CS-67720 Metric Embeddings Theory **Fall 2021/22**

Exercise 3

Lecturer: Yair Bartal

1. In class we have shown that the metric of every n-vertex weighted connected graph X embeds into its MST with distortion $n-1$. Denote this (non-contractive) embedding by f. (Note: This exercise should be solved directly, without the use of Theorem 9.8 of Lecture 9.)

   (a) Consider $X = C_n$, the n-point (unweighted) cycle, and let T be a spanning tree of X. Compute the scaling distortion of the embedding $f : C_n \to T$. Compute an upper bound on ℓ_q-*dist*(f) for every $1 \le q \le \infty$. (Provide the computation, not just the final result, all bounds should be expressed up to a constant using O notation).

   (b) Is it true that for every n-vertex weighted connected graph X, the embedding f is such that $\ell_1 - dist(f) = O(1)$? Prove the statement or provide a counterexample. In the case that X may have several MSTs, does it matter for this question if we choose the worst or best one (in terms of average distortion) ?

BONUS: How does the answer to the previous question change if we allow embeddings of X into a spanning tree of weight at most $w(\text{MST}) + c$, for some constant c ? *Extra challenging:* What if we allow a spanning tree of weight at most $(1+\rho)w(\text{MST})$, for some $0 < \rho < 1$?

2. (a) Prove the following claim:

   Claim 3.1. *Let $k > 1$. Any k-HST (U, d) whose defining tree (i.e., the labeled tree in the k-HST definition) has maximum degree s, embeds in $\ell_\infty^{O(\log s)}$, with distortion $1 + O(1/k)$.*

   Directions: Show that for some $c > 0$ (e.g. c=4), and $k > c$, that the following holds by induction over subspaces U_i: there exists $f_i : U_i \to \ell_\infty^{O(\log s)}$ such that for all $x \in U_i$ $\|f_i(x)\|_\infty \le diam(U_i)$, and such that for all $x, y \in U_i$ it holds $(1 - c/k)\, d_{U_i}(x, y) \le \|f_i(x) - f_i(y)\|_\infty \le (1 + c/k)\, d_{U_i}(x, y)$.
   Hint: Consider using results or ideas from the solution to Exercise 1, Question 2(a).

   (b) Deduce that any ultrametric space X embeds into $\ell_\infty^{O(dim(X))}$, with constant distortion, where $dim(X)$ denotes the doubling dimension of X. (Note: You may use part (a) even if you did not solve it.)

3. (a) Let L_n be the metric of an n-node path graph. Prove that for any $\alpha > 1$, there exists a Ramsey embedding of L_n into an ultrametric with distortion α, with core of size $n^{g(\alpha)}$, where $g(\alpha) = 1 - O(1/\alpha)$, and $g(\alpha) > 0$. (Note: This exercise should be solved directly, without the use of Theorem 10.7 of Lecture 10. Also note that for small $1 < \alpha < 8$ the claim does not follow from this theorem.)

   Directions: The target ultrametric can be a k-HST with some $k > 2$, whose defining tree is a complete binary tree. You can define the embedding recursively by partitioning L_n into two subspaces, where the points in the cut would not be in the core and the distortion is kept between the pairs remaining in the core (Similarly to the high level structure of the proof of Theorem 10.7). Then estimate the size of the resulting (ultrametric) tree.

(b) Let H^d denote the d-dimensional hypercube (of size $n = 2^d$). Prove that there exists a Ramsey embedding of H^d (with ℓ_1 metric) into an equilateral metric space with distortion $1 + \epsilon$, with core of size $\geq n^{\Omega(\epsilon^2)}$, for $0 < \epsilon < 1$.

Hint: Use results or ideas from the solution to Exercise 1, Question 2(b).

BONUS: Prove that there exists a Ramsey embedding of H^d into an equilateral metric space with distortion $\alpha > 1$, with core of size $\geq n^{1-g(\alpha)}$, where $g(\alpha) = O(\frac{\log(\alpha)}{\alpha})$. Alternatively, provide any bound satisfying $g(\alpha) \to 0$, when $\alpha \to \infty$. Hint: Extend ideas from the solution to part (b).