This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TSP.2016.2519005, IEEE Transactions on Signal Processing

A Unifying Framework for Adaptive Radar Detection in Homogeneous plus Structured Interference-Part II: Detectors Design

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Abstract

This paper deals with the problem of adaptive multidimensional/multichannel signal detection in homogeneous Gaussian disturbance with unknown covariance matrix and structured (unknown) deterministic interference. The aforementioned problem extends the well-known Generalized Multivariate Analysis of Variance (GMANOVA) tackled in the open literature. In a companion paper, we have obtained the Maximal Invariant Statistic (MIS) for the problem under consideration, as an enabling tool for the design of suitable detectors which possess the Constant False-Alarm Rate (CFAR) property. Herein, we focus on the development of several theoretically-founded detectors for the problem under consideration. First, all the considered detectors are shown to be function of the MIS, thus proving their CFARness property. Secondly, coincidence or statistical equivalence among some of them in such a general signal model is proved. Thirdly, strong connections to well-known (simpler) scenarios analyzed in adaptive detection literature are established. Finally, simulation results are provided for a comparison of the proposed receivers.

Index Terms

Manuscript received 17th July 2015; revised 17th December 2015; accepted 28th December 2015. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Wenwu Wang.

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Adaptive Radar Detection, CFAR, Invariance Theory, Maximal Invariants, Double-subspace Model, GMANOVA, Coherent Interference.

I. Introduction

A. Motivation and Related Works

The problem of adaptive detection has been subject of great interest in the last decades. Many works appeared in the open literature, dealing with the design and performance analysis of suitable detectors in several specific settings (see for instance [1] and references therein).

As introduced in a companion paper, herein we focus on a signal model which generalizes that of Generalized Multivariate Analysis of Variance (GMANOVA) [2] by considering an additional unknown double-subspace structured deterministic interference. Such model is here denoted as I-GMANOVA. The I-GMANOVA model is very general and comprises many adaptive detection setups as special instances, ranging from point-like targets (resp. interference) [3] to extended ones [4], from a single-steering assumption to a vector subspace one [5], [6], and the GMANOVA model itself [2], only to mention a few examples. We recall that attractive modifications of GMANOVA have also appeared in the recent literature [7], [8], focusing on the design of computationally-efficient approximate Maximum Likelihood (ML) estimators when the unknown signal matrix is constrained to be diagonal [7] or block-diagonal [8].

In the case of composite hypothesis testing, the three widely-used design criteria are the Generalized Likelihood Ratio Test (GLRT), the Rao test, and the Wald test [9]. Their use is well-established in the context of adaptive detection literature [4], [10]–[13]; more importantly they are known to share the same asymptotic performance [9]. However, in the finite-sample case their performance differs and their relative assessment depends on the specific hypothesis testing model being considered. Such statement holds true unless some specific instances occur, such as in [14], where it is proved that they are statistically equivalent in the case of point-like targets and a partially-homogeneous scenario. Other than the aforementioned detectors, in the context of radar adaptive detection it is also customary to consider their two-step variations, with the two-step GLRT (2S-GLRT) being the most common. Those are typically obtained by designing the detector under the assumption of a known disturbance covariance matrix and replacing it with a sample estimate based on a set of secondary (or signal-free) data [15].

Furthermore, a few interesting alternative detectors for composite hypothesis testing are the Durbin (naive) test [16] and the Terrell (Gradient) test [17]. These detectors have been shown to be asymptotically efficient as the aforementioned well-known criteria. The same rationale applies to the Lawley-Hotelling (LH) test [18]. Though these detectors are well-known in the statistics field, the development and application of these decision rules is *less frequently encountered in radar adaptive detection literature*,

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e.g., [15], [19]. The reason is that an important prerequisite for a wide-spread application of an adaptive detection algorithm consists in showing its CFARness with respect to the nuisance parameters; in this respect, the assessment of such property in radar adaptive detection literature has been somewhat lacking.

Of course, the use of GMANOVA model in the context of adaptive radar detection is not new and dates back to the the milestone study in [15], where the development and analysis of the GLRT was first proposed. A similar work was then presented years later in [20], where the focus was on the design of a compression matrix prior to the detection process, aimed at reducing the computational burden and minimizing the performance loss with respect to the standard processing. More recently, GLRT, Rao and Wald tests were developed and compared under the GMANOVA model [21], along with some other heuristic detectors. Unfortunately, albeit the CFARness of the proposed detectors was proved, no clear connection to the MIS was established. More importantly, no (structured) deterministic interference was considered in all the aforementioned works; the proposed I-GMANOVA model is aimed at filling such gap.

We point out that the closest study to ours (in terms of interference accounting) is the work in [12], where range-spread and vector subspace targets and interference are considered; however the sole GLRT and 2S-GLRT are derived and analyzed. Similarly, a Rao test (and its two-step version) is recently obtained in [22]. It is worth noticing that the model considered by the aforementioned works is included in the I-GMANOVA model, and can be readily obtained by assuming a canonical form for the right-subspace matrix of both the signal and the interference.

Summarizing, in our opinion the huge (but scattered) literature on adaptive detection in many case-specific signal models and the presence of several detectors (developed on theoretically-solid assumptions) for generic composite hypothesis testing problems lacks a comprehensive and systematic analysis. First, such analysis may help the generic designer in readily obtaining a plethora of suitable adaptive detectors in some relevant scenarios which can be fitted into the considered I-GMANOVA model. Secondly, the development of detectors closed-form expressions under I-GMANOVA model may allow to easily claim some general statistical equivalence results than those already noticed in some special instances (see e.g., [11], [19], [23]). Thirdly, the available explicit expression for each detector allows for a systematic analysis of its (possible) CFARness (under a quite general signal model). The latter task is greatly simplified when knowledge of the explicit form of the MIS is available for the considered problem; in this respect, the derivation of the MIS and its analysis, object of a companion paper, fulfills this need.

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B. Summary of the contributions and Paper Organization

The main contributions of the second part of this work are thus related to detectors development and CFAR property analysis and can be summarized as follows:

- Starting from the canonical form obtained in our companion paper, for the general model under investigation we derive closed-form expressions for the (i) GLRT, (ii) Rao test, (iii) Wald test, (iv) Gradient test (v) Durbin test, (vi) two-step GLRT (2S-GLRT), and (vii) LH test. As an interesting byproduct of our derivation, we show that Durbin test is *statistically equivalent* to the Rao test for the considered (adaptive) detection problem, thus extending the findings in [19], obtained for the simpler case of a point-like target without interference. Similarly, we demonstrate the statistical equivalence between Wald test and 2S-GLRT, thus extending the works in [11] and [23], concerning the special instances of point-like targets (no interference) and multidimensional signals, respectively;
- The general expressions of the receivers are exploited to analyze special cases of interest, such as: (a) vector subspace detection of point-like targets (with possible structured interference) [3], [6], [11], (b) multidimensional signals [10], [23], (c) range-spread (viz. extended) targets [4], [24], [25], and (d) standard GMANOVA (i.e., without structured interference) [15], [21]. In such special instances, possible coincidence or statistical equivalence is investigated among the considered detectors;
- Exploiting the matrix pair form of the MIS obtained in part one, we show that *all* the considered detector can be expressed as a function of the MIS, thus proving their CFARness with respect to both the covariance of the disturbance and the deterministic (structured) interference;
- Finally, a simulation results section is provided to compare the proposed detectors in terms of the relevant parameters and underline the common trends among them.

The remainder of the paper is organized as follows: in Sec. II, we describe the hypothesis testing problem under investigation; in Sec. III, we obtain the general expressions for the detectors considered in this paper and we express them as a function of the MIS; in Sec. IV, we particularize the obtained expressions to the aforementioned special instances of adaptive detection problems; then, in Sec. V we compare the obtained detectors through simulation results and finally in Sec. VI we draw some concluding remarks and indicate future research directions. Proofs and derivations are confined to an additional document containing supplemental material[1].

II. Problem Formulation

In a companion paper, we have shown that the considered problem admits an equivalent (but simpler) formulation by exploiting the so-called "canonical form", that is:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{B}_{t,0}^T & \boldsymbol{0}_{M \times r} \end{bmatrix}^T \boldsymbol{C} + \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{A}\, \boldsymbol{B}_s\, \boldsymbol{C} + \boldsymbol{N} \end{cases}, \tag{1}$$

where we have assumed that a data matrix $\boldsymbol{Z} \in \mathbb{C}^{N \times K}$ has been collected. Also, we have adopted the following definitions:

- $\boldsymbol{A} \triangleq \begin{bmatrix} \boldsymbol{E}_t & \boldsymbol{E}_r \end{bmatrix} \in \mathbb{C}^{N \times J}$, where $\boldsymbol{E}_t \triangleq \begin{bmatrix} \boldsymbol{I}_t & \boldsymbol{0}_{t \times (N-t)} \end{bmatrix}^T$ and $\boldsymbol{E}_r \triangleq \begin{bmatrix} \boldsymbol{0}_{r \times t} & \boldsymbol{I}_r & \boldsymbol{0}_{r \times (N-J)} \end{bmatrix}^T$ are the (known) left subspaces of the interference and useful signal (in canonical form), respectively (we have denoted $J \triangleq r + t$);
- $\boldsymbol{B}_s \triangleq \begin{bmatrix} \boldsymbol{B}_{t,1}^T & \boldsymbol{B}^T \end{bmatrix}^T$, where $\boldsymbol{B}_{t,i} \in \mathbb{C}^{t \times M}$ and $\boldsymbol{B} \in \mathbb{C}^{r \times M}$ are the (unknown) interference (under $\mathcal{H}_i$) and useful signal matrices, respectively;
- $\boldsymbol{C} \triangleq \begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M \times (K-M)} \end{bmatrix} \in \mathbb{C}^{M \times K}$ is the (known) right subspace matrix associated to *both* signal and interference in canonical form;
- $\boldsymbol{N}$ is a disturbance matrix such that $\boldsymbol{N} \sim \mathcal{CN}_{N \times K}(\boldsymbol{0}_{N \times K}, \boldsymbol{I}_K, \boldsymbol{R})$, where $\boldsymbol{R} \in \mathbb{C}^{N \times N}$ is an (unknown) positive definite covariance matrix [15].

We recall that the detection problem in (1) is tantamount to testing the null hypothesis $\boldsymbol{B} = \boldsymbol{0}_{r \times M}$ (viz. $||\boldsymbol{B}||_F = 0$, denoted with $\mathcal{H}_0$) against the alternative that $\boldsymbol{B}$ is unrestricted (viz. $||\boldsymbol{B}||_F > 0$, denoted with $\mathcal{H}_1$), along with the set of nuisance parameters $\boldsymbol{B}_{t,i}$ and $\boldsymbol{R}$.

[1]*Notation* - Lower-case (resp. Upper-case) bold letters denote vectors (resp. matrices), with a_n (resp. $A_{n,m}$) representing the n-th (resp. the (n,m)-th) element of the vector $\boldsymbol{a}$ (resp. matrix $\boldsymbol{A}$); $\mathbb{R}^N$, $\mathbb{C}^N$, and $\mathbb{H}^{N \times N}$ are the sets of N-dimensional vectors of real numbers, of complex numbers, and of $N \times N$ Hermitian matrices, respectively; upper-case calligraphic letters and braces denote finite sets; $\mathbb{E}\{\cdot\}$, $(\cdot)^T$, $(\cdot)^\dagger$, $(\cdot)^*$, $\mathrm{Tr}\,[\cdot]$, $||\cdot||$, $\Re\{\cdot\}$ and $\Im\{\cdot\}$, denote expectation, transpose, Hermitian, conjugate, matrix trace, Euclidean norm, real part, and imaginary part operators, respectively; $\boldsymbol{0}_{N \times M}$ (resp. $\boldsymbol{I}_N$) denotes the $N \times M$ null (resp. identity) matrix; $\boldsymbol{0}_N$ (resp. $\boldsymbol{1}_N$) denotes the null (resp. ones) column vector of length N; $\mathrm{vec}(\boldsymbol{M})$ stacks the first to the last column of the matrix $\boldsymbol{M}$ one under another to form a long vector; $\det(\boldsymbol{A})$ and $||\boldsymbol{A}||_F$ denote the determinant and Frobenius norm of matrix $\boldsymbol{A}$; $\boldsymbol{A} \otimes \boldsymbol{B}$ indicates the Kronecker product between $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices; $\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}}$ denotes the gradient of scalar valued function $f(\boldsymbol{x})$ w.r.t. vector $\boldsymbol{x}$ arranged in a column vector, while $\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}^T}$ its transpose (i.e. a row vector); the symbol "$\sim$" means "distributed as"; $\boldsymbol{x} \sim \mathcal{CN}_N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a complex (proper) Gaussian-distributed vector $\boldsymbol{x}$ with mean vector $\boldsymbol{\mu} \in \mathbb{C}^{N \times 1}$ and covariance matrix $\boldsymbol{\Sigma} \in \mathbb{C}^{N \times N}$; $\boldsymbol{X} \sim \mathcal{CN}_{N \times M}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ denotes a complex (proper) Gaussian-distributed matrix $\boldsymbol{X}$ with mean $\boldsymbol{A} \in \mathbb{C}^{N \times M}$ and $\mathrm{Cov}[\mathrm{vec}(\boldsymbol{X})] = \boldsymbol{B} \otimes \boldsymbol{C}$; $\boldsymbol{P}_A$ denotes the orthogonal projection of the full-column-rank matrix $\boldsymbol{A}$, that is $\boldsymbol{P}_{\boldsymbol{A}} \triangleq [\boldsymbol{A}(\boldsymbol{A}^\dagger \boldsymbol{A})^{-1} \boldsymbol{A}^\dagger]$, while $\boldsymbol{P}_A^\perp$ its complement, that is $\boldsymbol{P}_A^\perp \triangleq (\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{A}})$.

In the present manuscript we will consider decision rules which declare $\mathcal{H}_1$ (resp. $\mathcal{H}_0$) if $\Phi(\boldsymbol{Z}) \geq \eta$ (resp. $\Phi(\boldsymbol{Z}) < \eta$), where $\Phi(\cdot) \in \mathbb{C}^{N\times K} \to \mathbb{R}$ indicates the generic form of a statistic processing the received data $\boldsymbol{Z}$ and η denotes the threshold to be set in order to achieve a predetermined probability of false alarm (P_{fa}).

As a preliminary step towards the derivation of suitable detectors for the problem at hand, we also give the following auxiliary definitions:

- $\boldsymbol{b}_R \in \mathbb{R}^{rM\times 1}$ and $\boldsymbol{b}_I \in \mathbb{R}^{rM\times 1}$ are obtained as $\boldsymbol{b}_R \triangleq \Re\{\boldsymbol{b}\}$ and $\boldsymbol{b}_I \triangleq \Im\{\boldsymbol{b}\}$, respectively, where we have defined $\boldsymbol{b} \triangleq \mathrm{vec}(\boldsymbol{B})$;
- $\boldsymbol{\theta}_r \triangleq \begin{bmatrix} \boldsymbol{b}_R^T & \boldsymbol{b}_I^T \end{bmatrix}^T \in \mathbb{R}^{2rM\times 1}$ is the vector collecting the parameters of interest;
- $\boldsymbol{\theta}_s \triangleq \begin{bmatrix}\boldsymbol{\theta}_{s,a}^T & \boldsymbol{\theta}_{s,b}^T\end{bmatrix}^T \in \mathbb{R}^{(2tM+N^2)\times 1}$ is the vector of nuisance parameters containing: (a) $\boldsymbol{\theta}_{s,a} \triangleq \begin{bmatrix} \boldsymbol{b}_{t,R}^T & \boldsymbol{b}_{t,I}^T \end{bmatrix}^T \in \mathbb{R}^{2tM\times 1}$ where $\boldsymbol{b}_{t,R}$ and $\boldsymbol{b}_{t,I}$ are the vectors obtained as $\boldsymbol{b}_{t,R} \triangleq \Re\{\boldsymbol{b}_t\}$ and $\boldsymbol{b}_{t,I} \triangleq \Im\{\boldsymbol{b}_t\}$, respectively, where $\boldsymbol{b}_t \triangleq \mathrm{vec}(\boldsymbol{B}_t)$ (i.e. $\boldsymbol{B}_{t,i}$ under $\mathcal{H}_i$); (b) $\boldsymbol{\theta}_{s,b}$ contains in a given order[2] the real and imaginary parts of the off-diagonal entries together with the diagonal elements of $\boldsymbol{R}$;
- $\boldsymbol{\theta} \triangleq \begin{bmatrix} \boldsymbol{\theta}_r^T & \boldsymbol{\theta}_s^T \end{bmatrix}^T \in \mathbb{R}^{(2JM+N^2)\times 1}$ is the overall unknown parameter vector;
- $\widehat{\boldsymbol{\theta}}_0 \triangleq \begin{bmatrix} \boldsymbol{\theta}_{r,0}^T & \widehat{\boldsymbol{\theta}}_{s,0}^T \end{bmatrix}^T$, with $\boldsymbol{\theta}_{r,0} = \boldsymbol{0}_{2rM}$ (that is, the true value of $\boldsymbol{\theta}_r$ under $\mathcal{H}_0$) and $\widehat{\boldsymbol{\theta}}_{s,0}$ denoting the ML estimate of $\boldsymbol{\theta}_s$ under $\mathcal{H}_0$;
- $\widehat{\boldsymbol{\theta}}_1 \triangleq \begin{bmatrix} \widehat{\boldsymbol{\theta}}_{r,1}^T & \widehat{\boldsymbol{\theta}}_{s,1}^T \end{bmatrix}^T$, with $\widehat{\boldsymbol{\theta}}_{r,1}$ and $\widehat{\boldsymbol{\theta}}_{s,1}$ denoting the ML estimates of $\boldsymbol{\theta}_r$ and $\boldsymbol{\theta}_s$, respectively, under $\mathcal{H}_1$.

The probability density function (pdf) of $\boldsymbol{Z}$, when the hypothesis $\mathcal{H}_1$ is in force, is denoted with $f_1(\cdot)$ and it is given in closed form as:

$$f_1(\boldsymbol{Z};\boldsymbol{B}_s,\boldsymbol{R}) = \pi^{-NK}\det(\boldsymbol{R})^{-K} \times \exp\left(-\mathrm{Tr}\left[\boldsymbol{R}^{-1}(\boldsymbol{Z}-\boldsymbol{A}\boldsymbol{B}_s\boldsymbol{C})(\boldsymbol{Z}-\boldsymbol{A}\boldsymbol{B}_s\boldsymbol{C})^\dagger\right]\right), \quad (2)$$

while the corresponding pdf under $\mathcal{H}_0$, denoted in the following with $f_0(\cdot)$, is similarly obtained when replacing $\boldsymbol{A}\boldsymbol{B}_s\boldsymbol{C}$ with $\boldsymbol{E}_t\,\boldsymbol{B}_{t,0}\,\boldsymbol{C}$ in Eq. (2). In the following, in order for our analysis to apply, we will assume that the condition $(K-M) \geq N$ holds. Such condition is typically satisfied in practical adaptive detection setups [15].

[2]More specifically, $\boldsymbol{\theta}_{s,b} \triangleq \boldsymbol{\Xi}(\boldsymbol{R})$, where $\boldsymbol{\Xi}(\cdot)$ denotes the one-to-one mapping providing $\boldsymbol{\theta}_{s,b}$ from $\boldsymbol{R}$.

A. MIS for the considered problem

In what follows, we recall the MIS for the hypothesis testing under investigation, obtained in our companion paper. The mentioned statistic will be exploited in Sec. III to ascertain the CFARness of each considered detector. Before proceeding further, let

$$\boldsymbol{V}_{c,1} \triangleq \begin{bmatrix} \boldsymbol{I}_M \\ \boldsymbol{0}_{(K-M)\times M} \end{bmatrix}, \quad \boldsymbol{V}_{c,2} \triangleq \begin{bmatrix} \boldsymbol{0}_{M\times(K-M)} \\ \boldsymbol{I}_{K-M} \end{bmatrix}, \tag{3}$$

and observe that $\boldsymbol{P}_{\boldsymbol{C}^\dagger} = (\boldsymbol{V}_{c,1}\boldsymbol{V}_{c,1}^\dagger)$ and $\boldsymbol{P}^\perp_{\boldsymbol{C}^\dagger} = (\boldsymbol{V}_{c,2}\boldsymbol{V}_{c,2}^\dagger)$. Given these definitions, we denote (as in Part I): (i) $\boldsymbol{Z}_c \triangleq (\boldsymbol{Z}\boldsymbol{V}_{c,1}) \in \mathbb{C}^{N\times M}$, (ii) $\boldsymbol{Z}_{c,\perp} \triangleq (\boldsymbol{Z}\,\boldsymbol{V}_{c,2}) \in \mathbb{C}^{N\times(K-M)}$ and (iii) $\boldsymbol{S}_c \triangleq (\boldsymbol{Z}_{c,\perp}\boldsymbol{Z}_{c,\perp}^\dagger) = (\boldsymbol{Z}\boldsymbol{P}^\perp_{\boldsymbol{C}^\dagger}\boldsymbol{Z}^\dagger) \in \mathbb{C}^{N\times N}$.

It has been shown in our companion paper that the MIS is given by:

$$\boldsymbol{T}(\boldsymbol{Z}_c, \boldsymbol{S}_c) = \begin{cases} \begin{bmatrix} \boldsymbol{T}_a \triangleq \left\{ \boldsymbol{Z}_{2.3}^\dagger\, \boldsymbol{S}_{2.3}^{-1}\, \boldsymbol{Z}_{2.3} \right\} \\ \boldsymbol{T}_b \triangleq \left\{ \boldsymbol{Z}_3^\dagger\, \boldsymbol{S}_{33}^{-1} \boldsymbol{Z}_3 \right\} \end{bmatrix} & J < N \\ \boldsymbol{Z}_2^\dagger\, \boldsymbol{S}_{22}^{-1} \boldsymbol{Z}_2 & J = N \end{cases}, \tag{4}$$

where $\boldsymbol{Z}_{2.3} \triangleq (\boldsymbol{Z}_2 - \boldsymbol{S}_{23}\boldsymbol{S}_{33}^{-1}\boldsymbol{Z}_3)$ and $\boldsymbol{S}_{2.3} \triangleq (\boldsymbol{S}_{22} - \boldsymbol{S}_{23}\,\boldsymbol{S}_{33}^{-1}\boldsymbol{S}_{32})$. Also, we have exploited the following partitioning for matrices $\boldsymbol{Z}_c$ and $\boldsymbol{S}_c$:

$$\boldsymbol{Z}_c = \begin{bmatrix} \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \\ \boldsymbol{Z}_3 \end{bmatrix}, \quad \boldsymbol{S}_c = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{S}_{13} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{31} & \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix}, \tag{5}$$

where $\boldsymbol{Z}_1 \in \mathbb{C}^{t\times M}$, $\boldsymbol{Z}_2 \in \mathbb{C}^{r\times M}$, and $\boldsymbol{Z}_3 \in \mathbb{C}^{(N-J)\times M}$, respectively. Furthermore, $\boldsymbol{S}_{ij}$, $(i,j) \in \{1,2,3\}\times\{1,2,3\}$, is a sub-matrix whose dimensions can be obtained replacing 1, 2 and 3 with t, r and $(N-J)$, respectively[3]. Additionally, for notational convenience, we also give the following definitions that will be used throughout the manuscript:

$$\boldsymbol{Z}_{23} \triangleq \begin{bmatrix} \boldsymbol{Z}_2 \\ \boldsymbol{Z}_3 \end{bmatrix}, \quad \boldsymbol{S}_2 \triangleq \begin{bmatrix} \boldsymbol{S}_{22} & \boldsymbol{S}_{23} \\ \boldsymbol{S}_{32} & \boldsymbol{S}_{33} \end{bmatrix}. \tag{6}$$

Finally we recall that, for the I-GMANOVA model, the induced maximal invariant equals $\boldsymbol{T}_{\mathrm{p}} \triangleq \boldsymbol{B}^\dagger\,\boldsymbol{R}_{2.3}^{-1}\,\boldsymbol{B} \in \mathbb{C}^{M\times M}$, where $\boldsymbol{R}_{2.3}$ is analogously defined as $\boldsymbol{S}_{2.3}$ when $\boldsymbol{S}_c$ is replaced with the true covariance $\boldsymbol{R}$.

[3]Hereinafter, in the case $J = N$, the "3-components" are no longer present in the partitioning.

III. DETECTORS DESIGN

In this section we will consider several decision statistics designed according to well-founded design criteria. Initially, we will concentrate on the derivation of the well-known GLRT (including its two-step version), Rao and Wald tests [9]. Then, we will devise the explicit form of recently used detection statistics, such as the Gradient (Terrell) test [17], the Durbin (naive) test [16], which have been shown to be asymptotically distributed as the three aforementioned detectors (under very mild conditions). Finally, for the sake of completeness, we will obtain the LH test for the problem at hand, following the lead of [15].

A. GLR

The generic form of the GLR in terms of the complex-valued unknowns is given by [9]:

$$\frac{\max_{\{\boldsymbol{B}_s,\boldsymbol{R}\}} f_1(\boldsymbol{Z};\, \boldsymbol{B}_s, \boldsymbol{R})}{\max_{\{\boldsymbol{B}_{t,0},\boldsymbol{R}\}} f_0(\boldsymbol{Z};\, \boldsymbol{B}_{t,0}, \boldsymbol{R})}. \tag{7}$$

First, it can be readily shown that the ML estimate of $\boldsymbol{R}$ under $\mathcal{H}_1$ (resp. under $\mathcal{H}_0$), parametrized by $\boldsymbol{B}_s$ (resp. $\boldsymbol{B}_{t,0}$) is:

$$\hat{\boldsymbol{R}}_1(\boldsymbol{B}_s) \triangleq K^{-1}\,(\boldsymbol{Z}-\boldsymbol{A}\,\boldsymbol{B}_s\boldsymbol{C})(\boldsymbol{Z}-\boldsymbol{A}\,\boldsymbol{B}_s\boldsymbol{C})^{\dagger}, \tag{8}$$

$$\hat{\boldsymbol{R}}_0(\boldsymbol{B}_{t,0}) \triangleq K^{-1}\,(\boldsymbol{Z}-\boldsymbol{E}_t\,\boldsymbol{B}_{t,0}\boldsymbol{C})(\boldsymbol{Z}-\boldsymbol{E}_t\,\boldsymbol{B}_{t,0}\boldsymbol{C})^{\dagger}. \tag{9}$$

After substitution of Eqs. (8) and (9) in $f_1(\cdot)$ and $f_0(\cdot)$, respectively, the concentrated likelihoods are expressed as:

$$\begin{aligned} f_1(\boldsymbol{Z};\boldsymbol{B}_s,\hat{\boldsymbol{R}}_1(\boldsymbol{B}_s)) &= (K/(\pi e))^{KN} \\ &\times \det\left[(\boldsymbol{Z}-\boldsymbol{A}\,\boldsymbol{B}_s\boldsymbol{C})(\boldsymbol{Z}-\boldsymbol{A}\,\boldsymbol{B}_s\boldsymbol{C})^{\dagger}\right]^{-K}, \end{aligned} \tag{10}$$

$$\begin{aligned} f_0(\boldsymbol{Z};\boldsymbol{B}_{t,0},\hat{\boldsymbol{R}}_0(\boldsymbol{B}_{t,0})) &= (K/(\pi e))^{KN} \\ &\times \det\left[(\boldsymbol{Z}-\boldsymbol{E}_t\,\boldsymbol{B}_{t,0}\boldsymbol{C})(\boldsymbol{Z}-\boldsymbol{E}_t\,\boldsymbol{B}_{t,0}\boldsymbol{C})^{\dagger}\right]^{-K}. \end{aligned} \tag{11}$$

Then the ML estimates of $\boldsymbol{B}_s$ and $\boldsymbol{B}_{t,0}$ under $\mathcal{H}_1$ and $\mathcal{H}_0$, respectively, are obtained as the solutions to the following optimization problems:

$$\widehat{\boldsymbol{B}}_s \triangleq \arg\min_{\boldsymbol{B}_s} \det[(\boldsymbol{Z}-\boldsymbol{A}\,\boldsymbol{B}_s\boldsymbol{C})(\boldsymbol{Z}-\boldsymbol{A}\,\boldsymbol{B}_s\boldsymbol{C})^{\dagger}]\,; \tag{12}$$

$$\widehat{\boldsymbol{B}}_{t,0} \triangleq \arg\min_{\boldsymbol{B}_{t,0}} \det[(\boldsymbol{Z}-\boldsymbol{E}_t\boldsymbol{B}_{t,0}\boldsymbol{C})(\boldsymbol{Z}-\boldsymbol{E}_t\boldsymbol{B}_{t,0}\boldsymbol{C})^{\dagger}]\,. \tag{13}$$

It has been shown in [15] that the optimizers have the closed form:

$$\widehat{\boldsymbol{B}}_s = (\boldsymbol{A}^{\dagger}\,\boldsymbol{S}_c^{-1}\,\boldsymbol{A})^{-1}\boldsymbol{A}^{\dagger}\,\boldsymbol{S}_c^{-1}\,\boldsymbol{Z}\,\boldsymbol{C}^{\dagger}(\boldsymbol{C}\boldsymbol{C}^{\dagger})^{-1}; \tag{14}$$

$$\widehat{\boldsymbol{B}}_{t,0} = (\boldsymbol{E}_t^{\dagger}\,\boldsymbol{S}_c^{-1}\,\boldsymbol{E}_t)^{-1}\boldsymbol{E}_t^{\dagger}\,\boldsymbol{S}_c^{-1}\boldsymbol{Z}\,\boldsymbol{C}^{\dagger}(\boldsymbol{C}\boldsymbol{C}^{\dagger})^{-1}. \tag{15}$$

Substituting Eqs. (14) and (15) into (10) and (11), respectively, provides (after lengthy manipulations):

$$f_1(\boldsymbol{Z};\, \widehat{\boldsymbol{B}}_s, \hat{\boldsymbol{R}}_1) = (K/(\pi e))^{KN}\, \det[\boldsymbol{S}_c]^{-K}$$

$$\times \det\left[\boldsymbol{I}_M + (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^\dagger\, \boldsymbol{P}_{\boldsymbol{A}_1}^{\perp}\, (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})\right]^{-K}, \tag{16}$$

$$f_0(\boldsymbol{Z};\, \widehat{\boldsymbol{B}}_{t,0}, \hat{\boldsymbol{R}}_0) = (K/(\pi e))^{KN}\, \det[\boldsymbol{S}_c]^{-K}$$

$$\times \det\left[\boldsymbol{I}_M + (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^\dagger\, \boldsymbol{P}_{\boldsymbol{A}_0}^{\perp}\, (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})\right]^{-K}, \tag{17}$$

where we have defined $\boldsymbol{A}_1 \triangleq (\boldsymbol{S}_c^{-1/2}\boldsymbol{A})$, $\boldsymbol{A}_0 \triangleq (\boldsymbol{S}_c^{-1/2}\boldsymbol{E}_t)$ and $\boldsymbol{Z}_{W1} \triangleq (\boldsymbol{S}_c^{-1/2}\boldsymbol{Z})$, respectively. Finally, substituting Eqs. (16) and (17) into Eq. (7) and after taking the k-th root, the following explicit statistic is obtained:

$$t_{\text{glr}} \triangleq \frac{\det[\boldsymbol{I}_M + (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^\dagger\, \boldsymbol{P}_{\boldsymbol{A}_0}^{\perp}(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})]}{\det[\boldsymbol{I}_M + (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^\dagger \boldsymbol{P}_{\boldsymbol{A}_1}^{\perp}(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})]} \tag{18}$$

$$= \frac{\det[\boldsymbol{I}_K + \boldsymbol{Z}_{W1}^\dagger\, \boldsymbol{P}_{\boldsymbol{A}_0}^{\perp} \boldsymbol{Z}_{W1}\boldsymbol{P}_{\boldsymbol{C}^\dagger}]}{\det[\boldsymbol{I}_K + \boldsymbol{Z}_{W1}^\dagger\, \boldsymbol{P}_{\boldsymbol{A}_1}^{\perp}\, \boldsymbol{Z}_{W1}\boldsymbol{P}_{\boldsymbol{C}^\dagger}]}, \tag{19}$$

where the last expression follows from Sylvester's determinant theorem [26]. Furthermore, we observe that Eq. (18) can be also re-arranged in the following useful equivalent forms (again obtained via Sylvester's determinant theorem):

$$t_{\text{glr}} = \det[\boldsymbol{I}_M - \boldsymbol{D}_0^{-1/2}(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^\dagger \boldsymbol{\mathcal{P}}_\Delta (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})\boldsymbol{D}_0^{-1/2}]^{-1} \tag{20}$$

$$= \det[\boldsymbol{I}_M + \boldsymbol{D}_1^{-1/2}(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^\dagger \boldsymbol{\mathcal{P}}_\Delta (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})\boldsymbol{D}_1^{-1/2}], \tag{21}$$

where $\boldsymbol{\mathcal{P}}_\Delta \triangleq (\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})$ and $\boldsymbol{D}_i \triangleq [\boldsymbol{I}_M + (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^\dagger\, \boldsymbol{P}_{\boldsymbol{A}_i}^{\perp}\, (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})]$, respectively. Finally, it is worth noticing that Eq. (21) is in the well-known *Wilks' Lambda statistic* form [27]. Moreover, the latter expression generalizes the GLR in [15] to the interference scenario (i.e., $t \neq 0$).

For the sake of completeness, we also report the closed form ML estimates of $\boldsymbol{R}$ obtained under $\mathcal{H}_0$ and $\mathcal{H}_1$ (after back-substitution of (14) and (15) in Eqs. (8) and (9), respectively):

$$\hat{\boldsymbol{R}}_1(\widehat{\boldsymbol{B}}_s) = K^{-1}[\boldsymbol{S}_c + (\boldsymbol{Z} - \boldsymbol{S}_c^{1/2}\, \boldsymbol{P}_{\boldsymbol{A}_1}\, \boldsymbol{Z}_{W1})\boldsymbol{P}_{\boldsymbol{C}^\dagger}$$

$$\times (\boldsymbol{Z} - \boldsymbol{S}_c^{1/2}\, \boldsymbol{P}_{\boldsymbol{A}_1}\, \boldsymbol{Z}_{W1})^\dagger], \tag{22}$$

$$\hat{\boldsymbol{R}}_0(\widehat{\boldsymbol{B}}_{t,0}) = K^{-1}[\boldsymbol{S}_c + (\boldsymbol{Z} - \boldsymbol{S}_c^{1/2}\, \boldsymbol{P}_{\boldsymbol{A}_0}\, \boldsymbol{Z}_{W1})\boldsymbol{P}_{\boldsymbol{C}^\dagger}$$

$$\times (\boldsymbol{Z} - \boldsymbol{S}_c^{1/2}\, \boldsymbol{P}_{\boldsymbol{A}_0}\, \boldsymbol{Z}_{W1})^\dagger], \tag{23}$$

and underline that we will use the short-hand notation $\hat{\boldsymbol{R}}_i$ in what follows. Finally, before proceeding further, we state some useful properties of ML covariance estimates (later exploited in this paper) in the form of the following lemma.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TSP.2016.2519005, IEEE Transactions on Signal Processing

Lemma 1. *The ML estimates of $\boldsymbol{R}$ under $\mathcal{H}_1$ and $\mathcal{H}_0$ satisfy the following equalities:*

$$\hat{\boldsymbol{R}}_1^{-1}\boldsymbol{A} = K\,\boldsymbol{S}_c^{-1}\boldsymbol{A} = K\left[\boldsymbol{S}_c^{-1}\boldsymbol{E}_t \quad \boldsymbol{S}_c^{-1}\boldsymbol{E}_r\right], \tag{24}$$

$$\hat{\boldsymbol{R}}_0^{-1}\boldsymbol{E}_t = K\,\boldsymbol{S}_c^{-1}\boldsymbol{E}_t. \tag{25}$$

Proof: Provided as supplementary material. ■

CFARness of GLRT: Using the expression in Eq. (18), we here verify that t_{glr} can be expressed in terms of the MIS (cf. Eq. (4)), thus proving its CFARness. Indeed, it can be shown that[4]

$$(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^{\dagger}\boldsymbol{P}_{\boldsymbol{A}_0}^{\perp}(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1}) = \boldsymbol{Z}_{2.3}^{\dagger}\,\boldsymbol{S}_{2.3}^{-1}\boldsymbol{Z}_{2.3} + \boldsymbol{Z}_3^{\dagger}\,\boldsymbol{S}_{33}^{-1}\boldsymbol{Z}_3, \tag{26}$$

$$(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^{\dagger}\boldsymbol{P}_{\boldsymbol{A}_1}^{\perp}(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1}) = \boldsymbol{Z}_3^{\dagger}\,\boldsymbol{S}_{33}^{-1}\,\boldsymbol{Z}_3, \tag{27}$$

from which it follows

$$t_{\mathrm{glr}} = \frac{\det[\boldsymbol{I}_M + \boldsymbol{T}_a + \boldsymbol{T}_b]}{\det[\boldsymbol{I}_M + \boldsymbol{T}_b]}, \tag{28}$$

which demonstrates invariance of the GLR(T) with respect to the nuisance parameters and thus ensures CFAR property.

B. Rao statistic

The generic form for the Rao statistic is given by [9]:

$$\left.\frac{\partial \ln f_1(\boldsymbol{Z};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_r^T}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_0}[\boldsymbol{I}^{-1}(\widehat{\boldsymbol{\theta}}_0)]_{\boldsymbol{\theta}_r,\boldsymbol{\theta}_r}\left.\frac{\partial \ln f_1(\boldsymbol{Z};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_r}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}_0}, \tag{29}$$

where

$$\boldsymbol{I}(\boldsymbol{\theta}) \triangleq \mathbb{E}\left\{\frac{\partial \ln f_1(\boldsymbol{Z};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\frac{\partial \ln f_1(\boldsymbol{Z};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T}\right\}, \tag{30}$$

denotes the Fisher Information Matrix (FIM) and $[\boldsymbol{I}^{-1}(\boldsymbol{\theta})]_{\boldsymbol{\theta}_r,\boldsymbol{\theta}_r}$ indicates the sub-matrix obtained by selecting from the FIM inverse only the elements corresponding to the vector $\boldsymbol{\theta}_r$. It is shown (the proof is provided as supplementary material) that the aforementioned statistic is given in closed form as:

$$t_{\mathrm{rao}} \triangleq \mathrm{Tr}\left[\boldsymbol{Z}_{d,0}^{\dagger}\,\widehat{\boldsymbol{R}}_0^{-1}\,\boldsymbol{E}_r\,\widehat{\boldsymbol{\Gamma}}_{22}^{\circ}\,\boldsymbol{E}_r^{\dagger}\,\widehat{\boldsymbol{R}}_0^{-1}\,\boldsymbol{Z}_{d,0}\,\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right], \tag{31}$$

where we have partitioned $\widehat{\boldsymbol{\Gamma}}^{\circ} \triangleq (\boldsymbol{A}^{\dagger}\,\widehat{\boldsymbol{R}}_0^{-1}\,\boldsymbol{A})^{-1}$ as:

$$\widehat{\boldsymbol{\Gamma}}^{\circ} = \begin{bmatrix}\widehat{\boldsymbol{\Gamma}}_{11}^{\circ} & \widehat{\boldsymbol{\Gamma}}_{12}^{\circ}\\ \widehat{\boldsymbol{\Gamma}}_{21}^{\circ} & \widehat{\boldsymbol{\Gamma}}_{22}^{\circ}\end{bmatrix}. \tag{32}$$

[4]The proof of the aforementioned equalities is non-trivial and thus provided as supplementary material.

and, $\widehat{\mathbf{\Gamma}}^{\circ}_{ij}$, $(i,j) \in \{1,2\} \times \{1,2\}$, is a sub-matrix whose dimensions can be obtained replacing 1 and 2 with t and r, respectively. Additionally, we have defined:

$$\boldsymbol{Z}_{d,0} \triangleq \left(\boldsymbol{Z} - \boldsymbol{S}_c^{1/2}\, \boldsymbol{P}_{\boldsymbol{A}_0}\, \boldsymbol{S}_c^{-1/2}\, \boldsymbol{Z}\, \boldsymbol{P}_{\boldsymbol{C}^\dagger}\right) . \tag{33}$$

Eq. (31) can be rewritten in a more familiar (and convenient) way. Indeed, it is proved[5] that:

$$\widehat{\boldsymbol{R}}_0^{-1/2} \boldsymbol{Z}_{d,0}\, \boldsymbol{P}_{\boldsymbol{C}^\dagger} = \boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_0} \boldsymbol{Z}_{w0}\, \boldsymbol{P}_{\boldsymbol{C}^\dagger}, \tag{34}$$

where $\bar{\boldsymbol{A}}_0 \triangleq (\widehat{\boldsymbol{R}}_0^{-1/2} \boldsymbol{E}_t)$ and $\boldsymbol{Z}_{w0} \triangleq (\widehat{\boldsymbol{R}}_0^{-1/2} \boldsymbol{Z})$, respectively, and also

$$\boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_0}\, \widehat{\boldsymbol{R}}_0^{-1/2}\, \boldsymbol{E}_r\, \mathbf{\Gamma}^{\circ}_{22}\, \boldsymbol{E}_r^\dagger\, \widehat{\boldsymbol{R}}_0^{-1/2}\, \boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_0} = \left(\boldsymbol{P}_{\bar{\boldsymbol{A}}_1} - \boldsymbol{P}_{\bar{\boldsymbol{A}}_0}\right), \tag{35}$$

where $\bar{\boldsymbol{A}}_1 \triangleq \widehat{\boldsymbol{R}}_0^{-1/2} \boldsymbol{A}$. Therefore, an alternative form of t_{rao} is obtained substituting Eq. (34) into Eq. (31) and exploiting (35), thus leading to the compact expression:

$$t_{\text{rao}} = \operatorname{Tr}\left[\boldsymbol{Z}_{w0}^\dagger\, (\boldsymbol{P}_{\bar{\boldsymbol{A}}_1} - \boldsymbol{P}_{\bar{\boldsymbol{A}}_0})\, \boldsymbol{Z}_{w0}\, \boldsymbol{P}_{\boldsymbol{C}^\dagger}\right] . \tag{36}$$

CFARness of Rao Test: We now express the Rao statistic as a function of the MIS, aiming at showing its CFARness. To this end, we first notice that Eq. (36) can be rewritten as:

$$t_{\text{rao}} = \operatorname{Tr}\left[(\boldsymbol{Z}_{w0}\boldsymbol{V}_{c,1})^\dagger (\boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_0} - \boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_1})\, (\boldsymbol{Z}_{w0}\boldsymbol{V}_{c,1})\right] . \tag{37}$$

Moreover, exploiting the equalities[6]

$$\begin{aligned}(\boldsymbol{Z}_{w0}\boldsymbol{V}_{c,1})^\dagger\, \boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_0} (\boldsymbol{Z}_{w0}\boldsymbol{V}_{c,1}) = K\, \Big\{ &\boldsymbol{Z}_{23}^\dagger\, \boldsymbol{S}_2^{-1}\, \boldsymbol{Z}_{23} \\ -\boldsymbol{Z}_{23}^\dagger\, \boldsymbol{S}_2^{-1}\, \boldsymbol{Z}_{23} (\boldsymbol{I}_M + \boldsymbol{Z}_{23}^\dagger\, \boldsymbol{S}_2^{-1}\, \boldsymbol{Z}_{23})^{-1} &\boldsymbol{Z}_{23}^\dagger\, \boldsymbol{S}_2^{-1}\, \boldsymbol{Z}_{23} \Big\},\end{aligned} \tag{38}$$

$$\begin{aligned}(\boldsymbol{Z}_{w0}\boldsymbol{V}_{c,1})^\dagger\, \boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_1}\, (\boldsymbol{Z}_{w0}\boldsymbol{V}_{c,1}) = K\, \Big\{ &\boldsymbol{Z}_3^\dagger\, \boldsymbol{S}_{33}^{-1}\, \boldsymbol{Z}_3 \\ -\boldsymbol{Z}_3^\dagger\, \boldsymbol{S}_{33}^{-1}\, \boldsymbol{Z}_3 (\boldsymbol{I}_M + \boldsymbol{Z}_3^\dagger\, \boldsymbol{S}_{33}^{-1}\, \boldsymbol{Z}_3)^{-1} &\boldsymbol{Z}_3^\dagger\, \boldsymbol{S}_{33}^{-1}\, \boldsymbol{Z}_3 \Big\},\end{aligned} \tag{39}$$

Eq. (36) can be rewritten as:

$$\begin{aligned} t_{\text{rao}} = K \operatorname{Tr}[&\boldsymbol{T}_a - (\boldsymbol{T}_a + \boldsymbol{T}_b) \\ &\times (\boldsymbol{I}_M + \boldsymbol{T}_a + \boldsymbol{T}_b)^{-1} (\boldsymbol{T}_a + \boldsymbol{T}_b) \\ &+ \boldsymbol{T}_b (\boldsymbol{I}_M + \boldsymbol{T}_b)^{-1} \boldsymbol{T}_b], \end{aligned} \tag{40}$$

which is only function of the MIS, thus proving its CFARness.

[5]The proof is provided as supplementary material.

[6]Their proof is provided as supplementary material for this manuscript.

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C. Wald statistic

The generic form for the Wald statistic is given by [9]:

$$(\hat{\boldsymbol{\theta}}_{r,1}-\boldsymbol{\theta}_{r,0})^T\{[\boldsymbol{I}^{-1}(\widehat{\boldsymbol{\theta}}_1)]_{\boldsymbol{\theta}_r,\boldsymbol{\theta}_r}\}^{-1}\,(\hat{\boldsymbol{\theta}}_{r,1}-\boldsymbol{\theta}_{r,0})\,. \tag{41}$$

It is shown (the proof is provided as supplementary material) that the aforementioned statistic is given in closed form as:

$$t_{\text{wald}} \triangleq \mathrm{Tr}\left[\boldsymbol{Z}_{W1}^{\dagger} K \boldsymbol{P}_{\boldsymbol{A}_0}^{\perp} \boldsymbol{S}_c^{-1/2} \boldsymbol{E}_r\, \widehat{\boldsymbol{\Gamma}}_{22}^{1}\, \boldsymbol{E}_r^{\dagger}\, \boldsymbol{S}_c^{-1/2} \boldsymbol{P}_{\boldsymbol{A}_0}^{\perp} \boldsymbol{Z}_{W1}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right], \tag{42}$$

where $\widehat{\boldsymbol{\Gamma}}_{ij}^{1}$ indicates the (i,j)-th sub-matrix of $\widehat{\boldsymbol{\Gamma}}^{1} \triangleq (\boldsymbol{A}^{\dagger}\,\widehat{\boldsymbol{R}}_1^{-1}\,\boldsymbol{A})^{-1}$, obtained using the same partitioning as in Eq. (32) for matrix $\widehat{\boldsymbol{\Gamma}}^{\circ}$. The above expression can be rewritten in a more compact way, as shown in what follows. Indeed, the inner matrix in Eq. (42) is rewritten as[7]:

$$K\,\boldsymbol{P}_{\boldsymbol{A}_0}^{\perp} \boldsymbol{S}_c^{-1/2}\, \boldsymbol{E}_r\, \widehat{\boldsymbol{\Gamma}}_{22}^{1}\, \boldsymbol{E}_r^{\dagger} \boldsymbol{S}_c^{-1/2} \boldsymbol{P}_{\boldsymbol{A}_0}^{\perp} = \boldsymbol{\mathcal{P}}_{\Delta} \tag{43}$$

which then gives:

$$t_{\text{wald}} = \mathrm{Tr}\left[\boldsymbol{Z}_{W1}^{\dagger}\, \boldsymbol{\mathcal{P}}_{\Delta}\, \boldsymbol{Z}_{W1}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]. \tag{44}$$

CFARness of Wald Test: Finally we prove CFARness of Wald statistic. First, it is apparent that Eq. (44) can be rewritten as:

$$t_{\text{wald}} = \mathrm{Tr}\left[(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^{\dagger}(\boldsymbol{P}_{\boldsymbol{A}_0}^{\perp}-\boldsymbol{P}_{\boldsymbol{A}_1}^{\perp})\,\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1}\right]. \tag{45}$$

Secondly, exploiting Eqs. (26) and (27) (as in the GLR case), Eq. (45) is rewritten as:

$$t_{\text{wald}} = \mathrm{Tr}\left[\boldsymbol{Z}_{2.3}^{\dagger}\,\boldsymbol{S}_{2.3}^{-1}\,\boldsymbol{Z}_{2.3}\right] = \mathrm{Tr}[\,\boldsymbol{T}_a\,]\,. \tag{46}$$

Therefore t_{wald} depends on the data matrix uniquely through the MIS (actually, only through the first component).

D. Gradient statistic

The Gradient (Terrell) test requires the evaluation of the following statistic [17], [28]:

$$\left.\frac{\partial \ln f_1(\boldsymbol{Z};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_r^T}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_0} (\hat{\boldsymbol{\theta}}_{r,1}-\boldsymbol{\theta}_{r,0}). \tag{47}$$

The appeal of Eq. (47) arises from the fact that it does not require neither to invert the FIM nor to evaluate a compressed likelihood function under both hypotheses (as opposed to GLR, Wald, and Rao

[7] The proof is provided as supplementary material.

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statistics). As a consequence, this formal simplicity can make the Gradient statistic easy to compute. Moreover, under some mild technical conditions, such test is asymptotically equivalent to the GLR, Rao and Wald statistics [17].

It is shown (the proof is provided as supplementary material) that the Gradient statistic is given in closed form as:

$$t_{\text{grad}} \triangleq \Re\left\{\text{Tr}\left[\boldsymbol{Z}_{w1}^{\dagger} K \boldsymbol{P}_{\boldsymbol{A}_0}^{\perp} \boldsymbol{S}_c^{-1/2} \boldsymbol{E}_r \,\widehat{\boldsymbol{\Gamma}}_{22}^{1}\, \boldsymbol{E}_r^{\dagger} \widehat{\boldsymbol{R}}_0^{-1}\, \boldsymbol{Z}_{d,0}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\}, \tag{48}$$

where $\boldsymbol{Z}_{d,0}$ is given in Eq. (33). The expression in Eq. (48) can be cast in a more compact form, as shown below. First, we notice that the following equality holds[8]:

$$\widehat{\boldsymbol{R}}_0^{-1}\, \boldsymbol{Z}_{d,0}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} = \boldsymbol{S}_c^{-1/2} \boldsymbol{P}_{\boldsymbol{A}_0}^{\perp} (\boldsymbol{S}_c^{1/2}\, \widehat{\boldsymbol{R}}_0^{-1})\, \boldsymbol{Z}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \tag{49}$$

which, after substitution in Eq. (48) and exploitation of Eq. (43), gives the final form

$$t_{\text{grad}} = \Re\{\text{Tr}[\boldsymbol{Z}_{w1}^{\dagger}\, \boldsymbol{\mathcal{P}}_{\Delta}\, (\boldsymbol{S}_c^{1/2}\, \widehat{\boldsymbol{R}}_0^{-1/2})\, \boldsymbol{Z}_{w0}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}]\}, \tag{50}$$

where identical steps as for Wald test have been exploited.

CFARness of Gradient Test: First, Eq. (50) can be readily rewritten as:

$$\begin{aligned} t_{\text{grad}} = \Re\Big\{\text{Tr}\Big[&(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^{\dagger}\, (\boldsymbol{P}_{\boldsymbol{A}_0}^{\perp} - \boldsymbol{P}_{\boldsymbol{A}_1}^{\perp}) \\ &\times (\boldsymbol{S}_c^{1/2}\, \widehat{\boldsymbol{R}}_0^{-1/2})\, \boldsymbol{Z}_{w0} \boldsymbol{V}_{c,1}\Big]\Big\}\,. \end{aligned} \tag{51}$$

Moreover, exploiting the following equalities[9]

$$\begin{aligned} &\left((\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^{\dagger}\, \boldsymbol{P}_{\boldsymbol{A}_0}^{\perp}\, \boldsymbol{S}_c^{1/2}\, \widehat{\boldsymbol{R}}_0^{-1/2} \boldsymbol{Z}_{w0} \boldsymbol{V}_{c,1}\right) \\ &= K\,(\boldsymbol{T}_a + \boldsymbol{T}_b)[\boldsymbol{I}_M - (\boldsymbol{I}_M + \boldsymbol{T}_a + \boldsymbol{T}_b)^{-1}(\boldsymbol{T}_a + \boldsymbol{T}_b)], \end{aligned} \tag{52}$$

$$\begin{aligned} &\left((\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^{\dagger}\, \boldsymbol{P}_{\boldsymbol{A}_1}^{\perp}\, \boldsymbol{S}_c^{1/2}\, \widehat{\boldsymbol{R}}_0^{-1/2} \boldsymbol{Z}_{w0} \boldsymbol{V}_{c,1}\right) \\ &= K\,\boldsymbol{T}_b[\boldsymbol{I}_M - (\boldsymbol{I}_M + \boldsymbol{T}_a + \boldsymbol{T}_b)^{-1}(\boldsymbol{T}_a + \boldsymbol{T}_b)], \end{aligned} \tag{53}$$

it thus follows that:

$$\begin{aligned} t_{\text{grad}} = \Re\big\{\text{Tr}\big[&K\,\boldsymbol{T}_a \left(\boldsymbol{I}_M - (\boldsymbol{I}_M + \boldsymbol{T}_a + \boldsymbol{T}_b)^{-1}\right. \\ &\left.\times (\boldsymbol{T}_a + \boldsymbol{T}_b)\right)\big]\big\}\,, \end{aligned} \tag{54}$$

which shows that also the Gradient test satisfies the CFAR property.

[8] The proof is deferred to supplementary material.

[9] The proof is provided as supplementary material.

E. Durbin statistic

The Durbin test (also referred to as "Naive test") consists in the evaluation of the following decision statistic [16]:

$$(\hat{\boldsymbol{\theta}}_{r,01} - \boldsymbol{\theta}_{r,0})^T \left\{ \left[\boldsymbol{I}\left(\widehat{\boldsymbol{\theta}}_0\right)\right]_{\boldsymbol{\theta}_r,\boldsymbol{\theta}_r} \left[\boldsymbol{I}^{-1}\left(\widehat{\boldsymbol{\theta}}_0\right)\right]_{\boldsymbol{\theta}_r,\boldsymbol{\theta}_r} \times \right.$$
$$\left. \left[\boldsymbol{I}\left(\widehat{\boldsymbol{\theta}}_0\right)\right]_{\boldsymbol{\theta}_r,\boldsymbol{\theta}_r} \right\} (\hat{\boldsymbol{\theta}}_{r,01} - \boldsymbol{\theta}_{r,0})\,, \tag{55}$$

where the estimate $\hat{\boldsymbol{\theta}}_{r,01}$ is defined as:

$$\hat{\boldsymbol{\theta}}_{r,01} \triangleq \arg\max_{\boldsymbol{\theta}_r} f_1(\boldsymbol{Z}; \boldsymbol{\theta}_r, \hat{\boldsymbol{\theta}}_{s,0})\,. \tag{56}$$

In general, the Durbin statistic is asymptotically equivalent to GLR, Rao and Wald statistics, as shown in [16]. However, for the considered problem, a stronger result holds with respect to the Rao statistic, as stated by the following theorem.

Theorem 2. *The Durbin statistic for the hypothesis testing model considered in Eq. (1) is statistically equivalent to the Rao statistic. Therefore, the test is also CFAR.*

Proof: Provided as supplementary material. ■

It is worth noticing that the present result generalizes the statistical equivalence observed between Rao and Durbin statistics for the simpler scenario of point-like targets and single-steering assumption in [19]. On the other hand, Thm. 2 proves that such result holds for the (very general) hypothesis testing problem considered in this work.

F. Two-step GLR (2S-GLR)

It is also worth considering a two-step GLR (2S-GLR), which first consists in evaluating the GLR statistic under the assumption that $\boldsymbol{R}$ is known and then plugging-in a reasonable estimate of $\boldsymbol{R}$. The GLR statistic for known $\boldsymbol{R}$ can be expressed in implicit form as [15]:

$$\frac{\max_{\boldsymbol{B}_s} f_1(\boldsymbol{Z}; \boldsymbol{B}_s, \boldsymbol{R})}{\max_{\boldsymbol{B}_{t,0}} f_0(\boldsymbol{Z}; \boldsymbol{B}_{t,0}, \boldsymbol{R})}\,. \tag{57}$$

The ML estimates of $\boldsymbol{B}_s$ and $\boldsymbol{B}_{t,0}$ are more easily obtained from optimizing the logarithm of $f_1(\cdot)$ and $f_0(\cdot)$, respectively, that is:

$$\begin{aligned} &- K\,\ln(\pi^N \det[\boldsymbol{R}]) \\ &- \mathrm{Tr}[\boldsymbol{R}^{-1}(\boldsymbol{Z} - \boldsymbol{A}\boldsymbol{B}_s\boldsymbol{C})(\boldsymbol{Z} - \boldsymbol{A}\boldsymbol{B}_s\boldsymbol{C})^\dagger], \end{aligned} \tag{58}$$

and

$$
\begin{aligned}
&- K\,\ln(\pi^N \det[\boldsymbol{R}]) \\
&- \mathrm{Tr}[\boldsymbol{R}^{-1}(\boldsymbol{Z} - \boldsymbol{E}_t\,\boldsymbol{B}_{t,0}\,\boldsymbol{C})(\boldsymbol{Z} - \boldsymbol{E}_t\,\boldsymbol{B}_{t,0}\,\boldsymbol{C})^\dagger].
\end{aligned} \tag{59}
$$

Maximization of Eqs. (58) and (59) with respect to $\boldsymbol{B}_s$ and $\boldsymbol{B}_{t,0}$, respectively, can be obtained following the same steps employed in [15] and thus it is omitted for brevity. Therefore, after optimization, the following statistic is obtained (as the logarithm of Eq. (57)):

$$
\mathrm{Tr}[\boldsymbol{Z}^\dagger\,\boldsymbol{R}^{-1/2}\,(\boldsymbol{P}_{\breve{\boldsymbol{A}}_1} - \boldsymbol{P}_{\breve{\boldsymbol{A}}_0})\,\boldsymbol{R}^{-1/2}\,\boldsymbol{Z}\,\boldsymbol{P}_{\boldsymbol{C}^\dagger}], \tag{60}
$$

where we have defined $\breve{\boldsymbol{A}}_1 \triangleq (\boldsymbol{R}^{-1/2}\boldsymbol{A})$ and $\breve{\boldsymbol{A}}_0 \triangleq (\boldsymbol{R}^{-1/2}\boldsymbol{E}_t)$, respectively. We recall that the expression in Eq. (60) depends on $\boldsymbol{R}$. We now turn our attention on finding an estimate for the covariance $\boldsymbol{R}$. Clearly, in order to obtain a meaningful estimate to be be plugged in both the numerator and denominator of Eq. (57), such estimate should be based only on signal-free data (also commonly denoted as "secondary data").

It is not difficult to show that the covariance estimate based only on secondary data is given by[10]:

$$
\widehat{\boldsymbol{R}}_{sd} = (K - M)^{-1}\,\boldsymbol{S}_c\,. \tag{61}
$$

Thus, substitution of Eq. (61) into Eq. (60) leads to the final form of 2S-GLR:

$$
\begin{aligned}
&\mathrm{Tr}[\boldsymbol{Z}^\dagger\,\sqrt{K-M}\boldsymbol{S}_c^{-1/2}\,\boldsymbol{\mathcal{P}}_\Delta\,\boldsymbol{S}_c^{-1/2}\,\sqrt{K-M}\,\boldsymbol{Z}\,\boldsymbol{P}_{\boldsymbol{C}^\dagger}] \propto \\
&t_{\mathrm{2s-glr}} \triangleq \mathrm{Tr}\left[\boldsymbol{Z}^\dagger\,\boldsymbol{S}_c^{-1/2}\,\boldsymbol{\mathcal{P}}_\Delta\,\boldsymbol{S}_c^{-1/2}\,\boldsymbol{Z}\,\boldsymbol{P}_{\boldsymbol{C}^\dagger}\right].
\end{aligned} \tag{62}
$$

From direct comparison of Eqs. (44) and (62), a general equivalence result is obtained, stated in the form of the following lemma.

Lemma 3. *The 2S-GLR statistic is statistically equivalent to the Wald statistic. Therefore, the test is also CFAR.*

The aforementioned lemma extends the statistical equivalence observed between 2S-GLR and Wald statistics in the simpler cases of point-like targets [11], range-spread targets [24] and multidimensional signals [23].

[10]It should be noted that the same result would be obtained by considering $\widehat{\boldsymbol{R}}_1^{-1}$ (i.e., the ML estimate under $\mathcal{H}_1$) as the signal-free covariance estimate. Indeed, since Eq. (60) depends only on $\boldsymbol{R}$ through the quantities $(\boldsymbol{R}^{-1}\boldsymbol{A})$ and $(\boldsymbol{R}^{-1}\boldsymbol{E}_t)$ an thus Lem. 1 could be exploited to obtain the same final statistic.

G. Lawley-Hotelling (LH) statistic

Finally, for the sake of a complete comparison, we also consider (and generalize) the simpler statistic proposed in [15, pag. 37] as a reasonable approximation to GLR. Indeed, Wilks' Lambda form of GLR in Eq. (21) can be rewritten as:

$$\begin{aligned} t_{\mathrm{glr}} = \det[\boldsymbol{I}_M + \boldsymbol{D}_1^{-1/2} \{ \widehat{\boldsymbol{B}}_s^\dagger (\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{A}) \, \widehat{\boldsymbol{B}}_s \\ - \widehat{\boldsymbol{B}}_{t,0}^\dagger (\boldsymbol{E}_t^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{E}_t) \, \widehat{\boldsymbol{B}}_{t,0} \} \, \boldsymbol{D}_1^{-1/2}], \end{aligned} \tag{63}$$

where we exploited $(\boldsymbol{C}\boldsymbol{C}^\dagger)^{-1} = (\boldsymbol{C}\boldsymbol{C}^\dagger)^{-1/2}$ and closed-form estimates for $\widehat{\boldsymbol{B}}_s$ and $\widehat{\boldsymbol{B}}_{t,0}^\dagger$ (cf. Eqs. (14) and (15), respectively). As the number of samples K grows large, we can invoke approximation $\boldsymbol{S}_c \approx (K-M)\boldsymbol{R}$, that is, the sample covariance based on secondary data will accurately approximate the true covariance matrix. Accordingly, we can safely approximate

$$\begin{aligned} \boldsymbol{D}_1 &\approx \boldsymbol{I}_M \,, \\ \widehat{\boldsymbol{B}}_s &\approx (\boldsymbol{A}^\dagger \boldsymbol{R}^{-1} \boldsymbol{A})^{-1} \, \boldsymbol{A}^\dagger \boldsymbol{R}^{-1} \boldsymbol{C}^\dagger (\boldsymbol{C}\boldsymbol{C}^\dagger)^{-1} \,, \\ \widehat{\boldsymbol{B}}_{t,0} &\approx (\boldsymbol{E}_t^\dagger \boldsymbol{R}^{-1} \boldsymbol{E}_t)^{-1} \, \boldsymbol{E}^\dagger \boldsymbol{R}^{-1} \boldsymbol{C}^\dagger (\boldsymbol{C}\boldsymbol{C}^\dagger)^{-1} \,, \end{aligned} \tag{64}$$

since $(\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{A}) \approx (K-M)^{-1} (\boldsymbol{A}^\dagger \boldsymbol{R}^{-1} \boldsymbol{A})$ and $(\boldsymbol{E}_t^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{E}_t) \approx (K-M)^{-1} (\boldsymbol{E}_t^\dagger \boldsymbol{R}^{-1} \boldsymbol{E}_t)$, respectively. Therefore, based on these approximations, it is apparent that the second contribution within the determinant in Eq. (63) will be a vanishing term as the number of observations increases. Hence, GLR statistic will be given by the determinant of an *identity matrix plus a small perturbing term*.

Additionally, we remark that when $\boldsymbol{\Upsilon} \in \mathbb{H}^{M \times M}$ is a small perturbing matrix, $\det[\boldsymbol{I}_M + \boldsymbol{\Upsilon}]$ can be (accurately) approximated at first order as $\prod_{i=1}^M (1 + \upsilon_i) \approx 1 + \sum_{i=1}^M \upsilon_i = 1 + \mathrm{Tr}[\boldsymbol{\Upsilon}]$, where υ_i denotes the i-th eigenvalue of $\boldsymbol{\Upsilon}$. Based on those reasons, we formulate the LH statistic as:

$$\begin{aligned} t_{\mathrm{lh}} &\triangleq \mathrm{Tr}\left[\boldsymbol{D}_1^{-1/2} \, (\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1})^\dagger \, \mathcal{P}_\Delta \, (\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1}) \, \boldsymbol{D}_1^{-1/2} \right] \\ &= \mathrm{Tr}\left[(\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1})^\dagger \, \mathcal{P}_\Delta \, (\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1}) \, \boldsymbol{D}_1^{-1} \right] . \end{aligned} \tag{65}$$

CFARness of LH statistic: The CFARness is proved by using Eqs. (26) and (27) from Sec. III-A within Eq. (65), thus obtaining:

$$t_{\mathrm{lh}} = \mathrm{Tr}\left[\boldsymbol{T}_a (\boldsymbol{I}_M + \boldsymbol{T}_b)^{-1} \right] . \tag{66}$$

Finally, in Tab. I it is shown a recap table, summarizing all the considered detectors and their respective expressions in terms of the MIS in Eq. (4).

Table I

DETECTORS COMPARISON AND THEIR FUNCTIONAL DEPENDENCE OF THE MIS (VIZ. CFARNESS). AUXILIARY DEFINITIONS: $\boldsymbol{T}_{a+b} \triangleq (\boldsymbol{T}_a + \boldsymbol{T}_b)$ AND $\boldsymbol{D}_i \triangleq [\boldsymbol{I}_M + (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^\dagger \, \boldsymbol{P}^\perp_{\bar{\boldsymbol{A}}_i} \, (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})]$.

Detector	*Standard Expression*	*MIS function*
GLR	$\det[\boldsymbol{D}_0]/\det[\boldsymbol{D}_1]$	$\det[\boldsymbol{I}_M + \boldsymbol{T}_{a+b}]/\det[\boldsymbol{I}_M + \boldsymbol{T}_b]$
Rao/Durbin	$\mathrm{Tr}[\boldsymbol{Z}^\dagger_{W0}\,(\boldsymbol{P}_{\bar{\boldsymbol{A}}_1} - \boldsymbol{P}_{\bar{\boldsymbol{A}}_0})\,\boldsymbol{Z}_{W0}\,\boldsymbol{P}_{\boldsymbol{C}^\dagger}]$	$K\,\mathrm{Tr}[\boldsymbol{T}_a - \boldsymbol{T}_{a+b}(\boldsymbol{I}_M + \boldsymbol{T}_{a+b})^{-1}\boldsymbol{T}_{a+b} + \boldsymbol{T}_b(\boldsymbol{I}_M + \boldsymbol{T}_b)^{-1}\boldsymbol{T}_b]$
Wald/2S-GLR	$\mathrm{Tr}[\boldsymbol{Z}^\dagger_{W1}(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,\boldsymbol{Z}_{W1}\,\boldsymbol{P}_{\boldsymbol{C}^\dagger}]$	$\mathrm{Tr}[\boldsymbol{T}_a]$
Gradient	$\Re\left\{\mathrm{Tr}[\boldsymbol{Z}^\dagger_{W1}\,(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,(\boldsymbol{S}_c^{1/2}\,\widehat{\boldsymbol{R}}_0^{-1/2})\,\boldsymbol{Z}_{W0}\,\boldsymbol{P}_{\boldsymbol{C}^\dagger}]\right\}$	$\Re\left\{\mathrm{Tr}\left[K\,\boldsymbol{T}_a\left(\boldsymbol{I}_M - (\boldsymbol{I}_M + \boldsymbol{T}_{a+b})^{-1}\boldsymbol{T}_{a+b}\right)\right]\right\}$
LH	$\mathrm{Tr}[(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^\dagger\,(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})\,\boldsymbol{D}_1^{-1}]$	$\mathrm{Tr}\left[\boldsymbol{T}_a(\boldsymbol{I}_M + \boldsymbol{T}_b)^{-1}\right]$.

IV. DETECTORS IN SPECIAL CASES

A. Adaptive (Vector Subspace) Detection of a Point-like Target

In the present case we start from general formulation in Eq. (1) and assume that: (i) $t = 0$ (i.e., there is no interference, thus $J = r$ and $\boldsymbol{A} = \begin{bmatrix}\boldsymbol{I}_r & \boldsymbol{0}_{r\times(N-r)}\end{bmatrix}^T \in \mathbb{C}^{N\times r}$); (ii) $M = 1$, i.e., the matrix $\boldsymbol{B}$ collapses to a vector $\boldsymbol{b} \in \mathbb{C}^{J\times 1}$ and (iii) $\boldsymbol{c} \triangleq \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \in \mathbb{C}^{1\times K}$ (i.e., a row vector). Such case has been extensively dealt in adaptive detection literature [3], [5], [15], [29], [30]. The hypothesis testing in canonical form is then:

$$\begin{cases}\mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{A}\,\boldsymbol{b}\,\boldsymbol{c} + \boldsymbol{N}\end{cases}. \tag{67}$$

Clearly, since in this case $M = 1$ holds, $(K-1)$ vector components are assumed signal-free, that is, $\boldsymbol{Z}$ admits the partitioning $\boldsymbol{Z} = \begin{bmatrix}\boldsymbol{z}_p & \boldsymbol{Z}_s\end{bmatrix} = \begin{bmatrix}\boldsymbol{z}_c & \boldsymbol{Z}_{c,\perp}\end{bmatrix}$, where $\boldsymbol{z}_p$ denotes the signal vector related to the cell under test and the columns of $\boldsymbol{Z}_s$ represent the secondary (training) data. Also, $\boldsymbol{P}_{\boldsymbol{A}_0} = \boldsymbol{0}_{N\times N}$ (resp. $\boldsymbol{P}^\perp_{\boldsymbol{A}_0} = \boldsymbol{I}_N$) holds, because of the absence of the structured interference. In the latter case, it can be shown that the simplified projector form holds:

$$\boldsymbol{P}_{\boldsymbol{C}^\dagger} = \begin{bmatrix}1 & \boldsymbol{0}^T_{K-1} \\ \boldsymbol{0}_{K-1} & \boldsymbol{0}_{(K-1)\times(K-1)}\end{bmatrix}. \tag{68}$$

Given the results in Eq. (68), it can be shown that $\boldsymbol{S}_c = \boldsymbol{Z}_s\boldsymbol{Z}_s^\dagger$ and $\hat{\boldsymbol{R}}_0 = \frac{1}{K}\boldsymbol{S}_0$, where $\boldsymbol{S}_0 \triangleq (\boldsymbol{z}_p\,\boldsymbol{z}_p^\dagger + \boldsymbol{Z}_s\boldsymbol{Z}_s^\dagger)$ hold, respectively. In some cases we will also use the Sherman-Woodbury formula [26] applied to $\boldsymbol{S}_0^{-1}$, that is:

$$\boldsymbol{S}_0^{-1} = \boldsymbol{S}_c^{-1} - \frac{\boldsymbol{S}_c^{-1}\boldsymbol{z}_p\,\boldsymbol{z}_p^\dagger\boldsymbol{S}_c^{-1}}{1 + \boldsymbol{z}_p^\dagger\,\boldsymbol{S}_c^{-1}\boldsymbol{z}_p}. \tag{69}$$

GLR: In the specific case of $M = 1$, the following form of the GLR is obtained from Eq. (20):

$$t_{\mathrm{glr}} = \frac{1}{1-\eta}, \quad \eta \triangleq \frac{\boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1/2} \boldsymbol{P}_{\boldsymbol{A}_1} \boldsymbol{S}_c^{-1/2} \boldsymbol{z}_p}{1 + \boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{z}_p}, \tag{70}$$

since we have exploited $\boldsymbol{D}_0 \to d_0 = (1 + \boldsymbol{z}_{p,1}^\dagger \boldsymbol{z}_{p,1})$ and $(\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1})^\dagger \boldsymbol{\mathcal{P}}_\Delta (\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1}) \to (\boldsymbol{z}_{p,1}^\dagger \boldsymbol{P}_{\boldsymbol{A}_1} \boldsymbol{z}_{p,1})$, where $\boldsymbol{z}_{p,1} \triangleq (\boldsymbol{S}_c^{-1/2} \boldsymbol{z}_p)$. Clearly, t_{glr} is an increasing function of η, the latter thus being an equivalent form of the statistic and coinciding with the well-known multi-rank signal model GLR described in [3], [15].

Rao/Durbin statistic: For the present scenario, Eq. (36) specializes into:

$$\begin{aligned} t_{\mathrm{rao}} &= \mathrm{Tr}[\boldsymbol{Z}_{w0}^\dagger \boldsymbol{P}_{\bar{\boldsymbol{A}}_1} \boldsymbol{Z}_{w0} \boldsymbol{P}_{\boldsymbol{C}^\dagger}] = \mathrm{Tr}[\boldsymbol{z}_p^\dagger (\hat{\boldsymbol{R}}_0^{-1/2} \, \boldsymbol{P}_{\bar{\boldsymbol{A}}_1} \, \hat{\boldsymbol{R}}_0^{-1/2}) \boldsymbol{z}_p] \\ &\propto \boldsymbol{z}_p^\dagger \boldsymbol{S}_0^{-1} \boldsymbol{A} (\boldsymbol{A}^\dagger \boldsymbol{S}_0^{-1} \boldsymbol{A})^{-1} \boldsymbol{A}^\dagger \boldsymbol{S}_0^{-1} \boldsymbol{z}_p \triangleq \eta_{\mathrm{rao}}. \end{aligned} \tag{71}$$

Eq. (71) can be further simplified by exploiting the Woodbury identity in (69) (and similar steps as in [13]), thus obtaining the following simplified form of the Rao statistic:

$$\eta_{\mathrm{rao}} = \frac{1}{1 + \boldsymbol{z}_{p,1}^\dagger \boldsymbol{z}_{p,1}} \left[\frac{\boldsymbol{z}_{p,1}^\dagger \boldsymbol{P}_{\boldsymbol{A}_1} \boldsymbol{z}_{p,1}}{1 + \boldsymbol{z}_{p,1} \boldsymbol{P}_{\boldsymbol{A}_1}^\perp \boldsymbol{z}_{p,1}} \right]. \tag{72}$$

Finally, for $r = 1$ (i.e., a single-steering case) ($\boldsymbol{A} \to \boldsymbol{a} \in \mathbb{C}^{N \times 1}$), Eq. (72) reduces to:

$$\eta_{\mathrm{rao}} = \frac{\left| \boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{a} \right|^2 / \left(\boldsymbol{a}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{a} \right)}{[1 + \boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{z}_p] \left[1 + \boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{z}_p - \frac{\left| \boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{a} \right|^2}{(\boldsymbol{a}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{a})} \right]}, \tag{73}$$

which coincides with the well-known Rao statistic for the single-steering case developed in [13].

Wald/2S-GLR statistic: Starting from Eq. (44), we particularize the Wald statistic as follows:

$$\begin{aligned} t_{\mathrm{wald}} &= \mathrm{Tr}[\boldsymbol{Z}_{w1}^\dagger \boldsymbol{P}_{\boldsymbol{A}_1} \boldsymbol{Z}_{w1} \boldsymbol{P}_{\boldsymbol{C}^\dagger}] = \mathrm{Tr}[\boldsymbol{z}_{p,1}^\dagger \, \boldsymbol{P}_{\boldsymbol{A}_1} \, \boldsymbol{z}_{p,1}] \\ &= \boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{A} (\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{A})^{-1} \boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{z}_p. \end{aligned} \tag{74}$$

In the special case $r = 1$ (i.e., a single-steering case), Eq. (74) becomes:

$$t_{\mathrm{wald}} = \frac{\left| \boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{a} \right|^2}{\boldsymbol{a}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{a}}, \tag{75}$$

which is recognized as the well-known *Adaptive Matched Filter* (AMF) [11], [31].

Gradient statistic: In this case the gradient statistic in Eq. (50) specializes into:

$$\begin{aligned} t_{\mathrm{grad}} &= \Re \left\{ \mathrm{Tr} \left[\boldsymbol{Z}_{w1}^\dagger \boldsymbol{P}_{\boldsymbol{A}_1} (\boldsymbol{S}_c^{1/2} \widehat{\boldsymbol{R}}_0^{-1/2}) \boldsymbol{Z}_{w0} \boldsymbol{P}_{\boldsymbol{C}^\dagger} \right] \right\} \\ &= \Re \left\{ \boldsymbol{z}_{p,1}^\dagger \, \boldsymbol{P}_{\boldsymbol{A}_1} (\boldsymbol{S}_c^{1/2} \widehat{\boldsymbol{R}}_0^{-1/2}) \, \boldsymbol{z}_{p,0} \right\} \\ &= K \, \Re \left\{ \boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{A} (\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{A})^{-1} \boldsymbol{A}^\dagger \boldsymbol{S}_0^{-1} \boldsymbol{z}_p \right\}, \end{aligned} \tag{76}$$

where $\boldsymbol{z}_{p,0} \triangleq (\widehat{\boldsymbol{R}}_0^{-1/2} \boldsymbol{z}_p)$. It is interesting to note that, exploiting Eq. (69), the gradient statistic is rewritten as:

$$\begin{aligned} t_{\text{grad}} &= K\, \Re\left\{ \frac{\left(\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{z}_p\right)^\dagger (\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{A})^{-1} \left(\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{z}_p\right)}{1 + \boldsymbol{z}_p^\dagger\, \boldsymbol{S}_c^{-1} \boldsymbol{z}_p} \right\} \\ &= K\, \frac{\left(\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{z}_p\right)^\dagger (\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{A})^{-1} \left(\boldsymbol{A}^\dagger \boldsymbol{S}_c^{-1} \boldsymbol{z}_p\right)}{1 + \boldsymbol{z}_p^\dagger\, \boldsymbol{S}_c^{-1} \boldsymbol{z}_p}, \end{aligned} \tag{77}$$

where in last line we have omitted $\Re\{\cdot\}$ since Eq. (77) is formed by Hermitian quadratic forms (at both numerator and denominator); thus it is *always real-valued*. Therefore Eq. (77) is *statistically equivalent* to Kelly's GLR in Eq. (70).

LH statistic: We recall that for point-like targets, the condition $M = 1$ holds. Therefore the LH test is *statistically equivalent* to the GLRT since the operators $\mathrm{Tr}[\cdot]$ and $\det[\cdot]$ are non-influential when applied to a scalar value. This follows since the expressions in Eqs. (21) and (65) are thus related by a monotone transformation.

B. Adaptive Vector Subspace Detection with Structured Interference

In the present case we start from general formulation in Eq. (1) and assume that: (i) $M = 1$, i.e., the matrices $\boldsymbol{B}$ and $\boldsymbol{B}_{t,i}$ collapse to the vectors $\boldsymbol{b} \in \mathbb{C}^{r\times 1}$ and $\boldsymbol{b}_{t,i} \in \mathbb{C}^{t\times 1}$, respectively; (ii) $\boldsymbol{c} \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{1\times K}$ (i.e., a row vector). Such case has been dealt in [6]. Given the aforementioned assumptions, the problem in canonical form is given as:

$$\begin{cases} \mathcal{H}_0 : & \boldsymbol{Z} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{b}_{t,0}^T & \boldsymbol{0}_r^T \end{bmatrix}^T \boldsymbol{c} + \boldsymbol{N} \\ \mathcal{H}_1 : & \boldsymbol{Z} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{b}_{t,1}^T & \boldsymbol{b}^T \end{bmatrix}^T \boldsymbol{c} + \boldsymbol{N} \end{cases}. \tag{78}$$

Clearly, since in this case $M = 1$ holds, $(K-1)$ vector components are assumed signal-free, that is, $\boldsymbol{Z}$ admits the partitioning $\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{z}_p & \boldsymbol{Z}_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_c & \boldsymbol{Z}_{c,\perp} \end{bmatrix}$, where $\boldsymbol{z}_p$ denotes the signal vector related to the cell of interest and the columns of $\boldsymbol{Z}_s$ represent the secondary (or training) data. In the latter case, it can be shown that the same simplified projector form in Eq. (68) holds. Given the results in Eq. (68), it can be shown that $\boldsymbol{S}_c = \boldsymbol{Z}_s \boldsymbol{Z}_s^\dagger$ and $\hat{\boldsymbol{R}}_0 = \frac{1}{K} \boldsymbol{S}_0$, where $\boldsymbol{S}_0 \triangleq (\boldsymbol{S}_c + \boldsymbol{S}_c^{1/2} \boldsymbol{P}_{\boldsymbol{A}_0}^\perp \boldsymbol{S}_c^{-1/2} \boldsymbol{z}_p \boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1/2} \boldsymbol{P}_{\boldsymbol{A}_0}^\perp \boldsymbol{S}_c^{1/2})$ hold, respectively. In some cases we will also use the Sherman-Woodbury formula [26] applied to $\boldsymbol{S}_0^{-1}$ and consider the product $\boldsymbol{S}_0^{-1} \boldsymbol{z}_p$, which provides:

$$\begin{aligned} \boldsymbol{S}_0^{-1} \boldsymbol{z}_p \;=\; & \boldsymbol{S}_c^{-1} \boldsymbol{z}_p - (\boldsymbol{S}_c^{-1/2} \boldsymbol{P}_{\boldsymbol{A}_0}^\perp \boldsymbol{S}_c^{-1/2}) \boldsymbol{z}_p \\ & \times \frac{\boldsymbol{z}_p^\dagger \boldsymbol{S}_c^{-1/2} \boldsymbol{P}_{\boldsymbol{A}_0}^\perp \boldsymbol{S}_c^{-1/2} \boldsymbol{z}_p}{1 + \boldsymbol{z}_p^\dagger\, \boldsymbol{S}_c^{-1/2} \boldsymbol{P}_{\boldsymbol{A}_0}^\perp \boldsymbol{S}_c^{-1/2} \boldsymbol{z}_p}. \end{aligned} \tag{79}$$

GLR: In the specific case of $M = 1$, the following form of the GLRT is obtained from Eq. (20):

$$t_{\rm glr} = \frac{1}{1-\eta}, \quad \eta \triangleq \frac{\boldsymbol{z}_{p,1}^{\dagger}\,(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,\boldsymbol{z}_{p,1}}{1 + \boldsymbol{z}_{p,1}^{\dagger}\,\boldsymbol{P}_{\boldsymbol{A}_0}^{\perp}\,\boldsymbol{z}_{p,1}}, \tag{80}$$

since we have exploited $\boldsymbol{D}_0 \to d_0 = (1 + \boldsymbol{z}_p^{\dagger}\boldsymbol{S}_c^{-1/2}\,\boldsymbol{P}_{\boldsymbol{A}_0}^{\perp}\,\boldsymbol{S}_c^{-1/2}\boldsymbol{z}_p)$ and $(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^{\dagger}\,\boldsymbol{\mathcal{P}}_{\Delta}\,(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1}) \to \boldsymbol{z}_{p,1}^{\dagger}(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,\boldsymbol{z}_{p,1}$, where $\boldsymbol{z}_{p,1} \triangleq (\boldsymbol{S}_c^{-1/2}\boldsymbol{z}_p)$. Clearly, Eq. (80) is an increasing function of η, which can be thus seen as an equivalent form of the GLR.

Rao/Durbin statistic: For the present scenario, Eq. (36) specializes into:

$$t_{\rm rao} = \boldsymbol{z}_{p,0}^{\dagger}\,(\boldsymbol{P}_{\bar{\boldsymbol{A}}_1} - \boldsymbol{P}_{\bar{\boldsymbol{A}}_0})\,\boldsymbol{z}_{p,0}, \tag{81}$$

where $\boldsymbol{z}_{p,0} \triangleq (\widehat{\boldsymbol{R}}_0^{-1/2}\boldsymbol{z}_p)$.

Wald/2S-GLR statistic: Starting from Eq. (44), we particularize the Wald statistic as follows:

$$t_{\rm wald} = \boldsymbol{z}_{p,1}^{\dagger}\,(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,\boldsymbol{z}_{p,1}. \tag{82}$$

Gradient statistic: In this case the gradient statistic in Eq. (50) specializes into:

$$t_{\rm grad} = \Re\left\{\boldsymbol{z}_{p,1}^{\dagger}\,(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,(\boldsymbol{S}_c^{1/2}\widehat{\boldsymbol{R}}_0^{-1/2})\,\boldsymbol{z}_{p,0}\right\}. \tag{83}$$

We now rewrite Eq. (83) as:

$$t_{\rm grad} = K\,\Re\left\{\boldsymbol{z}_p^{\dagger}\,\boldsymbol{S}_c^{-1/2}\,(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,(\boldsymbol{S}_c^{1/2}\boldsymbol{S}_0^{-1})\,\boldsymbol{z}_p\right\}. \tag{84}$$

Exploiting Eq. (79) and observing that $(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\boldsymbol{P}_{\boldsymbol{A}_0}^{\perp} = \boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0}$ holds, Eq. (84) is expressed as:

$$\begin{aligned} t_{\rm grad} &= K\,\Re\left\{\frac{\boldsymbol{z}_p^{\dagger}\,\boldsymbol{S}_c^{-1/2}\,(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,\boldsymbol{S}_c^{-1/2}\,\boldsymbol{z}_p}{1 + \boldsymbol{z}_p^{\dagger}\,\boldsymbol{S}_c^{-1/2}\,\boldsymbol{P}_{\boldsymbol{A}_0}^{\perp}\,\boldsymbol{S}_c^{-1/2}\,\boldsymbol{z}_p}\right\} \\ &= K\,\frac{\boldsymbol{z}_p^{\dagger}\,\boldsymbol{S}_c^{-1/2}\,(\boldsymbol{P}_{\boldsymbol{A}_1} - \boldsymbol{P}_{\boldsymbol{A}_0})\,\boldsymbol{S}_c^{-1/2}\,\boldsymbol{z}_p}{1 + \boldsymbol{z}_p^{\dagger}\,\boldsymbol{S}_c^{-1/2}\,\boldsymbol{P}_{\boldsymbol{A}_0}^{\perp}\,\boldsymbol{S}_c^{-1/2}\,\boldsymbol{z}_p}, \end{aligned} \tag{85}$$

where in last line we have omitted $\Re\{\cdot\}$ since Eq. (85) is formed by Hermitian quadratic forms (at both numerator and denominator); thus it is *always real-valued*. Therefore Eq. (85) is *statistically equivalent* to GLR in Eq. (80).

LH statistic: As in the case of no-interference in Sec. IV-A, the condition $M = 1$ holds. Therefore the LH statistic is *statistically equivalent* to the GLR.

C. Multidimensional Signals

In the present case we start from formulation in Eq. (1) and assume that: (i) $t=0$ (i.e. there is no interference, meaning $J=r$), (ii) $\boldsymbol{A}=\boldsymbol{E}_r=\boldsymbol{I}_N$ (thus $J=r=N$) and (iii) $\boldsymbol{C}\triangleq\begin{bmatrix}\boldsymbol{I}_M & \boldsymbol{0}_{M\times(K-M)}\end{bmatrix}$. Such case has been dealt in [10], [23]. Thus, the hypothesis testing in canonical form is given by:

$$\begin{cases}\mathcal{H}_0: & \boldsymbol{Z}=\boldsymbol{N}\\ \mathcal{H}_1: & \boldsymbol{Z}=\boldsymbol{B}\,\boldsymbol{C}+\boldsymbol{N}\end{cases}. \tag{86}$$

Clearly, since in this case $J=N$ holds, $(K-M)$ vector components are assumed signal-free, that is, $\boldsymbol{Z}$ admits the partitioning $\boldsymbol{Z}=\begin{bmatrix}\boldsymbol{Z}_M & \boldsymbol{Z}_s\end{bmatrix}=\begin{bmatrix}\boldsymbol{Z}_c & \boldsymbol{Z}_{c,\perp}\end{bmatrix}$, where $\boldsymbol{Z}_M$ denotes the signal matrix collecting the cells containing the useful signals and the columns of $\boldsymbol{Z}_s$ are the training data. In the latter case, it can be shown that the simplified projector form holds:

$$\boldsymbol{P}_{\boldsymbol{C}^\dagger}=\begin{bmatrix}\boldsymbol{I}_M & \boldsymbol{0}_{M\times(K-M)}\\ \boldsymbol{0}_{(K-M)\times M} & \boldsymbol{0}_{(K-M)\times(K-M)}\end{bmatrix}. \tag{87}$$

Given the results in Eq. (68), it can be shown that $\boldsymbol{S}_c=\boldsymbol{Z}_s\boldsymbol{Z}_s^\dagger$ and $\hat{\boldsymbol{R}}_0=\frac{1}{K}\boldsymbol{S}_0$, where $\boldsymbol{S}_0\triangleq(\boldsymbol{Z}_M\boldsymbol{Z}_M^\dagger+\boldsymbol{Z}_s\boldsymbol{Z}_s^\dagger)$ holds, respectively. Also, it is not difficult to show that $\boldsymbol{P}_{\boldsymbol{A}_1}=\boldsymbol{P}_{\bar{\boldsymbol{A}}_1}=\boldsymbol{I}_N$ and $\boldsymbol{P}_{\boldsymbol{A}_1}^\perp=\boldsymbol{P}_{\bar{\boldsymbol{A}}_1}^\perp=\boldsymbol{0}_{N\times N}$, respectively.

GLR: In order to specialize GLR expression we start from Eq. (18). Indeed, it can be easily shown that:

$$\begin{aligned}t_{\mathrm{glr}}&=\frac{\det[\boldsymbol{I}_M+(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^\dagger(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})]}{\det[\boldsymbol{I}_M+(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^\dagger\boldsymbol{P}_{\bar{\boldsymbol{A}}_1}^\perp(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})]}\\ &=\det[\boldsymbol{I}_M+\boldsymbol{Z}_M^\dagger\boldsymbol{S}_c^{-1}\boldsymbol{Z}_M]\\ &=\det[\boldsymbol{I}_M+\boldsymbol{S}_c^{-1/2}\boldsymbol{Z}_M\,\boldsymbol{Z}_M^\dagger\,\boldsymbol{S}_c^{-1/2}]\\ &=\det[\boldsymbol{S}_c+\boldsymbol{Z}_M\boldsymbol{Z}_M^\dagger]\,/\,\det[\boldsymbol{S}_c],\end{aligned} \tag{88}$$

where we have exploited $\boldsymbol{P}_{\bar{\boldsymbol{A}}_1}^\perp=\boldsymbol{0}_{N\times N}$ and Sylvester's determinant theorem in third and fourth lines, respectively. It is apparent that the latter expressions coincide with those in [10, Eqs. (18) and (20)], respectively.

Rao/Durbin statistic: For the present setup Eq. (36) specializes into:

$$\begin{aligned}t_{\mathrm{rao}}&=\mathrm{Tr}[\boldsymbol{Z}_{w0}^\dagger\boldsymbol{P}_{\bar{\boldsymbol{A}}_1}\boldsymbol{Z}_{w0}\boldsymbol{P}_{\boldsymbol{C}^\dagger}]=\mathrm{Tr}[\boldsymbol{Z}^\dagger\widehat{\boldsymbol{R}}_0^{-1}\boldsymbol{Z}\,\boldsymbol{P}_{\boldsymbol{C}^\dagger}]\\ &=K\,\mathrm{Tr}[(\boldsymbol{Z}\,\boldsymbol{P}_{\boldsymbol{C}^\dagger})^\dagger\boldsymbol{S}_0^{-1}(\boldsymbol{Z}\,\boldsymbol{P}_{\boldsymbol{C}^\dagger})]=K\,\mathrm{Tr}[\boldsymbol{Z}_M^\dagger\boldsymbol{S}_0^{-1}\boldsymbol{Z}_M],\end{aligned} \tag{89}$$

which coincides with the specific result obtained in [23], which was originally derived as a modified two-step GLRT procedure in [10].

Wald/2S-GLR statistic: Starting from Eq. (44), we particularize the Wald statistic as follows:

$$t_{\text{wald}} = \text{Tr}[\boldsymbol{Z}_{w1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_1} \boldsymbol{Z}_{w1} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}] = \text{Tr}[\boldsymbol{Z}^{\dagger} \boldsymbol{S}_c^{-1} \boldsymbol{Z}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}}]$$
$$= \text{Tr}[(\boldsymbol{Z}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}})^{\dagger} \boldsymbol{S}_c^{-1} (\boldsymbol{Z}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}})] = \text{Tr}[\boldsymbol{Z}_M^{\dagger} \boldsymbol{S}_c^{-1} \boldsymbol{Z}_M], \tag{90}$$

which coincides with the specific result obtained in [23].

Gradient statistic: In this case the gradient statistic in Eq. (50) specializes into:

$$t_{\text{grad}} = \Re \left\{ \text{Tr} \left[\boldsymbol{Z}_{w1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_1} (\boldsymbol{S}_c^{1/2} \widehat{\boldsymbol{R}}_0^{-1/2}) \boldsymbol{Z}_{w0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \right] \right\}$$
$$= K\, \Re \left\{ \text{Tr} \left[\boldsymbol{Z}^{\dagger} \boldsymbol{S}_0^{-1}\, \boldsymbol{Z}\, \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \right] \right\} = K\, \text{Tr} \left[\boldsymbol{Z}_M^{\dagger}\, \boldsymbol{S}_0^{-1}\, \boldsymbol{Z}_M \right]. \tag{91}$$

It is interesting to observe that in this specific scenario, *Gradient statistic coincides with Rao statistic* in Eq. (89).

LH statistic: In this specific instance, LH statistic in Eq. (65) specializes into:

$$t_{\text{lh}} = \text{Tr} \left[(\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1})^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_1} (\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1}) \times \right. \tag{92}$$
$$\left. \left(\boldsymbol{I}_M + (\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1})^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_1}^{\perp} (\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1}) \right)^{-1} \right] = \text{Tr} \left[\boldsymbol{Z}_M^{\dagger} \boldsymbol{S}_c^{-1} \boldsymbol{Z}_M \right],$$

since $\boldsymbol{P}_{\boldsymbol{A}_1}^{\perp} = \boldsymbol{0}_{N \times N}$ (resp. $\boldsymbol{P}_{\boldsymbol{A}_1} = \boldsymbol{I}_N$) for multidimensional signal setup. From inspection of the last line, it is apparent that *LH statistic coincides with Wald/2S-GLR statistic* in Eq. (90) for this specific scenario.

D. Range-spread Targets

In the present case we start from general formulation in Eq. (1) and assume that: (i) $t = 0$ (i.e., there is no interference, thus $J = r$); (ii) $r = 1$, thus the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ collapse to $\boldsymbol{a} \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{C}^{N \times 1}$ and $\boldsymbol{b} \in \mathbb{C}^{1 \times M}$ (i.e., a row vector), respectively; (iii) $\boldsymbol{C} \triangleq \begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M \times K-M} \end{bmatrix}$. Such case has been dealt in [4], [24], [25]. Therefore, the hypothesis testing in canonical form is given by:

$$\begin{cases} \mathcal{H}_0 : & \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1 : & \boldsymbol{Z} = \boldsymbol{a}\, \boldsymbol{b}\, \boldsymbol{C} + \boldsymbol{N} \end{cases}. \tag{93}$$

Additionally, $(K-M)$ vector components are assumed signal-free, that is, $\boldsymbol{Z}$ admits the partitioning $\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}_e & \boldsymbol{Z}_s \end{bmatrix}$ where $\boldsymbol{Z}_e \in \mathbb{C}^{N \times M}$ comprises the cells containing the extended target and $\boldsymbol{Z}_s \in \mathbb{C}^{N \times (K-M)}$ collects the secondary (training) data. In the latter case, the following simplified projector form holds:

$$\boldsymbol{P}_{\boldsymbol{C}^{\dagger}} = \begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M \times (K-M)} \\ \boldsymbol{0}_{(K-M) \times M} & \boldsymbol{0}_{(K-M) \times M} \end{bmatrix}. \tag{94}$$

Based on the structure of Eq. (94), it follows that $\boldsymbol{S}_c = \boldsymbol{Z}_s \boldsymbol{Z}_s^\dagger$ and $\widehat{\boldsymbol{R}}_0 = \frac{1}{K}\boldsymbol{S}_0$, where $\boldsymbol{S}_0 \triangleq (\boldsymbol{Z}_e \boldsymbol{Z}_e + \boldsymbol{Z}_s \boldsymbol{Z}_s^\dagger)$. Moreover, it can be shown that $\boldsymbol{P}_{\boldsymbol{a}_1}$ and $\boldsymbol{P}_{\bar{\boldsymbol{a}}_1}$ (where we have analogously defined $\boldsymbol{a}_1 \triangleq (\boldsymbol{S}_c^{-1/2}\boldsymbol{a})$ and $\bar{\boldsymbol{a}}_1 \triangleq (\widehat{\boldsymbol{R}}_0^{-1/2}\boldsymbol{a})$) assumes the following simplified expression:

$$\boldsymbol{P}_{\boldsymbol{a}_1} = \frac{\boldsymbol{S}_c^{-1/2}\boldsymbol{a}\boldsymbol{a}^\dagger \boldsymbol{S}_c^{-1/2}}{\boldsymbol{a}^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{a}}; \qquad \boldsymbol{P}_{\bar{\boldsymbol{a}}_1} = \frac{\boldsymbol{S}_0^{-1/2}\boldsymbol{a}\boldsymbol{a}^\dagger \boldsymbol{S}_0^{-1/2}}{\boldsymbol{a}^\dagger \boldsymbol{S}_0^{-1}\boldsymbol{a}}. \tag{95}$$

In some cases, we will use the Woodbury identity applied to $\boldsymbol{S}_0^{-1}$, that is:

$$\boldsymbol{S}_0^{-1} = \boldsymbol{S}_c^{-1} - \boldsymbol{S}_c^{-1}\boldsymbol{Z}_e\,(\boldsymbol{I}_M + \boldsymbol{Z}_e^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{Z}_e)^{-1}\,\boldsymbol{Z}_e^\dagger\,\boldsymbol{S}_c^{-1}. \tag{96}$$

GLR: Aiming at particularizing the expression of the GLR for the present case, we follow the same derivation as in [15] to show that Eq. (20) can be specialized exploiting the equalities

$$(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^\dagger \boldsymbol{P}_{\boldsymbol{a}_1}(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1}) = \frac{(\boldsymbol{Z}_e^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{a})(\boldsymbol{Z}_e^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{a})^\dagger}{(\boldsymbol{a}^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{a})}, \tag{97}$$

$$\boldsymbol{D}_0 = \boldsymbol{I}_M + \boldsymbol{Z}_e^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{Z}_e, \tag{98}$$

thus obtaining $t_{\text{glr}} = [1/(1-\eta^{'})]$, where:

$$\eta^{'} \triangleq \frac{(\boldsymbol{a}^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{Z}_e)\,[\boldsymbol{I}_M + \boldsymbol{Z}_e^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{Z}_e]^{-1}\,(\boldsymbol{Z}_e^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{a})}{(\boldsymbol{a}^\dagger \boldsymbol{S}_c^{-1}\boldsymbol{a})}. \tag{99}$$

The result in Eq. (99) is obtained after substitution of Eqs. (97) and (98) into Eq. (20) and exploiting Sylvester's determinant theorem. Such GLR form[11] (as t_{glr} is a monotone function of $\eta^{'}$) corresponds to that found in [15].

Rao/Durbin statistic: In the present case Eq. (36) specializes into:

$$\begin{aligned} t_{\text{rao}} &= \text{Tr}[\boldsymbol{Z}_{w0}^\dagger \boldsymbol{P}_{\bar{\boldsymbol{a}}_1}\boldsymbol{Z}_{w0}\boldsymbol{P}_{\boldsymbol{C}^\dagger}] \\ &= \text{Tr}[(\boldsymbol{Z}\boldsymbol{P}_{\boldsymbol{C}^\dagger})^\dagger \widehat{\boldsymbol{R}}_0^{-1/2}\boldsymbol{P}_{\bar{\boldsymbol{a}}_1}\widehat{\boldsymbol{R}}_0^{-1/2}(\boldsymbol{Z}\boldsymbol{P}_{\boldsymbol{C}^\dagger})] \\ &= \text{Tr}[\boldsymbol{Z}_e^\dagger\,\widehat{\boldsymbol{R}}_0^{-1/2}\boldsymbol{P}_{\bar{\boldsymbol{a}}_1}\widehat{\boldsymbol{R}}_0^{-1/2}\boldsymbol{Z}_e] \\ &= \frac{K\,\text{Tr}[\boldsymbol{Z}_e^\dagger\,\boldsymbol{S}_0^{-1}\boldsymbol{a}\boldsymbol{a}^\dagger \boldsymbol{S}_0^{-1}\boldsymbol{Z}_e]}{(\boldsymbol{a}^\dagger \boldsymbol{S}_0^{-1}\boldsymbol{a})} = K\frac{\left\|\boldsymbol{Z}_e^\dagger \boldsymbol{S}_0^{-1}\boldsymbol{a}\right\|^2}{(\boldsymbol{a}^\dagger \boldsymbol{S}_0^{-1}\boldsymbol{a})}. \end{aligned} \tag{100}$$

Eq. (100) is recognized as the result found in [24].

[11]It is worth pointing out that an alternative (equivalent) form of GLR was obtained in [4], [32] for the range-spread case. The aforementioned expression can be simply obtained starting from general formula in Eq. (18), straightforward application of Sylvester's determinant theorem and exploitation of the simplified assumptions of range-spread scenario.

Wald/2S-GLR statistic: Starting from Eq. (44), we particularize the test as follows:

$$\begin{aligned} t_{\text{wald}} &= \text{Tr}[\boldsymbol{Z}_{w1}^{\dagger}\boldsymbol{P}_{\boldsymbol{a}_1}\boldsymbol{Z}_{w1}\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}] \\ &= \text{Tr}[(\boldsymbol{Z}\boldsymbol{P}_{\boldsymbol{C}^{\dagger}})^{\dagger}\boldsymbol{S}_c^{-1/2}\boldsymbol{P}_{\boldsymbol{a}_1}\boldsymbol{S}_c^{-1/2}(\boldsymbol{Z}\boldsymbol{P}_{\boldsymbol{C}^{\dagger}})] \\ &= \text{Tr}[\boldsymbol{Z}_e^{\dagger}\,\boldsymbol{S}_c^{-1/2}\boldsymbol{P}_{\boldsymbol{a}_1}\boldsymbol{S}_c^{-1/2}\,\boldsymbol{Z}_e] \\ &= \frac{\text{Tr}[\boldsymbol{Z}_e^{\dagger}\,\boldsymbol{S}_c^{-1}\boldsymbol{a}\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{Z}_e]}{(\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})} = \frac{\left\|\boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a}\right\|^2}{(\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})}, \end{aligned} \tag{101}$$

which agrees with the result in [24] and can be shown to coincide with the generalized AMF proposed in [4], thus extending the theoretical findings in [11].

Gradient statistic: In this case Eq. (50) reduces to:

$$\begin{aligned} t_{\text{grad}} &= \Re\left\{\text{Tr}\left[\boldsymbol{Z}_{w1}^{\dagger}\boldsymbol{P}_{\boldsymbol{a}_1}(\boldsymbol{S}_c^{1/2}\widehat{\boldsymbol{R}}_0^{-1/2})\boldsymbol{Z}_{w0}\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}\right]\right\} \\ &= K\frac{\Re\left\{\text{Tr}\left[\boldsymbol{Z}_e^{\dagger}\,\boldsymbol{S}_c^{-1}\,\boldsymbol{a}\boldsymbol{a}^{\dagger}\,\boldsymbol{S}_0^{-1}\,\boldsymbol{Z}_e\right]\right\}}{(\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})} \\ &= K\frac{\Re\left\{\left[\boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a}\right]^{\dagger}\left[\boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_0^{-1}\boldsymbol{a}\right]\right\}}{(\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})}, \end{aligned} \tag{102}$$

where we have used $\Re\{\text{Tr}[\boldsymbol{m}\boldsymbol{n}^{\dagger}]\} = \Re[\boldsymbol{m}^{\dagger}\boldsymbol{n}]$, with $\boldsymbol{m}$ and $\boldsymbol{n}$ being two column vectors of proper size. Furthermore, by exploiting Eq. (96), the following equality holds

$$\left(\boldsymbol{Z}_e^{\dagger}\,\boldsymbol{S}_0^{-1}\boldsymbol{a}\right) = [\boldsymbol{I}_M + \boldsymbol{Z}_e^{\dagger}\,\boldsymbol{S}_c^{-1}\boldsymbol{Z}_e]^{-1}(\boldsymbol{Z}_e^{\dagger}\,\boldsymbol{S}_c^{-1}\boldsymbol{a}) \tag{103}$$

which, substituted into Eq. (102), gives:

$$t_{\text{grad}} = K\frac{\left[\boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a}\right]^{\dagger}[\boldsymbol{I}_M + \boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{Z}_e]^{-1}\left[\boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a}\right]}{(\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})}, \tag{104}$$

where we have omitted $\Re\{\cdot\}$ since Eq. (104) is an Hermitian quadratic form (i.e., it is always real-valued). Therefore, the *Gradient statistic is statistically equivalent to the GLR in Eq. (99).*

LH statistic: In this case the general LH statistic form in Eq. (65) specializes into:

$$\begin{aligned} t_{\text{lh}} &= \text{Tr}\left[(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^{\dagger}\boldsymbol{P}_{\boldsymbol{a}_1}(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})\boldsymbol{D}_1^{-1}\right] \\ &= \text{Tr}\left[\frac{\boldsymbol{Z}_e^{\dagger}\,\boldsymbol{S}_c^{-1}\boldsymbol{a}\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{Z}_e}{(\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})}\boldsymbol{D}_1^{-1}\right], \end{aligned} \tag{105}$$

where $\boldsymbol{D}_1 = \boldsymbol{I}_M + (\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})^{\dagger}\boldsymbol{P}_{\boldsymbol{a}_1}^{\perp}(\boldsymbol{Z}_{w1}\boldsymbol{V}_{c,1})$ in this specific case. Matrix $\boldsymbol{D}_1$ can be further rewritten as:

$$\boldsymbol{D}_1 = (\boldsymbol{I}_M + \boldsymbol{Z}_e^{\dagger}\,\boldsymbol{S}_c^{-1}\,\boldsymbol{Z}_e) - \frac{\boldsymbol{Z}_e^{\dagger}\,\boldsymbol{S}_c^{-1}\boldsymbol{a}\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{Z}_e}{(\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})}\,. \tag{106}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TSP.2016.2519005, IEEE Transactions on Signal Processing

Applying the Woodbury identity on $\boldsymbol{D}_1^{-1}$, we obtain:

$$\boldsymbol{D}_1^{-1} = \Big\{\boldsymbol{D}_0^{-1} + \frac{\boldsymbol{D}_0^{-1}\left(\boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a}\right)(\boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})^{\dagger}\boldsymbol{D}_0^{-1}}{\left(\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a}\right)\left[1 - \frac{(\boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})^{\dagger}\boldsymbol{D}_0^{-1}(\boldsymbol{Z}_e^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a})}{\boldsymbol{a}^{\dagger}\boldsymbol{S}_c^{-1}\boldsymbol{a}}\right]}\Big\}, \tag{107}$$

where we exploited the definition of $\boldsymbol{D}_0$ in Eq. (98). Thus, after substitution into Eq. (105), we obtain

$$t_{\mathrm{lh}} = \eta^{'} + \frac{(\eta^{'})^2}{1-\eta^{'}} = \frac{\eta'}{1-\eta^{'}} \propto \eta^{'} \tag{108}$$

with η' given by Eq. (99). Thus LH statistic is *statistically equivalent to the GLR* for range-spread targets.

E. Standard GMANOVA

In the present case no interference is present ($t = 0$, thus $J = r$). This reduces to the standard adaptive detection problem via the GMANOVA model considered in [15], [20], [21] and whose canonical form is:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{Z} = \boldsymbol{N} \\ \mathcal{H}_1: & \boldsymbol{Z} = \boldsymbol{A}\boldsymbol{B}\boldsymbol{C} + \boldsymbol{N} \end{cases}. \tag{109}$$

Clearly, under the above assumptions, it holds $\boldsymbol{P}_{\boldsymbol{A}_0} = \boldsymbol{P}_{\bar{\boldsymbol{A}}_0} = \boldsymbol{0}_{N\times N}$. Therefore, the ML covariance estimate under $\mathcal{H}_0$ simplifies into $\widehat{\boldsymbol{R}}_0 = K^{-1}\boldsymbol{S}_0$, where $\boldsymbol{S}_0 \triangleq \boldsymbol{Z}\boldsymbol{Z}^{\dagger}$(cf. Eq. (23)).

GLR: Direct specialization of Eq. (18) gives the explicit statistic:

$$t_{\mathrm{glr}} = \frac{\det[\boldsymbol{I}_M + (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^{\dagger}(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})]}{\det[\boldsymbol{I}_M + (\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^{\dagger}\,\boldsymbol{P}^{\perp}_{\bar{\boldsymbol{A}}_1}(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})]}, \tag{110}$$

which coincides with the classical expression[12] of GLR obtained in [15].

Rao/Durbin statistic: Direct particularization of Eq. (36) gives:

$$t_{\mathrm{rao}} = \mathrm{Tr}[\boldsymbol{Z}_{W0}^{\dagger}\boldsymbol{P}_{\bar{\boldsymbol{A}}_1}\boldsymbol{Z}_{W0}\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}], \tag{111}$$

which provides the result obtained in [21].

Wald/2S-GLR statistic: Direct specialization of Eq. (44) gives leads to:

$$t_{\mathrm{wald}} = \mathrm{Tr}[\boldsymbol{Z}_{W1}^{\dagger}\boldsymbol{P}_{\boldsymbol{A}_1}\boldsymbol{Z}_{W1}\boldsymbol{P}_{\boldsymbol{C}^{\dagger}}], \tag{112}$$

which is the same result obtained in [21].

[12]We point out that Eq. (110) can be also re-arranged in a similar form as Eq. (21) (i.e., a Wilks' Lambda statistic form). Such expression, being equal to $t_{\mathrm{glr}} = \det[\boldsymbol{I}_M + \boldsymbol{D}_1^{-1/2}\,(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})^{\dagger}\,\boldsymbol{P}_{\boldsymbol{A}_1}(\boldsymbol{Z}_{W1}\boldsymbol{V}_{c,1})\,\boldsymbol{D}_1^{-1/2}]$, represents the alternative GLR form obtained in [15].

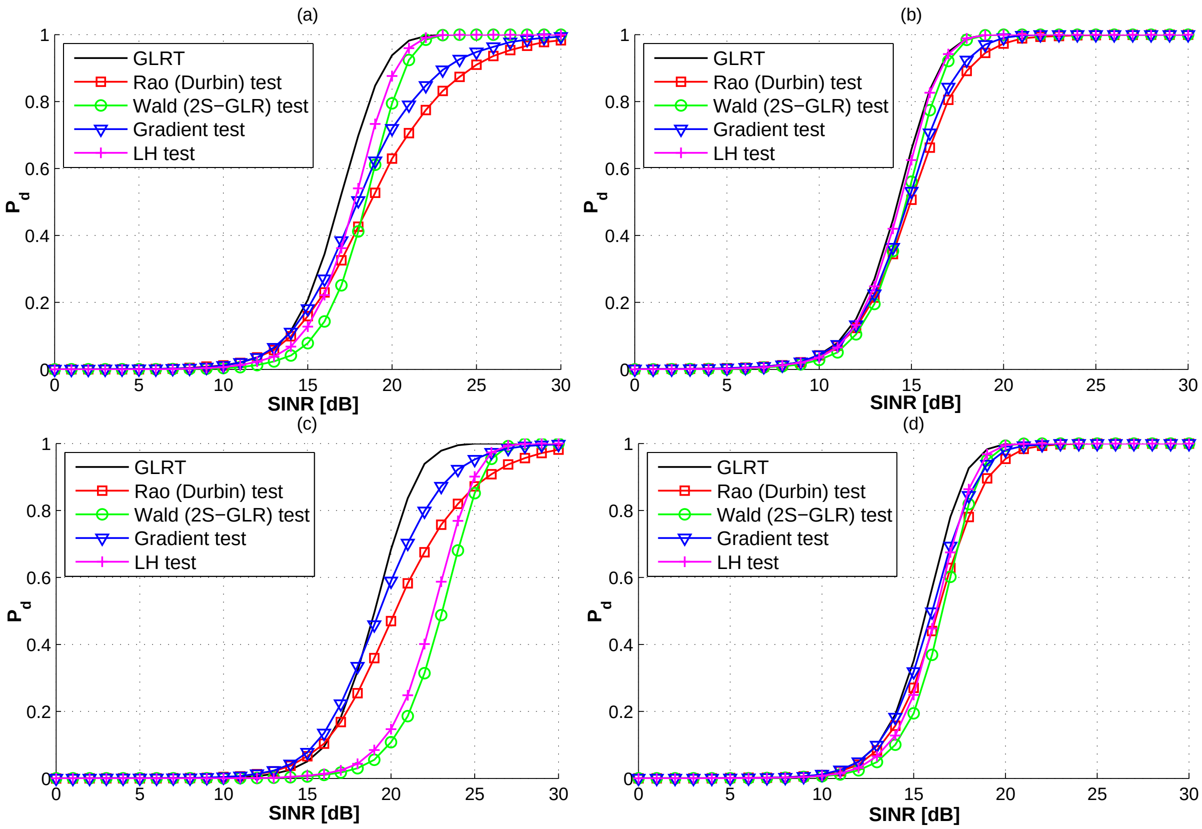

Figure 1. P_d vs. ρ for all the considered detectors; common parameters: $M = 3$ and $N = 8$. Case (a) (top-left) $r = 2$, $t = 4$ and $K = 12$; Case (b) (top-right) $r = 2$, $t = 4$ and $K = 19$; Case (c) (bottom-left) $r = 4$, $t = 2$ and $K = 12$; Case (d) (bottom-right) $r = 4$, $t = 2$ and $K = 19$.

Gradient statistic: Direct application of Eq. (50) provides:

$$t_{\text{grad}} = \Re\left\{ \text{Tr}\left[\boldsymbol{Z}_{w1}^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_1} (\boldsymbol{S}_c^{1/2} \widehat{\boldsymbol{R}}_0^{-1/2}) \boldsymbol{Z}_{w0} \boldsymbol{P}_{\boldsymbol{C}^{\dagger}} \right] \right\}. \tag{113}$$

LH statistic: In this case the LH statistic specializes into:

$$t_{\text{lh}} = \text{Tr}\left[(\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1})^{\dagger} \boldsymbol{P}_{\boldsymbol{A}_1} (\boldsymbol{Z}_{w1} \boldsymbol{V}_{c,1}) \boldsymbol{D}_1^{-1} \right]. \tag{114}$$

where $\boldsymbol{D}_1$ is defined as in Sec. III-A. Eq. (114) clearly coincides with the statistic obtained in [15, pag. 37].

V. Simulation results

In Fig. 1, we report the probability of detection P_d vs. the SINR for all the considered detectors, defined as $\rho \triangleq \text{Tr}[\boldsymbol{B}^{\dagger} \boldsymbol{R}_{2.3}^{-1} \boldsymbol{B}]$, that is, the trace of the induced maximal invariant (cf. Sec. II-A). We underline that such term is also proportional to the non-centrality parameter $\lambda \triangleq (\boldsymbol{\theta}_{r,1}-\boldsymbol{\theta}_{r,0})^T\{[\boldsymbol{I}^{-1}(\boldsymbol{\theta}_0)]_{\boldsymbol{\theta}_r,\boldsymbol{\theta}_r}\}^{-1}(\boldsymbol{\theta}_{r,1}-\boldsymbol{\theta}_{r,0})$, representing the synthetic parameter on which the asymptotic performances of all the considered test depend [9]. The curves have been obtained via standard Monte Carlo counting techniques. More specifically, the thresholds necessary to ensure a preassigned value of P_{fa} have been evaluated exploiting $100/P_{fa}$ independent trials, while the P_d values are estimated over $5 \cdot 10^3$ independent trials. As to the disturbance, it is modeled as an exponentially-correlated Gaussian vector with covariance matrix (in canonical form) $\boldsymbol{R} = \sigma_n^2 \boldsymbol{I}_N + \sigma_c^2 \boldsymbol{R}_c$, where $\sigma_n^2 > 0$ is the thermal noise power, $\sigma_c^2 > 0$ is the clutter power, and the (i,j)-th element of $\boldsymbol{R}_c$ is given by $0.95^{|i-j|}$. The clutter-to-noise ratio σ_c^2/σ_n^2 is set here to $30\,\mathrm{dB}$, with $\sigma_n^2 = 1$. We point out that the specific value of the deterministic interference $\boldsymbol{B}_t$ does not need to be specified at each trial considered (for both P_{fa} and P_d evaluation); the reason is that the performance of each detector depends on the unknown parameters solely through the induced maximal invariant, which is *independent* on $\boldsymbol{B}_t$ (cf. Sec. II-A). Finally, all the numerical examples assume $P_{fa} = 10^{-4}$.

In order to average the performance of P_d with respect to $\boldsymbol{B}$, for each independent trial we generate the signal matrix as $\boldsymbol{B} = \alpha_B \boldsymbol{B}_g$, where $\boldsymbol{B}_g \sim \mathcal{CN}(\boldsymbol{0}_{r\times M}, \boldsymbol{I}_M, \boldsymbol{I}_r)$ and $\alpha_B \in \mathbb{R}$. The latter coefficient is a scaling factor used to achieve the desired SINR value, that is, $\alpha_B \triangleq \sqrt{\rho\,/\,\mathrm{Tr}[\boldsymbol{B}_g^\dagger\, \boldsymbol{R}_{2.3}^{-1}\, \boldsymbol{B}_g]}$.

For our simulations[13] we assume $M = 3$ (i.e., an extended target), $N = 8$, and two different scenarios of signal and interference lying in a vector subspace, that is, (i) $r = 2$ and $t = 4$ (sub-plots (a) and (b)) and (ii) $r = 4$ and $t = 2$ (sub-plots (c) and (d)). Additionally, for each of these setups, the cases corresponding to $K = 12$ and $K = 19$ columns for $\boldsymbol{Z}$ have been considered, representing two extreme case-studies. Indeed, the first case clearly corresponds to a *sample-starved* scenario (i.e. the number of signal-free data required to achieve a consistent (invertible) estimate of $\boldsymbol{R}$ is just satisfied, that is, $(K - M) = 9$) while the second case to a setup where an adequate number of samples needed to obtain an accurate estimate for $\boldsymbol{R}$ is provided (i.e., in this case $(K - M) = 2N = 16$, with a consequent loss of $3\,\mathrm{dB}$ in estimating $\boldsymbol{R}$ with the sample covariance approach, with respect to the known covariance case, as dictated from [33]).

The following observations can be made from inspection of the results. First, as K grows large, all the considered detectors converge to the same performance, corresponding to the non-adaptive case.

[13]Of course, due to the high number of setup parameters involved in the detection problem (i.e., N, K, M, r, t), we do not claim the following conclusions to be general for any type of setup. Nonetheless, we illustrate a generic setup in order to show some common trends observed among the detectors. A general numerical comparison is omitted due to the lack of space and since performance comparison in some specific setups (such as those considered in Sec. IV) can be found in the related literature. Nonetheless, the supplementary material attached to this paper contains some additional numerical results aimed at confirming the statistical equivalence results obtained for the considered special scenarios.

Differently, in the sample-starved case (viz. the difference $K - M$ is close to N) a significant difference in detection performance can be observed among them. First of all, the GLRT has the best performance in the medium-high SNR range. Differently, the Rao and Gradient tests perform significantly better than Wald and LH tests for a moderate number of K (i.e., $K = 12$) in the case $r > t$ (cf. sub-plot (c), corresponding to $r = 4$ and $t = 2$). On the other hand, for the same case $K = 12$, but $r = 2$ and $t = 4$, Wald and LH tests outperform Rao and Gradient tests when ρ is higher than $\approx 18\,\mathrm{dB}$. This is easily explained since both Wald (viz. 2S-GLRT) and LH tests both rely on an accurate estimate of true covariance $\boldsymbol{R}$ based on the sole signal-free data (cf. Secs. III-C-III-F and III-G, respectively). Differently, both Gradient and Rao tests employ a covariance estimate under the hypothesis $\mathcal{H}_0$ (that is, $\widehat{\boldsymbol{R}}_0$). The latter covariance estimate also relies on the use of the additional contributions of $\boldsymbol{Z}$ corrupted by the signal $\boldsymbol{B}$. Although using them to evaluate $\widehat{\boldsymbol{R}}_0$ may be detrimental when the number of signal-free samples is adequate or the SINR is high (cf. sub-plots (a) and (b)), when the SINR is low (i.e., the energy spread among the different columns is not so high) and the number of signal-free samples is not sufficient to guarantee a reliable estimate of $\boldsymbol{R}$, the degradation of using signal-corrupted terms is overcome by the (beneficial) availability of additional samples for covariance estimation.

VI. Conclusions

In this second part of this work, we have derived several detectors for adaptive detection in a GMANOVA signal model with structured interference (viz. I-GMANOVA). We derived the GLR, Rao, Wald, 2S-GLR, Durbin, Gradient, and LH statistics. All the aforementioned statistics have been shown to be CFAR with respect to the nuisance parameters, by proving that all can be written in terms of the MIS (obtained in the first part of this work). For the considered general model, we also established statistical equivalence between: (i) Wald and 2S-GLR statistics and (ii) Durbin and Rao statistics.

Furthermore, the following statistical-equivalence results have been proved in the following special setups:

- For point-like targets (with possible point-like interference), we have shown that Gradient and LH tests are statistically equivalent to Kelly's GLRT;
- For multidimensional signals, we have shown that: (a) Rao test is statistically equivalent to Gradient test and (b) Wald test (2S-GLRT) is statistically equivalent to LH test;
- For range-spread targets and rank-one subspace ($r = 1$), we have shown that Gradient and LH tests are statistically equivalent to the GLRT.

Finally, simulation results were provided to compare the performance of the aforementioned detectors.

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