MathTools HW 10

1. Let $C_n = [0,1]^n$. Show that C_n is the convex hull of all the binary vectors: $C_n = \text{conv}\left(\{0,1\}^n\right)$.

2. Let $B_n = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1$, the $\ell 1$ unit ball.
   (a) Show that B_n is a polytope.
   (b) Show that $B_n = \text{conv}\left(\pm e_1, \ldots, \pm e_n\right)$, where e_i is the i-th standard basis vector.
   (c) Find all the vertices of B_n. Prove your answer!

3. Let $\Delta_n = \left\{x \in [0,1]^{n+1} : \sum_{i=0}^{n} x_i = 1\right\}$ be the n-simplex. Δ_n is clearly a polytope.
   (a) Find the vertices of Δ_n.
   (b) Let V be the set of all vertices. Show that $\Delta_n = \text{conv}(V)$.
   (c) Take any subset $S \subset V$ of vertices. Show that $\text{conv}(S)$ is a face of Δ_n.

4. Let $x_1, \ldots, x_m \in \mathbb{R}^n$ be vectors.

   Fact: The set $\mathcal{P} = \text{conv}(x_1, \ldots, x_m)$ is a polytope.

   We will not prove this fact right now (perhaps at a later point in the course), but I hope that at this point your "geometric intuition" tells you that indeed this should be the case.

   Prove the following:
   (a) Any face of $\mathcal{P}$ *contains* one of the points $\{x_1, \ldots, x_m\}$.
   (b) $\text{vertices}(\mathcal{P}) \subset \{x_1, \ldots, x_m\}$.

5. Denote by $HS(a,b) = \{x : a^\top x \leq b\}$ a halfspace, and $H(a,b) = \{x : a^\top x = b\}$ its boundary (corresponding hyperplane).

   Let $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$, and consider the polyhedron $\mathcal{P} = \bigcap_{i=1}^{m} HS(a_i, b_i)$. Throughout, assume that $\mathcal{P}$ is not empty, that is, $\mathcal{P} \neq \emptyset$.
   (a) Suppose that $\mathcal{P} \neq \bigcap_{i=1}^{m-1} HS(a_i, b_i)$, that is, that the constraint $a_m^\top x \leq b_m$ is *not* redundant. Prove that $H(a_m, b_m)$ is a supporting hyperplane of $\mathcal{P}$.
   (b) Suppose that $a_m \notin \text{span}(a_1, \ldots, a_{m-1})$. Prove that $H(a_m, b_m)$ is a supporting hyperplane.