Computability - Exercise 8

All questions should be answered *formally* and *rigorously* unless stated otherwise

Due: June 7, 2020

Question 1

For each of the following languages, determine whether they are in R, RE $\setminus$ coRE, coRE $\setminus$ RE, or $\overline{\text{RE} \cup \text{coRE}}$. Prove your answer.

1. $SUB_{TM} = \{\langle M_1, M_2\rangle : L(M_1) \subseteq L(M_2)\}$.
2. **(Not for submission)** $EQ_{TM} = \{\langle M_1, M_2\rangle : L(M_1) = L(M_2)\}$.
3. $L = \{\langle M_1, M_2, w\rangle : M_1 \text{ and } M_2 \text{ agree on } w\}$, where two TMs M_1 and M_2 agree on a word $w \in \Sigma^*$ if both accept, both reject, or both do not halt when run on w.
4. $L = \{\langle M, w\rangle : \text{in the run of } M \text{ on } w, M \text{ never modifies the portion of the tape that contains } w\}$.
5. **(Not for submission)** $L = \{\langle M, w_1, w_2\rangle : M \text{ accepts } w_1 \text{ and rejects } w_2\}$.

Question 2

Recall that:

$E_{TM} = \{\langle M\rangle : L(M) = \emptyset\}$.
$ALL_{TM} = \{\langle M\rangle : L(M) = \Sigma^*\}$.
$FINITE_{TM} = \{\langle M\rangle : \exists n \in \mathbb{N} \cup \{0\} \text{ such that } |L(M)| = n\}$.
$INFINITE_{TM} = \overline{FINITE_{TM}}$.
$REG_{TM} = \{\langle M\rangle : L(M) \in \text{REG}\}$.

Prove that:

1. $E_{TM} \leq_m FINITE_{TM}$.
2. $FINITE_{TM} \leq_m REG_{TM}$.
3. **(Not for submission)** $ALL_{TM} \leq_m INFINITE_{TM}$.

Question 3 (Not for submission)

Recall that a language $L \subseteq \Sigma^*$ is RE-*hard* if for every language $L' \in \text{RE}$, it holds that $L' \leq_m L$. We say that an RE-hard language L is RE-*complete* if $L \in \text{RE}$. For a language L, let $\text{Pref}(L) = \{x \in \Sigma^* : \exists y \in \Sigma^* \text{ such that } x \cdot y \in L\}$.

Describe a language $L \in \text{R}$ such that $\text{Pref}(L)$ is RE-complete.

Hint: We know that $HALT_{TM}$ is RE-hard.

Question 4

1. Prove that there are uncountably many RE-hard languages.

   Hint: Recall the proof of Rice's theorem.

2. Show that there are no $\overline{\text{RE} \cup \text{coRE}}$-hard languages. That is, there does not exist a language L such that for every language $L' \in \overline{\text{RE} \cup \text{coRE}}$ we have $L' \leq_m L$.

   Hint: How many reductions exist?

3. For each of the following languages, determine whether they are in R, RE $\setminus$ coRE, coRE $\setminus$ RE, or $\overline{\text{RE} \cup \text{coRE}}$. Prove your answer. You may use Rice's theorem. Note that the languages are defined with a parameter $n \in \mathbb{N} \cup \{0\}$, and that the answer may depend on n. For example (beware, spoiler), $L_{=n}$ is in co-RE for $n = 0$ and is not in co-RE for all $n > 0$.

   (a) $L_{\leq n} = \{\langle M \rangle : |L(M)| \leq n\}$.

   (b) $L_{=n} = \{\langle M \rangle : |L(M)| = n\}$.

   (c) $L_{\geq n} = \{\langle M \rangle : |L(M)| \geq n\}$.

Question 5

For each of the following languages, determine whether they are in R, RE $\setminus$ coRE, coRE $\setminus$ RE, or $\overline{\text{RE} \cup \text{coRE}}$. Prove your answer.

1. **(Not for submission)** Consider a modified version of the Tiling Problem in which we replace the vertical and horizontal constraints V and H on the set T of tiles, by a set $S \subseteq T^6$ of constraints on 3×2 rectangles in the tiling. Formally, a function $f : [n] \times [n] \to T$ is a legal $n \times n$ modified tiling if for all $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$, we have that

   $$(f(i,j), f(i+1,j), f(i+2,j), f(i,j+1), f(i+1,j+1), f(i+2,j+1)) \in S,$$

   and $f(1,1), f(1,2) \neq t_0$.

   That is, every 3×2 rectangle agrees with some 6-tuple in S, and the two leftmost tiles on the bottom row are different from t_0.

   $L_1 = \{\langle T, S, t_0)\rangle : \forall n \in \mathbb{N}$ there exists a legal $n \times n$ modified tiling$\}$.

2. $L_2 = \{\langle T, H, V, t_0 \rangle :$ there exists exactly one tiling of the quarter plane$\}$.

   Hint: In order to show that $L_2 \notin$ coRE, you can use a reduction from $HALT^{\epsilon}_{TM}$ with an output that induces one option to tile the quarter plane regardless of the run of M on ε, and another possible option (its existence depends on that run). In this solution, think of a clever (and direct) use of the initial tile defined in the reduction from $\overline{HALT^{\epsilon}_{TM}}$ that was shown in class. Make sure you understand the details of that proof, and in particular, the use of the tiles in the first row of the tiling.

   Alternatively, you can reduce from $\overline{TILE}$, by adding tiles and constraints, in a way that induces a tiling that does not depend on the input to the reduction.

3. $L_3 = \{\langle T, H, V, t_0 \rangle : \forall n \in \mathbb{N}$ there exists a legal $1 \times n$ tiling$\}$.

Question 6 (Not for submission)

Let N be an NTM over Σ with $\# \notin \Sigma$. Consider the language

$$L' = \{x \cdot \# \cdot 1^t \ : x \in \Sigma^*, \text{ and there is an accepting run of } N \text{ on } x \text{ of length at most } t\}.$$

Prove that $L' \in$ R. Note it implies that the length of an accepting run of N on w can serve as a witness to the fact that $w \in L(N)$.