Some Special Matrices

1. **Diagonal Matrix**: A diagonal Matrix where the entries outside the main diagonal are all zero; the main diagonal is from the top left to the bottom right of the square matrix.

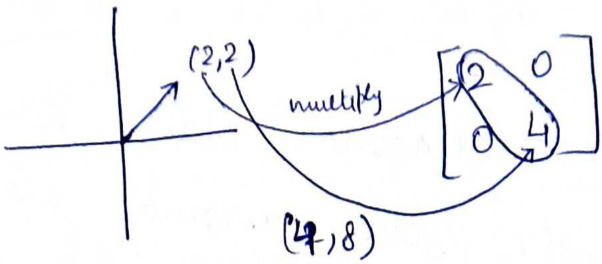
a. **Powers**: The nth power of a diagonal matrix (where n is a non-negative integer) can be obtained by raising each diagonal element to the power of n.

b. **Eigenvalues**: The eigenvalues of a diagonal matrix are just the value on the diagonal. The corresponding of the eigen vectors are the standard basis vectors.

c. **Multiplication by a vector**: When a diagonal matrix multiplies a vector, it scales each component of the vector by the corresponding element on the diagonal.

d. **Matrix Multiplication**: The product of two diagonal matrices is just the diagonal matrix with the corresponding elements on the diagonals multiplied.

c



d $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \times \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$

2. Orthogonal Matrix

An orthogonal matrix is a square matrix whose columns and rows are orthogonal unit vectors (i.e. orthonormal vectors), meaning that they are all of unit length and are at right angles to each other.

Perfect rotation, no scaling or shearing

a. Inverse Equals Transpose: The transpose of an orthogonal matrix equals its inverse i.e. A^T = A^(-1). This property makes calculation with orthogonal matrices computationally efficient. $\boxed{A^T = A^{-1}}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ → orthogonal

90°

$\boxed{ab + cd = 0}$

$\sqrt{a^2+c^2} = 1$ $\sqrt{b^2+d^2} = 1$

$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

↳ orthogonal → θ

$\theta \to 30^\circ$

$$\begin{bmatrix} \sqrt{3}/2 & \xrightarrow{\times} & 1/2 \\ -1/2 & \xrightarrow{\times} & \sqrt{3}/2 \end{bmatrix}$$

$$\frac{\sqrt{3}}{2} \times \frac{1}{2} - \frac{1}{2}\frac{\sqrt{3}}{2} = 0$$

$$\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = 1$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to$ also orthogonal.

Identity Matrix

3. Symmetric Matrix

A symmetric matrix is a type of square matrix that is equal to its own transpose. In other words, if you swap its rows with columns, you get the same matrix.

a. Real Eigenvalue: The eigenvalues of a real symmetric matrix are always, real, not complex.

b. Orthogonal Eigenvector: For a real symmetric matrix, the eigenvectors corresponding to different eigenvalues are ~~values~~ always orthogonal to each other. If one eigenvalues are

$$\boxed{A = A^T}$$

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} \xrightarrow{\text{Transpose}} \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Matrix Decomposition

Matrix Decomposition

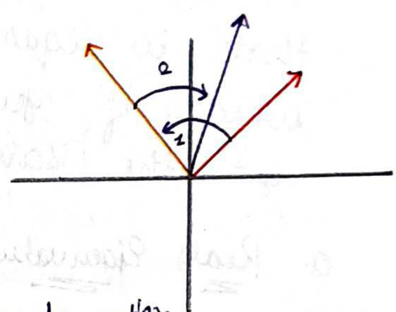
- LU Decomp
- QR decomp
- Eigen decomp
- SVD

Matrix Decomposition ⟶ before study (Matrix Composition)

Matrix Composition

$$\overset{(a)}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \overset{(b)}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$$

L ⟵ R

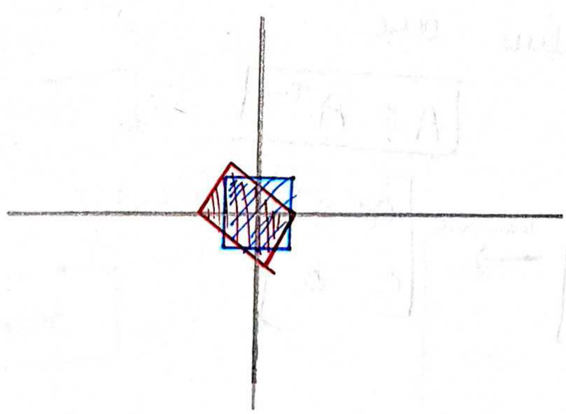


When we perform tranformation on this then result is equal to the result of putting transformation b and then a

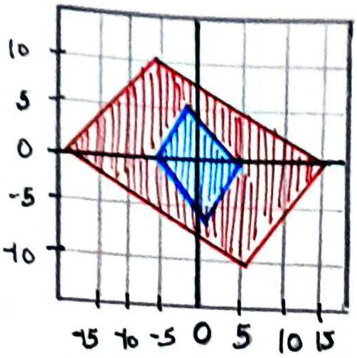
3 metrices ↓

A ⟵ B ⟵ C

L ⟵ R

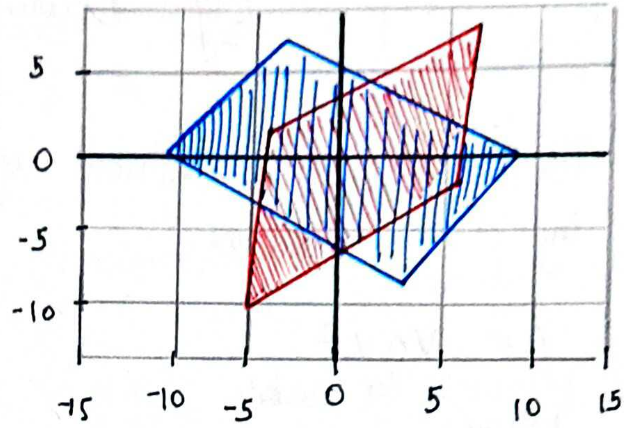


Transform #1 (C)



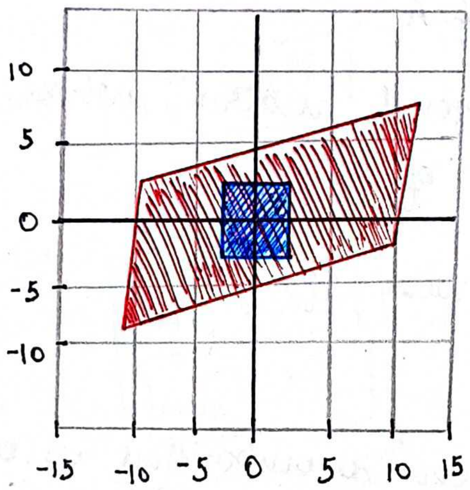
Transformation #2

C → B



Transformation # 3

B → A



Combined Transformation

$\underbrace{A\ B\ C}$ = D → Composition

combine and make D

D = A B C → Decomposition

↳ convert into single individual

Eigen Decomposition

The eigen decomposition of a matrix A is given by the equation:

$A = V \Lambda V^{-1}$

↳ lambda

where:

- V is a matrix whose columns are the eigen vectors of A
- Λ is a diagonal matrix whose entries are the eigenvalues of A
- V^{-1} is the inverse of V

Assuming

1. **Square matrix**: Eigen decomposition is only defined for square matrices

2. **Diagonalizability**: For a $n * n$ matrix it should have n linearly independent eigen vectors.

$$A = V \Lambda V^{-1}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\lambda_1 \quad \lambda_2$

Eigen values

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Eigen Vectors

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}} \underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1}}$$

decomposition

Diagonalizability → $n * n$ → n linearly independent eigen vectors

(2×2) $\quad$ (2)

different direction vectors

eigenvalue → $\boxed{A\vec{V} = \lambda\vec{V}}$

↓

for 2 Eigen Value →

$$\boxed{\begin{array}{c} \boxed{A\vec{V_1} = \lambda\vec{V_1}} \\ \boxed{A\vec{V_2} = \lambda\vec{V_2}} \end{array}}$$

Matrix form

$\boxed{AV = V\Lambda}$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\vec{V_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$V = [\vec{V_1} \ \ \vec{V_2}]$$

$$V = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

$$\left.\begin{aligned} A\vec{v}_1 &= \lambda_1\vec{v}_1 \\ A\vec{v}_2 &= \lambda_2\vec{v}_2 \end{aligned}\right] \xrightarrow{\text{Matrix form}} \boxed{AV = V\Lambda}$$

Proof

$$A\vec{v}_1 = \lambda_1\vec{v}_1 \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\underset{2\times 2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}} = \lambda_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\begin{bmatrix} ax_1 & by_1 \\ cx_1 & dy_1 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda y_1 \end{bmatrix} \rightarrow \boxed{\begin{aligned} ax_1 + by_1 &= \lambda x_1 \\ cx_1 + dy_1 &= \lambda y_1 \\ \\ ax_2 + bx_2 &= \lambda_2 x_2 \\ cx_2 + dy_2 &= \lambda_2 y_2 \end{aligned}}$$

$$A\vec{v}_2 = \lambda_2\vec{v}_2 \xrightarrow{\text{follow same steps above}}$$

$$AV = V\Lambda \rightarrow \text{matrix form}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{bmatrix}$$

$$\boxed{\begin{aligned} ax_1 + by_1 &= \lambda_1 x_1 & \quad ax_2 + by_2 &= \lambda_2 x_2 \\ cx_1 + dy_1 &= \lambda_1 y_1 & \quad cx_2 + dy_2 &= \lambda_2 y_2 \end{aligned}}$$

Hence Prooved

$$\left.\begin{aligned} Av_1 &= \lambda\vec{v}_1 \\ A\vec{v}_2 &= \lambda_2\vec{v}_2 \end{aligned}\right] \xrightarrow{\text{Matrix form}} \boxed{AV = V\Lambda}$$

$A\vec{V_1} = \lambda_1 \vec{V_1}$ $\longrightarrow$ $AV = V\Lambda$

$A\vec{V_2} = \lambda_2 \vec{V_2}$

Multiply both sid (V^{-1})

$$\boxed{A = V\Lambda V^{-1}}$$ → eigen decomposition

V → eigen vectors

Λ → diagonal eigen values

Eigen Decomposition of symmetric Matrix

$A \rightarrow \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ (symmetric)

$$\boxed{A = V\Lambda V^{-1}}$$ → Spectral decomposition

V → eigen vector

$\boxed{V \rightarrow \text{orthogonal}}$ $\boxed{\Lambda \rightarrow \text{diagonal}}$

$A = V\Lambda V^{-1}$

V → rotate, Λ → scaling, V^{-1} → Rotate

Advantage of Eigen Decomposition

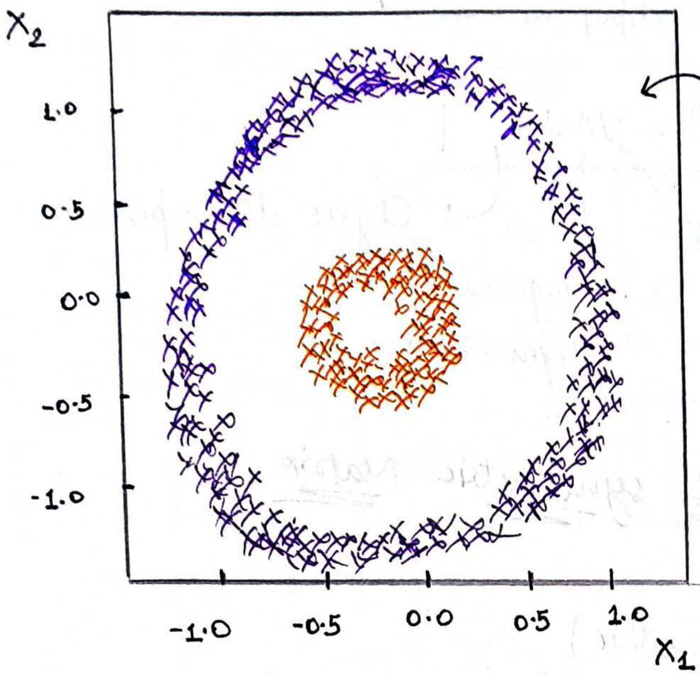
→ ML → pca

→ Physics

→ Signal theory

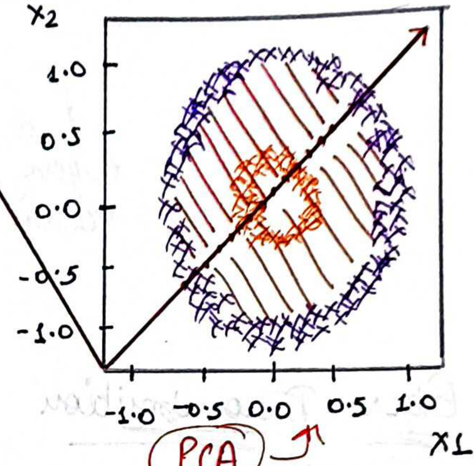
→ quantum theory

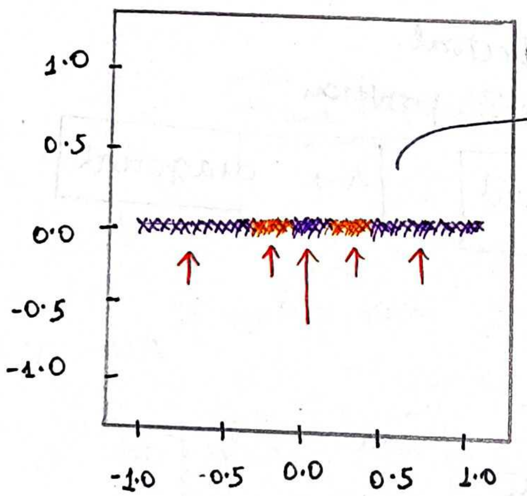
Kernel PCA



Original Data

concentric circles





Data after PCA in 1D

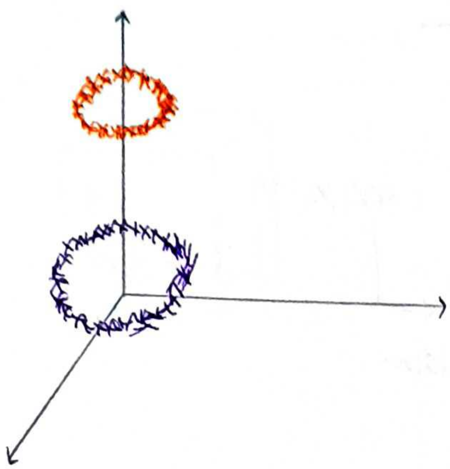
Here's not ~~dif~~ make difference betn two category in 1D and 2D data convert into 1D data.

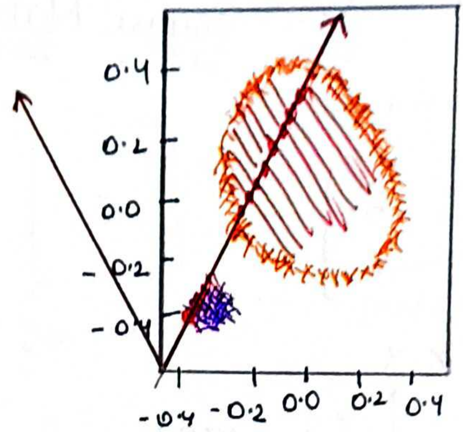
PCA fails

Kernel PCA use for this

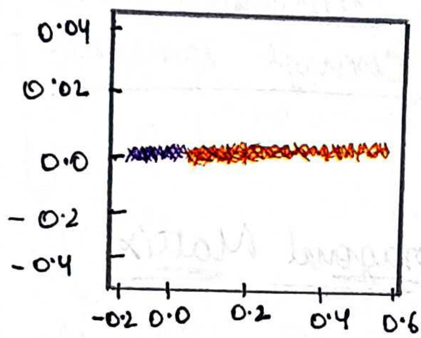
Kernel → Data → higher dim → lower dim

2D → 3D → 2D → 1D





Data after Kernel PCA in 2D

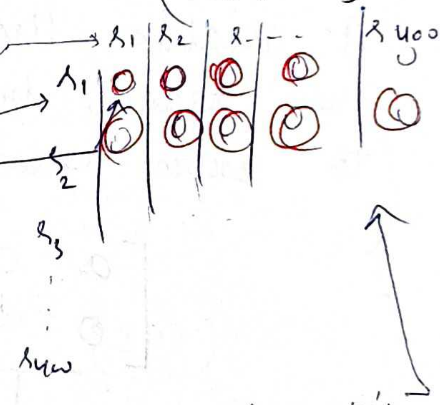


Data after Kernel PCA in 1D

Kernel = $e^{-(\text{gama} * \text{dis}^2)}$

(400,2) → Kernel

(400, 400)



* Find the euclidean distance betn
* distance square and then put in kernel

Similarity matrix

Q→ if Number ↑↑ then similarity ↑↑
if Number ↓↓ then similarity ↓↓

[400 × 400]
↳ $\boxed{\text{symmetric}}$ → eigen decomposition

400 dim matrix → 400 eigen vectors
↓
best 2 column extract

Non-Square Matrix

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

2×3

Output 2D (2), input (3D) (3)

$m \times n$

m dim, n dim

* Square Matrix not change dimension but Non Square Matrix can change Dimension.

Rectangular Diagonal Matrix

A matrix that would be diagonal if it were square, but instead is rectangular due to extra rows or columns of zeros.

$$\begin{bmatrix} a & 0 & \boxed{0} \\ 0 & b & \boxed{0} \end{bmatrix} \qquad \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

rect diag matrix → remove → square

2 Transformation

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix} \longrightarrow \begin{array}{cc} x & y \\ \end{array} \begin{bmatrix} \boxed{a} & 0 \\ 0 & \boxed{b} \end{bmatrix} \begin{array}{ccc} x & y & z \\ \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2×3

B ← A

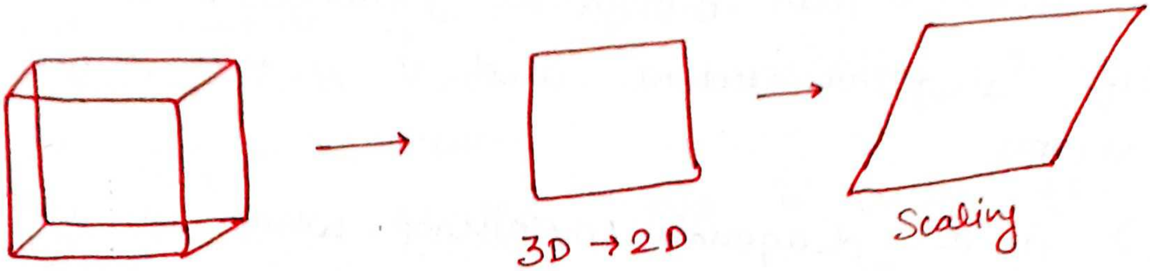
2×2 → Diagonal matrix → Scaling

$x \xrightarrow{\text{scale}} a$

$y \xrightarrow{\text{scale}} b$

2×3 → rectangle and change dim $3D \to 2D$

rectangular diagonal → Linear Transformation



2.

$$\underset{3\times 2}{\begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix}} \rightarrow \text{Linear tranformation}$$

$$\rightarrow \underbrace{\underset{}{\overset{2\times 2}{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}} \quad \overset{2\times 3}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}}$$

↳ we cannot do this

$$\overset{(B)}{\underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}} \xleftarrow{} \overset{(A)}{\underset{2\times 2}{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}}$$

↳ Scale

2D → 3D

$$(2\times 3) \xrightarrow{\text{Totaly opposite}} (3\times 2)$$

What is SVD ?

↳ Singular value decomp

SVD is matrix decomposition/ factorization method that decomposes a matrix into three other matrices. Given a matrix A, the singular value decomposition of A is usually written as:

$$A = U\Sigma V^T$$

~~where~~ $A = V\Lambda V^{-1}$
only square matrix

↳ A could be any matrix but A is definite not n time

Here:

- U and V are orthogonal matrices. U is the left singular vectors and V is the right singular vectors.
- Σ is a diagonal containing what we call the singular values.

Application of SVD

1. Machine Learning and Data Science : SVD is used in Principal component Analysis (PCA), a technique for dimensionality reduction. This is helpful when dealing with high-dimensional data. It's also used in various recommedation system, for example in collaborative filtering which is used in Netflix movie recommendation.

2. Natural Language Processing (NLP) : SVD is used in Latent Semantic Analysis (LSA), a technique for extracting the underlying meaning (semantic information) from textual data. LSA uses SVD to reduce the dimensionality of a term - document matrix, which helps identify relationships between terms and documents.

3. Computer Vison : In computer vision, SVD is used in image compression. By keeping only the largest singular values and corresponding singular vectors, we can ~~respect~~ represent an image using less data without losing too much information.

4. Signal Processing : SVD is used to separate useful signals from noise. This is useful in applications like mobile communication and audio signal processing.

5. Numerical Linear Algebra : SVD is used for matrix inversion and solving system of linear equations. It is often a numerically stable way to solve ill- conditioned system.

6. Psychometrics : In psychometrics and education, SVD is used in the construction and scoring of psychological and educational test, where it is often important to extract underlying latent traits.

7. Bioinformatics : SVD and related techniques are often used to analyze gene expression data, where it is important to identify the underlying patterns of gene activity.

8. Quantum Computing : SVD is also used in quantum state tomography to understand the state of a quantum system.

SVD The Equation

$$A = U \Sigma V^T$$

A: any matrix definite

U: Orthogonal

Σ: Rectangular diagonal

V^T: Orthogonal

$$A = U \Sigma V^T$$

A → $(m \times n)$

U → $(m \times m)$

Σ → $(m \times n)$

V^T → $(n \times n)$

$(m \times m)$ $(m \times n)$ → $(m \times n)$

$(m \times n)$ $(n \times n)$ → $\boxed{m \times n}$

SVD Relationship with Eigen Decomposition

$A = U \Sigma V^T$

$(m \times n)$

non-square

$A = V \Lambda V^{-1}$

↓

Symmetric

we can make non-square → Square matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
(2×3)

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$
3×2

$$\begin{bmatrix} 6 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \text{Symmetric Square matrix}$$
2×2

$$AA^T = U\Sigma V^T (U\Sigma V^T)^T$$

$$AA^T = U\Sigma V^T V \Sigma^T U^T \qquad (\because VV^T = \underset{\text{identity matrix}}{I}, V^T = V^{-1})$$

$$= U\Sigma\Sigma^T U^T$$

$$\boxed{AA^T = \underbrace{UXU^T}_{\text{symmetric}}} \quad \text{where } X = \Sigma\Sigma^T$$

if $A^TA = (U\Sigma V^T)^T U\Sigma V^T$

$$= V\Sigma^T U^T U \Sigma V^T$$

$$= V\Sigma^T \Sigma V^T$$

$$\boxed{A^TA = VYV^T} \quad \text{where} = Y = \Sigma^T\Sigma$$

$U^T = U^{-1}$

$AA^T = U\overset{\text{eigen value}}{X}U^T \rightarrow X = \Sigma\Sigma^T$
$A^TA = V\underset{\text{eigen value}}{Y}V^T \rightarrow Y = \Sigma^T\Sigma$ } both eqn represent eigen decomposition

Symmetric

$U \rightarrow$ matrix where cols contain eigenvectors of AA^T

$V \rightarrow$ matrix whose cols contain eigenvector of AA^T

$A \rightarrow u$ (left singular vector)

$A \rightarrow v$ (singular vectors) Right

$X = \Sigma \Sigma^T$

$Y = \Sigma^T \Sigma$

$A = U \Sigma V^T$

$(2\times3) \quad (2\times2)(2\times3)(3\times3)$

$$\Sigma = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} = a^2 b^2$$

eigenvalue AAT

$$Y = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = a^2 b^2 0$$

eigen value A^TA

$X, Y \rightarrow a^2 b^2$

$\sqrt{a^2}, \sqrt{b^2}$

$\underbrace{a, b}$ $\rightarrow$ SVD eqn

$\begin{pmatrix} AA^T \\ A^TA \end{pmatrix} \rightarrow a, b \rightarrow$ SVD

$\rightarrow$ Singular value

$\downarrow$

sqrt (eigenvalue)

$A = U \Sigma V^T \leftarrow$ right singular vector (A^TA)

left singular vector (AAT)

$(a, b) \rightarrow \underbrace{\sqrt{a^2}, \sqrt{b^2}}$

sqrt (eigen value)

$\rightarrow A^TA$ AAT

Geometric Intuition

$A = U \Sigma V^T$ → geometric intuition

↑

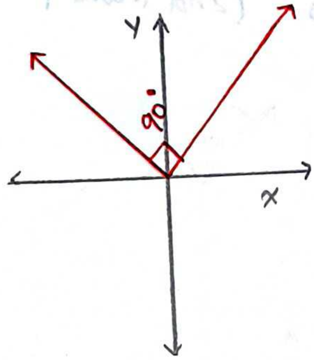
matrix
$(m \times n)$

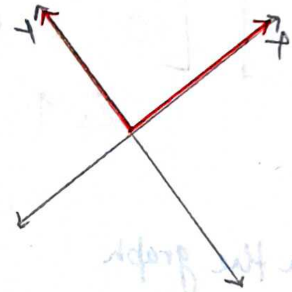
$A = V \Lambda V^{-1}$

Symmetric Orthogonal diagonal

L ← R

V^{-1} → V mean Orthogonal 90° and V^{-1} inverse of 90°





Λ → means strech of chaing the size or magnitude

V → Again become 90° but change the size.

size change

eigenvector

$V \rightarrow \boxed{A^TA}$ ∴ $A = 2 \times 3$

$A^TA = 3 \times 3$

→ 3×3

Orthogonal → 3Dim

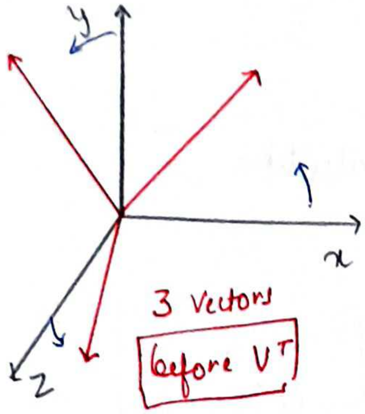
$A = U \Sigma V^T$

↓

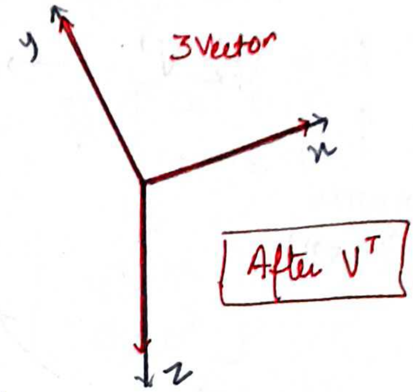
matrix
(2×3)
→ input (3D)

Output
(2D)

V^T
Antirotate or Anticlockwise rotate.



Ist Tranformation



$$\Sigma = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$ $\because$ a,b some value

↓ break down

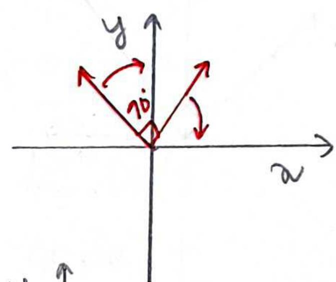
$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow$$ Reduce The dimension in 2D (2nd Transformation)

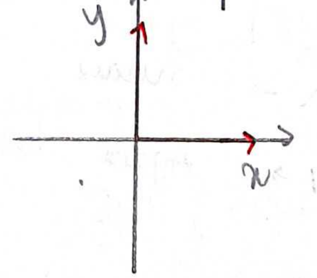
$\Sigma_1 \uparrow$

$\Sigma_2 \uparrow$ ↓

Stretch the graph

U → Create orthogonal vector and chang the position of vector in 2D.



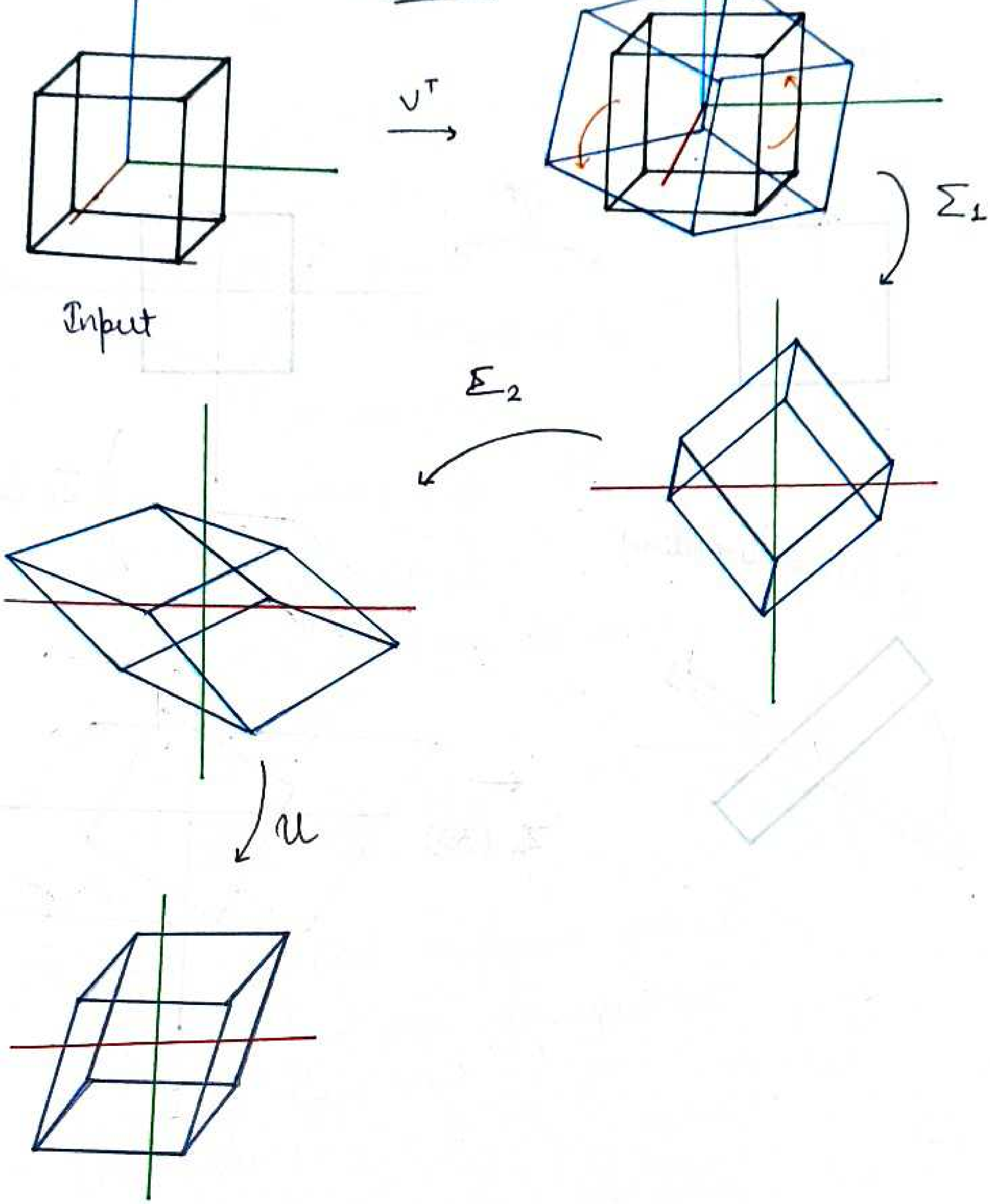


Step1:- $\underset{(m \times n)}{A}$ → Anticlock wise rotation input space

Step2:- Dimension reduction $(n>m)$, Dimension increase $(m>n)$
Shrink Stretch

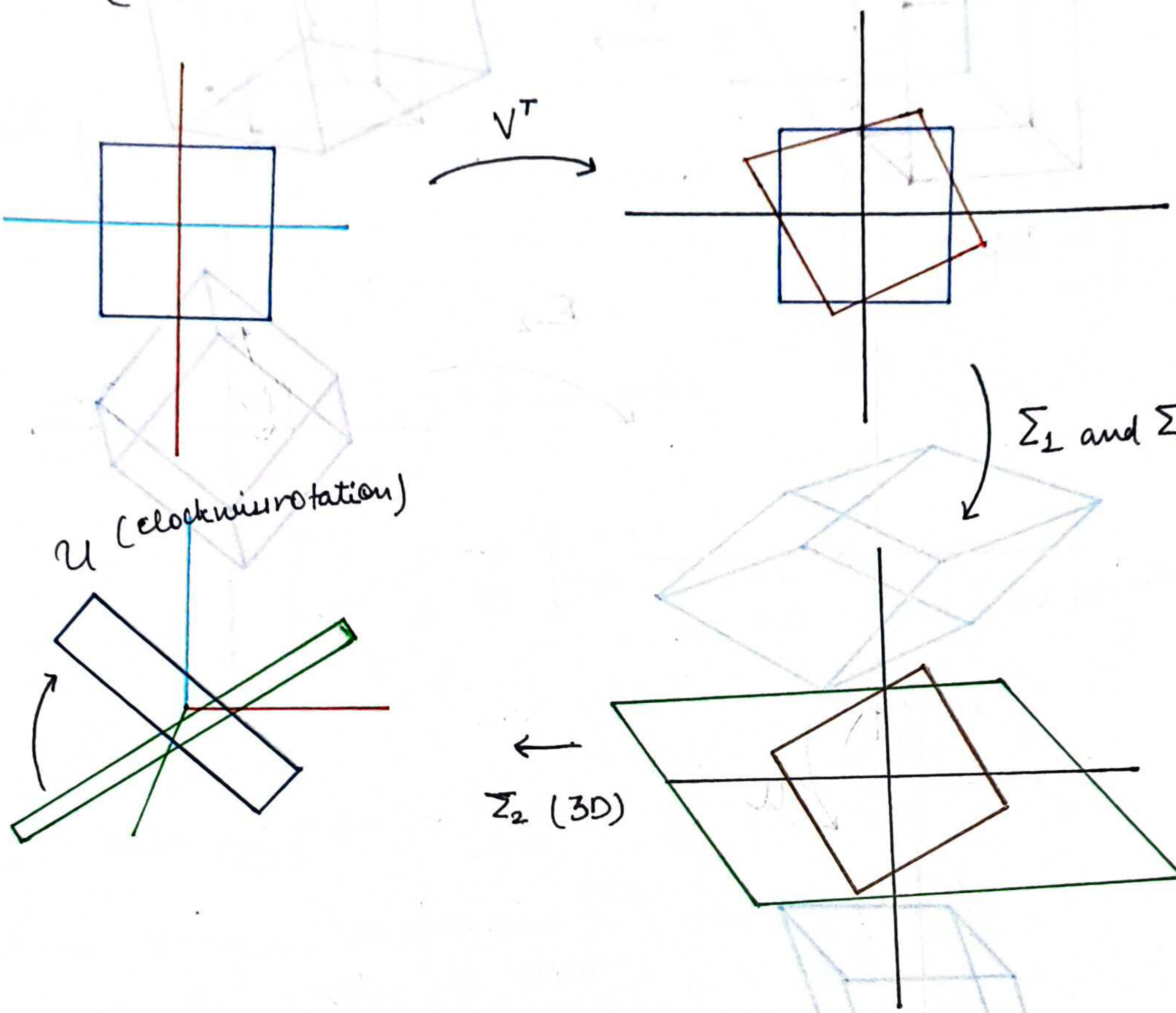
Step3:- Rotation clock wise (vectors rotate)

Demo 1



Demo 2

(3x2) matrix



How to calculate SVD

$A = U \Sigma V^T$ → ? How to calculate?

→ Use eigen decomposition

→ $AA^T = U \Lambda U^T$

↳ Sqrt → Σ

$A^T A = V \Lambda V^T$

* But this method is numerically unstable.

Use :- $\boxed{\text{np.linalg.svd(A)}}$

↳ U, Σ, V^T (give all value)

SVD in PCA

PCA → Principal component extract

↳ eigen decomposition

↳ SVD

cgpa | iq | result

↳ covariance matrix (3x3) → eigen decomposition

→ SVD is faster than eigen Decomposition

SVD for larger Dataset.

SVD use in Sklearn

iris

$X = (150, 4)$ → cov → (4,4)

SL	SW	PL	PW

1) mean center all the cols (X_c)

2) $\frac{X_c^T X_c}{n-1}$ = Cov matrix

Square of eigen vector

eigen vector

$X_c = U \Sigma \boxed{V}^T$ → SVD

$\boxed{X_c^T X_c = V Y V^T}$ → eigen decomposition

→ eigen vector of $X_c^T X_c$ → covariance

* In SVD we can calculate eigen vector, and eigen value without finding covariance matrix.

1. Increamental PCA
2. Randomized PCA