Chapter 1

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Theorem 4. The sum of two odd numbers is even.

Proof. Recall the definition of an odd number to be a number, n, that can be written as $2k-1$ where k is an integer. Recall also that the definition of an even number is a number that can be written as $2m$ where m is an integer.
Let n_1 and n_2 both be odd numbers. Then,

$$\begin{aligned} n_1 + n_2 &= (2k-1) + (2j-1) \\ &= 2k - 1 + 2j - 1 \\ &= 2k + 2j - 2 \\ &= 2(k + j - 1) \\ &= 2m \end{aligned}$$

where $m = k + j - 1$. As integers are closed under addition and subtraction m itself must be an integer. □

Result. Therefore, $k + j - 1$ can be written as m, and we are left with $2m$.

Theorem 7. Every binary number, n, that ends in 0 is even.

Proof. Recall the definition of an even number to be a number that can be written as $2m$. The definition of a binary number is a number that can be written as

$$n = a_{n-1} \cdot 2^{n-1} + ... + a_1 \cdot 2^1 + a_0 \cdot 2^0$$

Let $m = a_{n-1} \cdot 2^{n-1} + ... + a_1 \cdot 2^1$
Thus, we have

$$n = 2m + 0 \cdot 1 = 2m$$

. □

Result. Thus, through substitution, any binary number n ending in 0 can be written as $2m$.

Theorem 10. The difference of two consecutive perfect squares is odd.

Proof. Recall the definition of an odd number to be any number that can be written $2n+1$. Recall also the definition of a perfect square is any number that can be written n^2.
Then,

$$\begin{aligned}(n+1)^2 - (n^2)\\ n^2+n+n+1-n^2\\ 2n+1\end{aligned}$$

□

Result. Thus, the difference between any two consecutive perfect squares is always odd since it can always be written $2n+1$.

Theorem 17. At least 2 of 1,185 daily flights at Chicago O'Hare International Airport must take off within 90 seconds of each other

Proof. Recall that the pigeonhole principle states that if k items are put into n containers, and $k > n$, then at least one container must have more than one item.
Let $k =$ flights per day and $n =$ 90 second time intervals
Then,

$$\begin{aligned}24 \cdot 60 \cdot 60 &= 86,400 \text{ seconds per day}\\ n = 86,400/90 &= 960\\ k &= 1,185\end{aligned}$$

Since $1,185 > 960$ at least one 90 second interval must contain more than one flight. □

Result. Thus, per the pigeonhole principle, there must be at least 2 flights that take off within 90 seconds of each other

Theorem 22a. In a list of five digit distinct numbers of length 20 there are no two subsets that share the same sum.

Proof. Recall that the pigeonhole principle states that if k items are put into n containers, and $k > n$, then at least one container must have more than one item.
Let $k =$ subsets and $n =$ range of possible sums
Then,

$$\begin{aligned}k &= 2^{20} - 1\\ &= 1,048,575\\ n &= 20 \cdot 99,999\\ &= 1,999,980\end{aligned}$$

□

Result. Thus, through pigeonhole principle, there is no guarantee that there are two subsets with the same sum in a list of five distinct numbers of length 20.

Theorem 22b. The smallest list of five digit numbers in which two subsets have the same sum are of length 22.

Proof. Recall that the pigeonhole principle states that if k items are put into n containers, and $k > n$, then at least one container must have more than one item.
Let $k =$ subsets and $n =$ range of possible sums
For a list of length 22, we calculate,

$$\begin{aligned} k &= 2^{22} - 1 \\ &= 4,194,303 \\ n &= 22 \cdot 99,999 \\ &= 2,199,978 \end{aligned}$$

The pigeonhole principle guarantees at least two subsets share the same sum in this case. For a list of length 21, we calculate,

$$\begin{aligned} k &= 2^{21} - 1 \\ &= 2,097,151 \\ n &= 21 \cdot 99,999 \\ &= 2,099,979 \end{aligned}$$

The pigeonhole principle fails to guarantee two subsets share the same sum in this case. □

Result. Hence, $n = 22$ is the smallest integer for which the number of subsets must exceed the range of possible sums, ensuring at least two subsets share a common sum.

Theorem 23. If n is any integer, then $3n^3 + n + 5$ is odd.

Proof. Recall the definition of an odd number to be a number, n, that can be written as $2k + 1$ where k is an integer.
Then,

$$\begin{aligned} 3n^3 + n + 5 &= 2(2k+1)^3 + (2k+1) + 5 \\ &= 3(8k^3 + 12k^2 + 6k + 1) + 2k + 1 + 5 \\ &= 24k^3 + 36k^2 + 18k + 3 + 2k + 6 \\ &= 24k^3 + 36k^2 + 20k + 9 \\ &= 2(12k^3 + 18k^2 + 10k + 4) + 1 \end{aligned}$$

Let $m = 12k^3 + 18k^2 + 10k + 4$. Thus, the expression translates to $2m + 1$, which is odd.

Recall the definition of an even number to be a number, n, that can be written as $2k$

where k is an integer.
Then,

$$\begin{aligned} 3(2k)^3 + 2k + 5 \\ 3(8k^3) + 2k + 5 \\ 24k^3 + 2k + 5 \\ 2(12k^3 + k + 2) + 1 \end{aligned}$$

Let $m = 12k^3 + k + 2$. By substitution, we get $2m + 1$, which is odd. $\square$

Result. Therefore, through substitution of either an odd or even integer, we find the expression always takes the form $2m + 1$.