A Causal Zoo

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1 A Causal Zoo

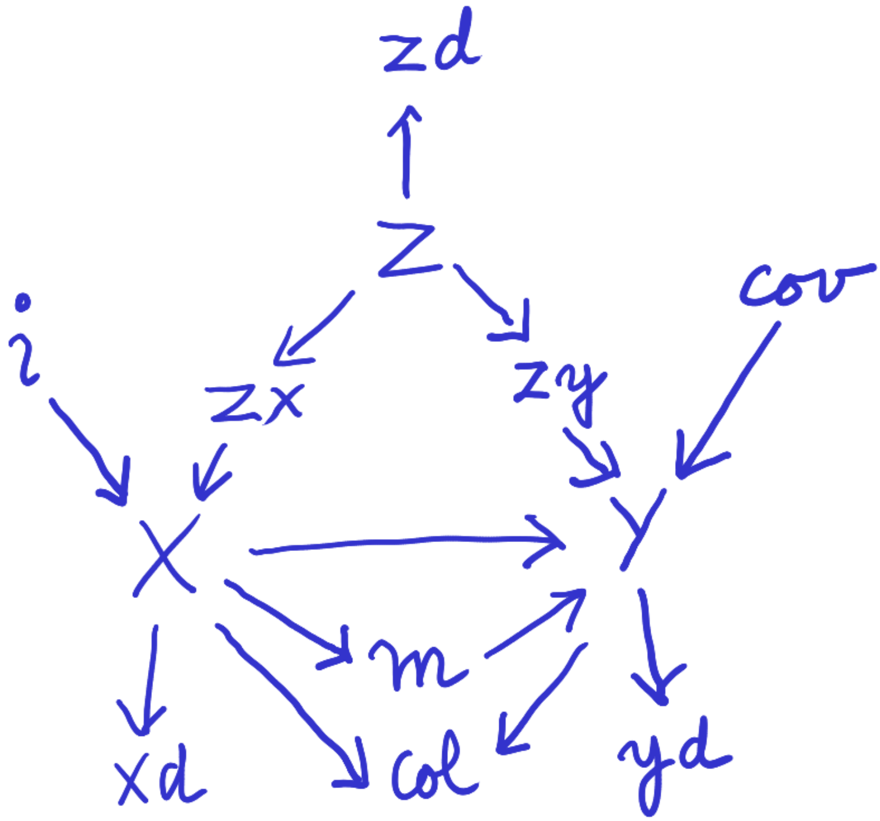

Figure 1: A DAG

```
nams <- c('z','zx','zy','cov','x','y',
          'm','i', 'xd', 'yd', 'zd',
          'col')
mat <- matrix(0, length(nams), length(nams))
rownames(mat) <- nams
colnames(mat) <- nams

# confounding back-door path

mat['zx','z'] <- 3
mat['zy','z'] <- 3
mat['x','zx'] <- 1
mat['y','zy'] <- 2

# direct effect of x on y

mat['y','x'] <- 3

# indirect effect: Note that the causal effect is 3 + 1 x 1 = 4

mat['m','x'] <- 1
mat['y','m'] <- 1
```

```r
# Instrumental variable

mat['x','i'] <- 1

# 'Covariate'

mat['y','cov'] <- 2

# descendant of X

mat['xd','x'] <- 1

# descendant of Y

mat['yd','y'] <- 1

# descendant of z -- imperfect control

mat['zd','z'] <- 2

# collider

mat['col','y'] <- 1
mat['col','x'] <- 1

# independent SD of error for every variable

diag(mat) <- 1

# but make SD of Y different

mat['y','y'] <- 4
mat['i','i'] <- 2.4
```

2 DAG - not in lower triangular form

```r
mat
```

```
    z zx zy cov x y m   i xd yd zd col
z   1  0  0   0 0 0 0 0.0  0  0  0   0
zx  3  1  0   0 0 0 0 0.0  0  0  0   0
zy  3  0  1   0 0 0 0 0.0  0  0  0   0
cov 0  0  0   1 0 0 0 0.0  0  0  0   0
x   0  1  0   0 1 0 0 1.0  0  0  0   0
y   0  0  2   2 3 4 1 0.0  0  0  0   0
m   0  0  0   0 1 0 1 0.0  0  0  0   0
i   0  0  0   0 0 0 0 2.4  0  0  0   0
xd  0  0  0   0 1 0 0 0.0  1  0  0   0
yd  0  0  0   0 0 1 0 0.0  0  1  0   0
zd  2  0  0   0 0 0 0 0.0  0  0  1   0
col 0  0  0   0 1 1 0 0.0  0  0  0   1
```

3 DAG - in lower triangular form

Expressing the DAG in lower triangular form makes it easy to iteratively work out the variance matrix.

```
dag <- permute_to_dag(mat) # can be permuted to lower-diagonal form
dag
```

```
      i z zx x m cov zy y col zd yd xd
i   2.4 0  0 0 0   0  0 0   0  0  0  0
z   0.0 1  0 0 0   0  0 0   0  0  0  0
zx  0.0 3  1 0 0   0  0 0   0  0  0  0
x   1.0 0  1 1 0   0  0 0   0  0  0  0
m   0.0 0  0 1 1   0  0 0   0  0  0  0
cov 0.0 0  0 0 0   1  0 0   0  0  0  0
zy  0.0 3  0 0 0   0  1 0   0  0  0  0
y   0.0 0  0 3 1   2  2 4   0  0  0  0
col 0.0 0  0 1 0   0  0 1   1  0  0  0
zd  0.0 2  0 0 0   0  0 0   0  1  0  0
yd  0.0 0  0 0 0   0  0 1   0  0  1  0
xd  0.0 0  0 1 0   0  0 0   0  0  0  1
attr(,"class")
[1] "dag"    "matrix" "array"
```

4 Variance matrix

```
covld(dag)
```

```
        i  z zx      x      m cov zy      y   col zd     yd     xd
i    5.76  0  0   5.76   5.76   0  0  23.04  28.8  0  23.04   5.76
z    0.00  1  3   3.00   3.00   0  3  18.00  21.0  2  18.00   3.00
zx   0.00  3 10  10.00  10.00   0  9  58.00  68.0  6  58.00  10.00
x    5.76  3 10  16.76  16.76   0  9  85.04 101.8  6  85.04  16.76
m    5.76  3 10  16.76  17.76   0  9  86.04 102.8  6  86.04  16.76
cov  0.00  0  0   0.00   0.00   1  0   2.00   2.0  0   2.00   0.00
zy   0.00  3  9   9.00   9.00   0 10  56.00  65.0  6  56.00   9.00
y   23.04 18 58  85.04  86.04   2 56 473.16 558.2 36 473.16  85.04
col 28.80 21 68 101.80 102.80   2 65 558.20 661.0 42 558.20 101.80
zd   0.00  2  6   6.00   6.00   0  6  36.00  42.0  5  36.00   6.00
yd  23.04 18 58  85.04  86.04   2 56 473.16 558.2 36 474.16  85.04
xd   5.76  3 10  16.76  16.76   0  9  85.04 101.8  6  85.04  17.76
```

5 Some models to try

```r
fmlas <- list(
  y ~ x,              # with confounding
  y ~ x + z,          # unconfounded
  y ~ x + zy,         # unconfounded using generating model
  y ~ x + zx,         # unconfounded using assignment model
  y ~ x + zx + zy,    # 'doubly robust'
  y ~ x + zy + cov,   # adding a covariate unrelated to x
  y ~ x + z + m,      # adding a mediator
  y ~ x + z + xd,     # adding a descendant of X
  y ~ x + z + yd,     # adding a descendant of Y
  y ~ x + z + yd + cov,     # adding a descendant of Y and a covariate
  y ~ x + z + col,    # adding a collider
  y ~ x + z + i,      # adding an instrumental variable
  y ~ x + z + i + cov,  # adding an instrumental variable and a covariate
  y ~ x + xd,         # adding a descendant of x
  y ~ x + i,          # using an instrumental variable as a control
  y ~ x + zd,         # imperfect control for confounding
  y ~ x + zd + cov,   # imperfect control + covariate
  y ~ x + zd + xd,    # imperfect control + descendant of x
  y ~ x + zd + i,     # bias amplification
  y ~ x | i           # instrumental variable using two-stage least squares
)
```

6 'Fitting' the models

```r
fmlas %>%
  lapply(coefx, dag) %>%
  lapply(as.data.frame) %>%
  do.call(rbind.data.frame, .) -> df

df <- within(
  df,
  { # label positions for plotting
    pos <- ifelse(grepl('IV|zy|m|zd.*z', label), 2, 4)
    pos2 <- ifelse(grepl('yd$|z . xd|zd$', label), 2, pos)
    pos2 <- ifelse(grepl('yd', label), 3, pos2)
    pos2 <- ifelse(grepl('yd.*cov$', label), 1, pos2)
  }
)
pdf <- df
sapply(pdf, is.numeric) %>%
  {pdf[,.] <<- round(pdf[,.], 3)}
pdf[, c(6,1,4,2,3,5)] %>% print(row.names=F)
```

label	beta_x	sd_factor	sd_e	sd_x_avp	var_e_adj
y ~ x	5.074	1.577	6.455	4.094	43.156
y ~ x + z	4.000	1.795	5.000	2.786	26.852
y ~ x + zy	4.000	1.557	4.583	2.943	22.556
y ~ x + zx	4.000	2.057	5.348	2.600	30.719
y ~ x + zx + zy	4.000	1.763	4.583	2.600	23.423
y ~ x + zy + cov	4.000	1.401	4.123	2.943	18.962
y ~ x + z + m	3.000	5.205	4.899	0.941	26.769
y ~ x + z + xd	4.000	5.312	5.000	0.941	27.885
y ~ x + z + yd	0.154	0.846	0.981	1.159	1.072
y ~ x + z + yd + cov	0.182	0.904	0.977	1.081	1.107
y ~ x + z + col	-0.808	1.024	0.981	0.958	1.072
y ~ x + z + i	4.000	3.536	5.000	1.414	27.885
y ~ x + z + i + cov	4.000	3.240	4.583	1.414	24.360
y ~ x + xd	5.074	6.645	6.455	0.971	44.755
y ~ x + i	5.636	1.693	5.617	3.317	33.882
y ~ x + zd	4.377	1.796	5.554	3.092	33.129
y ~ x + zd + cov	4.377	1.676	5.181	3.092	29.942
y ~ x + zd + xd	4.377	5.837	5.554	0.951	34.403
y ~ x + zd + i	4.947	2.752	5.366	1.949	32.111
y ~ x (IV = i)	4.000	3.254	7.810	2.400	63.179

```r
cap = "$Var(\\hat{\\beta}_x)$ and $E(\\hat{\\beta}_x)$: variance versus bias."
```

```r
df %>%
  xyplot(log2(sd_factor^2) ~ beta_x, . , font = 2, cex = 1,
         scales = list(y = list(at=seq(-3,10), labels=2^(5+seq(-3,10)))),
         xlim = c(-1.6, 6.5),
         pch = 20,
         xlab = TeX('$E(\\hat{\\beta})$'),
         ylab = TeX('$Var(\\hat{\\beta})$ using relative $n$ to get equivalent power'),
         labs = sub('y ~ ','',.$label),
         pos = .$pos) +
  layer(panel.grid(h=-1,v=-1)) +
  layer(panel.abline(v = 4, lty = 3)) +
  layer(panel.text(..., labels = labs, pos = pos))
```

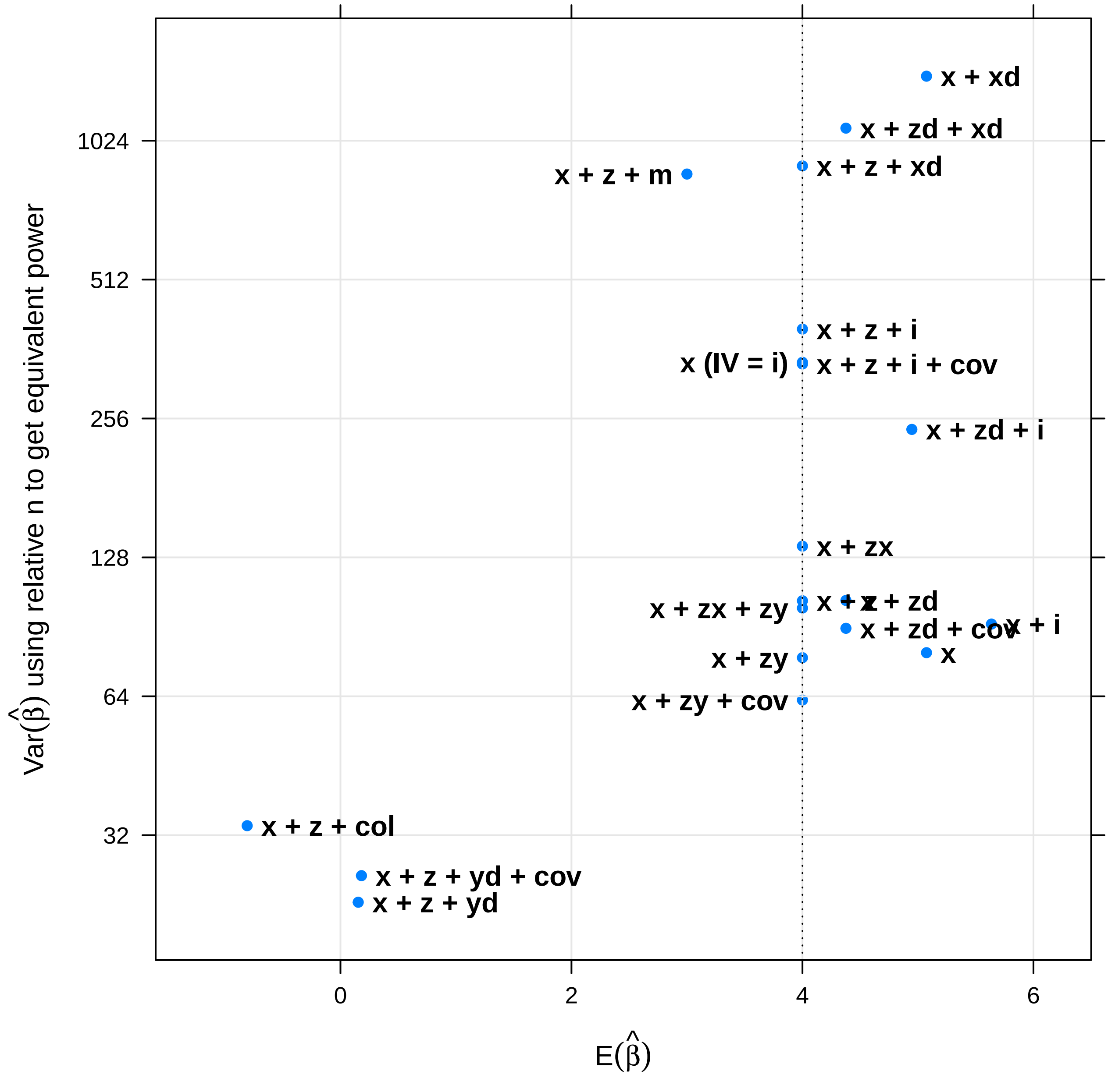

Figure 2: $Var(\hat{\beta}_x)$ and $E(\hat{\beta}_x)$: variance versus bias.

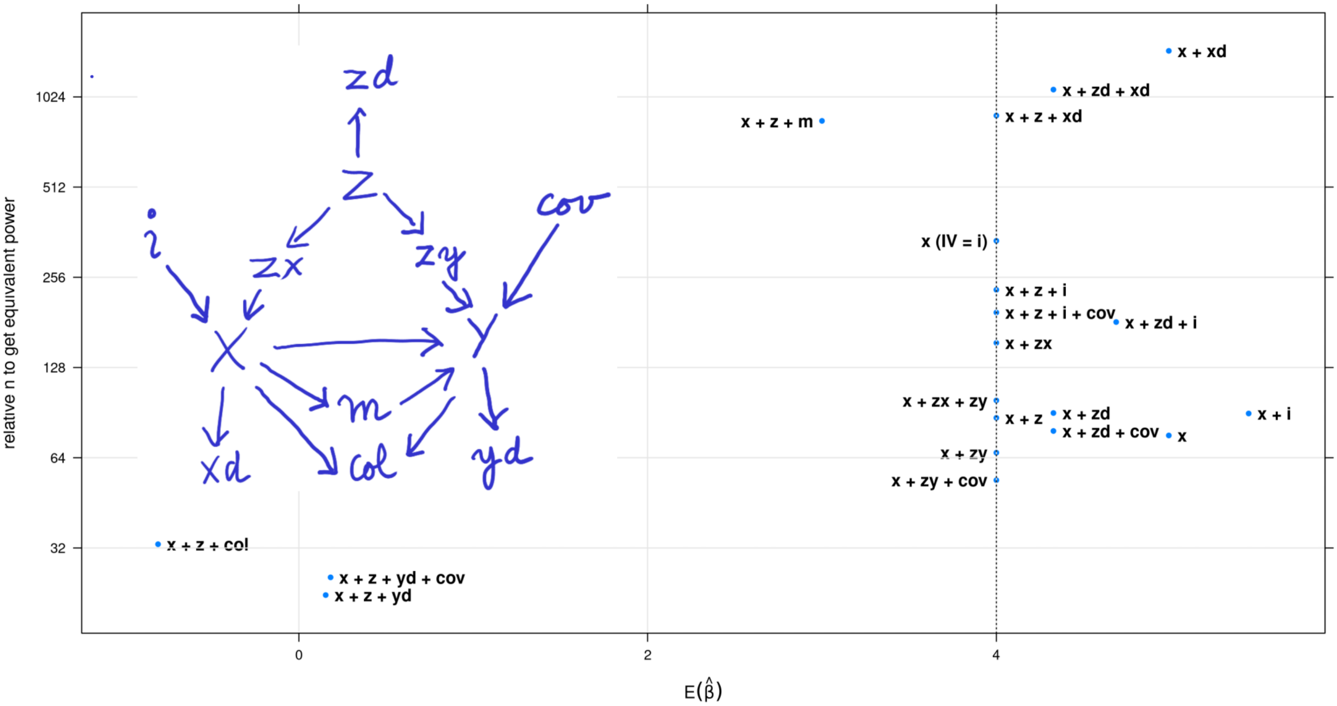

Figure 3: Variance versus bias with DAG

7 Which model is best?

It depends on the purpose of the analysis! Thanks to Hugh McCague for the idea of including the following figure to illustrate how focusing on predictive power does not lead to a suitable model to estimate the causal effect of X.

```
caption <- "Adjusted measure of fit versus bias. Which model(s) would you choose?"
```

```
df %>%
  subset(!grepl('i2', label)) %>%
  xyplot(var_e_adj ~ beta_x, . , font = 2,
         #scales = list(y = list(at=seq(-3,10), labels=2^(5+seq(-3,10)))),
         # xlim = c(-.6, 6),
         pch = 16,
         xlab = TeX('$E(\\hat{\\beta})$'),
         ylab = TeX('Goodness of fit using an equivalent to adjusted R-squared (smaller is better)'),
         labs = sub('y ~ ','',.$label),
         pos = .$pos2) +
  layer(panel.text(..., labels = labs, pos = pos)) +
  layer(panel.grid(h=-1,v=-1)) +
  layer(panel.abline(v = 4, lty = 3))
```

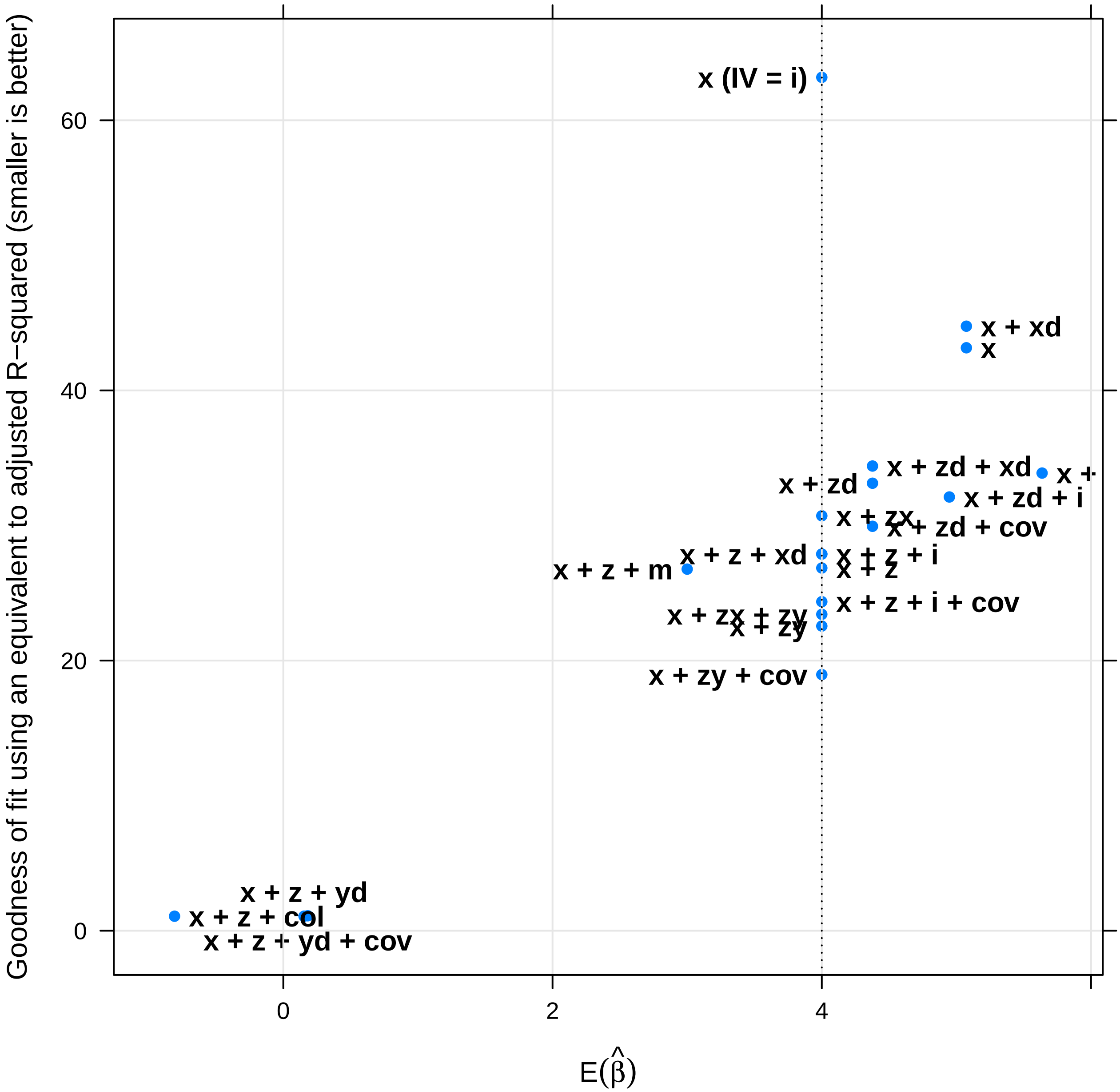

Figure 4: Adjusted measure of fit versus bias. Which model(s) would you choose?

8 What's happening with IVs?

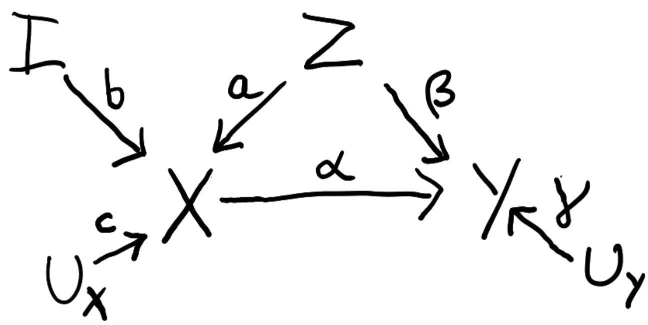

Figure 5: A simple DAG with an IV

Let's assume multivariate normality and build a variance matrix for Z, I, X, Y.

We can scale I, Z and X so they have unit variance and zero means. This eliminates irrelevant nuisance parameters.

Since I is an instrument for the confounding effect of Z:

$$Var\begin{pmatrix} Z \\ I \\ X \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ a & b & 1 \end{pmatrix}$$

with $a^2 + b^2 \leq 1$.

Focus first on the *assignment model*, i.e. the model that determines the value of X from the values of Z, I and U_X.

Letting $c^2 = 1 - a^2 - b^2$, c^2 represent the portion of the variance in X that is not attributed to the instrument, I, nor to the confounder, Z, define

$$\rho_I = \frac{b^2}{b^2 + c^2}$$

the proportion of the variance in X not due to Z that is 'explained' by I.

For an instrument that captures all of the variation not due to the confounder, $c^2 = 0$ and $\rho_I = 1$.

Focusing next on the model generating Y, let

$$Y = \alpha X + \beta Z + \gamma \varepsilon$$

with $\varepsilon \sim \mathrm{N}(0, 1)$, independent of other variables.

The variance matrix is:

$$Var\begin{pmatrix} Z \\ I \\ X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 0 & a & a\alpha + \beta \\ 0 & 1 & b & b\alpha \\ a & b & 1 & \alpha + a\beta \\ a\alpha + \beta & b\alpha & \alpha + a\beta & v_{yy} \end{pmatrix}$$

where $v_{yy} = \alpha^2 + \beta^2 + 2a\alpha\beta + \sigma_\varepsilon^2$

We can verify that the regression coefficients for the regression of Y on X and Z are

$$\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha + a\beta \\ a\alpha + \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The variance of the least-squares estimator of α based on a regression on X and the confounder Z is:

$$\begin{aligned}\operatorname{Var}(\hat{\alpha}) &\approx \frac{1}{n}\,\frac{\sigma_\epsilon^2}{1-a^2}\\ &= \frac{1}{n}\,\frac{\gamma^2}{b^2+c^2}\end{aligned}$$

The asymptotic expectation of the instrumental variable estimator $\tilde{\alpha}$ is

$$\sigma_{IX}^{-1}\sigma_{IY} = \frac{1}{b} \times b\alpha = \alpha$$

The variance of $\tilde{\alpha}$ is (Fox 2016, 241):

$$\operatorname{Var}(\tilde{\alpha}) \approx \frac{1}{n}\,\sigma_{\epsilon IV}^2\sigma_{IX}^{-1}\sigma_{II}\sigma_{XI}^{-1} \quad = \frac{1}{n}\,(\beta^2+\gamma^2)\frac{1}{b^2}$$

Thus, the variance inflation factor – which is the same as the 'sample size inflation factor to achieve the same power' – using IV estimation instead of controlling for a confounder (assuming that both approaches are available) is:

$$\begin{aligned}IVVIF &= \frac{\operatorname{Var}(\tilde{\alpha})}{\operatorname{Var}(\hat{\alpha})}\\ &= \frac{\beta^2+\gamma^2}{b^2}\Big/\frac{\gamma^2}{b^2+c^2}\\ &= \frac{\beta^2+\gamma^2}{\gamma^2}\Big/\frac{b^2}{b^2+c^2}\\ &= 1\Big/\left(\frac{\gamma^2}{\gamma^2+\beta^2}\times\frac{b^2}{b^2+c^2}\right)\\ &= \left(1+\frac{\beta^2}{\gamma^2}\right)\times\left(1+\frac{c^2}{b^2}\right)\\ &= \frac{1}{1-R_{Y,Z|X}^2}\times\frac{1}{R_{X,I|Z}^2}\end{aligned}$$

The first term is structural in the sense that it is a consequence of the problem, specifically the degree of confounding relative to the residual error variance in the model. For a given problem, the IV has no impact on this, so it represents a lower bound for the IVVIF. The second term clarifies that it is not the *correlation of the IV with X* directly that affects the IVVIF, but its **partial correlation** adjusted for the relationship of X with confounders.

In conclusion: In any situation where you have a choice, controlling for confounders will do better than using a corresponding IV, i.e. an IV that annihilates the confounder. Fitting with IVs does not take the same advantage of a model with a small error variance in the same way that a regression model does.

The lower bound for error variance created by the confounder could swamp the benefit of small residual variance in the generating model. In contrast, a regression model takes full proportional advantage of a reduction in residual error.

References

Fox, John. 2016. *Applied Regression Analysis and Generalized Linear Models*. 3rd ed. Sage Publications.