MathTools HW 7

1. **Weyl's interlacing inequality.** Let $A, B \in M_{n\times n}(\mathbb{R})$ be two symmetric matrices. Denote by $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$ the eigenvalues of A in decreasing order (similarly for B etc.).

   (a) Prove the following bounds on the eigenvalues of $A+B$:

   $$\lambda_{k+t-1}(A+B) \leq \lambda_k(A) + \lambda_t(B)\,,$$

   and

   $$\lambda_{i+j-n}(A+B) \geq \lambda_i(A) + \lambda_j(B)\,.$$

   (b) Deduce that

   $$\lambda_k(A) + \lambda_n(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_1(B)\,.$$

   (c) Deduce that if $\operatorname{rank}(B) \leq r$, then

   $$\lambda_{k+r}(A) \leq \lambda_k(A+B) \leq \lambda_{k-r}(A)$$

   for all $r+1 \leq k \leq n-r$.

   Hint: Use the Courant-Fischer theorem.

2. Let $A \in M_{n\times m}(\mathbb{R})$. Define the symmetric $(n+m)$-by-$(n+m)$ matrix

   $$\tilde{A} = \begin{bmatrix} 0 & A \\ A^\top & 0 \end{bmatrix}.$$

   (a) Explain why $\operatorname{rank}(\tilde{A}) = 2\operatorname{rank}(A)$.

   (b) Let $r = \operatorname{rank}(A)$. Show that the non-zero eigenvalues of $\tilde{A}$ are *exactly* $\pm\sigma_1(A), \ldots, \pm\sigma_r(A)$.

3. Recall that if $A \in M_n(\mathbb{R})$ is symmetric, then the singular values of A are $|\lambda_1|, \ldots, |\lambda_n|$.

   In particular, $\|A\|_{2,2} = \max(\lambda_1, |\lambda_n|)$; is this still true in general, when A is diagonalizable but not symmetric? Prove or give a counter-example.

4. Let $A, B \in M_{n\times n}(\mathbb{R})$ be two symmetric matrices. Prove that $\operatorname{tr}(BABA) \leq \operatorname{tr}(B^2A^2)$.