SCS 2017: Longitudinal and Nested Data

Statistics on Trial(s)

Frequentist vs Bayesian Inference

– Objective p-values versus subjective posteriors

Georges Monette, March 2017

P-values on trial: Sir Roy Meadow and Sally Clark

Sally Clark

- Young lawyer, gives birth to first son in September 1996
- son dies, apparently of SIDS, at 10 weeks
- second son born a year later
- dies, apparently of SIDS, at 8 weeks
- only evidence of trauma consistent with resuscitation attempts
- charged with two counts of murder

Sir Roy Meadow

- distinguished pediatrician
- as expert witness testifies:
  - probability of one SIDS death: $\frac{1}{8,500}$
  - probability of two: $\left(\frac{1}{8,500}\right)^2 = \frac{1}{72,250,000}$
  - 'if she's innocent, the chances of this happening are 1 in 72 million'
- jury convicts Sally Clark of murder in November 1999
- first appeal lost in October 2000
- second appeal succeeds and Sally Clark is released in January 2003
- she dies in 2007 at the age of 42

H_0: Sally is innocent

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Y: 2 children die for no apparent cause

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Meadow's calculation

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Criticism : 1) assumes independence

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Correct p-value is larger - maybe $\frac{1}{10{,}000}$!

BUT:

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Do we really want $P(y^+ \mid H_0)$?

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Do we really want $P(Y^+ | H_0)$?

Don't we really want $P(H_0 | Y)$?

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- Must be close!?
- Is $P(Y^+|H_0)$ a good proxy for $P(H_0|Y)$?

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Does it establish guilt beyond a reasonable doubt?

BUT:

Do we really want $P(Y^+ | H_0)$?

Don't we really want $P(H_0 | Y)$?

- Must be close!?
- Is $P(Y^+ | H_0)$ a good proxy for $P(H_0 | Y)$?

Let's find out:

$$\underline{P(H_0 | Y)} = \frac{P(H_0, Y)}{P(Y)} = \frac{P(Y | H_0)\,\underline{P(H_0)}}{P(Y)}$$

Bayesian tree:

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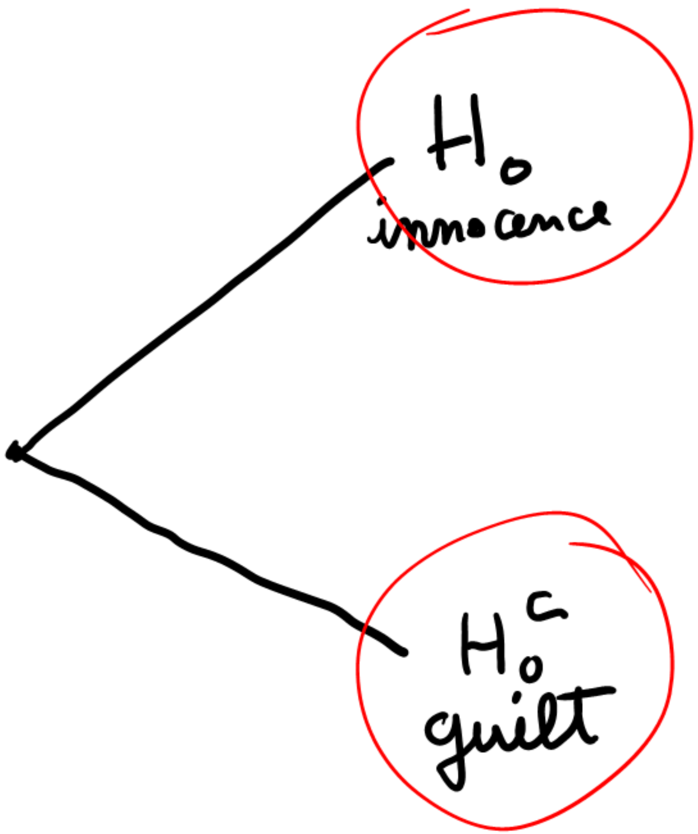

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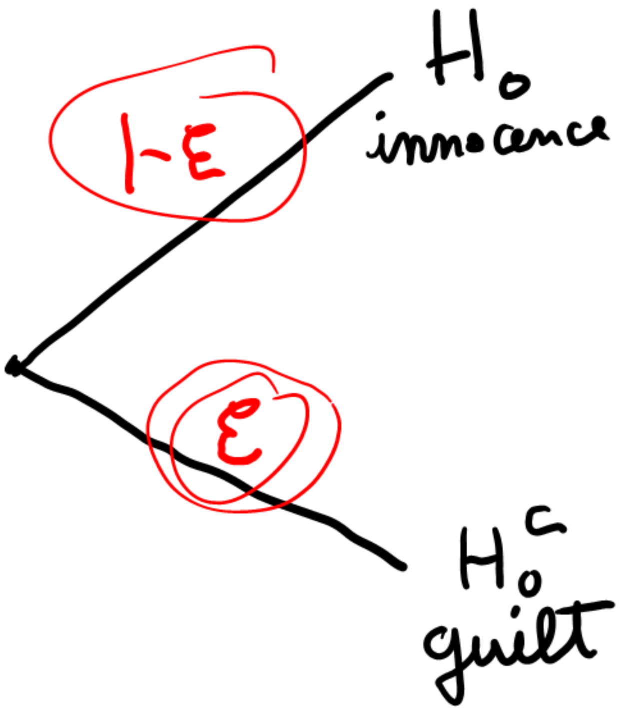

ε = very small number

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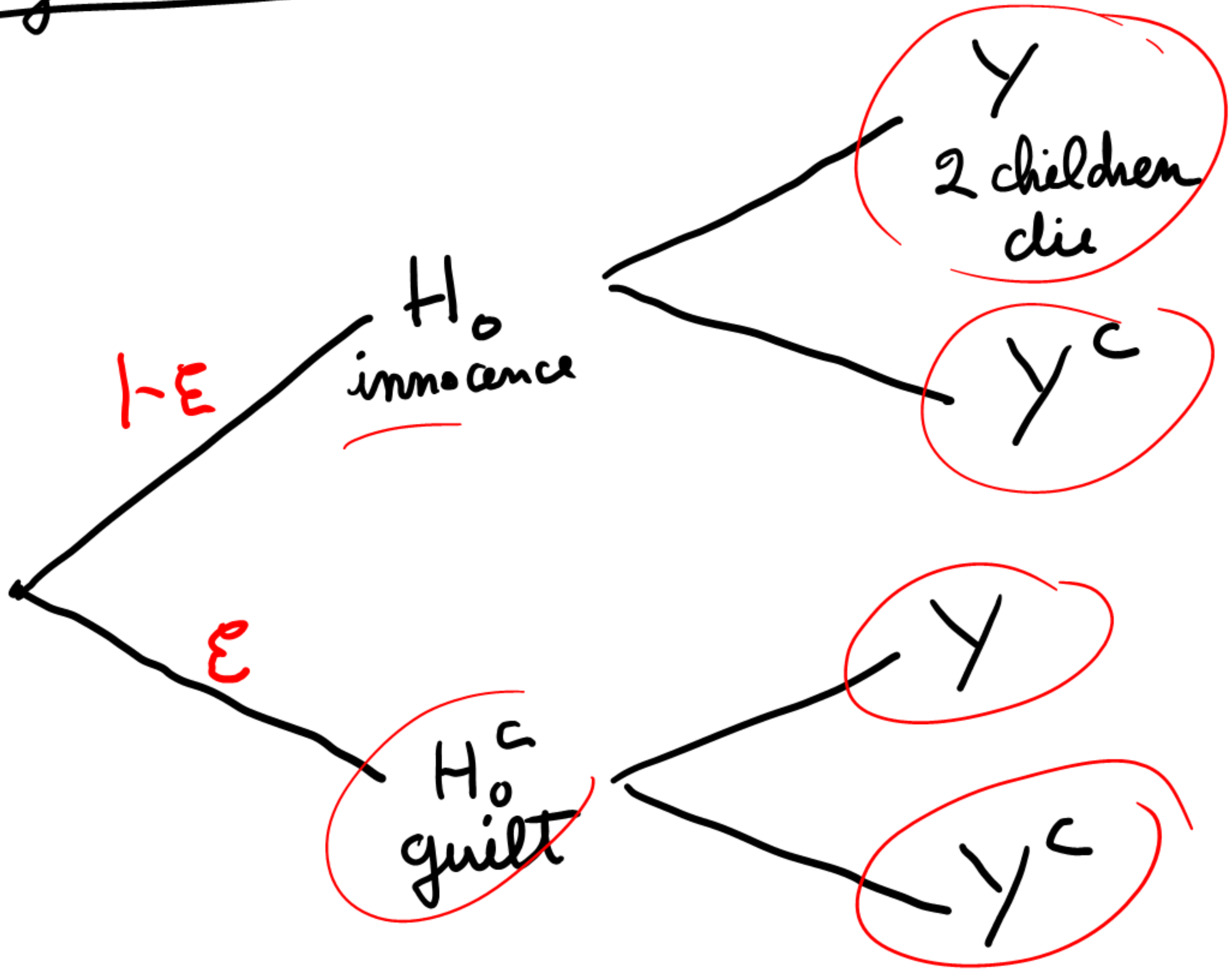

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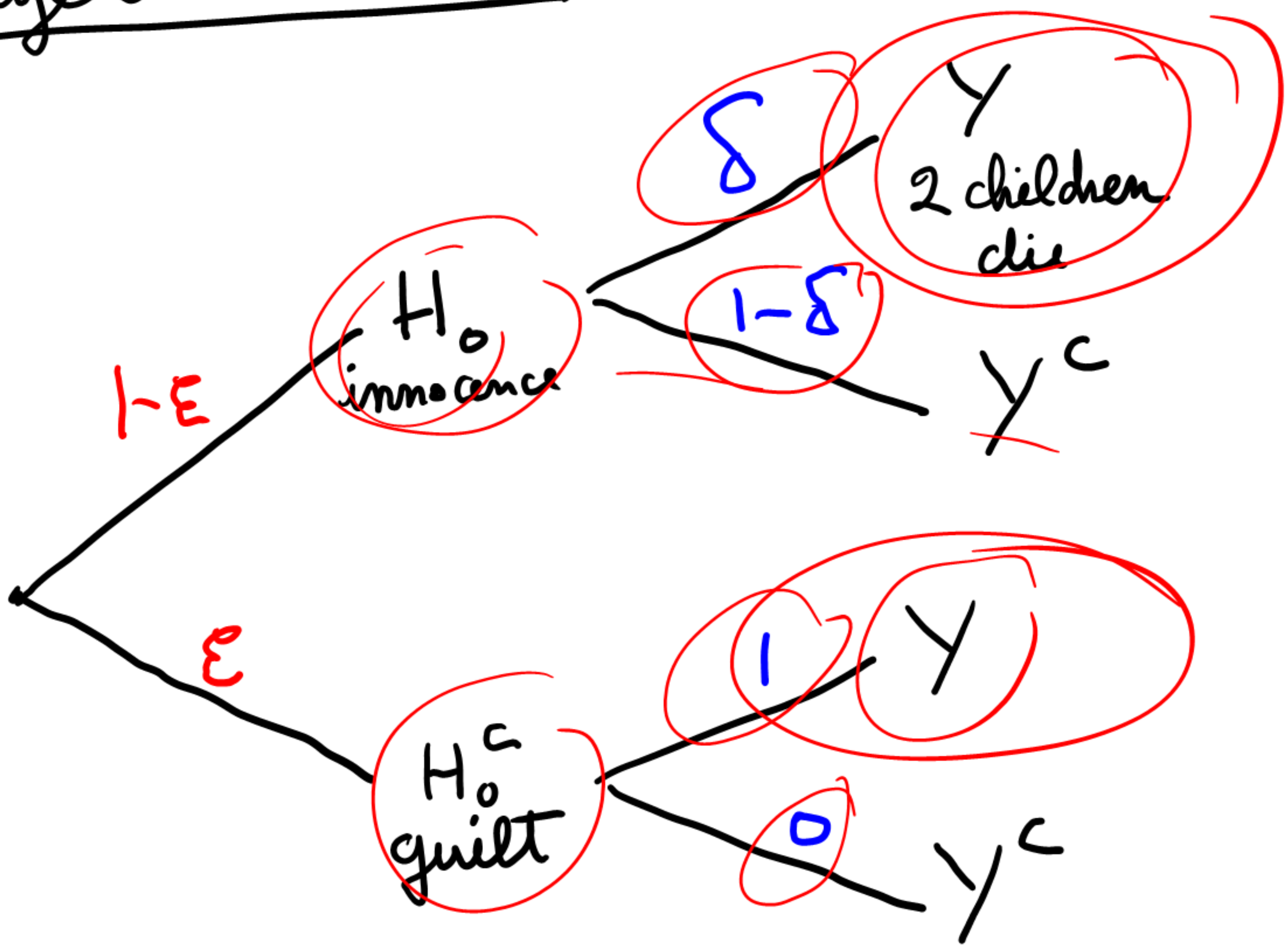

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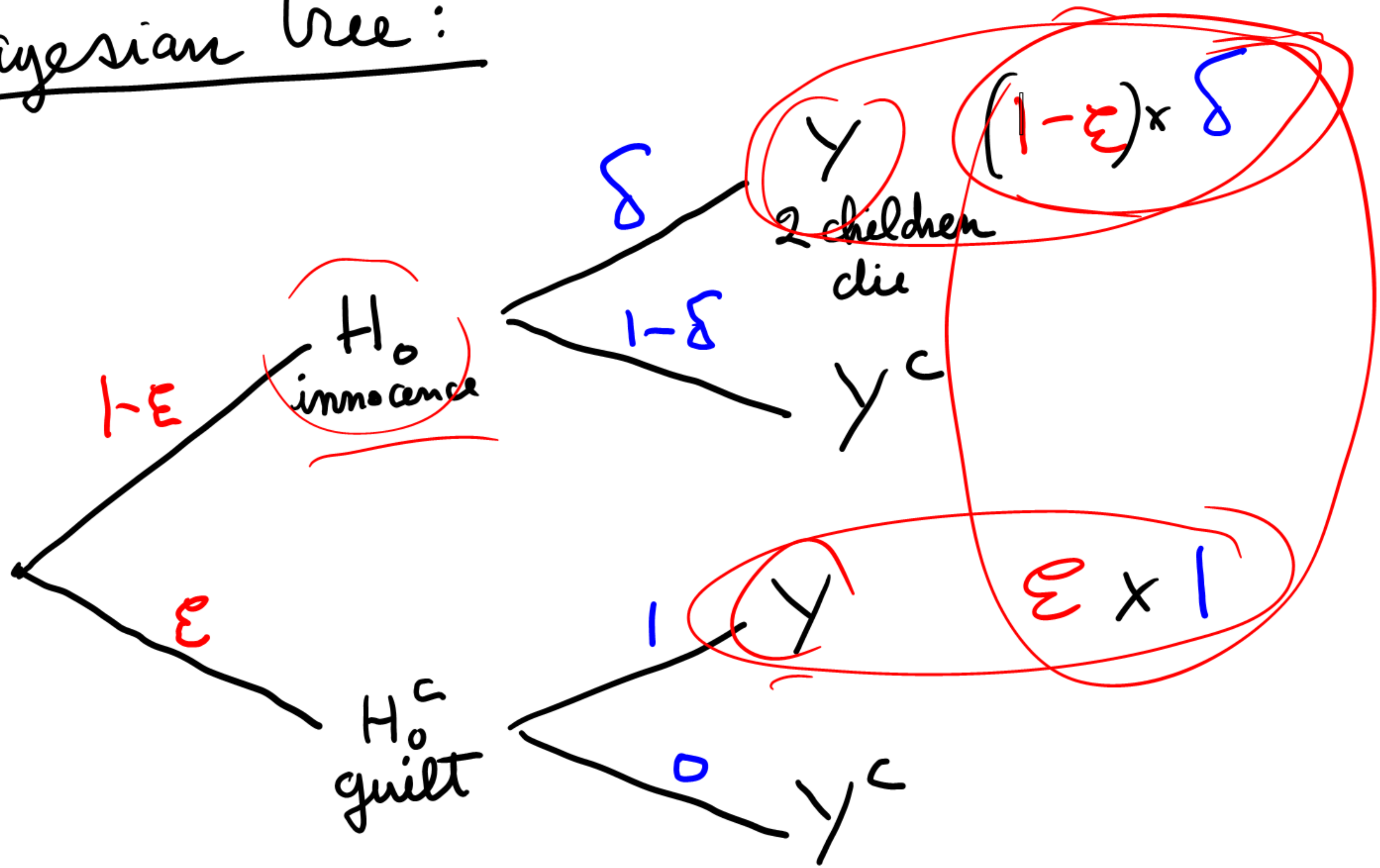

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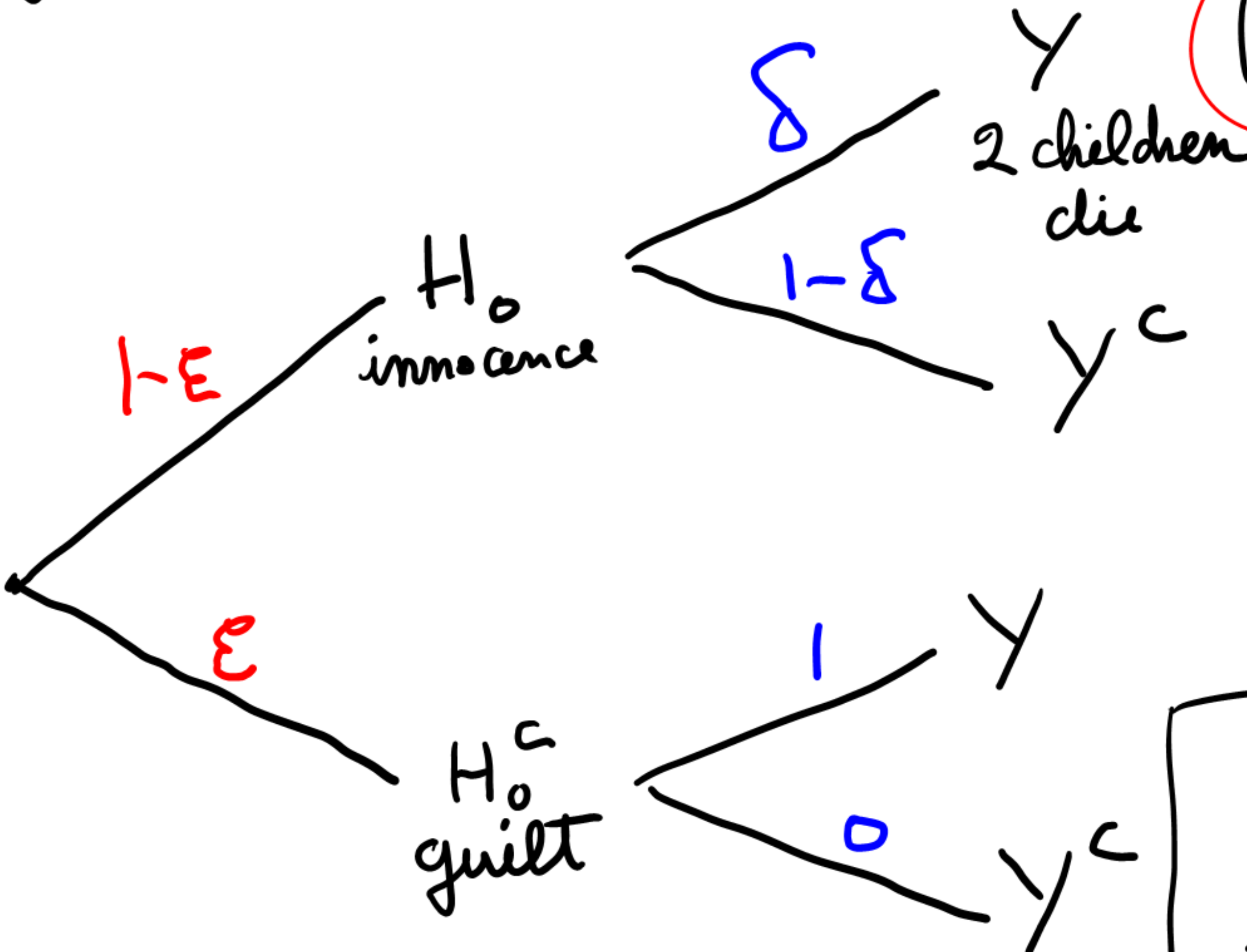
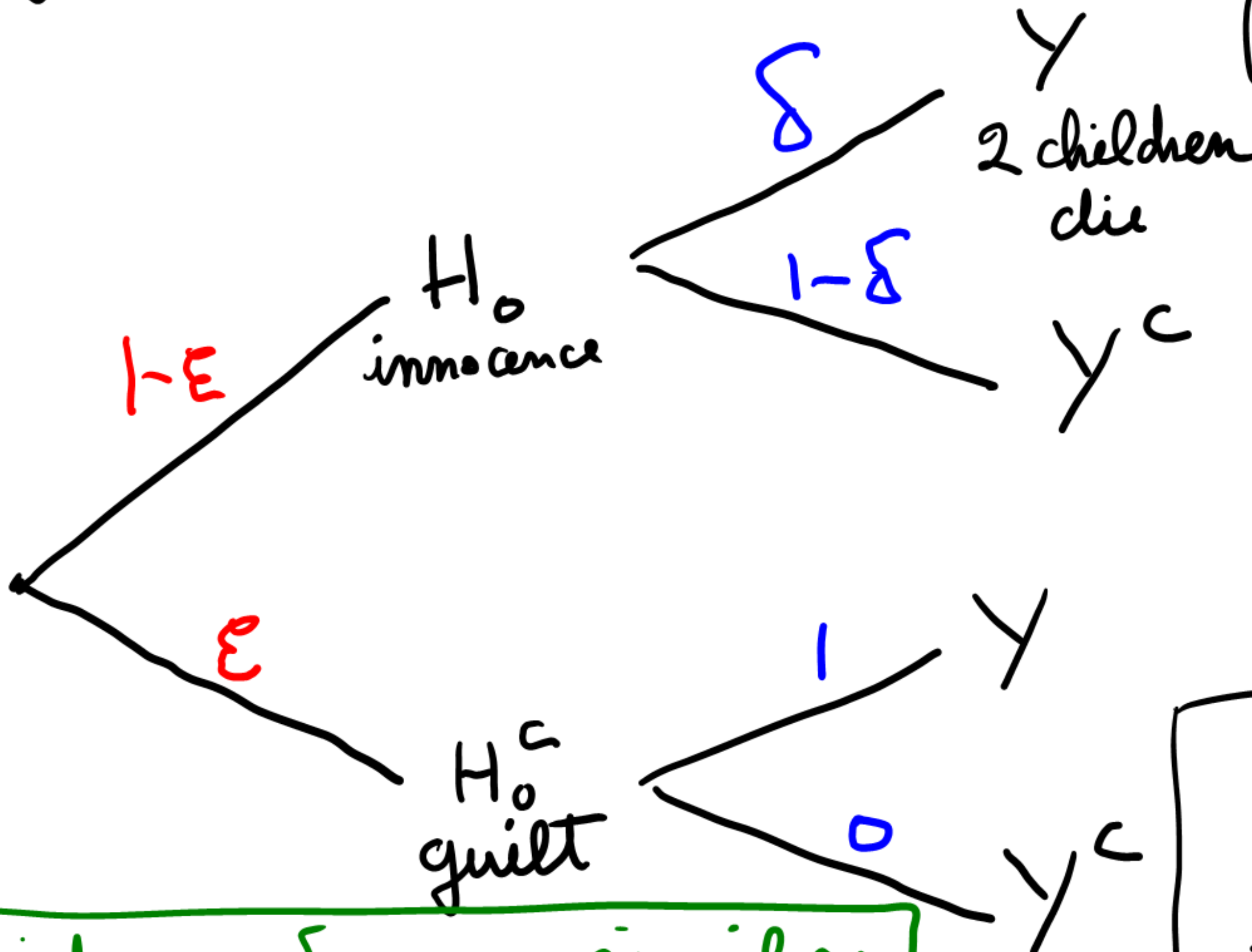


$$(1-\varepsilon)\times\delta$$

$$\varepsilon\times 1$$

$$P(H_0 \mid Y) = \frac{(1-\varepsilon)\,\delta}{(1-\varepsilon)\,\delta + \varepsilon} \approx \frac{\delta}{\delta+\varepsilon}$$

Bayesian tree:



$(1-\varepsilon)\times\delta$

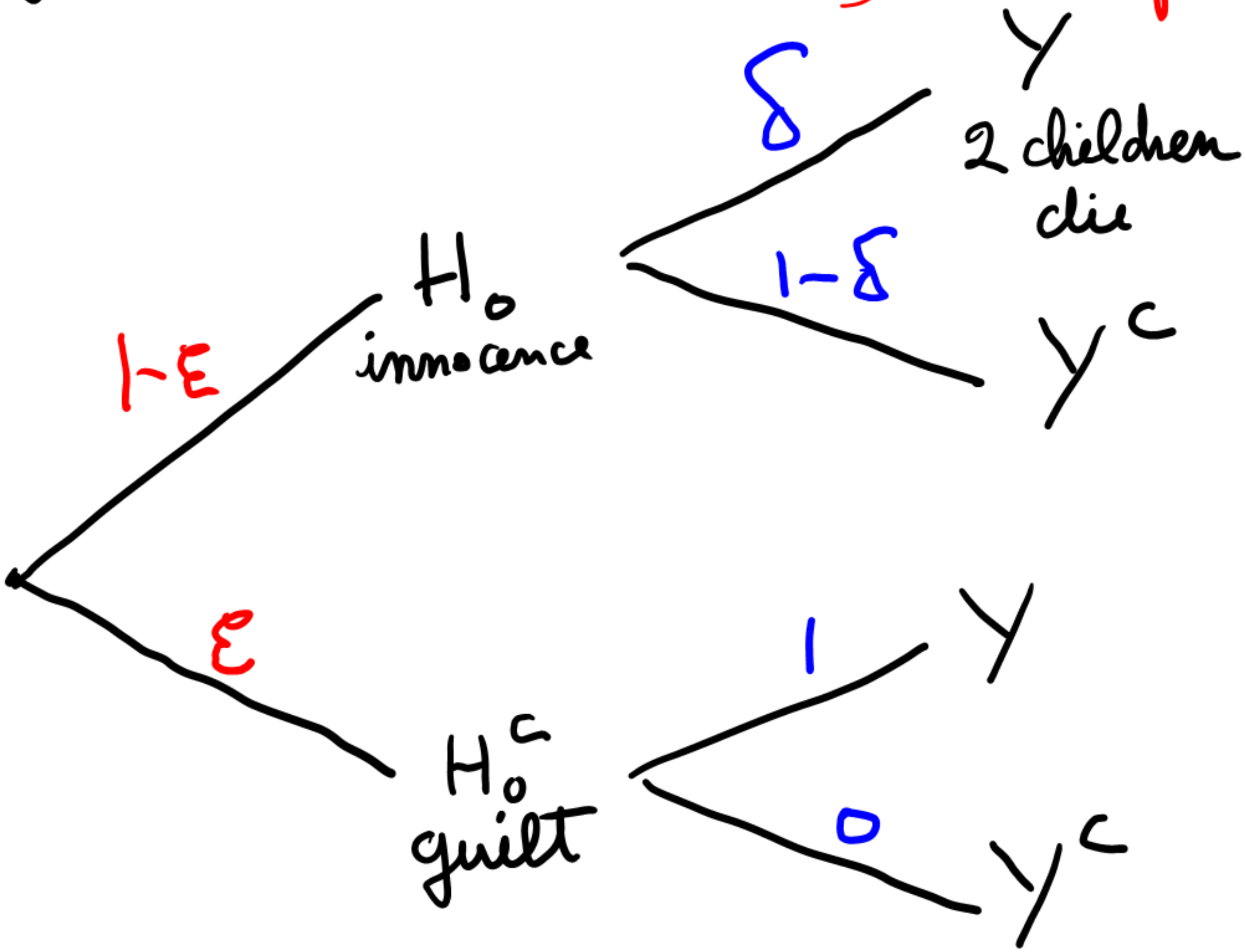
$\varepsilon\times 1$

$$P(H_0|Y) = \frac{(1-\varepsilon)\,\delta}{(1-\varepsilon)\,\delta+\varepsilon} \approx \frac{\delta}{\delta+\varepsilon}$$

So if ε & δ are similar then $P(H_0|Y) \approx \frac{1}{2}$

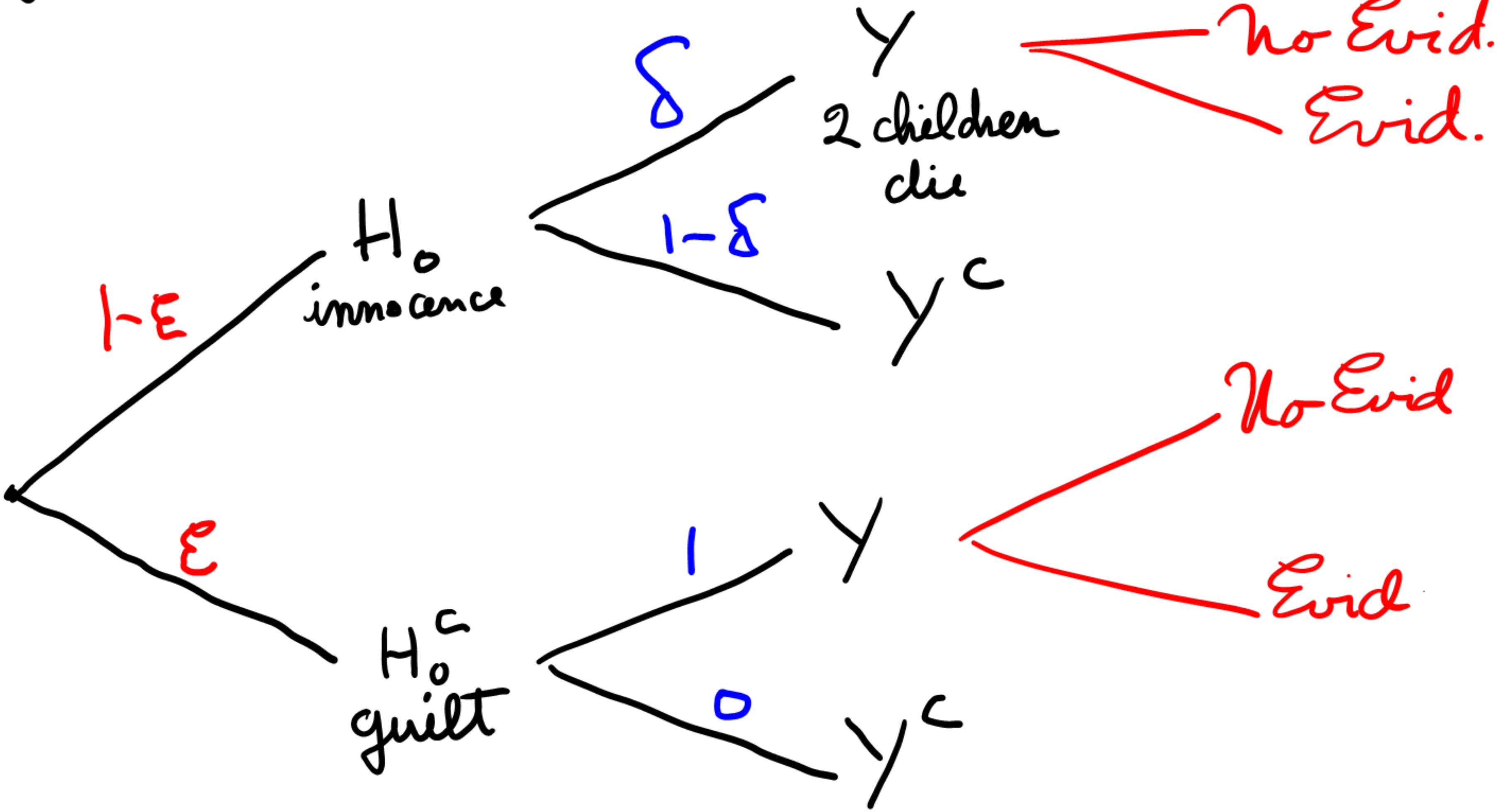
Bayesian tree:

But this ignores lack of evidence



If it's a crime, it's a "perfect" crime – and, presumably, only a small proportion of crimes are perfect crimes.

Bayesian tree:



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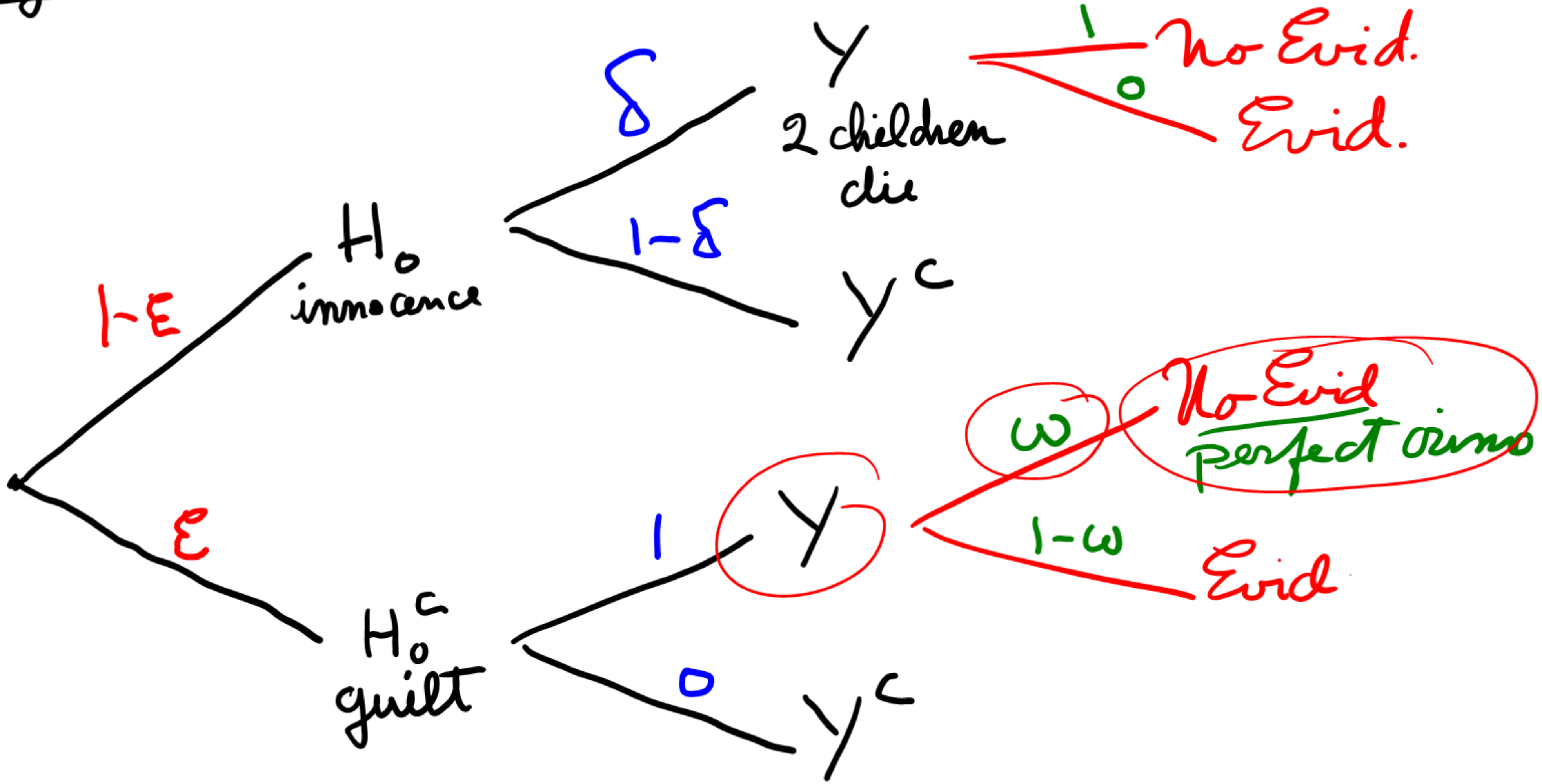

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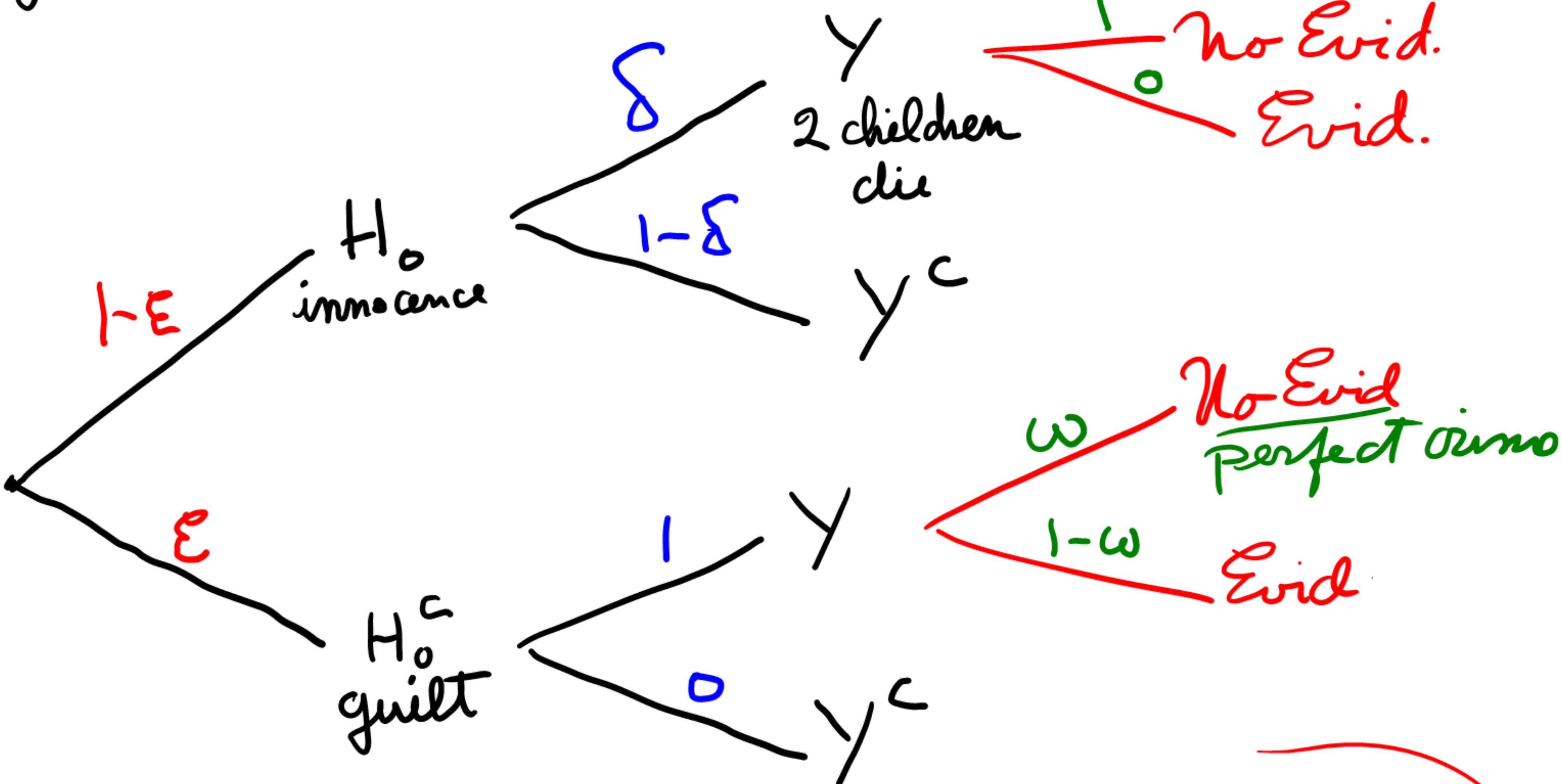

$$\Pr(H_0 \mid Y \,\&\, \text{No Evid}) \approx \frac{\delta}{\delta + \cancel{\varepsilon\omega}} \approx \text{close to } 1 \,!!$$

Sally Clark was innocent beyond a reasonable doubt.

$P(Y^+ \mid H_0)$

was a very poor proxy for

$P(H_0 \mid Y)$

$P(Y^+ | H_0)$

was a very poor proxy for

$P(H_0 | Y)$

- But many (most?) scientists only know about $P(Y^+ | H_0)$.
- Many have been trained to be adamantly opposed to using "subjective" Bayesian methods instead of "objective" Frequentist methods.

— Using the p-value as a proxy for $P(H_0 \mid Y)$ has come to be known as the "Prosecutor's Fallacy"

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- But many (most?) scientists testifying don't know of other ways

— Using the p-value as a proxy for $P(H_0 \mid y)$ has come to be known as the "Prosecutor's Fallacy"

So, how did we get to this?

Why are we using p-values if they are so bad?

Dilemma

Dilemma

Θ : unknown state of nature

Dilemma

Θ : unknown state of nature

Y : what's observed

Dilemma

Θ : unknown state of nature

Y : what's observed

What does Y say about Θ?

Dilemma

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Model : $P(Y \mid \Theta)$

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From this we can get p-values, confidence intervals, size-α tests.

Dilemma

Θ : unknown state of nature

Y : what's observed

What does Y say about Θ ?

Model : $p(Y \mid \Theta)$

From this we can get p-values, confidence intervals, size-α tests,

But not $p(\Theta \mid Y)$ posterior

For that we need to start with $p(\Theta)$ prior

To get what you really want, you need to pay a price many are reluctant to pay.

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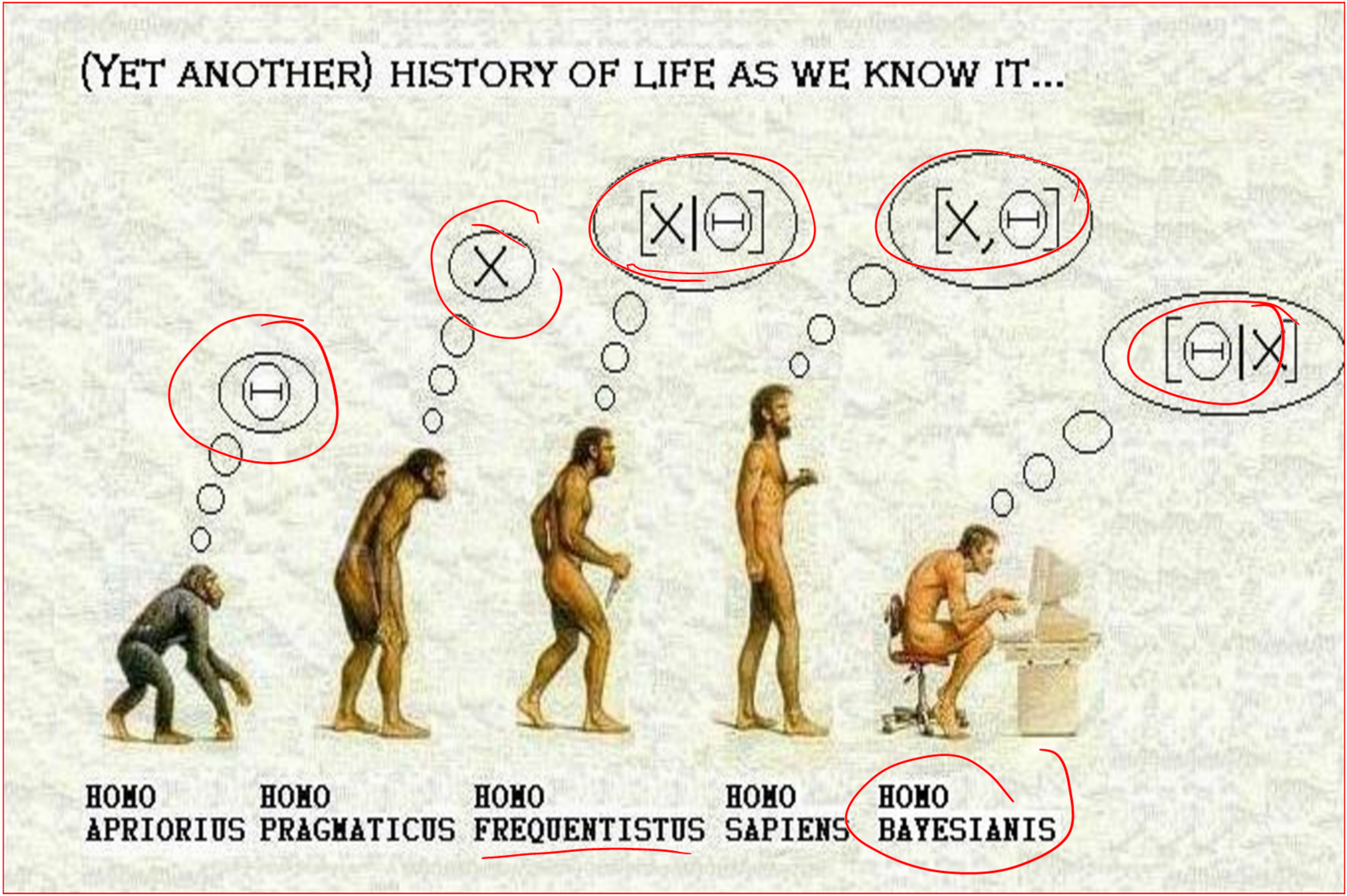
A historical perspective


(YET ANOTHER) HISTORY OF LIFE AS WE KNOW IT...
Θ
X
[X|Θ]
[X,Θ]
[Θ|X]
HOMO APRIORIUS
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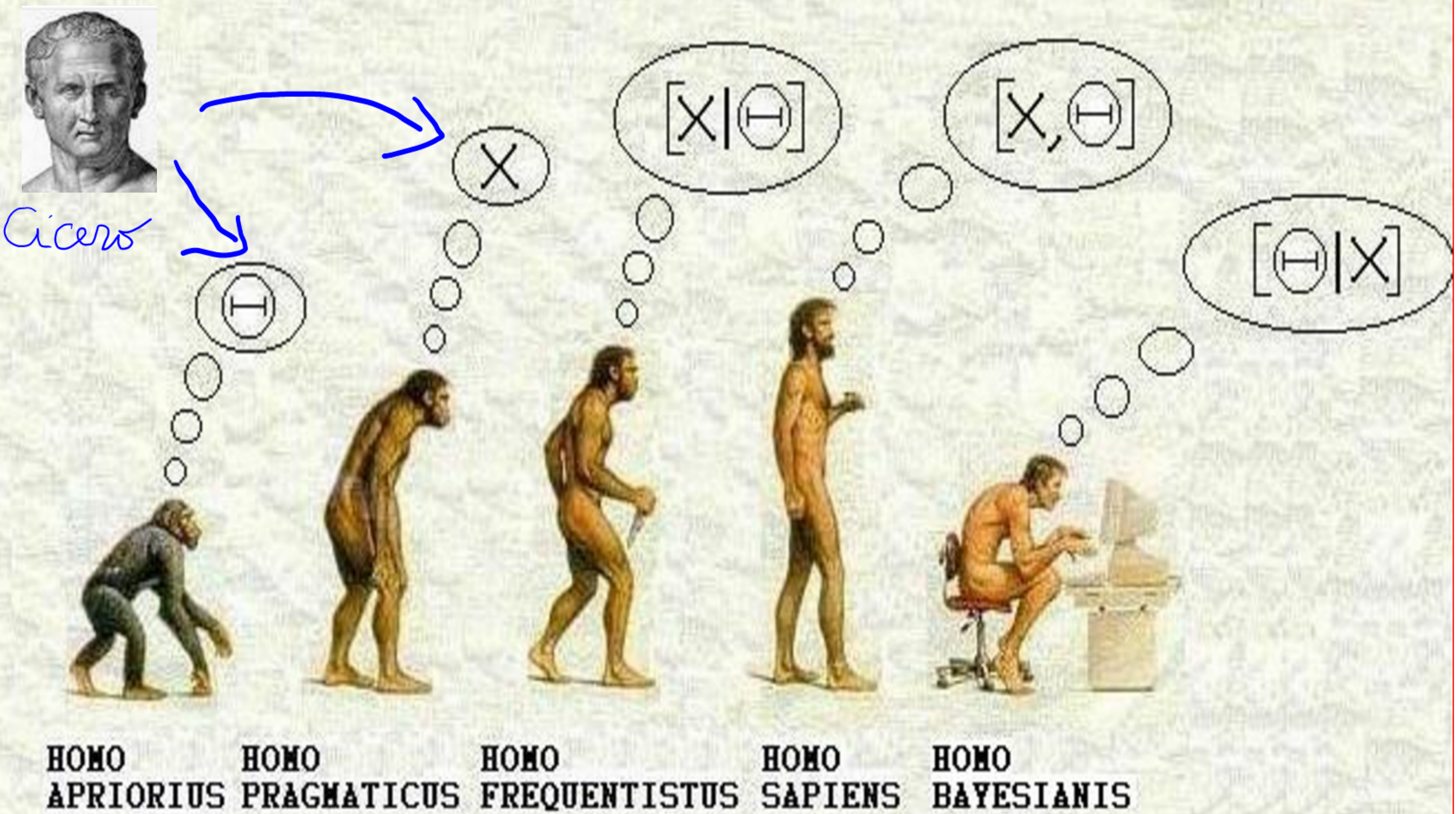

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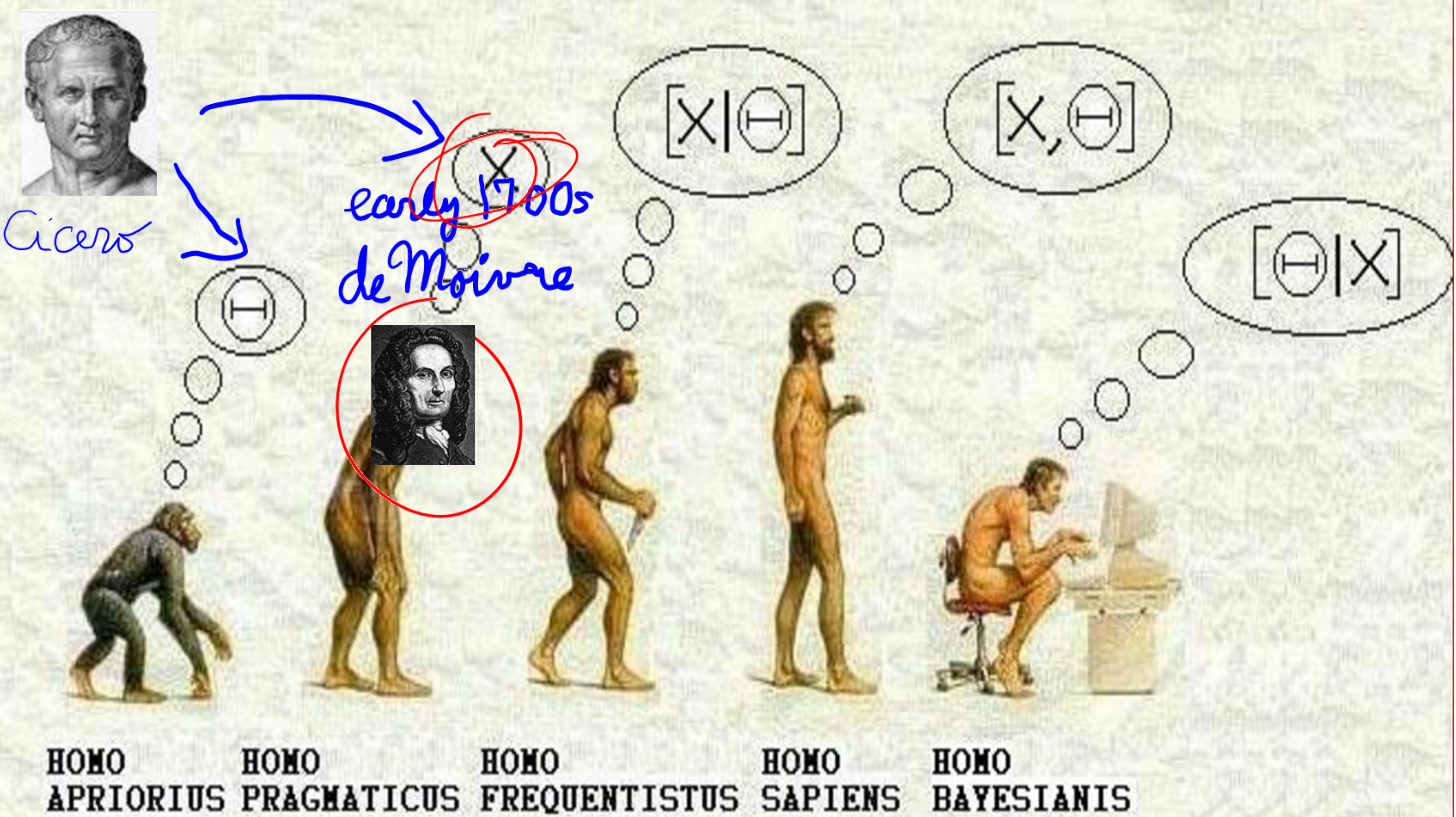

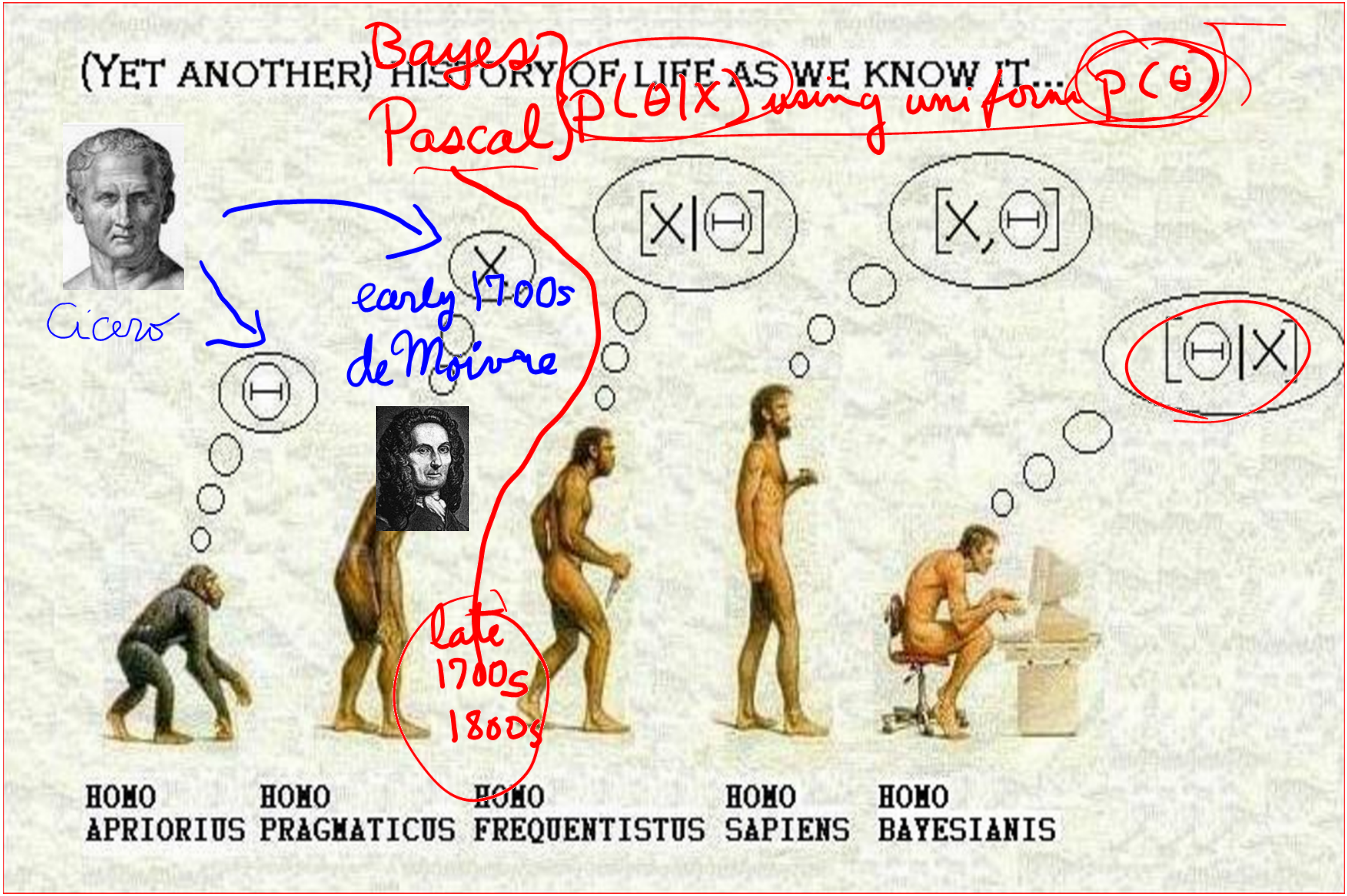

(YET ANOTHER) HISTORY OF LIFE AS WE KNOW IT...
Bayes
Pascal
P(θ|X) using uniform P(θ)
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[Θ]
early 1700s
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[X,Θ]
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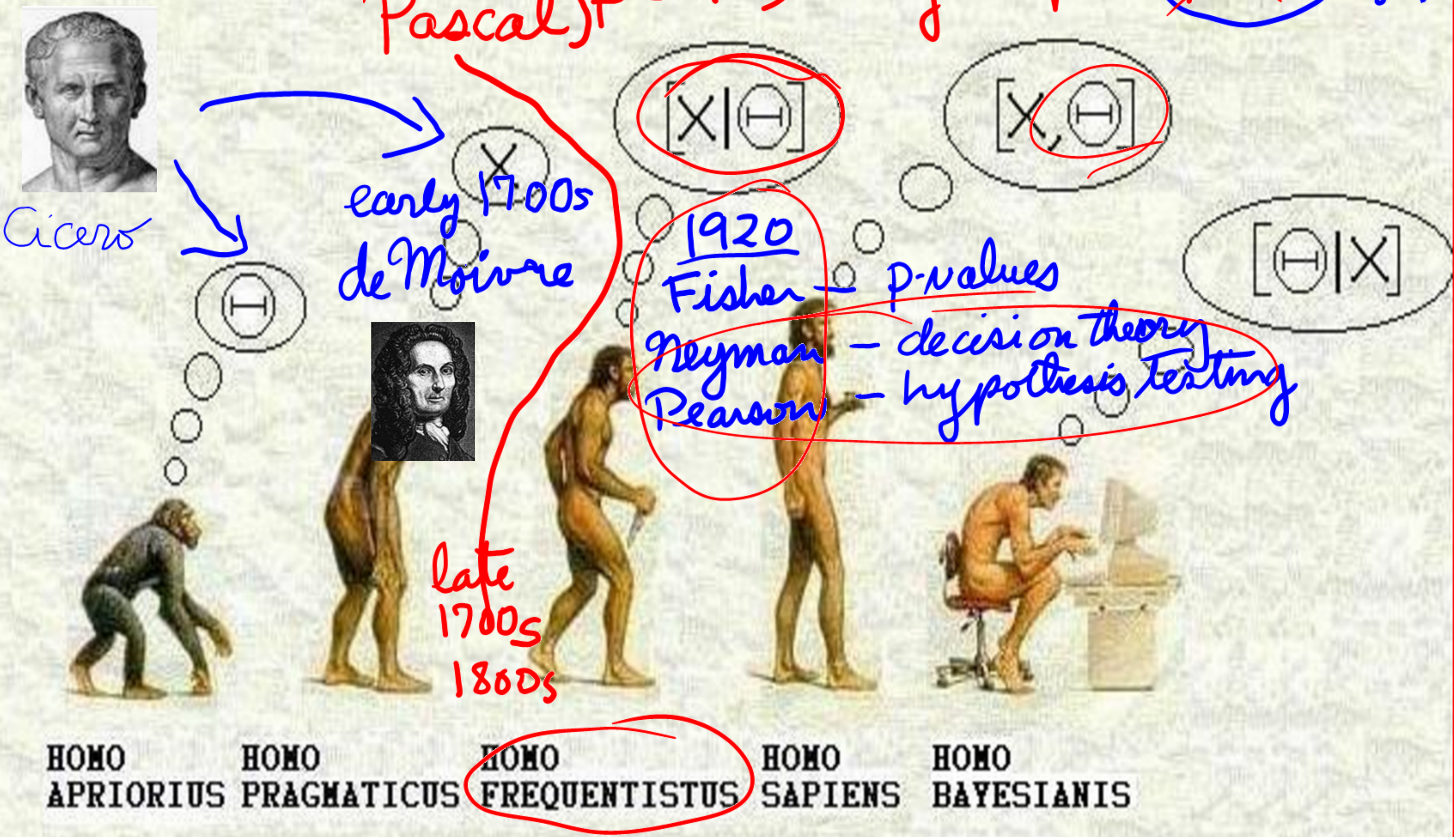

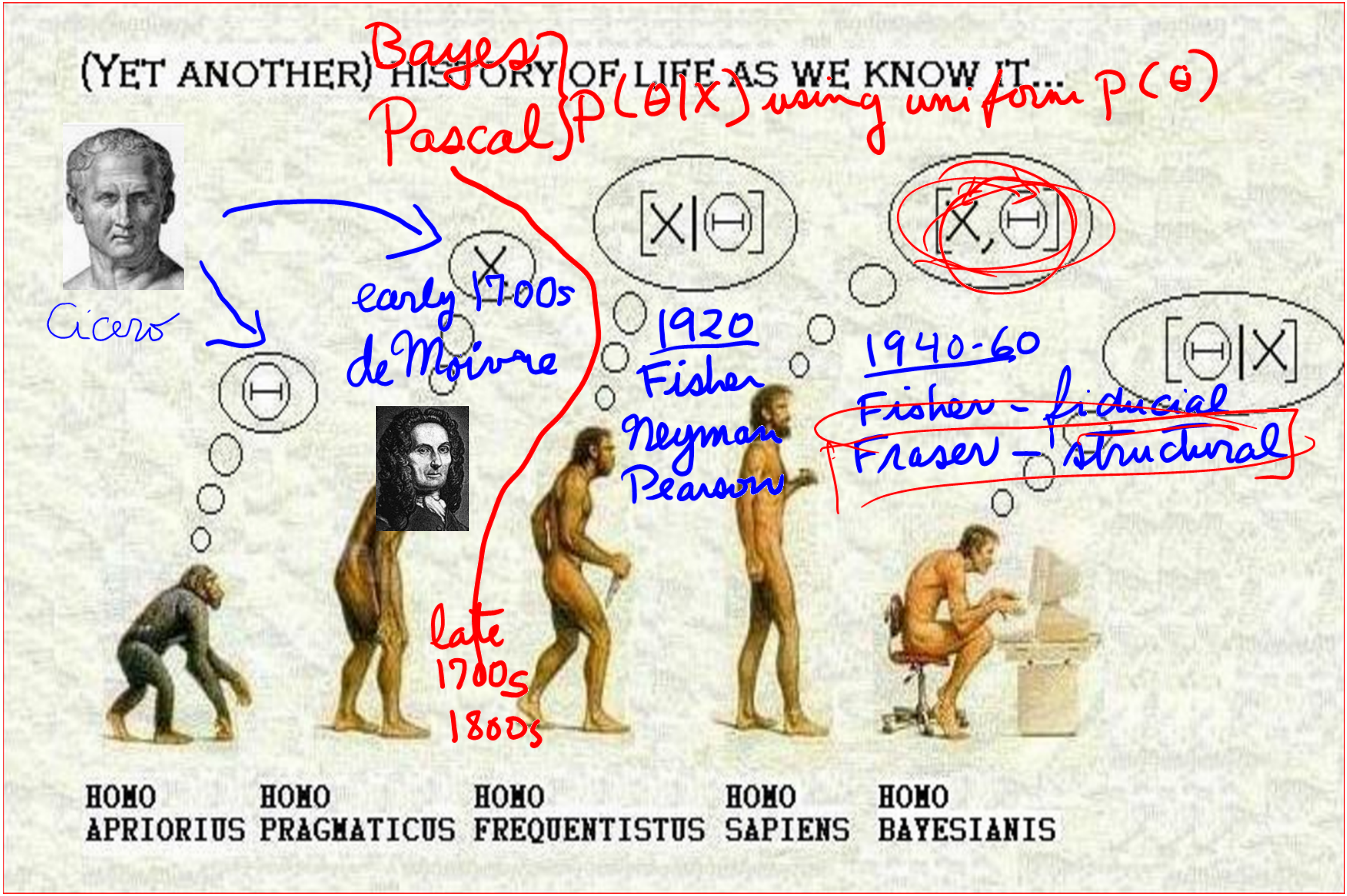
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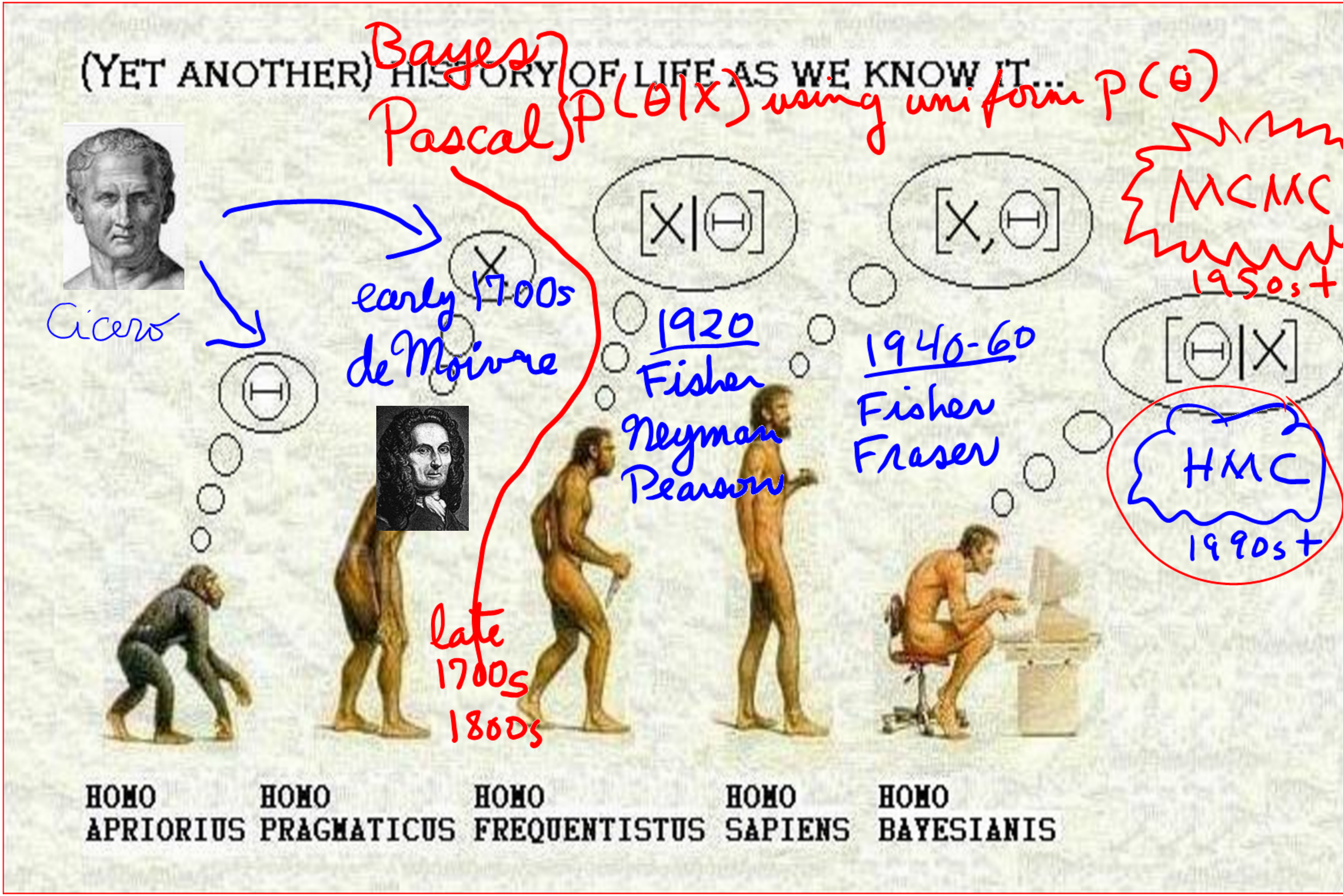
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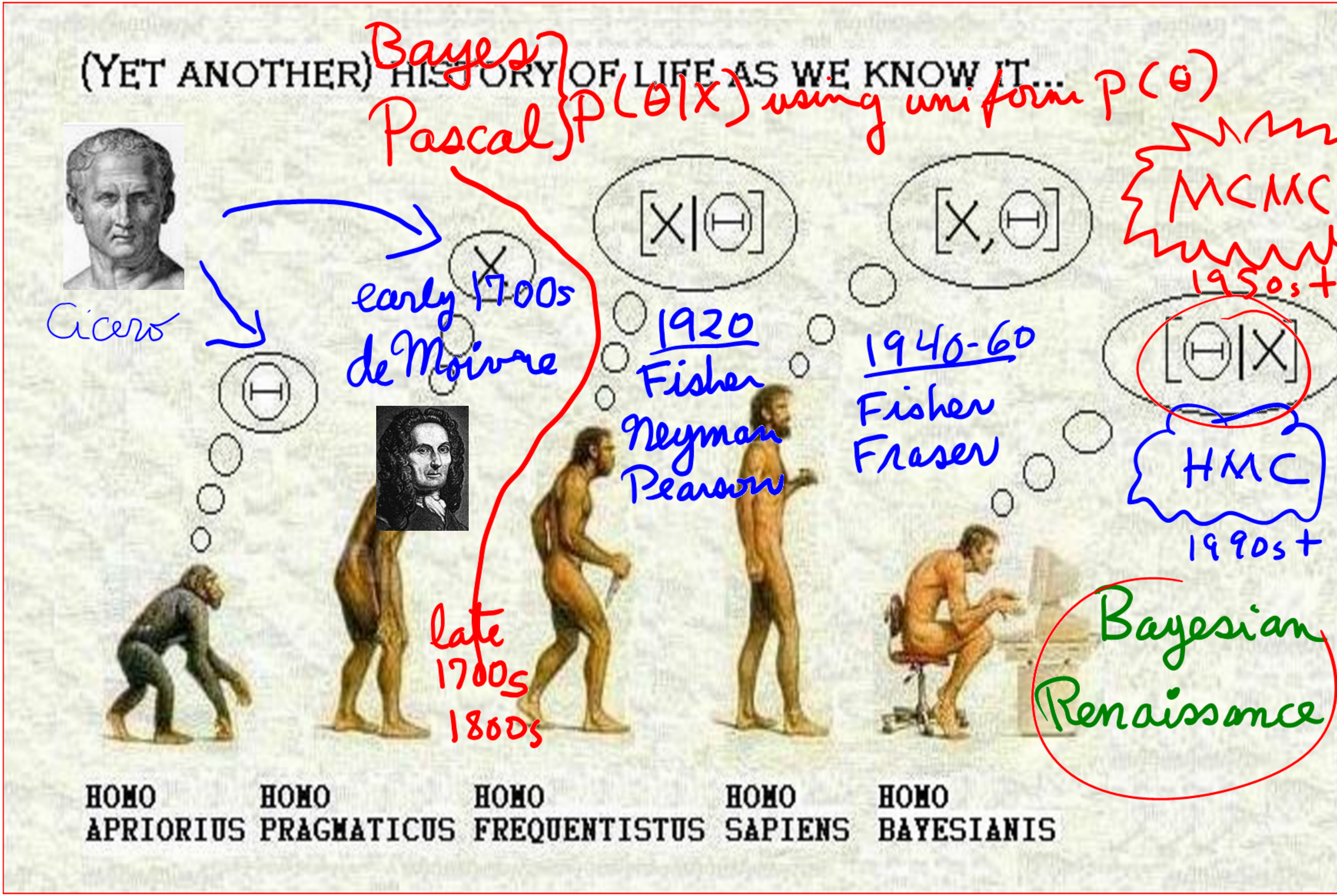
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~~Basic~~ Philosophical problem

Philosophical ~~Basic~~ problem

Given a model $P(X | \theta)$

Philosophical ~~Basic~~ problem

Given a model $P(X|\theta)$

To get $P(\theta|X)$

you need to be willing

to specify $P(\theta)$

Philosophical ~~Basic~~ problem

Given a model $P(X|\theta)$

To get $P(\theta|X)$

you need to be willing

to specify $P(\theta)$

Then $P(X,\theta) = P(X|\theta)P(\theta)$

and $P(\theta|X) = \dfrac{P(X,\theta)}{P(X)}$

$P(A|B) = \dfrac{P(A,B)}{P(B)}$

Philosophical ~~Basic~~ problem

Given a model $P(X|\theta)$ Model

To get $P(\theta|X)$

you need to be willing

to specify $P(\theta)$ Prior

Then $P(X,\theta) = P(X|\theta)P(\theta)$

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Posterior

~~Basic~~ Philosophical problem

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To get $P(\theta|X)$

you need to be willing to specify $P(\theta)$ Prior

Then $P(X,\theta) = P(X|\theta)P(\theta)$

and $\underbrace{P(\theta|X)}_{\text{Posterior}} = \dfrac{P(X,\theta)}{P(X)}$

- You need a prior to get a posterior.
- Can we justify a particular prior?

Frequentists only use $P(X|\theta)$
and don't need $P(\theta)$

Frequentists only use $P(X \mid \theta)$ and don't need $P(\theta)$

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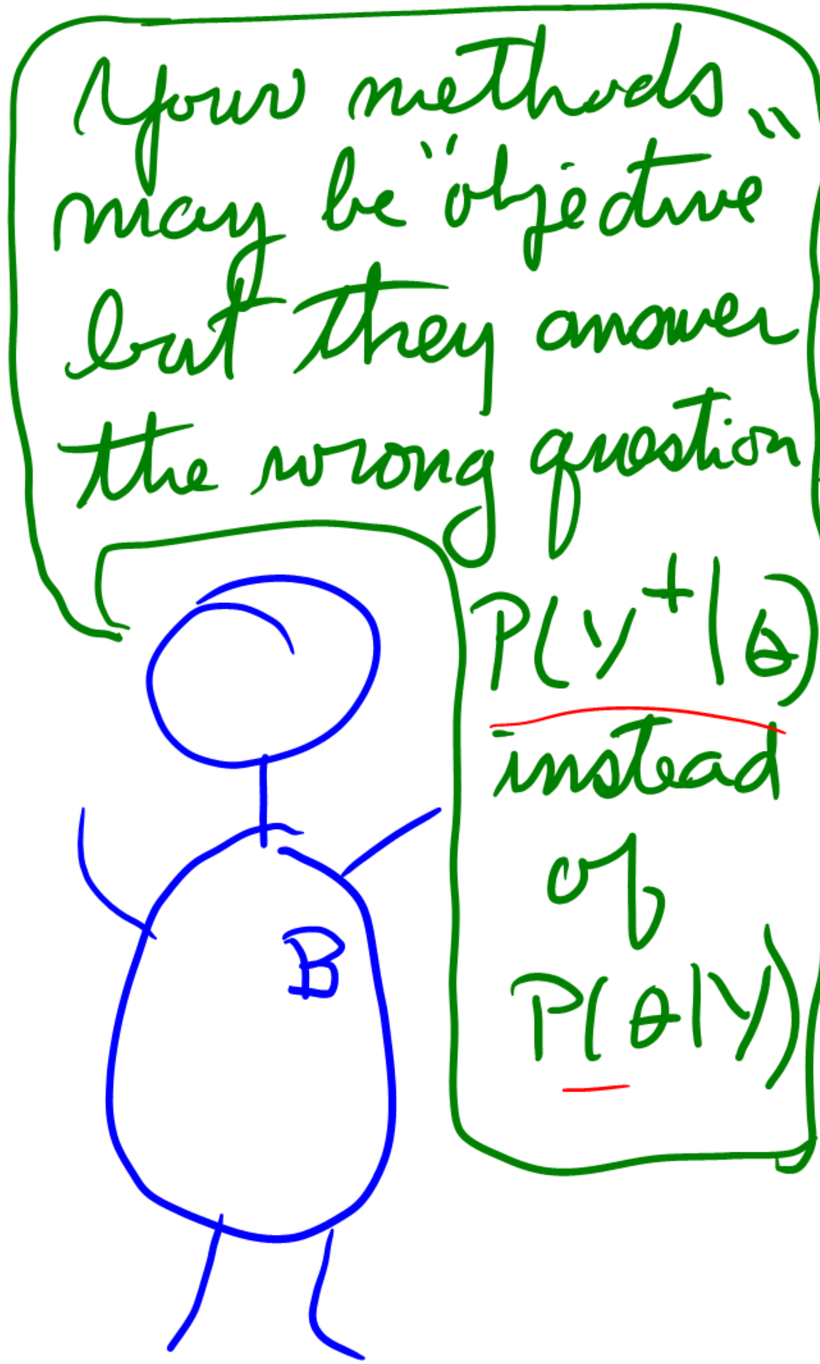

Frequentists only use $P(X|\theta)$ and don't need $P(\theta)$

Your methods are subjective. You have no objective justification for your prior

Fisher

Your methods may be "objective" but they answer the wrong question

$P(Y^+|\theta)$ instead of $P(\theta|Y)$

Practical problem:

$$p(X, \theta) = p(X \mid \theta)\, p(\theta)$$

Practical problem:

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$$\int P(X,\theta)\,d\theta$$

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$$p(X,\theta) = p(X|\theta)\,p(\theta)$$

$$p(\theta|X) = \frac{p(X,\theta)}{p(X)}$$

$$\int p(X,\theta)\,d\theta$$

If θ has high dimension this becomes easily impossible.

Practical problem:

$$P(X, \theta) = P(X \mid \theta)\, P(\theta)$$

$$P(\theta \mid X) = \frac{P(X, \theta)}{P(X)}$$

MCMC (mid 20th C.)

comes to the rescue:

It's possible to sample from $P(\theta \mid X)$ knowing only $P(X, \theta)$

Posteriors without priors?

Fisher - Fiducial inference
Fraser - Structural inference
Objective Bayesian inference

Baking the Bayesian omelette without breaking the Bayesian egg.

Emerging practice:
Using weakly informative priors.

Posteriors without priors?

Fisher - Fiducial inference

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Objective Bayesian inference

The reason why thinking of a 95% CI for a linear regression parameter as a 95% "probability" interval is harmless.

Baking the Bayesian omelette without breaking the Bayesian egg.

Emerging practice:

Using weakly informative priors.

Posteriors without priors?

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The reason why thinking of a 95% CI for a linear regression parameter as a 95% "probability" interval is harmless.

Baking the Bayesian omelette without breaking the Bayesian egg.

But don't take it for granted
Remember Sally Clark

Emerging practice:
Using weakly informative priors.

Markov Chain Monte Carlo

Use

$$\underline{P(\theta, X)} = \underline{P(X|\theta)\, P(\theta)}$$

Markov Chain Monte Carlo

Use $P(\theta, X) = P(X|\theta)\, P(\theta)$

joint model × prior

Samples from $P(\theta|X)$ using only $P(\theta, X)$

i.e. no need to find elusive $P(X)$

Markov Chain Monte Carlo

Use $P(\theta, X) = P(X|\theta)\, P(\theta)$

joint model × prior

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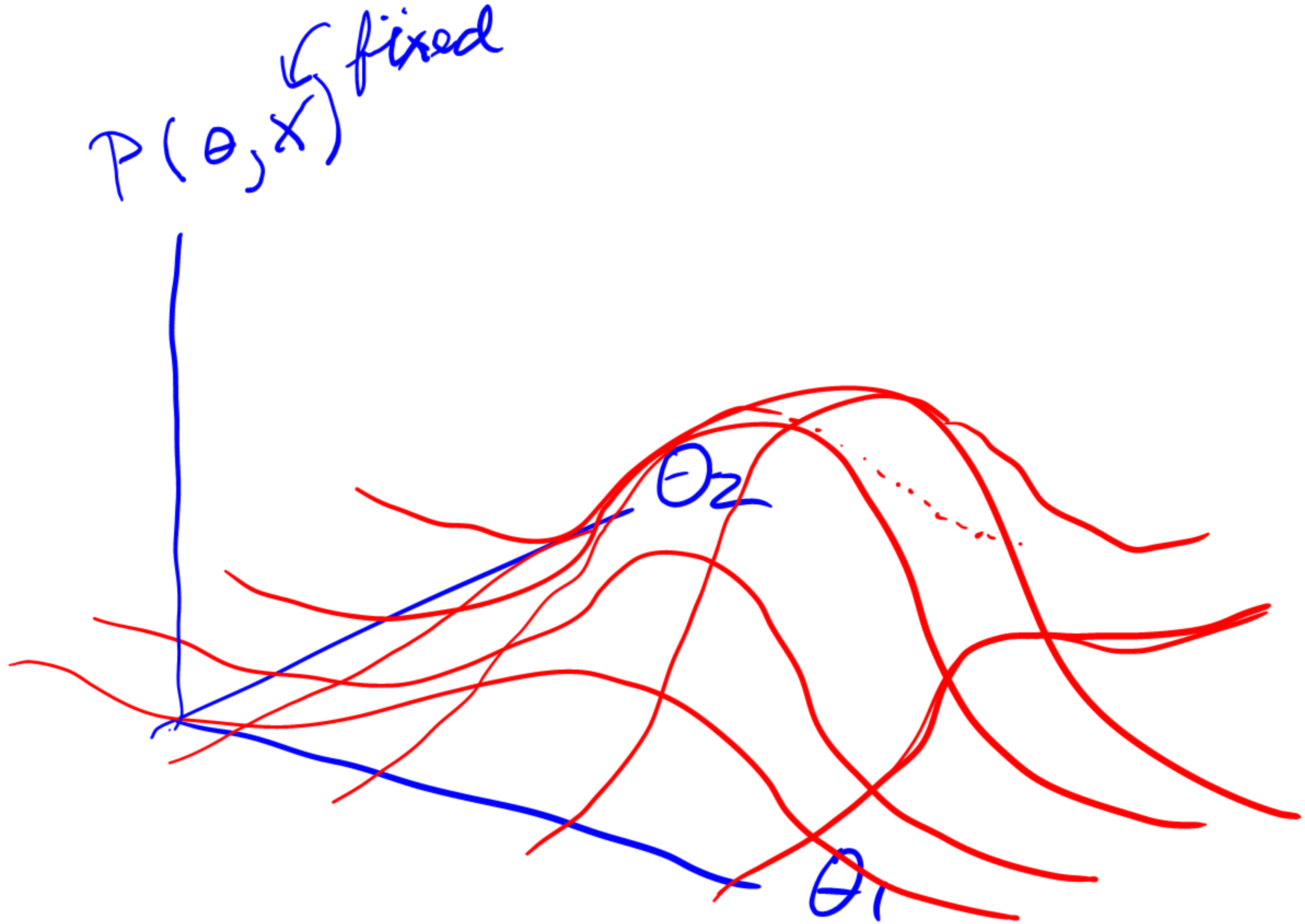
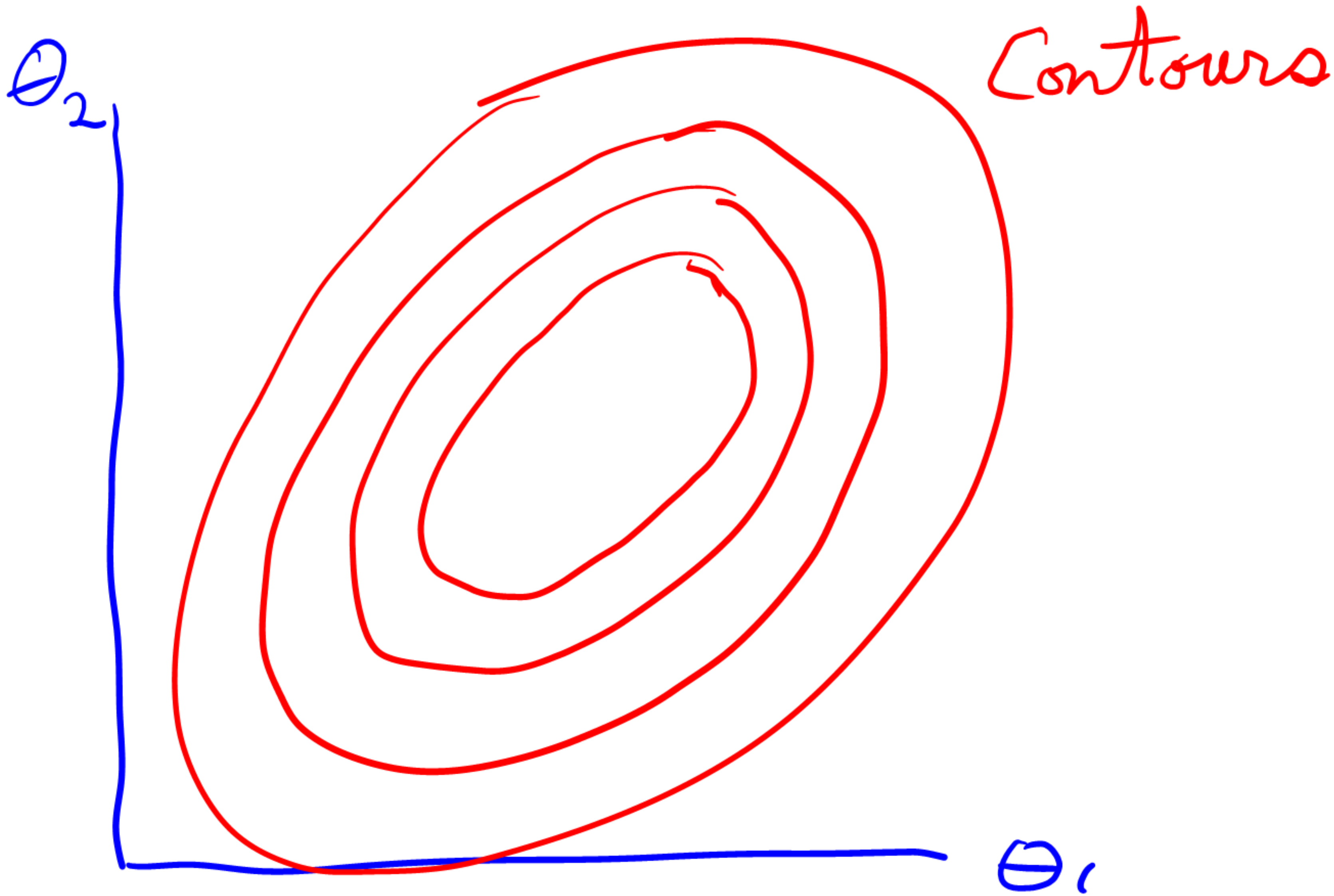
With X fixed, think of $P(\theta, X)$

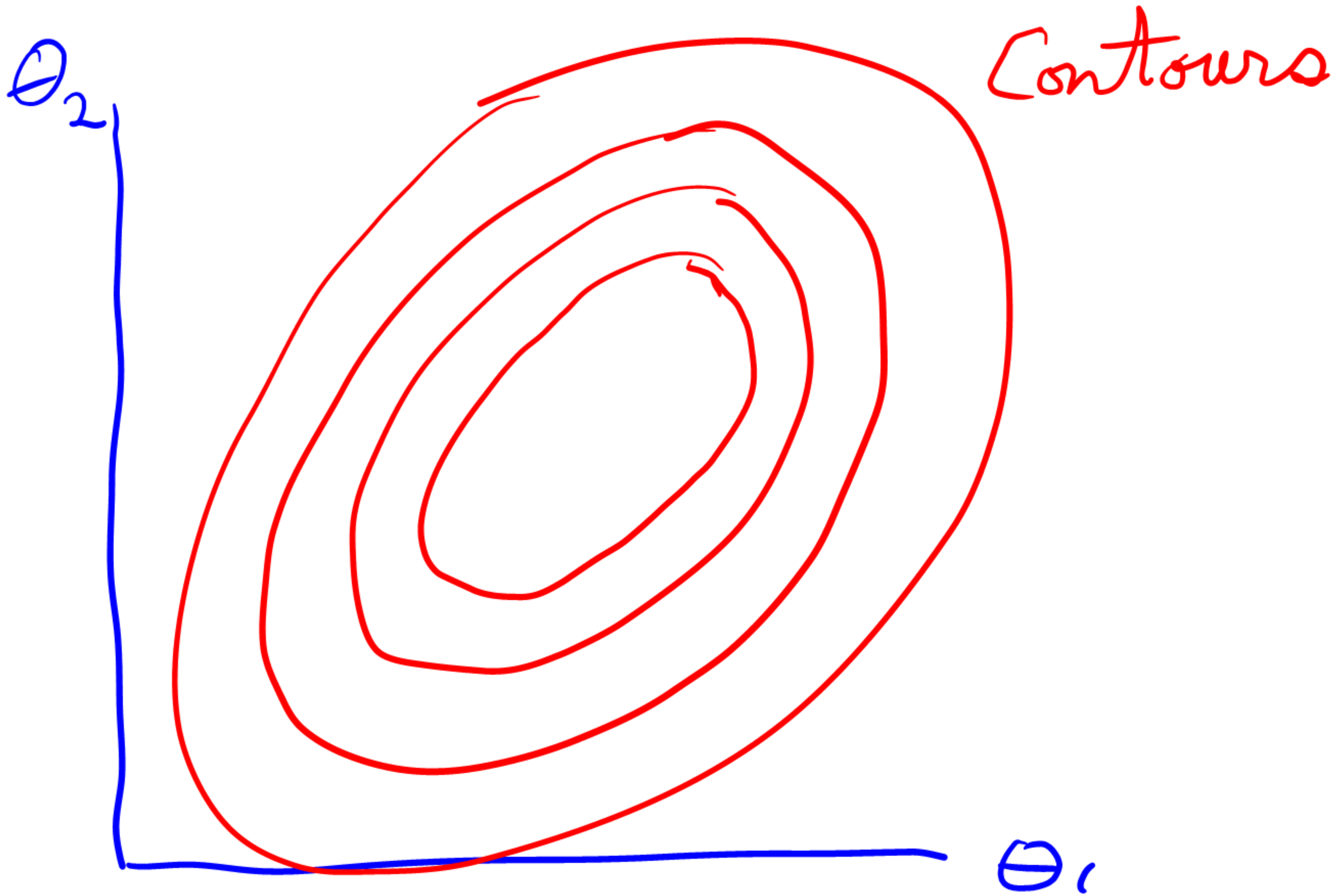
as defining a mountain over θ space



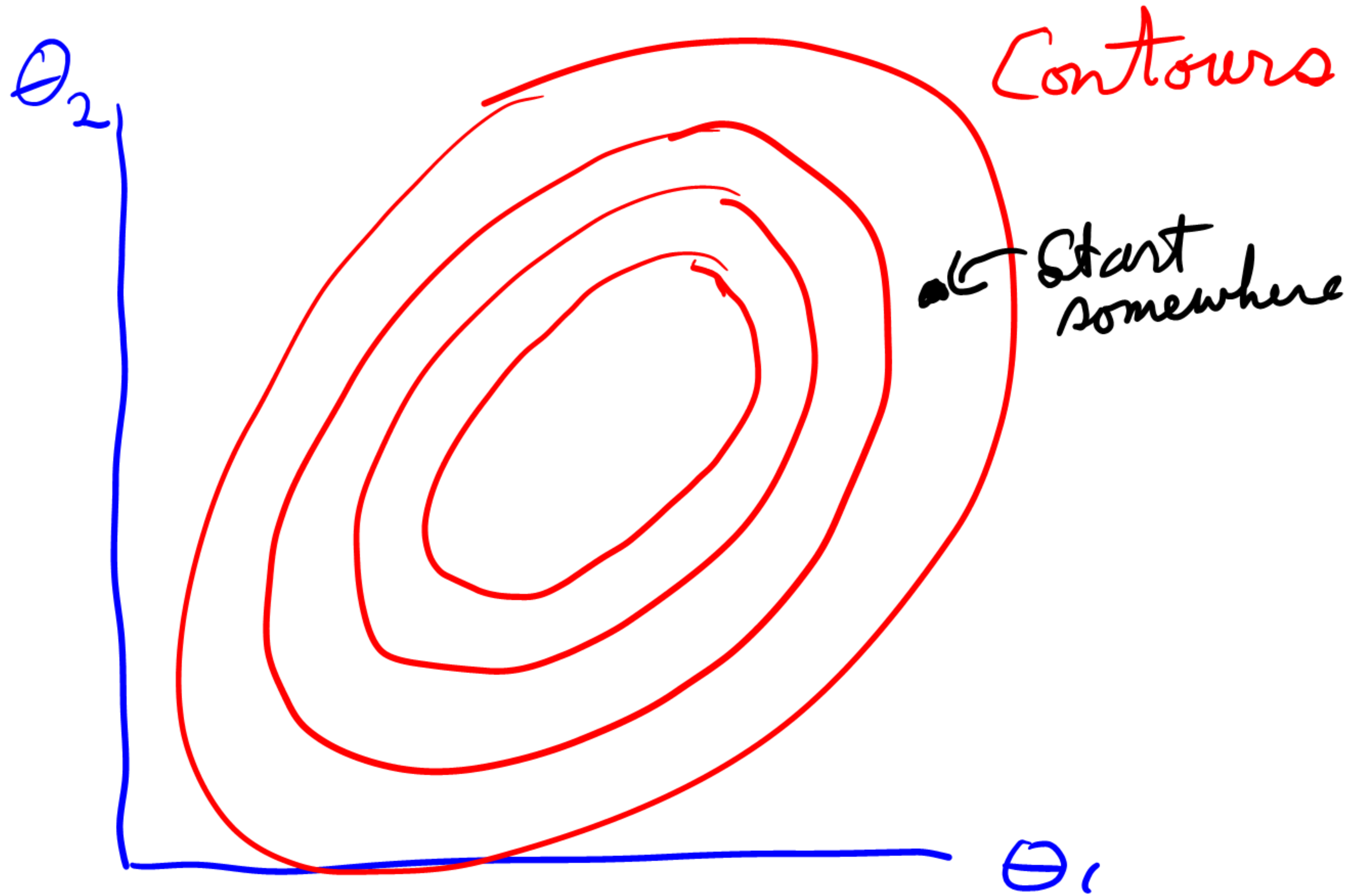
$P(\theta, x)$
fixed
θ_2
θ_1



Contours
θ_2
θ_1



Metropolis-Hastings algorithm



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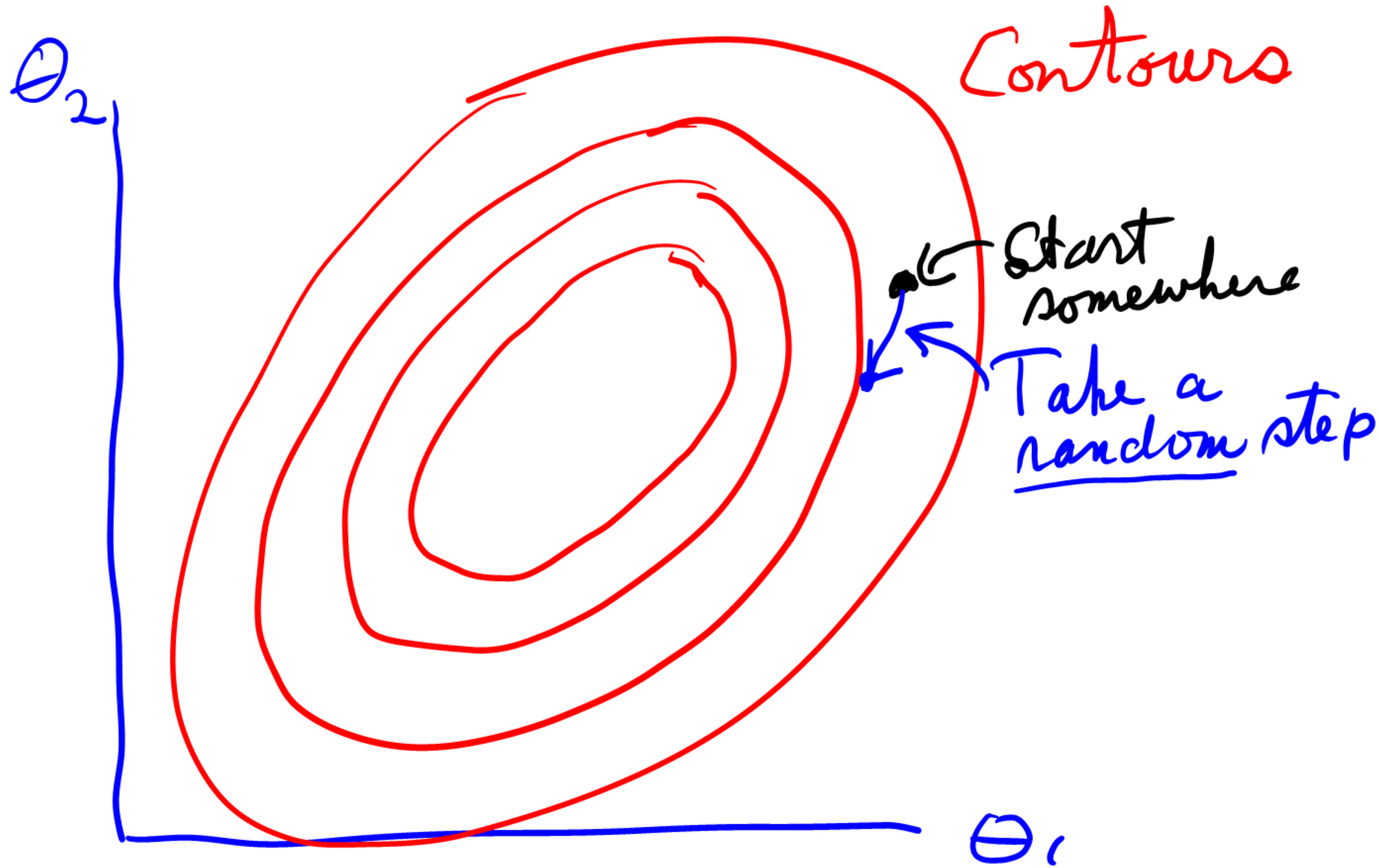

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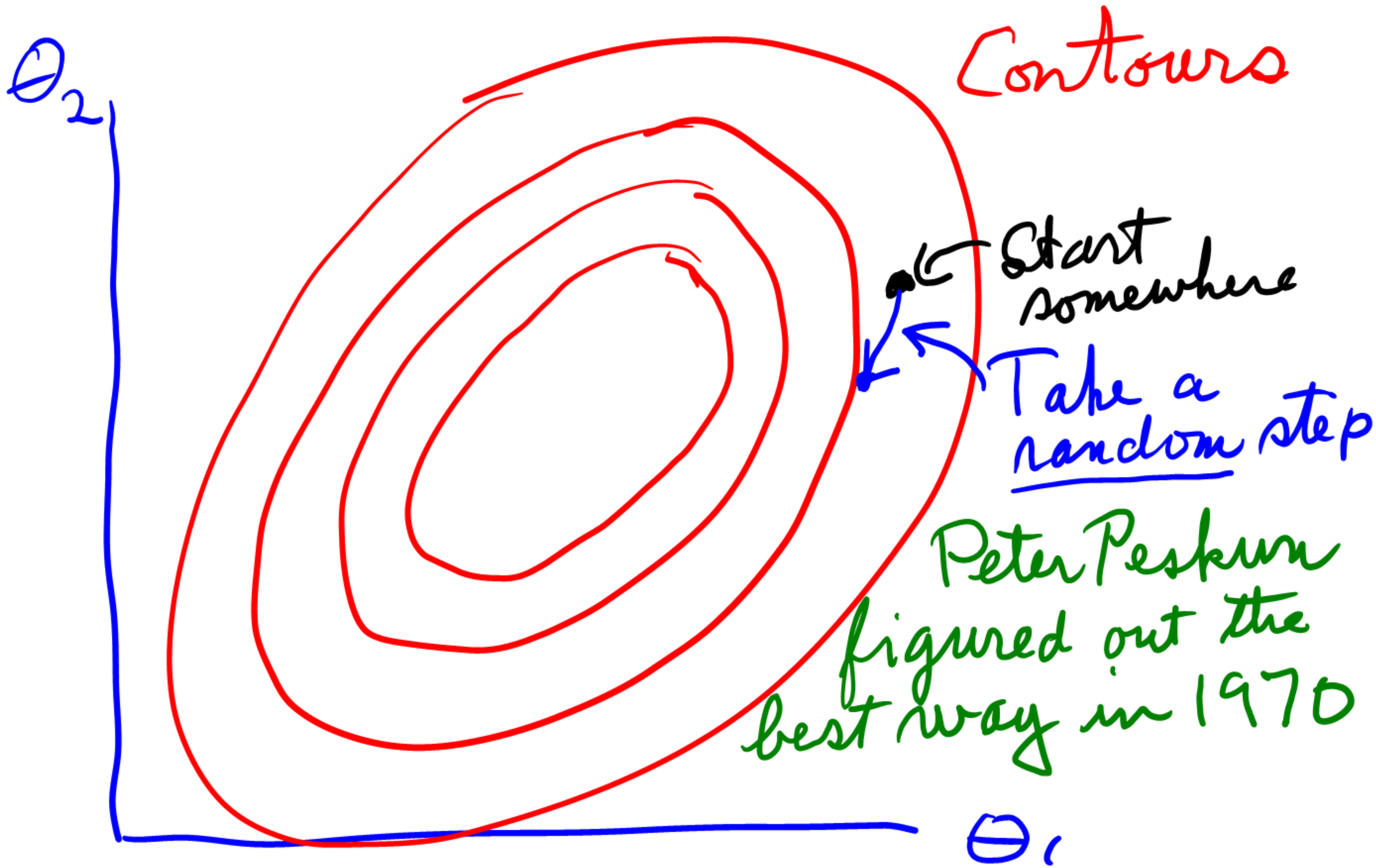

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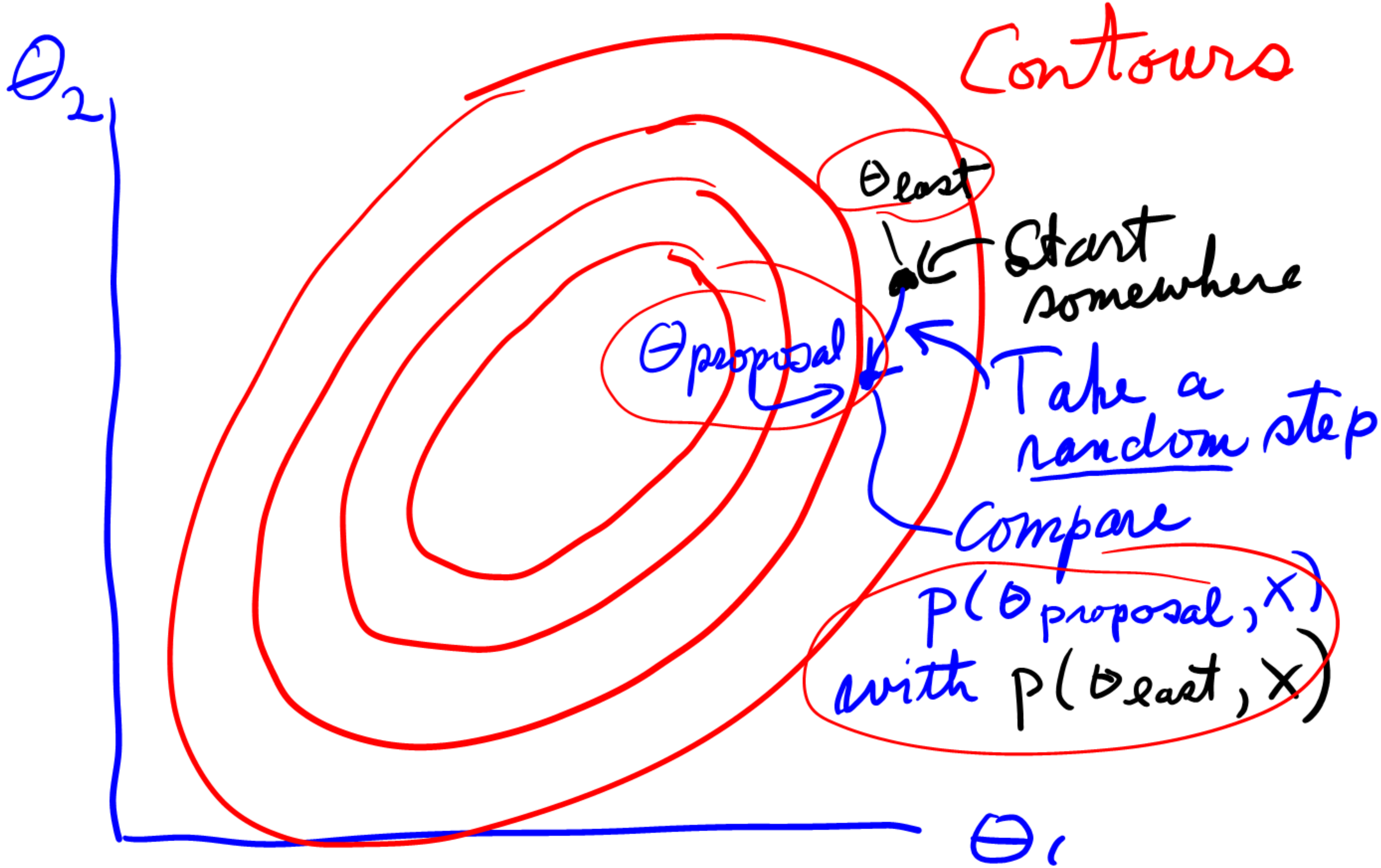

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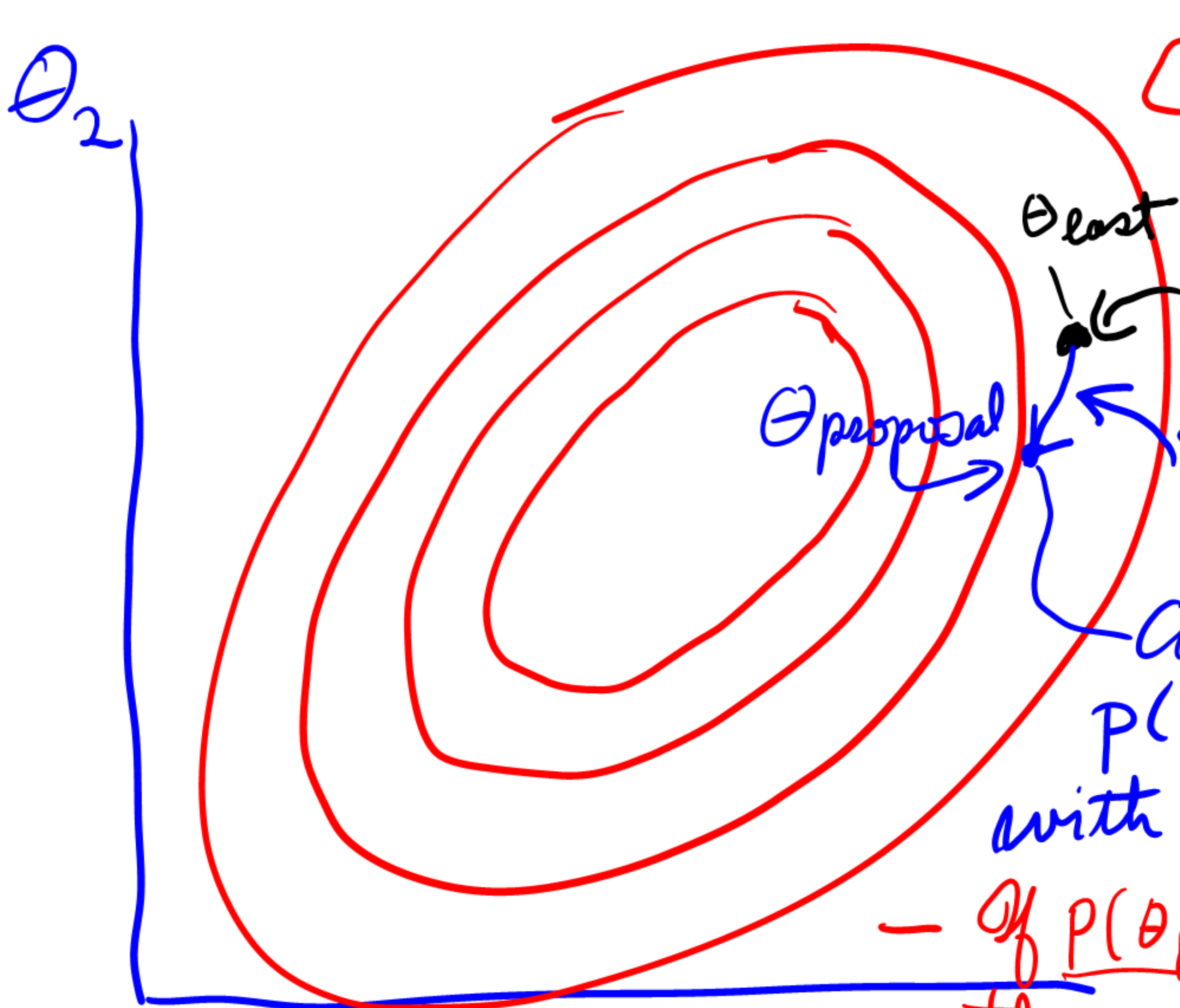
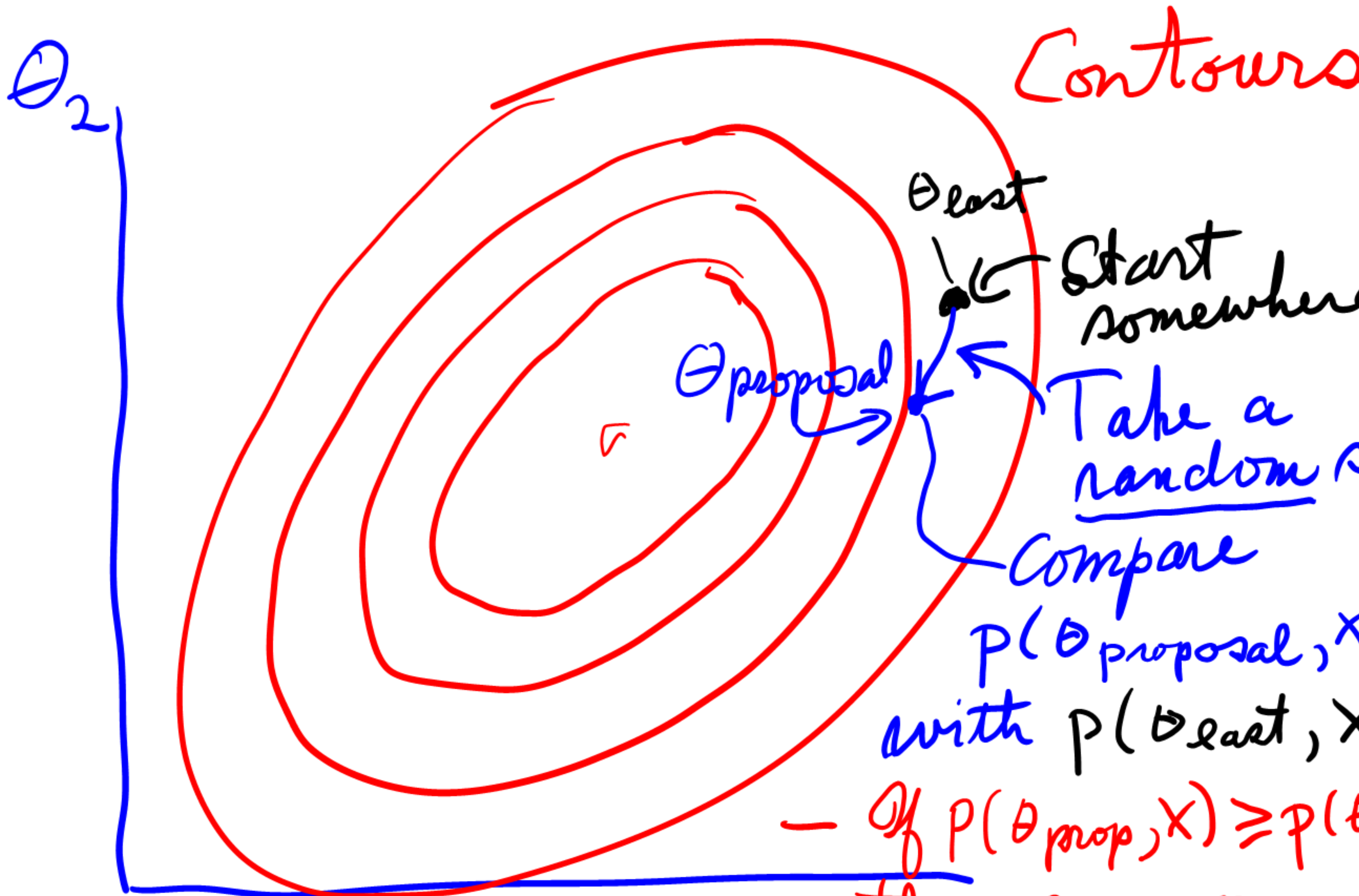


Start somewhere

Take a random step

Compare $p(\theta_{proposal}, X)$
with $p(\theta_{last}, X)$

– If $p(\theta_{prop}, X) \geq p(\theta_{last}, X)$
then $\theta_{new} = \theta_{proposal}$



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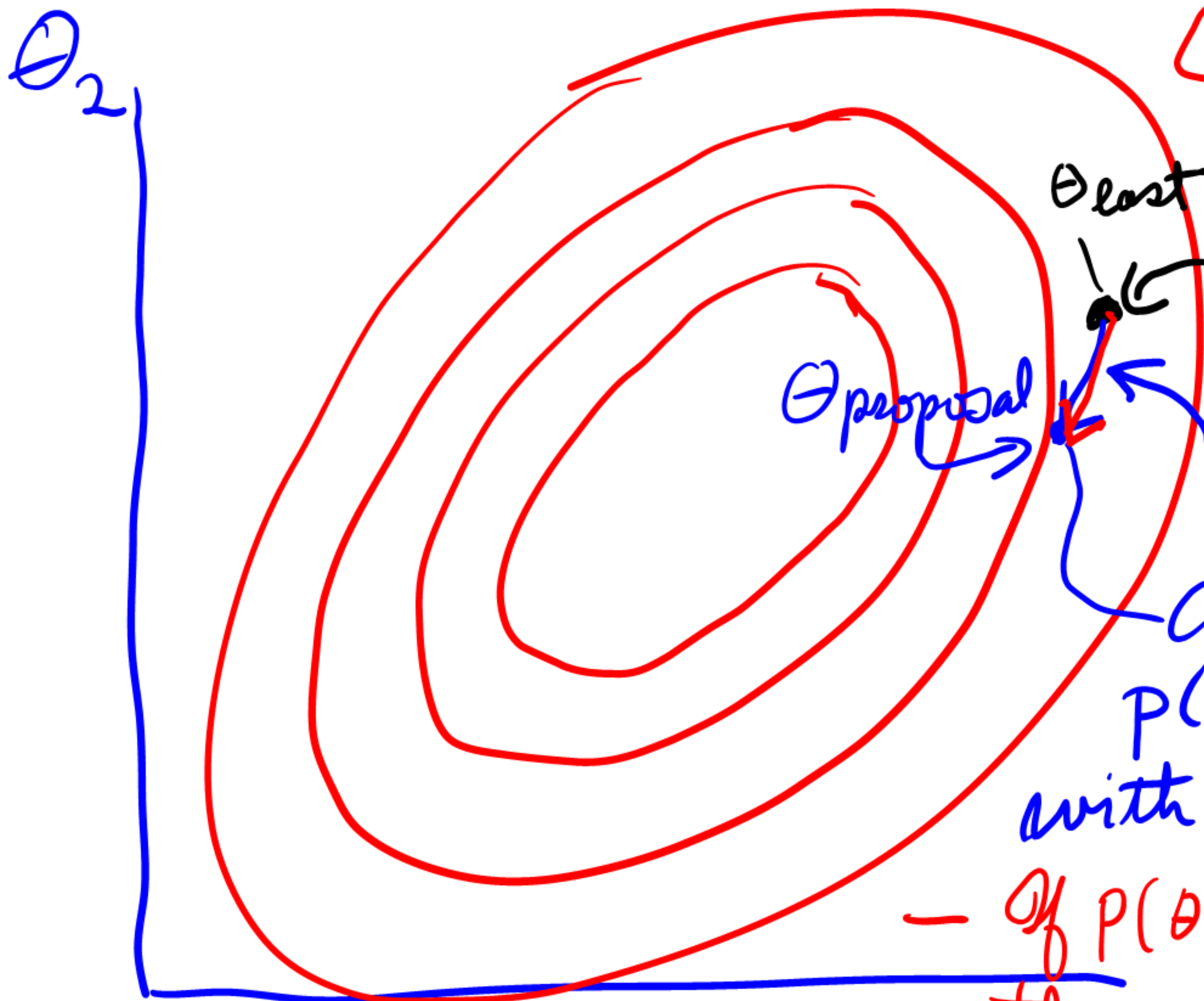
- Otherwise, i.e. if you have moved down in probability, let $\theta_{new} = \theta_{proposal}$ with prob $\dfrac{p(\theta_{prop}, X)}{p(\theta_{last}, X)}$ else $\theta_{new} = \theta_{last}$



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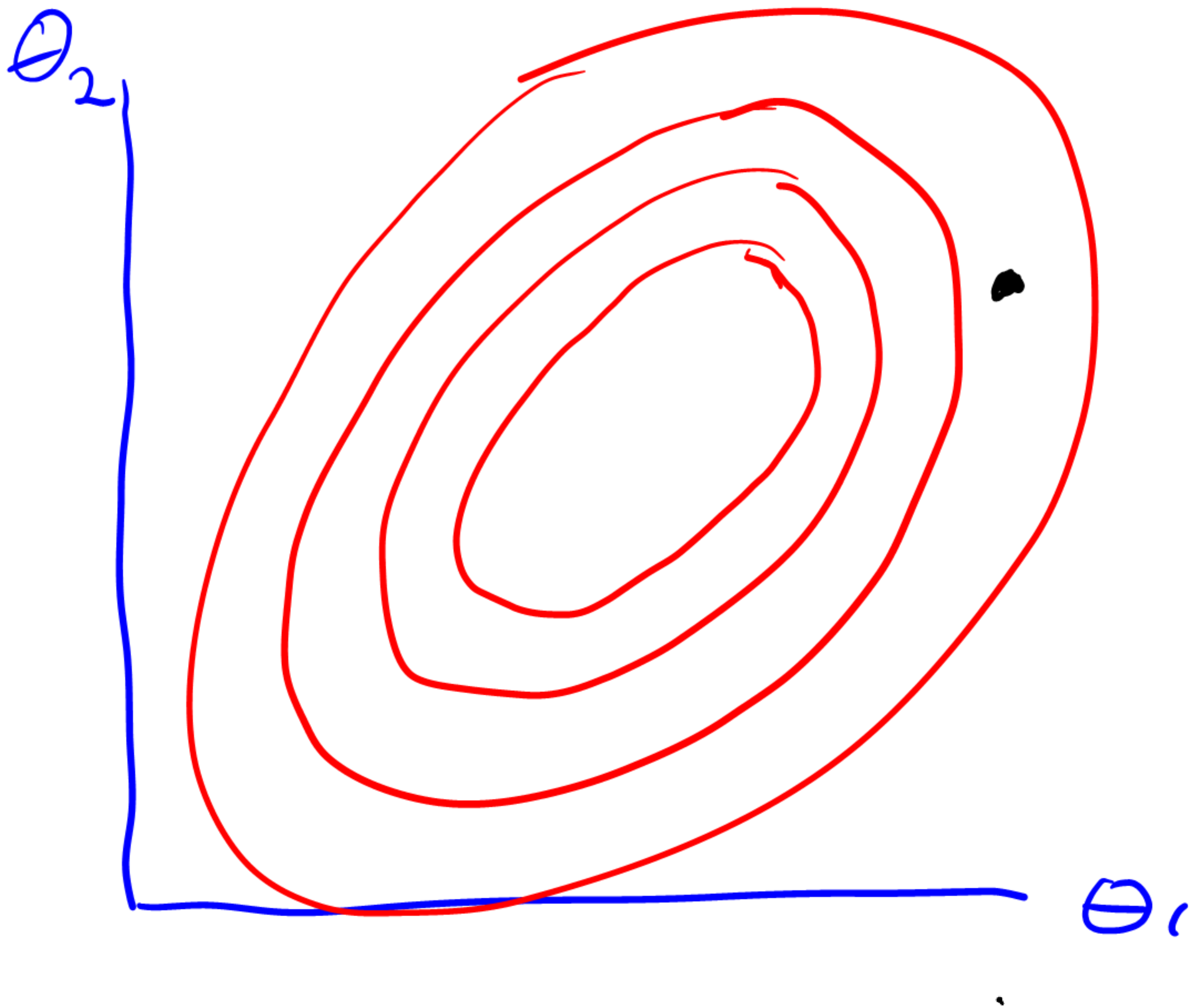
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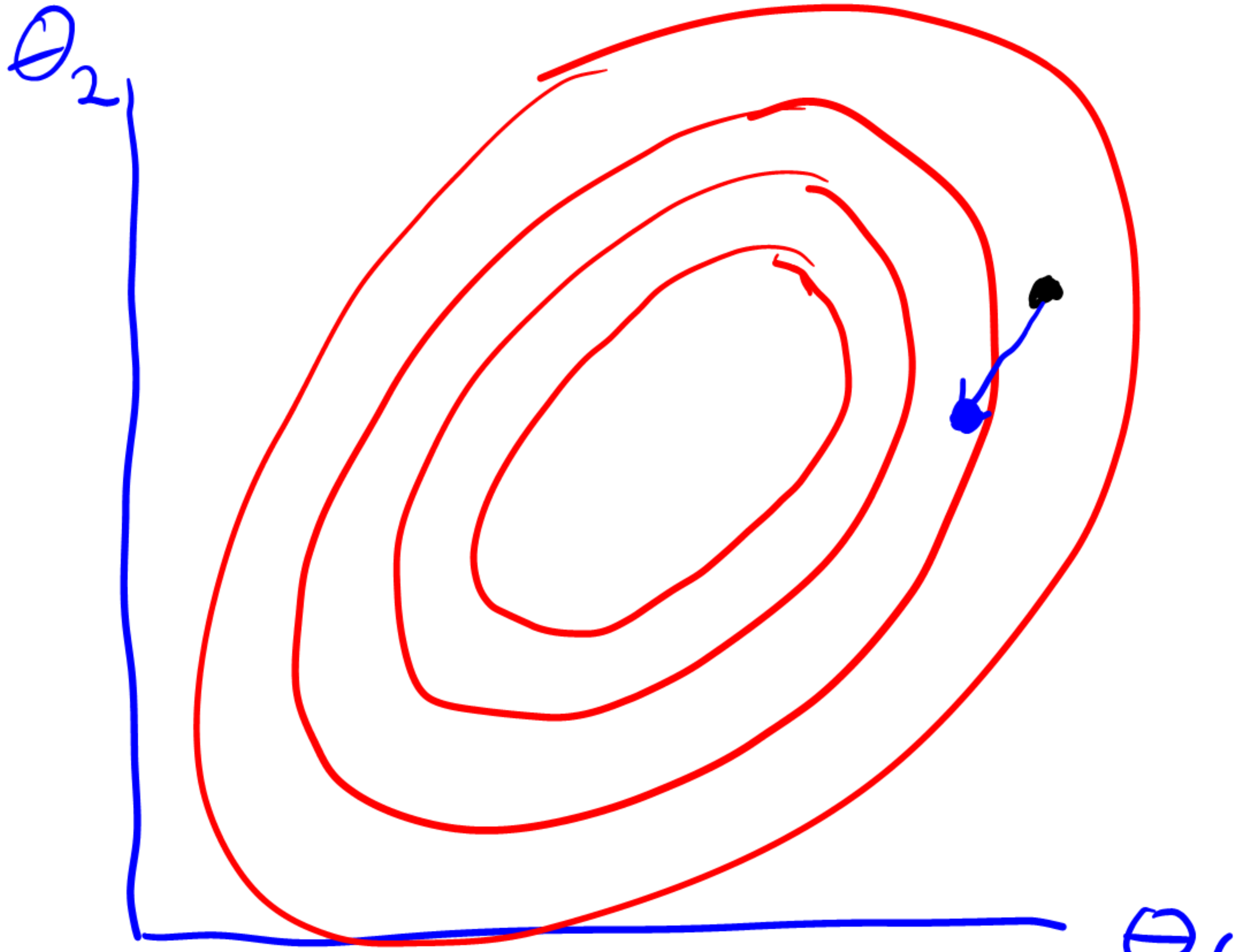
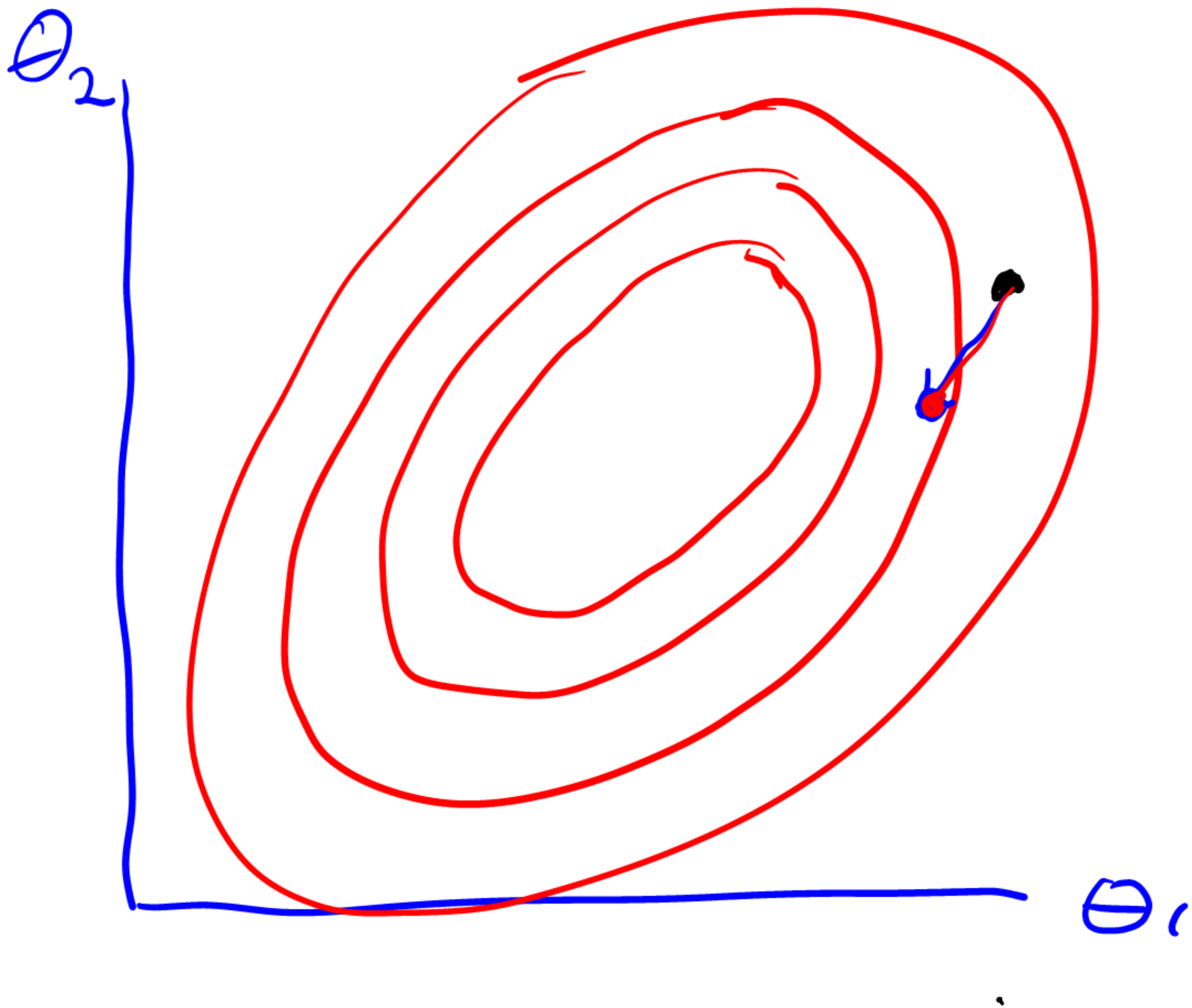
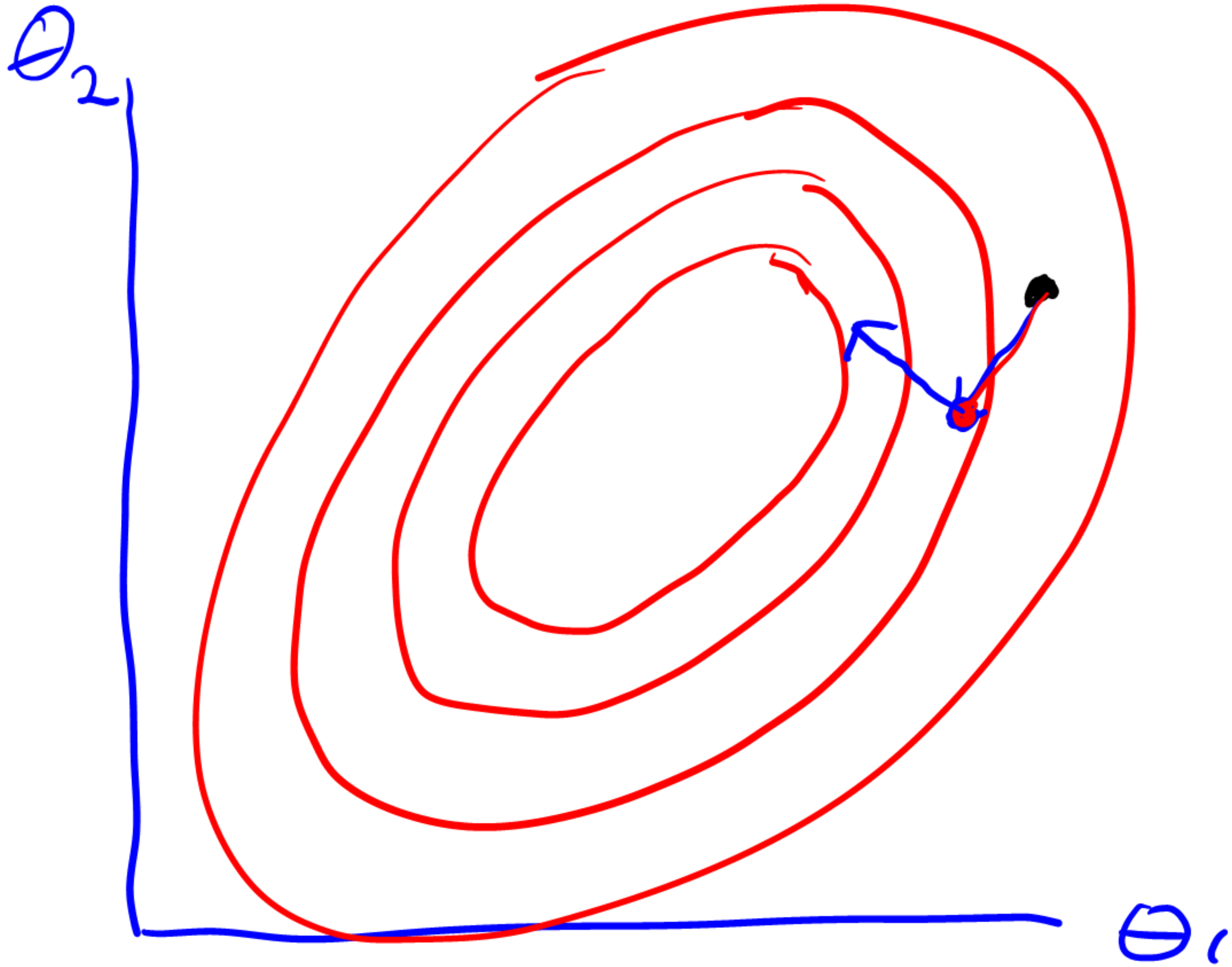
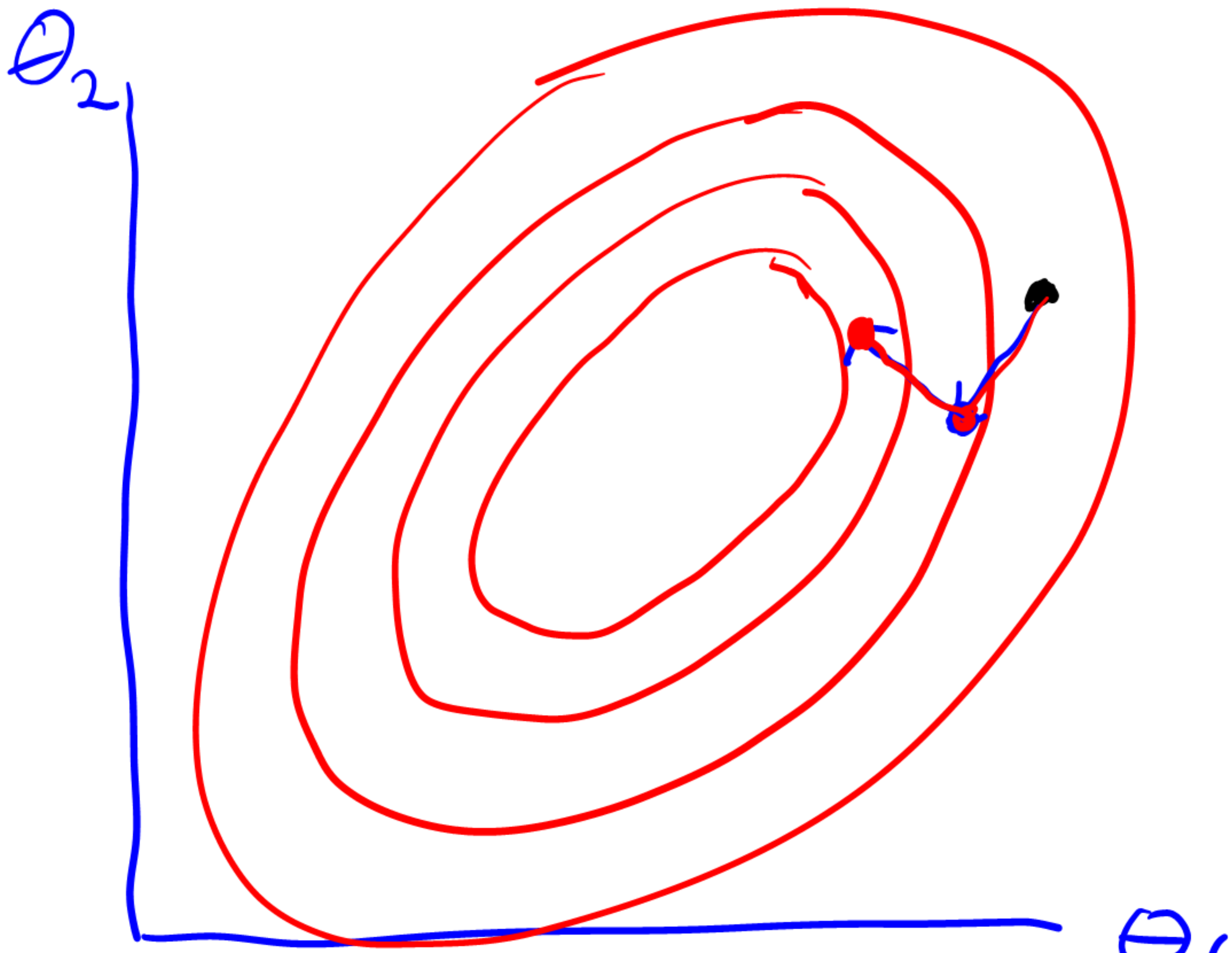
θ_1


θ_2
θ_1

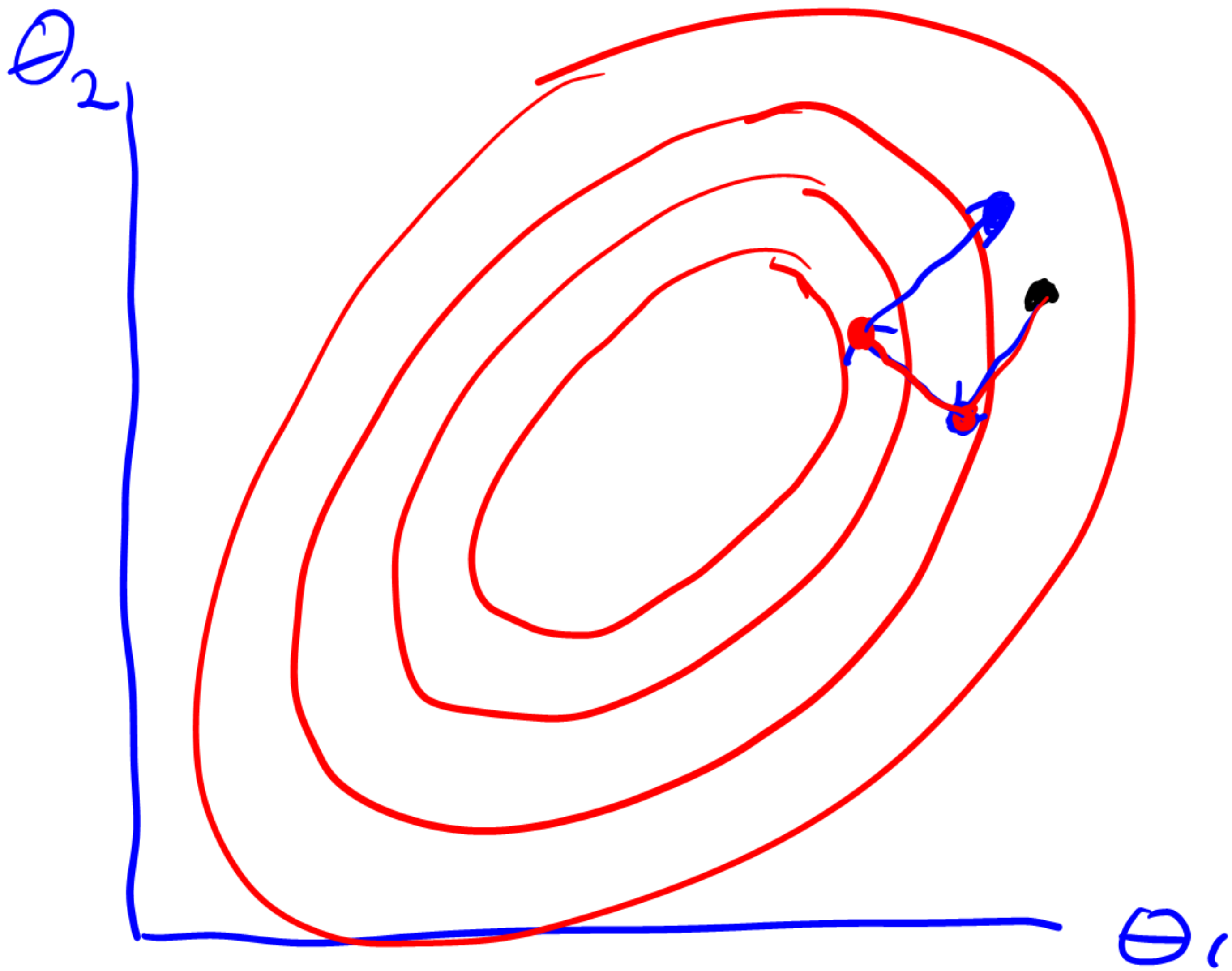

θ_2
θ_1



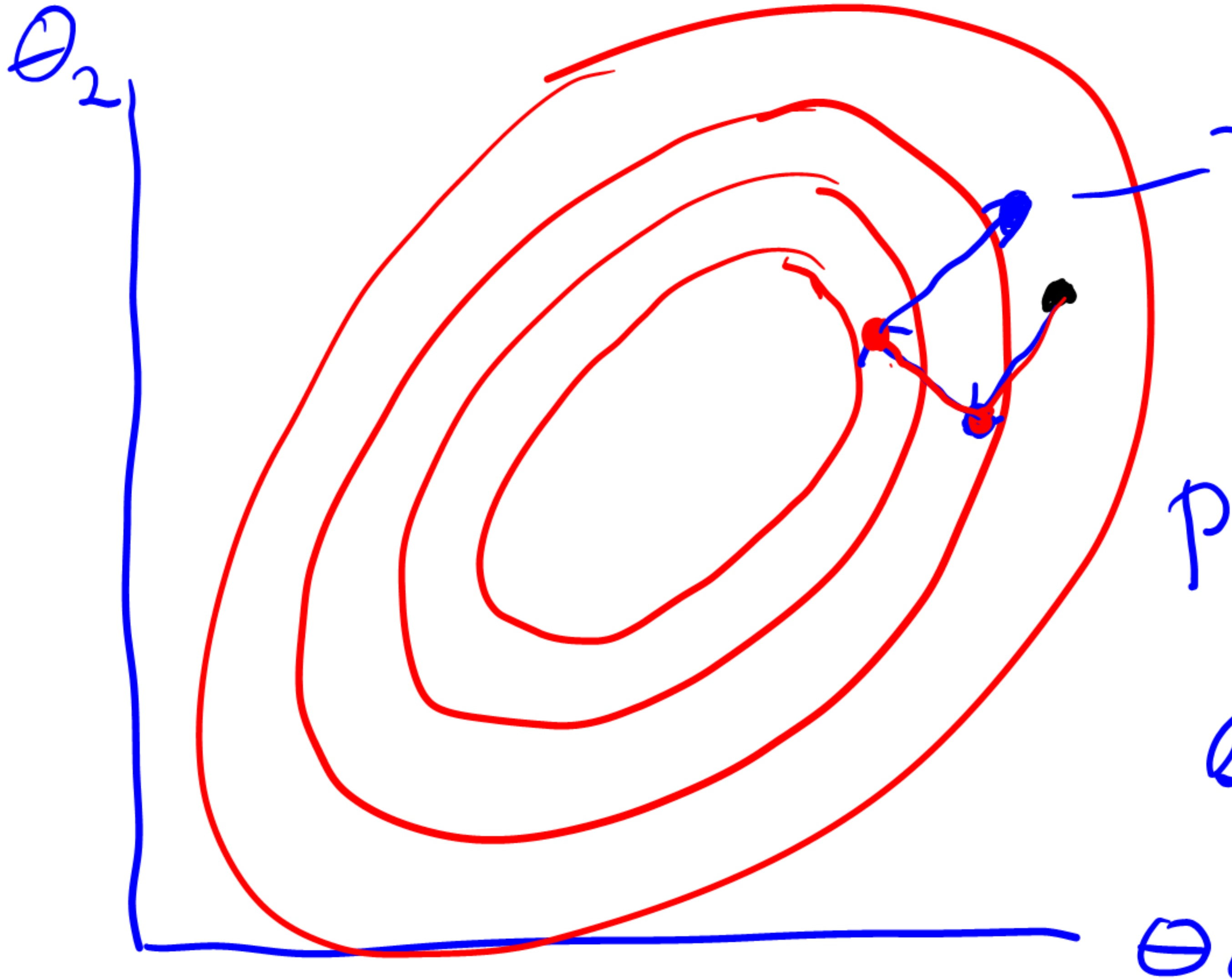
θ2
θ1



θ_2
θ_1



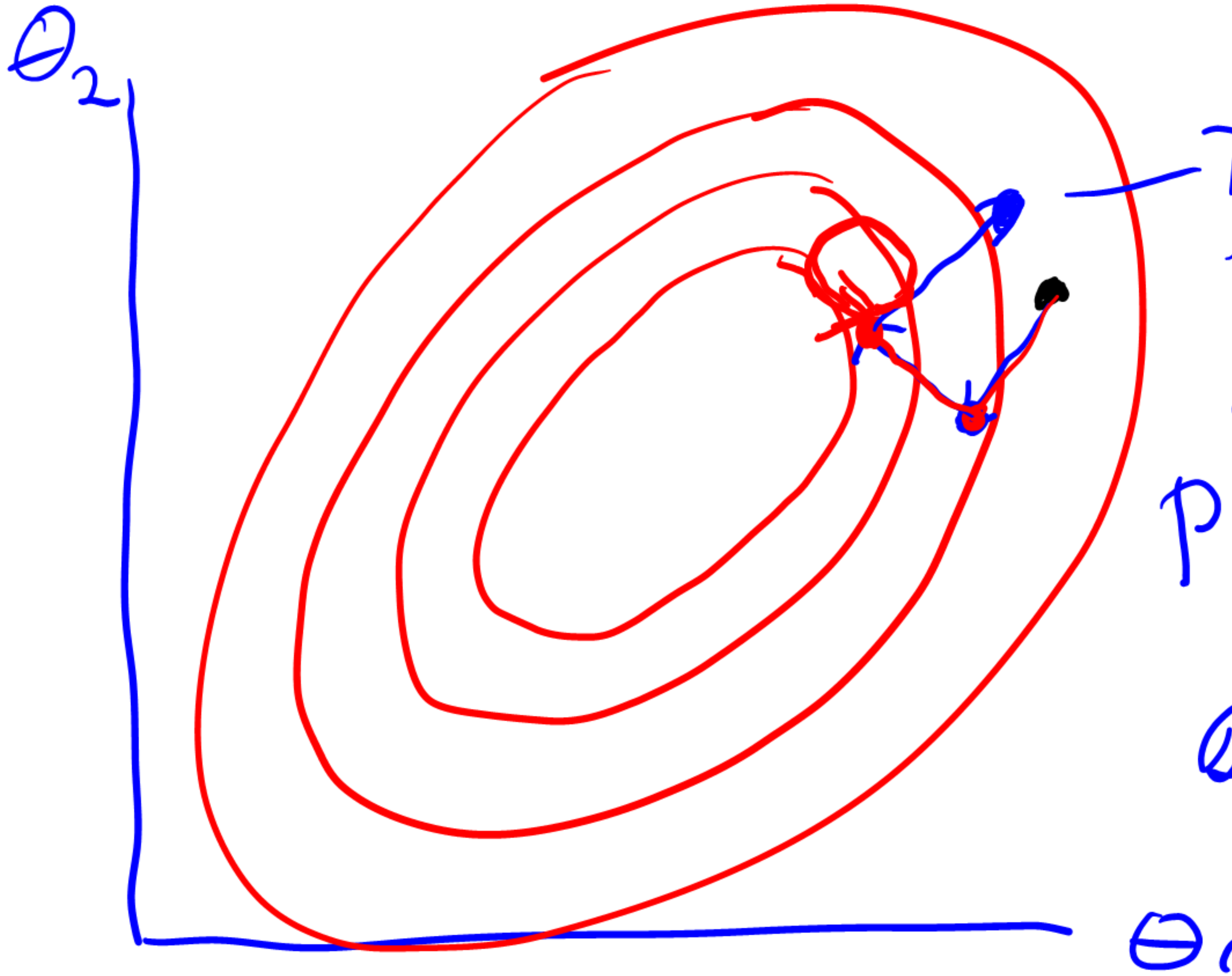
θ2
θ1



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move
if Tails stay



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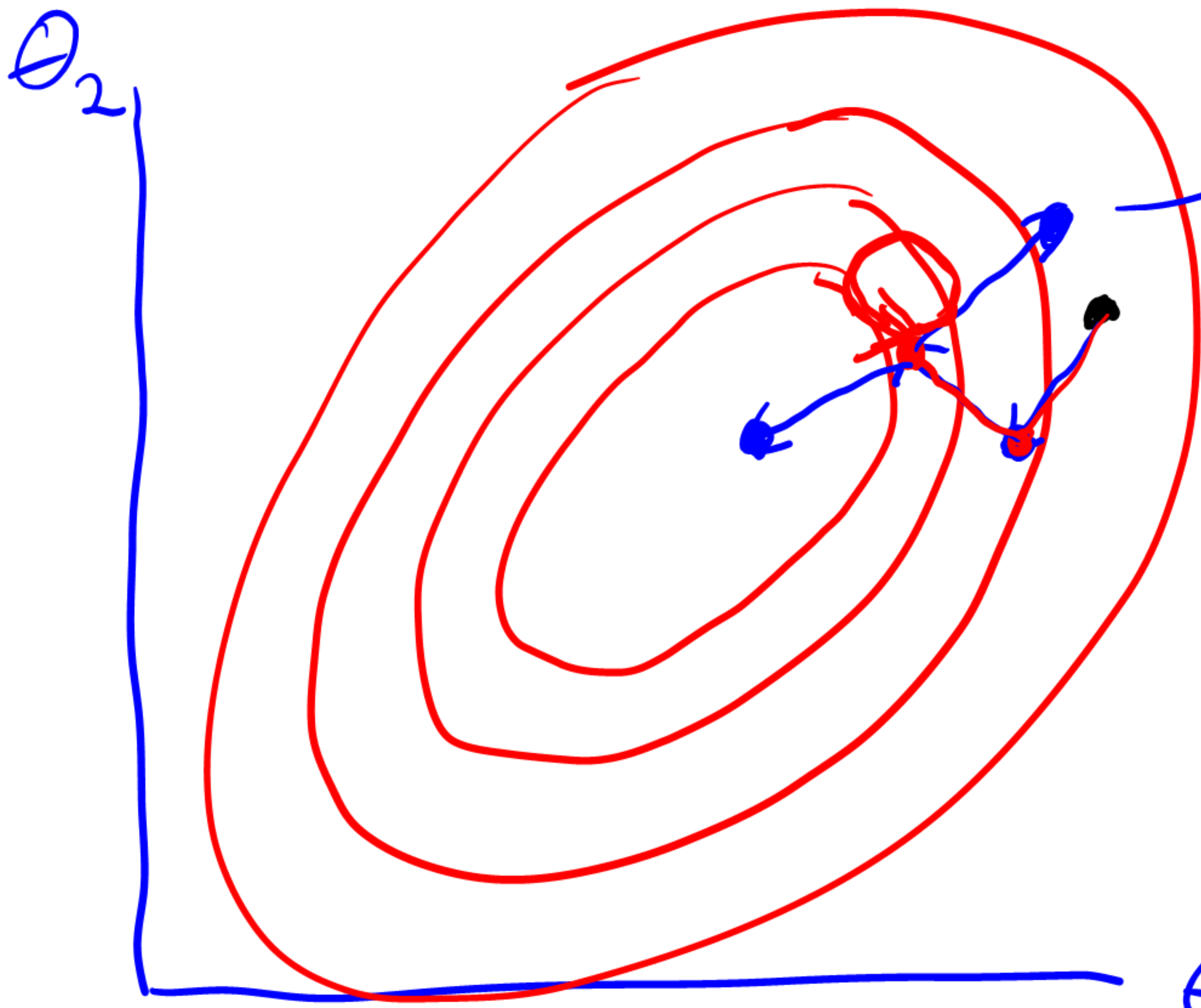
$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move
if Tails stay



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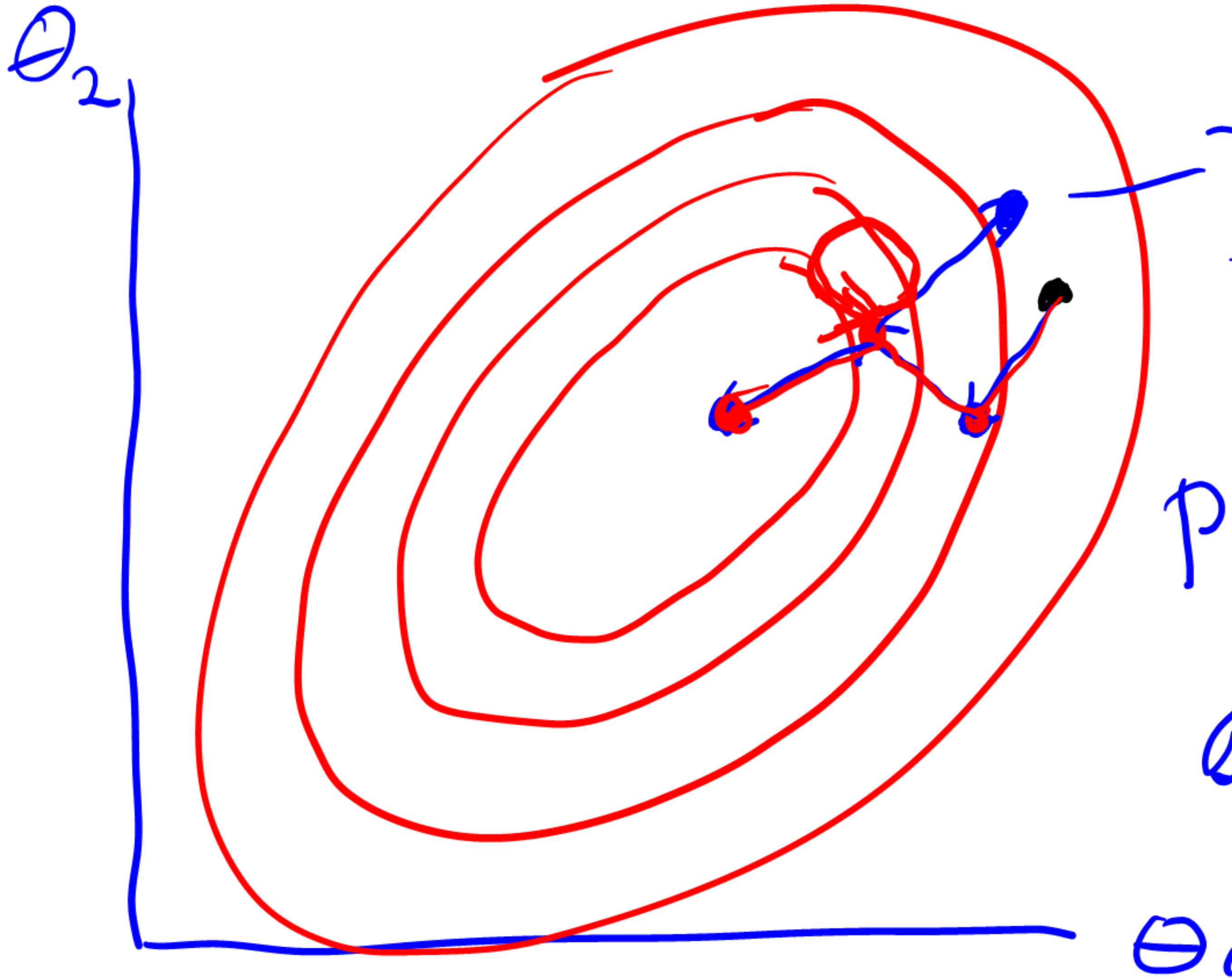
If Heads move

if Tails stay



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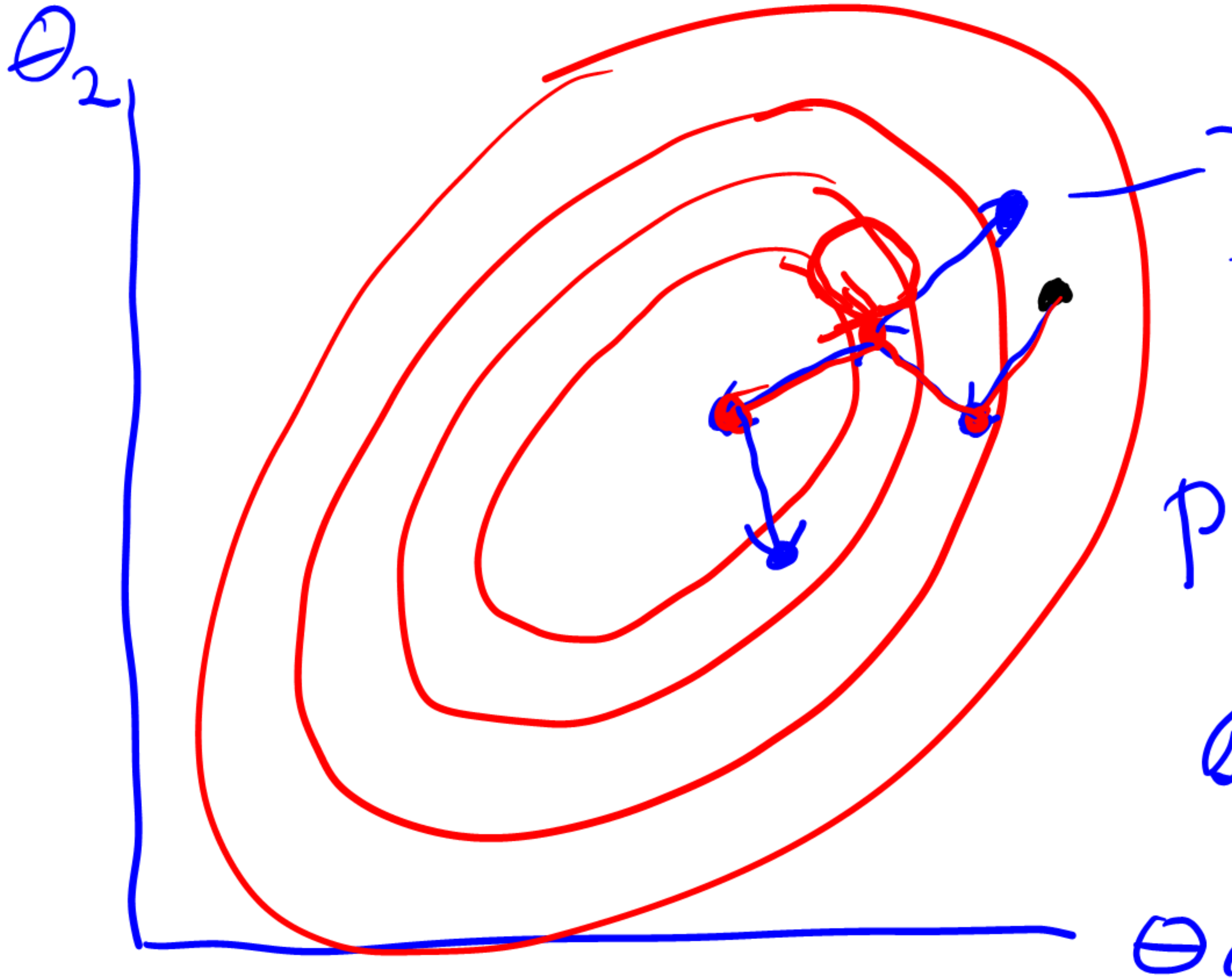

toss a biased coin with

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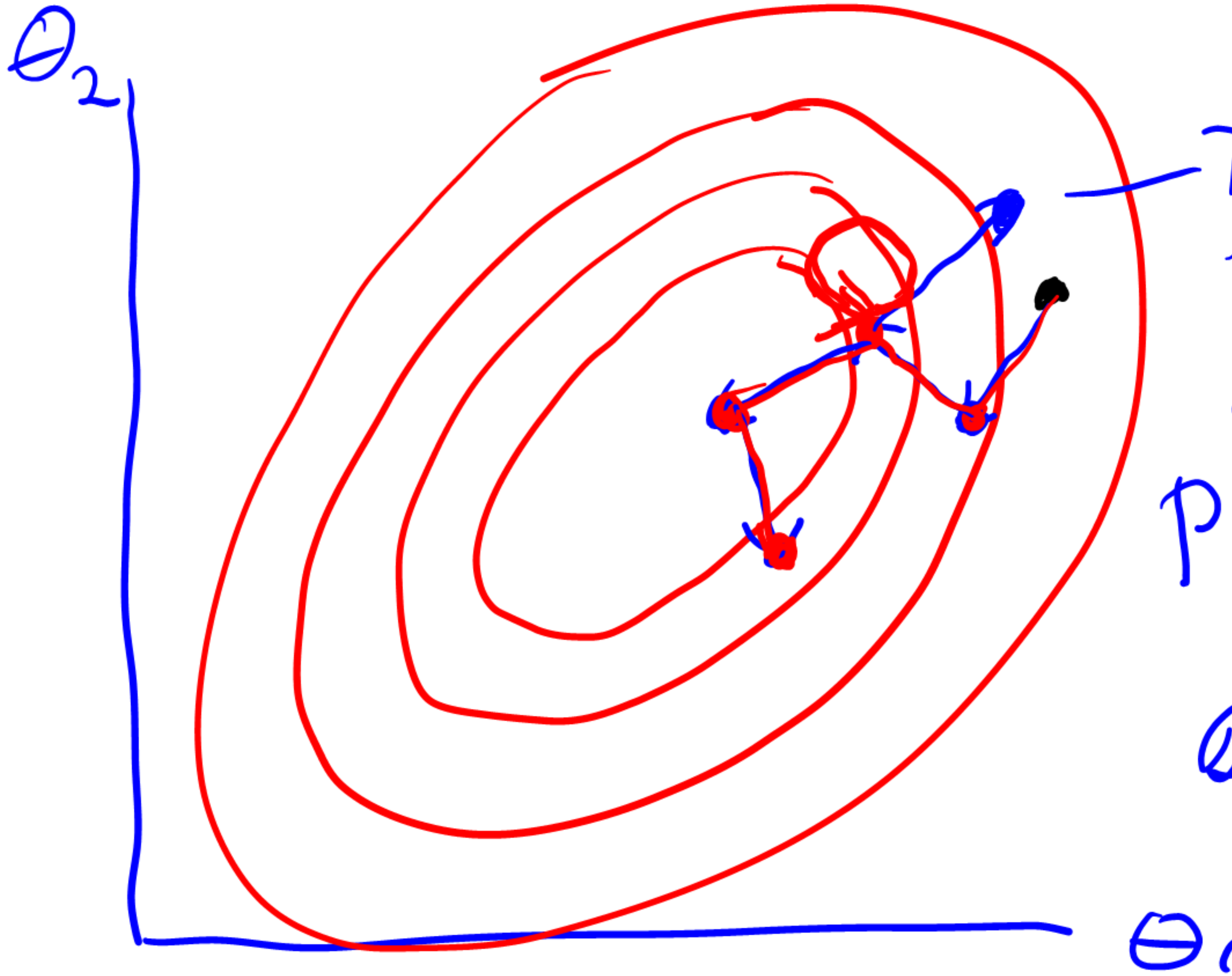
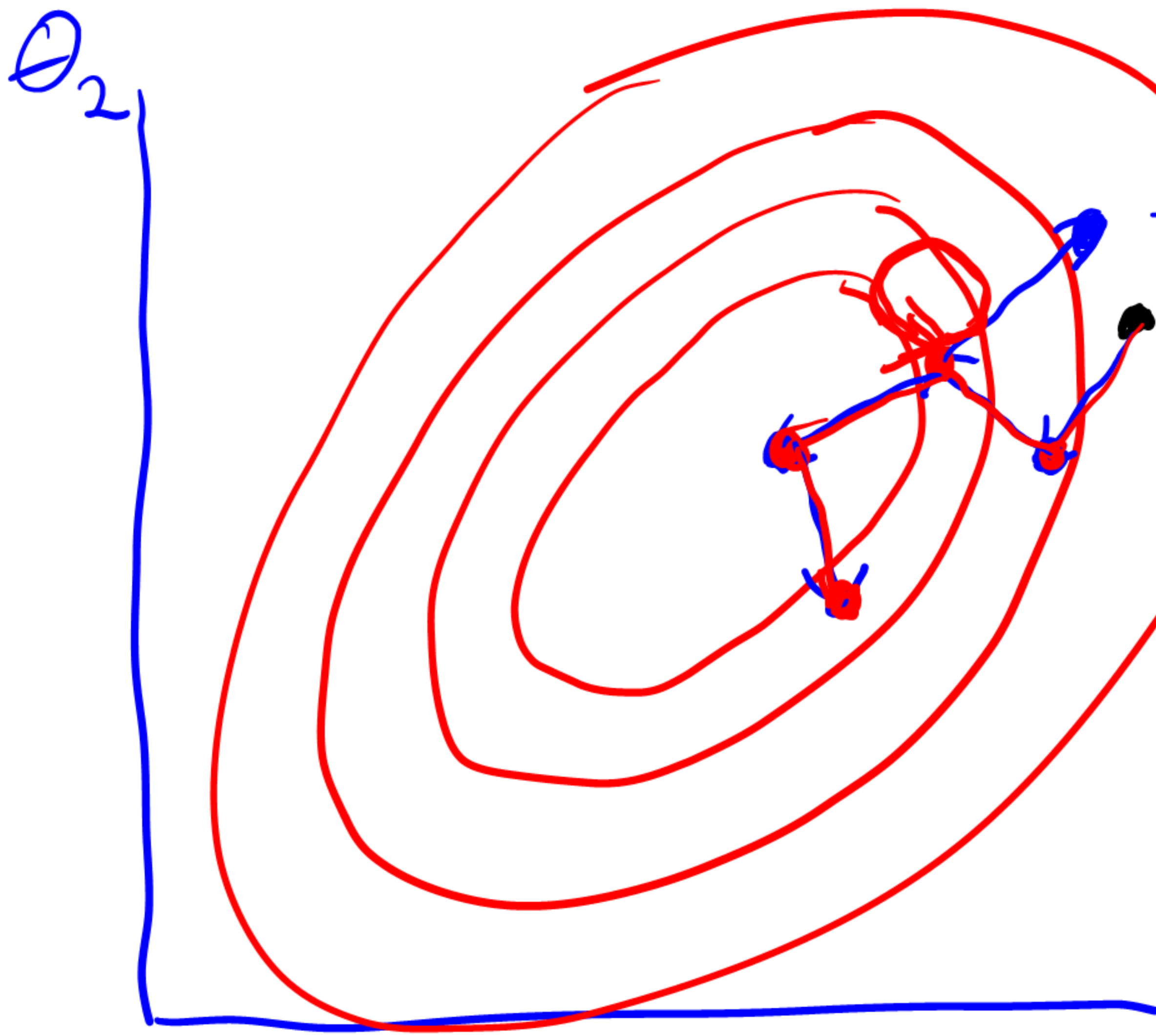
If Heads move
if Tails stay



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move
if Tails stay

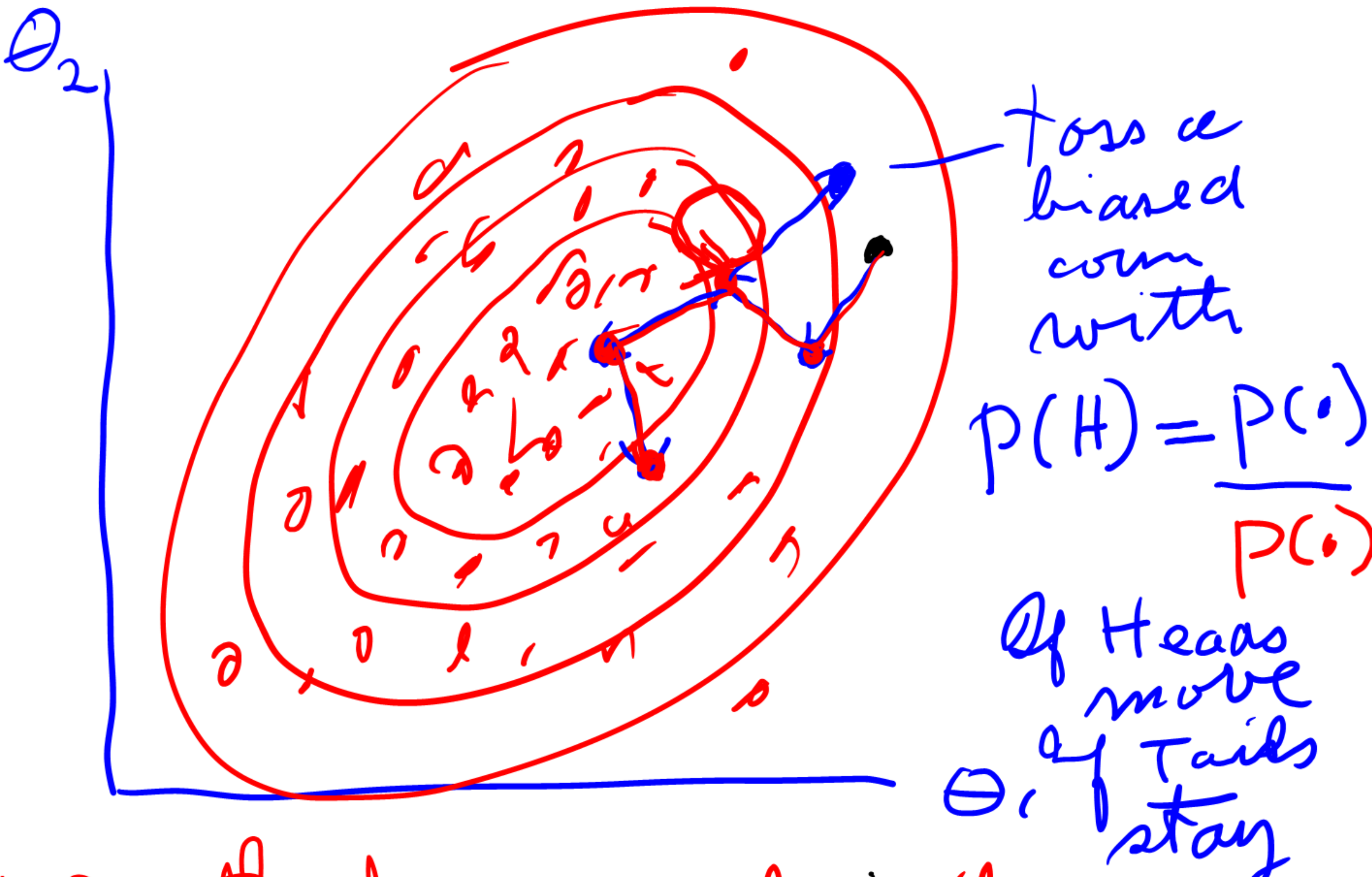


toss a biased coin with

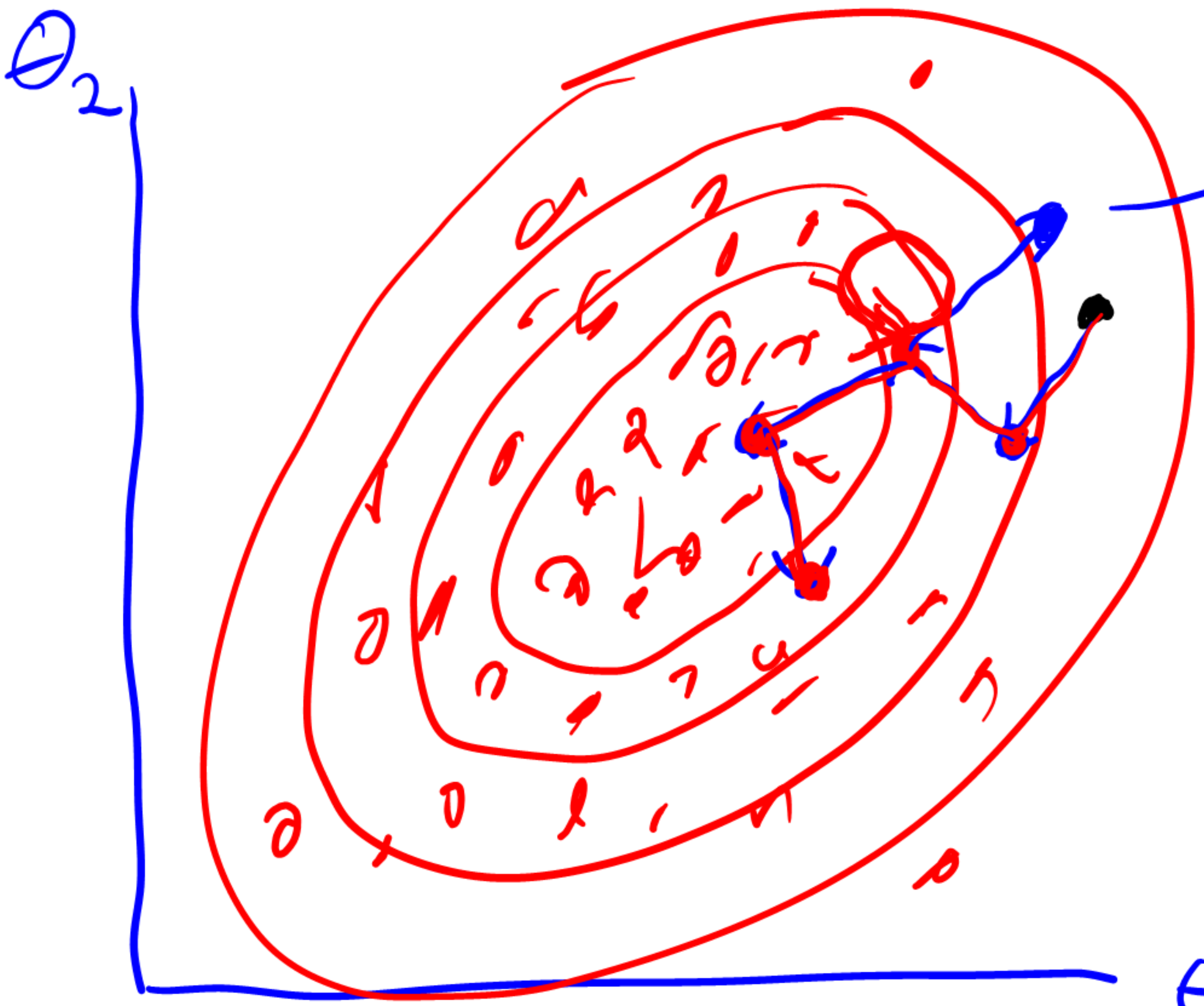
$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move
if Tails stay

Keep doing this for a very long time



Keep doing this for a very long time



toss a biased coin with

$$P(H) = \frac{P(\cdot)}{P(\cdot)}$$

If Heads move
if Tails stay

Generate a sample from $P(\theta|X)$

That's the XX-H algorithm

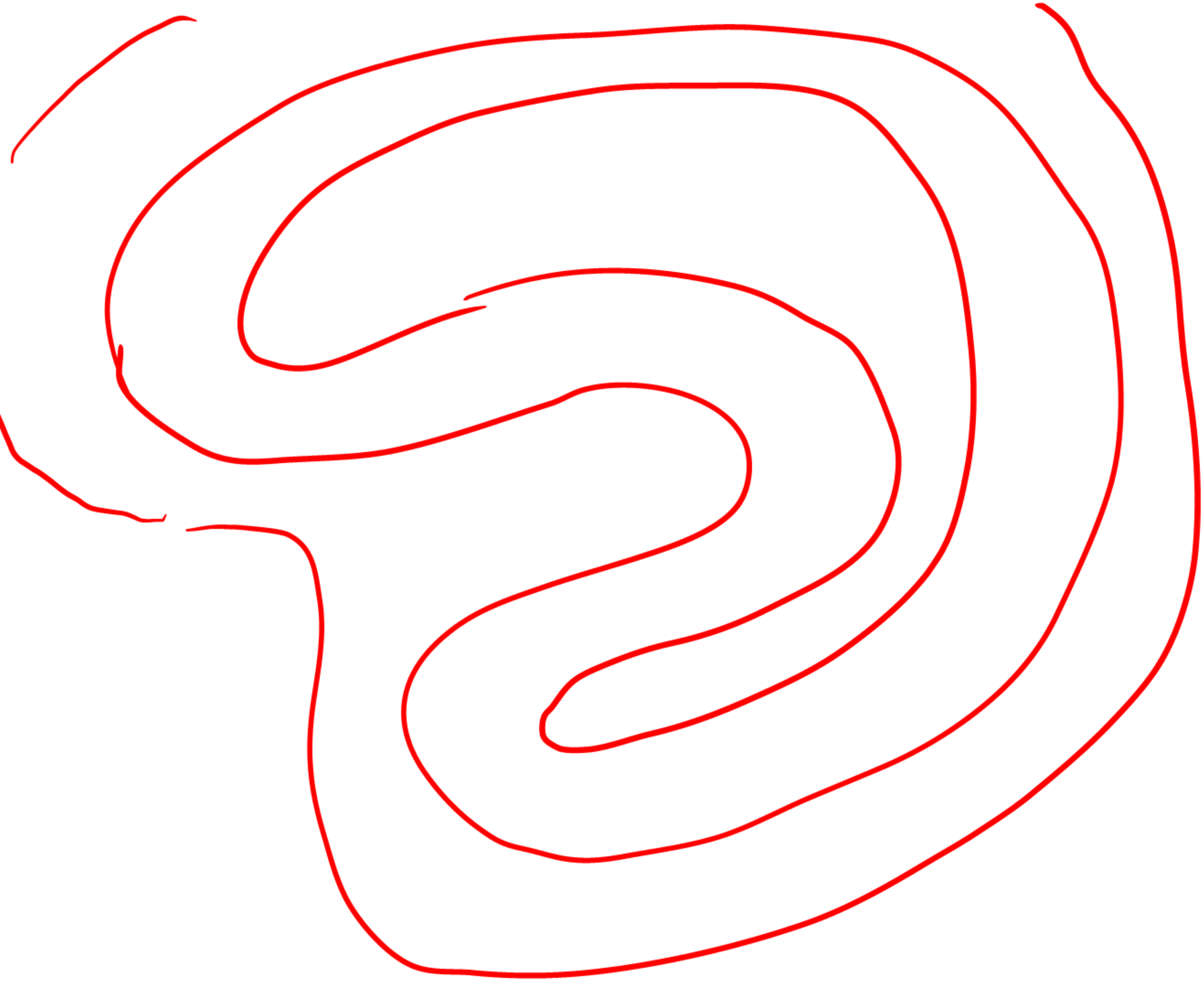
That's the M-H algorithm

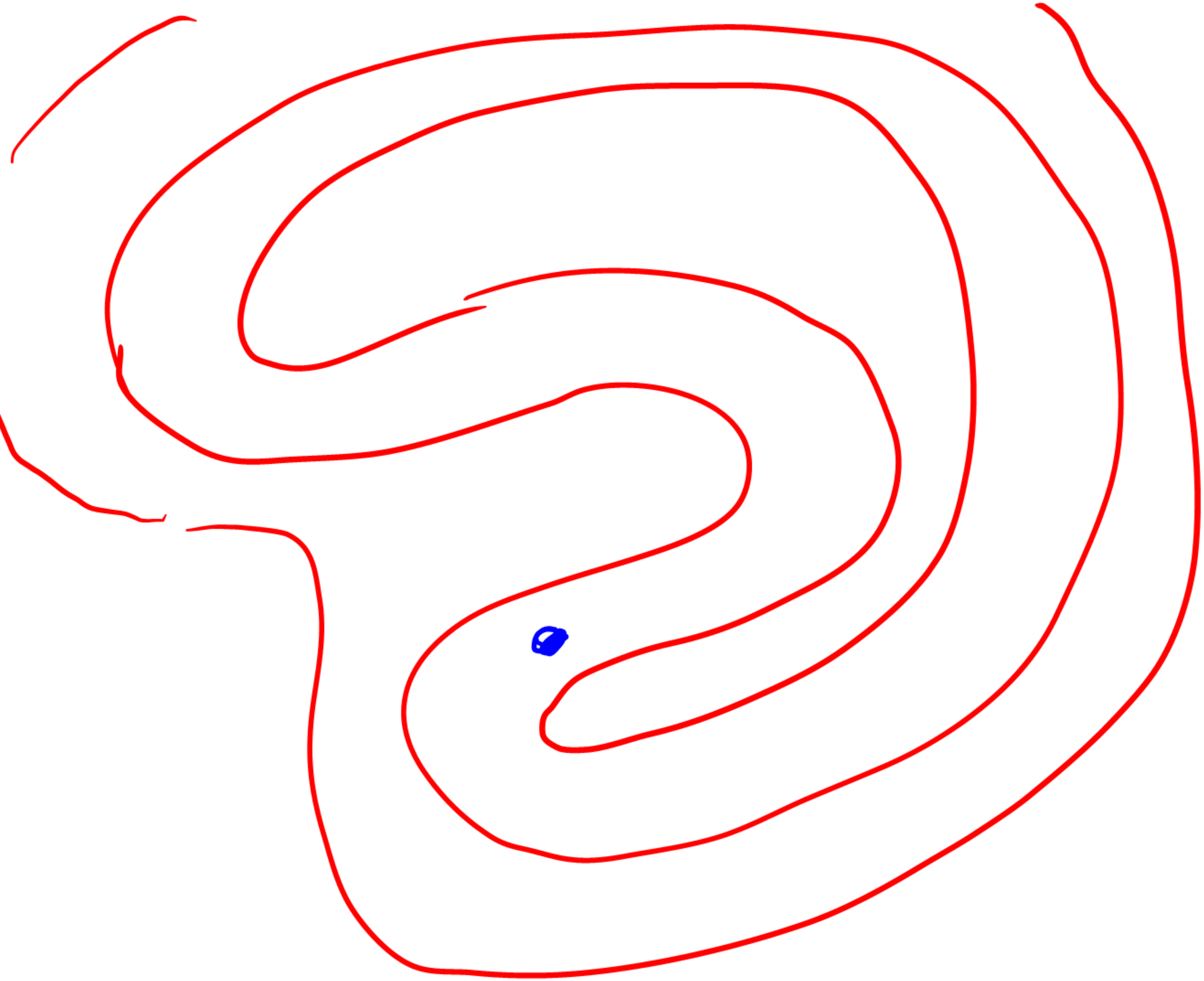
Can be very slow in high dimensions with non-elliptical contours, e.g.

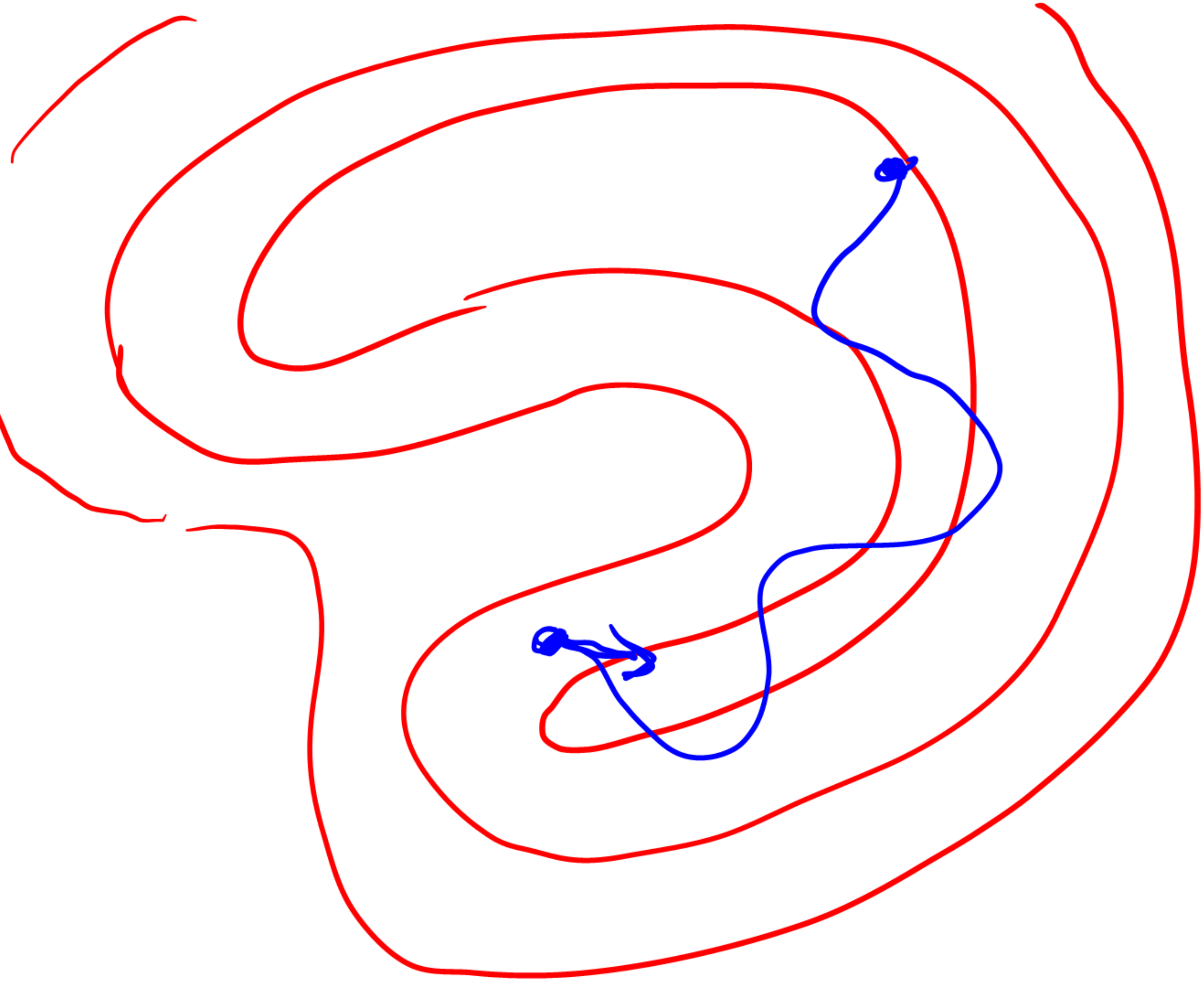
Can get stuck in corners for a long time!

Hamiltonian Monte Carlo

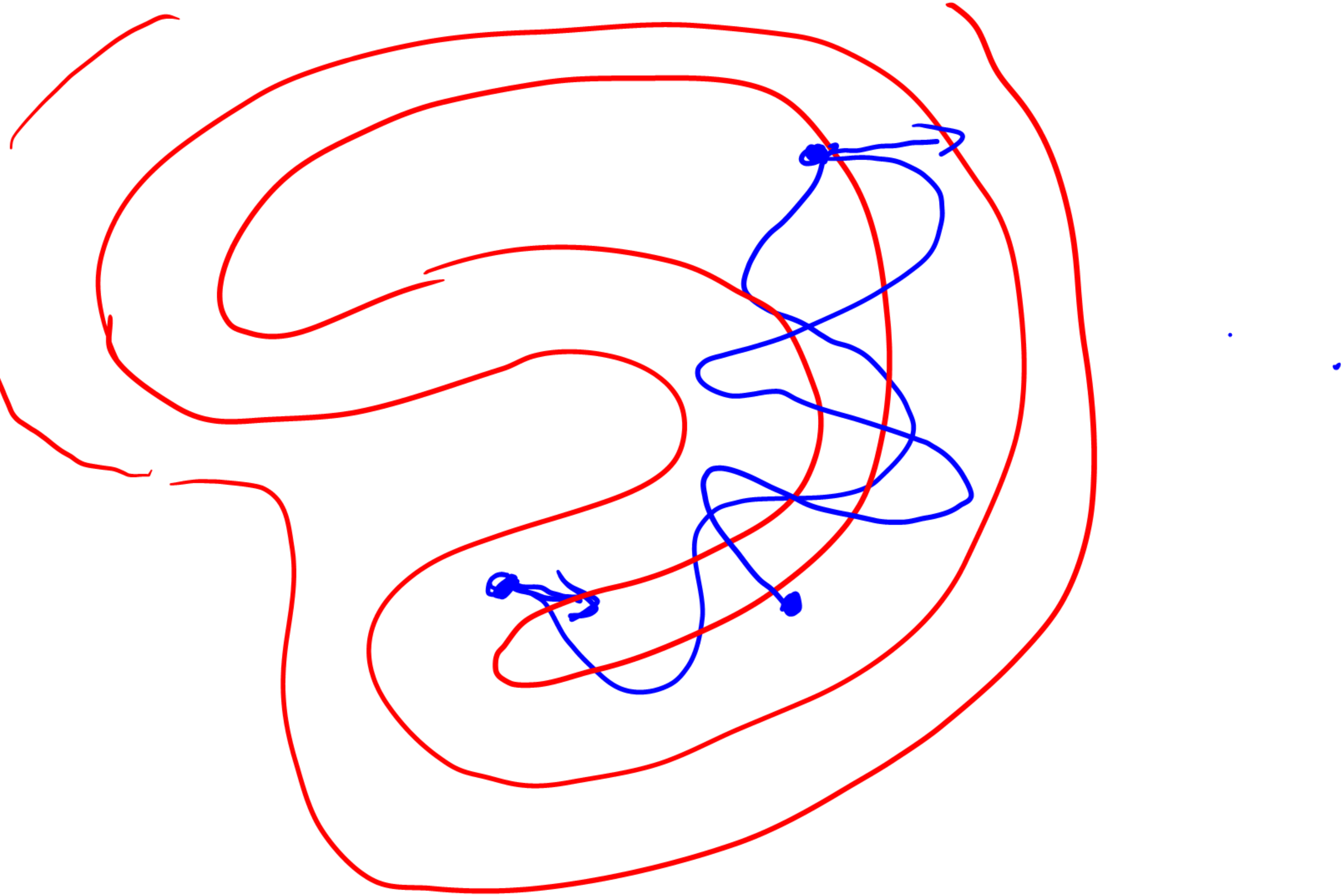
- Turn the mountain into a bowl by using $-\log p(\theta, x)$
- Instead of random steps, go for a ride on a frictionless skateboard with swivel wheels — starting with a random push.

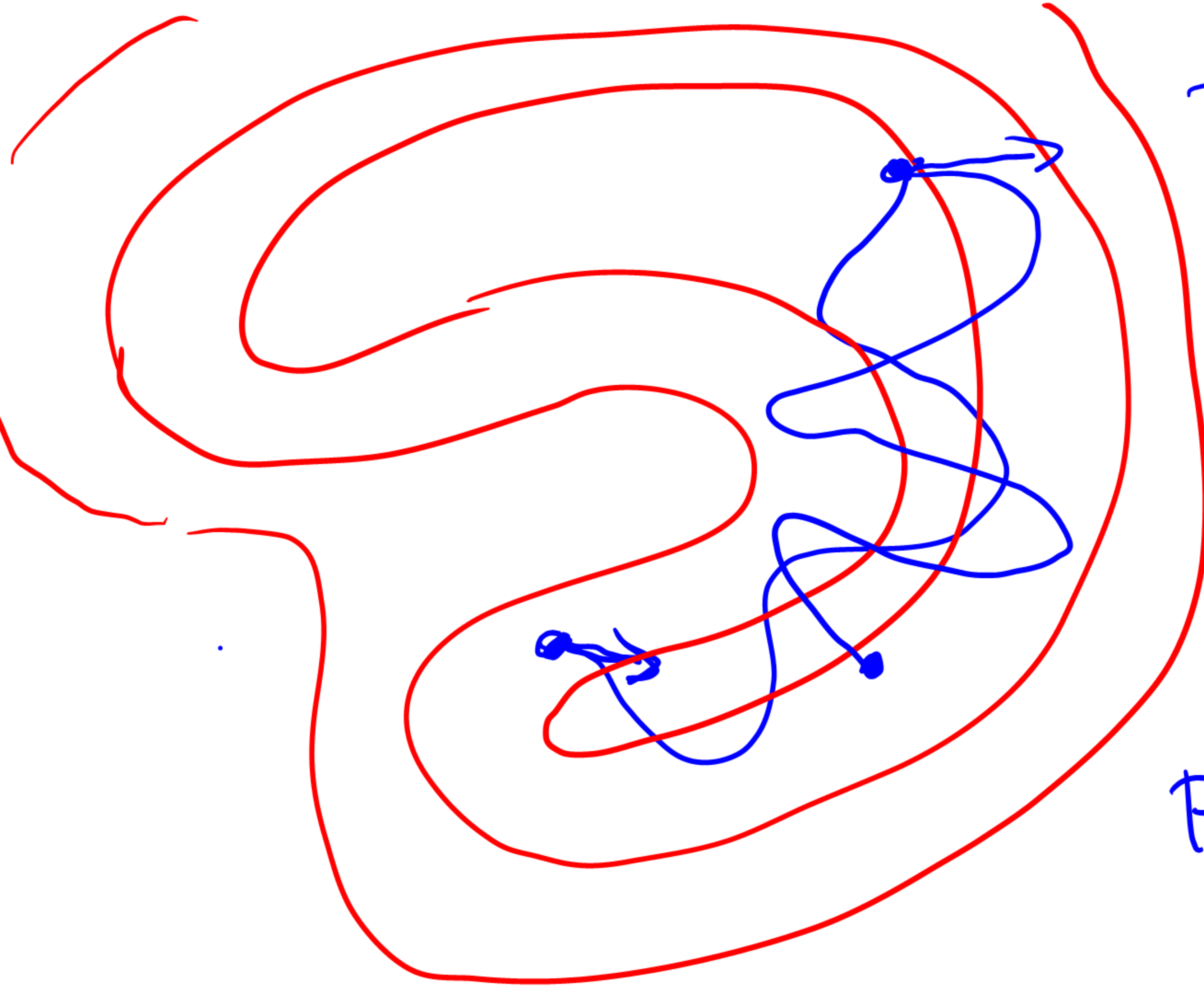




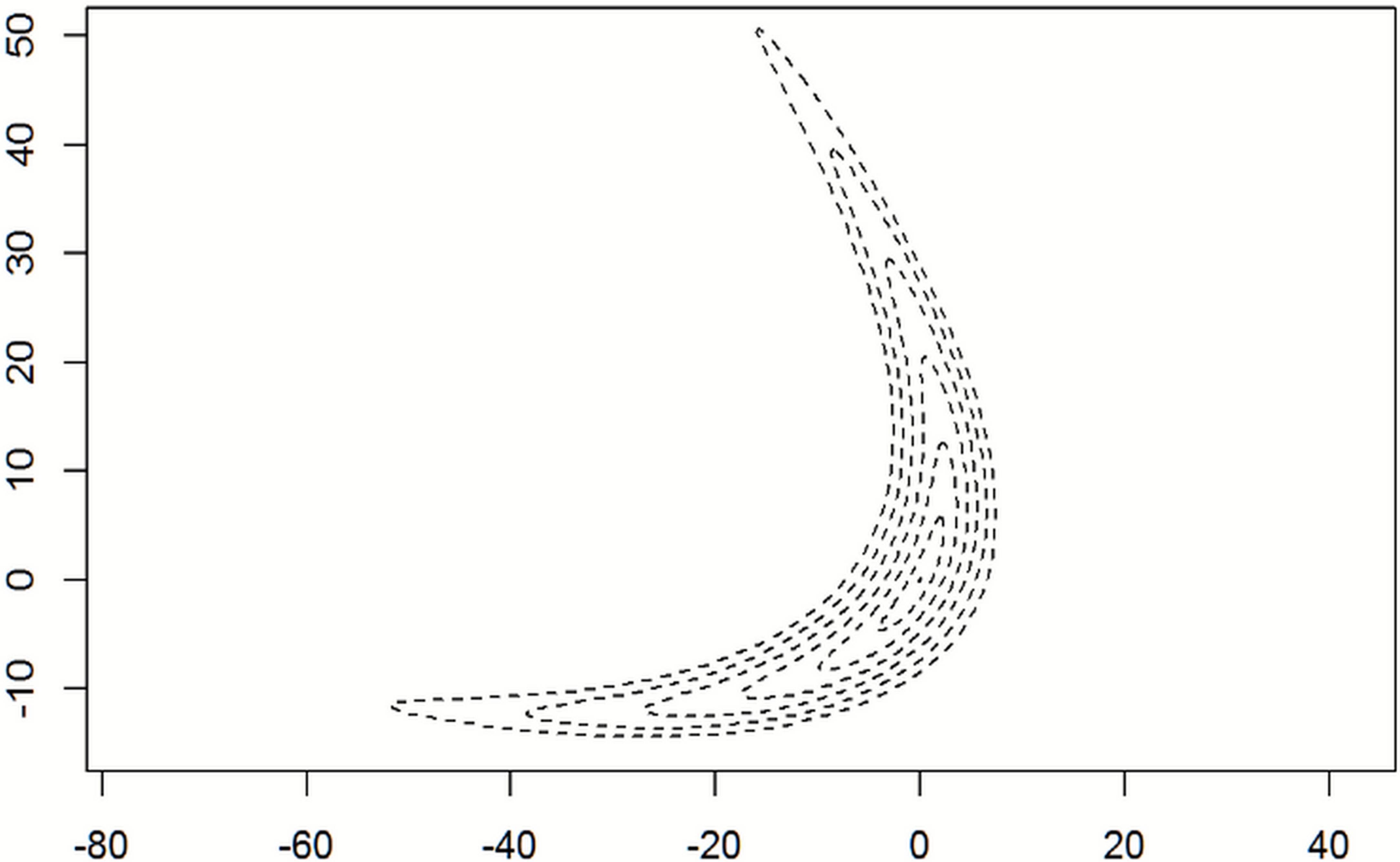


Stop after a set time.

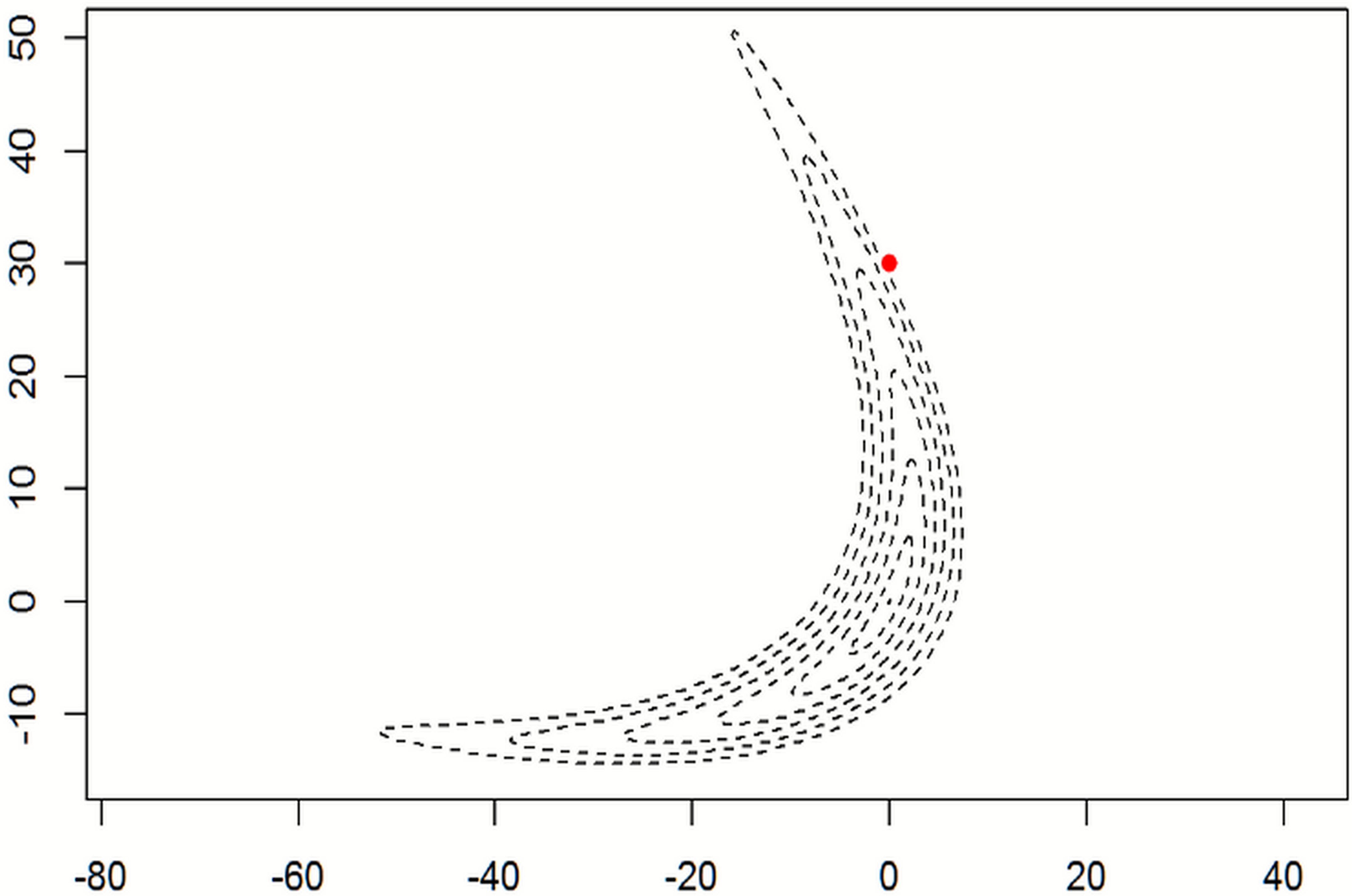




This can do a much better job of covering $p(\theta|X)$.



50
40
30
20
10
0
-10
-80
-60
-40
-20
0
20
40



50
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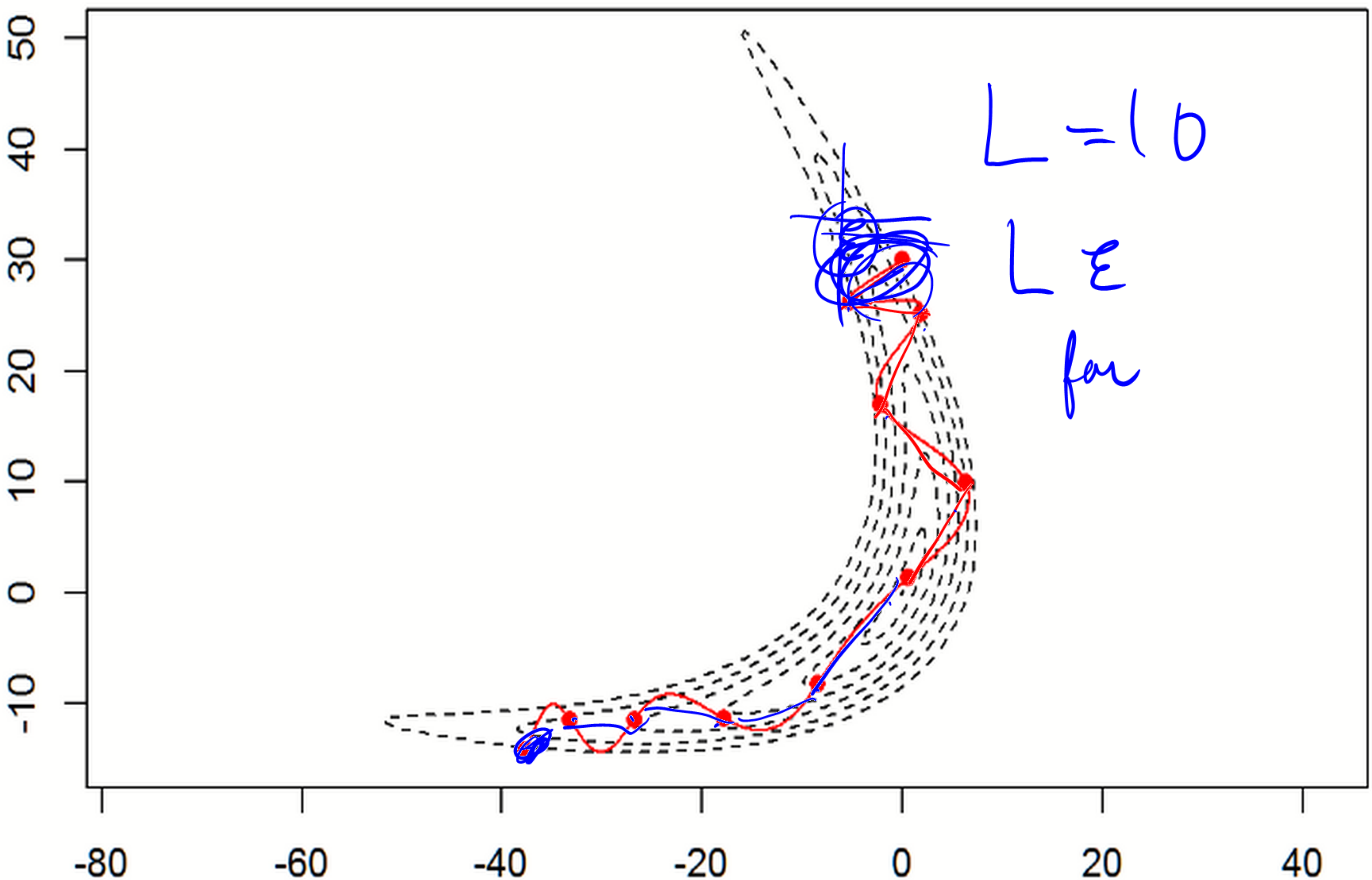
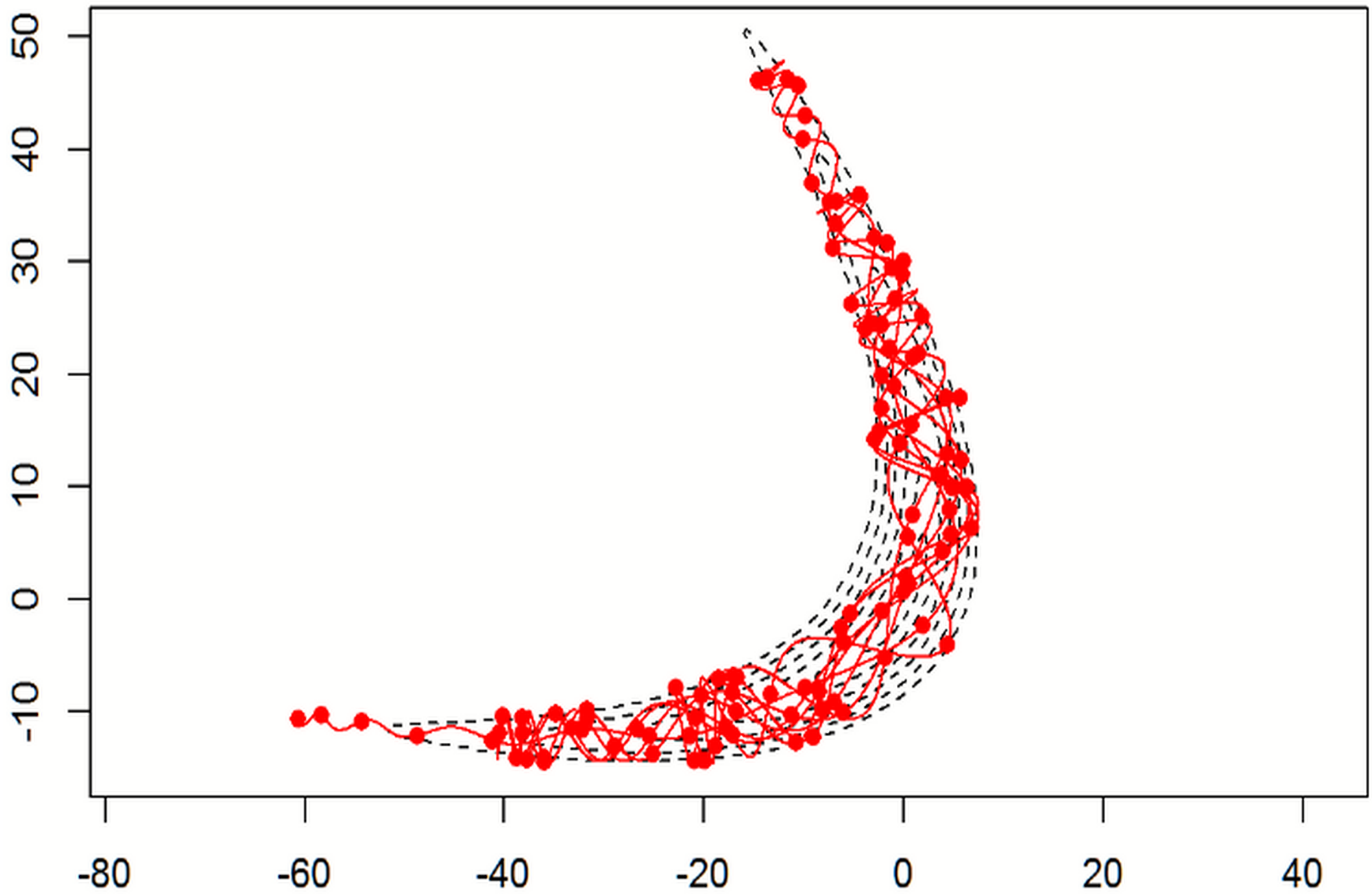


L=10
LE
far
-80
-60
-40
-20
0
20
40
50
40
30
20
10
0
-10


50
40
30
20
10
0
-10
-80
-60
-40
-20
0
20
40

L : # of leaps (leapfrog) steps

ε : size of each step

$L\varepsilon$: "distance" travelled in θ-space

If too small – too much like random walk

If L too large – too much computing time

If ε too large – bad numerical approximation to continuous path and too high likelihood of rejecting proposal.

From Neal (2011)

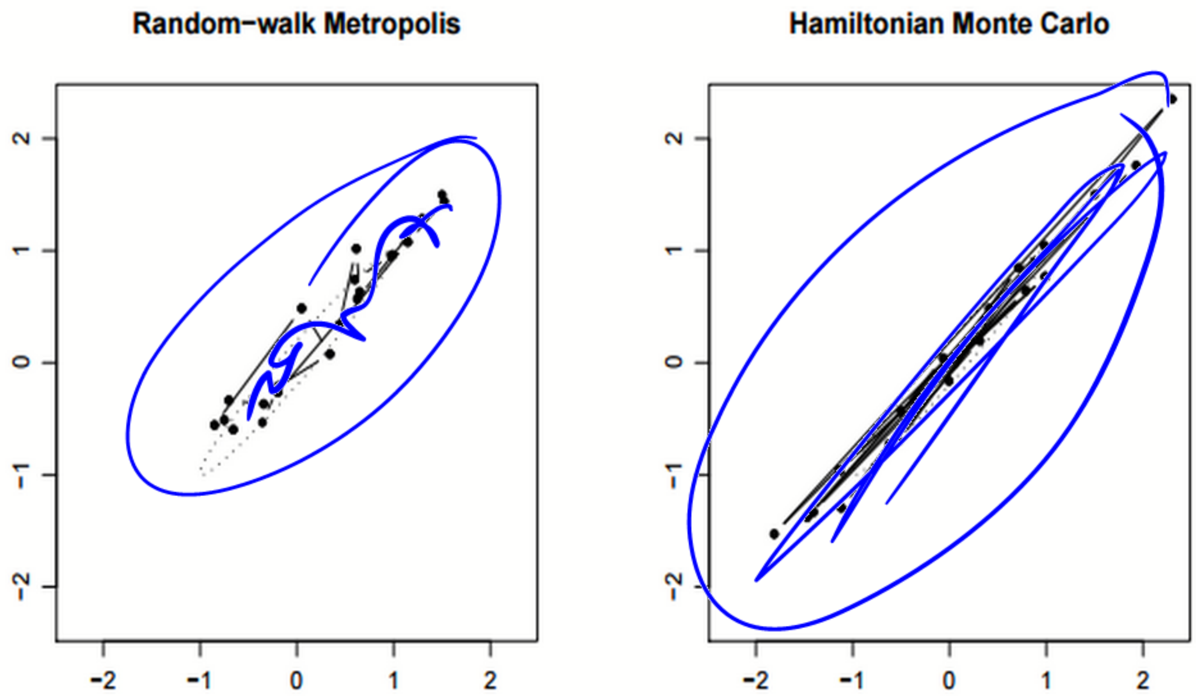


Figure 4: Twenty iterations of the random-walk Metropolis method (with 20 updates per iteration) and of the Hamiltonian Monte Carlo method (with 20 leapfrog steps per trajectory) for a 2D Gaussian distribution with marginal standard deviations of one and correlation 0.98. Only the two position coordinates are plotted, with ellipses drawn one standard deviation away from the mean.

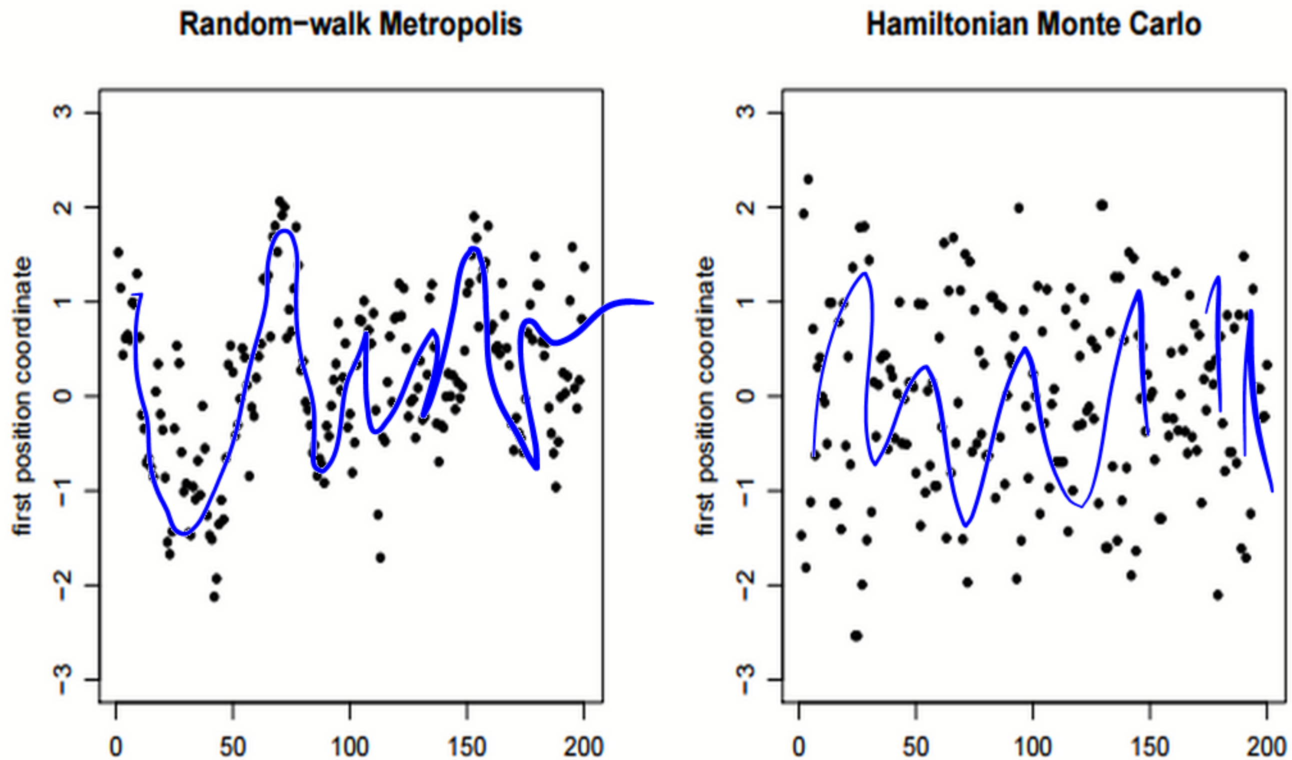


Figure 5: Two hundred iterations, starting with the twenty iterations shown above, with only the first position coordinate plotted.

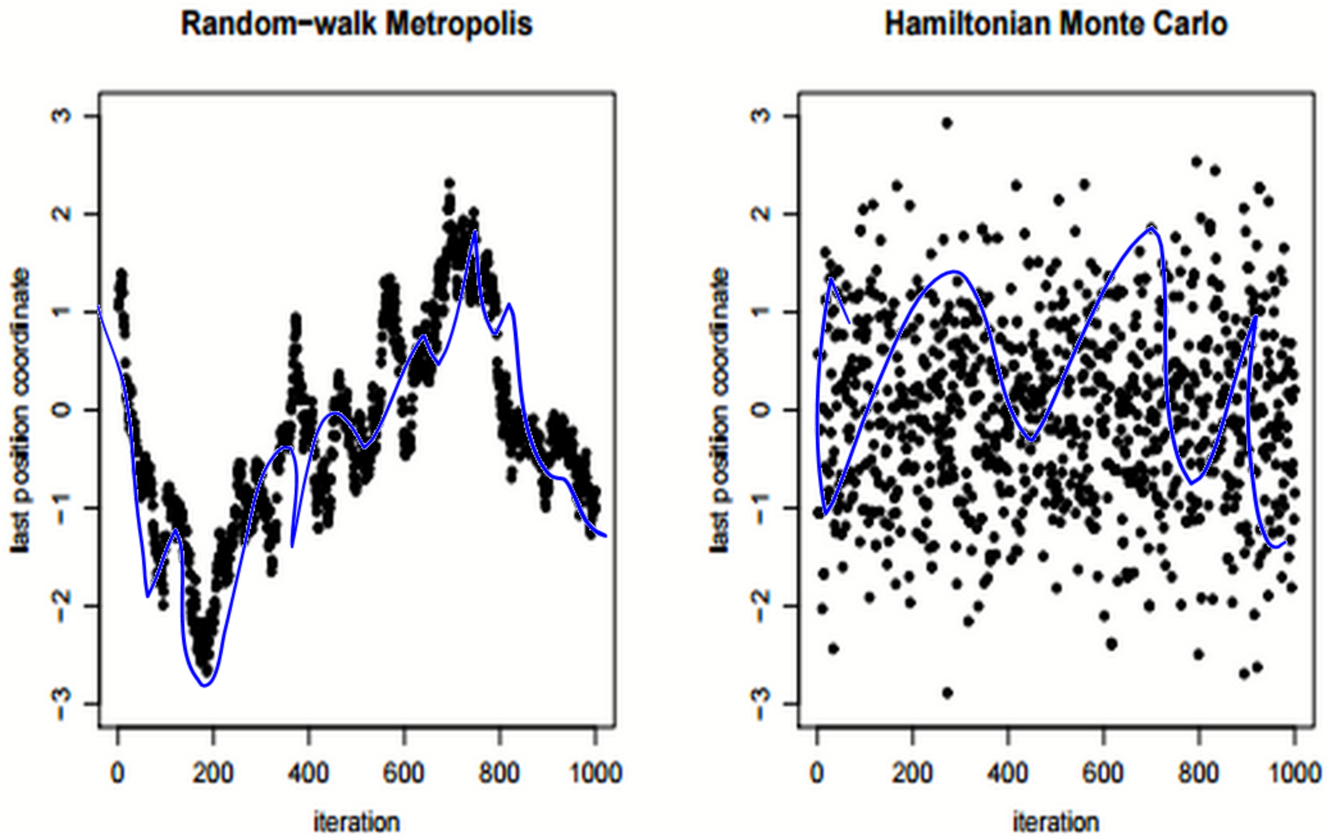


Figure 6: Values for the variable with largest standard deviation for the 100-dimensional example, from a random-walk Metropolis run and an HMC run with $L = 150$. To match computation time, 150 updates were counted as one iteration for random-walk Metropolis.