Matrix Exponentiation

This is very helpful in solving recurrence relations. One of whose examples is fibonacci numbers.

$$f(n) = f(n-1) + f(n-2)$$

In matrix exponentiation , we try to find a matrix which can convert k th state to $(k + 1)^{th}$ state, i.e.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix} = \begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix}$$

On solving we get a = b = c = 1 and d = 0. Our matrix becomes

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Since by multiplying once we reach on we reach on $(n + 1)^{th}$ state from nth state. In the same way we reach on $(n+k)^{th}$ state from nth state by multiplying M matrix k times with our original matrix.

$$M^k * \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix} = \begin{bmatrix} f(n+k) \\ f(n+k-1) \end{bmatrix}$$

$$M^{k-1} * \begin{bmatrix} f(1) \\ f(0) \end{bmatrix} = \begin{bmatrix} f(k) \\ f(k-1) \end{bmatrix}$$

We can compute M^{k-1} in log(k) time. Using these equations this can be done

$$A^n = A^{n/2} * A^{n/2} \qquad (n = even)$$
$$A^n = A^{n/2} * A^{n/2} * A \qquad (n = odd)$$

Various helpful Recurrence Relations

Recurrence relations can be solved very easily. We can find the M matrix just by analyzing.

Various examples are

1. **f(n) = a*f(n-1) + b*f(n-2)**

$$\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix} = \begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix}$$

2. **f(n) = f(n-1) + f(n-2) + c**

   Here 3 variables are involved , f(n-1) , f(n-2) and c. Therefore our M matrix will be of 3x3.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} f(n) \\ f(n-1) \\ c \end{bmatrix} = \begin{bmatrix} f(n+1) \\ f(n) \\ c \end{bmatrix}$$

3. **f(n) = f(n-1) + f(n-3)**

   This can be further written as f(n) = f(n-1) + 0*f(n-2) + f(n-3). So here also 3 variables are involved, that are, f(n-1) , f(n-2) , f(n-3). Hence a 3x3 matrix will be formed.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} f(n) \\ f(n-1) \\ f(n-2) \end{bmatrix} = \begin{bmatrix} f(n+1) \\ f(n) \\ f(n-1) \end{bmatrix}$$

4. **f(n) = a*f(n-1) + c*f(n-3) + d*f(n-4) + e**

   This can be written as f(n) = a*f(n-1) + 0*f(n-2) + c*f(n-3) + d*f(n-4) + e

   Therefore, here 5 variables are involved viz f(n-1) , f(n-2) , f(n-3) , f(n-4) , e. Hence a 5x5 matrix will be formed.

$$\begin{bmatrix} a & 0 & c & d & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} f(n) \\ f(n-1) \\ f(n-2) \\ f(n-3) \\ e \end{bmatrix} = \begin{bmatrix} f(n+1) \\ f(n) \\ f(n-1) \\ f(n-2) \\ e \end{bmatrix}$$