e \in \Gamma$). Which configurations are obtained from this configuration by the transitions $\delta(q, c) = (q', x, RR)$ and $\delta(q, c) = (q', x, LL)$?
2. Which configuration is obtained from qcv by $\delta(q, c) = (q', x, LL)$ (assume the machine stays at the leftmost cell)?
3. Show that for every JTM $M_1 = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}\rangle$, there exists an equivalent TM M_2 (that is, M_1 and M_2 accept/reject/do not halt on the same words).

   In your answer, write the components of M_2 formally (and in particular, the states and transition function) and explain the idea behind them. There is no need to formally prove the correctness of your construction.

Question 2

Answer the following questions. Shortly prove your answers.

1. Is it true that $\text{RE} = \overline{\text{coRE}}$?
2. Is RE closed under complementation?
3. Are there languages $L_1 \in \text{RE}$ and $L_2 \in \text{coRE}$ such that $L_1 \cup L_2 \in \text{R}$?
4. If $L_1 \subseteq L_2$, and $L_1 \notin \text{RE}$, is it possible that $L_2 \in \text{RE}$?

Question 3

1. Consider the language $E_{TM} = \{\langle M \rangle : M \text{ is a TM, and } L(M) = \emptyset\}$. That is, E_{TM} is the set of all encodings of TMs such that their language is empty. Prove the following claims.

   (a) $E_{TM} \in \text{coRE}$.

   (b) $E_{TM} \notin \text{RE}$.
   Guidance: Use the fact that $HALT_{TM} = \{\langle M, w\rangle : M \text{ halts on } w\}$ is undecidable (that is, $HALT_{TM} \notin \text{R}$). Assume by way of contradiction that $E_{TM} \in \text{R}$, and use a machine that decides E_{TM} in order to decide $HALT_{TM}$. Deduce that $E_{TM} \notin \text{RE}$.

2. **(Not for submission)** We say that a Turing machine is *nice* if it decides its language, has no more than 100 states, and its tape alphabet is $\{0, 1, \sqcup\}$.

   Prove that $L_{nice} = \{\langle x, y\rangle : \text{There exists a nice TM that accepts } x \text{ and rejects } y\} \in \text{R}$.

Question 4 (Not for submission)

Let A and B be two disjoint languages ($A \cap B = \emptyset$). We say that a language C *separates* A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. In Exercise 5 you have seen that if $A, B \in \text{coRE}$, then there exists a language $C \in \text{R}$ that separates A and B.

Show that if we replace the requirement that $A, B \in \text{coRE}$ with the requirement that $A, B \in \text{RE}$, then the claim is false. **We provide a guidance, but you are encouraged to try and solve the question without it**:

- Show that if we assume by contradiction that the claim is correct, then there exists a TM K that given an input $\langle M, w\rangle$, where M is a TM and w is a word, always halts and acts as follows.
  - If M accepts w, then K accepts $\langle M, w\rangle$,
  - if M rejects w, then K rejects $\langle M, w\rangle$, and
  - if M does not halt on w, then K may accept or reject $\langle M, w\rangle$.
- Show by a diagonal argument, that such machine K cannot exist.

Question 5

Definition: We say that a function $f : \Sigma^* \to \Sigma^*$ is *computable* if there exists a TM M_f such that for every input x, M_f halts with $f(x)$ written on the tape. Let $f : \Sigma^* \to \Sigma^*$ be a computable function and let $L \subseteq \Sigma^*$ be a language over Σ such that $L \in \text{RE}$.

Prove the following claims.

1. $f(L) = \{f(x) : x \in L\} \in \text{RE}$
2. **(Not for submission)** $f^{-1}(L) = \{y : f(y) \in L\} \in \text{RE}$