A Unifying Framework for Adaptive Radar Detection in Homogeneous Plus Structured Interference—Part I: On the Maximal Invariant Statistic

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***Abstract*—This paper deals with the problem of adaptive multidimensional/multichannel signal detection in homogeneous Gaussian disturbance with unknown covariance matrix and structured deterministic interference. The aforementioned problem corresponds to a generalization of the well-known Generalized Multivariate Analysis of Variance (GMANOVA). In this Part I of the paper, we formulate the considered problem in canonical form and, after identifying a desirable group of transformations for the considered hypothesis testing, we derive a Maximal Invariant Statistic (MIS) for the problem at hand. Furthermore, we provide the MIS distribution in the form of a stochastic representation. Finally, strong connections to the MIS obtained in the open literature in simpler scenarios are underlined.**

***Index Terms*—Adaptive radar detection, CFAR, statistical invariance, maximal invariants, double-subspace model, GMANOVA, coherent interference.**

I. Introduction

A. Motivation and Related Works

The problem of adaptive detection of targets embedded in Gaussian interference is an active research field which has been subject of great interest in the last decades. Many works appeared in the open literature, dealing with the design and performance analysis of several detectors handling many specific detection problems (the interested reader is referred to [1] and references therein for further details).

It can be shown that most of the aforementioned models can be seen as special cases of the model considered by Kelly and Forsythe [2], which is very general and encompasses point-like and extended targets as special instances. The considered model allows for training samples which contain random interference modeled as an unknown covariance matrix that accounts for both clutter and thermal noise, with the implicit assumption that signal plus noise and noise-only vector samples share the same covariance matrix, thus determining a so-called *homogeneous environment*.

The signal model considered in the aforementioned report is the well-known Generalized Multivariate Analysis of Variance (GMANOVA) in statistics literature [3], also referred to as a "double-subspace" signal model (see for example [4], [5]). The standard GMANOVA model was first formulated by Potthoff and Roy [6] and consists in a generic patterned mean problem with a data matrix whose columns are normal random vectors with a common unknown covariance matrix. The GMANOVA model was later studied in more detail in [7], where maximum likelihood estimates of unknown parameters were obtained. For a detailed introduction to estimation and detection in GMANOVA model (along with few interesting application examples) the interested reader may refer to the excellent tutorial [8].

Differently, in this paper we will study a modified version of GMANOVA with respect to its classical formulation [8], referred to as I-GMANOVA in what follows. The considered model allows for the presence of a structured (partially known) non-zero mean under both hypotheses. Such disturbance is collectively represented as an unknown deterministic matrix, which determines an additional set of nuisance parameters for the considered hypothesis testing (i.e., other than the covariance matrix). The aforementioned model easily accounts for the presence of structured subspace interference affecting the target detection task. Thus it is clear that taking such interference into consideration enables the application of this model to adaptive radar detection; for instance it may accommodate the presence of multiple pulsed coherent jammers impinging on the radar antenna from some directions.

Although several different decision criteria can be considered to attack composite hypothesis testing problems [9], [10], an elegant and systematic way consists in resorting to the so-called *Principle of Invariance* [10], [11]. Indeed, the aforementioned principle, when exploited at the design stage, allows to focus on decision rules enjoying some desirable practical features. The preliminary step consists in individuating a suitable group of transformations which leaves the formal structure of the hypothesis testing problem and data distribution unaltered. With reference to radar adaptive detection, the mentioned principle represents an effective tool for obtaining a statistic which is invariant with respect to the set of nuisance parameters, therefore constituting the basis for Constant False Alarm Rate (CFAR) rules. Indeed, every invariant decision rule can be written in terms of the

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Maximal Invariant Statistic (MIS) [10]. Therefore, with reference to I-GMANOVA model, the principle of invariance allows for imposing CFARness property with respect to (*i*) the clutter plus noise (disturbance) covariance matrix and (*ii*) the jammer location parameters.

It is worth remarking that the use of the invariance principle for generic composite hypothesis testing problems [10], [11] (and, more specifically, in the context of radar adaptive detection) *is not new*. Indeed, starting from the seminal paper [12], many works focused on adaptive radar detection problems with the use of *invariance theory*. For example, in [13]–[15], invariance theory was exploited to study the problem of single-subspace (adaptive) detection of point-like targets. Later, similar works appeared in the open literature dealing with the case of a target spread among more range cells [16], [17]. More recently, the same statistical tool has been employed to address the problem of adaptive single-subspace detection problem (of point-like targets) in the joint presence of random and subspace structured interference in [18]. In this respect, we build upon the aforementioned results in order to develop an exhaustive study for the considered I-GMANOVA model under the lens of the invariance.

B. *Summary of the Contributions and Paper Organization*

The main contributions of the first part of the present study are summarized as follows:

- We first show that the problem at hand admits a more intuitive representation, by exploiting the *canonical form*. Such representation helps obtaining the maximal invariant statistics and gaining insights for the problem under investigation;
- The group of transformations which leaves the problem invariant is identified, thus allowing the search for a MIS.
- Given the aforementioned group of transformations, the canonical form is exploited in order to obtain the MIS, which, for the I-GMANOVA model is represented by *two matrices* which compress the original data. Such result can be interpreted as the generalization of the two-components scalar MIS obtained in the classical references [12], [14].
- A theoretical performance analysis of the MIS is obtained, in terms of its distribution. Even though in the considered setup the MIS does not generally admit an explicit expression for its probability density function (pdf), a simpler form of the statistic distribution, by means of a suitable stochastic representation, is provided.
- Finally, the obtained MIS expression is compared with similar findings obtained in the literature for simpler scenarios, thus showing that the aforementioned cases can be seen as special instances of the obtained MIS.

The explicit expression of the MIS obtained in this first part is then exploited to show CFARness of all the detectors considered in Part II of this work.

The remainder of the paper is organized as follows: in Section II we introduce the hypothesis testing problem under investigation; in Section III we describe the desirable invariance properties and derive the MIS. Section IV is devoted to the statistical characterization of the MIS, while in Section V we particularize the MIS to specific instances and compare it with previously obtained results in the open literature. Some concluding remarks and future research directions are given in Section VI; finally, proofs and derivations are confined to the Appendices.

Notation: Lower-case (resp. Upper-case) bold letters denote vectors (resp. matrices), with a_n (resp. $A_{n,m}$) representing the nth (resp. the (n,m)th) element of the vector $\boldsymbol{a}$ (resp. matrix $\boldsymbol{A}$); $\mathbb{R}^{N\times 1}$ (resp. $\mathbb{R}^{N\times M}$), $\mathbb{C}^{N\times 1}$ (resp. $\mathbb{C}^{N\times M}$), and $\mathbb{H}^{N\times N}$ are the sets of N-dimensional column vectors (resp. $N\times M$ matrices) of real numbers, of complex numbers, and of $N\times N$ Hermitian matrices, respectively; upper-case calligraphic letters and braces denote finite sets; $\mathbb{E}\{\cdot\}$, $\mathrm{Cov}[\cdot]$, $(\cdot)^T$, $(\cdot)^\dagger$, $\angle\cdot$, $\mathrm{Tr}[\cdot]$, denote expectation, covariance, transpose, Hermitian, phase and matrix trace operators, respectively; $\boldsymbol{0}_{N\times M}$ (resp. $\boldsymbol{I}_N$) denotes the $N\times M$ null (resp. $N\times N$ dimensional identity) matrix; $\boldsymbol{0}_N$ (resp. $\boldsymbol{1}_N$) denotes the null (resp. ones) column vector of length N; $\mathrm{vec}(\boldsymbol{M})$ stacks the first to the last column of the matrix $\boldsymbol{M}$ one under another to form a long vector; $\det(\boldsymbol{A})$ and $\|\boldsymbol{A}\|_F$ denote the determinant and Frobenius norm of matrix $\boldsymbol{A}$; $\boldsymbol{A}\otimes\boldsymbol{B}$ indicates the Kronecker product between $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices; $\mathrm{diag}(\boldsymbol{A},\boldsymbol{B})$ denotes the block-diagonal matrix obtained by placing matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ along the main diagonal; the symbol "$\sim$" means "distributed as"; $\boldsymbol{x}\sim\mathcal{CN}_N(\boldsymbol{\mu},\boldsymbol{\Sigma})$ denotes a complex (proper) Gaussian-distributed vector $\boldsymbol{x}$ with mean vector $\boldsymbol{\mu}\in\mathbb{C}^{N\times 1}$ and covariance matrix $\boldsymbol{\Sigma}\in\mathbb{C}^{N\times N}$; $\boldsymbol{X}\sim\mathcal{CN}_{N\times M}(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ denotes a complex (proper) Gaussian-distributed matrix $\boldsymbol{X}$ with mean $\boldsymbol{A}\in\mathbb{C}^{N\times M}$ and $\mathrm{Cov}[\mathrm{vec}(\boldsymbol{X})]=\boldsymbol{B}\otimes\boldsymbol{C}$; $\boldsymbol{S}\sim\mathcal{CW}_N(K,\boldsymbol{A})$ denotes a complex central Wishart distributed matrix $\boldsymbol{S}$ with parameters $K\in\mathbb{N}$ and $\boldsymbol{A}\in\mathbb{C}^{N\times N}$ positive definite matrix; $\boldsymbol{M}\sim\mathcal{CF}_a(\boldsymbol{A},\ell,m)$ is a non-central multivariate complex F distributed matrix $\boldsymbol{M}$ with mean $\boldsymbol{A}$ and parameters a, ℓ, and m; $\boldsymbol{P}_{\boldsymbol{A}}$ denotes the orthogonal projection of the full-column-rank matrix $\boldsymbol{A}$, that is $\boldsymbol{P}_{\boldsymbol{A}}\triangleq[\boldsymbol{A}(\boldsymbol{A}^\dagger\boldsymbol{A})^{-1}\boldsymbol{A}^\dagger]$, while $\boldsymbol{P}_{\boldsymbol{A}}^{\perp}$ its complement, that is $\boldsymbol{P}_{\boldsymbol{A}}^{\perp}\triangleq(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{A}})$.

II. Problem Formulation

We assume that a matrix of complex-valued samples $\boldsymbol{X}\in\mathbb{C}^{N\times K}$ is collected, accounting for both primary (signal-bearing) and secondary (signal-free) data. The hypothesis testing problem under investigation can be formulated as:

$$\begin{cases}\mathcal{H}_0: & \boldsymbol{X}=\widetilde{\boldsymbol{A}}_t\widetilde{\boldsymbol{B}}_t\widetilde{\boldsymbol{C}}+\boldsymbol{N}_0\\ \mathcal{H}_1: & \boldsymbol{X}=(\widetilde{\boldsymbol{A}}_t\widetilde{\boldsymbol{B}}_t+\widetilde{\boldsymbol{A}}_r\widetilde{\boldsymbol{B}}_r)\widetilde{\boldsymbol{C}}+\boldsymbol{N}_0\end{cases}, \tag{1}$$

where:

- $\boldsymbol{N}_0\in\mathbb{C}^{N\times K}$ is a matrix whose columns are independent and identically distributed (iid) proper complex normal random vectors with zero mean and (unknown) positive definite covariance matrix $\boldsymbol{R}_\star\in\mathbb{C}^{N\times N}$, that is $\boldsymbol{N}_0\sim\mathcal{CN}_{N\times K}(\boldsymbol{0}_{N\times K},\boldsymbol{I}_K,\boldsymbol{R}_\star)$;
- $\widetilde{\boldsymbol{B}}_t\in\mathbb{C}^{t\times M}$ and $\widetilde{\boldsymbol{B}}_r\in\mathbb{C}^{r\times M}$ denote the (unknown) deterministic matrix coordinates, representing the interference and the useful signal, respectively;
- $\widetilde{\boldsymbol{A}}_t\in\mathbb{C}^{N\times t}$ and $\widetilde{\boldsymbol{A}}_r\in\mathbb{C}^{N\times r}$ represent the (known) left subspace of the interference and the useful signal, respectively. The matrices $\widetilde{\boldsymbol{A}}_t$ and $\widetilde{\boldsymbol{A}}_r$ are both assumed full-column rank, with their columns being linearly independent;

- Similarly, $\widetilde{\boldsymbol{C}} \in \mathbb{C}^{M\times K}$ is a known matrix describing the right subspace associated to *both* signal and interference; the matrix $\widetilde{\boldsymbol{C}}$ is assumed full-row rank.

Additionally, aiming at a compact notation, we define $\widetilde{\boldsymbol{A}} \triangleq [\widetilde{\boldsymbol{A}}_t \quad \widetilde{\boldsymbol{A}}_r] \in \mathbb{C}^{N\times J}$ and $\widetilde{\boldsymbol{B}} \triangleq [\widetilde{\boldsymbol{B}}_t^T \quad \widetilde{\boldsymbol{B}}_r^T]^T \in \mathbb{C}^{J\times M}$, where we have denoted $J \triangleq r + t$. In the following, our analysis is carried out assuming that $(K - M) \geq N$. Such condition is typically satisfied in practical adaptive detection setups [2].

From inspection of (1), we notice that the considered test has a complicated structure, which is thus difficult to analyze. Therefore, before proceeding further, we first show that a simpler equivalent formulation of the considered problem can be obtained in the so-called canonical form [2].

With this intent, we first consider the QR-decomposition of $\widetilde{\boldsymbol{A}} = \boldsymbol{Q}_\alpha \boldsymbol{R}_\alpha$, where $\boldsymbol{Q}_\alpha \in \mathbb{C}^{N\times J}$ is a slice of a unitary matrix (i.e., $\boldsymbol{Q}_\alpha^\dagger \boldsymbol{Q}_\alpha = \boldsymbol{I}_J$) and $\boldsymbol{R}_\alpha \in \mathbb{C}^{J\times J}$ a non-singular upper triangular matrix. It can be readily shown that $\boldsymbol{Q}_\alpha$ and $\boldsymbol{R}_\alpha$ can be conveniently partitioned as:

$$\boldsymbol{Q}_\alpha = [\boldsymbol{Q}_{\alpha,t} \quad \boldsymbol{Q}_{\alpha,r}], \quad \boldsymbol{R}_\alpha = \begin{bmatrix} \boldsymbol{R}_{\alpha,t} & \boldsymbol{R}_{\alpha,x} \\ \boldsymbol{0}_{r\times t} & \boldsymbol{R}_{\alpha,r} \end{bmatrix}, \tag{2}$$

where $\boldsymbol{Q}_{\alpha,t} \in \mathbb{C}^{N\times t}$ and $\boldsymbol{R}_{\alpha,t} \in \mathbb{C}^{t\times t}$ arise from the QR-decomposition of $\widetilde{\boldsymbol{A}}_t$, namely $\widetilde{\boldsymbol{A}}_t = \boldsymbol{Q}_{\alpha,t}\boldsymbol{R}_{\alpha,t}$, with $\boldsymbol{Q}_{\alpha,t}$ such that $\boldsymbol{Q}_{\alpha,t}^\dagger \boldsymbol{Q}_{\alpha,t} = \boldsymbol{I}_t$ and $\boldsymbol{R}_{\alpha,t}$ a non-singular upper triangular matrix. Furthermore, $\boldsymbol{R}_{\alpha,x} \in \mathbb{C}^{t\times r}$ and $\boldsymbol{R}_{\alpha,r} \in \mathbb{C}^{r\times r}$ is another non-singular upper triangular matrix. Similarly, $\boldsymbol{Q}_{\alpha.r} \in \mathbb{C}^{N\times r}$ is such that $\boldsymbol{Q}_{\alpha.r}^\dagger \boldsymbol{Q}_{\alpha.r} = \boldsymbol{I}_r$. Equalities in (2) are almost immediate consequences of the well-known Gram-Schmidt procedure [19]. Now, let us define a unitary matrix $\boldsymbol{U}_\alpha \in \mathbb{C}^{N\times N}$ whose first J columns are collectively equal to $\boldsymbol{Q}_\alpha$. Then, it follows that:

$$\boldsymbol{A} \triangleq \underbrace{\boldsymbol{U}_\alpha^\dagger \boldsymbol{Q}_\alpha}_{\in\mathbb{C}^{N\times J}} = \begin{bmatrix} \boldsymbol{I}_t & \boldsymbol{0}_{t\times r} \\ \boldsymbol{0}_{r\times t} & \boldsymbol{I}_r \\ \boldsymbol{0}_{(N-J)\times t} & \boldsymbol{0}_{(N-J)\times r} \end{bmatrix} = [\boldsymbol{E}_t \quad \boldsymbol{E}_r], \tag{3}$$

where

$$\boldsymbol{E}_t \triangleq [\boldsymbol{I}_t \quad \boldsymbol{0}_{t\times r} \quad \boldsymbol{0}_{t\times(N-J)}]^T$$

and

$$\boldsymbol{E}_r \triangleq [\boldsymbol{0}_{r\times t} \quad \boldsymbol{I}_r \quad \boldsymbol{0}_{r\times(N-J)}]^T,$$

respectively. Also, let $\widetilde{\boldsymbol{C}}$ be expressed in terms of its Singular Value Decomposition (SVD) as

$$\widetilde{\boldsymbol{C}} = \boldsymbol{U}_\gamma \boldsymbol{\Lambda}_\gamma \boldsymbol{V}_\gamma^\dagger, \tag{4}$$

where $\boldsymbol{U}_\gamma \in \mathbb{C}^{M\times M}$ and $\boldsymbol{V}_\gamma \in \mathbb{C}^{K\times K}$ are both unitary matrices, and the matrix of the singular values $\boldsymbol{\Lambda}_\gamma \in \mathbb{R}^{M\times K}$ has the following noteworthy form (since $K \geq M$):

$$\boldsymbol{\Lambda}_\gamma = [\widetilde{\boldsymbol{\Lambda}}_\gamma \quad \boldsymbol{0}_{M\times(K-M)}], \tag{5}$$

with $\widetilde{\boldsymbol{\Lambda}}_\gamma \in \mathbb{C}^{M\times M}$ being a diagonal matrix. Therefore

$$\widetilde{\boldsymbol{C}}\boldsymbol{V}_\gamma = \boldsymbol{M}_\gamma \, [\boldsymbol{I}_M \quad \boldsymbol{0}_{M\times(K-M)}] \tag{6}$$

holds, where $\boldsymbol{M}_\gamma \triangleq \boldsymbol{U}_\gamma \widetilde{\boldsymbol{\Lambda}}_\gamma$.

Given the aforementioned definitions, without loss of generality we will consider the transformed data matrix $\boldsymbol{Z} \triangleq (\boldsymbol{U}_\alpha^\dagger \boldsymbol{X} \boldsymbol{V}_\gamma) \in \mathbb{C}^{N\times K}$ in what follows. Such transformation does not alter the hypothesis testing problem being considered, as it simply applies left and right rotations to data matrix $\boldsymbol{X}$ (viz. multiplications by unitary matrices). The new data matrix, when $\mathcal{H}_1$ is in force, can be expressed as:

$$\begin{aligned} \boldsymbol{Z} &= \boldsymbol{U}_\alpha^\dagger(\boldsymbol{Q}_\alpha \boldsymbol{R}_\alpha \widetilde{\boldsymbol{B}} \boldsymbol{U}_\gamma \boldsymbol{\Lambda}_\gamma \boldsymbol{V}_\gamma^\dagger)\boldsymbol{V}_\gamma + \boldsymbol{N} & (7) \\ &= \boldsymbol{A}(\boldsymbol{R}_\alpha \widetilde{\boldsymbol{B}} \boldsymbol{M}_\gamma)\,[\boldsymbol{I}_M \quad \boldsymbol{0}_{M\times(K-M)}] + \boldsymbol{N} & (8) \\ &= \boldsymbol{A} \begin{bmatrix} \boldsymbol{B}_{t,1} \\ \boldsymbol{B} \end{bmatrix} [\boldsymbol{I}_M \quad \boldsymbol{0}_{M\times(K-M)}] + \boldsymbol{N}, & (9) \end{aligned}$$

where we have defined $\boldsymbol{B}_{t,1} \triangleq ((\boldsymbol{R}_{\alpha,t}\widetilde{\boldsymbol{B}}_t + \boldsymbol{R}_{\alpha,x}\widetilde{\boldsymbol{B}}_r)\boldsymbol{M}_\gamma) \in \mathbb{C}^{t\times M}$, $\boldsymbol{B} \triangleq (\boldsymbol{R}_{\alpha,r}\widetilde{\boldsymbol{B}}_r \boldsymbol{M}_\gamma) \in \mathbb{C}^{r\times M}$, and $\boldsymbol{N} \triangleq (\boldsymbol{U}_\alpha^\dagger \boldsymbol{N}_0 \boldsymbol{V}_\gamma) \in \mathbb{C}^{N\times K}$, respectively. Furthermore, for the sake of notational convenience, we have defined $\boldsymbol{B}_s \triangleq [\boldsymbol{B}_{t,1}^T \quad \boldsymbol{B}^T]^T$. On the other hand, when $\mathcal{H}_0$ holds true, the matrix $\boldsymbol{Z}$ can be expressed as:

$$\begin{aligned} \boldsymbol{Z} &= \boldsymbol{U}_\alpha^\dagger \left(\boldsymbol{Q}_\alpha \boldsymbol{R}_\alpha \begin{bmatrix} \widetilde{\boldsymbol{B}}_t \\ \boldsymbol{0}_{r\times M} \end{bmatrix} \boldsymbol{U}_\gamma \boldsymbol{\Lambda}_\gamma \boldsymbol{V}_\gamma^\dagger\right)\boldsymbol{V}_\gamma + \boldsymbol{N} & (10) \\ &= \boldsymbol{A} \begin{bmatrix} \boldsymbol{B}_{t,0} \\ \boldsymbol{0}_{r\times M} \end{bmatrix} [\boldsymbol{I}_M \quad \boldsymbol{0}_{M\times(K-M)}] + \boldsymbol{N}, & (11) \end{aligned}$$

where $\boldsymbol{B}_{t,0} \triangleq (\boldsymbol{R}_{\alpha,t}\widetilde{\boldsymbol{B}}_t \boldsymbol{M}_\gamma) \in \mathbb{C}^{t\times M}$. Furthermore, aiming at keeping a compact notation, we will employ the definition $\boldsymbol{C} \triangleq [\boldsymbol{I}_M \quad \boldsymbol{0}_{M\times(K-M)}]$ in what follows. Gathering all the above results, the problem in (1) can be equivalently rewritten in terms of $\boldsymbol{Z}$ as:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{A}\begin{bmatrix} \boldsymbol{B}_{t,0} \\ \boldsymbol{0}_{r\times M} \end{bmatrix}\boldsymbol{C} + \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{A}\boldsymbol{B}_s\boldsymbol{C} + \boldsymbol{N} \end{cases} \tag{12}$$

Finally we recall that, since $\boldsymbol{N}_0 \sim \mathcal{CN}_{N\times K}(\boldsymbol{0}_{N\times K}, \boldsymbol{I}_K, \boldsymbol{R}_\star)$, $\boldsymbol{N}$ is distributed as $\boldsymbol{N} \sim \mathcal{CN}_{N\times K}(\boldsymbol{0}_{N\times K}, \boldsymbol{I}_K, \boldsymbol{R})$, where $\boldsymbol{R} \triangleq (\boldsymbol{U}_\alpha^\dagger \boldsymbol{R}_\star \boldsymbol{U}_\alpha)$ [2].

An important remark is now in order. Specifically, for the problem in (1), the relevant parameter to decide for the presence of a target is $\widetilde{\boldsymbol{B}}_r$. Otherwise stated, if the hypothesis $\mathcal{H}_1$ holds true, then $\|\widetilde{\boldsymbol{B}}_r\|_F > 0$, while $\|\widetilde{\boldsymbol{B}}_r\|_F = 0$ under the target-absent hypothesis ($\mathcal{H}_0$). As a consequence, since $\boldsymbol{R}_{\alpha,r}$ is non-singular, problem in (12) is equivalent to:

$$\begin{cases} \mathcal{H}_0: & \|\boldsymbol{B}\|_F = 0, \\ \mathcal{H}_1: & \|\boldsymbol{B}\|_F > 0, \end{cases} \tag{13}$$

which partitions the relevant-signal parameter space, say $\boldsymbol{\Theta}_r$, as:

$$\boldsymbol{\Theta}_r = \underbrace{\{\boldsymbol{0}_{r\times M}\}}_{\boldsymbol{\Theta}_{r,0}} \cup \underbrace{\{\boldsymbol{B} \in \mathbb{C}^{r\times M} : \|\boldsymbol{B}\|_F > 0\}}_{\boldsymbol{\Theta}_{r,1}}. \tag{14}$$

The canonical form in (12) will be exploited hereinafter in our analysis.

III. Maximal Invariant Statistic

In what follows, we will search for functions of the data sharing invariance with respect to those parameters (namely the nuisance parameters, $\boldsymbol{R}$, $\boldsymbol{B}_{t,1}$, and $\boldsymbol{B}_{t,0}$) which are irrelevant for the specific decision problem. To this end, we resort to the so-called "Principle of Invariance" [10], whose main idea consists in finding transformations that properly cluster data without altering

- the formal structure of the hypothesis testing problem given by (14);
- the Gaussian assumption for the received data matrix under each hypothesis;
- the double-subspace structure containing the useful signal components.

The following subsection is thus devoted to the definition of a suitable group which fulfills the above requirements.

A. Desired Invariance Properties

Let

$$\boldsymbol{V}_{c,1} \triangleq \begin{bmatrix} \boldsymbol{I}_M \\ \boldsymbol{0}_{(K-M)\times M} \end{bmatrix}, \quad \boldsymbol{V}_{c,2} \triangleq \begin{bmatrix} \boldsymbol{0}_{M\times(K-M)} \\ \boldsymbol{I}_{K-M} \end{bmatrix}, \tag{15}$$

and observe that $\boldsymbol{P}_{\boldsymbol{C}^\dagger} = (\boldsymbol{V}_{c,1}\boldsymbol{V}_{c,1}^\dagger)$ and $\boldsymbol{P}^{\perp}_{\boldsymbol{C}^\dagger} = (\boldsymbol{V}_{c,2}\boldsymbol{V}_{c,2}^\dagger)$.

Also, let us consider the sufficient statistic[1] $\{\boldsymbol{Z}_c, \boldsymbol{S}_c\}$, where the mentioned quantities are defined as

$$\boldsymbol{Z}_c \triangleq (\boldsymbol{Z}\boldsymbol{V}_{c,1}) \in \mathbb{C}^{N\times M}, \tag{16}$$

$$\boldsymbol{Z}_{c,\perp} \triangleq (\boldsymbol{Z}\,\boldsymbol{V}_{c,2}) \in \mathbb{C}^{N\times(K-M)}, \tag{17}$$

$$\boldsymbol{S}_c \triangleq (\boldsymbol{Z}_{c,\perp}\boldsymbol{Z}_{c,\perp}^\dagger) = (\boldsymbol{Z}\boldsymbol{P}^{\perp}_{\boldsymbol{C}^\dagger}\boldsymbol{Z}^\dagger) \in \mathbb{C}^{N\times N}. \tag{18}$$

Clearly, given the simplified structure of $\boldsymbol{C}$, $\boldsymbol{Z}_c$ (resp. $\boldsymbol{Z}_{c,\perp}$) is simply obtained by taking the first M (resp. the last $K-M$) columns of the transformed data matrix $\boldsymbol{Z}$.

Now, denote by $\mathcal{GL}(N)$ the linear group of $N\times N$ non singular matrices and introduce the following sets

$$\mathcal{G} \triangleq \left\{ \boldsymbol{G} \triangleq \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} & \boldsymbol{G}_{13} \\ \boldsymbol{0}_{r\times t} & \boldsymbol{G}_{22} & \boldsymbol{G}_{23} \\ \boldsymbol{0}_{(N-J)\times t} & \boldsymbol{0}_{(N-J)\times r} & \boldsymbol{G}_{33} \end{bmatrix} \in \mathcal{GL}(N) \right.$$
$$\left. : \boldsymbol{G}_{11} \in \mathcal{GL}(t),\, \boldsymbol{G}_{22} \in \mathcal{GL}(r),\, \boldsymbol{G}_{33} \in \mathcal{GL}(N-J) \right\} \tag{19}$$

$$\mathcal{F} \triangleq \left\{ \boldsymbol{F} \triangleq \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{0}_{r\times M} \\ \boldsymbol{0}_{(N-J)\times M} \end{bmatrix} \in \mathbb{C}^{N\times M} \;:\; \boldsymbol{F}_1 \in \mathbb{C}^{t\times M} \right\} \tag{20}$$

along with the composition operator "$\circ$", defined as:

$$(\boldsymbol{G}_a, \boldsymbol{F}_a) \circ (\boldsymbol{G}_b, \boldsymbol{F}_b) = (\boldsymbol{G}_b\boldsymbol{G}_a, \boldsymbol{G}_b\boldsymbol{F}_a + \boldsymbol{F}_b)\,. \tag{21}$$

The sets and the composition operator are here represented compactly as $\mathcal{L} \triangleq (\mathcal{G}\times\mathcal{F}, \circ)$. Then, it is not difficult to show that $\mathcal{L}$ constitutes a *group*, since it satisfies the following elementary axioms:

- $\mathcal{L}$ is *closed* with respect to the operation "$\circ$", defined in (21);
- $\forall(\boldsymbol{G}_a, \boldsymbol{F}_a)$, $(\boldsymbol{G}_b, \boldsymbol{F}_b)$, and $(\boldsymbol{G}_c, \boldsymbol{F}_c) \in \mathcal{L}$: $[(\boldsymbol{G}_a, \boldsymbol{F}_a) \circ (\boldsymbol{G}_b, \boldsymbol{F}_b)] \circ (\boldsymbol{G}_c, \boldsymbol{F}_c) = (\boldsymbol{G}_a, \boldsymbol{F}_a) \circ [(\boldsymbol{G}_b, \boldsymbol{F}_b) \circ (\boldsymbol{G}_c, \boldsymbol{F}_c)]$ (*Associative property*);
- there exists a unique $(\boldsymbol{G}_I, \boldsymbol{F}_I) \in \mathcal{L}$ such that $\forall(\boldsymbol{G}, \boldsymbol{F}) \in \mathcal{L}$: $(\boldsymbol{G}_I, \boldsymbol{F}_I) \circ (\boldsymbol{G}, \boldsymbol{F}) = (\boldsymbol{G}, \boldsymbol{F}) \circ (\boldsymbol{G}_I, \boldsymbol{F}_I) = (\boldsymbol{G}, \boldsymbol{F})$ (*Existence of the Identity element*);
- $\forall(\boldsymbol{G}, \boldsymbol{F}) \in \mathcal{L}$, there exists $(\boldsymbol{G}_{-1}, \boldsymbol{F}_{-1}) \in \mathcal{L}$ such that $(\boldsymbol{G}_{-1}, \boldsymbol{F}_{-1}) \circ (\boldsymbol{G}, \boldsymbol{F}) = (\boldsymbol{G}, \boldsymbol{F}) \circ (\boldsymbol{G}_{-1}, \boldsymbol{F}_{-1}) = (\boldsymbol{G}_I, \boldsymbol{F}_I)$ (*Existence of the Inverse element*).

Also, the aforementioned group leaves the hypothesis testing problem in (12) invariant under the action $\ell(\cdot,\cdot)$ defined by:

$$\ell(\boldsymbol{Z}_c, \boldsymbol{S}_c) = \left(\boldsymbol{G}\boldsymbol{Z}_c + \boldsymbol{F}, \boldsymbol{G}\,\boldsymbol{S}_c\,\boldsymbol{G}^\dagger\right) \quad \forall(\boldsymbol{G}, \boldsymbol{F}) \in \mathcal{L}\,. \tag{22}$$

The proof of the aforementioned statement is given in Appendix A. Moreover, it is important to point out that $\mathcal{L}$ preserves the family of distributions, and, at the same time, includes those transformations which are relevant from a practical point of view, as they allow claiming the CFAR property (with respect to $\boldsymbol{R}$ and $\boldsymbol{B}_{t,i}$) as a consequence of the invariance.

B. Derivation of the MIS

In Section III.A we have identified a group $\mathcal{L}$ which leaves unaltered the problem under investigation. It is thus reasonable finding decision rules that are invariant under $\mathcal{L}$. Toward this goal, the Principle of Invariance is invoked because it allows to construct statistics that organize data into distinguishable equivalence classes. Such functions of the data are called Maximal Invariant Statistics and, given the group of transformations, every invariant test may be written as a function of the maximal invariant [11].

Before presenting the explicit expression of the MIS, we give the following preliminary definitions based on the partitioning of matrices $\boldsymbol{Z}_c$ and $\boldsymbol{S}_c$:

$$\boldsymbol{Z}_c = \begin{bmatrix} \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \\ \boldsymbol{Z}_3 \end{bmatrix}; \quad \boldsymbol{S}_c = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{S}_{13} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{31} & \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix}. \tag{23}$$

where $\boldsymbol{Z}_1 \in \mathbb{C}^{t\times M}$, $\boldsymbol{Z}_2 \in \mathbb{C}^{r\times M}$, and $\boldsymbol{Z}_3 \in \mathbb{C}^{(N-J)\times M}$, respectively; $\boldsymbol{S}_{ij}$, $(i,j) \in \{1,2,3\}\times\{1,2,3\}$, is a sub-matrix whose dimensions can be obtained replacing 1, 2, and 3 with t, r, and $(N-J)$, respectively[2]. Furthermore, we also define the following partitioning for $\boldsymbol{Z}_{c,\perp}$, which will be used throughout the manuscript:

$$\boldsymbol{Z}_{c,\perp} = [\,\boldsymbol{Z}_{\perp,1}^T \quad \boldsymbol{Z}_{\perp,2}^T \quad \boldsymbol{Z}_{\perp,3}^T\,]^T, \tag{24}$$

where $\boldsymbol{Z}_{\perp,1} \in \mathbb{C}^{t\times(K-M)}$, $\boldsymbol{Z}_{\perp,2} \in \mathbb{C}^{r\times(K-M)}$ and $\boldsymbol{Z}_{\perp,3} \in \mathbb{C}^{(N-J)\times(K-M)}$, respectively. We underline that each sub-matrix of $\boldsymbol{S}_c$ in (23) can be expressed in terms of (24), that is, $\boldsymbol{S}_{ij} = (\boldsymbol{Z}_{\perp,i}\boldsymbol{Z}_{\perp,j}^\dagger)$. We are thus ready to present the proposition providing the expression of a maximal invariant for the problem at hand.

Proposition 1: A MIS with respect to $\mathcal{L}$ for the problem in (12) is given by:

$$\boldsymbol{T}(\boldsymbol{Z}_c, \boldsymbol{S}_c) = \begin{cases} \begin{bmatrix} \boldsymbol{T}_a \triangleq \left\{ \boldsymbol{Z}_{2.3}^\dagger\,\boldsymbol{S}_{2.3}^{-1}\,\boldsymbol{Z}_{2.3} \right\} \\ \boldsymbol{T}_b \triangleq \left\{ \boldsymbol{Z}_3^\dagger\,\boldsymbol{S}_{33}^{-1}\boldsymbol{Z}_3 \right\} \end{bmatrix}, & J < N \\ \boldsymbol{Z}_2^\dagger\,\boldsymbol{S}_{22}^{-1}\,\boldsymbol{Z}_2, & J = N \end{cases} \tag{25}$$

[1]Indeed, Fisher-Neyman factorization theorem ensures that deciding from $\{\boldsymbol{Z}_c, \boldsymbol{S}_c\}$ is tantamount to deciding from raw data $\boldsymbol{Z}$ [3].

[2]Hereinafter, in the case $J = N$, the "3-components" are no longer present in the partitioning.

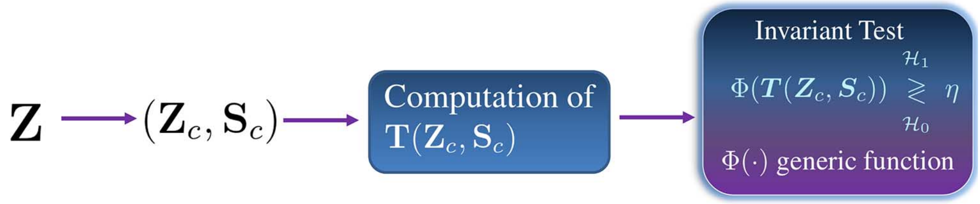

Fig. 1. Block diagram of processing leading to a generic invariant test.

where $\boldsymbol{Z}_{2.3} \triangleq (\boldsymbol{Z}_2 - \boldsymbol{S}_{23}\boldsymbol{S}_{33}^{-1}\boldsymbol{Z}_3)$ and $\boldsymbol{S}_{2.3} \triangleq (\boldsymbol{S}_{22} - \boldsymbol{S}_{23}\boldsymbol{S}_{33}^{-1}\boldsymbol{S}_{32})$.

Proof: The proof is given in Appendix B. ■

Some important remarks are now in order.

- In the case $J < N$, the MIS is given by a pair of matrices (namely $\boldsymbol{T}_a$ and $\boldsymbol{T}_b$) where the second component ($\boldsymbol{T}_b$) represents an *ancillary part*, that is, its distribution does not depend on the hypothesis in force;
- In the case $J < N$, the matrices $\boldsymbol{T}_a \in \mathbb{C}^{M\times M}$ and $\boldsymbol{T}_b \in \mathbb{C}^{M\times M}$ have rank equal to $\min\{M, r\}$ and $\min\{M, N - J\}$, respectively;
- It is of certain interest comparing the general expression in (25) with the MIS instances obtained in [12], [14], [16], [18] for specific adaptive detection scenarios. Accordingly, Section V will be devoted to comparisons and exhaustive discussion of the specialized forms in some relevant scenarios;
- Finally, exploiting [10, Thm. 6.2.1], every invariant test may be written as a function of (25) (see Fig. 1 for a schematic representation). Therefore, it naturally follows that every CFAR test can be expressed in terms of the MIS. Part II of this study will be devoted to the design of theoretically-founded detectors whose CFARness will be proved by showing their dependence on the data solely through the obtained MIS.

IV. Statistical Characterization of the MIS

In this section, we provide the statistical characterization of the MIS for the case $J < N$, as the case referring to $J = N$ can be obtained as a simple corollary. To this end, we show that the MIS can be written as a function of whitened random vectors and matrices and then we find a suitable stochastic representation by means of a one-to-one transformation.

First, we consider the following transformation $(\boldsymbol{G}^\circ, \boldsymbol{F}^\circ) \in \mathcal{L}$, which leads to:

$$\boldsymbol{Z}_c^\circ = \boldsymbol{G}^\circ \boldsymbol{Z}_c + \boldsymbol{F}^\circ = [\, \boldsymbol{Z}_1^{\circ T} \quad \boldsymbol{Z}_2^{\circ T} \quad \boldsymbol{Z}_3^{\circ T} \,]^T \tag{26}$$

and

$$\boldsymbol{S}_c^\circ = \boldsymbol{G}^\circ \boldsymbol{S}_c \boldsymbol{G}^{\circ\dagger} = \begin{bmatrix} \boldsymbol{S}_{11}^\circ & \boldsymbol{S}_{12}^\circ & \boldsymbol{S}_{13}^\circ \\ \boldsymbol{S}_{21}^\circ & \boldsymbol{S}_{22}^\circ & \boldsymbol{S}_{23}^\circ \\ \boldsymbol{S}_{31}^\circ & \boldsymbol{S}_{32}^\circ & \boldsymbol{S}_{33}^\circ \end{bmatrix}, \tag{27}$$

where $\boldsymbol{Z}_i^\circ$ and $\boldsymbol{S}_{\ell m}^\circ$ ($i, \ell, m \in \{1, 2, 3\}$) are similarly defined as in (23) and the pair $(\boldsymbol{G}^\circ, \boldsymbol{F}^\circ)$ is suitably defined as:

$$\boldsymbol{G}^\circ \triangleq \begin{bmatrix} \boldsymbol{G}_{11}^\circ & \boldsymbol{G}_{12}^\circ & \boldsymbol{G}_{13}^\circ \\ \boldsymbol{0}_{r\times t} & \boldsymbol{R}_{2.3}^{-1/2} & -\boldsymbol{R}_{2.3}^{-1/2}\boldsymbol{R}_{23}\,\boldsymbol{R}_{33}^{-1} \\ \boldsymbol{0}_{(N-J)\times t} & \boldsymbol{0}_{(N-J)\times r} & \boldsymbol{R}_{33}^{-1/2} \end{bmatrix}; \tag{28}$$

$$\boldsymbol{F}^\circ \triangleq \begin{bmatrix} \boldsymbol{F}_1^\circ \\ \boldsymbol{0}_{r\times M} \\ \boldsymbol{0}_{(N-J)\times M} \end{bmatrix}; \tag{29}$$

where $\boldsymbol{G}_{1j}^\circ$, $j \in \{1, 2, 3\}$, and $\boldsymbol{F}_1^\circ$ are generic matrices of proper dimensions, while $\boldsymbol{R}_{22} \in \mathbb{C}^{r\times r}$, $\boldsymbol{R}_{23} \in \mathbb{C}^{r\times(N-J)}$, and $\boldsymbol{R}_{33} \in \mathbb{C}^{(N-J)\times(N-J)}$ are obtained partitioning the true covariance matrix $\boldsymbol{R}$ in the same way as done for $\boldsymbol{S}$ in (23). Finally, we have defined $\boldsymbol{R}_{2.3} \triangleq \boldsymbol{R}_{22} - \boldsymbol{R}_{23}\,\boldsymbol{R}_{33}^{-1}\,\boldsymbol{R}_{23}^\dagger$.

Hereinafter, we will study MIS statistical characterization after the transformation $(\boldsymbol{G}^\circ, \boldsymbol{F}^\circ)$. This will simplify the subsequent analysis and does not affect the obtained results since the MIS is (by definition) invariant with respect to every transformation belonging to $\mathcal{L}$.

Now observe that, under $\mathcal{H}_i$, $i \in \{0, 1\}$, it holds:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{Z}_2^\circ \\ \boldsymbol{Z}_3^\circ \end{bmatrix} |\mathcal{H}_i &= \boldsymbol{G}_3^\circ \begin{bmatrix} \boldsymbol{Z}_2 \\ \boldsymbol{Z}_3 \end{bmatrix} |\mathcal{H}_i \\ &\sim \mathcal{CN}_{(N-t)\times M}\left(i\,\boldsymbol{G}_3^\circ \begin{bmatrix} \boldsymbol{I}_r \\ \boldsymbol{0}_{(N-J)\times r} \end{bmatrix} \boldsymbol{B}, \boldsymbol{I}_M, \boldsymbol{I}_{N-t} \right), \end{aligned} \tag{30}$$

$$\begin{aligned} \begin{bmatrix} \boldsymbol{S}_{22}^\circ & \boldsymbol{S}_{23}^\circ \\ \boldsymbol{S}_{32}^\circ & \boldsymbol{S}_{33}^\circ \end{bmatrix} &= \left\{ \boldsymbol{G}_3^\circ \begin{bmatrix} \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix} (\boldsymbol{G}_3^\circ)^\dagger \right\} \\ &\sim \mathcal{CW}_{N-t}(K - M, \boldsymbol{I}_{N-t}), \end{aligned} \tag{31}$$

where we have defined $\boldsymbol{G}_3^\circ \in \mathbb{C}^{(N-t)\times(N-t)}$ as

$$\boldsymbol{G}_3^\circ \triangleq \begin{bmatrix} \boldsymbol{R}_{2.3}^{-1/2} & -\boldsymbol{R}_{2.3}^{-1/2}\,\boldsymbol{R}_{23}\,\boldsymbol{R}_{33}^{-1} \\ \boldsymbol{0}_{(N-J)\times r} & \boldsymbol{R}_{33}^{-1/2} \end{bmatrix}. \tag{32}$$

Thus, exploiting the invariance property, we can equivalently rewrite the two components of $\boldsymbol{T}(\boldsymbol{Z}_c, \boldsymbol{S}_c)$ (cf. (25)) in terms of the *whitened* quantities

$$\boldsymbol{T}_a = (\boldsymbol{Z}_{2.3}^\circ)^\dagger\,(\boldsymbol{S}_{2.3}^\circ)^{-1}\,\boldsymbol{Z}_{2.3}^\circ, \tag{33}$$

$$\boldsymbol{T}_b = (\boldsymbol{Z}_3^\circ)^\dagger\,(\boldsymbol{S}_{33}^\circ)^{-1}\,\boldsymbol{Z}_3^\circ, \tag{34}$$

where $\boldsymbol{Z}_{2.3}^\circ \triangleq \boldsymbol{Z}_2^\circ - \boldsymbol{S}_{23}^\circ\,(\boldsymbol{S}_{33}^\circ)^{-1}\,\boldsymbol{Z}_3^\circ$ and $\boldsymbol{S}_{2.3}^\circ \triangleq \boldsymbol{S}_{22}^\circ - \boldsymbol{S}_{23}^\circ(\boldsymbol{S}_{33}^\circ)^{-1}\boldsymbol{S}_{32}^\circ$, respectively. Let us focus on $\boldsymbol{Z}_{2.3}^\circ$ and rewrite

$$\boldsymbol{Z}_2^\circ = [\, \boldsymbol{z}_{2,1}^\circ \quad \cdots \quad \boldsymbol{z}_{2,M}^\circ \,] \tag{35}$$

$$\boldsymbol{S}_{23}^\circ\,(\boldsymbol{S}_{33}^\circ)^{-1}\,\boldsymbol{Z}_3^\circ = \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k}\,\boldsymbol{r}_{3,k}^\dagger\,(\boldsymbol{S}_{33}^\circ)^{-1}\,\boldsymbol{Z}_3^\circ, \tag{36}$$

where $\boldsymbol{r}_{j,k}$ generically denotes the k-th column of $\boldsymbol{Z}_{\perp,j}^\circ$, the latter matrix being obtained from partitioning $\boldsymbol{Z}_{c,\perp}^\circ \triangleq \boldsymbol{G}^\circ \boldsymbol{Z}_{c,\perp}$ similarly as $\boldsymbol{Z}_{c,\perp}$ in (24). Given the aforementioned definitions, we obtain the explicit form given in (37) for $\boldsymbol{Z}_{2.3}^\circ$ in terms of its columns $\boldsymbol{q}_\ell$, $\ell \in \{1, \ldots M\}$, where we have further defined $\boldsymbol{\rho}_\ell \triangleq \{(\boldsymbol{S}_{33}^\circ)^{-1}\,\boldsymbol{z}_{3,\ell}^\circ\}$ and $\boldsymbol{z}_{3,\ell}^\circ$ similarly represents the ℓ-th column of $\boldsymbol{Z}_3^\circ$, that is, $\boldsymbol{Z}_3^\circ = [\, \boldsymbol{z}_{3,1}^\circ \quad \cdots \quad \boldsymbol{z}_{3,M}^\circ \,]$. [See (37) at the bottom of the page.]

First, it holds $\boldsymbol{r}_{2,k} \sim \mathcal{CN}_r(\boldsymbol{0}_r, \boldsymbol{I}_r)$ and $\boldsymbol{r}_{3,k} \sim \mathcal{CN}_{N-J}(\boldsymbol{0}_{N-J}, \boldsymbol{I}_{N-J})$; also it is apparent that these vectors are all mutually independent. Before proceeding, we define the short-hand notation "#3" to denote the conditioning with respect to all the terms with subscript "3". Then, it can be shown that $\boldsymbol{q}_\ell|(\#3, \mathcal{H}_0)$ is Gaussian distributed (recall that $\boldsymbol{z}_{2,\ell}^\circ|\mathcal{H}_0 \sim \mathcal{CN}_r(\boldsymbol{0}_r, \boldsymbol{I}_r)$) with mean vector $\boldsymbol{0}_r$ and covariance:

$$\mathbb{E}\left\{\left(\boldsymbol{z}_{2,\ell}^\circ - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k}\boldsymbol{r}_{3,k}^\dagger \boldsymbol{\rho}_\ell\right)\left(\boldsymbol{z}_{2,\ell}^\circ - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k}\boldsymbol{r}_{3,k}^\dagger \boldsymbol{\rho}_\ell\right)^\dagger\right\} = \boldsymbol{I}_r\left(1 + \boldsymbol{z}_{3,\ell}^\dagger\,(\boldsymbol{S}_{33}^\circ)^{-1}\boldsymbol{z}_{3,\ell}\right). \quad (38)$$

Similarly, the cross-covariance between $\boldsymbol{q}_m|(\#3, \mathcal{H}_0)$ and $\boldsymbol{q}_\ell|(\#3, \mathcal{H}_0)$ is given by:

$$\mathbb{E}\left\{\left(\boldsymbol{z}_{2,m}^\circ - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k}\boldsymbol{r}_{3,k}^\dagger \boldsymbol{\rho}_m\right)\left(\boldsymbol{z}_{2,\ell}^\circ - \sum_{k=1}^{K-M} \boldsymbol{r}_{2,k}\boldsymbol{r}_{3,k}^\dagger \boldsymbol{\rho}_\ell\right)^\dagger\right\} = \boldsymbol{I}_r\left(\boldsymbol{z}_{3,m}^\dagger\,(\boldsymbol{S}_{33}^\circ)^{-1}\boldsymbol{z}_{3,\ell}\right). \quad (39)$$

Therefore, in view of the aforementioned results, it follows that $\boldsymbol{\zeta}_{2.3} \triangleq \mathrm{vec}(\boldsymbol{Z}_{2.3}^\circ)$ is conditionally distributed as:

$$\boldsymbol{\zeta}_{2.3}|(\#3, \mathcal{H}_0) \sim \mathcal{CN}_{rM}(\boldsymbol{0}_{rM}, (\boldsymbol{I}_M + (\boldsymbol{Z}_3^\circ)^\dagger(\boldsymbol{S}_{33}^\circ)^{-1}\boldsymbol{Z}_3^\circ) \otimes \boldsymbol{I}_r). \quad (40)$$

Then, we whiten $\boldsymbol{\zeta}_{2.3}$, that is, we define:

$$\boldsymbol{x} \triangleq \left[(\boldsymbol{I}_M + \boldsymbol{Z}_3^{\circ\dagger}(\boldsymbol{S}_{33}^\circ)^{-1}\boldsymbol{Z}_3^\circ) \otimes \boldsymbol{I}_r\right]^{-1/2} \boldsymbol{\xi}_{2.3}, \quad (41)$$

which evidently gives $\boldsymbol{x}|(\#3, \mathcal{H}_0) \sim \mathcal{CN}_{rM}(\boldsymbol{0}_{rM}, \boldsymbol{I}_{rM})$. Also, we will exploit the equality

$$\left[\left(\boldsymbol{I}_M + \boldsymbol{Z}_3^{\circ\dagger}(\boldsymbol{S}_{33}^\circ)^{-1}\boldsymbol{Z}_3^\circ\right) \otimes \boldsymbol{I}_r\right]^{-1/2} = \left[\left(\boldsymbol{I}_M + \boldsymbol{Z}_3^{\circ\dagger}(\boldsymbol{S}_{33}^\circ)^{-1}\boldsymbol{Z}_3^\circ\right)^{-1/2}\right] \otimes \boldsymbol{I}_r \quad (42)$$

which readily follows from the distributive property of Kronecker product. As a consequence, we have that $\boldsymbol{x} = \mathrm{vec}(\boldsymbol{X})$, where we have defined

$$\boldsymbol{X} \triangleq (\boldsymbol{Z}_{2.3}^\circ\,\boldsymbol{K}_{s3}) \quad (43)$$

and $\boldsymbol{K}_{s3} \triangleq [(\boldsymbol{I}_M + \boldsymbol{Z}_3^{\circ\dagger}\,(\boldsymbol{S}_{33}^\circ)^{-1}\boldsymbol{Z}_3^\circ)^{-1/2}]^T$. Also, it is straightforward to show that $\boldsymbol{X}|(\mathcal{H}_0, \#3) = \boldsymbol{X}|\mathcal{H}_0 \sim \mathcal{CN}_{r\times M}(\boldsymbol{0}_{r\times M}, \boldsymbol{I}_M, \boldsymbol{I}_r)$ (i.e., it does not depend on the components with subscript "3"). We then consider a one-to-one transformation of $\boldsymbol{T}(\boldsymbol{Z}_c^\circ, \boldsymbol{S}_c^\circ)$, defined as:

$$\boldsymbol{T}_1(\boldsymbol{Z}_c^\circ, \boldsymbol{S}_c^\circ) \triangleq \begin{bmatrix} \boldsymbol{K}_{s3}^\dagger\,(\boldsymbol{Z}_{2.3}^\circ)^\dagger\,(\boldsymbol{S}_{2.3}^\circ)^{-1}\,\boldsymbol{Z}_{2.3}^\circ\,\boldsymbol{K}_{s3} \\ (\boldsymbol{Z}_3^\circ)^\dagger\,(\boldsymbol{S}_{33}^\circ)^{-1}\,\boldsymbol{Z}_3^\circ \end{bmatrix} \quad (44)$$

$$= \begin{bmatrix} \boldsymbol{X}^\dagger\,(\boldsymbol{S}_{2.3}^\circ)^{-1}\,\boldsymbol{X} \\ (\boldsymbol{Z}_3^\circ)^\dagger\,(\boldsymbol{S}_{33}^\circ)^{-1}\,\boldsymbol{Z}_3^\circ \end{bmatrix} \triangleq \begin{bmatrix} \boldsymbol{T}_{1,a} \\ \boldsymbol{T}_{1,b} \end{bmatrix}. \quad (45)$$

It is clear that, since $\boldsymbol{T}_1(\cdot)$ is a one-to-one transformation of the MIS, it is a MIS itself [10]. Therefore, without loss of generality, we will concentrate on the statistical characterization of $\boldsymbol{T}_1(\cdot)$.

We start by recalling that $\boldsymbol{S}_{2.3}^\circ$ is independent of $\{\boldsymbol{S}_{22}^\circ, \boldsymbol{S}_{33}^\circ\}$ [20, Thm. A.11]. Also, we notice that $\boldsymbol{S}_{33}^\circ \sim \mathcal{CW}_{N-J}(K - M, \boldsymbol{I}_{N-J})$ and $\boldsymbol{S}_{2.3}^\circ \sim \mathcal{CW}_r((K-M)-(N-J), \boldsymbol{I}_r)$. These results hold under both the hypotheses. Furthermore, conditioned on $\mathcal{H}_0$, $\boldsymbol{X}$ is independent on $\boldsymbol{T}_{1,b}$ (as it is independent on terms with subscript "3").

Therefore, it follows that conditioned on $\mathcal{H}_0$, $\boldsymbol{T}_{1,a}$ and $\boldsymbol{T}_{1,b}$ are *statistically independent matrices*, which means that the joint pdf can be written as $f_0(\boldsymbol{T}_{1,a}, \boldsymbol{T}_{1,b}) = f_0(\boldsymbol{T}_{1,a})\,f(\boldsymbol{T}_{1,b})$ (as $\boldsymbol{T}_{1,b}$ denotes the ancillary part of the MIS and thus its pdf is independent on the specific hypothesis). Finally, it is worth noticing that in the case $M \leq r$, we obtain the explicit pdf of $\boldsymbol{T}_{1,a}|\mathcal{H}_0 \sim \mathcal{CF}_M(\boldsymbol{0}_{M\times M}, r, (K-M)-(N-J))$ and, for $M \leq (N-J)$, $\boldsymbol{T}_{1,b} \sim \mathcal{CF}_M(\boldsymbol{0}_{M\times M}, N-J, (K-M)-(N-J))$, following [21].

On the other hand, when $\mathcal{H}_1$ holds true, it follows that $\boldsymbol{Z}_{2.3}^\circ|(\mathcal{H}_1, \#3) \sim \mathcal{CN}_{r\times M}(\boldsymbol{R}_{2.3}^{-1/2}\boldsymbol{B}, (\boldsymbol{I}_M + (\boldsymbol{Z}_3^\circ)^\dagger\,(\boldsymbol{S}_{33}^\circ)^{-1}\boldsymbol{Z}_3^\circ), \boldsymbol{I}_r)$ and consequently $\boldsymbol{X}|(\mathcal{H}_1, \#3) \sim \mathcal{CN}_{r\times M}(\boldsymbol{R}_{2.3}^{-1/2}\boldsymbol{B}\,\boldsymbol{K}_{s3}, \boldsymbol{I}_M, \boldsymbol{I}_r)$. A direct inspection of the last result reveals that

$$\boldsymbol{X}|(\#3, \mathcal{H}_1) = \boldsymbol{X}|(\boldsymbol{T}_{1,b}, \mathcal{H}_1) \sim \mathcal{CN}_{r\times M}\left(\boldsymbol{R}_{2.3}^{-1/2}\boldsymbol{B}\left((\boldsymbol{I}_M + \boldsymbol{T}_{1,b})^{-1/2}\right)^T, \boldsymbol{I}_M, \boldsymbol{I}_r\right), \quad (46)$$

which underlines that $\boldsymbol{T}_{1,a}$ and $\boldsymbol{T}_{1,b}$ are statistically dependent under $\mathcal{H}_1$, thus leading to $f_1(\boldsymbol{T}_{1,a}, \boldsymbol{T}_{1,b}) = f_1(\boldsymbol{T}_{1,a}|\boldsymbol{T}_{1,b})\,f(\boldsymbol{T}_{1,b})$. Again, in the specific case $M \leq r$, it holds $\boldsymbol{T}_{1.a}|(\mathcal{H}_1, \boldsymbol{T}_{1,b}) \sim \mathcal{CF}_M(\boldsymbol{\Omega}, r, (K-M)-(N-J))$, where we have denoted $\boldsymbol{\Omega} \triangleq (\boldsymbol{K}_{s3}^\dagger\,\boldsymbol{B}^\dagger\,\boldsymbol{R}_{2.3}^{-1}\,\boldsymbol{B}\,\boldsymbol{K}_{s3})$, following [21].

Finally, we conclude the section with a discussion on the induced maximal invariant in the parameter space [10]. The induced maximal invariant represents the reduced set of unknown parameters on which the hypothesis testing in the invariant domain depends. It can be readily shown that for I-GMANOVA model this equals $\boldsymbol{T}_{\mathrm{p}} \triangleq \boldsymbol{B}^\dagger\,\boldsymbol{R}_{2.3}^{-1}\,\boldsymbol{B} \in \mathbb{C}^{M\times M}$. In addition, the induced maximal invariant is not full rank in the general case, with the corresponding rank being equal to $\min\{r, M\}$. It is worth remarking that such result applies in general, that is, the

$$\boldsymbol{Z}_{2.3}^\circ = [\,\boldsymbol{q}_1 \quad \cdots \quad \boldsymbol{q}_M\,] = \left[\,\left(\boldsymbol{z}_{2,1}^\circ - \sum_{k=1}^{K-M}\boldsymbol{r}_{2,k}\boldsymbol{r}_{3,k}^\dagger\boldsymbol{\rho}_1\right) \quad \cdots \quad \left(\boldsymbol{z}_{2,M}^\circ - \sum_{k=1}^{K-M}\boldsymbol{r}_{2,k}\boldsymbol{r}_{3,k}^\dagger\boldsymbol{\rho}_M\right)\,\right] \quad (37)$$

distribution of the MIS will depend on the parameter space only through $\boldsymbol{T}_{\mathrm{p}}$, following classic results from [10].

V. MIS in Special Cases

A. Adaptive Detection of a Point-Like Target

In the present case we start from general formulation in (1) and assume that: (*i*) $t = 0$ (i.e., there is no interference); (*ii*) $r = 1$ (resp. $J = 1$), thus the matrix $\widetilde{\boldsymbol{A}}_r$ collapses to $\tilde{\boldsymbol{a}}_r \in \mathbb{C}^{N\times 1}$; (*iii*) $M = 1$, i.e., the matrix $\widetilde{\boldsymbol{B}}_r$ collapses to a scalar $\widetilde{b}_r \in \mathbb{C}$ and (*iv*) $\tilde{\boldsymbol{c}} \triangleq [1 \quad 0 \quad \cdots \quad 0] \in \mathbb{C}^{1\times K}$ (viz. a row vector). Such case has been dealt in [12]. Therefore, the hypothesis testing in canonical form is given by:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{a}\, b\, \boldsymbol{c} + \boldsymbol{N} \end{cases}, \tag{47}$$

where $\boldsymbol{a} = [1 \quad 0 \quad \cdots \quad 0]^T \in \mathbb{C}^{N\times 1}$ and $\boldsymbol{c} = \tilde{\boldsymbol{c}}$. By looking at the general MIS statistic expression in (25), the present problem admits the following simplified partitioning:

$$\boldsymbol{z}_c = \begin{bmatrix} z_2 \\ \boldsymbol{z}_3 \end{bmatrix}, \quad \boldsymbol{Z}_{c,\perp} = \begin{bmatrix} \boldsymbol{z}_{\perp,2} \\ \boldsymbol{Z}_{\perp,3} \end{bmatrix}, \quad \boldsymbol{S}_c = \begin{bmatrix} s_{22} & \boldsymbol{s}_{23} \\ \boldsymbol{s}_{32} & \boldsymbol{S}_{33} \end{bmatrix}, \tag{48}$$

where $z_2 \in \mathbb{C}$, $\boldsymbol{z}_3 \in \mathbb{C}^{(N-1)\times 1}$, $\boldsymbol{z}_{\perp,2} \in \mathbb{C}^{1\times(K-1)}$ (i.e., a row vector), $\boldsymbol{Z}_{\perp,3} \in \mathbb{C}^{(N-1)\times(K-1)}$, $s_{22} \in \mathbb{C}$, $\boldsymbol{s}_{23} \in \mathbb{C}^{1\times(N-1)}$ (i.e., a row vector), $\boldsymbol{s}_{32} \in \mathbb{C}^{(N-1)\times 1}$, and $\boldsymbol{S}_{33} \in \mathbb{C}^{(N-1)\times(N-1)}$, respectively. Exploiting the above partitioning, gives the simplified expressions:

$$s_{2.3} = \left(s_{22} - \boldsymbol{s}_{23}\, \boldsymbol{S}_{33}^{-1}\, \boldsymbol{s}_{32}\right), \tag{49}$$

$$z_{2.3} = \left(z_2 - \boldsymbol{s}_{23}\, \boldsymbol{S}_{33}^{-1}\, \boldsymbol{z}_3\right), \tag{50}$$

which are both scalar valued. Therefore, as an immediate consequence the two components of the MIS are both *scalar valued* and equal to:

$$t_a = \frac{\left|z_2 - \boldsymbol{s}_{23}\, \boldsymbol{S}_{33}^{-1}\, \boldsymbol{z}_3\right|^2}{\left[s_{22} - \boldsymbol{s}_{23}\, \boldsymbol{S}_{33}^{-1}\, \boldsymbol{s}_{32}\right]}, \tag{51}$$

$$t_b = \boldsymbol{z}_3^{\dagger}\, \boldsymbol{S}_{33}^{-1}\, \boldsymbol{z}_3, \tag{52}$$

which can be rewritten in the more familiar form:

$$t_a = \frac{|z_2 - \boldsymbol{z}_{\perp,2}\boldsymbol{Z}_{\perp,3}^{\dagger}\,(\boldsymbol{Z}_{\perp,3}\boldsymbol{Z}_{\perp,3}^{\dagger})^{-1}\boldsymbol{z}_3|^2}{\boldsymbol{z}_{\perp,2}[\boldsymbol{I}_{K-1} - \boldsymbol{Z}_{\perp,3}^{\dagger}(\boldsymbol{Z}_{\perp,3}\boldsymbol{Z}_{\perp,3}^{\dagger})^{-1}\boldsymbol{Z}_{\perp,3}]\boldsymbol{z}_{\perp,2}^{\dagger}}, \tag{53}$$

$$t_b = \boldsymbol{z}_3^{\dagger}(\boldsymbol{Z}_{\perp,3}\boldsymbol{Z}_{\perp,3}^{\dagger})^{-1}\boldsymbol{z}_3, \tag{54}$$

which are those obtained in [12].

Finally, in such case the (scalar-valued) induced maximal invariant simplifies to $t_{\mathrm{p}} = (|b|^2/r_{2.3})$, where $r_{2.3} \triangleq (r_{22} - \boldsymbol{r}_{23}\,\boldsymbol{R}_{33}^{-1}\,\boldsymbol{r}_{32})$, where we exploited similar simplified partitioning for $\boldsymbol{R}$ as for $\boldsymbol{S}_c$. The induced maximal invariant clearly coincides with the Signal-to-Noise plus Interference Ratio (SINR).

B. Adaptive Vector Subspace Detection

In the present case we start from the general formulation in (1) and assume that: (*i*) $t = 0$ (i.e., there is no interference, thus $J = r$); (*ii*) $M = 1$, thus the matrix $\widetilde{\boldsymbol{B}}_r$ collapses to a vector $\widetilde{\boldsymbol{b}}_r \in \mathbb{C}^{J\times 1}$ and (*iii*) $\tilde{\boldsymbol{c}} \triangleq [1 \quad 0 \quad \cdots \quad 0] \in \mathbb{C}^{1\times K}$ (i.e., a row vector). Such case has been dealt in [13], [14]. Therefore, the hypothesis testing in canonical form is given by:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{A}\,\boldsymbol{b}\,\boldsymbol{c} + \boldsymbol{N} \end{cases}, \tag{55}$$

where $\boldsymbol{A} = [\,\boldsymbol{I}_r \quad \boldsymbol{0}_{r\times(N-r)}\,]^T \in \mathbb{C}^{N\times r}$ and $\boldsymbol{c} = \tilde{\boldsymbol{c}}$. Considering the general MIS statistic expression in (25), the present problem admits the following simplified partitioning:

$$\boldsymbol{z}_c = \begin{bmatrix} \boldsymbol{z}_2 \\ \boldsymbol{z}_3 \end{bmatrix}, \quad \boldsymbol{Z}_{c,\perp} = \begin{bmatrix} \boldsymbol{Z}_{\perp,2} \\ \boldsymbol{Z}_{\perp,3} \end{bmatrix}, \quad \boldsymbol{S}_c = \begin{bmatrix} \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix}, \tag{56}$$

where $\boldsymbol{z}_2 \in \mathbb{C}^{J\times 1}$, $\boldsymbol{z}_3 \in \mathbb{C}^{(N-J)\times 1}$, $\boldsymbol{Z}_{\perp,2} \in \mathbb{C}^{J\times(K-1)}$, $\boldsymbol{Z}_{\perp,3} \in \mathbb{C}^{(N-J)\times(K-1)}$, $\boldsymbol{S}_{22} \in \mathbb{C}^{J\times J}$, $\boldsymbol{S}_{23} \in \mathbb{C}^{J\times(N-J)}$, $\boldsymbol{S}_{32} \in \mathbb{C}^{(N-J)\times J}$, and $\boldsymbol{S}_{33} \in \mathbb{C}^{(N-J)\times(N-J)}$, respectively.

Exploiting the above partitioning, gives $\boldsymbol{S}_{2.3} \in \mathbb{C}^{J\times J}$ and the simplified expression:

$$\boldsymbol{z}_{2.3} = \left(\boldsymbol{z}_2 - \boldsymbol{S}_{23}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{z}_3\right) \in \mathbb{C}^{J\times 1}. \tag{57}$$

Since $\boldsymbol{z}_{2.3}$ and $\boldsymbol{z}_3$ are both column vectors, the two components of the MIS both simplify to *scalars* and are equal to:

$$t_a = \boldsymbol{z}_{2.3}^{\dagger}\,\boldsymbol{S}_{2.3}^{-1}\,\boldsymbol{z}_{2.3}, \quad t_b = \boldsymbol{z}_3^{\dagger}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{z}_3, \tag{58}$$

which is the classic result obtained in [14].

Finally, the (scalar-valued) induced maximal invariant equals $t_{\mathrm{p}} = \boldsymbol{b}^{\dagger}\boldsymbol{R}_{2.3}^{-1}\boldsymbol{b}$ which is the result obtained in [14], being equal to the SINR.

C. Adaptive Vector Subspace Detection With Structured Interference

In the present case we start from general formulation in (1) and assume that: (*i*) $M = 1$, i.e., the matrices $\widetilde{\boldsymbol{B}}_r$ and $\widetilde{\boldsymbol{B}}_t$ collapse to the vectors $\widetilde{\boldsymbol{b}}_r \in \mathbb{C}^{r\times 1}$ and $\widetilde{\boldsymbol{b}}_t \in \mathbb{C}^{t\times 1}$, respectively; (*ii*) $\tilde{\boldsymbol{c}} \triangleq [1 \quad 0 \quad \cdots \quad 0] \in \mathbb{C}^{1\times K}$ (i.e., a row vector). Such case has been dealt in [18]. Given the aforementioned assumptions, the problem in canonical form is given as:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{A}\,[\,\boldsymbol{b}_{t,0}^T \quad \boldsymbol{0}_r^T\,]^T\,\boldsymbol{c} + \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{A}\,[\,\boldsymbol{b}_{t,1}^T \quad \boldsymbol{b}^T\,]^T\,\boldsymbol{c} + \boldsymbol{N} \end{cases}, \tag{59}$$

where $\boldsymbol{b}_{t,i} \in \mathbb{C}^{t\times 1}$, $\boldsymbol{b} \in \mathbb{C}^{r\times 1}$, and $\boldsymbol{c} = \tilde{\boldsymbol{c}}$. By looking at the general MIS statistic expression in (25) we deduce that the present problem admits the following simplified expression:

$$\boldsymbol{z}_c^T = [\,\boldsymbol{z}_1^T \quad \boldsymbol{z}_2^T \quad \boldsymbol{z}_3^T\,]^T, \tag{60}$$

where $\boldsymbol{z}_1 \in \mathbb{C}^{t\times 1}$, $\boldsymbol{z}_2 \in \mathbb{C}^{r\times 1}$, and $\boldsymbol{z}_3 \in \mathbb{C}^{(N-J)\times 1}$, respectively. Therefore, it is readily shown that $\boldsymbol{S}_{2.3} \in \mathbb{C}^{r\times r}$ and $\boldsymbol{z}_{2.3} = (\boldsymbol{z}_2 - \boldsymbol{S}_{23}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{z}_3) \in \mathbb{C}^{r\times 1}$. Since $\boldsymbol{z}_{2.3}$ and $\boldsymbol{z}_3$ are both column vectors, both components of the MIS become scalar-valued (*similarly* to the "no-interference" case) and equal to:

$$t_a = \boldsymbol{z}_{2.3}^{\dagger}\,\boldsymbol{S}_{2.3}^{-1}\,\boldsymbol{z}_{2.3}\,; \quad t_b = \boldsymbol{z}_3^{\dagger}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{z}_3\,; \tag{61}$$

which can be recognized as the result obtained in [18]. It is worth noticing that this result seems identical to that obtained in the previous sub-section (i.e., the interference-free case). However, we specify that, as opposed to the expression in (58), definition of constituents of MIS in (61) is obtained by discarding terms with subscript "1". In other terms, (61) is analogous to (58) *only* after projection in the complementary subspace of the interference.

Finally, the (scalar-valued) induced maximal invariant is $t_{\mathrm{p}} = \boldsymbol{b}^\dagger \boldsymbol{R}_{2.3}^{-1} \boldsymbol{b}$, which coincides with the result obtained in [18], being equal to the SINR in the complementary interference subspace.

D. Multidimensional Signals

We obtain the present setup starting from general formulation in (1) and assuming that: (*i*) $t = 0$ (i.e., there is no interference, thus $J = r$), (*ii*) $\widetilde{\boldsymbol{A}}_r = \boldsymbol{I}_N$ (thus $J = r = N$) and (*iii*) $\widetilde{\boldsymbol{C}} \triangleq [\, \boldsymbol{I}_M \quad \boldsymbol{0}_{M\times(K-M)}\,]$. Such case has been dealt in [16]. Therefore, the hypothesis testing problem in canonical form[3] is given by:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{B}\,\boldsymbol{C} + \boldsymbol{N} \end{cases}, \tag{62}$$

By looking at the general MIS statistic expression in (25) we notice that the present problem admits the following simplified partitioning:

$$\boldsymbol{Z}_c = \boldsymbol{Z}_2, \quad \boldsymbol{Z}_{c,\perp} = \boldsymbol{Z}_{\perp,2}, \quad \boldsymbol{S}_c = \boldsymbol{S}_{22}, \tag{63}$$

where $\boldsymbol{Z}_2 \in \mathbb{C}^{N\times M}$, $\boldsymbol{Z}_{\perp,2} \in \mathbb{C}^{N\times(K-M)}$, and $\boldsymbol{S}_{22} \in \mathbb{C}^{N\times N}$, respectively. Since in this particular setting $J = N$ holds, we exploit the alternative expression for the MIS in (25), which shows that the MIS reduces to a single matrix, being equal to:

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{Z}_2, \boldsymbol{S}_{22}) &= \boldsymbol{Z}_2^\dagger \boldsymbol{S}_{22}^{-1} \boldsymbol{Z}_2 \\ &= \boldsymbol{Z}_2^\dagger\, (\boldsymbol{Z}_{\perp,2}\boldsymbol{Z}_{\perp,2}^\dagger)^{-1}\, \boldsymbol{Z}_2\,. \end{aligned} \tag{64}$$

In the latter case, the maximal invariant induced in the parameter space reduces analogously to $\boldsymbol{T}_{\mathrm{p}} = \boldsymbol{B}^\dagger \boldsymbol{R}^{-1} \boldsymbol{B}$.

It is now interesting to compare the result in (64) with that obtained in [16]. Indeed, in the aforementioned work, the elementary action $\ell_M(\cdot,\cdot)$ is defined as:

$$\begin{aligned} \ell_M(\boldsymbol{Z}_2, \boldsymbol{S}_{22}) = (&\boldsymbol{G}_{22}\, \boldsymbol{Z}_2\, \boldsymbol{U}_d, \boldsymbol{G}_{22}\, \boldsymbol{S}_{22}\, \boldsymbol{G}_{22}^\dagger) \\ &\forall \boldsymbol{G}_{22} \in \mathcal{GL}(N)\,, \forall \boldsymbol{U}_d \in \mathcal{U}(M). \end{aligned} \tag{65}$$

which, compared to (22), enforces an *additional invariance* with respect to a right subspace rotation, via the unitary matrix $\boldsymbol{U}_d$. Clearly, this restricts further the class of invariant tests. Moreover, in (65) we have used $\mathcal{U}(M)$ to denote the group of unitary $M \times M$ matrices. It was shown in [16] that the MIS for the elementary action defined in (65) is given by the non-zero eigenvalues of the matrix

$$\begin{aligned} \boldsymbol{T}_c &\triangleq \boldsymbol{S}_{22}^{-1/2}\, (\boldsymbol{Z}_2\boldsymbol{Z}_2^\dagger)\, \boldsymbol{S}_{22}^{-1/2} \\ &= (\boldsymbol{Z}_{\perp,2}\boldsymbol{Z}_{\perp,2}^\dagger)^{-1/2}\, (\boldsymbol{Z}_2\boldsymbol{Z}_2^\dagger)\, (\boldsymbol{Z}_{\perp,2}\boldsymbol{Z}_{\perp,2}^\dagger)^{-1/2}, \end{aligned} \tag{66}$$

denoted with $\mathrm{eig}(\boldsymbol{T}_c)$ in what follows. Remarkably, we show hereinafter that the MIS in (66) can be directly linked to the expression in (64). We first notice that, after defining $\boldsymbol{Z}_m \triangleq \{(\boldsymbol{Z}_{\perp,2}\boldsymbol{Z}_{\perp,2}^\dagger)^{-1/2}\, \boldsymbol{Z}_2\} \in \mathbb{C}^{N\times M}$, the following equalities hold:

$$\boldsymbol{T} = (\boldsymbol{Z}_m^\dagger\, \boldsymbol{Z}_m), \quad \boldsymbol{T}_c = (\boldsymbol{Z}_m\, \boldsymbol{Z}_m^\dagger). \tag{67}$$

[3]It is worth noticing that in this case the original formulation in (1) coincides with the canonical form.

Therefore, by construction, the matrices $\boldsymbol{T} \in \mathbb{C}^{M\times M}$ and $\boldsymbol{T}_c \in \mathbb{C}^{N\times N}$ are such that $\mathrm{eig}(\boldsymbol{T}_c) = \mathrm{eig}(\boldsymbol{T})$ holds (and the vector length equals $\min\{M, N\}$), where we have expressed the non-zero eigenvalues through the implicit vector-valued function $\mathrm{eig}(\,\cdot\,)$. Then, we notice that the action $\ell_M(\cdot,\cdot)$ can be re-interpreted as the composition of the following sub-actions:

$$\begin{aligned} \ell_{M,a}(\boldsymbol{Z}_2, \boldsymbol{S}_{22}) &= (\boldsymbol{G}_{22}\, \boldsymbol{Z}_2, \boldsymbol{G}_{22}\, \boldsymbol{S}_{22}\, \boldsymbol{G}_{22}^\dagger) \quad \forall \boldsymbol{G}_{22} \in \mathcal{GL}(N) \\ \ell_{M,b}(\boldsymbol{Z}_2, \boldsymbol{S}_{22}) &= (\boldsymbol{Z}_2 \boldsymbol{U}_d, \boldsymbol{S}_{22}) \quad \forall \boldsymbol{U}_d \in \mathcal{U}(M). \end{aligned} \tag{68}$$

It is then recognized that $\ell_{M,a}(\cdot,\cdot) = \ell(\cdot,\cdot)$ for the case of multidimensional signals. Previously, we have shown that the MIS for the elementary action $\ell_{M,a}(\cdot,\cdot) = \ell(\cdot,\cdot)$ is simply given by the matrix $\boldsymbol{T}$ in (64).

Additionally, we notice that, for each $\boldsymbol{U}_d \in \mathcal{U}(M)$,

$$\begin{aligned} \boldsymbol{T}(\bar{\boldsymbol{Z}}_2, \bar{\boldsymbol{S}}_{22}) = \boldsymbol{T}(\boldsymbol{Z}_2, \boldsymbol{S}_{22}) &\Rightarrow \\ \boldsymbol{T}(\bar{\boldsymbol{Z}}_2 \boldsymbol{U}_d, \bar{\boldsymbol{S}}_{22}) = \boldsymbol{T}(\boldsymbol{Z}_2\, &\boldsymbol{U}_d, \boldsymbol{S}_{22})\,. \end{aligned} \tag{69}$$

Now, define the action $\ell_{M,b}^\star(\,\cdot\,)$ as:

$$\ell_{M,b}^\star(\boldsymbol{T}) = (\boldsymbol{U}_d^\dagger \boldsymbol{T}\, \boldsymbol{U}_d) \quad \forall \boldsymbol{U}_d \in \mathcal{U}(M), \tag{70}$$

where $\boldsymbol{T} \in \mathbb{H}^{M\times M}$. It is not difficult to show that a MIS for the elementary operation $\ell_{M,b}^\star(\,\cdot\,)$ in (70) is given by $\mathrm{eig}(\boldsymbol{T})$. Therefore, exploiting [10, p. 217, Thm. 6.2.2], it follows that a MIS for the action $\ell_M(\cdot,\cdot)$ is the composite function $\mathrm{eig}(\boldsymbol{T}(\boldsymbol{Z}_2, \boldsymbol{S}_{22}))$. However, since as underlined in (67), we have $\mathrm{eig}(\boldsymbol{T}) = \mathrm{eig}(\boldsymbol{T}_c)$, this clearly coincides with the result in [16].

Finally, by similar reasoning we obtain that, in such a case, the induced maximal invariant is given by $\mathrm{eig}(\boldsymbol{T}_{\mathrm{p}}) = \mathrm{eig}(\boldsymbol{B}^\dagger \boldsymbol{R}^{-1} \boldsymbol{B}) = \mathrm{eig}(\boldsymbol{R}^{-1/2} \boldsymbol{B}\, \boldsymbol{B}^\dagger \boldsymbol{R}^{-1/2})$, thus obtaining the result in [16].

E. Range-Spread Targets

In the present case we start from general formulation in (1) and assume that: (*i*) $t = 0$ (i.e., there is no interference, thus $J = r$); (*ii*) $r = 1$, thus the matrices $\widetilde{\boldsymbol{A}}_r$ and $\widetilde{\boldsymbol{B}}_r$ collapse to $\widetilde{\boldsymbol{a}}_r \in \mathbb{C}^{N\times 1}$ and $\widetilde{\boldsymbol{b}}_r \in \mathbb{C}^{1\times M}$ (i.e., a row vector), respectively; (*iii*) $\widetilde{\boldsymbol{C}} \triangleq [\, \boldsymbol{I}_M \quad \boldsymbol{0}_{M\times K-M}\,]$. Such case has been dealt in [17], [22]. Therefore, the hypothesis testing in canonical form is given by:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{C} + \boldsymbol{N} \end{cases}, \tag{71}$$

where $\boldsymbol{a} \triangleq [1 \quad 0 \quad \cdots \quad 0]^T \in \mathbb{C}^{N\times 1}$, $\boldsymbol{b} \in \mathbb{C}^{1\times M}$ and $\boldsymbol{C} = \widetilde{\boldsymbol{C}}$, respectively. By looking at the general MIS expression in (25), the present problem admits the following simplified partitioning:

$$\boldsymbol{Z}_c = \begin{bmatrix} \boldsymbol{z}_2 \\ \boldsymbol{Z}_3 \end{bmatrix}, \quad \boldsymbol{Z}_{c,\perp} = \begin{bmatrix} \boldsymbol{z}_{\perp,2} \\ \boldsymbol{Z}_{\perp,3} \end{bmatrix}, \quad \boldsymbol{S}_c = \begin{bmatrix} s_{22} & \boldsymbol{s}_{23} \\ \boldsymbol{s}_{32} & \boldsymbol{S}_{33} \end{bmatrix}, \tag{72}$$

where $\boldsymbol{z}_2 \in \mathbb{C}^{1\times M}$ (i.e., a row vector), $\boldsymbol{Z}_3 \in \mathbb{C}^{(N-1)\times M}$, $\boldsymbol{z}_{\perp,2} \in \mathbb{C}^{1\times(K-M)}$ (i.e., a row vector), $\boldsymbol{Z}_{\perp,3} \in \mathbb{C}^{(N-1)\times(K-M)}$, $s_{22} \in \mathbb{C}$, $\boldsymbol{s}_{23} \in \mathbb{C}^{1\times(N-1)}$ (i.e., a row vector), $\boldsymbol{s}_{32} \in \mathbb{C}^{(N-1)\times 1}$, and $\boldsymbol{S}_{33} \in \mathbb{C}^{(N-1)\times(N-1)}$, respectively. Exploiting the above partitioning, gives

$s_{2.3} = (s_{22} - \boldsymbol{s}_{23}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{s}_{32}) \in \mathbb{C}$ (i.e., a scalar) and the simplified expression:

$$\boldsymbol{z}_{2.3} = \left(\boldsymbol{z}_2 - \boldsymbol{s}_{23}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{Z}_3\right) \in \mathbb{C}^{1\times M}. \tag{73}$$

Given the simplified expressions for $\boldsymbol{z}_{2.3}$ (row vector) and $s_{2.3}$ (scalar), it automatically follows that the two matrix components of the MIS are given by:

$$\boldsymbol{T}_a = \left(\frac{1}{s_{2.3}}\right)\boldsymbol{z}_{2.3}^{\dagger}\boldsymbol{z}_{2.3}, \quad \boldsymbol{T}_b = \boldsymbol{Z}_3^{\dagger}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{Z}_3, \tag{74}$$

where the matrix $\boldsymbol{T}_a$ is rank-one in this specific case (as it is the output of a dyadic product). Also, the induced maximal invariant in the parameter space equals $\boldsymbol{T}_{\mathrm{p}} = (\frac{1}{r_{2.3}})\,\boldsymbol{b}^{\dagger}\boldsymbol{b}$, i.e., a rank-one matrix.

It is now of interest comparing the MIS represented by (74) with that obtained in [17]. The approach taken in the following is similar to that used for multidimensional signals in Section V.D. However, due to the more tedious mathematics involved, we confine the proof to Appendix C and we only state the results hereinafter.

Indeed, in the aforementioned work, the elementary action $\ell_R(\cdot,\cdot)$ is defined as:

$$\begin{aligned}\ell_R(\boldsymbol{Z}_c, \boldsymbol{S}_c) = (\boldsymbol{G}\,\boldsymbol{Z}_c\,\boldsymbol{U}_d, \boldsymbol{G}\,\boldsymbol{S}_c\,\boldsymbol{G}^{\dagger}),\\ \forall \boldsymbol{G} \in \mathcal{G}, \quad \forall \boldsymbol{U}_d \in \mathcal{U}(M),\end{aligned} \tag{75}$$

which, compared to (22), enforces an additional invariance with respect to a right subspace rotation of primary data, via the unitary matrix $\boldsymbol{U}_d$. It was shown in [17] that the MIS for the elementary action defined in (75) is given by the eigenvalues of the matrices

$$(\boldsymbol{T}_a + \boldsymbol{T}_b), \quad \boldsymbol{T}_b, \tag{76}$$

denoted with $\mathrm{eig}(\boldsymbol{T}_a + \boldsymbol{T}_b)$ and $\mathrm{eig}(\boldsymbol{T}_b)$ in what follows. Remarkably, we show hereinafter that the MIS in (76) can be directly linked to the expression in (74). We first notice that the action $\ell_R(\cdot,\cdot)$ can be re-interpreted as the composition of the following sub-actions:

$$\begin{aligned}\ell_{R,a}(\boldsymbol{Z}_c, \boldsymbol{S}_c) &= (\boldsymbol{G}\,\boldsymbol{Z}_c, \boldsymbol{G}\,\boldsymbol{S}_c\,\boldsymbol{G}^{\dagger}), \quad \forall \boldsymbol{G} \in \mathcal{G},\\ \ell_{R,b}(\boldsymbol{Z}_c, \boldsymbol{S}_c) &= (\boldsymbol{Z}_c\,\boldsymbol{U}_d, \boldsymbol{S}_c), \quad \forall \boldsymbol{U}_d \in \mathcal{U}(M).\end{aligned} \tag{77}$$

It is then recognized that $\ell_{R,a}(\cdot,\cdot) = \ell(\cdot,\cdot)$ for the case of range-spread targets. Also, we have previously shown that a MIS for the elementary action $\ell_{R,a}(\cdot,\cdot) = \ell(\cdot,\cdot)$ is given by (74).

Additionally, we notice that, for each $\boldsymbol{U}_d \in \mathcal{U}(M)$,

$$\begin{aligned}\boldsymbol{T}(\bar{\boldsymbol{Z}}_c, \bar{\boldsymbol{S}}_c) = \boldsymbol{T}(\boldsymbol{Z}_c, \boldsymbol{S}_c) \Rightarrow\\ \boldsymbol{T}(\bar{\boldsymbol{Z}}_c\,\boldsymbol{U}_d, \bar{\boldsymbol{S}}_c) = \boldsymbol{T}(\boldsymbol{Z}_c\,\boldsymbol{U}_d, \boldsymbol{S}_c).\end{aligned} \tag{78}$$

Now, define the action $\ell_{R,b}^{\star}(\cdot,\cdot)$ as:

$$\ell_{R,b}^{\star}(\boldsymbol{T}_a, \boldsymbol{T}_b) = (\boldsymbol{U}_d^{\dagger}\,\boldsymbol{T}_a\,\boldsymbol{U}_d, \boldsymbol{U}_d^{\dagger}\,\boldsymbol{T}_b\,\boldsymbol{U}_d), \tag{79}$$

$\forall \boldsymbol{U}_d \in \mathcal{U}(M)$, where $\boldsymbol{T}_b \in \mathbb{H}^{M\times M}$ and $\boldsymbol{T}_a = (\boldsymbol{a}\boldsymbol{a}^{\dagger})$ (that is, a rank-one matrix). It is shown in Appendix C that the MIS for the elementary operation $\ell_{R,b}^{\star}(\cdot,\cdot)$ in (79) is given by $\{\mathrm{eig}(\boldsymbol{T}_b), \mathrm{eig}(\boldsymbol{T}_a + \boldsymbol{T}_b)\}$. Therefore, by exploiting [10, p. 217, Thm. 6.2.2], it follows that the MIS for the action $\ell_2(\cdot)$ is the composite function

$$\begin{cases}\mathrm{eig}(\boldsymbol{Z}_3^{\dagger}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{Z}_3)\\ \mathrm{eig}\left(\boldsymbol{Z}_3^{\dagger}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{Z}_3 + \left(\frac{1}{s_{2.3}}\right)\boldsymbol{z}_{2.3}^{\dagger}\boldsymbol{z}_{2.3}\right)\end{cases}, \tag{80}$$

which clearly coincides with the result in [17].

Finally, the induced maximal invariant in such a case can be readily obtained as $\mathrm{eig}(\boldsymbol{T}_{\mathrm{p}}) = \frac{\|\boldsymbol{b}\|^2}{r_{2.3}} = \|\boldsymbol{b}\|^2(\boldsymbol{a}^{\dagger}\boldsymbol{R}^{-1}\boldsymbol{a})$ (since the rank-one induced maximal invariant has only one non-zero eigenvalue), which represents the overall SINR over the M cells, as defined in [17].

F. Standard GMANOVA

Finally, in the present case we start from general formulation in (1) and assume that: (*i*) $t = 0$ (i.e., there is no interference, thus $J = r$). Such model clearly coincides with that analyzed in [2], [4], *unfortunately not dealing with the derivation of the MIS*. Therefore, the hypothesis testing in canonical form is given by:

$$\begin{cases}\mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{N}\\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{A}\,\boldsymbol{B}\,\boldsymbol{C} + \boldsymbol{N}\end{cases}, \tag{81}$$

where $\boldsymbol{A} = [\,\boldsymbol{I}_J \quad \boldsymbol{0}_{J\times(N-J)}\,]^T$ and $\boldsymbol{B} \in \mathbb{C}^{J\times M}$, respectively. By looking at the general MIS statistic expression in (25) the present problem admits the following simplified partitioning:

$$\boldsymbol{Z}_c = \begin{bmatrix}\boldsymbol{Z}_2\\ \boldsymbol{Z}_3\end{bmatrix}, \boldsymbol{Z}_{c,\perp} = \begin{bmatrix}\boldsymbol{Z}_{\perp,2}\\ \boldsymbol{Z}_{\perp,3}\end{bmatrix}, \boldsymbol{S}_c = \begin{bmatrix}\boldsymbol{S}_{22} & \boldsymbol{S}_{23}\\ \boldsymbol{S}_{32} & \boldsymbol{S}_{33}\end{bmatrix}, \tag{82}$$

where $\boldsymbol{Z}_2 \in \mathbb{C}^{J\times M}$, $\boldsymbol{Z}_3 \in \mathbb{C}^{(N-J)\times M}$, $\boldsymbol{Z}_{\perp,2} \in \mathbb{C}^{J\times(K-M)}$, $\boldsymbol{Z}_{\perp,3} \in \mathbb{C}^{(N-J)\times(K-M)}$, $\boldsymbol{S}_{22} \in \mathbb{C}^{J\times J}$, $\boldsymbol{S}_{23} \in \mathbb{C}^{J\times(N-J)}$, $\boldsymbol{S}_{32} \in \mathbb{C}^{(N-J)\times J}$, and $\boldsymbol{S}_{33} \in \mathbb{C}^{(N-J)\times(N-J)}$, respectively. Given the simplified definitions in (82), the MIS is readily obtained via the standard formula in (25).

Finally, the induced maximal invariant is obtained through the standard formula $\boldsymbol{T}_{\mathrm{p}} = (\boldsymbol{B}^{\dagger}\,\boldsymbol{R}_{2.3}^{-1}\,\boldsymbol{B})$. The sole difference consists in the rank of matrix $\boldsymbol{T}_{\mathrm{p}}$, being equal to $\min\{J, M\}$, i.e., there is no reduction in the observation space due to structured interference.

VI. Conclusion

In the first part of this work, we have studied a generalization of GMANOVA model (denoted as I-GMANOVA) which comprises additional (deterministic) structured interference, modeling possible jamming interference. The study has been conducted with the help of the statistical theory of invariance. For the present problem, the group of transformations leaving the hypothesis testing problem invariant was derived, thus allowing identification of transformations which enforce CFARness. Then, a MIS was derived for the aforementioned group, thus explicitly underlining the basic structure of a generic CFAR receiver (several examples of CFAR receivers, based on theoretically-founded criteria, will be derived in Part II of the present work).

Furthermore, a statistical characterization of the considered MIS under both hypotheses was obtained, thus allowing for an efficient stochastic representation. As a byproduct, the general form of the induced maximal invariant in the parameter space was obtained for the considered hypothesis testing. Finally, the general MIS expression was particularized and compared with MIS obtained in specific instances found in the open literature. Analogies to other expressions of the MIS, obtained by enforcing invariance to a wider class of transformations (cf. Section V.E and V.D), were underlined and discussed.

APPENDIX A
INVARIANCE OF THE PROBLEM WITH RESPECT TO THE GROUP $\mathcal{L}$

In this Appendix we prove the invariance of the hypothesis testing problem in (12) with respect to the group of transformations $\mathcal{L}$ defined in Section III.A. Let $(\boldsymbol{G},\boldsymbol{F}) \in \mathcal{L}$ and notice that, under $\mathcal{H}_1$, the columns of $\boldsymbol{G}\boldsymbol{Z}_c + \boldsymbol{F}$ are independent complex normal vectors with covariance matrix $\boldsymbol{G}\boldsymbol{R}\boldsymbol{G}^\dagger$ and mean:

$$\boldsymbol{G}\boldsymbol{A}\boldsymbol{B}_s + \boldsymbol{F} = \begin{bmatrix} \boldsymbol{G}_{11}\boldsymbol{B}_{t,1} + \boldsymbol{G}_{12}\boldsymbol{B} + \boldsymbol{F}_1 \\ \boldsymbol{G}_{22}\boldsymbol{B} \\ \boldsymbol{0}_{(N-J)\times M} \end{bmatrix} \tag{83}$$

$$= \begin{bmatrix} \boldsymbol{B}'_{t,1} \\ \boldsymbol{B}' \\ \boldsymbol{0}_{(N-J)\times M} \end{bmatrix} = \boldsymbol{A}\boldsymbol{B}'_s, \tag{84}$$

where we have employed the definitions $\boldsymbol{B}'_{t,1} \triangleq (\boldsymbol{G}_{11}\boldsymbol{B}_{t,1} + \boldsymbol{G}_{12}\boldsymbol{B} + \boldsymbol{F}_{11}) \in \mathbb{C}^{t\times M}$ and $\boldsymbol{B}' \triangleq (\boldsymbol{G}_{22}\boldsymbol{B}) \in \mathbb{C}^{r\times M}$, respectively. Also, aiming at compact notation, we have denoted $\boldsymbol{B}'_s \triangleq [\,(\boldsymbol{B}'_{t,1})^T \quad (\boldsymbol{B}')^T\,]^T$. Furthermore, it is not difficult to show that $\boldsymbol{G}\boldsymbol{S}_c\boldsymbol{G}^\dagger = (\boldsymbol{G}\boldsymbol{Z}_{c,\perp})(\boldsymbol{G}\boldsymbol{Z}_{c,\perp})^\dagger$, with $(\boldsymbol{G}\boldsymbol{Z}_{c,\perp}) \sim \mathcal{CN}_{N\times(K-M)}(\boldsymbol{0}_{N\times(K-M)}, \boldsymbol{I}_{K-M}, \boldsymbol{G}\boldsymbol{R}\boldsymbol{G}^\dagger)$.

On the other hand, when $\mathcal{H}_0$ holds true, $\boldsymbol{G}\boldsymbol{Z}_c + \boldsymbol{F}$ shares the same covariance structure as in the case of $\mathcal{H}_1$, except for the mean, which becomes

$$\boldsymbol{G}\boldsymbol{A}\begin{bmatrix} \boldsymbol{B}_{t,0} \\ \boldsymbol{0}_{r\times M} \end{bmatrix} + \boldsymbol{F} = \begin{bmatrix} \boldsymbol{G}_{11}\boldsymbol{B}_{t,0} + \boldsymbol{F}_1 \\ \boldsymbol{0}_{r\times M} \\ \boldsymbol{0}_{(N-J)\times M} \end{bmatrix} \tag{85}$$

$$= \boldsymbol{A}\begin{bmatrix} \boldsymbol{B}'_{t,0} \\ \boldsymbol{0}_{r\times M} \end{bmatrix}, \tag{86}$$

where $\boldsymbol{B}'_{t,0} \triangleq (\boldsymbol{G}_{11}\boldsymbol{B}_{t,0} + \boldsymbol{F}_1) \in \mathbb{C}^{t\times M}$. Again, it can be shown that $(\boldsymbol{G}\boldsymbol{Z}_{c,\perp}) \sim \mathcal{CN}_{N\times(K-M)}(\boldsymbol{0}_{N\times(K-M)}, \boldsymbol{I}_{K-M}, \boldsymbol{G}\boldsymbol{R}\boldsymbol{G}^\dagger)$.

Therefore, it is apparent that the original partition of the parameter space, the data distribution, and the structure of the subspace containing the useful signal components are preserved after the transformation $(\boldsymbol{G},\boldsymbol{F})$. Indeed, the following equivalence holds between the original and the transformed test:

$$\begin{cases} \mathcal{H}_0 : & \|\boldsymbol{B}\|_F = 0 \iff \|\boldsymbol{B}'\|_F = 0, \\ \mathcal{H}_1 : & \|\boldsymbol{B}\|_F > 0 \iff \|\boldsymbol{B}'\|_F > 0, \end{cases} \tag{87}$$

where the nuisance parameters in the transformed space are $\boldsymbol{B}'_{t,i}$ and $(\boldsymbol{G}\boldsymbol{R}\boldsymbol{G}^\dagger)$.

APPENDIX B
DERIVATION OF THE MAXIMAL INVARIANT STATISTIC

In the present Appendix we provide a proof for Prop. 1. In particular, hereinafter we will focus on the case $J < N$, as to the derivation for $J = N$ can be obtained through identical steps. Before proceeding further, we recall that a statistic $\boldsymbol{T}(\boldsymbol{Z}_c, \boldsymbol{S}_c)$ is said to be a maximal invariant with respect to the group of transformations $\mathcal{L}$ iff

$$\text{(a)} \quad \boldsymbol{T}(\boldsymbol{Z}_c,\boldsymbol{S}_c) = \boldsymbol{T}[\ell(\boldsymbol{Z}_c,\boldsymbol{S}_c)], \quad \forall \ell \in \mathcal{L}\,; \tag{88}$$

$$\text{(b)} \quad \boldsymbol{T}(\boldsymbol{Z}_c,\boldsymbol{S}_c) = \boldsymbol{T}(\bar{\boldsymbol{Z}}_c,\bar{\boldsymbol{S}}_c) \Rightarrow \exists\, \ell \in \mathcal{L} \,:\, (\boldsymbol{Z}_c,\boldsymbol{S}_c) = \ell(\bar{\boldsymbol{Z}}_c,\bar{\boldsymbol{S}}_c)\,. \tag{89}$$

Conditions (a) and (b) correspond to the *invariance* and *maximality* properties, respectively. In order to prove (a), we first consider the following partitioning of matrix $\boldsymbol{G}$ and sub-matrix of $\boldsymbol{S}_c$:

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{G}_1 & \boldsymbol{G}_2 \\ \boldsymbol{0}_{(N-t)\times t} & \boldsymbol{G}_3 \end{bmatrix}, \quad \boldsymbol{S}_2 \triangleq \begin{bmatrix} \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix}, \tag{90}$$

where $\boldsymbol{S}_2 \in \mathbb{C}^{(N-t)\times(N-t)}$, $\boldsymbol{G}_1 \triangleq \boldsymbol{G}_{11} \in \mathbb{C}^{t\times t}$, $\boldsymbol{G}_2 \triangleq [\,\boldsymbol{G}_{12} \quad \boldsymbol{G}_{13}\,] \in \mathbb{C}^{t\times(N-t)}$, and

$$\boldsymbol{G}_3 \triangleq \begin{bmatrix} \boldsymbol{G}_{22} & \boldsymbol{G}_{23} \\ \boldsymbol{0}_{(N-J)\times r} & \boldsymbol{G}_{33} \end{bmatrix} \in \mathbb{C}^{(N-t)\times(N-t)}. \tag{91}$$

Then, let $(\bar{\boldsymbol{Z}}_c,\bar{\boldsymbol{S}}_c) \triangleq \ell(\boldsymbol{Z}_c,\boldsymbol{S}_c)$, with

$$\bar{\boldsymbol{Z}}_c = \boldsymbol{G}\boldsymbol{Z}_c + \boldsymbol{F}, \quad \bar{\boldsymbol{S}}_c = \boldsymbol{G}\,\boldsymbol{S}_c\,\boldsymbol{G}^\dagger. \tag{92}$$

It is apparent that the following equalities hold, when exploiting the specific structure of $\boldsymbol{G}$ and $\boldsymbol{F}$ (cf. ((19)–(20))):

$$\bar{\boldsymbol{Z}}_2 = \boldsymbol{G}_{22}\,\boldsymbol{Z}_2 + \boldsymbol{G}_{23}\,\boldsymbol{Z}_3, \tag{93}$$

$$\bar{\boldsymbol{Z}}_3 = \boldsymbol{G}_{33}\,\boldsymbol{Z}_3, \tag{94}$$

and

$$\bar{\boldsymbol{S}}_2 = \boldsymbol{G}_3\,\boldsymbol{S}_2\,\boldsymbol{G}_3^\dagger\,, \tag{95}$$

where $\bar{\boldsymbol{S}}_2$ is defined similarly as in (90). From (95), it can be inferred that:

$$\bar{\boldsymbol{S}}_{22} = [\,\boldsymbol{G}_{22} \quad \boldsymbol{G}_{23}\,]\,\boldsymbol{S}_2\,[\,\boldsymbol{G}_{22} \quad \boldsymbol{G}_{23}\,]^\dagger, \tag{96}$$

$$\bar{\boldsymbol{S}}_{23} = \boldsymbol{G}_{22}\,\boldsymbol{S}_{23}\,\boldsymbol{G}_{33}^\dagger + \boldsymbol{G}_{23}\,\boldsymbol{S}_{33}\,\boldsymbol{G}_{33}^\dagger, \tag{97}$$

$$\bar{\boldsymbol{S}}_{33} = \boldsymbol{G}_{33}\,\boldsymbol{S}_{33}\,\boldsymbol{G}_{33}^\dagger. \tag{98}$$

Additionally, exploiting the appropriate substitutions, it can be shown that:

$$\bar{\boldsymbol{Z}}_{2.3} = \left(\bar{\boldsymbol{Z}}_2 - \bar{\boldsymbol{S}}_{23}\,\bar{\boldsymbol{S}}_{33}^{-1}\,\bar{\boldsymbol{Z}}_3\right) = \boldsymbol{G}_{22}\,\boldsymbol{Z}_{2.3}, \tag{99}$$

$$\bar{\boldsymbol{S}}_{2.3} = \left(\bar{\boldsymbol{S}}_{22} - \bar{\boldsymbol{S}}_{23}\,\bar{\boldsymbol{S}}_{33}^{-1}\,\bar{\boldsymbol{S}}_{32}\right) = \boldsymbol{G}_{22}\,\boldsymbol{S}_{2.3}\,\boldsymbol{G}_{22}^\dagger. \tag{100}$$

Finally, substituting (94), (98), (99), and (100) into (25), we obtain:

$$\boldsymbol{T}(\ell(\boldsymbol{Z}_c,\boldsymbol{S}_c)) = \begin{bmatrix} \bar{\boldsymbol{Z}}_{2.3}^\dagger\,\bar{\boldsymbol{S}}_{2.3}^{-1}\,\bar{\boldsymbol{Z}}_{2.3} \\ \bar{\boldsymbol{Z}}_3^\dagger\,\bar{\boldsymbol{S}}_{33}^{-1}\,\bar{\boldsymbol{Z}}_3 \end{bmatrix} \tag{101}$$

$$= \begin{bmatrix} \boldsymbol{Z}_{2.3}^\dagger\,\boldsymbol{G}_{22}^\dagger\,(\boldsymbol{G}_{22}^\dagger)^{-1}\,\boldsymbol{S}_{2.3}^{-1}\,\boldsymbol{G}_{22}^{-1}\,\boldsymbol{G}_{22}\,\boldsymbol{Z}_{2.3} \\ \boldsymbol{Z}_3^\dagger\,\boldsymbol{G}_{33}^\dagger\,(\boldsymbol{G}_{33}^\dagger)^{-1}\,\boldsymbol{S}_{33}^{-1}\,(\boldsymbol{G}_{33}^{-1})\,\boldsymbol{G}_{33}\,\boldsymbol{Z}_3 \end{bmatrix} \tag{102}$$

$$= \begin{bmatrix} \boldsymbol{Z}_{2.3}^\dagger\,\boldsymbol{S}_{2.3}^{-1}\,\boldsymbol{Z}_{2.3} \\ \boldsymbol{Z}_3^\dagger\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{Z}_3 \end{bmatrix}, \tag{103}$$

which thus proves (a).

Now, in order to prove (b), assume that:

$$\boldsymbol{T}(\boldsymbol{Z}_c,\boldsymbol{S}_c) = \boldsymbol{T}(\bar{\boldsymbol{Z}}_c,\bar{\boldsymbol{S}}_c), \tag{104}$$

$$\begin{bmatrix} \boldsymbol{Z}_{2.3}^{\dagger}\,\boldsymbol{S}_{2.3}^{-1}\,\boldsymbol{Z}_{2.3} \\ \boldsymbol{Z}_{3}^{\dagger}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{Z}_{3} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{Z}}_{2.3}^{\dagger}\,\bar{\boldsymbol{S}}_{2.3}^{-1}\,\bar{\boldsymbol{Z}}_{2.3} \\ \bar{\boldsymbol{Z}}_{3}^{\dagger}\,\bar{\boldsymbol{S}}_{33}^{-1}\,\bar{\boldsymbol{Z}}_{3} \end{bmatrix}. \tag{105}$$

The last equality can be recast as the following pair of equalities

$$\boldsymbol{Y}_{2.3}\,\boldsymbol{Y}_{2.3}^{\dagger} = \bar{\boldsymbol{Y}}_{2.3}\,\bar{\boldsymbol{Y}}_{2.3}^{\dagger}, \quad \boldsymbol{Y}_{3}\,\boldsymbol{Y}_{3}^{\dagger} = \bar{\boldsymbol{Y}}_{3}\,\bar{\boldsymbol{Y}}_{3}^{\dagger}, \tag{106}$$

where $\boldsymbol{Y}_{2.3} \triangleq (\boldsymbol{S}_{2.3}^{-1/2}\boldsymbol{Z}_{2.3})^{\dagger}$, $\bar{\boldsymbol{Y}}_{2.3} \triangleq (\bar{\boldsymbol{S}}_{2.3}^{-1/2}\bar{\boldsymbol{Z}}_{2.3})^{\dagger}$, $\boldsymbol{Y}_{3} \triangleq (\boldsymbol{S}_{33}^{-1/2}\boldsymbol{Z}_{3})^{\dagger}$, and $\bar{\boldsymbol{Y}}_{3} = (\bar{\boldsymbol{S}}_{33}^{-1/2}\bar{\boldsymbol{Z}}_{3})^{\dagger}$. It follows from direct inspection of (106) that there exist unitary matrices[4] $\boldsymbol{U}_{2.3} \in \mathbb{C}^{r\times r}$ and $\boldsymbol{U}_{3} \in \mathbb{C}^{(N-J)\times(N-J)}$ such that $\boldsymbol{Y}_{2.3} = \bar{\boldsymbol{Y}}_{2.3}\boldsymbol{U}_{2.3}$ and $\boldsymbol{Y}_{3} = \bar{\boldsymbol{Y}}_{3}\boldsymbol{U}_{3}$.

First, let us define the following block-triangular decompositions for matrices $\boldsymbol{S}_2 = \boldsymbol{L}_2^{\dagger}\,\boldsymbol{L}_2$ and $\bar{\boldsymbol{S}}_2 = \bar{\boldsymbol{L}}_2^{\dagger}\,\bar{\boldsymbol{L}}_2$, where:

$$\boldsymbol{L}_2 \triangleq \begin{bmatrix} \boldsymbol{S}_{2.3}^{1/2} & \boldsymbol{0}_{r\times(N-J)} \\ \boldsymbol{S}_{33}^{-1/2}\boldsymbol{S}_{32} & \boldsymbol{S}_{33}^{1/2} \end{bmatrix}, \tag{107}$$

$$\bar{\boldsymbol{L}}_2 \triangleq \begin{bmatrix} \bar{\boldsymbol{S}}_{2.3}^{1/2} & \boldsymbol{0}_{r\times(N-J)} \\ \bar{\boldsymbol{S}}_{33}^{-1/2}\bar{\boldsymbol{S}}_{32} & \bar{\boldsymbol{S}}_{33}^{1/2} \end{bmatrix}. \tag{108}$$

Therefore, given the aforementioned definitions, it can be shown that:

$$\begin{bmatrix} \boldsymbol{Y}_{2.3}^{\dagger} \\ \boldsymbol{Y}_{3}^{\dagger} \end{bmatrix} = (\boldsymbol{L}_2^{\dagger})^{-1}\boldsymbol{Z}_{23} = \tag{109}$$

$$\begin{bmatrix} \boldsymbol{U}_{2.3}^{\dagger}\,\bar{\boldsymbol{Y}}_{2.3}^{\dagger} \\ \boldsymbol{U}_{3}^{\dagger}\,\bar{\boldsymbol{Y}}_{3}^{\dagger} \end{bmatrix} = \boldsymbol{U}_1(\bar{\boldsymbol{L}}_2^{\dagger})^{-1}\bar{\boldsymbol{Z}}_{23}, \tag{110}$$

where $\boldsymbol{Z}_{23} \triangleq [\,\boldsymbol{Z}_2^T \quad \boldsymbol{Z}_3^T\,]^T$ and $\boldsymbol{U}_1 \triangleq \mathrm{diag}(\boldsymbol{U}_{2.3}^{\dagger}, \boldsymbol{U}_3^{\dagger})$, respectively. From comparison of right hand side of (109) and (110), it readily follows that

$$\boldsymbol{Z}_{23} = \boldsymbol{L}_2^{\dagger}\,\boldsymbol{U}_1(\bar{\boldsymbol{L}}_2^{\dagger})^{-1}\bar{\boldsymbol{Z}}_{23}\,. \tag{111}$$

From inspection of (111), it is apparent that selecting the transformation $\boldsymbol{G}_3 = \boldsymbol{L}_2^{\dagger}\,\boldsymbol{U}_1(\bar{\boldsymbol{L}}_2^{\dagger})^{-1}$ (which is block-triangular as dictated by (91)) automatically verifies the set of equations:

$$\begin{cases} (i) & \boldsymbol{G}_3\,\bar{\boldsymbol{Z}}_{23} = \boldsymbol{Z}_{23} \\ (ii) & \boldsymbol{G}_3\,\bar{\boldsymbol{S}}_2\,\boldsymbol{G}_3^{\dagger} = \boldsymbol{S}_2 \end{cases}, \tag{112}$$

since it also holds

$$\begin{aligned} &(\boldsymbol{L}_2^{\dagger})\boldsymbol{U}_1(\bar{\boldsymbol{L}}_2^{\dagger})^{-1}\bar{\boldsymbol{S}}_2(\bar{\boldsymbol{L}}_2)^{-1}\boldsymbol{U}_1^{\dagger}\boldsymbol{L}_2 \\ &= (\boldsymbol{L}_2^{\dagger})\boldsymbol{U}_1(\bar{\boldsymbol{L}}_2^{\dagger})^{-1}\,\bar{\boldsymbol{L}}_2^{\dagger}\,\bar{\boldsymbol{L}}_2\,(\bar{\boldsymbol{L}}_2)^{-1}\boldsymbol{U}_1^{\dagger}\boldsymbol{L}_2 = \boldsymbol{L}_2^{\dagger}\,\boldsymbol{L}_2 = \boldsymbol{S}_2. \end{aligned} \tag{113}$$

Finally, we show how to build the remaining blocks of $\boldsymbol{G}$. To this end, let us denote $\boldsymbol{S}_3 \triangleq [\,\boldsymbol{S}_{12} \quad \boldsymbol{S}_{13}\,]$, $\bar{\boldsymbol{S}}_3 \triangleq [\,\bar{\boldsymbol{S}}_{12} \quad \bar{\boldsymbol{S}}_{13}\,]$, $\boldsymbol{S}_1 \triangleq \boldsymbol{S}_{11}$, and $\bar{\boldsymbol{S}}_1 \triangleq \bar{\boldsymbol{S}}_{11}$, and consider the block-triangular decompositions for matrices $\boldsymbol{S}_c = \boldsymbol{L}_c^{\dagger}\,\boldsymbol{L}_c$ and $\bar{\boldsymbol{S}}_c = \bar{\boldsymbol{L}}_c^{\dagger}\,\bar{\boldsymbol{L}}_c$ as:

$$\boldsymbol{L}_c \triangleq \begin{bmatrix} \boldsymbol{S}_{.1}^{1/2} & \boldsymbol{0}_{t\times(N-t)} \\ (\boldsymbol{L}_2^{\dagger}\boldsymbol{U}_1)^{-1}\boldsymbol{S}_3^{\dagger} & (\boldsymbol{U}_1^{\dagger}\boldsymbol{L}_2) \end{bmatrix} \tag{114}$$

$$\bar{\boldsymbol{L}}_c \triangleq \begin{bmatrix} \bar{\boldsymbol{S}}_{.1}^{1/2} & \boldsymbol{0}_{t\times(N-t)} \\ (\bar{\boldsymbol{L}}_2^{\dagger})^{-1}\bar{\boldsymbol{S}}_3^{\dagger} & \bar{\boldsymbol{L}}_2 \end{bmatrix} \tag{115}$$

where $\boldsymbol{S}_{.1} \triangleq \boldsymbol{S}_1 - \boldsymbol{S}_3\boldsymbol{S}_2^{-1}\boldsymbol{S}_3^{\dagger} \in \mathbb{C}^{t\times t}$ and analogously $\bar{\boldsymbol{S}}_{.1} \triangleq \bar{\boldsymbol{S}}_1 - \bar{\boldsymbol{S}}_3\bar{\boldsymbol{S}}_2^{-1}\bar{\boldsymbol{S}}_3^{\dagger} \in \mathbb{C}^{t\times t}$. Also, since we need to ensure $(\boldsymbol{G}\,\bar{\boldsymbol{S}}_c\,\boldsymbol{G}^{\dagger}) = \boldsymbol{S}_c$, it suffices that

$$\bar{\boldsymbol{L}}_c\,\boldsymbol{G}^{\dagger} = \boldsymbol{L}_c \tag{116}$$

$$\begin{aligned} &\begin{bmatrix} \bar{\boldsymbol{S}}_{.1}^{1/2} & \boldsymbol{0}_{t\times(N-t)} \\ (\bar{\boldsymbol{L}}_2^{\dagger})^{-1}\bar{\boldsymbol{S}}_3^{\dagger} & \bar{\boldsymbol{L}}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{G}_1^{\dagger} & \boldsymbol{0}_{t\times(N-t)} \\ \boldsymbol{G}_2^{\dagger} & \boldsymbol{G}_3^{\dagger} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{S}_{.1}^{1/2} & \boldsymbol{0}_{t\times(N-t)} \\ (\boldsymbol{L}_2^{\dagger}\boldsymbol{U}_1)^{-1}\boldsymbol{S}_3^{\dagger} & (\boldsymbol{U}_1^{\dagger}\boldsymbol{L}_2) \end{bmatrix} \end{aligned} \tag{117}$$

from which the following set of independent equations arises:

$$\begin{cases} (i) & \bar{\boldsymbol{S}}_{.1}^{1/2}\,\boldsymbol{G}_1^{\dagger} = \boldsymbol{S}_{.1}^{1/2} \\ (ii) & (\bar{\boldsymbol{L}}_2^{\dagger})^{-1}\,\bar{\boldsymbol{S}}_3^{\dagger}\,\boldsymbol{G}_1^{\dagger} + \bar{\boldsymbol{L}}_2\,\boldsymbol{G}_2^{\dagger} = (\boldsymbol{L}_2^{\dagger}\,\boldsymbol{U}_1)^{-1}\boldsymbol{S}_3^{\dagger} \end{cases}, \tag{118}$$

which provides the "completing" solutions for matrix $\boldsymbol{G}$:

$$\boldsymbol{G}_1 = \boldsymbol{S}_{.1}^{1/2}\bar{\boldsymbol{S}}_{.1}^{-1/2}, \tag{119}$$

$$\boldsymbol{G}_2^{\dagger} = \bar{\boldsymbol{S}}_2^{-1}\left(\boldsymbol{G}_3^{-1}\boldsymbol{S}_3^{\dagger} - \bar{\boldsymbol{S}}_3^{\dagger}\boldsymbol{G}_1^{\dagger}\right). \tag{120}$$

Up to now, we have shown how matrices $\boldsymbol{G}_i$ can be constructed. Finally, it can be easily shown that matrix $\boldsymbol{F}_1$ should be chosen as $\boldsymbol{F}_1 = \boldsymbol{Z}_1 - \sum_{i=1}^{3}\boldsymbol{G}_{1,i}\bar{\boldsymbol{Z}}_i$. This concludes the proof for (b).

Appendix C
MIS Obtained by Enforcing Additional Invariance in Range-Spread Case

In this Appendix we show that the statistic

$$\begin{cases} \boldsymbol{x}_1 \triangleq \mathrm{eig}(\boldsymbol{T}_b) \\ \boldsymbol{x}_2 \triangleq \mathrm{eig}(\boldsymbol{T}_b + \boldsymbol{T}_a) \end{cases}, \tag{121}$$

where $\boldsymbol{T}_a = (\boldsymbol{a}\boldsymbol{a}^{\dagger})$ ($\boldsymbol{a} \in \mathbb{C}^{M\times 1}$) and $\boldsymbol{T}_b \in \mathbb{H}^{M\times M}$, is a MIS for the elementary action

$$\ell_{R,b}^{\star}(\boldsymbol{T}_a, \boldsymbol{T}_b) = (\boldsymbol{U}_d^{\dagger}\,\boldsymbol{T}_a\,\boldsymbol{U}_d,\, \boldsymbol{U}_d^{\dagger}\,\boldsymbol{T}_b\,\boldsymbol{U}_d)\,, \tag{122}$$

where $\boldsymbol{U}_d \in \mathcal{U}(M)$. First, we observe that (121) is in one-to-one mapping with:

$$\begin{cases} \boldsymbol{x}_1 = \mathrm{eig}(\boldsymbol{T}_b) \\ \bar{\boldsymbol{x}}_2 \triangleq |\boldsymbol{k}| \end{cases}, \tag{123}$$

where $\boldsymbol{k} \triangleq (\boldsymbol{U}_b^{\dagger}\,\boldsymbol{a})$ and the modulus $|\cdot|$ in (123) should be intended element-wise. Also, $\boldsymbol{U}_b$ denotes the eigenvector matrix of $\boldsymbol{T}_b$, that is, $\boldsymbol{T}_b = \boldsymbol{U}_b\,\boldsymbol{\Lambda}_b\,\boldsymbol{U}_b^{\dagger}$. The existence of the aforementioned mapping can be proved as follows. We start by observing that $\mathrm{eig}(\boldsymbol{T}_b + \boldsymbol{a}\boldsymbol{a}^{\dagger})$ can be obtained as the zeros (with respect to the variable s) of the rational function [23]:

$$w(s) = (1 + \boldsymbol{k}^{\dagger}\,(\boldsymbol{\Lambda}_b - s\,\boldsymbol{I}_M)^{-1}\,\boldsymbol{k})\,. \tag{124}$$

Also, since $(\boldsymbol{\Lambda}_b - s\boldsymbol{I}_M)^{-1}$ is a diagonal matrix, $w(s)$ depends only on $|\boldsymbol{k}|$. Therefore $\mathrm{eig}(\boldsymbol{T}_b + \boldsymbol{a}\boldsymbol{a}^{\dagger})$ can be obtained starting

[4] Such property can be verified as follows: given the equality $\boldsymbol{A}\boldsymbol{A}^{\dagger} = \boldsymbol{B}\boldsymbol{B}^{\dagger}$ between two generic matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ (both $\in \mathbb{C}^{a_1\times a_2}$ and with rank $c < \min\{a_1, a_2\}$) and, after defining the reduced eigenvalue decompositions $(\boldsymbol{A}\boldsymbol{A}^{\dagger}) = \boldsymbol{U}_A\,\boldsymbol{\Lambda}_A\,\boldsymbol{U}_A^{\dagger}$ and $(\boldsymbol{B}\boldsymbol{B}^{\dagger}) = \boldsymbol{U}_B\,\boldsymbol{\Lambda}_B\,\boldsymbol{U}_B^{\dagger}$ (that is, $\boldsymbol{U}_A, \boldsymbol{U}_B \in \mathbb{C}^{a_1\times c}$ and $\boldsymbol{\Lambda}_A, \boldsymbol{\Lambda}_B \in \mathbb{R}^{c\times c}$), and the reduced SVDs $\boldsymbol{A} = \boldsymbol{U}_A\,\boldsymbol{\Sigma}_A\,\boldsymbol{V}_A^{\dagger}$ and $\boldsymbol{B} = \boldsymbol{U}_B\,\boldsymbol{\Sigma}_B\,\boldsymbol{V}_B^{\dagger}$ (that is, $\boldsymbol{V}_A, \boldsymbol{V}_B \in \mathbb{C}^{a_2\times c}$), it is apparent that such equality implies: (*i*) $\boldsymbol{\Sigma}_A = \boldsymbol{\Sigma}_B$; (*ii*) $\boldsymbol{U}_A = \boldsymbol{U}_B\,\boldsymbol{D}$, where $\boldsymbol{D} \in \mathbb{C}^{c\times c}$ denotes a diagonal matrix of phasors (recall that $\boldsymbol{\Lambda}_A = \boldsymbol{\Sigma}_A^2$ and $\boldsymbol{\Lambda}_B = \boldsymbol{\Sigma}_B^2$). Thus, it follows from substitution that $\boldsymbol{A} = \boldsymbol{U}_B\boldsymbol{D}\,\boldsymbol{\Sigma}_B\,\boldsymbol{V}_A^{\dagger}$. Therefore, after defining the unitary matrix $\boldsymbol{U}^{\star} \triangleq [\,\boldsymbol{V}_A \quad \bar{\boldsymbol{V}}_A\,]\,\mathrm{diag}(\boldsymbol{D}^{*}, \bar{\boldsymbol{D}})\,[\,\boldsymbol{V}_B \quad \bar{\boldsymbol{V}}_B\,]^{\dagger}$, where $\bar{\boldsymbol{D}} \in \mathbb{C}^{(a_2-c)\times(a_2-c)}$ is an arbitrary diagonal matrix of phasors and $\bar{\boldsymbol{V}}_A \in \mathbb{C}^{a_2\times(a_2-c)}$ and $\bar{\boldsymbol{V}}_B \in \mathbb{C}^{a_2\times(a_2-c)}$ are the "completing" unitary matrices of $\boldsymbol{V}_A$ and $\boldsymbol{V}_B$, it is finally demonstrated that $\boldsymbol{A}\boldsymbol{U}^{\star} = \boldsymbol{B}$.

from $\boldsymbol{\Lambda}_b$ (viz. $\mathrm{eig}(\boldsymbol{T}_b)$) and $|\boldsymbol{k}|$. Vice versa, the vector $|\boldsymbol{x}|$ is obtained from $\mathrm{eig}(\boldsymbol{T}_b + \boldsymbol{a}\boldsymbol{a}^\dagger)$ and $\mathrm{eig}(\boldsymbol{T}_b)$ by inverting (124), that is:

$$\boldsymbol{k}^\dagger (\boldsymbol{\Lambda}_b - x_{2,i}\boldsymbol{I}_M)^{-1}\boldsymbol{k} = -1, \tag{125}$$

$$\sum_{n=1}^{M} \frac{|k_n|^2}{(\lambda_{b,n} - x_{2,i})} = -1, \tag{126}$$

$$\boldsymbol{\alpha}_i^T \boldsymbol{\epsilon} = -1, \quad i \in \{1, \dots, M\} \tag{127}$$

where $\lambda_{b,n}$ is the n-th diagonal element of $\boldsymbol{\Lambda}_b$ (also, recall that $x_{2,i}$ is i-th eigenvalue of $\boldsymbol{T}_b + \boldsymbol{T}_a$) and

$$\boldsymbol{\epsilon} \triangleq [\, |k_1|^2 \quad \cdots \quad |k_M|^2 \,]^T, \tag{128}$$

$$\boldsymbol{\alpha}_i \triangleq [\, (\lambda_{b,1} - x_{2,i})^{-1} \quad \cdots \quad (\lambda_{b,M} - x_{2,i})^{-1} \,]^T. \tag{129}$$

It is shown hereinafter that the linear system in (127) (with respect to the unknown vector $\boldsymbol{\epsilon}$) admits a unique solution.

Indeed, the generic $\boldsymbol{\alpha}_i$ represents a scaled version of $(\boldsymbol{E}_p \boldsymbol{v}_{a+b,i})$, where $\boldsymbol{E}_p \triangleq \mathrm{diag}\{\boldsymbol{\epsilon}\}$ and $\boldsymbol{v}_{a+b,i}$ denotes the i-th eigenvector of $\boldsymbol{T}_b + \boldsymbol{a}\boldsymbol{a}^\dagger$ [23, (2.1)]. However, since we assume that the eigenvalues are distinct with probability one, the eigenvectors $\boldsymbol{v}_{a+b,i}$ will be linearly independent. Therefore, it follows that also the set $\{\boldsymbol{\alpha}_i\}_{i=1}^M$ constitutes a linearly independent basis. Such conclusion clearly implies that the system is invertible and admits a unique solution; therefore there exists a one-to-one correspondence between the statistics in (121) and (123).

Once established the correspondence between (121) and (123), it suffices to show that (123) is a MIS for the group of transformations specified in (122). In order to accomplish this task, we first prove *invariance* of statistic in (123). Indeed, given the transformations

$$\widetilde{\boldsymbol{T}}_a = (\boldsymbol{U}_d^\dagger \boldsymbol{a}\boldsymbol{a}^\dagger \boldsymbol{U}_d), \quad \widetilde{\boldsymbol{T}}_b = (\boldsymbol{U}_d^\dagger \boldsymbol{T}_b \boldsymbol{U}_d), \tag{130}$$

it is readily shown that $\mathrm{eig}(\widetilde{\boldsymbol{T}}_b)$ can obtained as the zeros of

$$\det(s\boldsymbol{I}_M - \boldsymbol{U}_d^\dagger \boldsymbol{T}_b \boldsymbol{U}_d) = 0 \Leftrightarrow \det(s\boldsymbol{I}_M - \boldsymbol{T}_b) = 0 \tag{131}$$

as a result of Sylvester's determinant theorem, thus coinciding with $\mathrm{eig}(\boldsymbol{T}_b)$. Also, it holds

$$\begin{aligned} |\widetilde{\boldsymbol{U}}_b^\dagger \tilde{\boldsymbol{a}}| &= |\boldsymbol{U}_b^\dagger \boldsymbol{U}_d \boldsymbol{U}_d^\dagger \boldsymbol{a}| \\ &= |\boldsymbol{U}_b^\dagger \boldsymbol{a}|. \end{aligned} \tag{132}$$

Therefore the statistic in (123) is *invariant*. We then prove *maximality*. Under the assumption

$$\begin{cases} \mathrm{eig}(\boldsymbol{T}_b) = \mathrm{eig}(\widetilde{\boldsymbol{T}}_b) \\ |\boldsymbol{U}_b^\dagger \boldsymbol{a}| = |\widetilde{\boldsymbol{U}}_b^\dagger \tilde{\boldsymbol{a}}| \end{cases}, \tag{133}$$

it can be readily shown that there exists a unitary matrix $\boldsymbol{V}$ that ensures the equality $(\boldsymbol{V}^\dagger \boldsymbol{T}_b \boldsymbol{V}) = \widetilde{\boldsymbol{T}}_b$, namely $\boldsymbol{V} \triangleq (\boldsymbol{U}_b \boldsymbol{D}_b \widetilde{\boldsymbol{U}}_b^\dagger)$, where $\boldsymbol{D}_b$ is a diagonal matrix of arbitrary phasors. Similarly, we have employed the eigendecomposition $\widetilde{\boldsymbol{T}}_b = (\widetilde{\boldsymbol{U}}_b \widetilde{\boldsymbol{\Lambda}}_b \widetilde{\boldsymbol{U}}_b^\dagger)$. Additionally, in order to complete proof of maximality, we need to prove that the aforementioned transformation, when applied to $\boldsymbol{T}_a = \boldsymbol{a}\boldsymbol{a}^\dagger$, can be adjusted to satisfy:

$$\boldsymbol{V}^\dagger (\boldsymbol{a}\boldsymbol{a}^\dagger) \boldsymbol{V} = \tilde{\boldsymbol{a}}\tilde{\boldsymbol{a}}^\dagger. \tag{134}$$

After substitution, such condition can be rewritten as:

$$\widetilde{\boldsymbol{U}}_b \boldsymbol{D}_b^\dagger \boldsymbol{U}_b^\dagger (\boldsymbol{a}\boldsymbol{a}^\dagger) \boldsymbol{U}_b \boldsymbol{D}_b \widetilde{\boldsymbol{U}}_b^\dagger = \tilde{\boldsymbol{a}}\tilde{\boldsymbol{a}}^\dagger, \tag{135}$$

$$\boldsymbol{D}_b^\dagger \boldsymbol{U}_b^\dagger (\boldsymbol{a}\boldsymbol{a}^\dagger) \boldsymbol{U}_b \boldsymbol{D}_b = \widetilde{\boldsymbol{U}}_b^\dagger \tilde{\boldsymbol{a}}\tilde{\boldsymbol{a}}^\dagger \widetilde{\boldsymbol{U}}_b, \tag{136}$$

$$[\boldsymbol{D}_b^\dagger (\boldsymbol{U}_b^\dagger \boldsymbol{a})][\boldsymbol{D}_b^\dagger (\boldsymbol{U}_b^\dagger \boldsymbol{a})]^\dagger = (\widetilde{\boldsymbol{U}}_b^\dagger \tilde{\boldsymbol{a}})(\widetilde{\boldsymbol{U}}_b^\dagger \tilde{\boldsymbol{a}})^\dagger. \tag{137}$$

The above rank-one matrix equality can be achieved by enforcing the vector equality

$$[\boldsymbol{D}_b^\dagger (\boldsymbol{U}_b^\dagger \boldsymbol{a})] = (\widetilde{\boldsymbol{U}}_b^\dagger \tilde{\boldsymbol{a}}) \tag{138}$$

by choosing each element of the diagonal matrix $\boldsymbol{D}_b^\dagger$ in order to rotate each phase term of $(\boldsymbol{U}_b^\dagger \boldsymbol{a})$ aiming at imposing $\angle(\boldsymbol{U}_b^\dagger \boldsymbol{a}) = \angle(\widetilde{\boldsymbol{U}}_b^\dagger \tilde{\boldsymbol{a}})$, since $|\boldsymbol{U}_b^\dagger \boldsymbol{a}| = |\widetilde{\boldsymbol{U}}_b^\dagger \tilde{\boldsymbol{a}}|$ by definition (cf. (133)). Therefore (123) (resp. (121)) is a MIS for the aforementioned group of transformations.

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