Statistics on Trial(s)

Frequentist vs Bayesian Inference

– Objective p-values versus subjective posteriors

Georges Monette, March 2017

P-values on trial: Sir Roy Meadow and Sally Clark

Sally Clark

- Young lawyer, gives birth to first son in September 1996
- son dies, apparently of SIDS, at 10 weeks
- second son born a year later
- dies, apparently of SIDS, at 8 weeks
- only evidence of trauma consistent with resuscitation attempts
- charged with two counts of murder

Sir Roy Meadow

- distinguished pediatrician
- as expert witness testifies:
    - probability of one SIDS death: $\frac{1}{8{,}500}$
    - probability of two: $\left(\frac{1}{8{,}500}\right)^2 = \frac{1}{72{,}250{,}000}$
    - 'if she's innocent, the chances of this happening are 1 in 72 million'
- jury convicts Sally Clark of murder in November 1999
- first appeal lost in October 2000
- second appeal succeeds and Sally Clark is released in January 2003
- she dies in 2007 at the age of 42

H_0: Sally is innocent

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Correct p-value is larger – maybe $\frac{1}{10{,}000}$!

BUT:

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Do we really want $P(y^+ \mid H_0)$?

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Does it establish guilt beyond a reasonable doubt?

BUT:

Do we really want $P(Y^+ \mid H_0)$?

Don't we really want $P(H_0 \mid Y)$?

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Let's find out:

$$P(H_0 \mid Y) = \frac{P(H_0, Y)}{P(Y)} = \frac{P(Y \mid H_0)\,P(H_0)}{P(Y)}$$

Bayesian tree:

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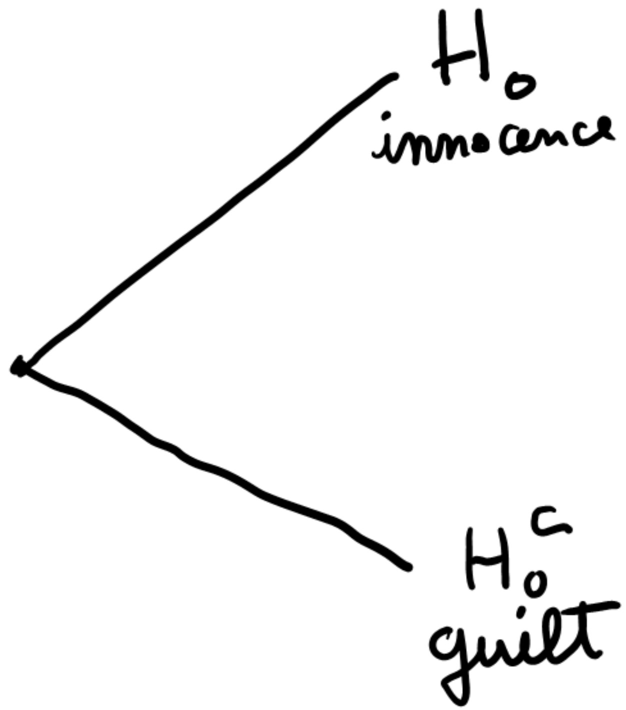

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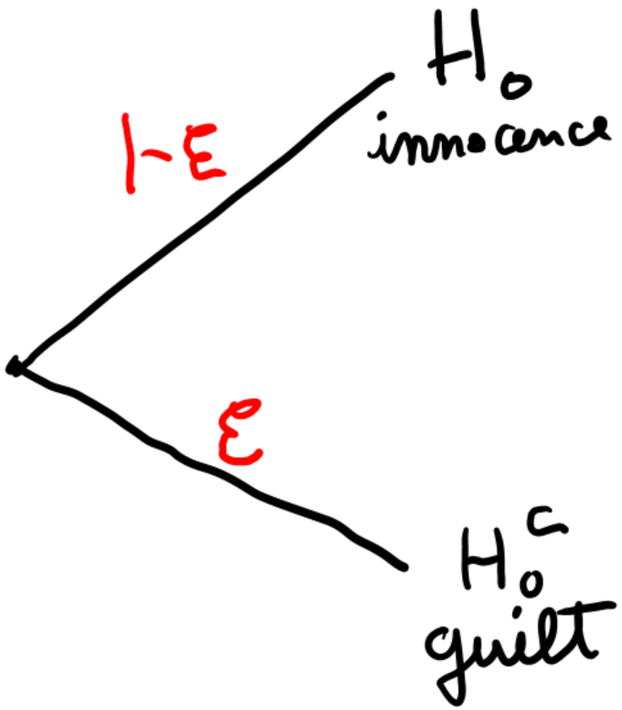

ε = very small number

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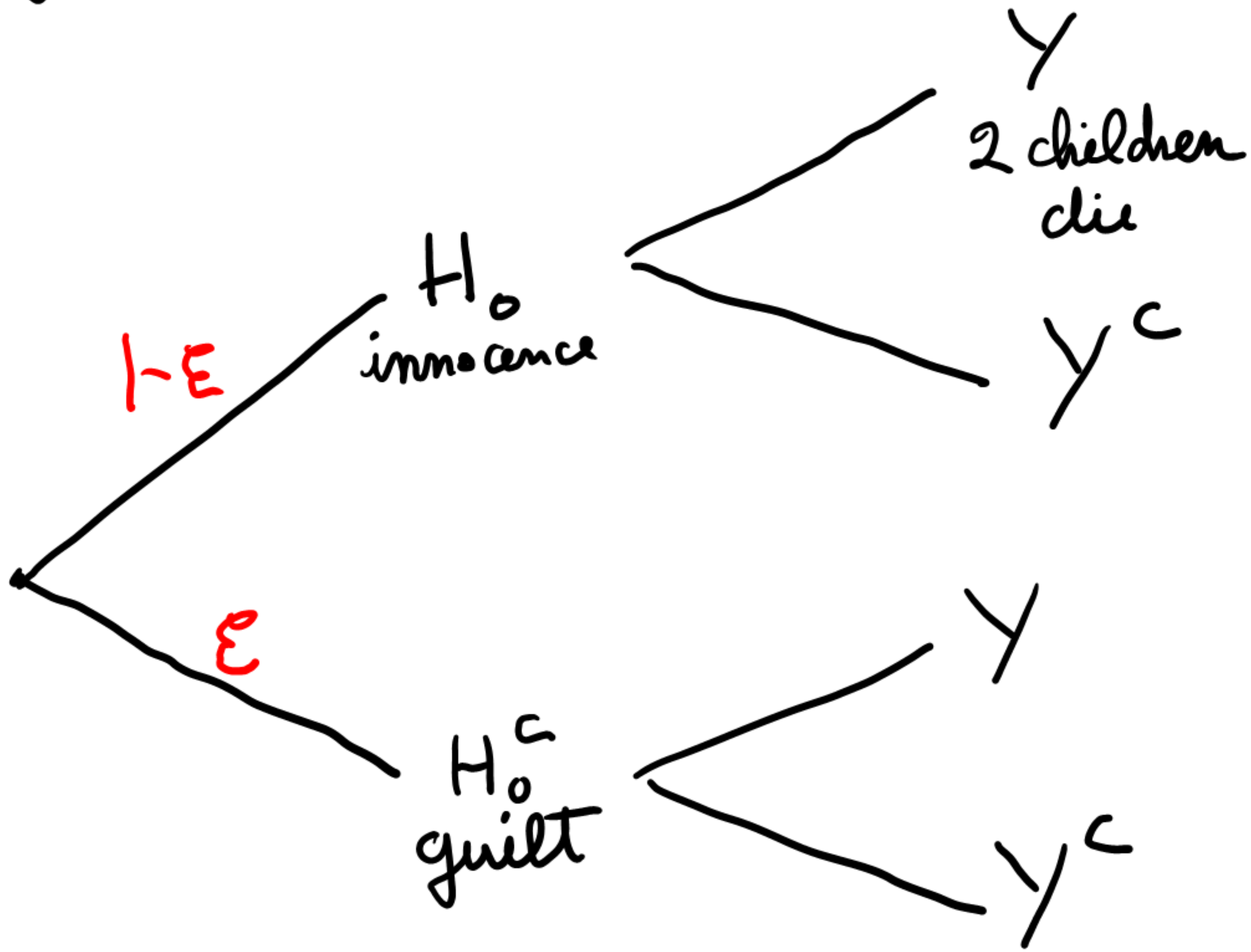

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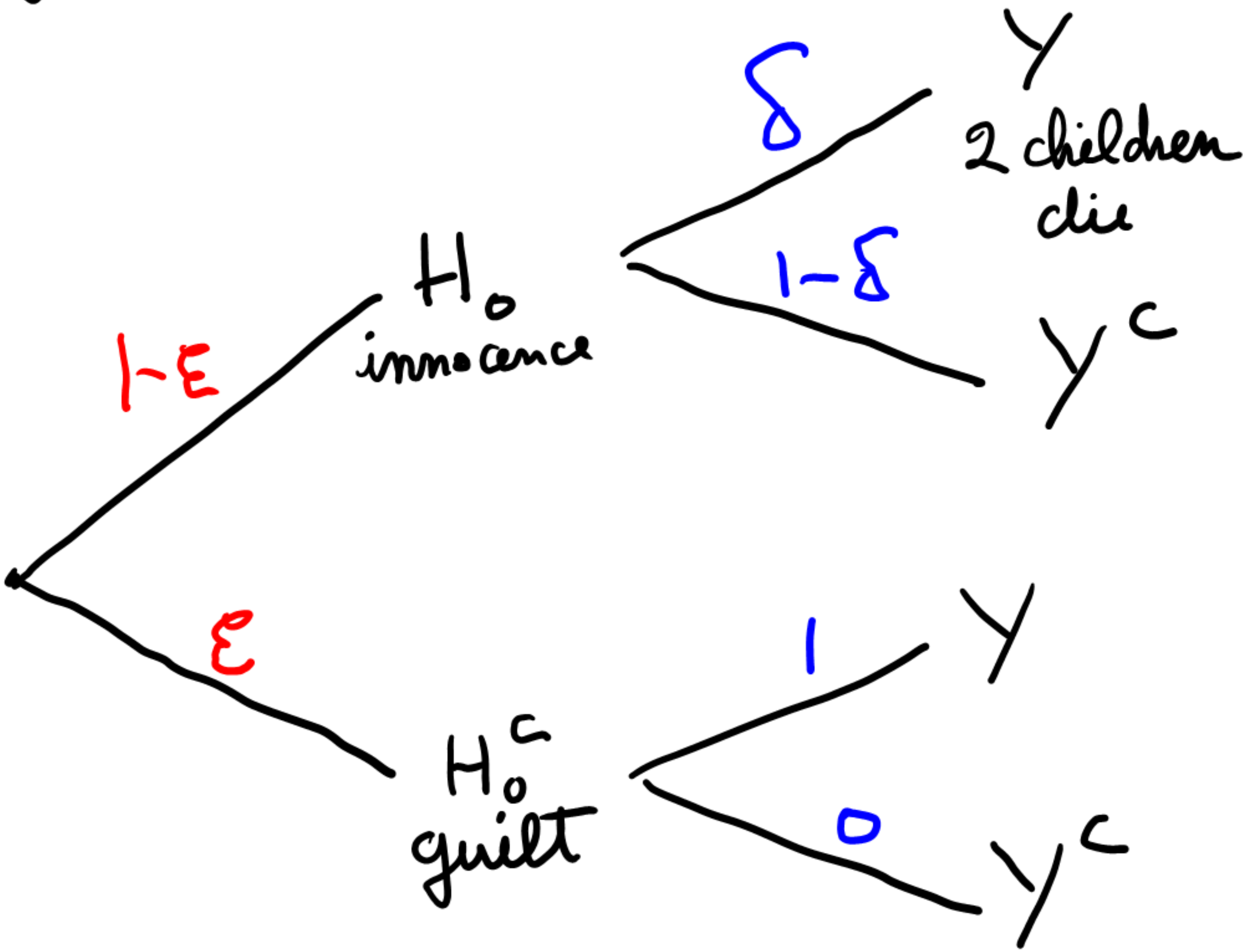

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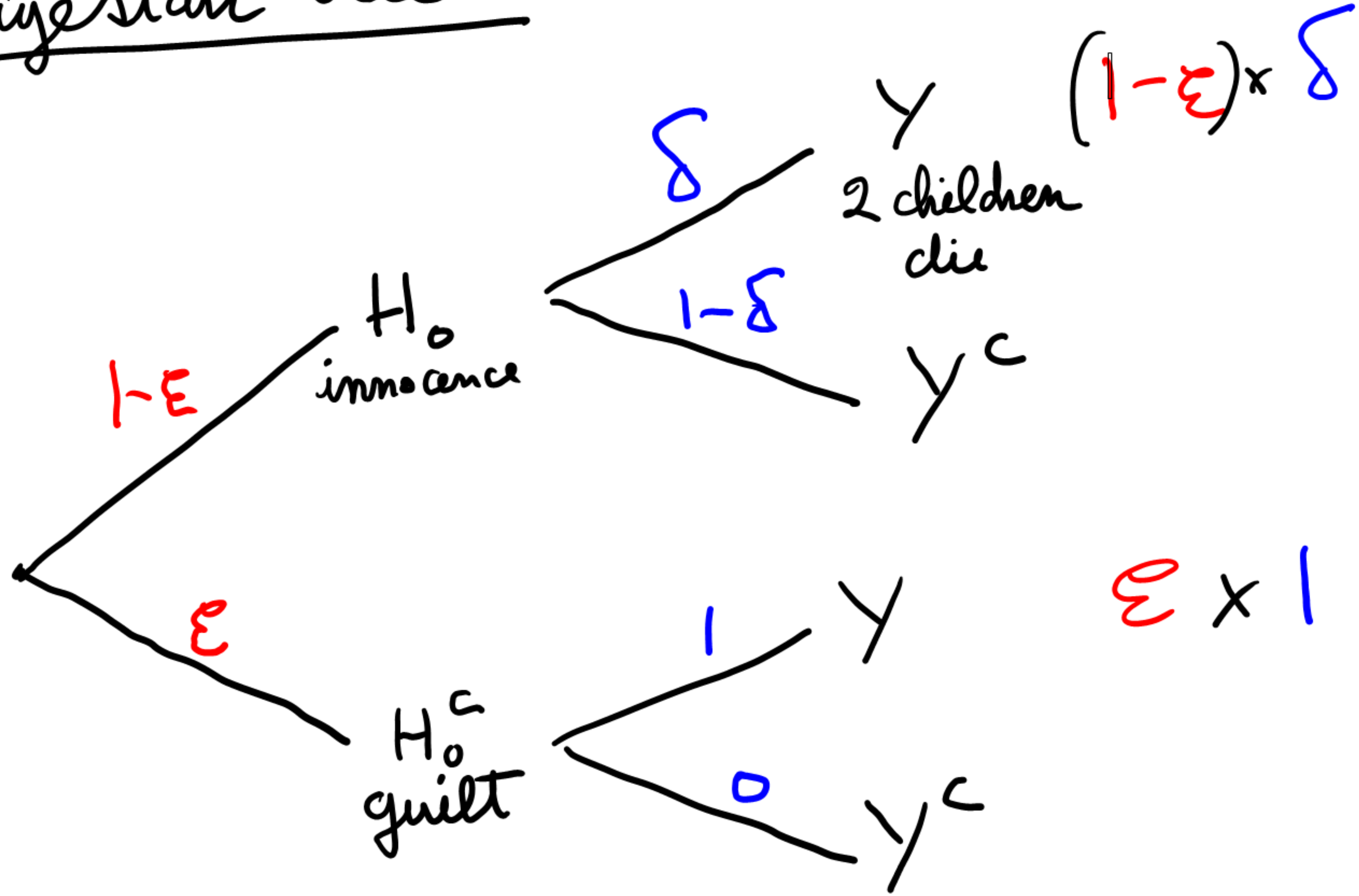

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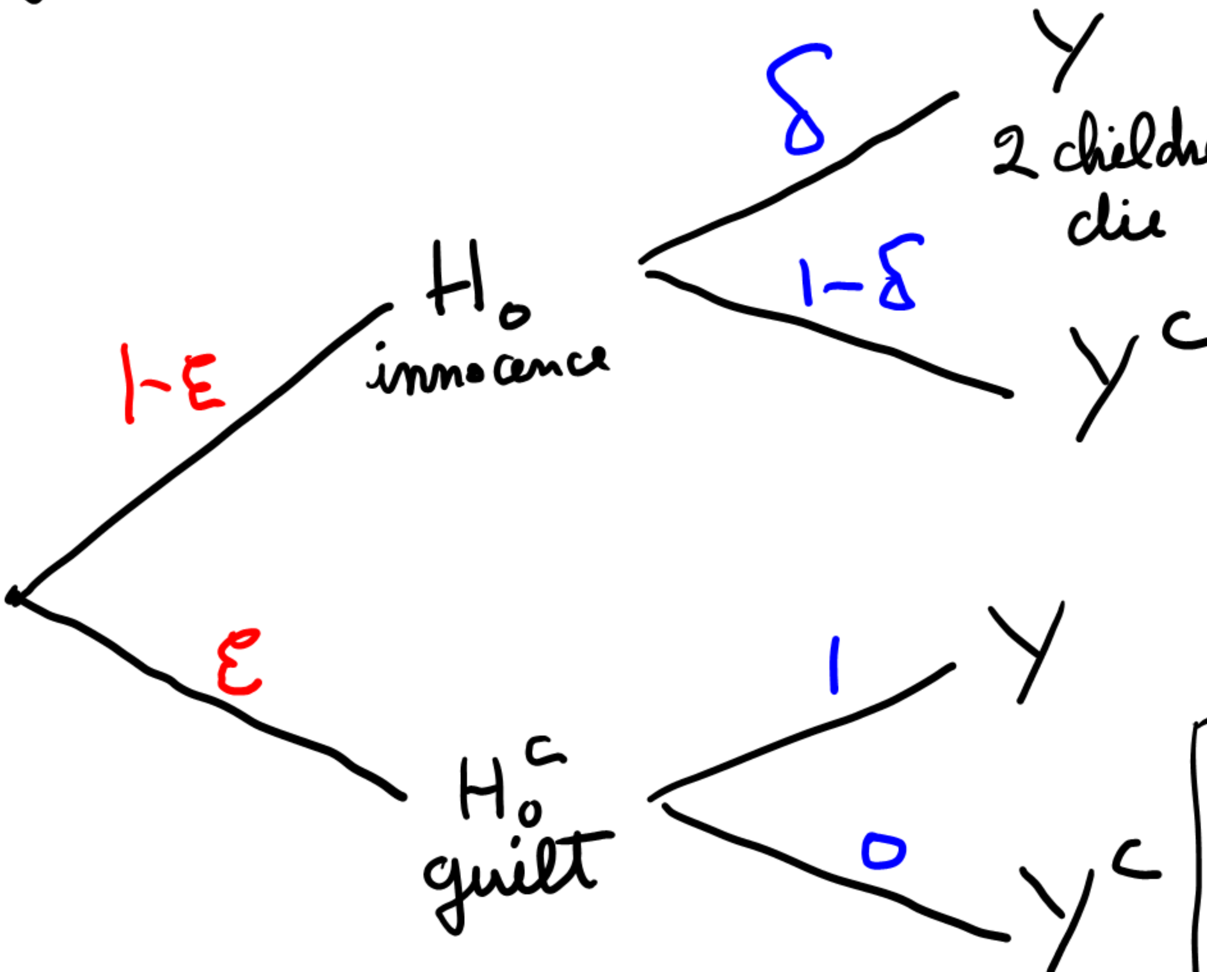
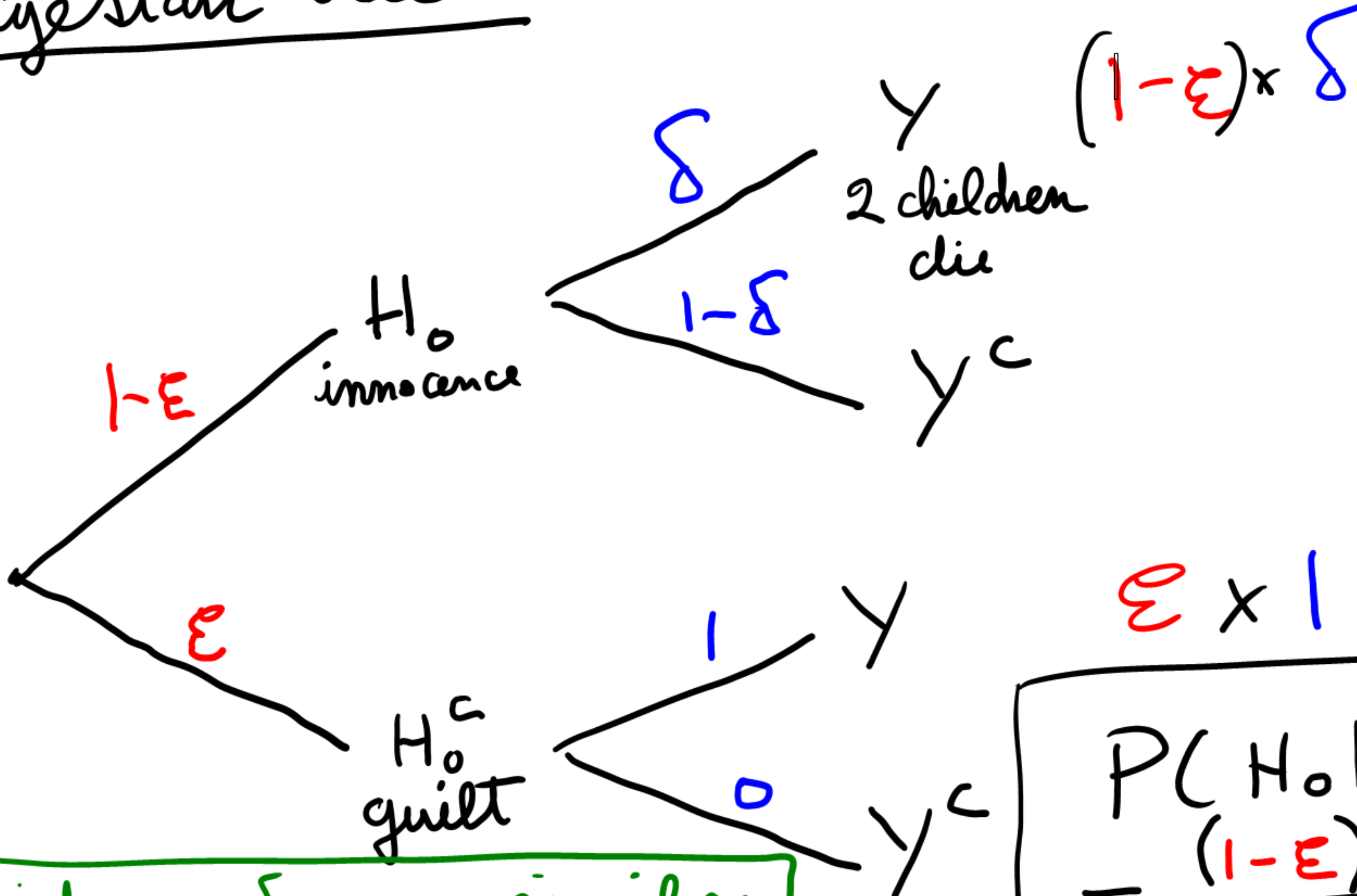


$(1-\varepsilon) \times \delta$

$\varepsilon \times 1$

$$\boxed{P(H_0 | Y) = \frac{(1-\varepsilon)\,\delta}{(1-\varepsilon)\,\delta + \varepsilon} \approx \frac{\delta}{\delta + \varepsilon}}$$

Bayesian tree:

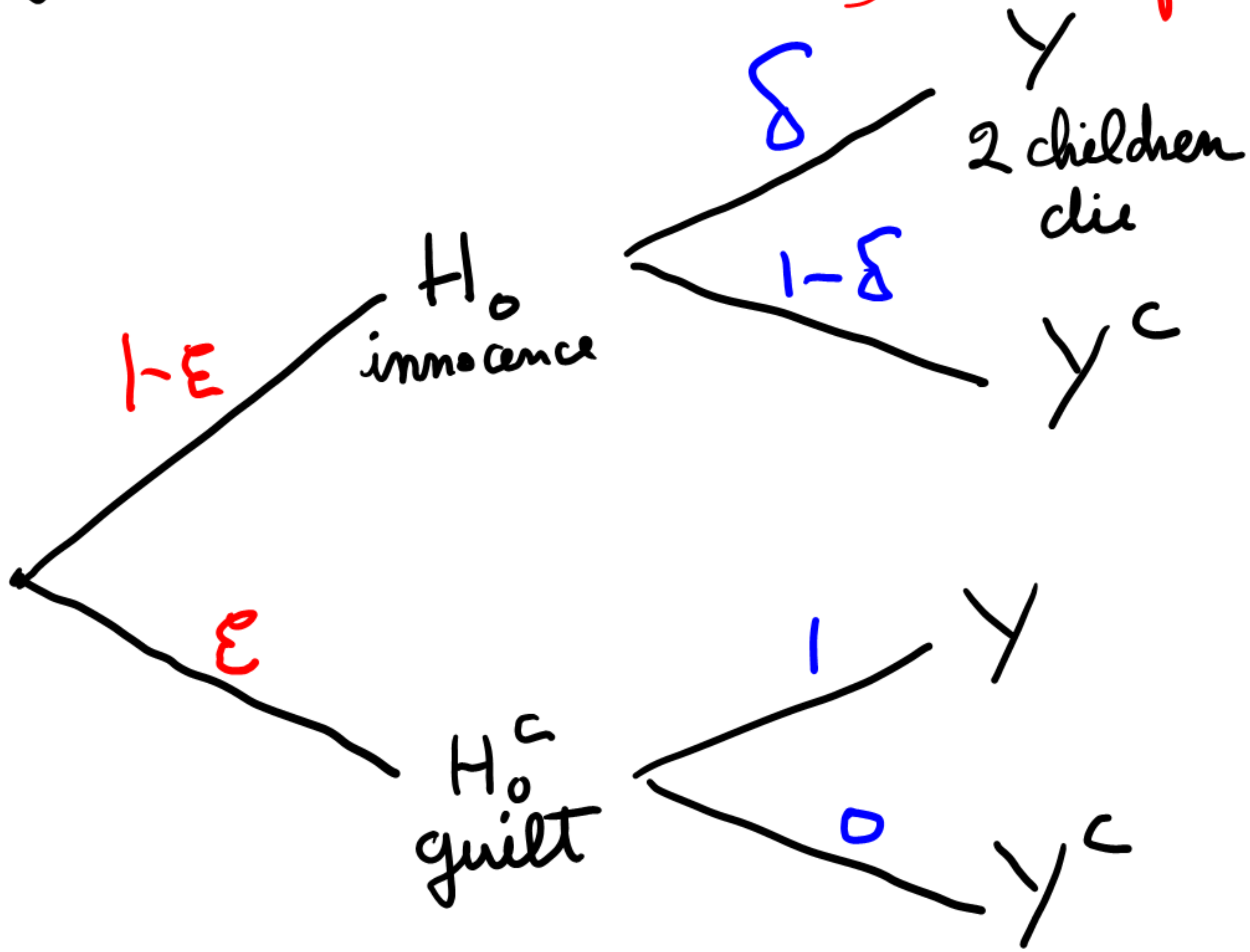


$$P(H_0|Y) = \frac{(1-\varepsilon)\,\delta}{(1-\varepsilon)\delta + \varepsilon} \approx \frac{\delta}{\delta+\varepsilon}$$

$\boxed{\text{So if } \varepsilon \text{ \& } \delta \text{ are similar then } P(H_0|Y) \approx \frac{1}{2}}$

Bayesian tree:

But this ignores lack of evidence



If it's a crime, it's a "perfect" crime — and, presumably, only a small proportion of crimes are perfect crimes.

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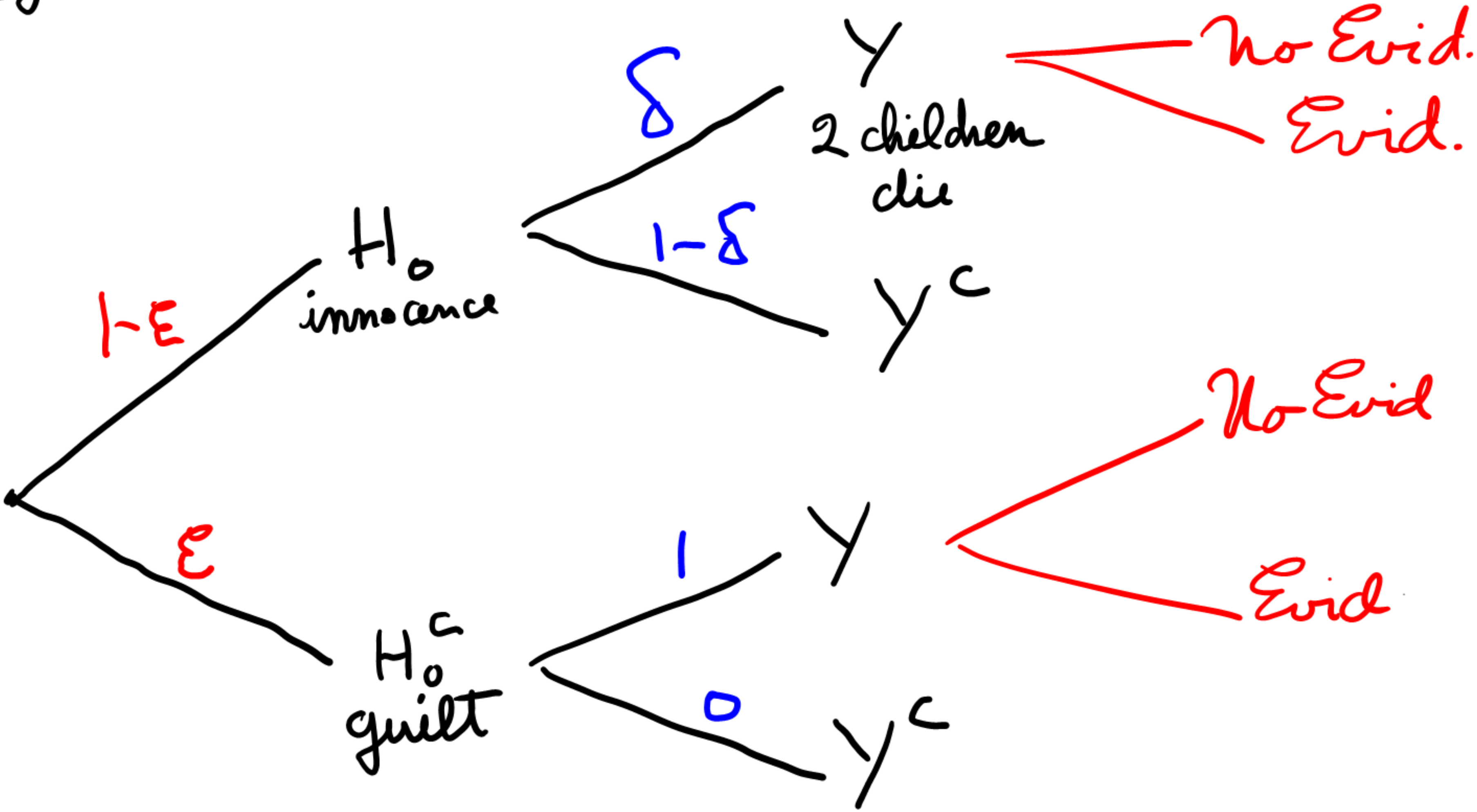

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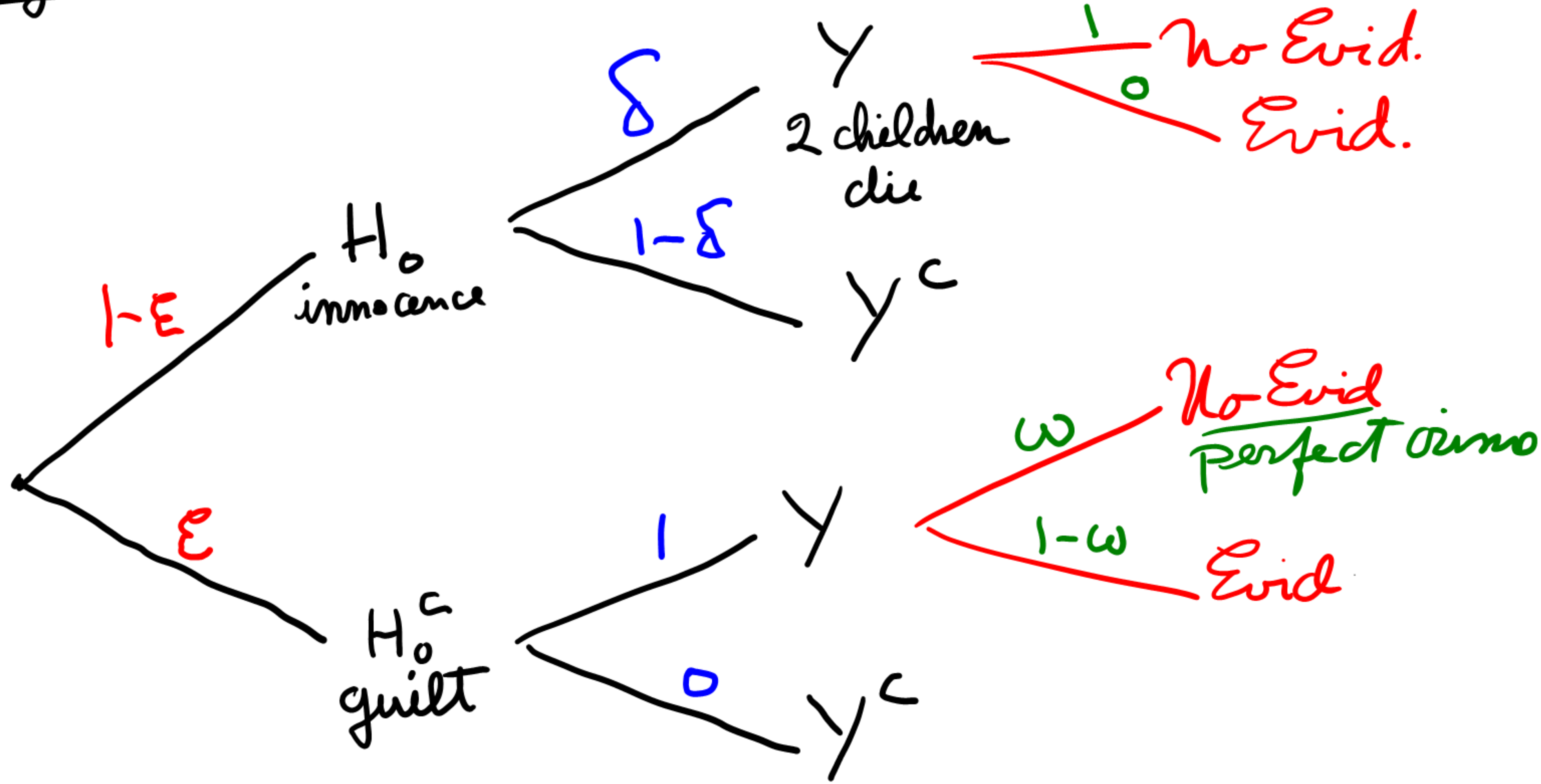

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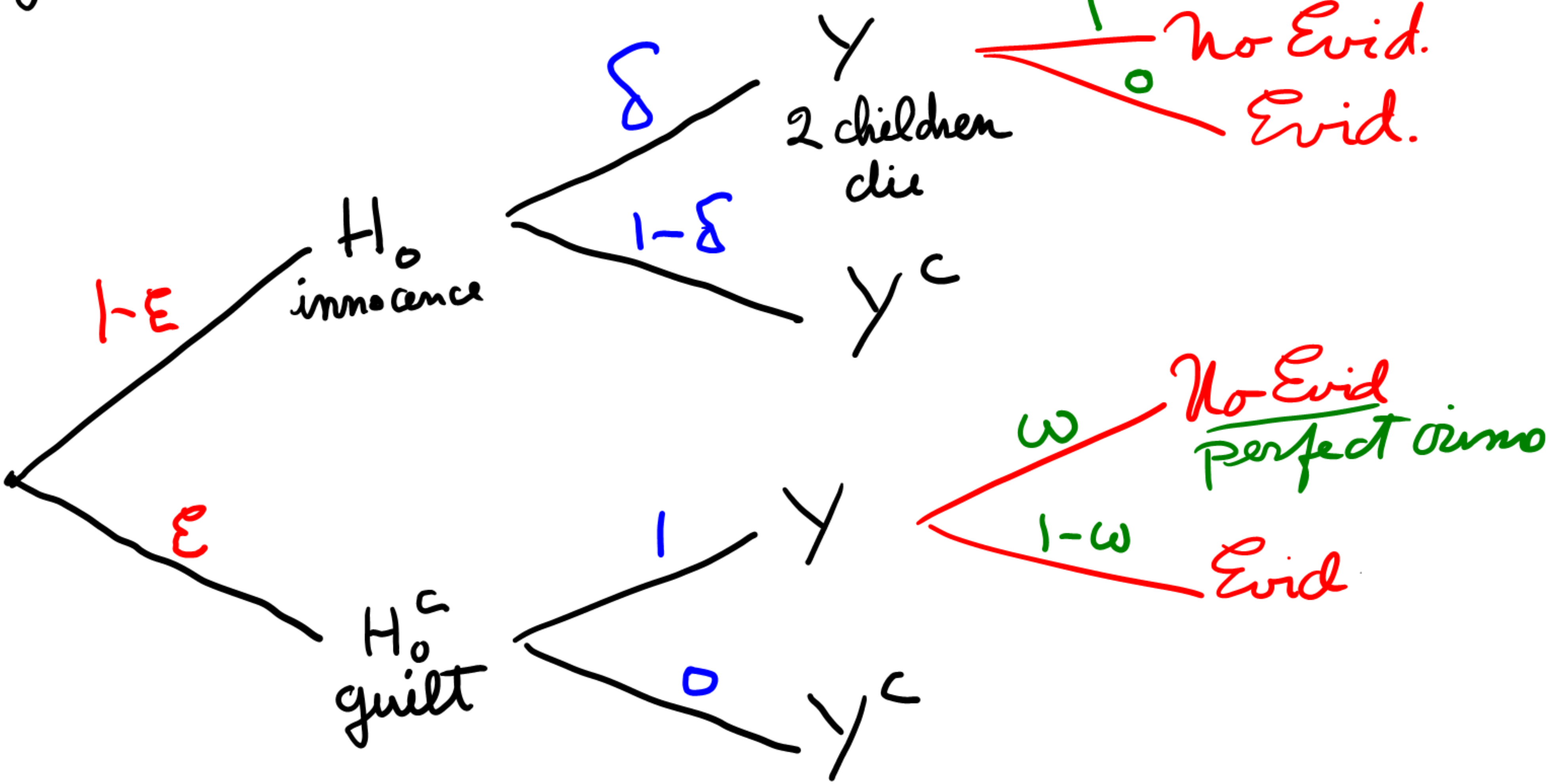

$$\Pr(H_0 \mid Y \& \text{No Evid}) \approx \frac{\delta}{\delta + \varepsilon\omega} \approx \text{close to } 1 \,!!$$

Sally Clark was
innocent beyond
a reasonable doubt.

$$P(Y^+ | H_0)$$

was a very poor proxy for

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— But many (most?) scientists only know about $P(Y^+ | H_0)$.

— Many have been trained to be adamantly opposed to using "subjective" Bayesian methods instead of "objective" Frequentist methods.

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- But many (most?) scientists testifying don't know of other ways

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So, how did we get to this?

Why are we using p-values if they are so bad?

Dilemma

Dilemma

Θ : unknown state of nature

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Θ : unknown state of nature

Y : what's observed

Dilemma

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What does Y say about Θ?

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Model : $P(Y \mid \Theta)$

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From this we can get p-values, confidence intervals, size-α tests.

Dilemma

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What does Y say about Θ ?

Model : $p(Y \mid \Theta)$

From this we can get p-values, confidence intervals, size-α tests.

But not $p(\Theta \mid Y)$ posterior

For that we need to start with $p(\Theta)$ prior

To get what you really want, you need to pay a price many are reluctant to pay.

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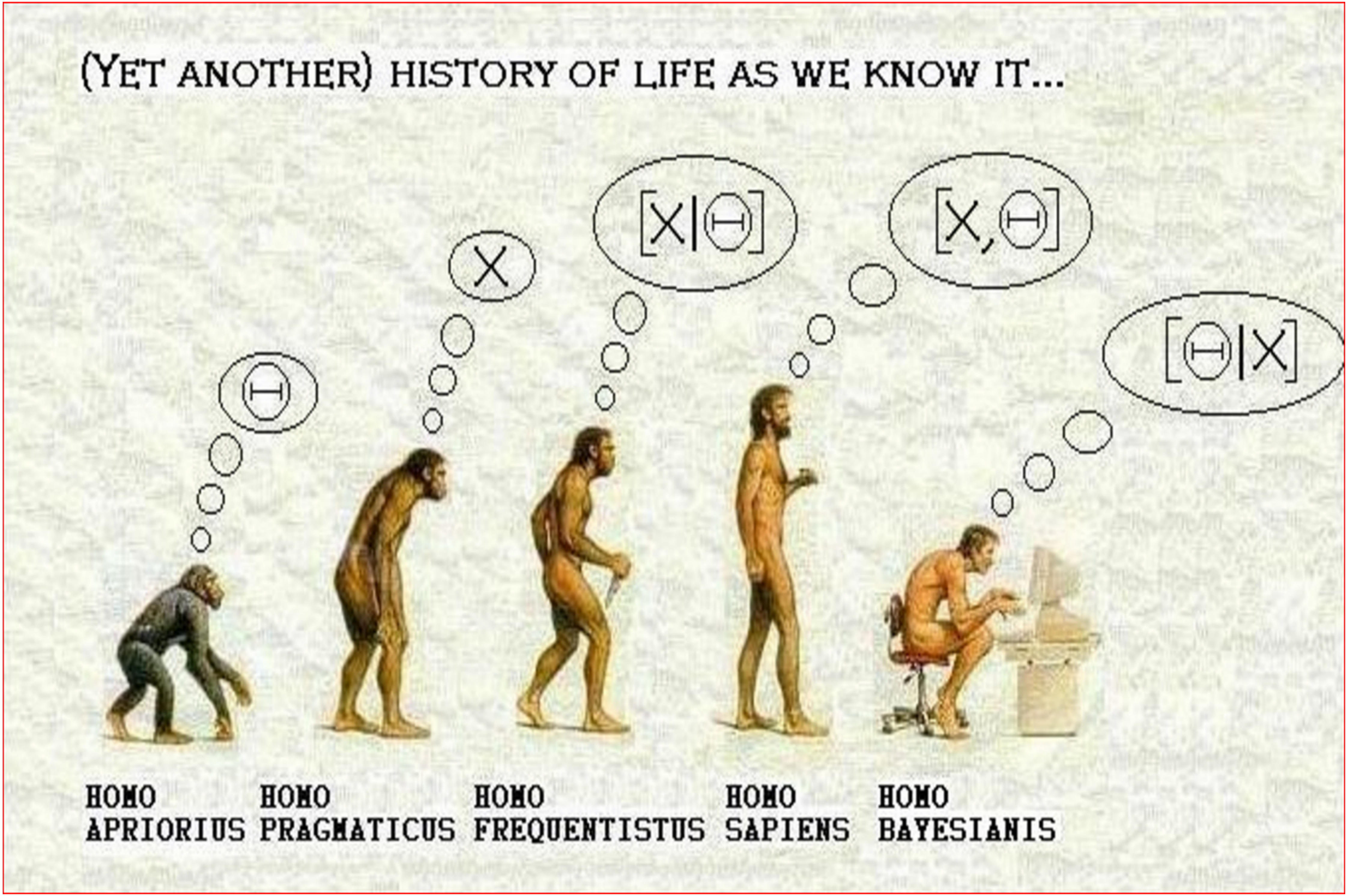
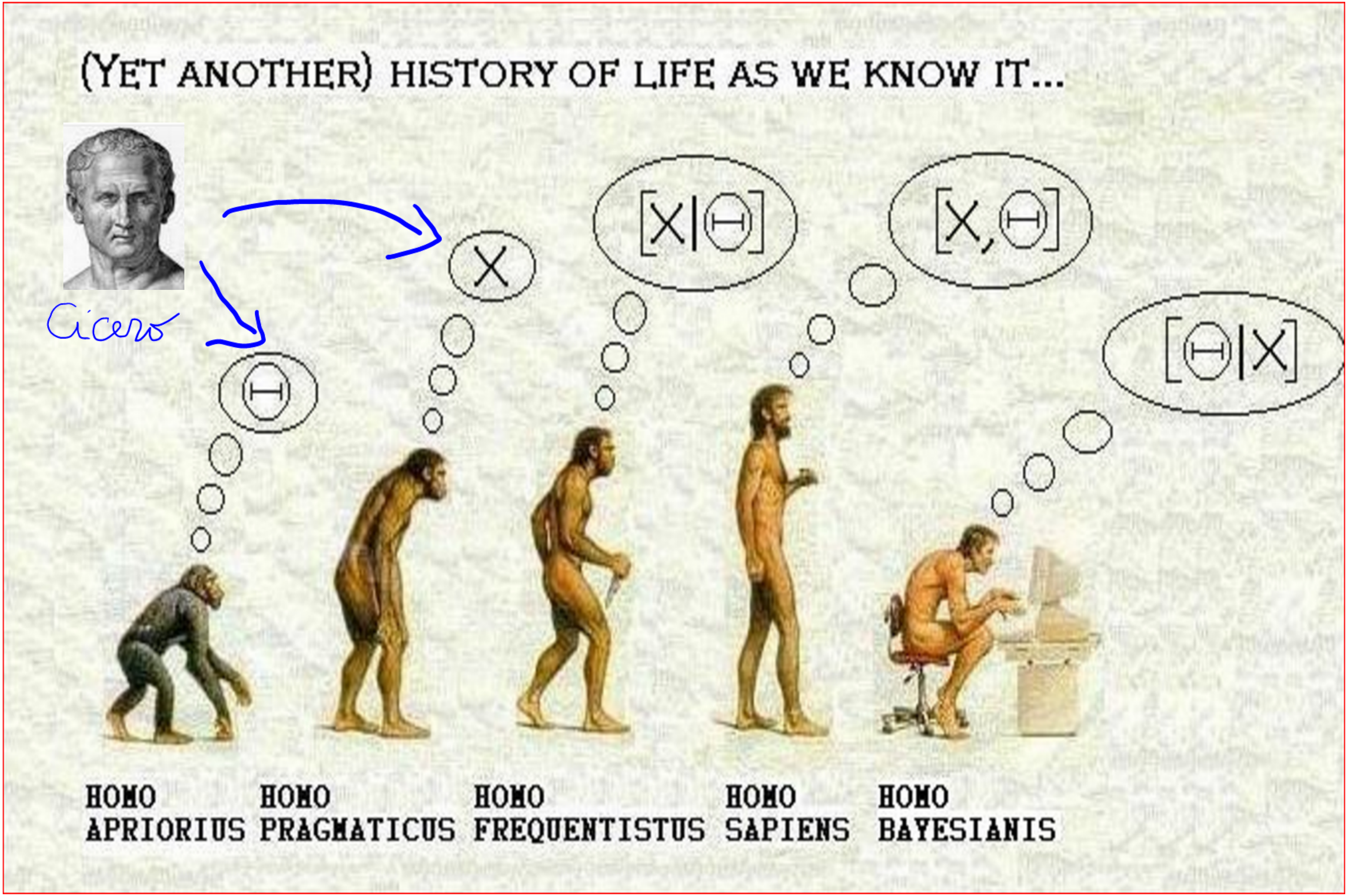
A historical perspective


(YET ANOTHER) HISTORY OF LIFE AS WE KNOW IT...
Θ
X
$[X|\Theta]$
$[X,\Theta]$
$[\Theta|X]$
HOMO APRIORIUS
HOMO PRAGMATICUS
HOMO FREQUENTISTUS
HOMO SAPIENS
HOMO BAYESIANIS


(YET ANOTHER) HISTORY OF LIFE AS WE KNOW IT...
Cicero
Θ
X
[X|Θ]
[X,Θ]
[Θ|X]
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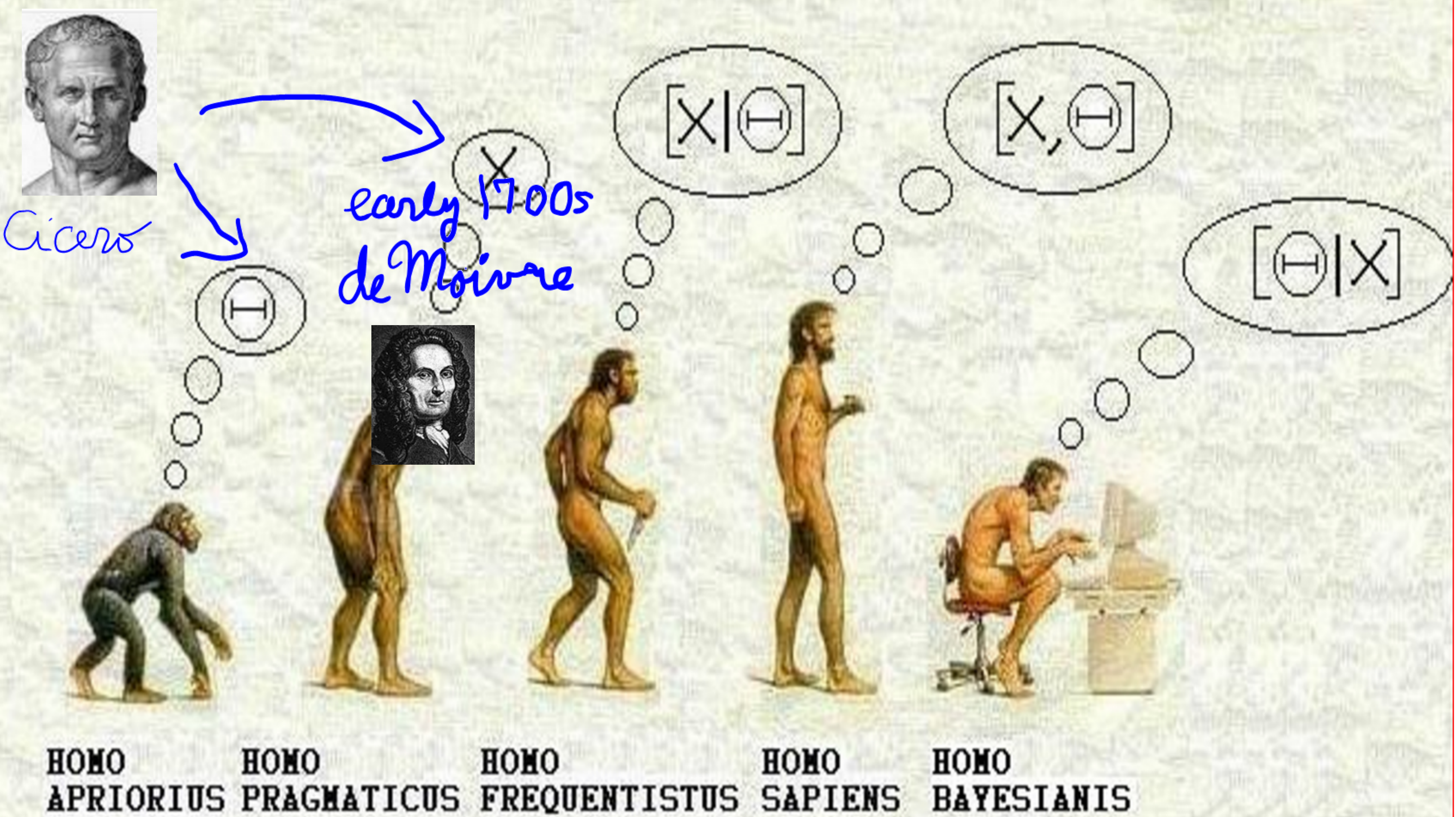

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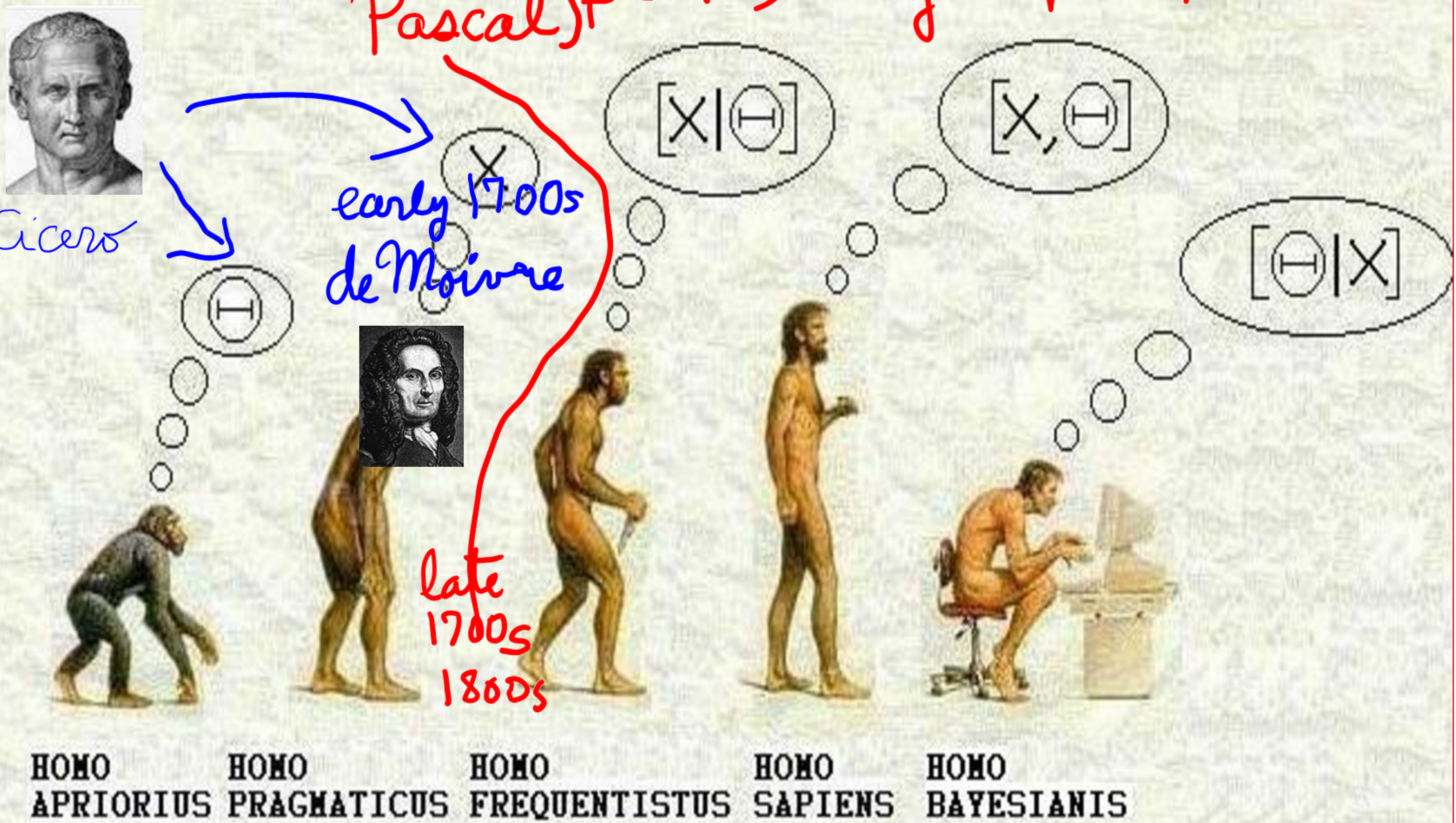
Bayes
Pascal $\Big\}$ $P(\theta|X)$ using uniform $p(\theta)$



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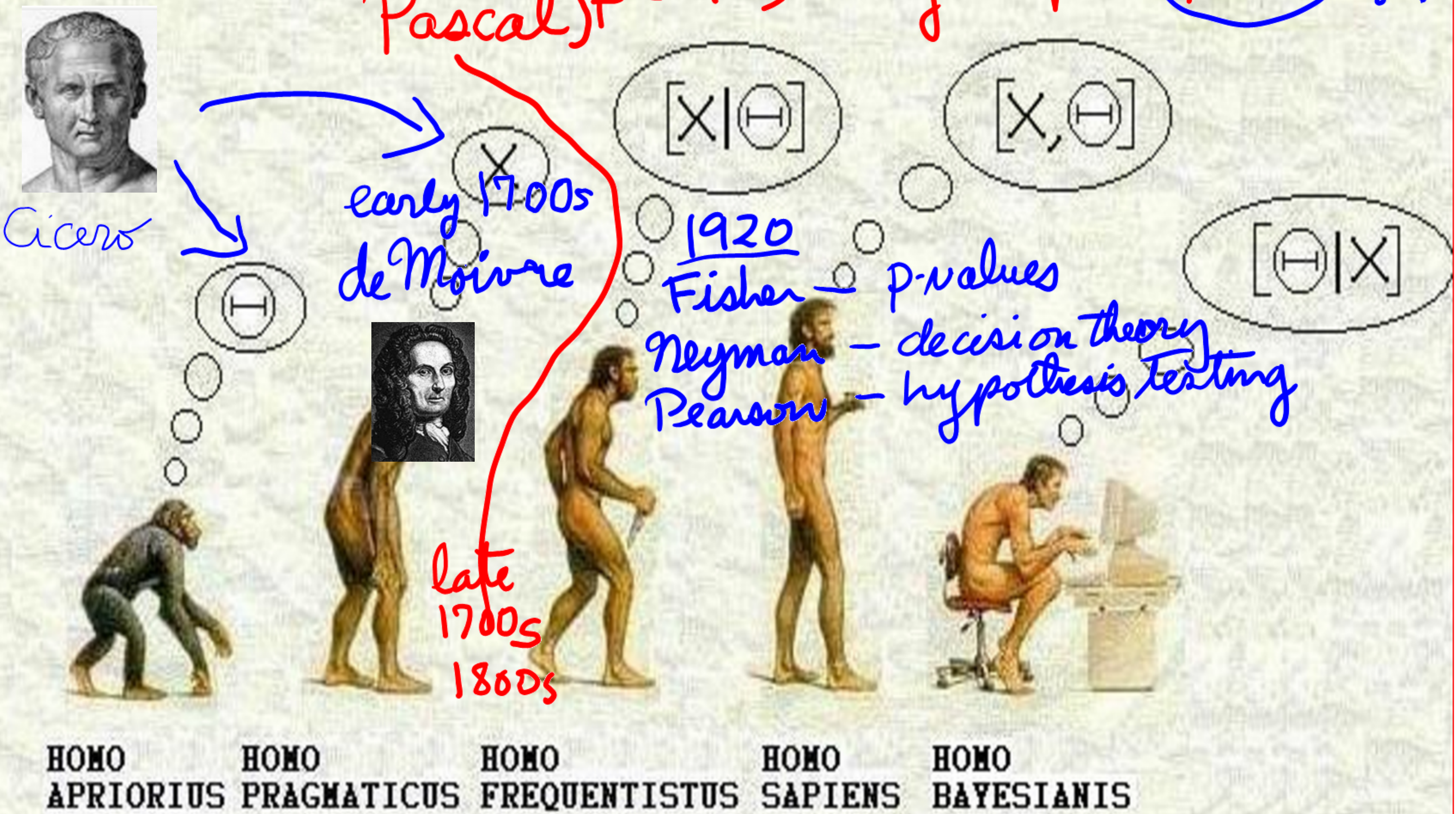

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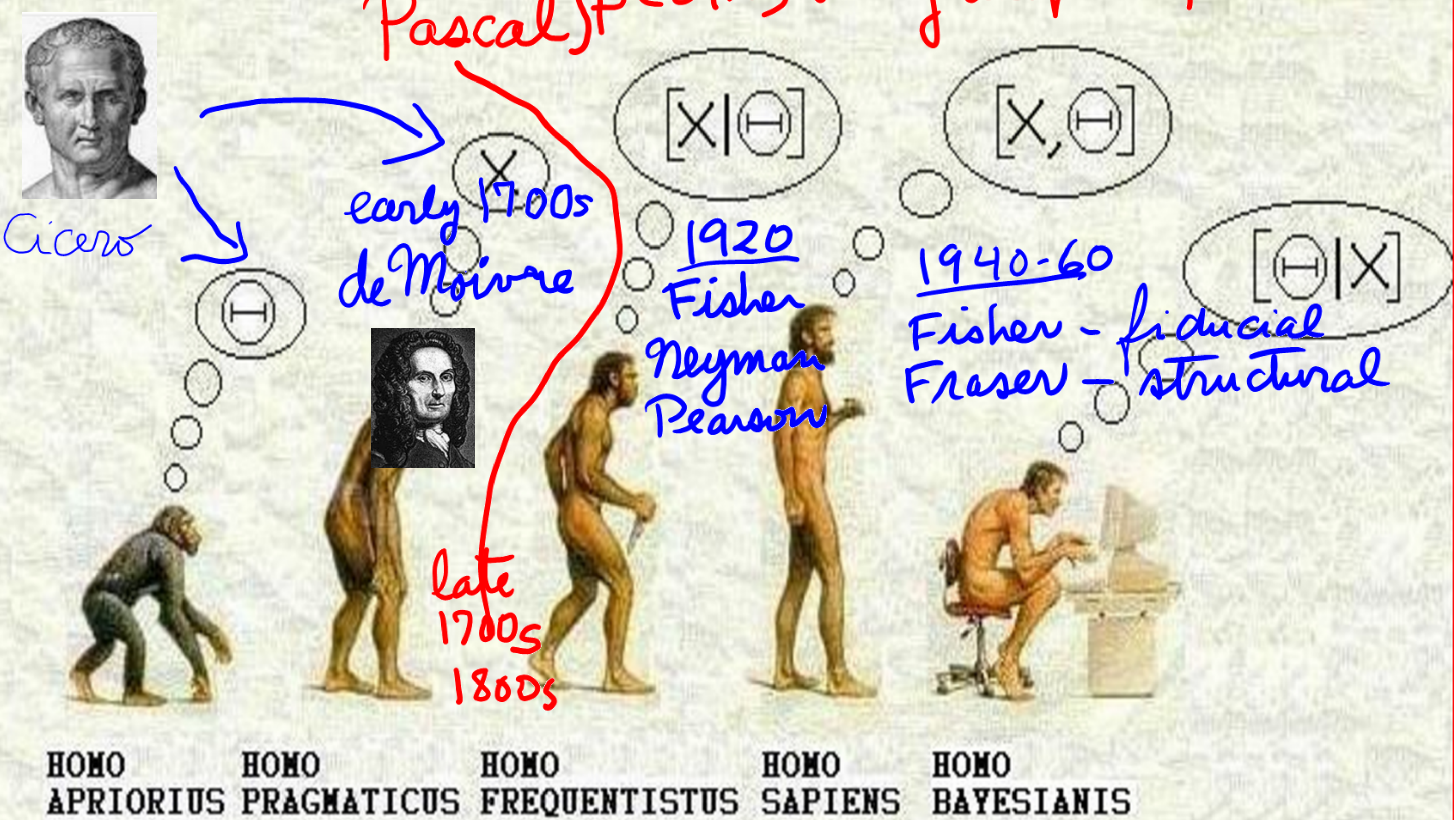

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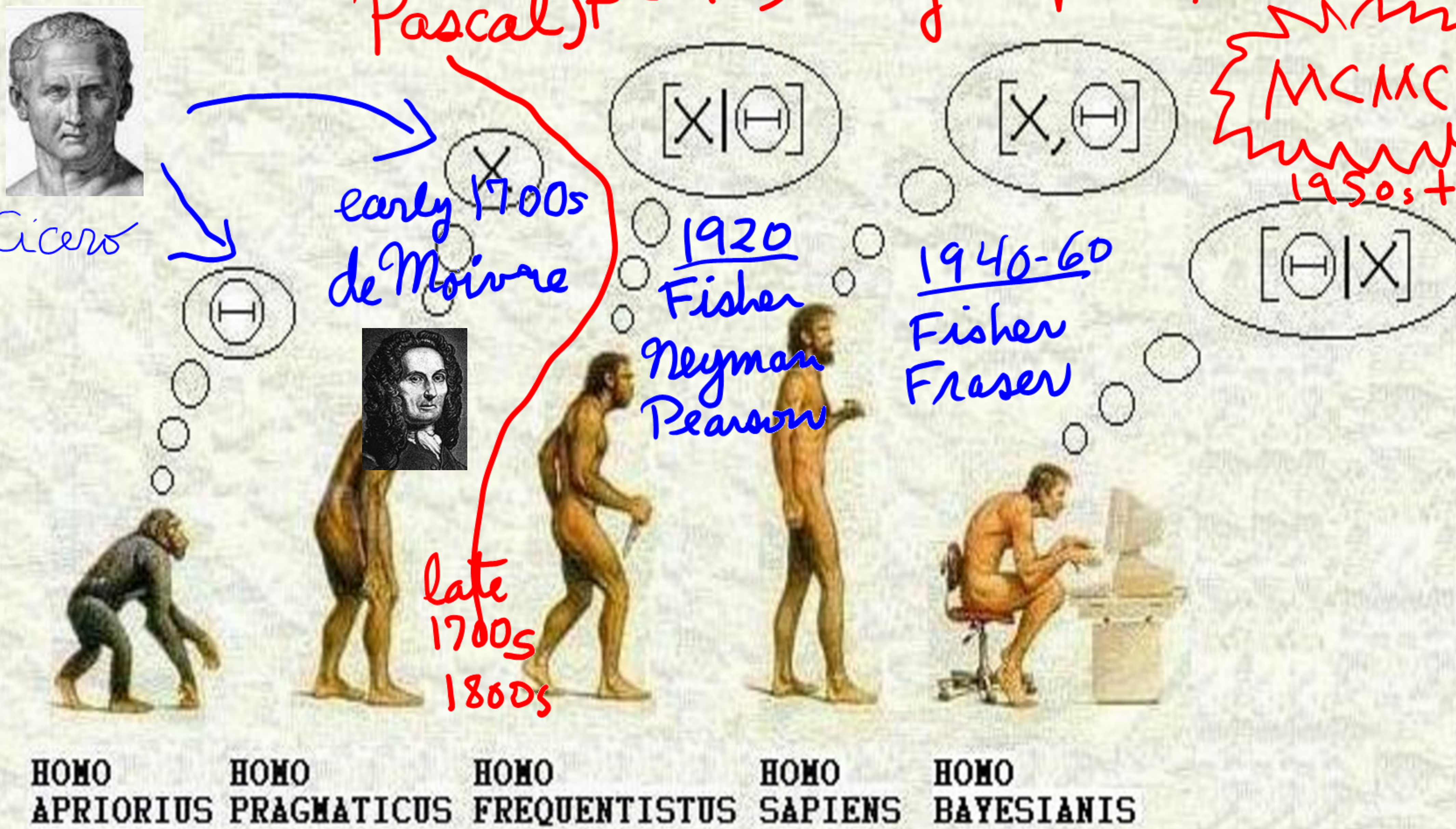

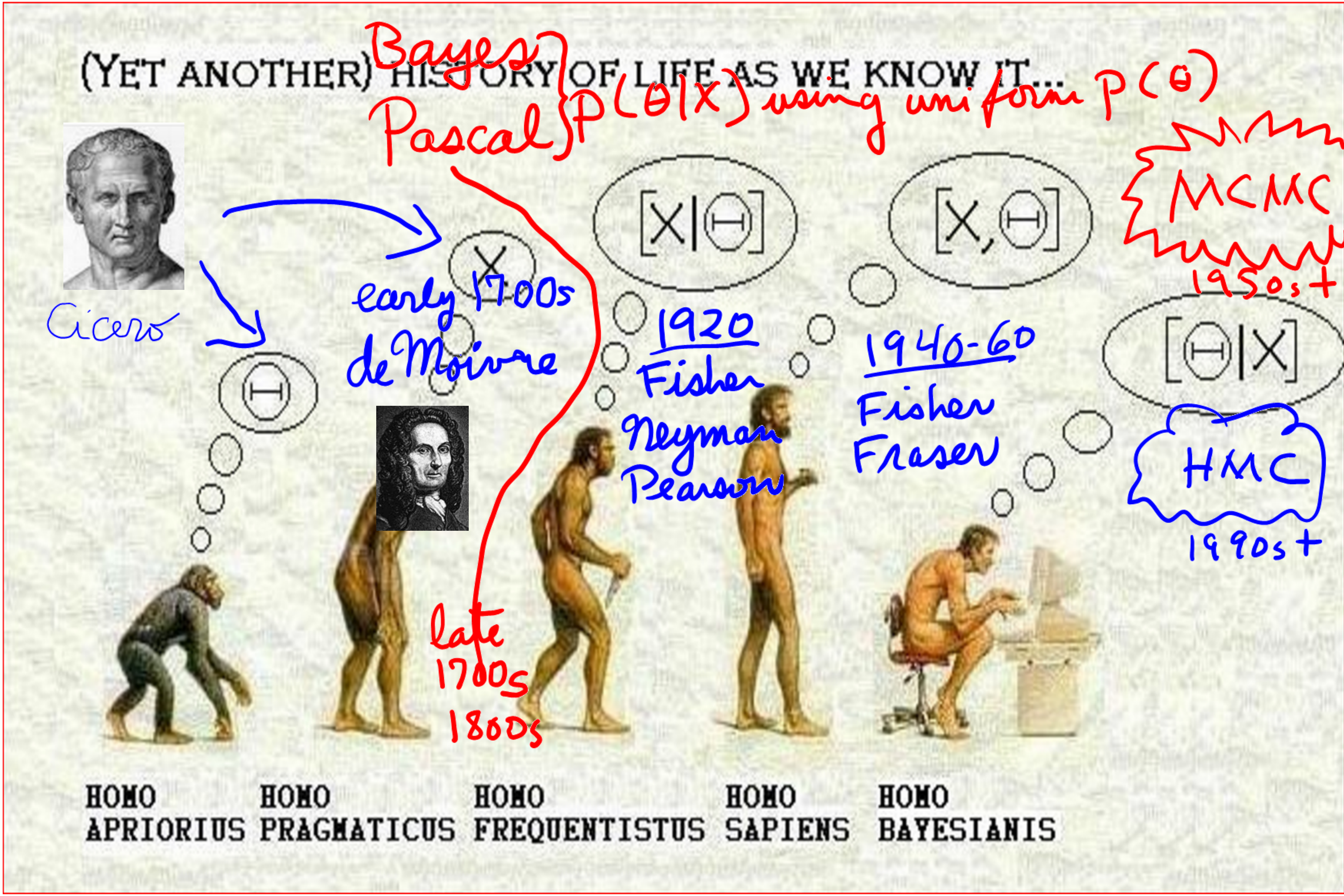

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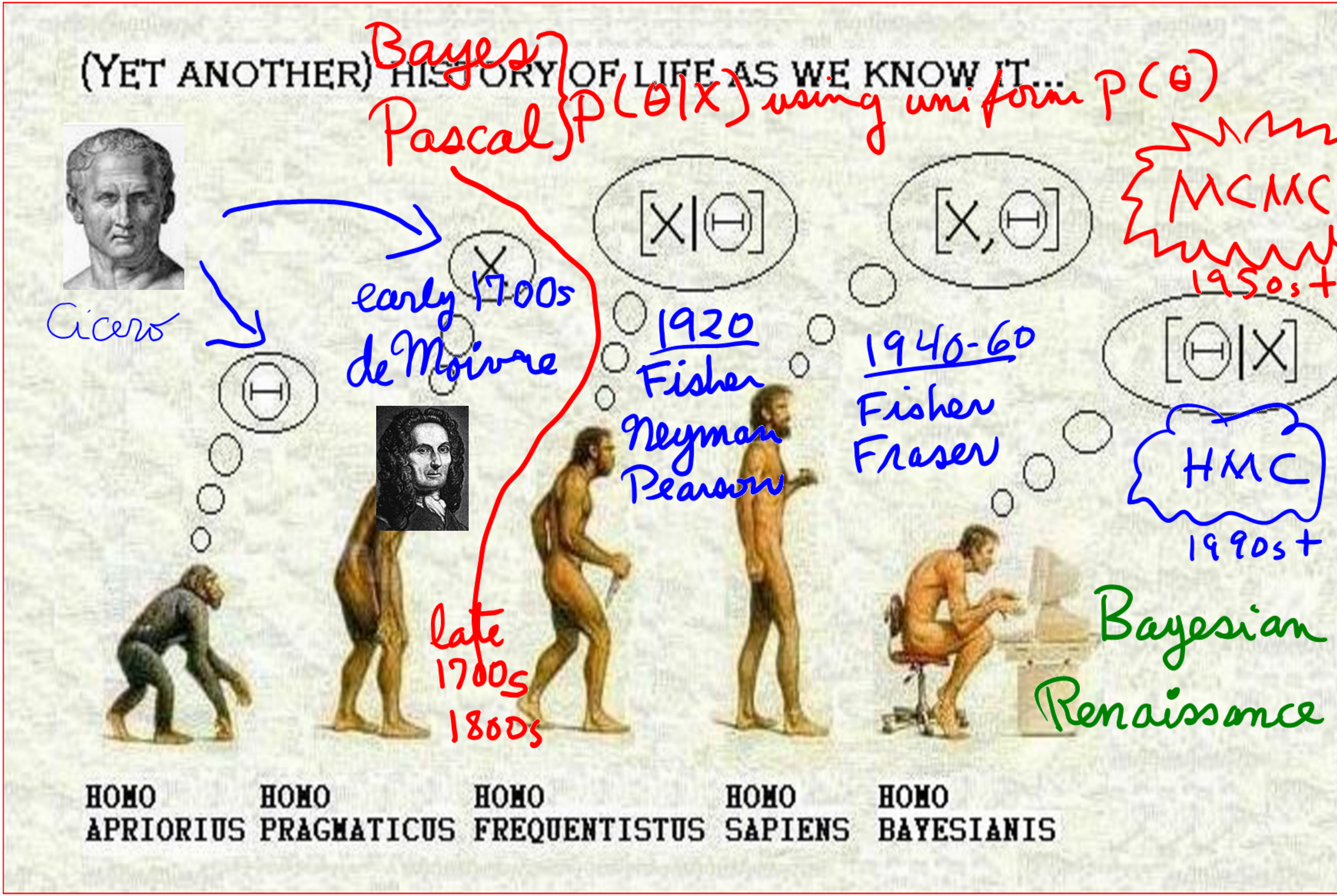
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Philosophical ~~Basic~~ problem

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Given a model $P(X|\theta)$

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To get $P(\theta|X)$

you need to be willing

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Then $P(X,\theta) = P(X|\theta)P(\theta)$

and $P(\theta|X) = \dfrac{P(X,\theta)}{P(X)}$

~~Basic~~ Philosophical problem

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Posterior

~~Basic~~ Philosophical problem

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Posterior

- You need a prior to get a posterior.
- Can we justify a particular prior?

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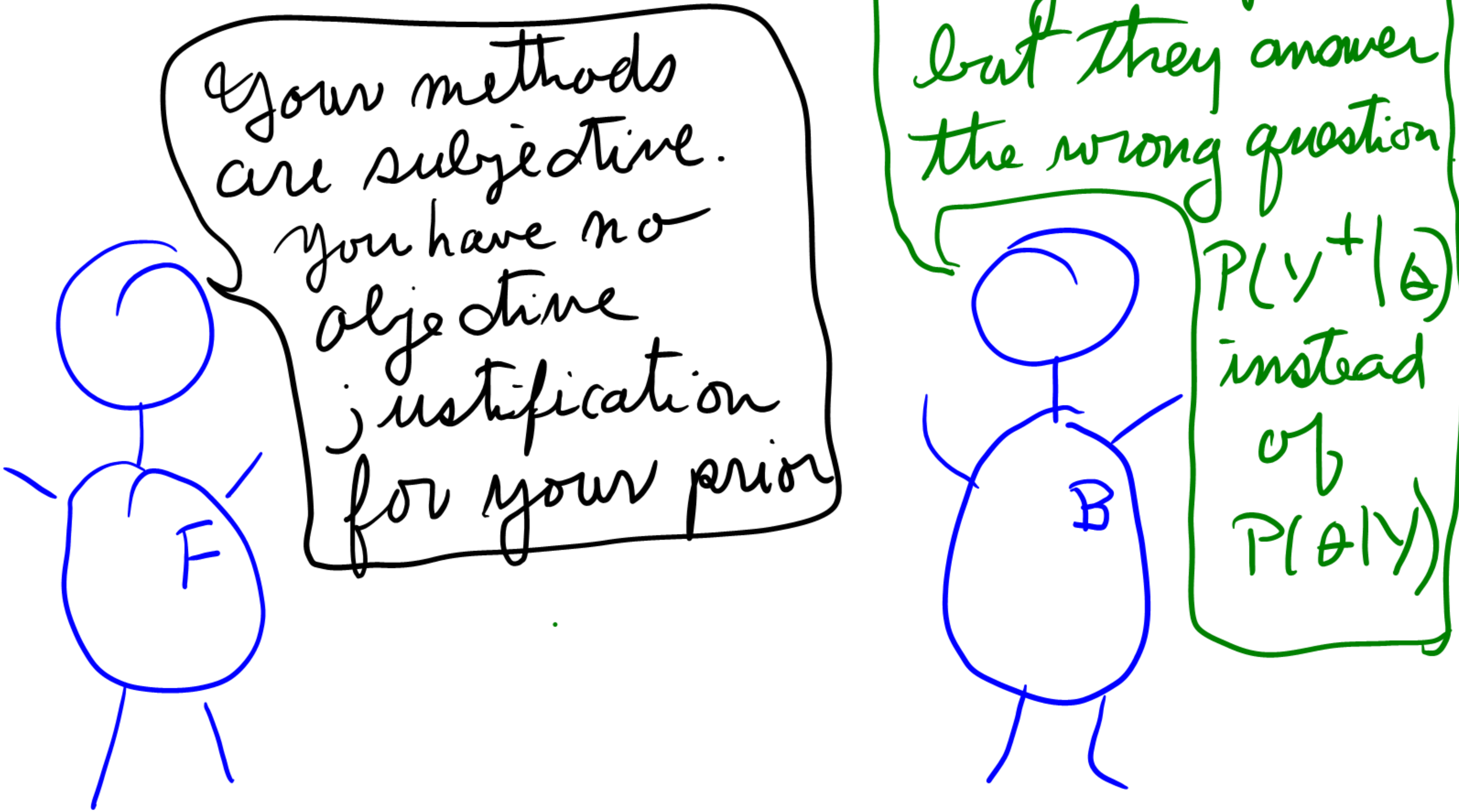

Frequentists only use $P(X|\theta)$ and don't need $P(\theta)$

Your methods are subjective. You have no objective justification for your prior

Your methods may be "objective" but they answer the wrong question $P(Y^+|\theta)$ instead of $P(\theta|Y)$

Practical problem:

$$p(X, \theta) = p(X \mid \theta)\, p(\theta)$$

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$$\int p(x, \theta)\, d\theta$$

If θ has high dimension this becomes easily impossible.

Practical problem:

$$p(X, \theta) = p(X|\theta)\, p(\theta)$$

$$p(\theta|X) = \frac{p(X, \theta)}{p(X)}$$

MCMC (mid 20th C.) comes to the rescue:
It's possible to sample from $p(\theta|X)$ knowing only $p(X, \theta)$

Posteriors without priors?

Fisher - Fiducial inference
Fraser - Structural inference
Objective Bayesian inference

Baking the Bayesian omelette without breaking the Bayesian egg.

Emerging practice:
Using weakly informative priors.

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The reason why thinking of a 95% CI for a linear regression parameter as a 95% "probability" interval is harmless.

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The reason why thinking of a 95% CI for a linear regression parameter as a 95% "probability" interval is harmless.

Emerging practice:
Using weakly informative priors.

But don't take it for granted
Remember Sally Clark

Improper vs proper priors
posteriors.
Problems
HMC

Markov Chain Monte Carlo

Use $\quad p(\theta, X) = p(X \mid \theta)\, p(\theta)$

Markov Chain Monte Carlo

Use $P(\theta, X) = P(X|\theta)\, P(\theta)$

joint model × prior

Samples from $P(\theta|X)$ using only $P(\theta, X)$

i.e. no need to find elusive $P(X)$

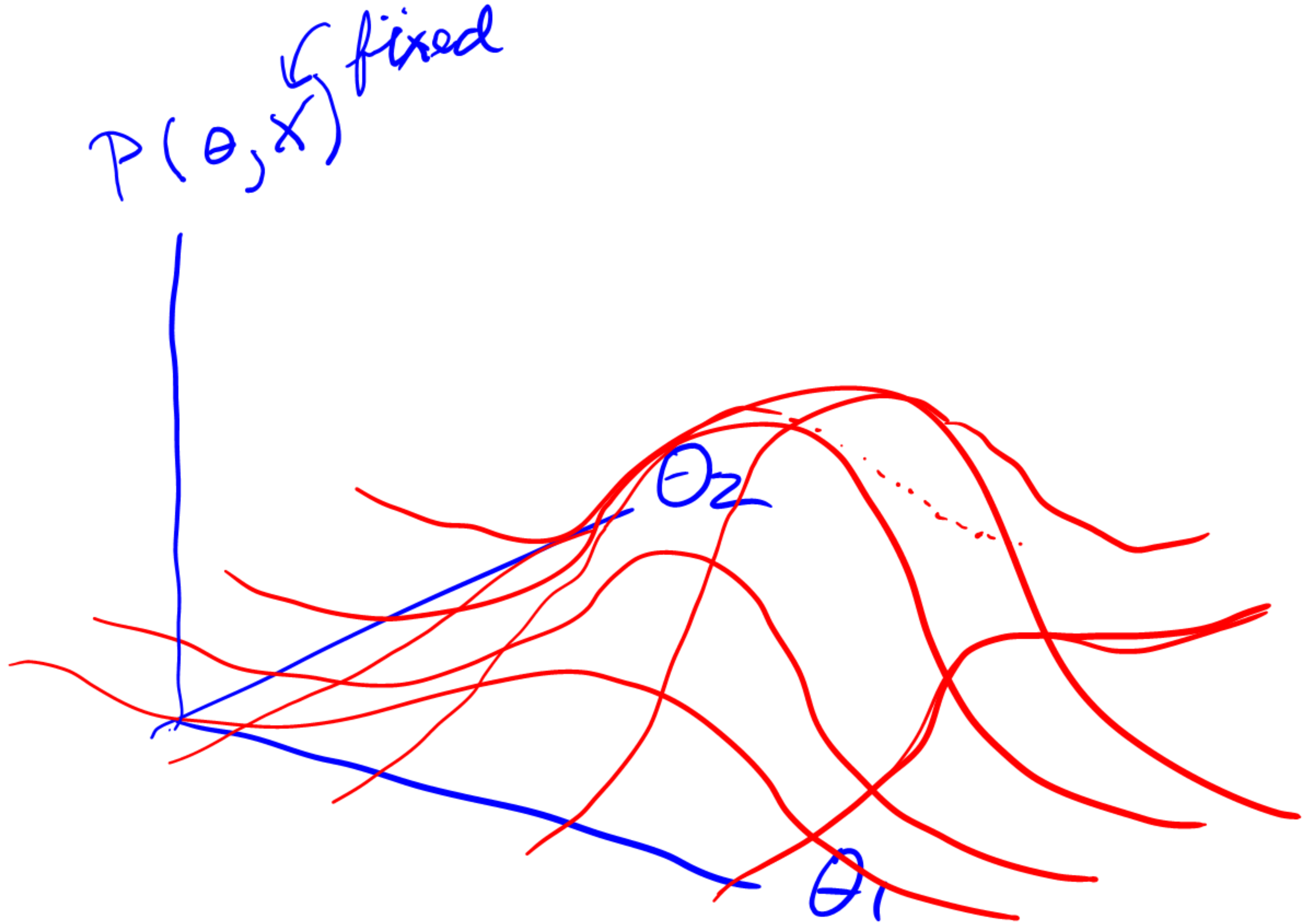
Markov Chain Monte Carlo

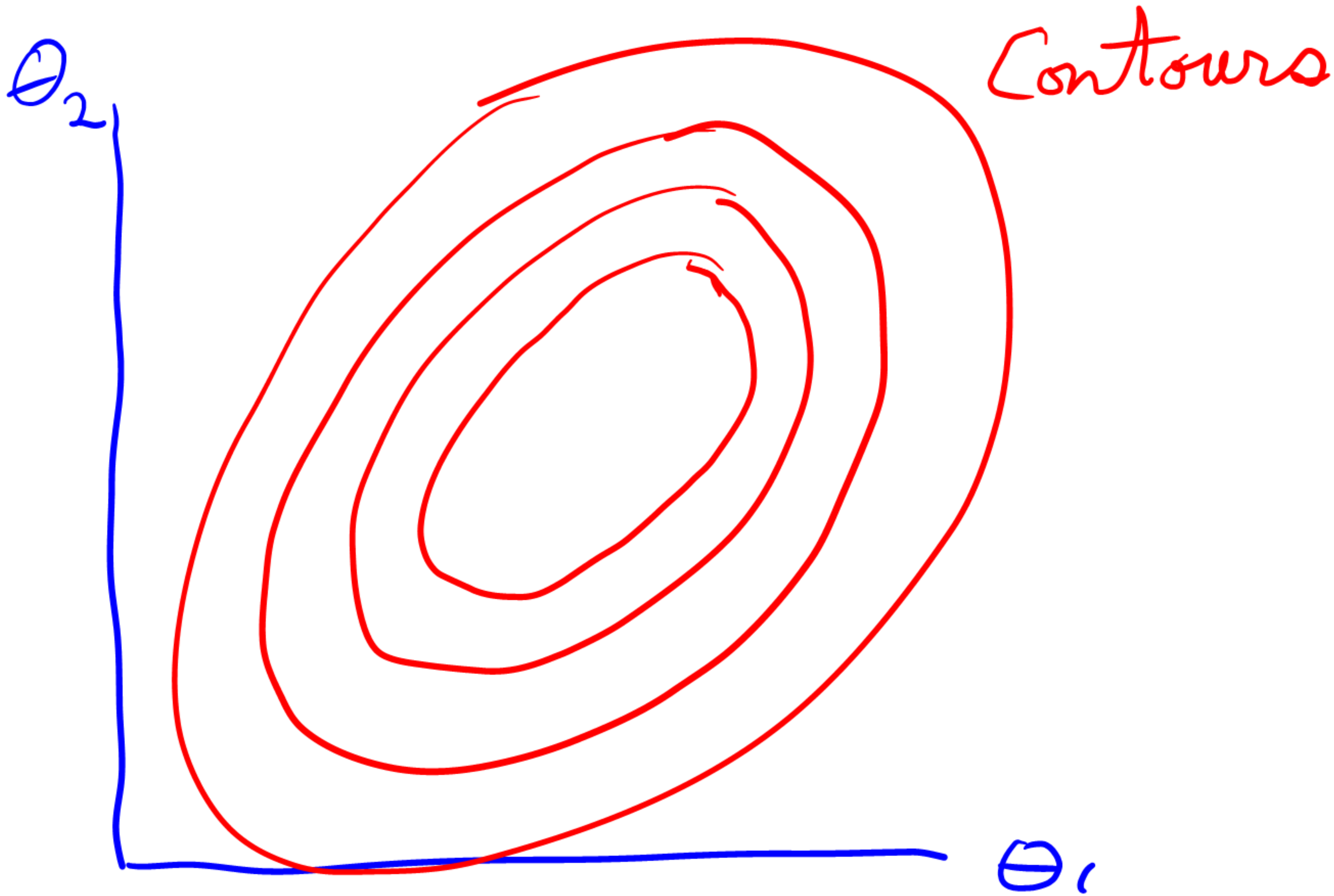
Use $\underbrace{P(\theta, X)}_{\text{joint}} = \underbrace{P(X|\theta)}_{\text{model}} \times \underbrace{P(\theta)}_{\text{prior}}$

Samples from $P(\theta|X)$ using only $P(\theta, X)$

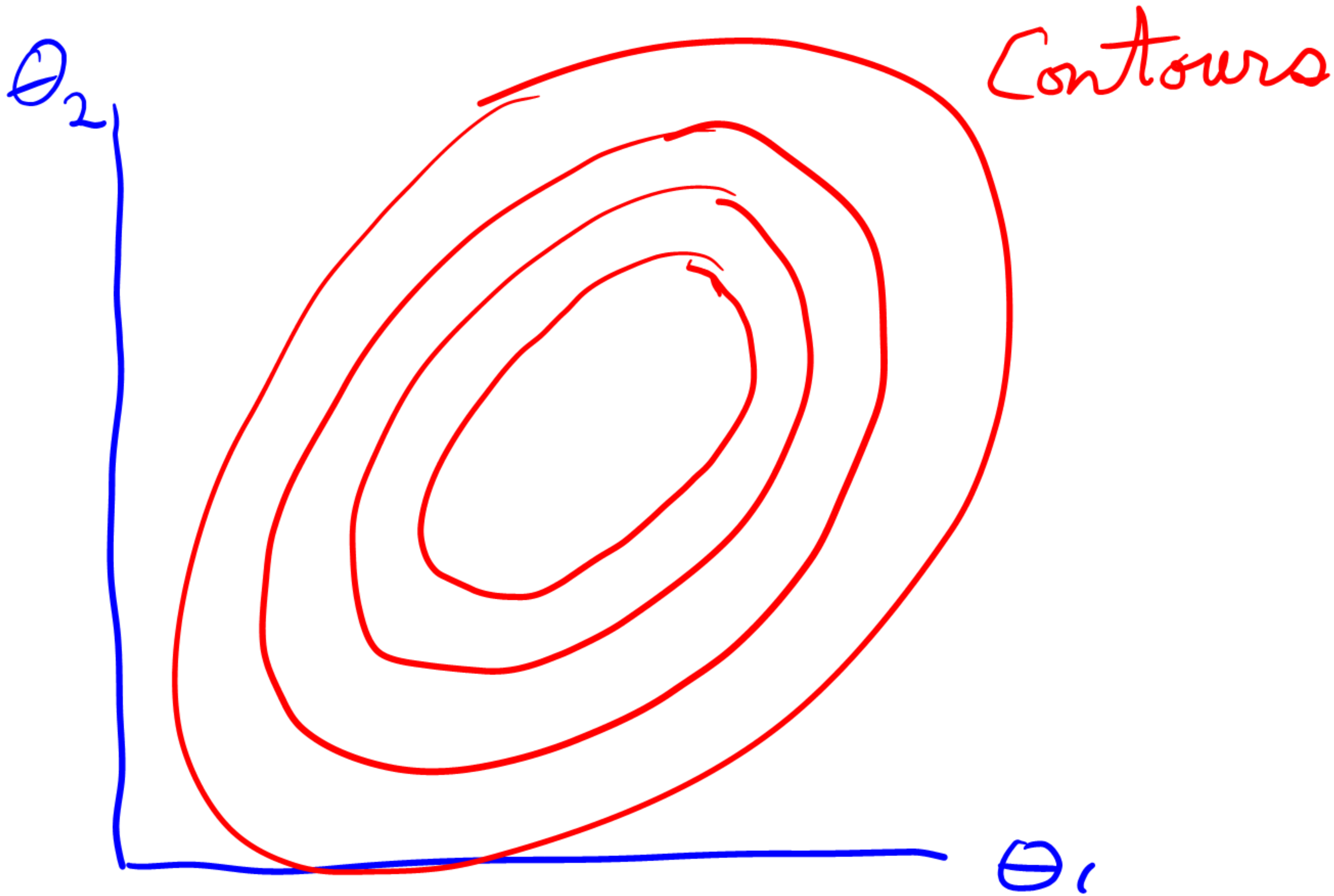
i.e. no need to find elusive $P(X)$

With X fixed, think of $P(\theta, X)$ as defining a mountain over θ space

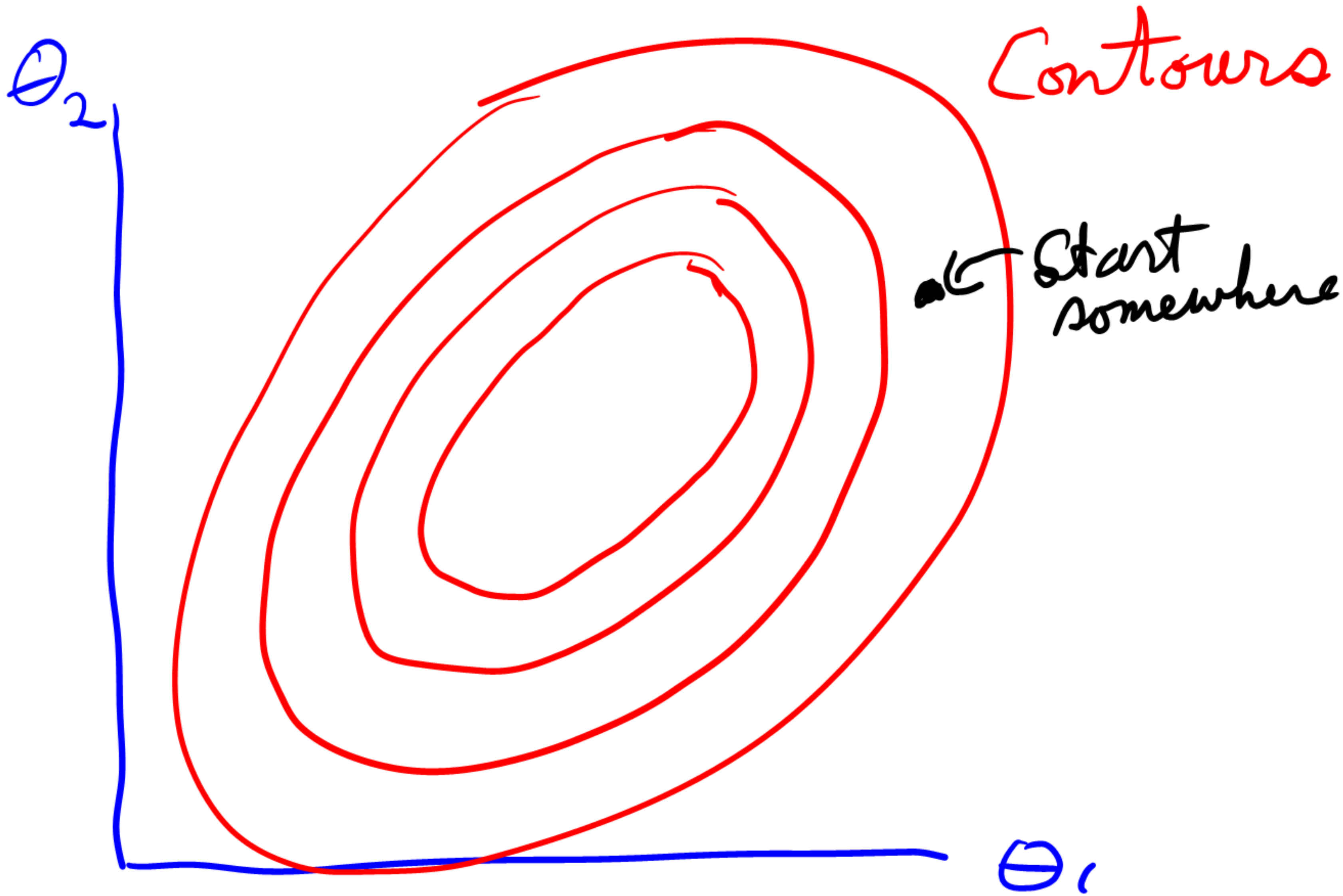

$P(\theta, x)$
fixed
θ_2
θ_1



Contours
θ_2
θ_1



Metropolis-Hastings algorithm



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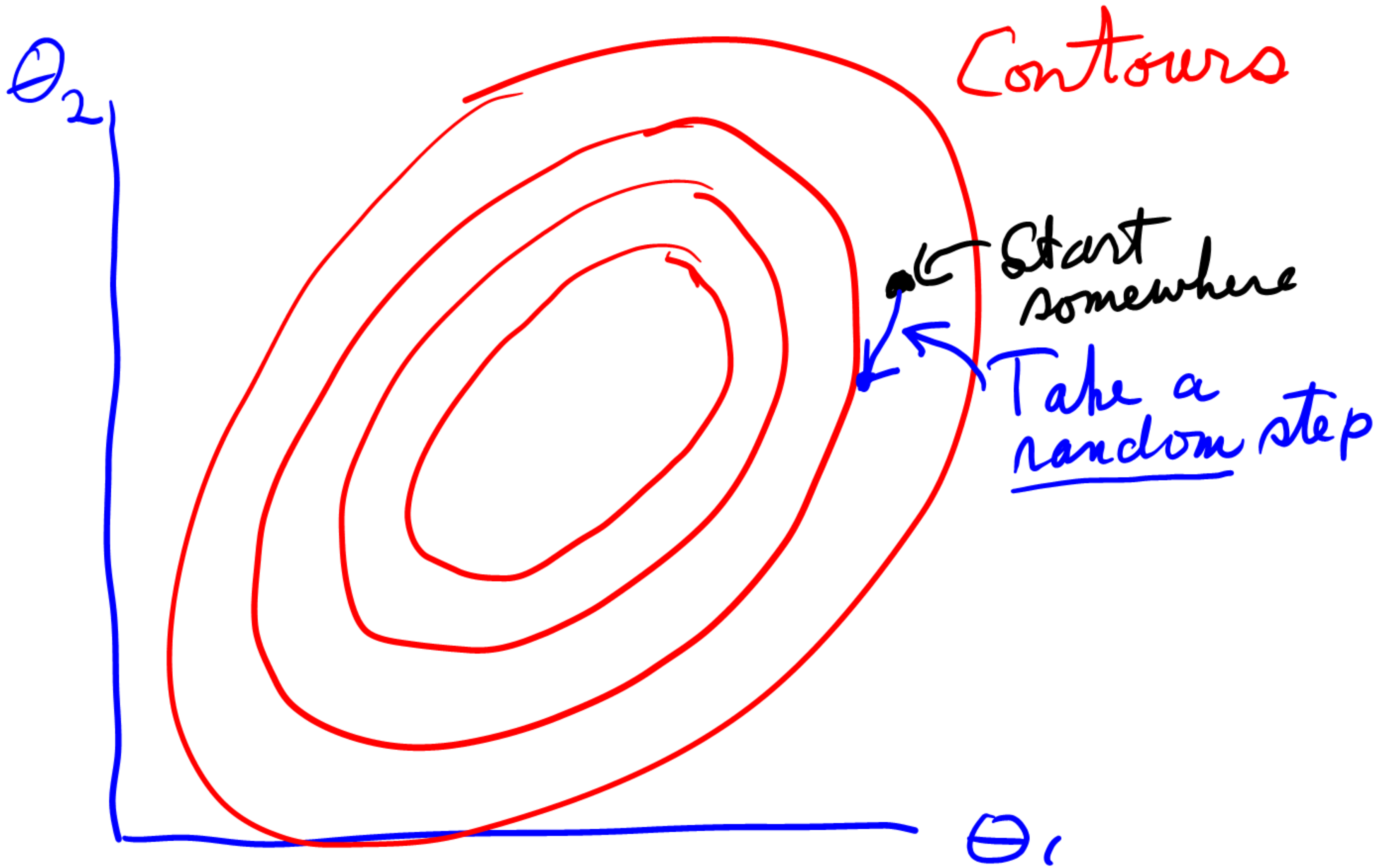

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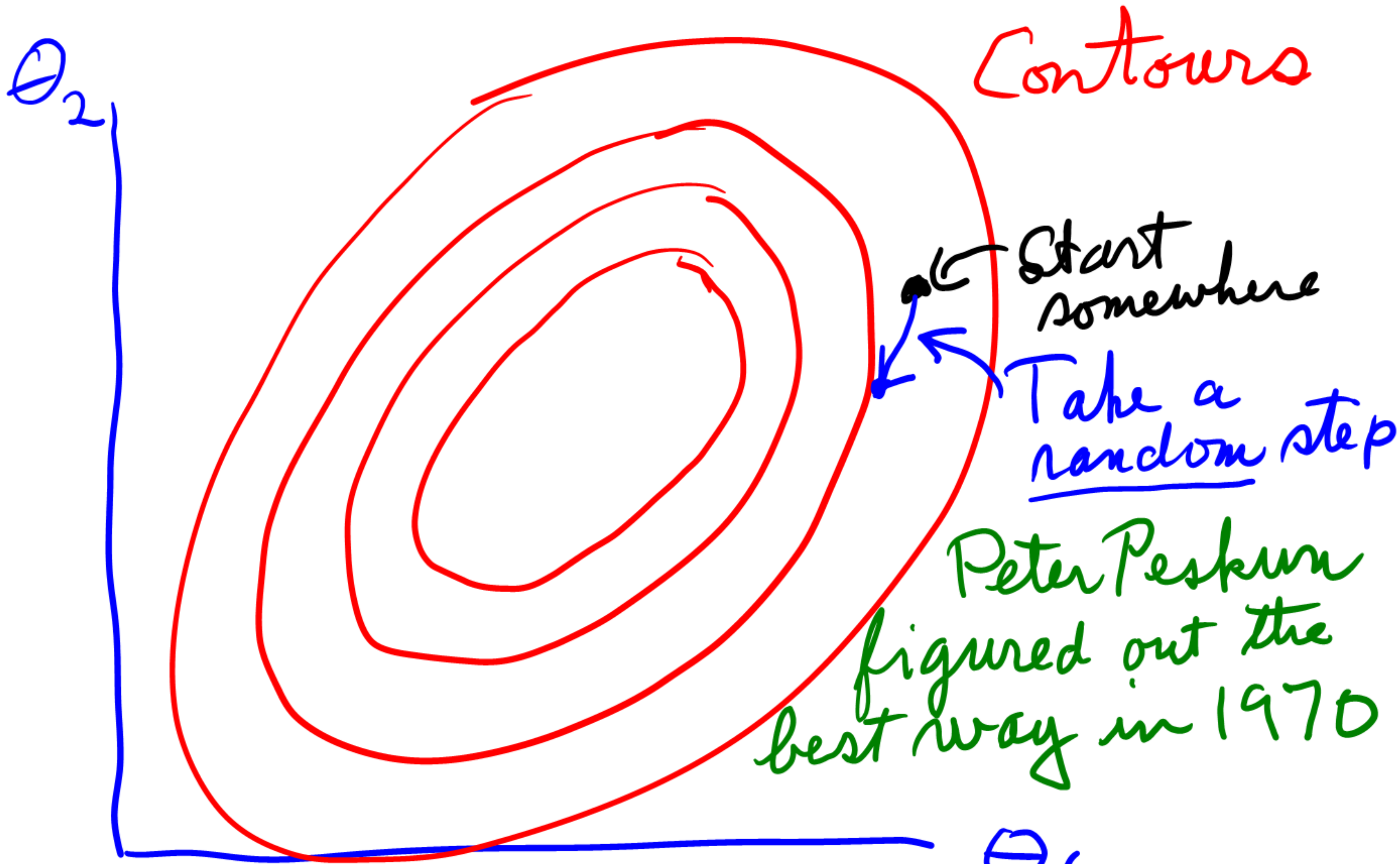

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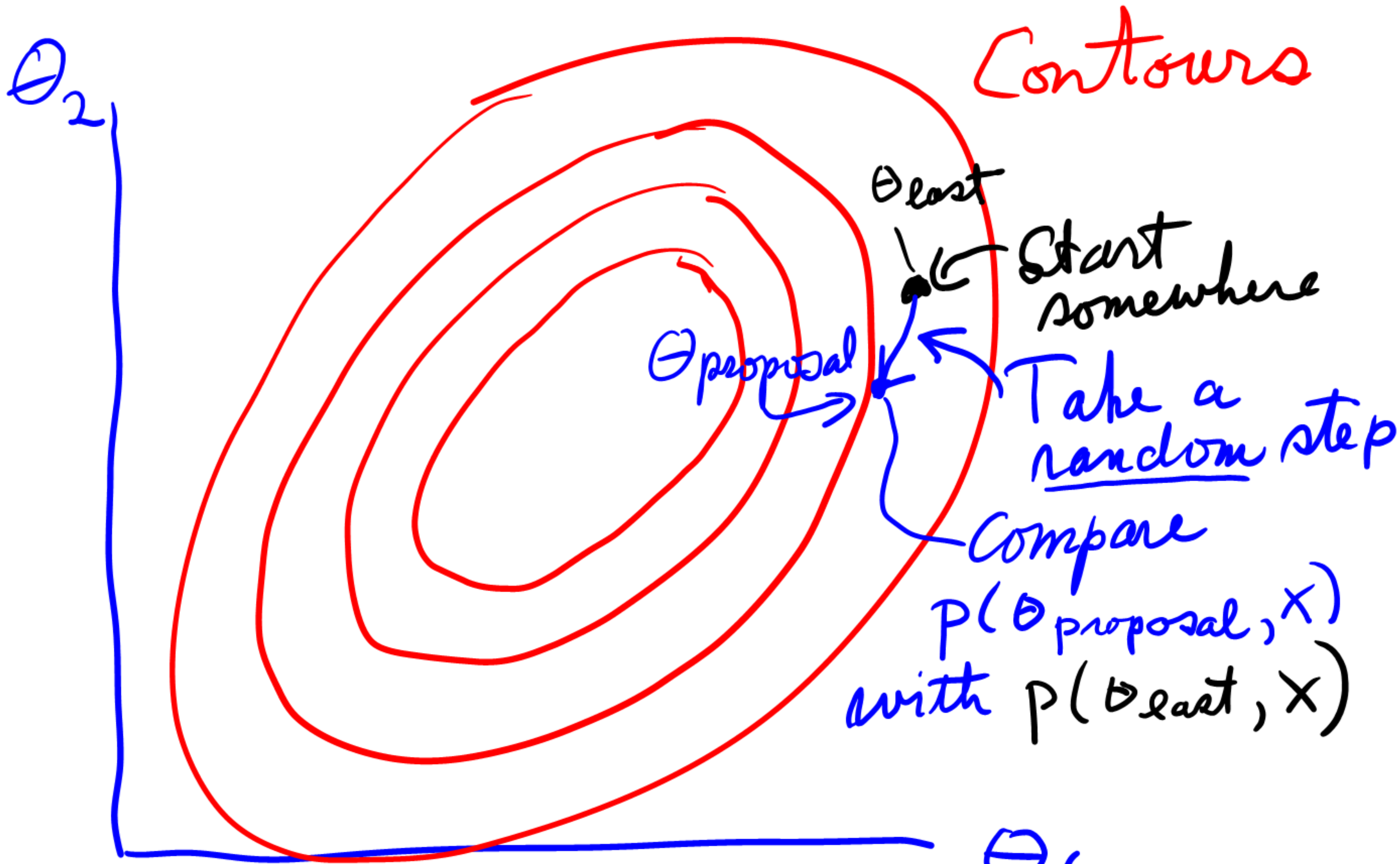

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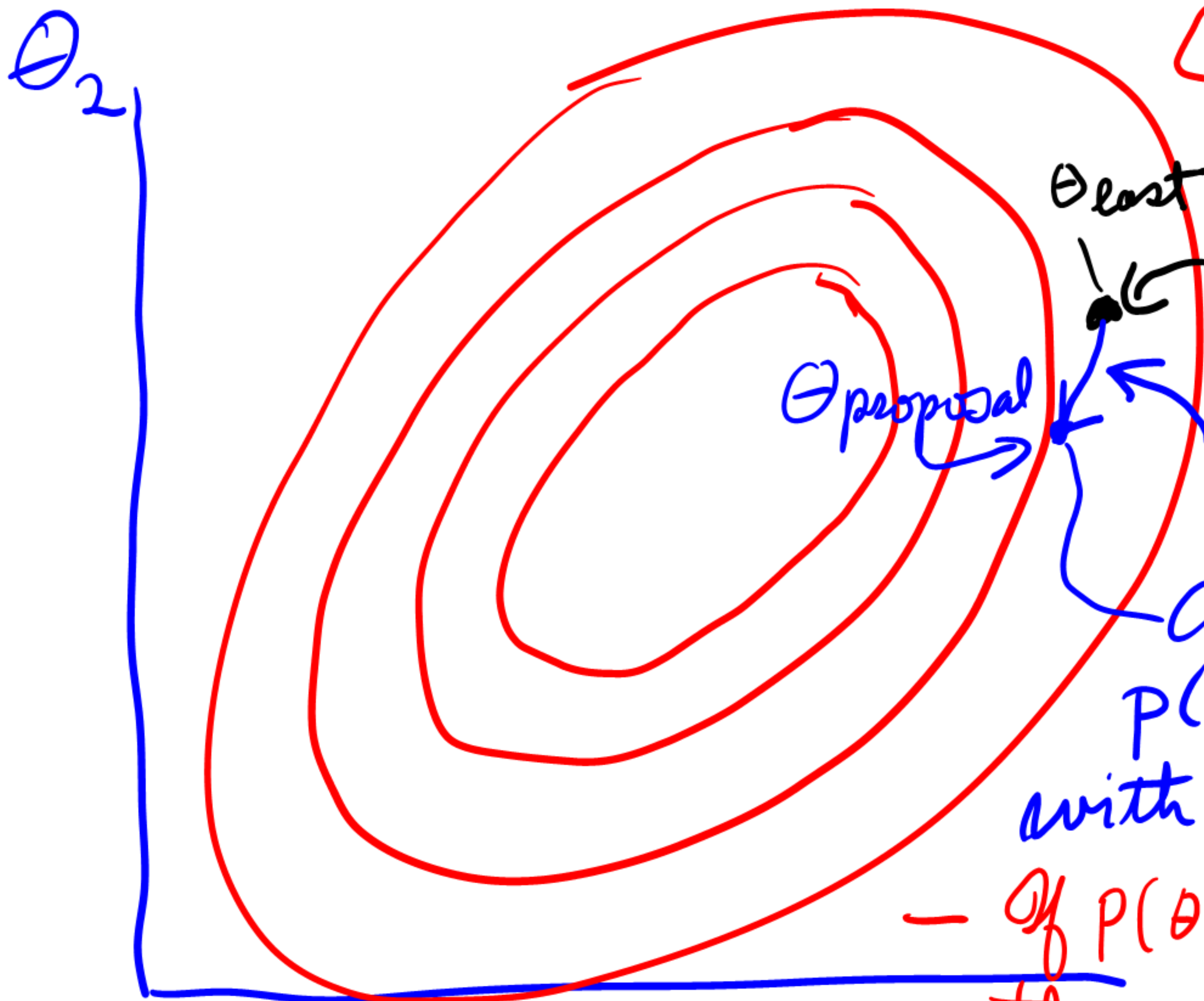


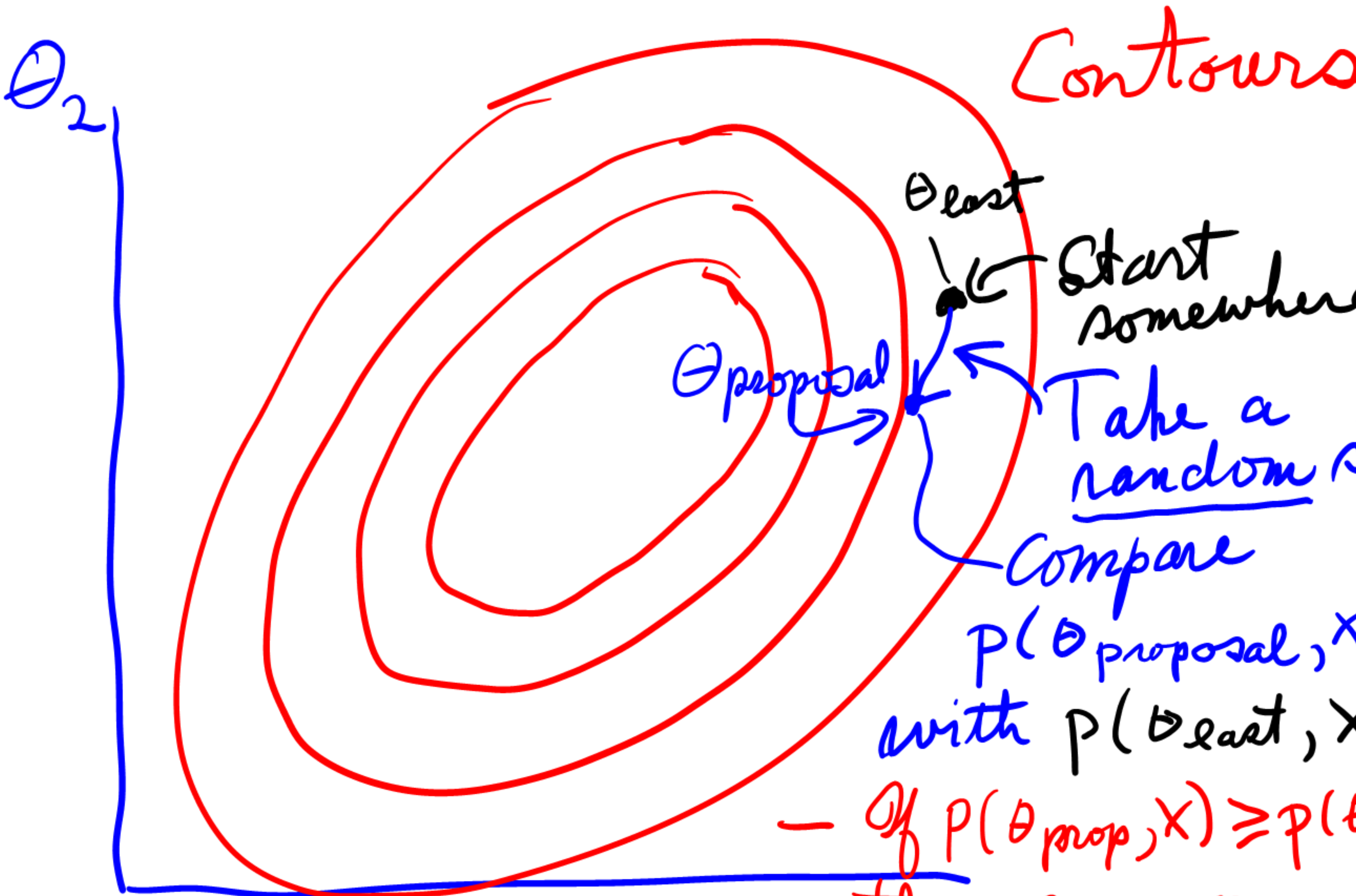
Contours

Start somewhere

Take a <u>random</u> step

Compare $p(\theta_{proposal}, x)$ with $p(\theta_{last}, x)$

– If $p(\theta_{prop}, x) \geq p(\theta_{last}, x)$ then $\theta_{new} = \theta_{proposal}$



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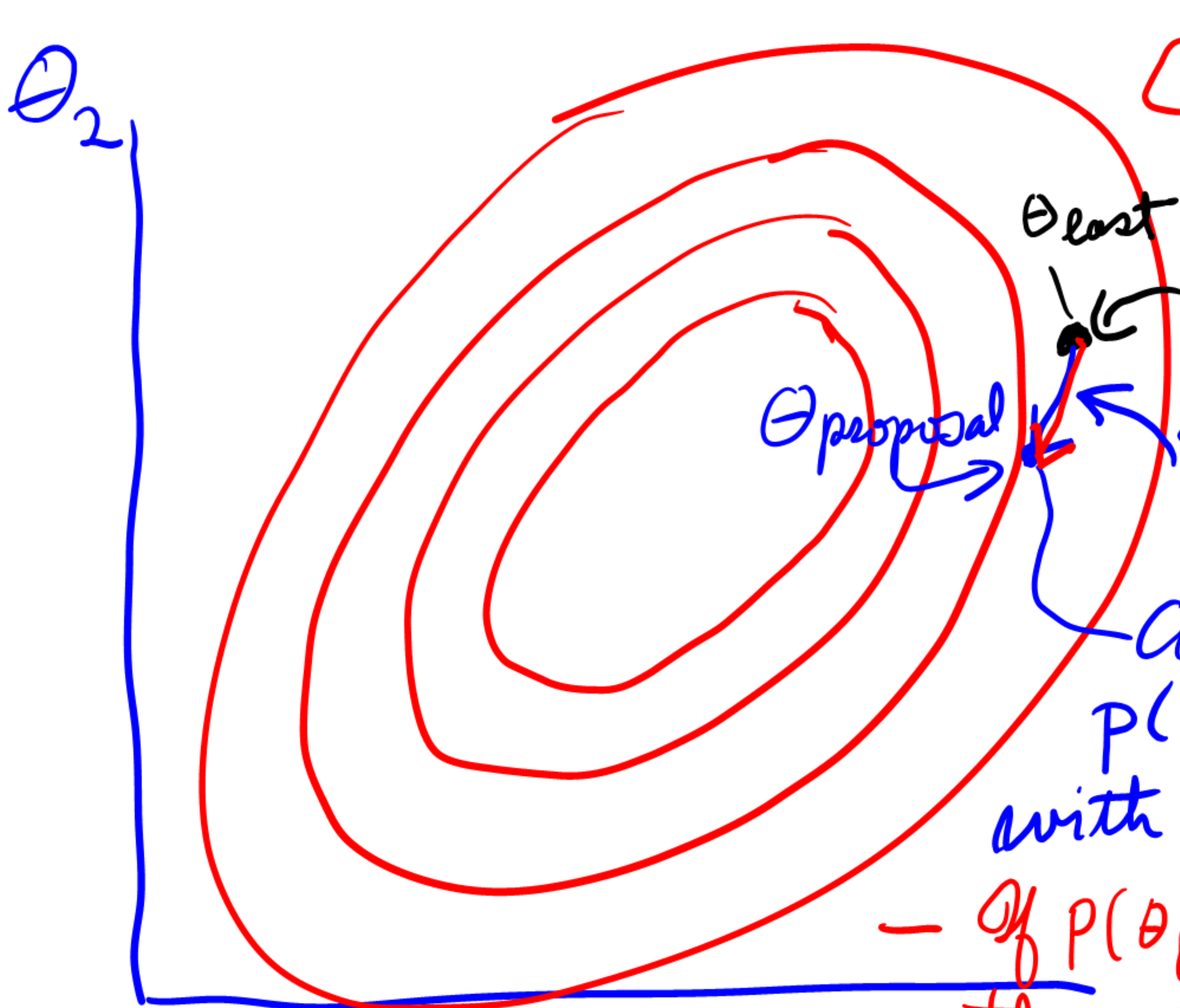

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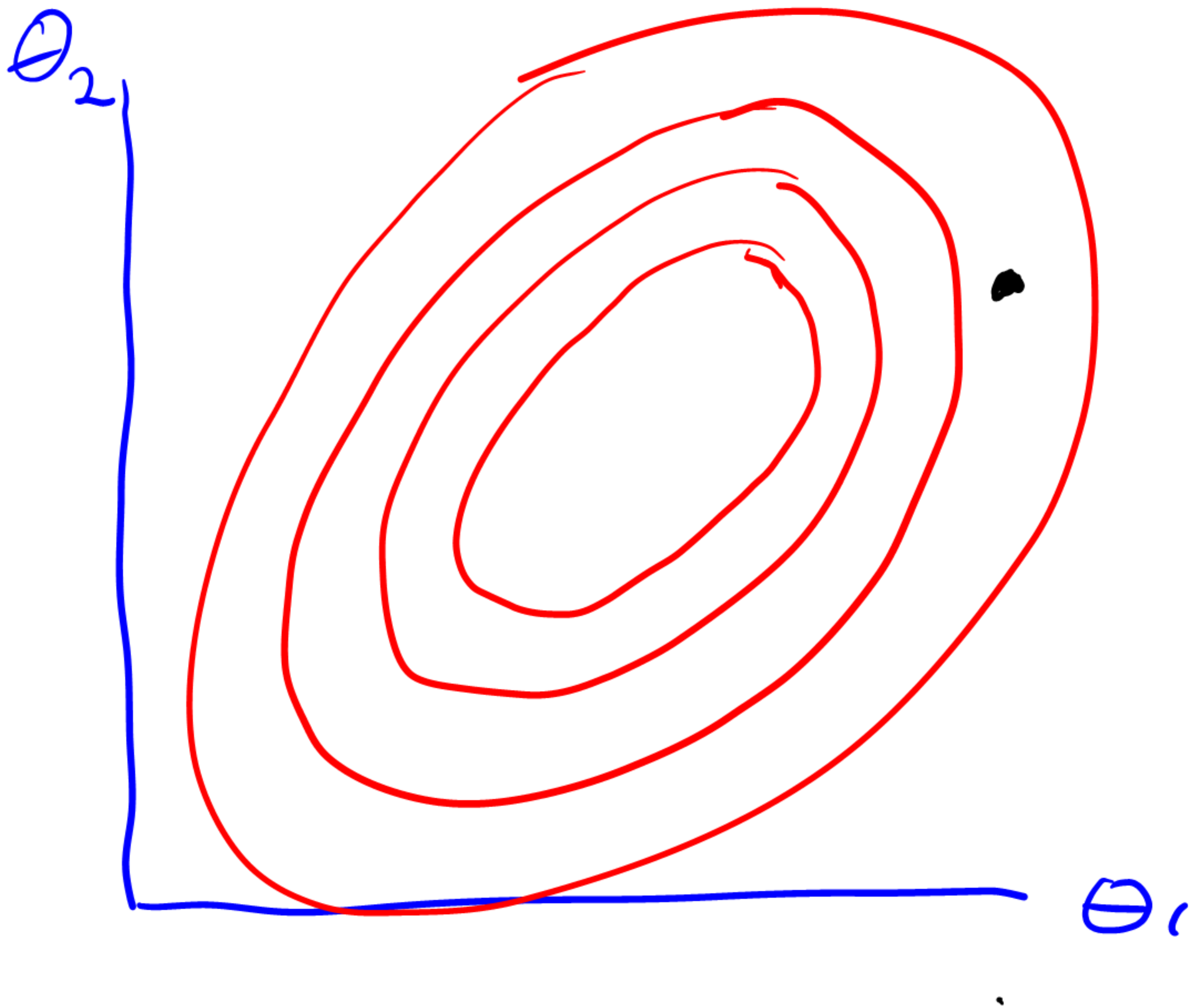
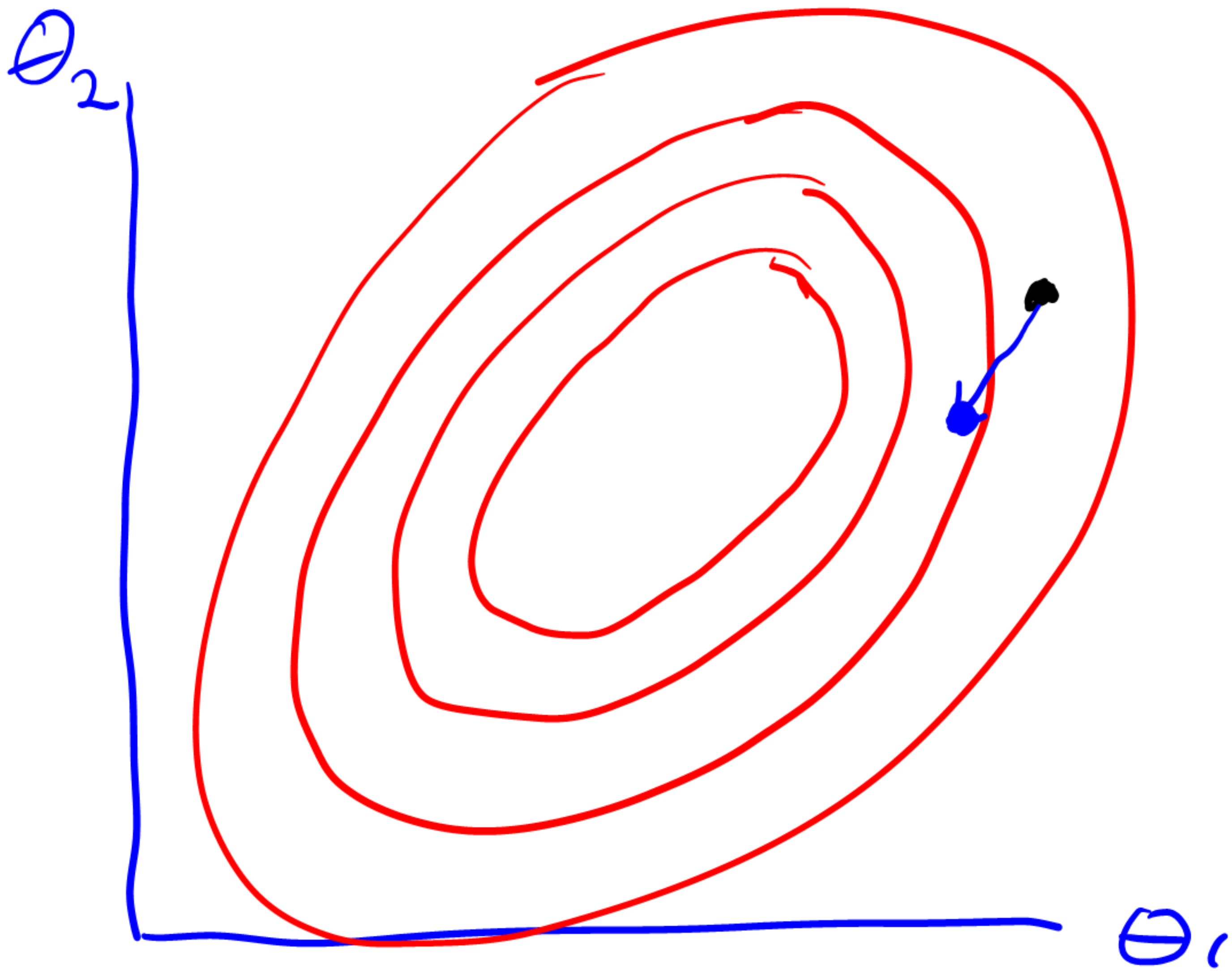
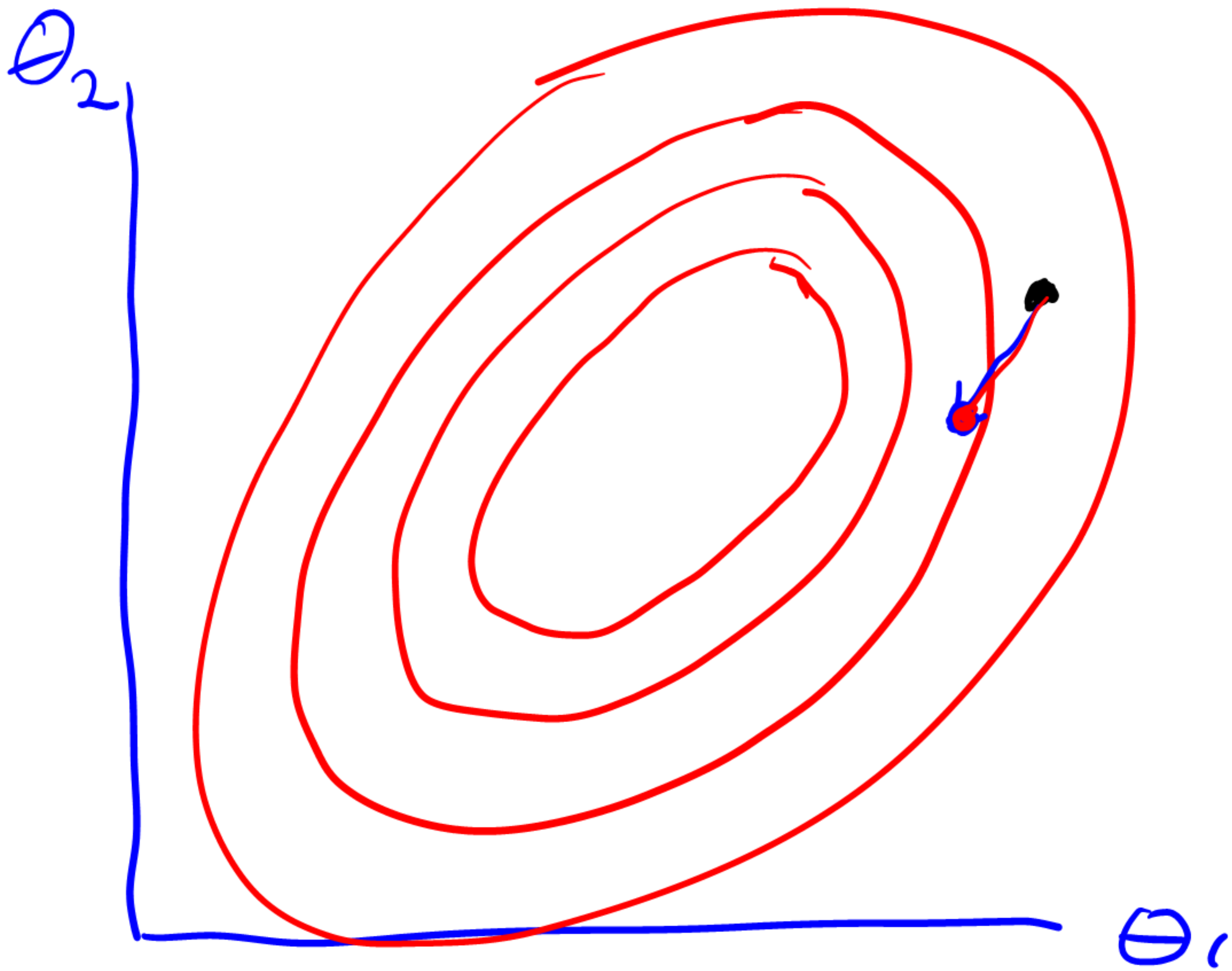

θ_2
θ_1



θ_2
θ_1



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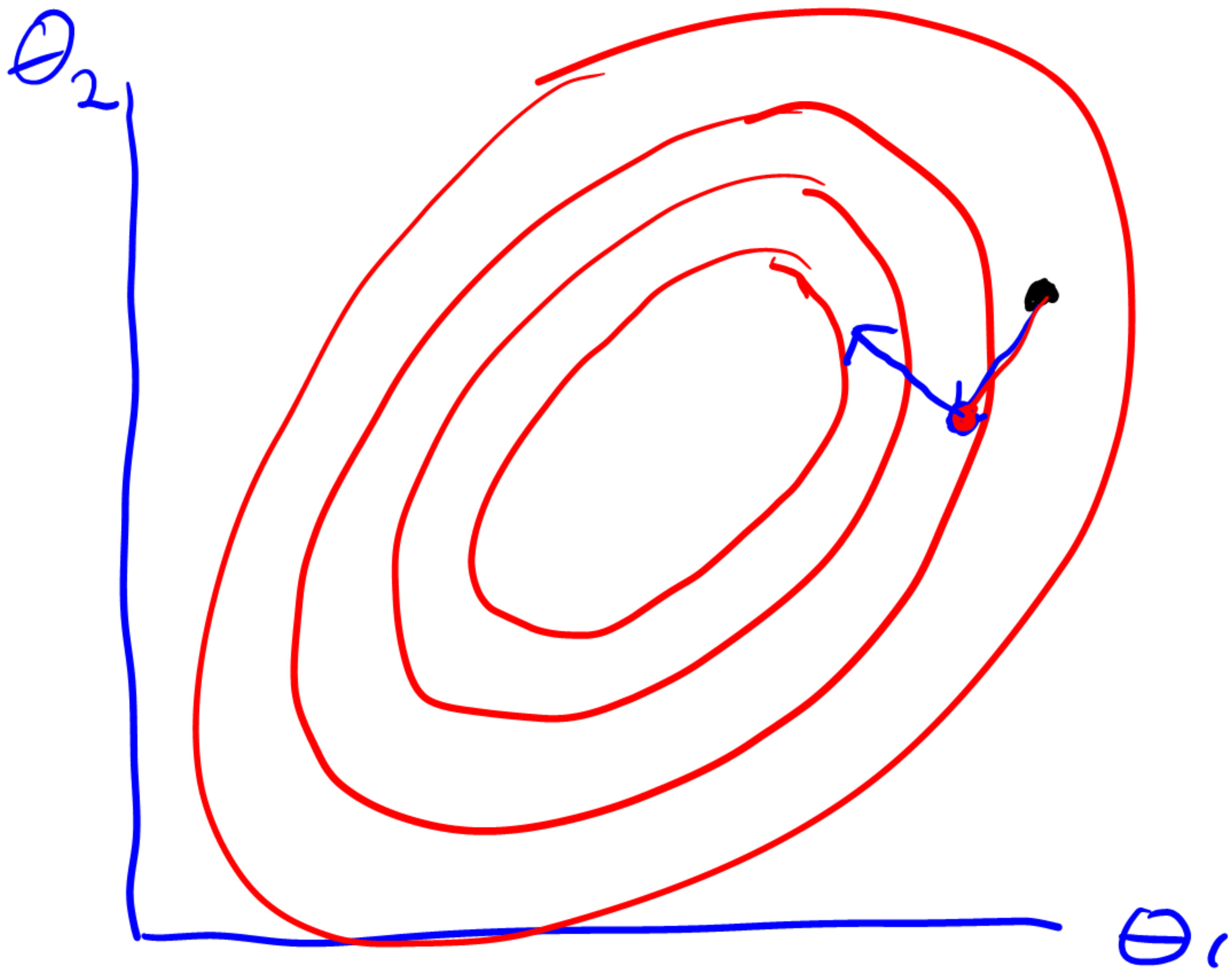

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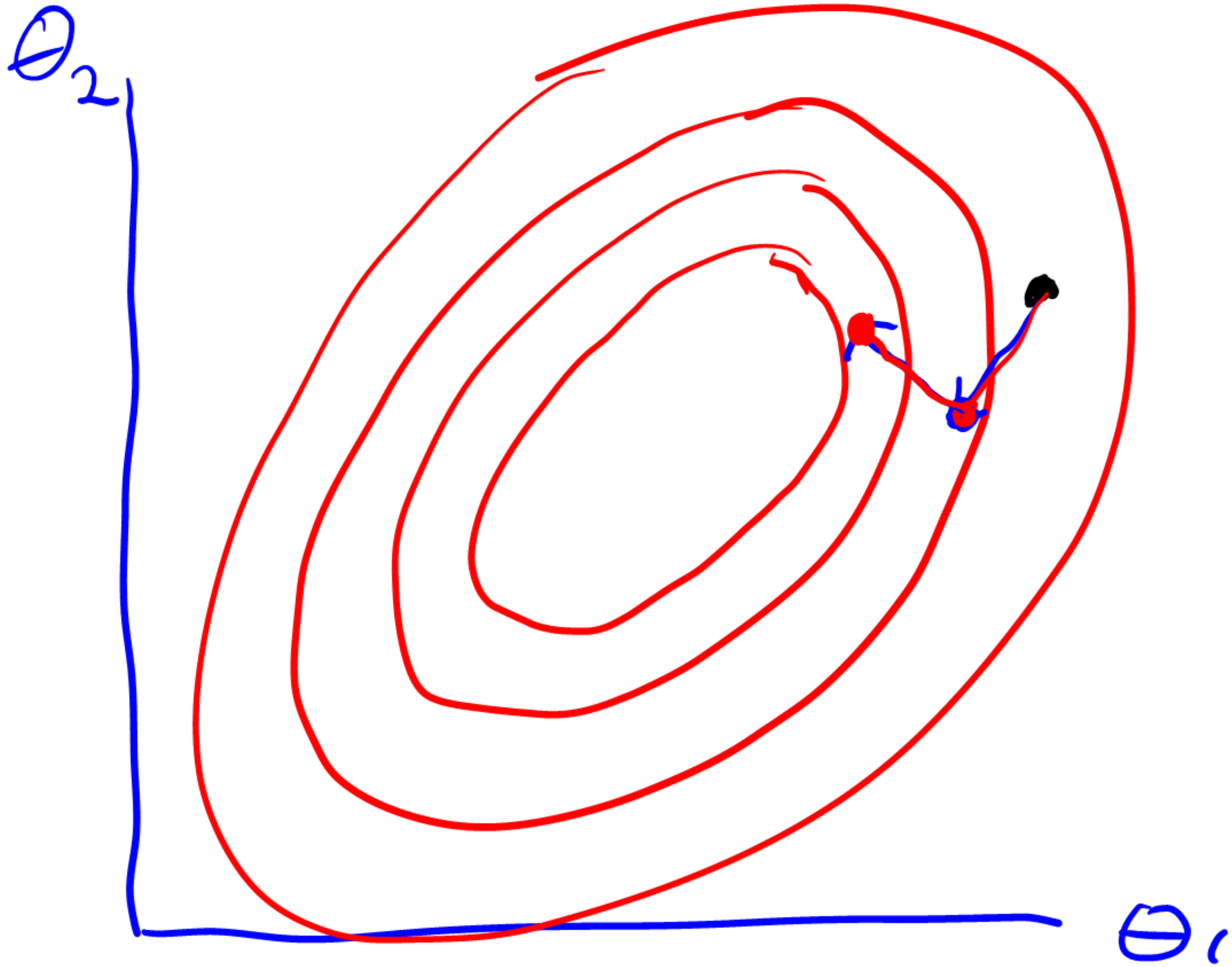

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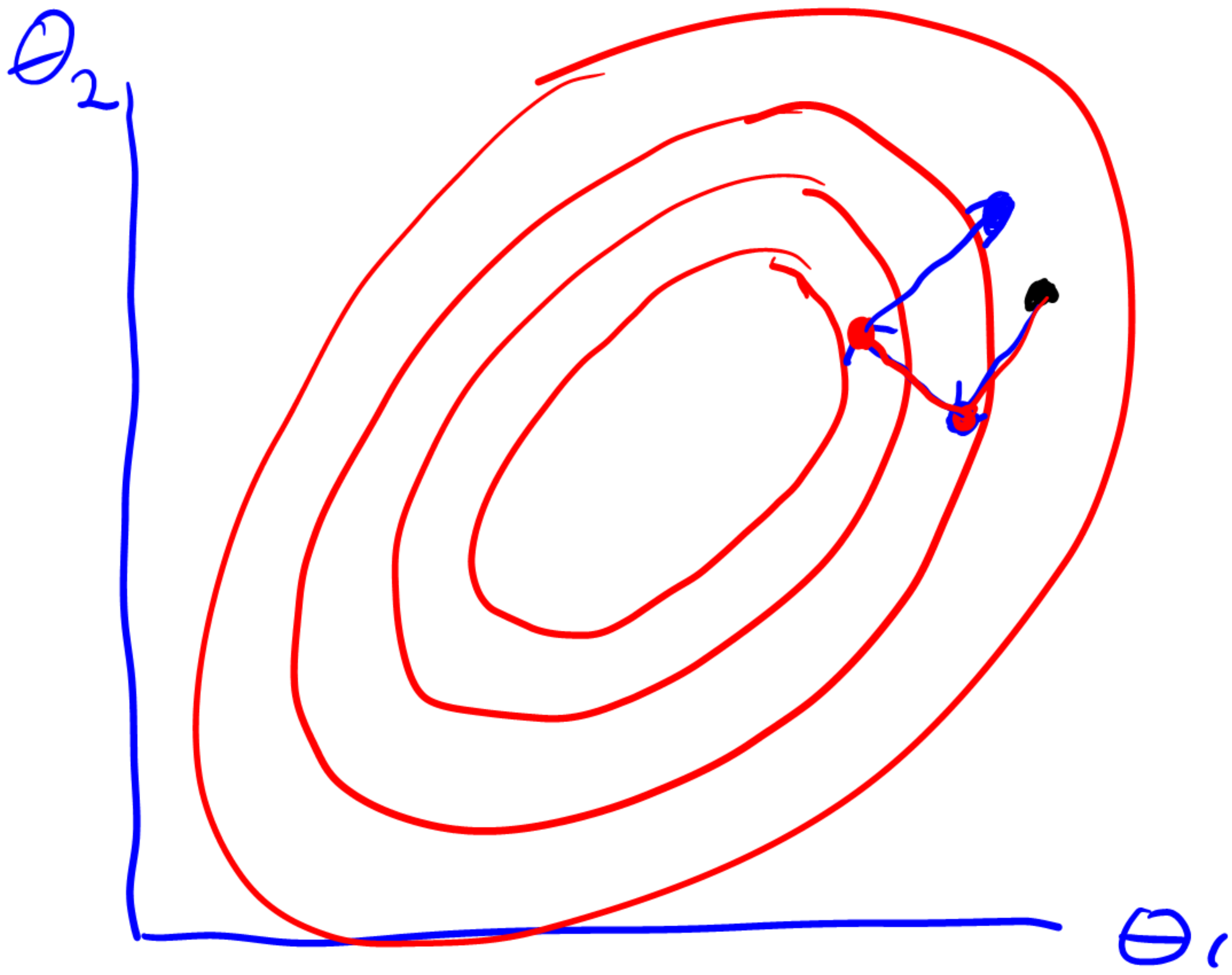

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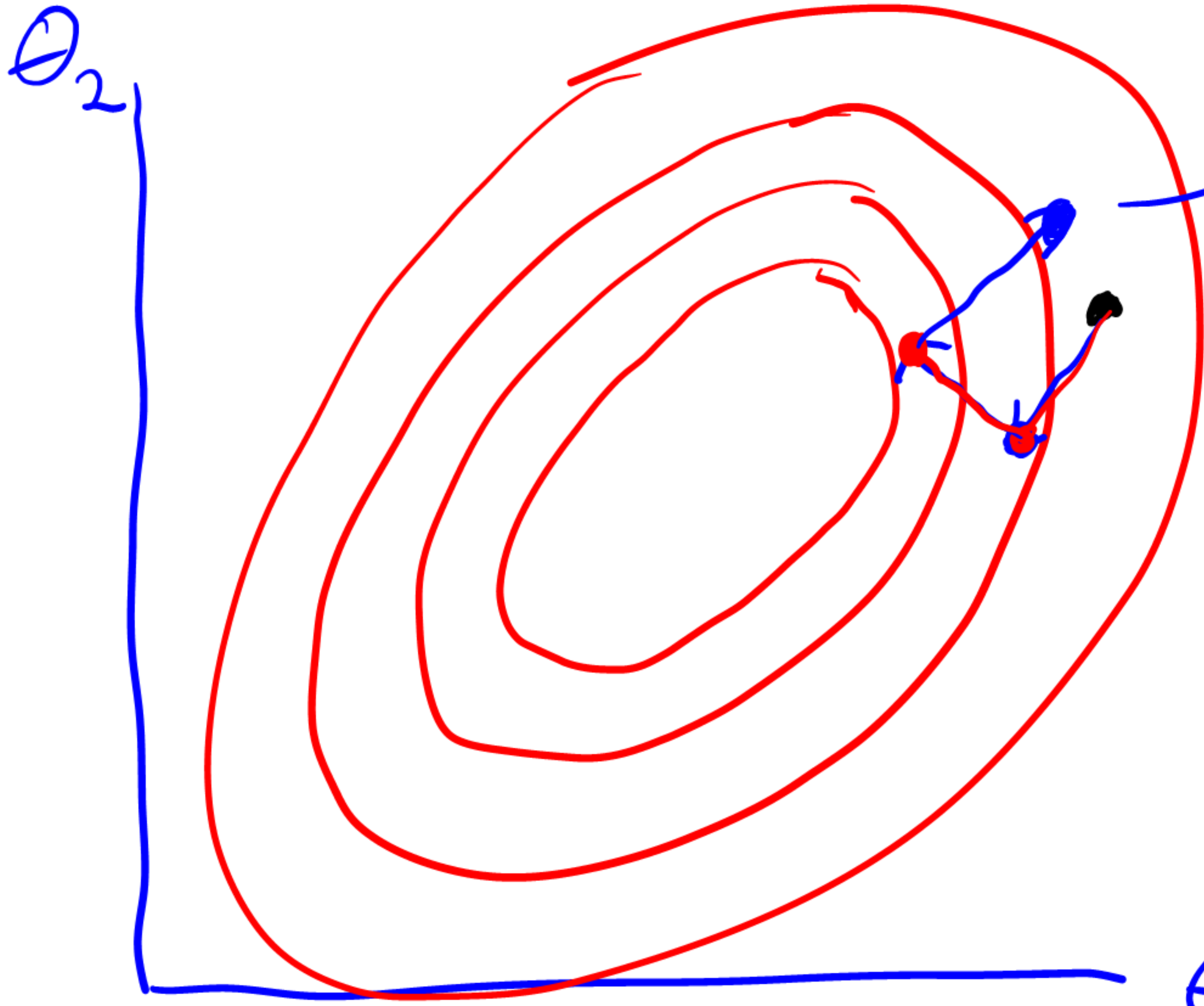
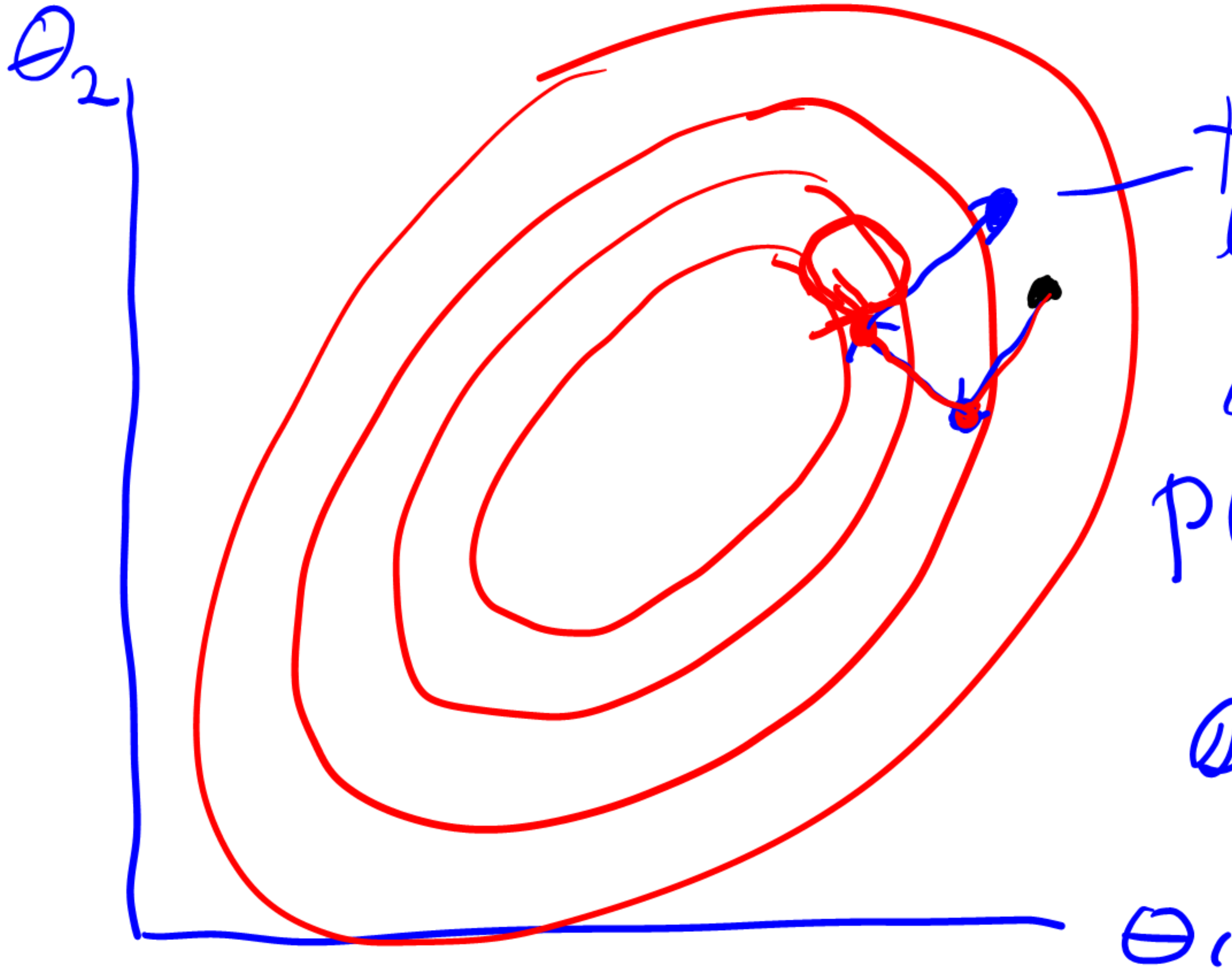


toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move

if Tails stay



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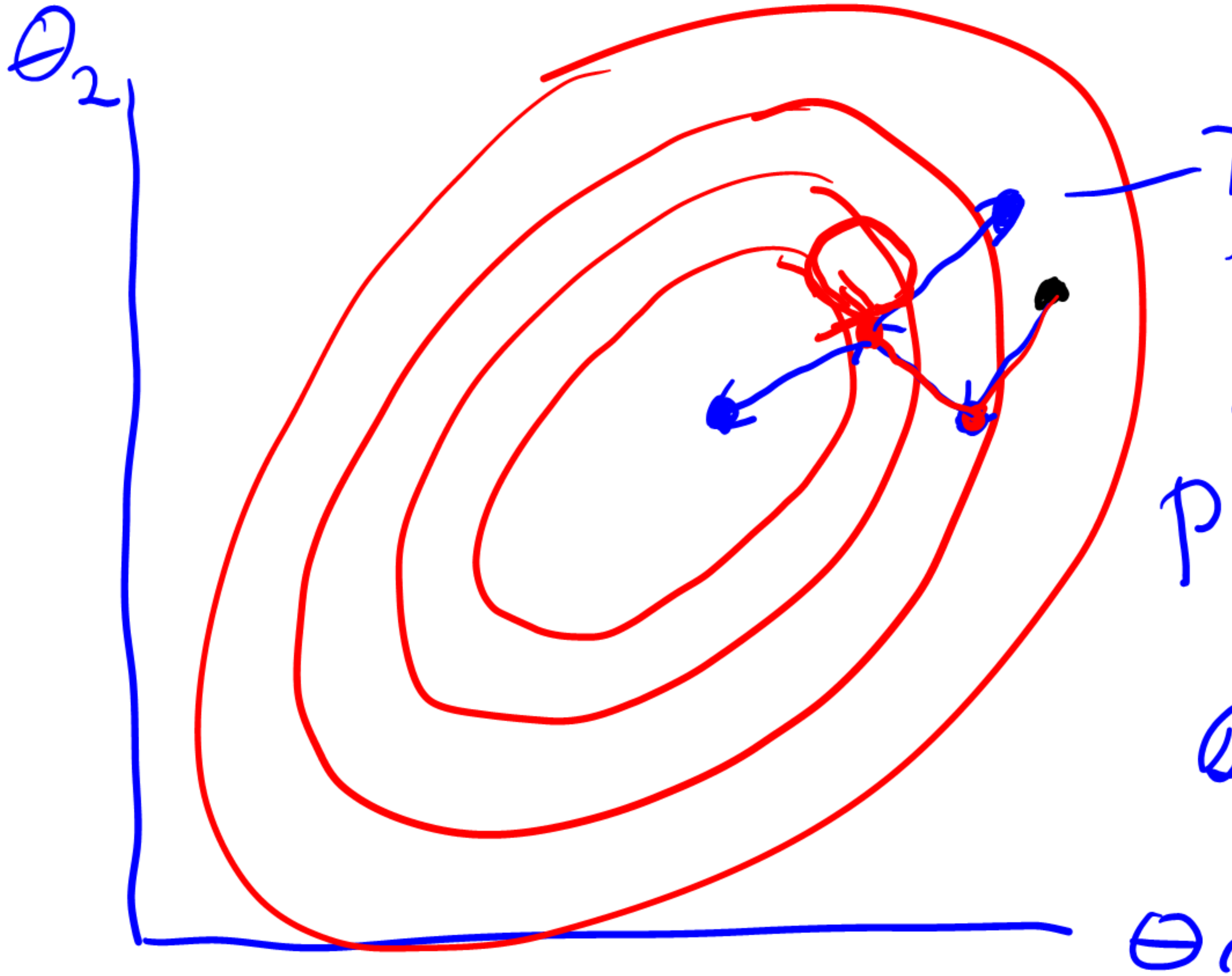

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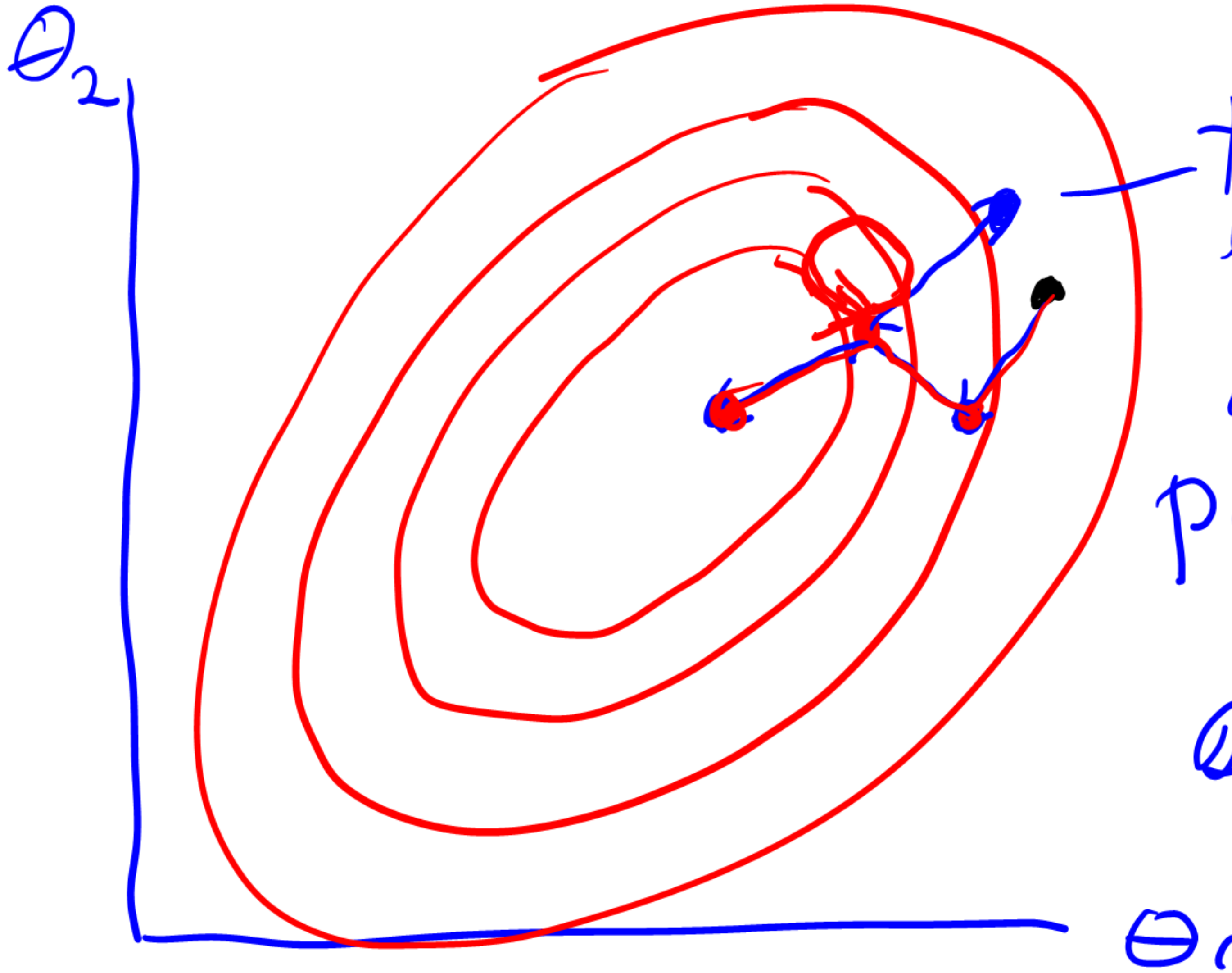
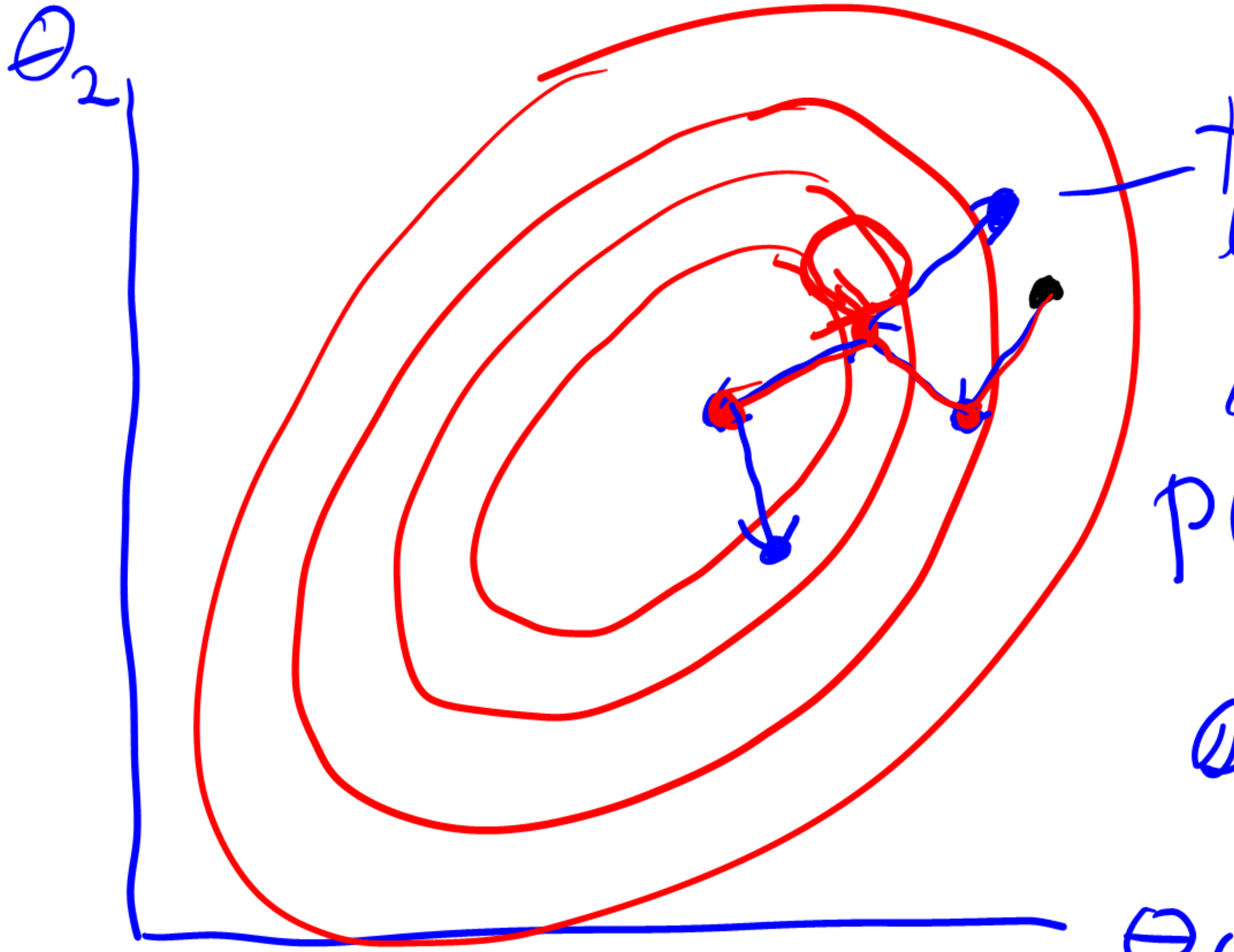
if Tails stay



toss a biased coin with

$$P(H) = \frac{P(\bullet)}{P(\bullet)}$$

If Heads move
If Tails stay



toss a biased coin with

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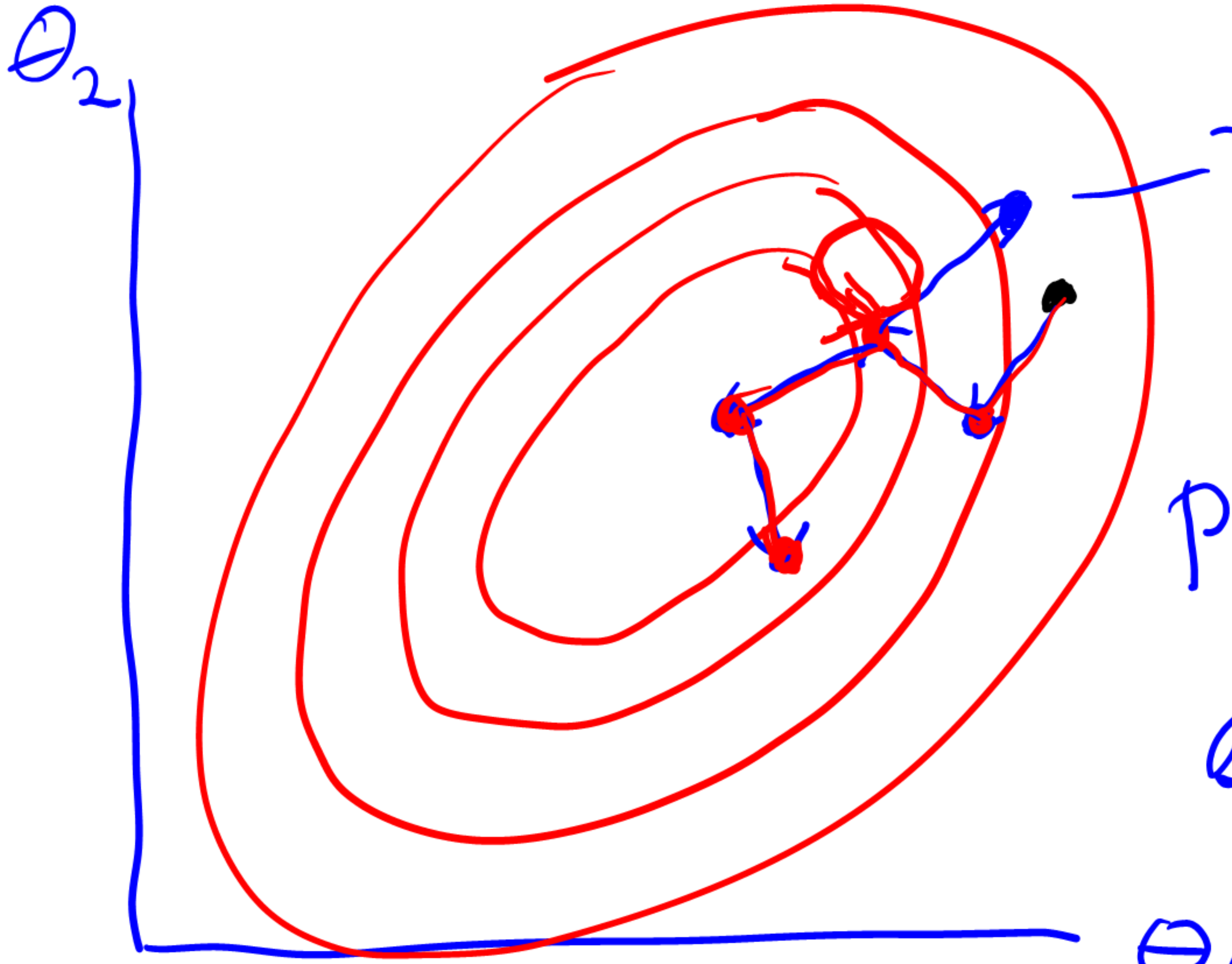
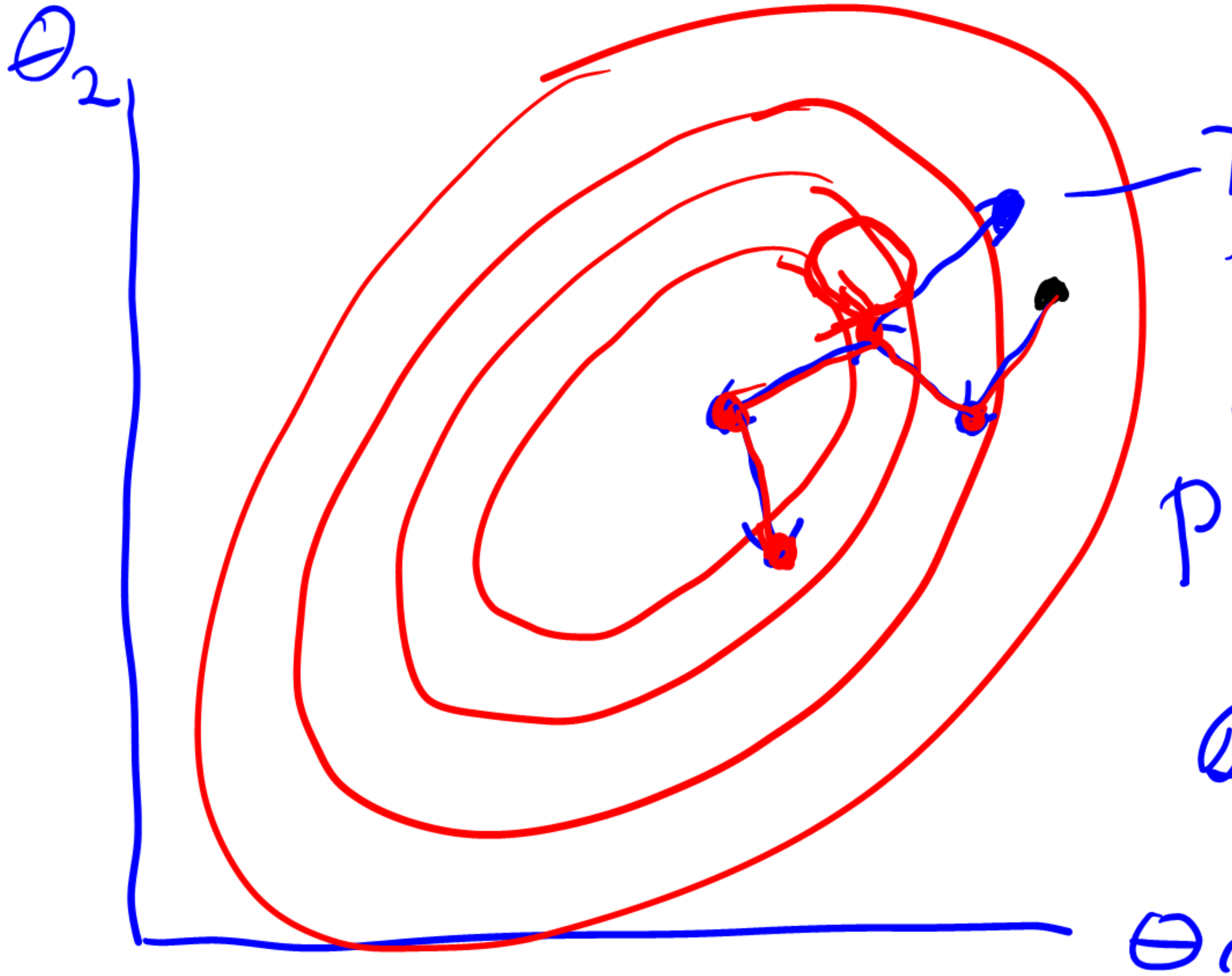
If Heads move
if Tails stay



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move
if Tails stay



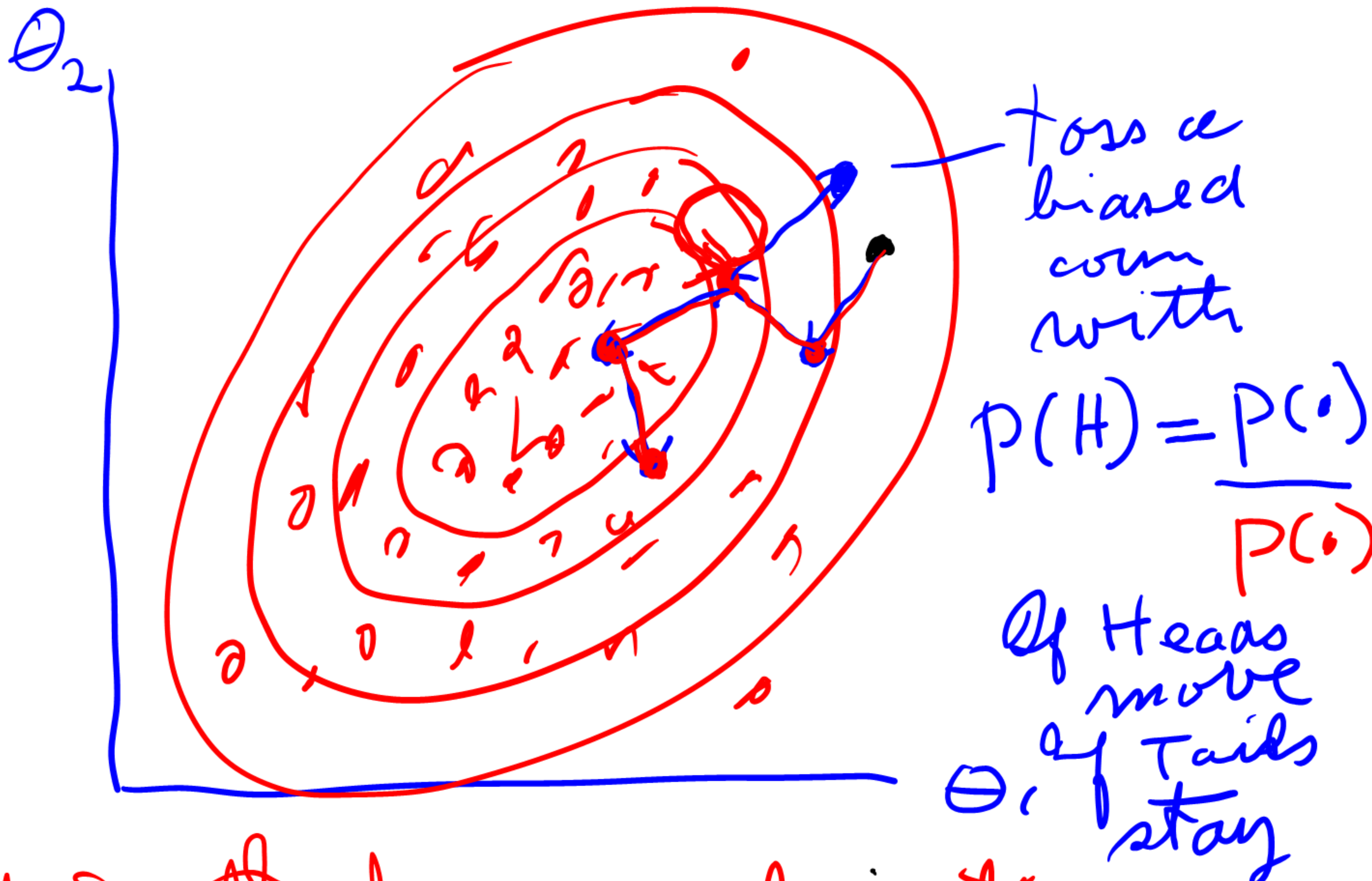
toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

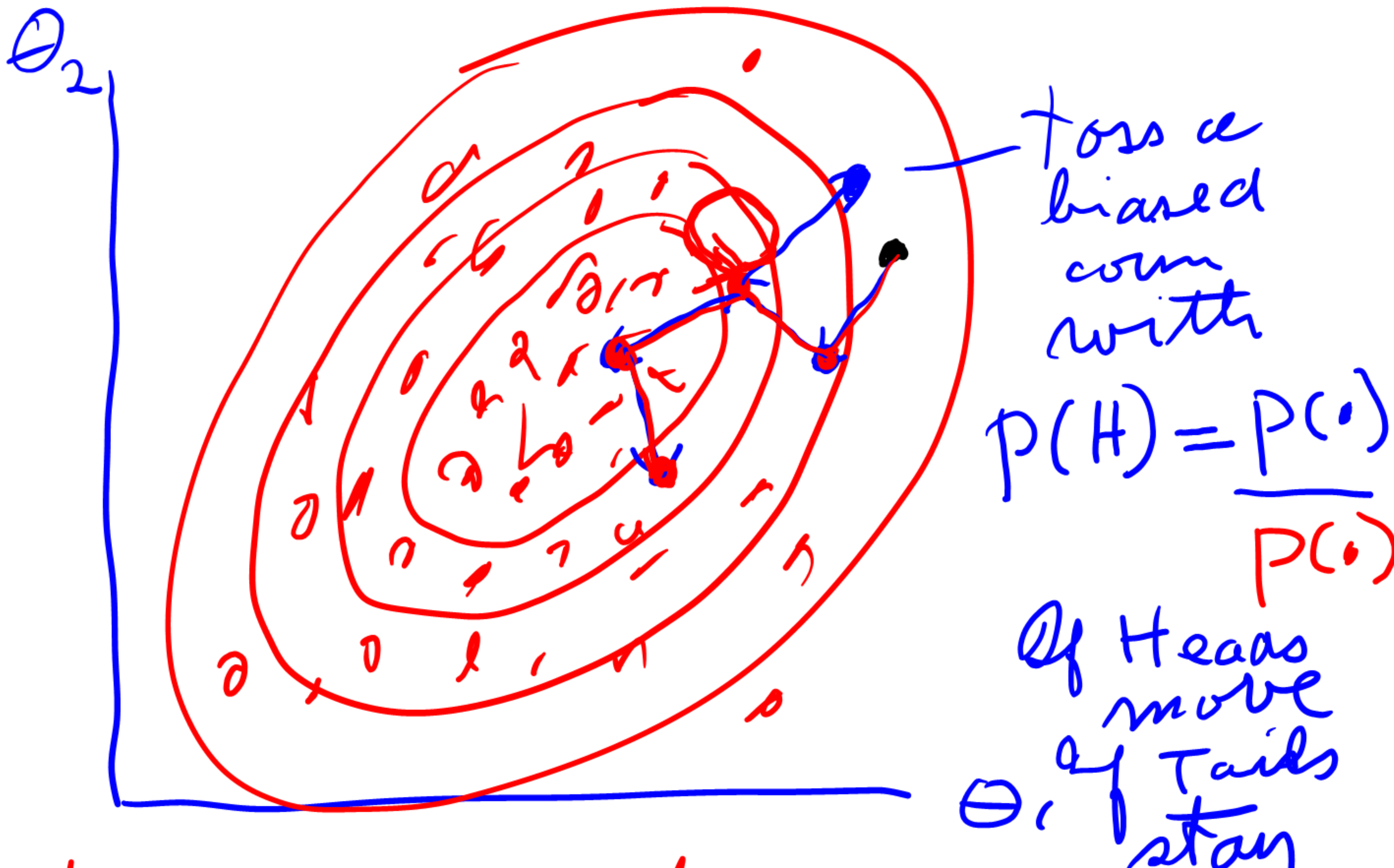
If Heads move

If Tails stay

Keep doing this for a very long time



Keep doing this for a very long time



Generate a sample from $P(\theta|X)$

That's the XX-H algorithm

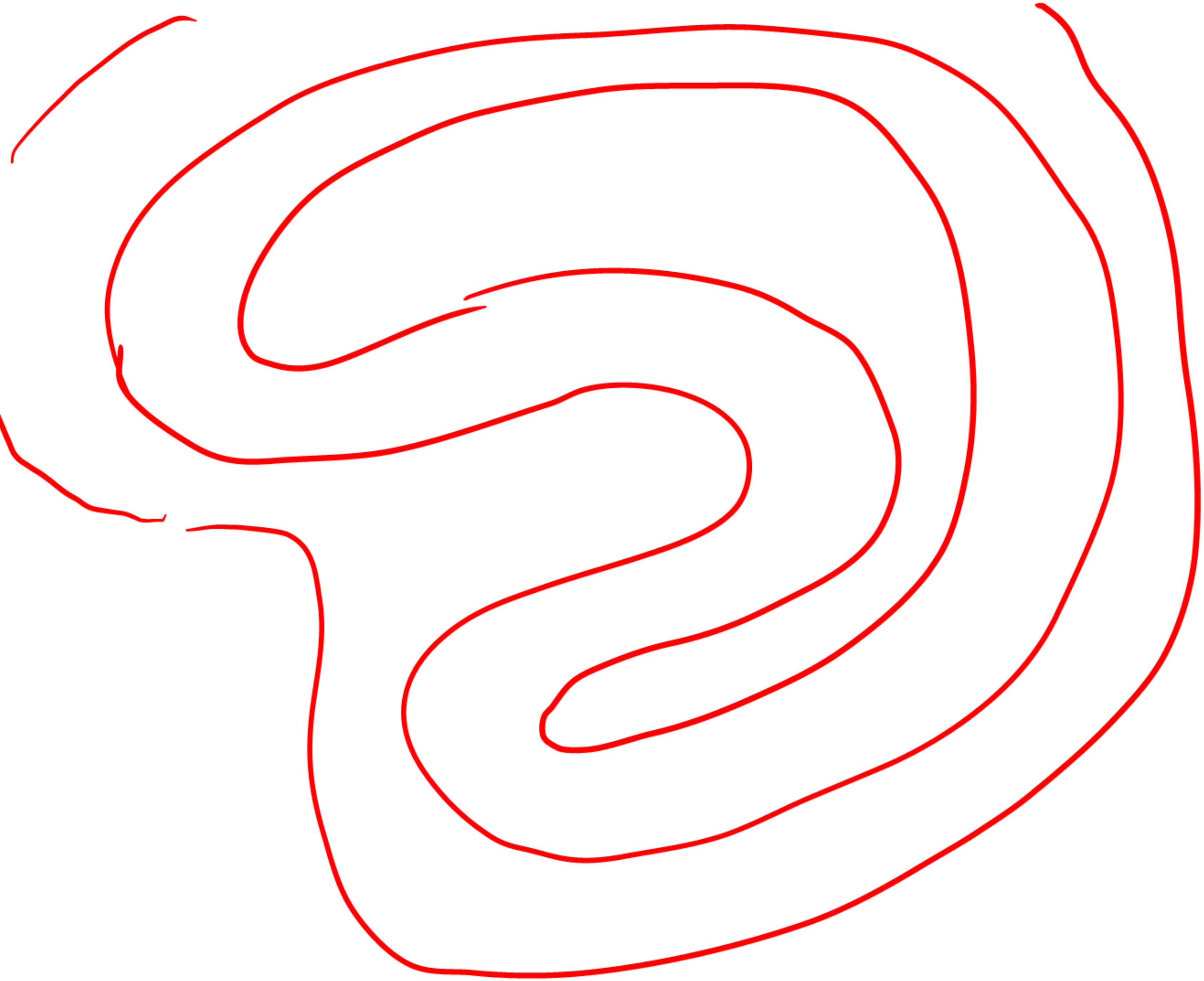
That's the MM-H algorithm

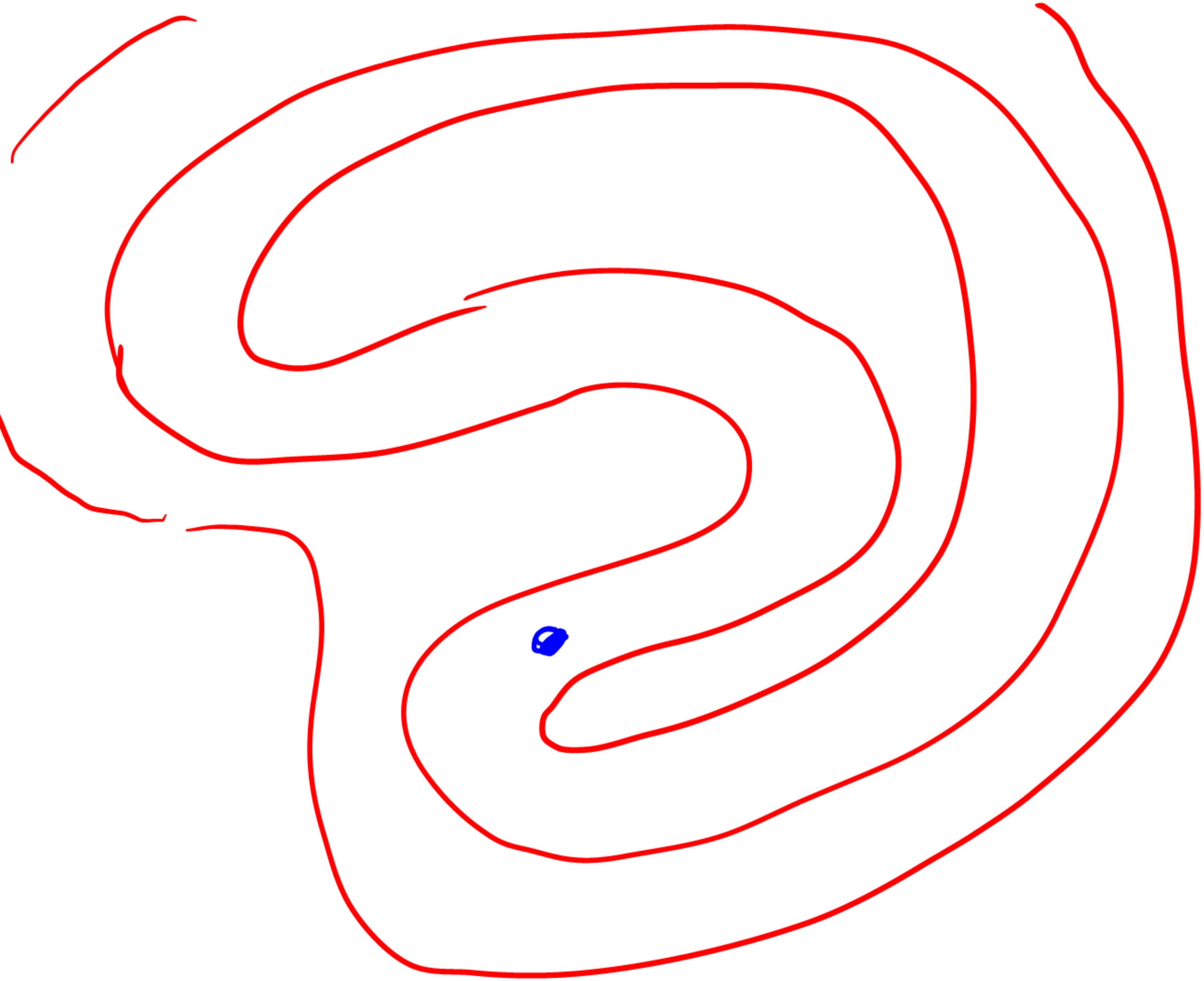
Can be very slow in high dimensions with non-elliptical contours, e.g.

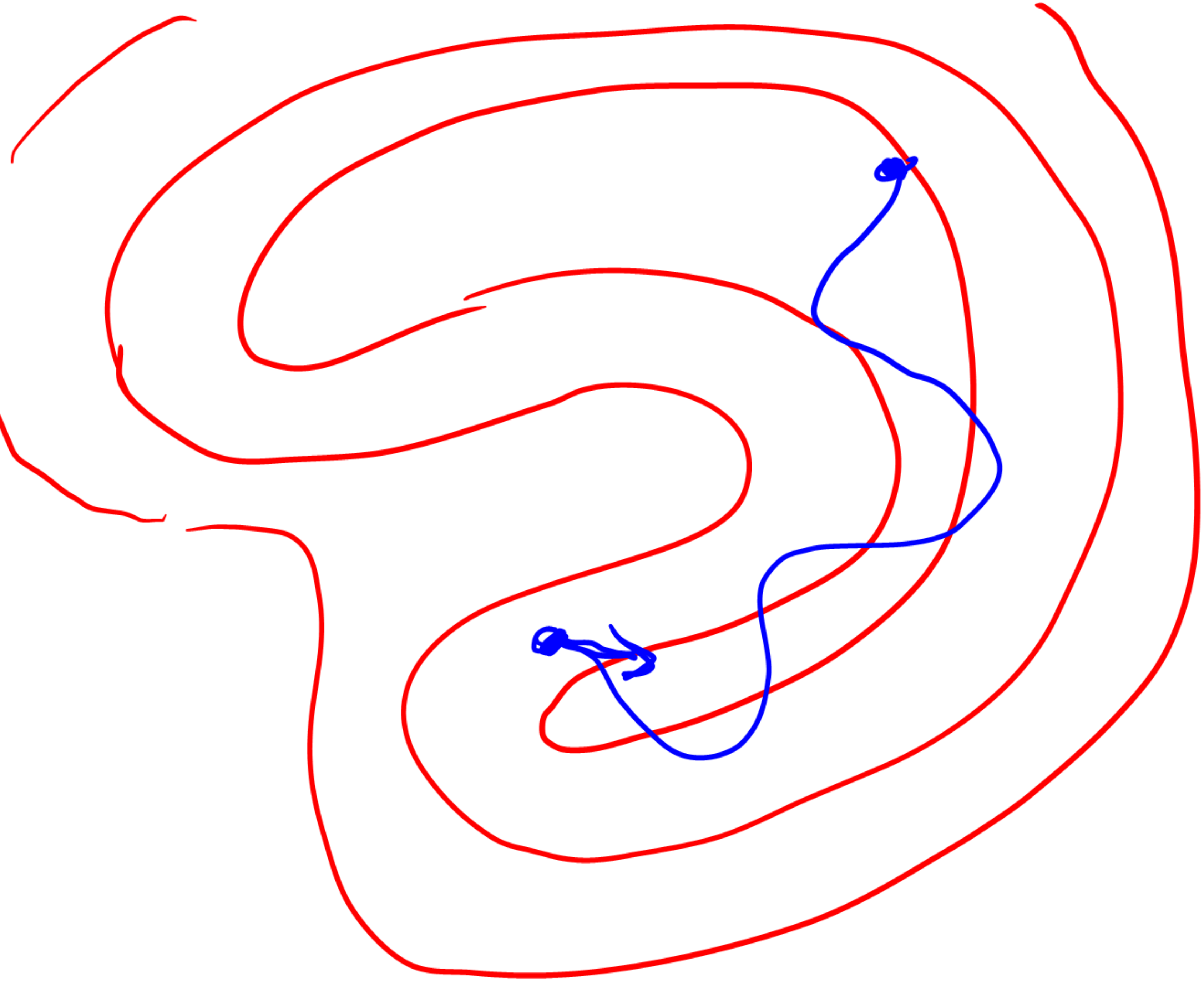
Can get stuck in corners for a long time!

Hamiltonian Monte Carlo

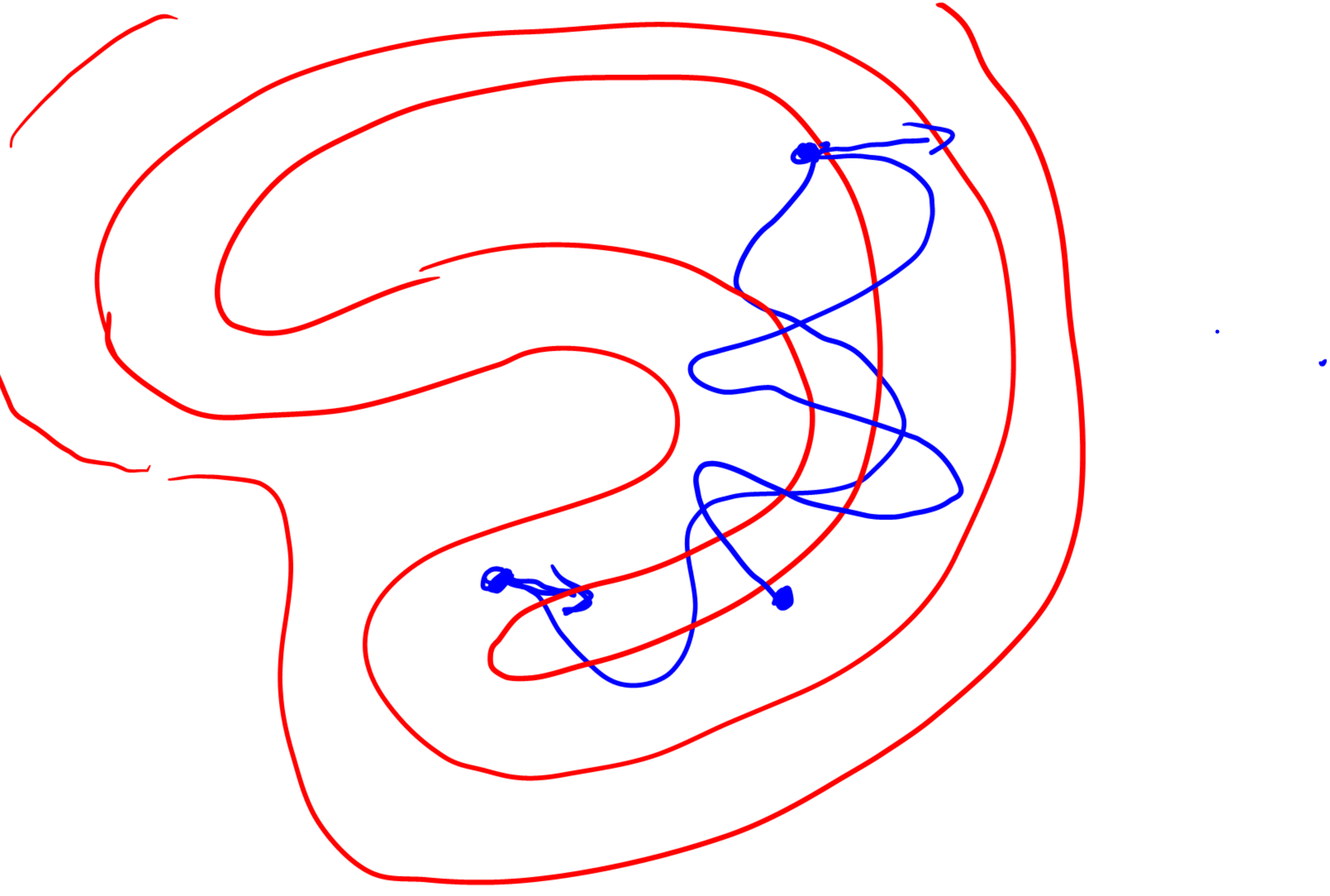
- Turn the mountain into a bowl by using $-\log p(\theta, x)$
- Instead of random steps, go for a ride on a frictionless skateboard with swivel wheels – starting with a random push.

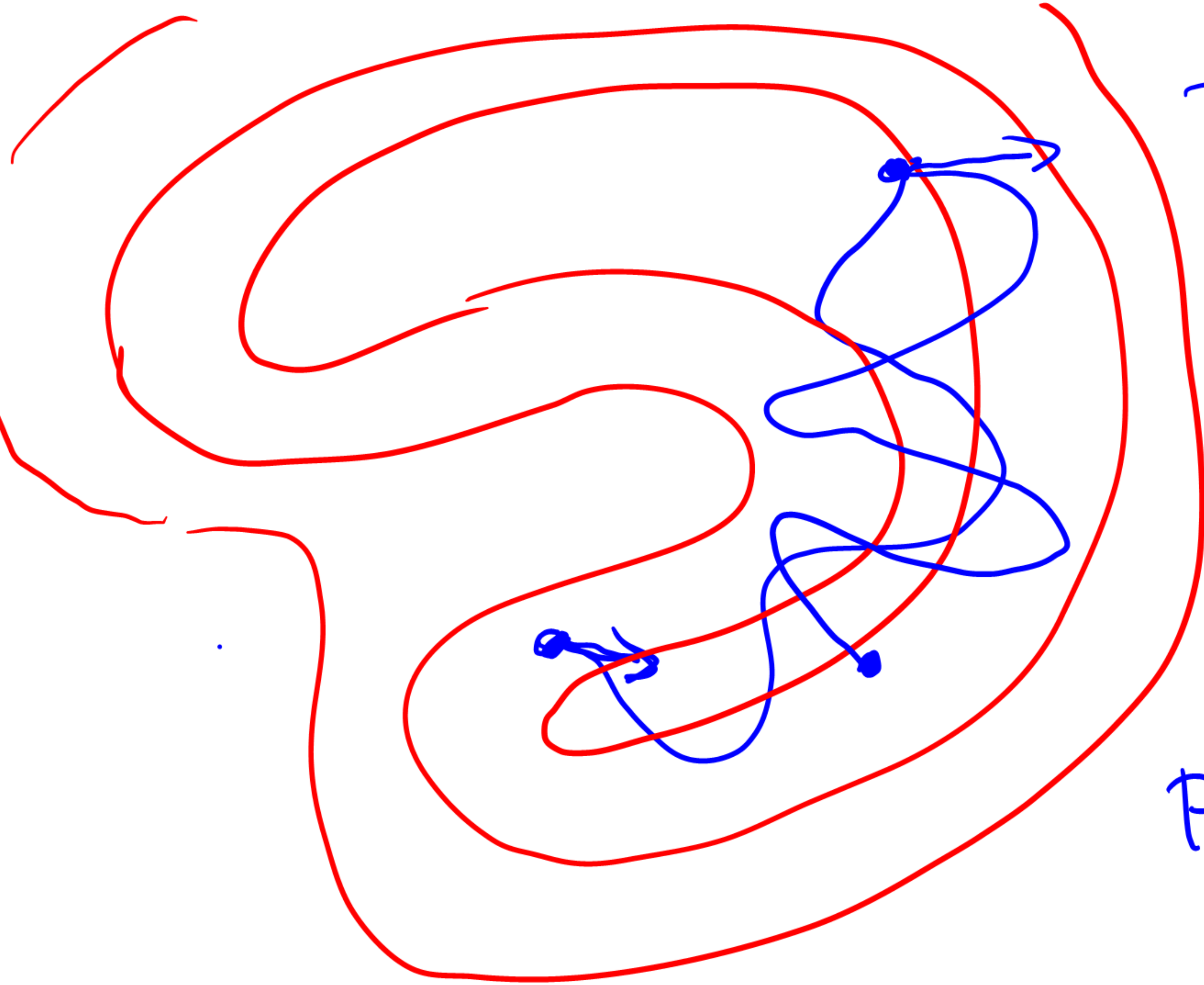




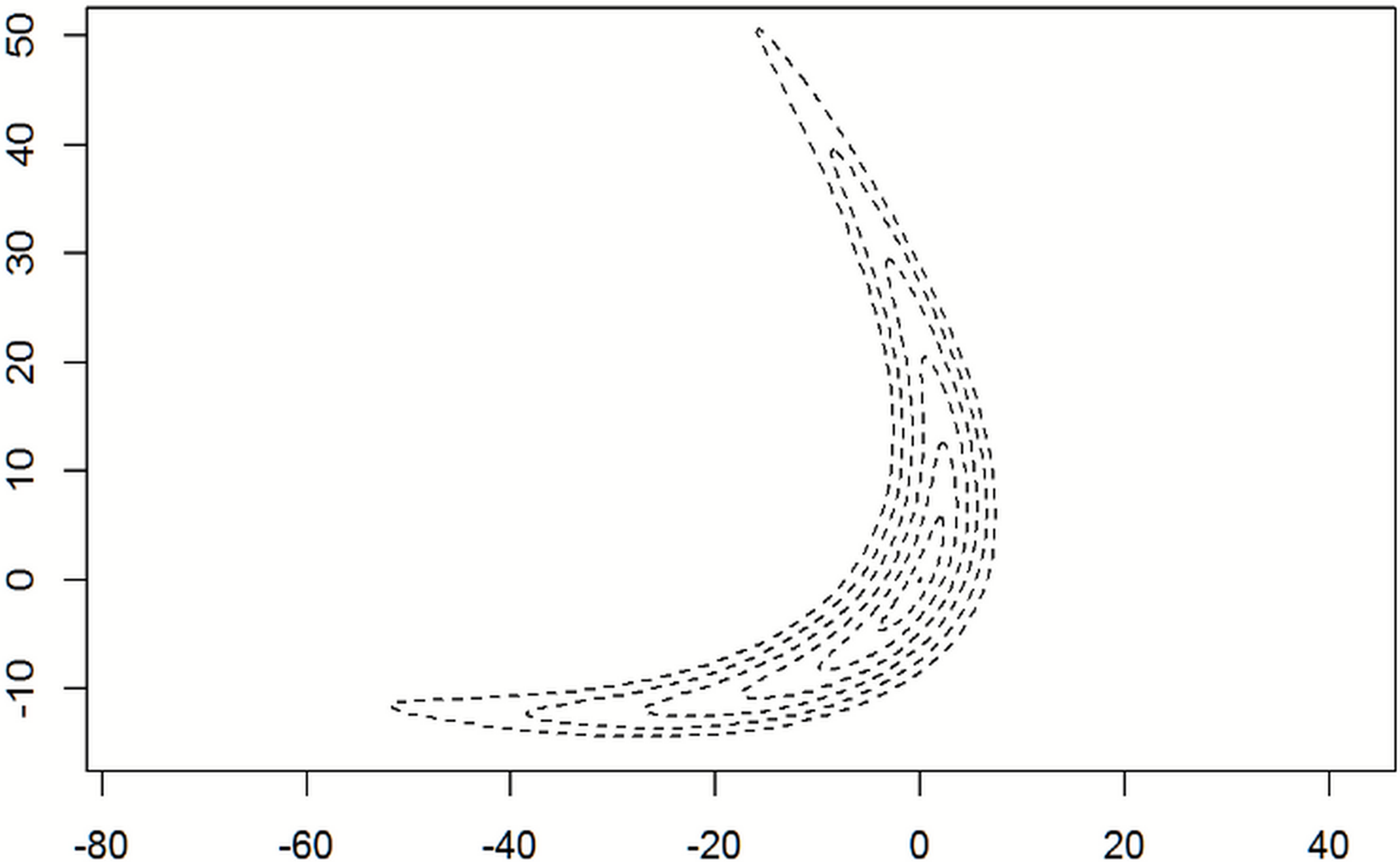


Stop after a set time.





This can do a much better job of covering $p(\theta|X)$.



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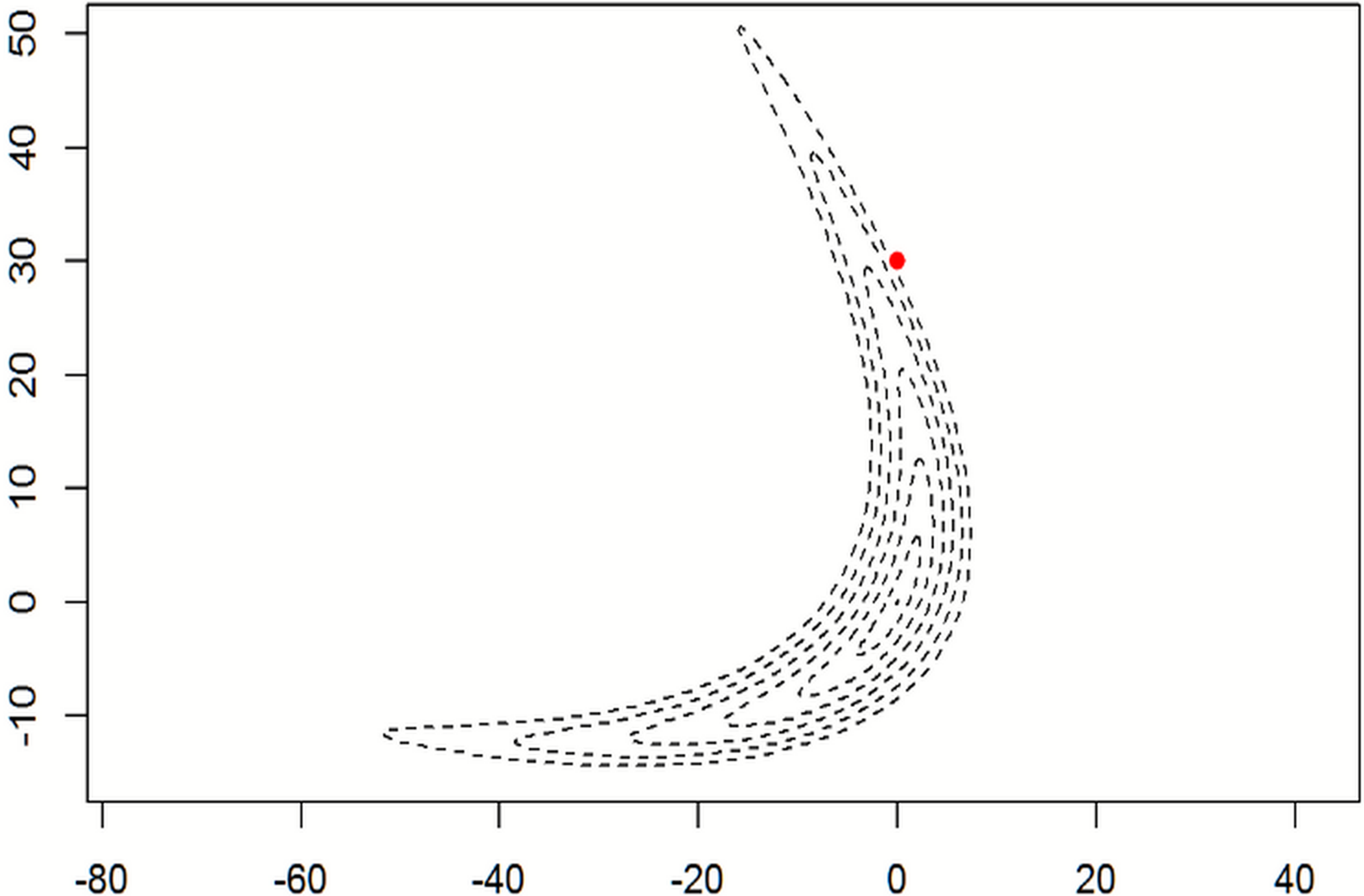
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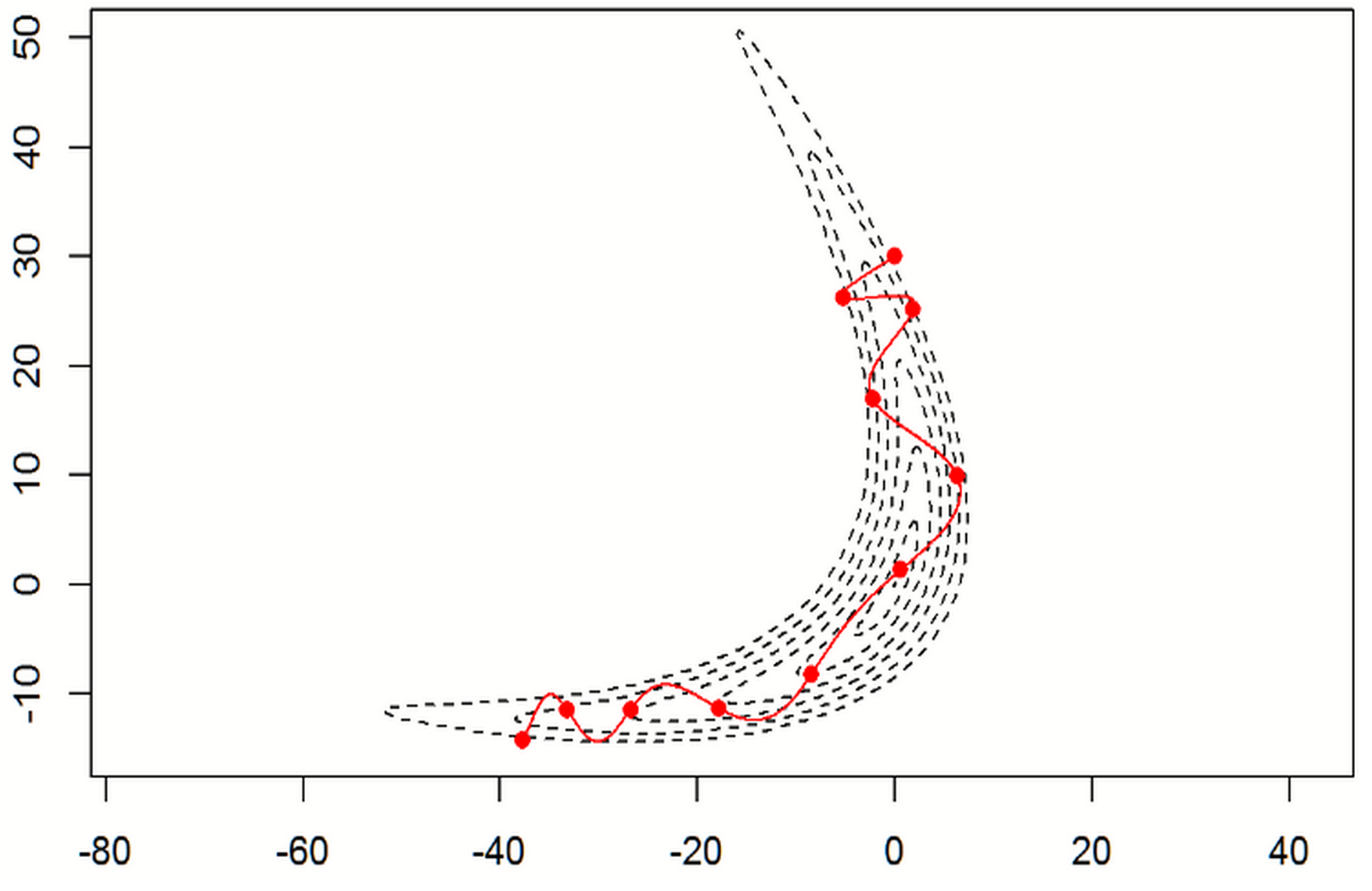

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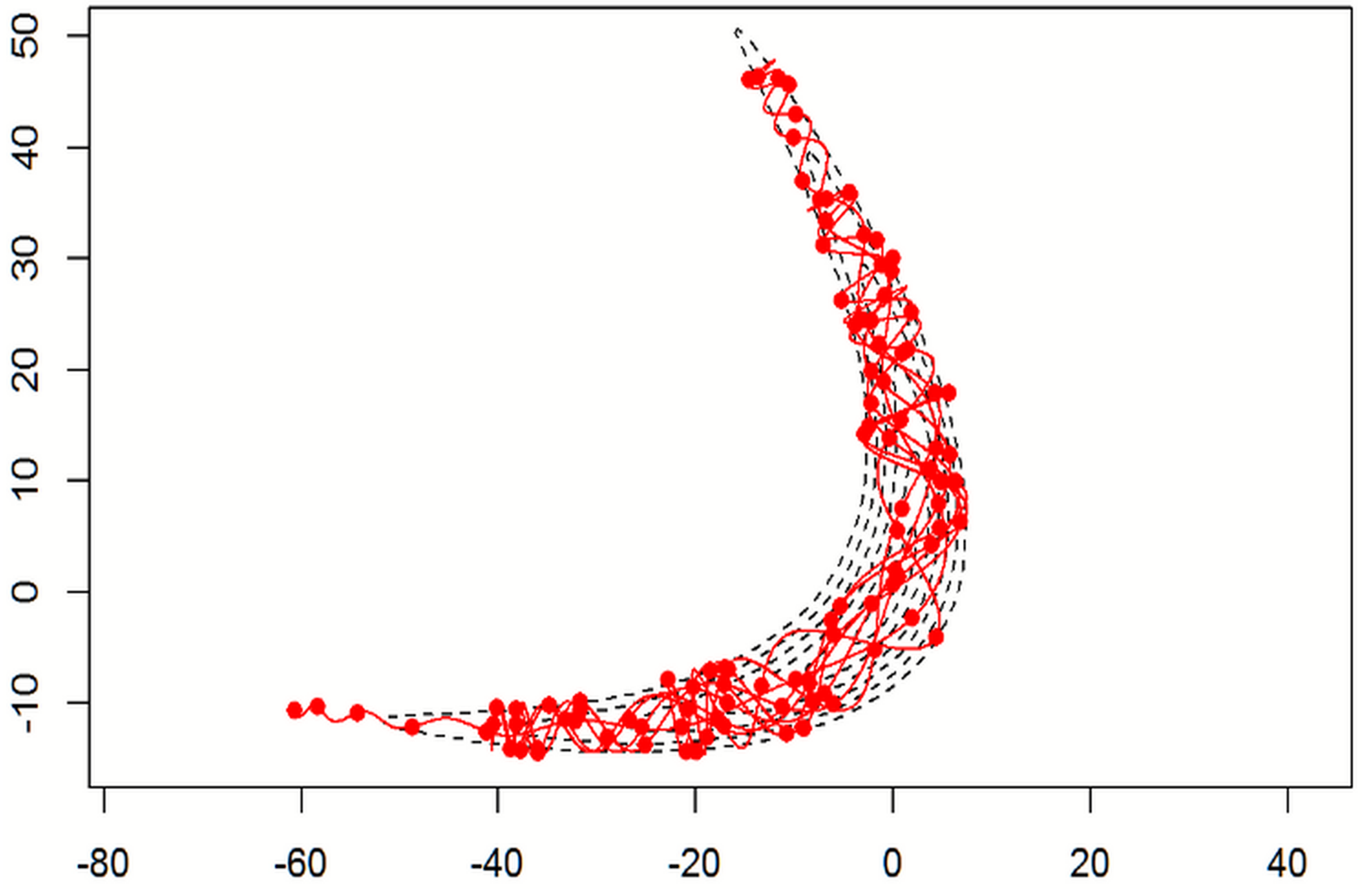

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-40
-20
0
20
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L : # of leaps (leapfrog) steps

ε : size of each step

$L\varepsilon$: "distance" travelled in θ-space

If too small – too much like random walk

If L too large – too much computing time

If ε too large – bad numerical approximation to continuous path and too high likelihood of rejecting proposal.

From Neal (2011)

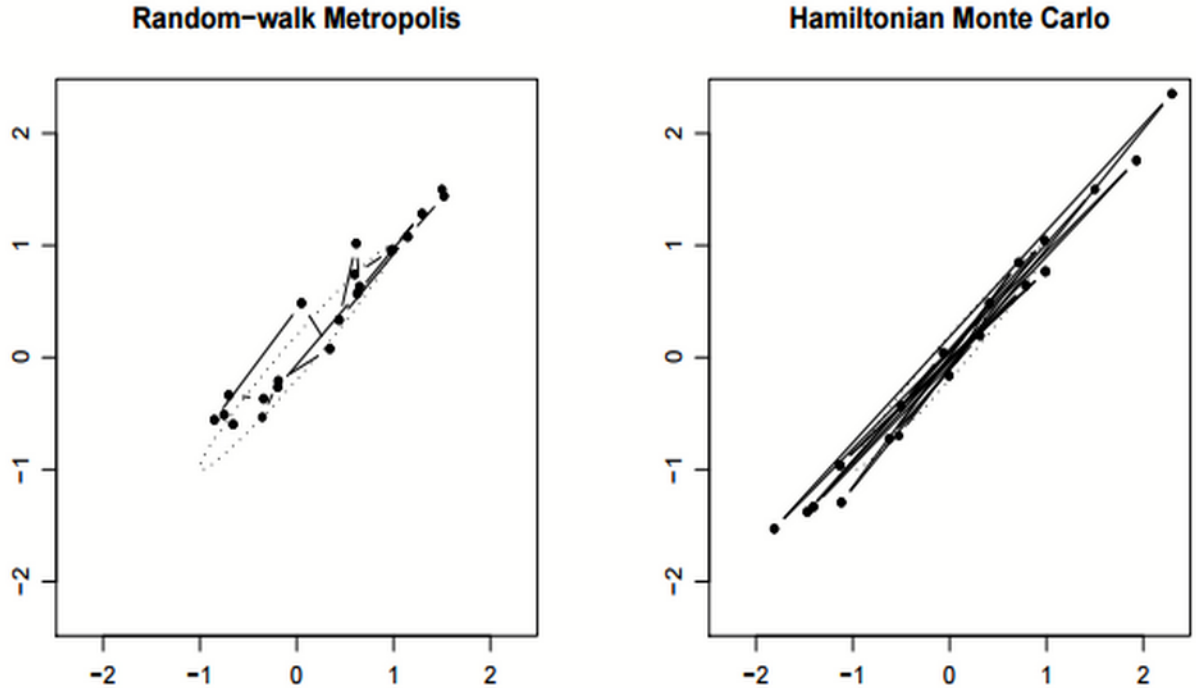

Figure 4: Twenty iterations of the random-walk Metropolis method (with 20 updates per iteration) and of the Hamiltonian Monte Carlo method (with 20 leapfrog steps per trajectory) for a 2D Gaussian distribution with marginal standard deviations of one and correlation 0.98. Only the two position coordinates are plotted, with ellipses drawn one standard deviation away from the mean.

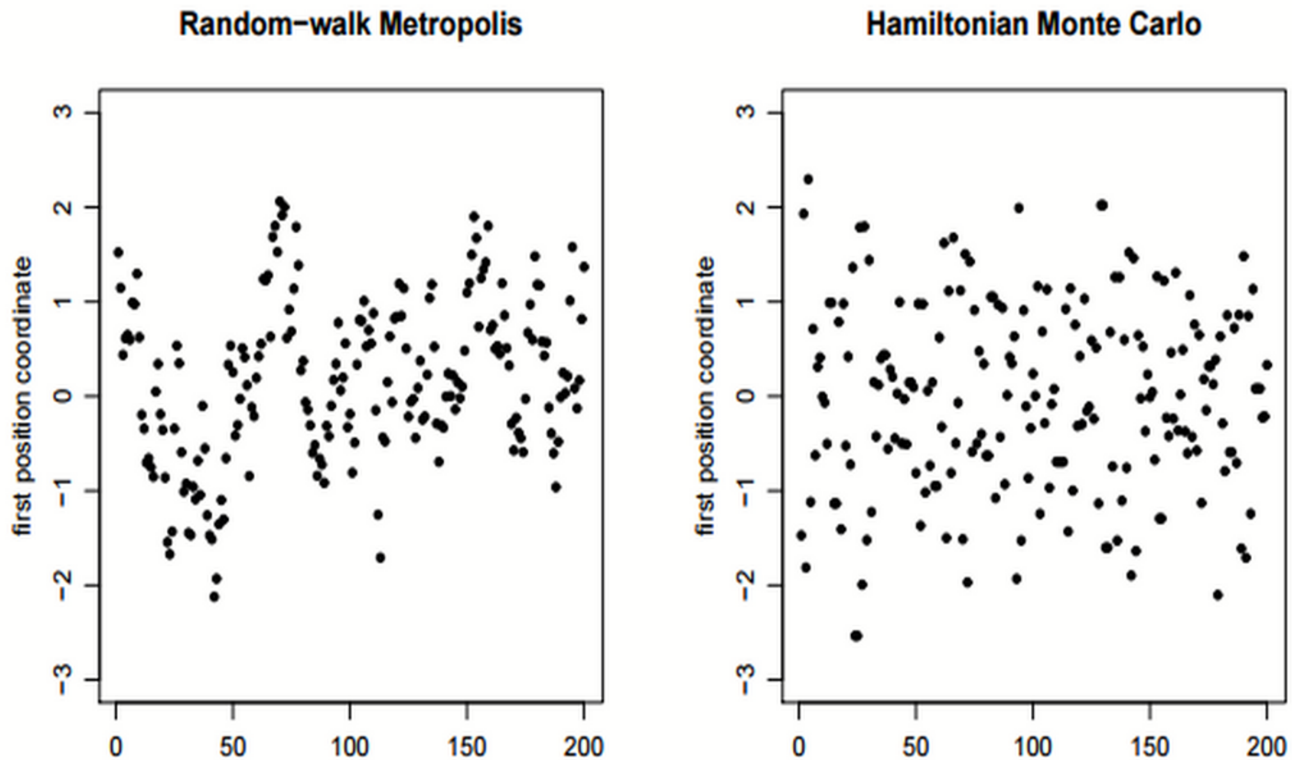

Figure 5: Two hundred iterations, starting with the twenty iterations shown above, with only the first position coordinate plotted.

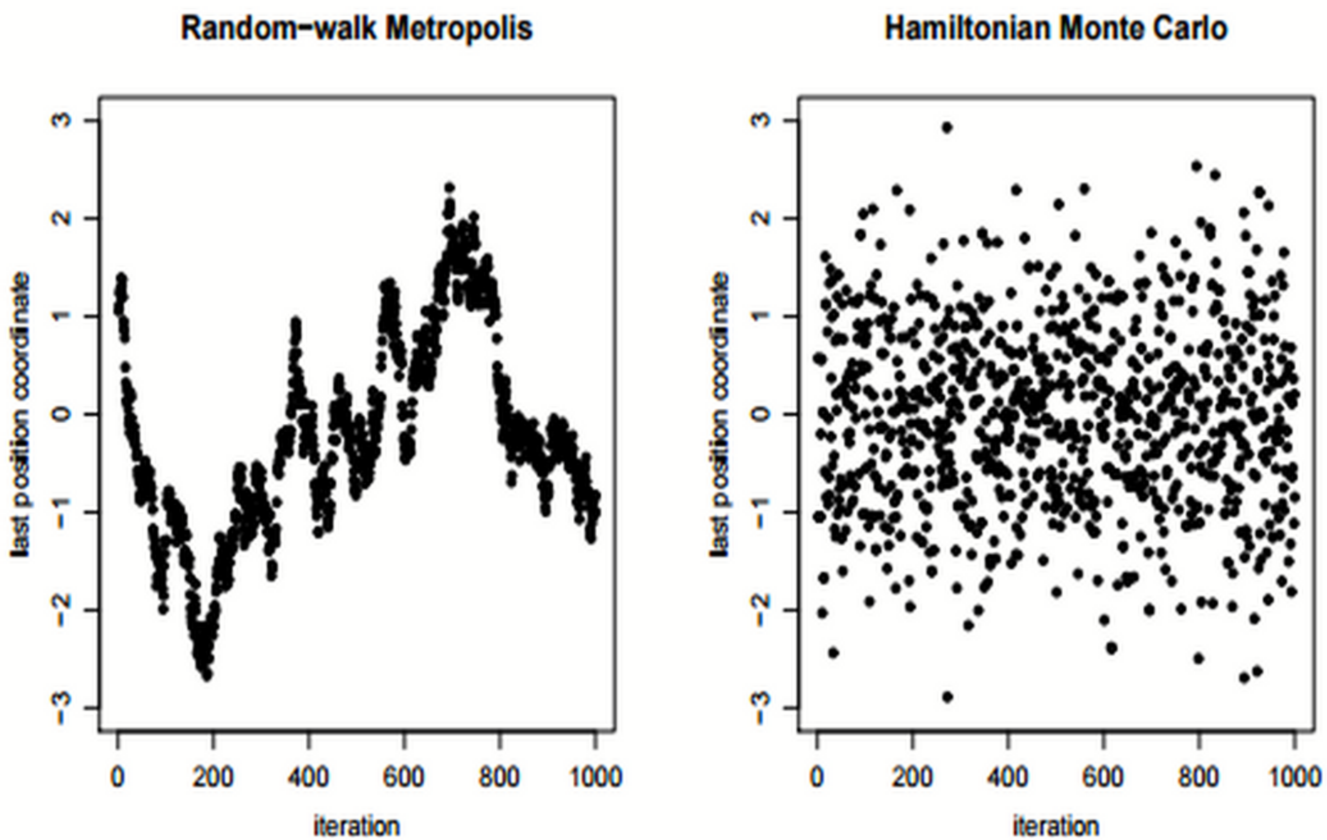

Figure 6: Values for the variable with largest standard deviation for the 100-dimensional example, from a random-walk Metropolis run and an HMC run with $L = 150$. To match computation time, 150 updates were counted as one iteration for random-walk Metropolis.