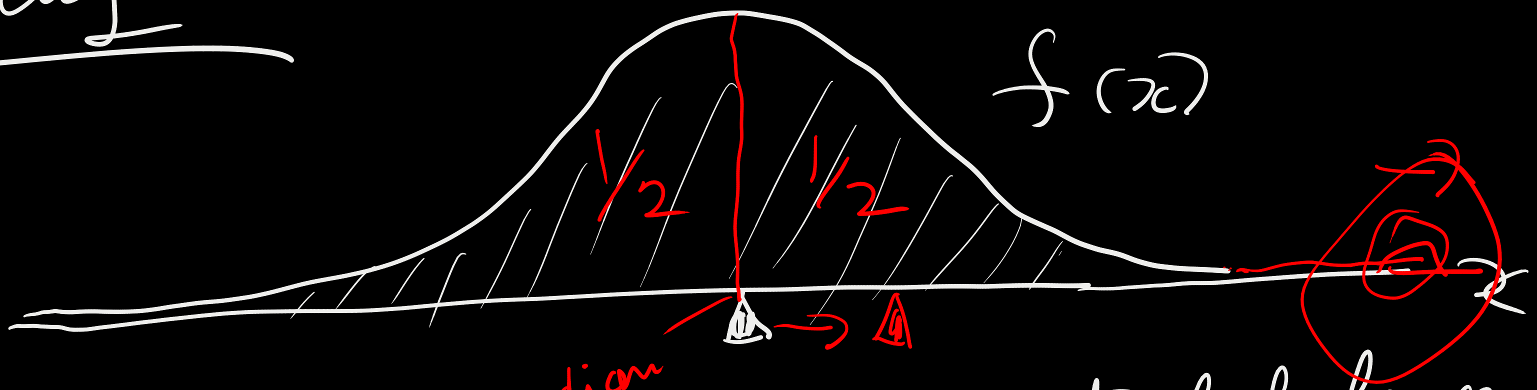
Chapter 4 Expected Value

Geometrically

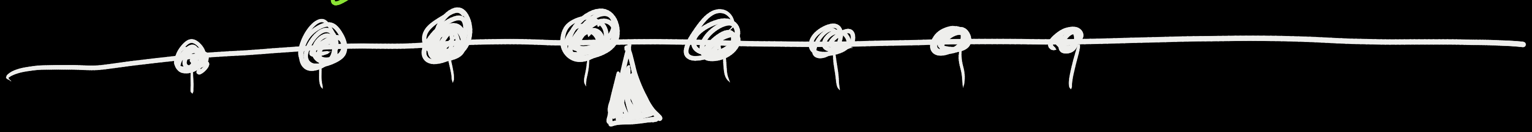
$X \sim f$



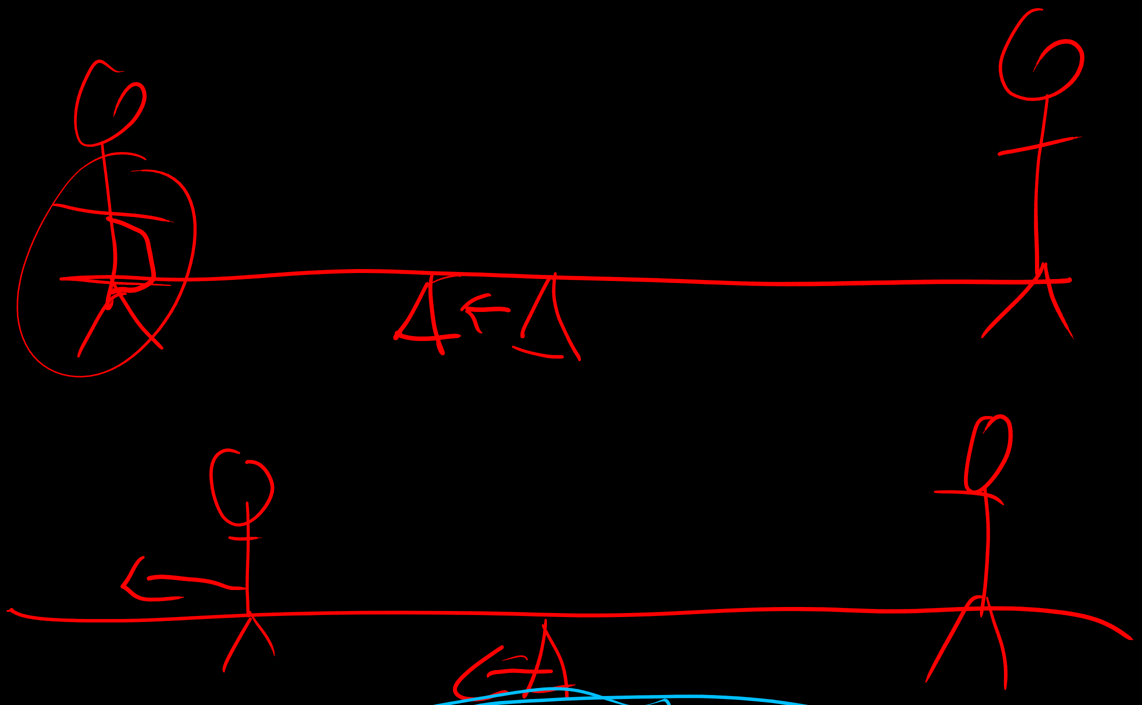
median $\approx$ E(X) is point of balance

weights = p(x)

$X \sim p$



E(X)



$$\lim_{a \to \infty} \int_{-a}^{2a} x\, f(x)\, dx \to 2\ 0$$

Defn $E(X) = \sum x \, p(x)$ provided $\sum |x| \, p(x) < \infty$

otherwise "not defined"

$$E(X) = \int_{-\infty} x \, f(x) \, dx \quad \text{provided} \quad \int |x| f(x) \, dx < \infty$$

Examples:

Exponential (λ)

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0, \; \lambda > 0$$

$\Gamma(2)$

$$E(X) = \frac{\Gamma(\alpha)}{\lambda} \boxed{\int_0^\infty \frac{x^{\alpha - 1} \lambda^2 e^{-\lambda x}}{\Gamma(\alpha)} \, dx} = \frac{1}{\lambda}$$

$\alpha = 2$

$$= \Gamma(2)\,\frac{1}{\lambda}\int \underbrace{\frac{x^{2-1}\lambda^{2} e^{-\lambda x}}{\Gamma(2)}}_{\text{Gamma}(2,\lambda)}\,dx \qquad \alpha = 2$$

$$= \frac{1}{\lambda}$$

$\text{Gamma}(\alpha, \lambda)$

Use Gamma distribution

$$\int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}\, dx = 1$$

$$\frac{\Gamma(\alpha)}{\lambda}$$

Poisson: $E(X) = \displaystyle\sum_{x=0}^{\infty} \underbrace{x \frac{\lambda^x}{x!} e^{-\lambda}}$

first term is 0

$$= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

We know $\displaystyle\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{\lambda}$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$= e^{-\lambda}\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

but $= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$

$$= e^{-\lambda}\lambda e^{\lambda}$$

$$= \lambda$$

St. Petersburg Paradox

How to make a sure \$1

Find someone willing to make fair bets on the toss of a coin.

i.e. if coin is H they give me \$N

" " " T & give them \$N

Let X = amount I win.

$$\underline{E(X)} = \frac{1}{2}N + \frac{1}{2}(-N) = 0$$

"Fair bet"

My strategy:

- Bet \$1
- If I win stop
- If I lose bet double the amount
- Keep doubling until I win

If my first win is at the kth bet

my winnings are:

$$\boxed{-1-2-4\ \dots\ -2^j-\dots-2^{k-1}}+2^k$$

$= \$ 1$

I am sure to eventually win $1

Recall for geometric series

$$\sum_{i=0}^{n-1} a^i = \frac{a^n-1}{a-1} \quad \text{if } a \neq 1$$

next − first

ratio − 1

- works for all finite n
- For "$n=\infty$" need $|a|<1$

Let W = amount won

It looks like $E(W) = 1$

not much, but sure!

Formally

$$E(W) = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} + \cdots +$$

$$= 1 \left(\frac{1}{2} + \frac{1}{4} + \cdots \right)$$

$$= 1$$

Zeno's Px

But let X be amount won on first winning bet:

$$E(X) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \times \frac{1}{8} + 8 \cdot \frac{1}{16} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots$$

$$= \infty$$

Let L be amount lost until first winning bet.

$$L = X - 1$$

$$E(L) = \infty.$$

$$\boxed{\infty - \infty = 1 \quad ?}$$ ✓ ?

Problem: You need ∞ capital to play this game

- If you have finite capital you might go bankrupt before winning
- Suppose you only have enough money to lose K times, then

$$P\left(W = -\sum_{i=1}^{K} 2^{i-1}\right) = 1 - \sum_{i=1}^{k} \left(\frac{1}{2}\right)^{i}$$

$$P\left(W = -\frac{2^k - 1}{2 - 1}\right) = 1 - \frac{\left(\left(\frac{1}{2}\right)^{k+1} - \frac{1}{2}\right)}{\frac{1}{2} - 1}$$

$$P\left\{W = -\left(2^k - 1\right)\right\} = 1 - \left(1 - \left(\frac{1}{2}\right)^k\right) = \frac{1}{2^k}$$

$$E(W) = -\left(2^k - 1\right) \times \frac{1}{2^k} + 1\left(1 - \frac{1}{2^k}\right)$$

$$= -1 + \frac{1}{2^k} + 1 - \frac{1}{2^k}$$

$$= 0$$

So a "fair" bet with a high probability of winning small at the cost of a small probability of losing big.

Scam strategy: Play the game with IOU's.

Gamma

$$E(X) = \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx$$

$$= \int_0^\infty \frac{\lambda^\alpha \lambda}{\lambda \Gamma(\alpha)} x^{\overset{(\alpha+1)-1}{\alpha}} e^{-\lambda x}\,dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\overset{\alpha}{\Gamma(\alpha+1)}}{\lambda^{\alpha+1}} \underbrace{\int_0^\infty \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x}\,dx}_{\text{Gamma}(\alpha+1)\text{ density}} = 1$$

$$= \frac{\alpha}{\lambda}$$

function

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\,\frac{1}{\lambda} = \frac{\alpha}{\lambda}$$ since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

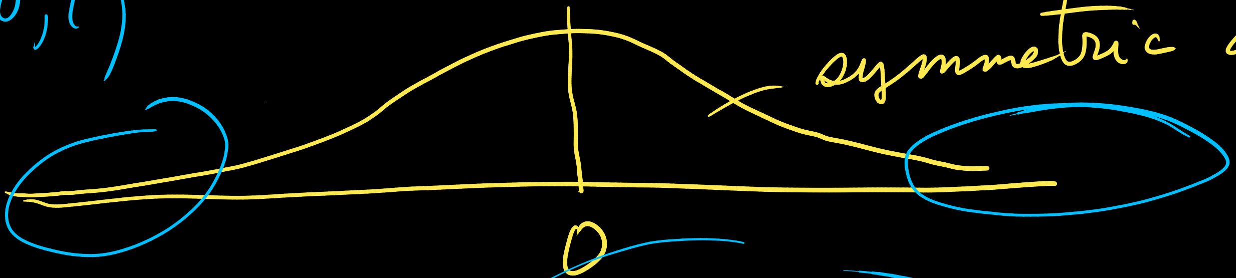
Cauchy:

$$f(x) = \boxed{\frac{1}{\pi}\left(\frac{1}{1+x^2}\right)}, \quad -\infty < x < +\infty$$

$X, Y \sim N(0,1)$

$\frac{X}{Y} \sim$ Cauchy

symmetric around 0



$e^{-\frac{1}{2}x^2}$

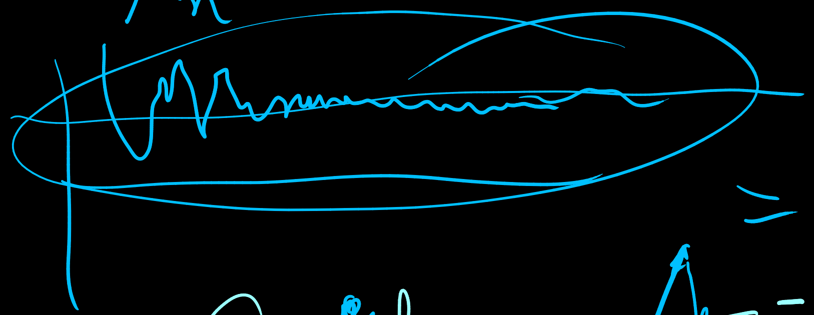
But $E(|X|) = \int_{-\infty}^{\infty} |x| \frac{1}{\pi} \frac{1}{1+x^2}\, dx$

$$= 2\int_0^\infty \frac{1}{\pi}\frac{x}{1+x^2}\,dx$$

$E(X)$

$$= \frac{2}{\pi}\left[\frac{1}{2}\log(x^2+1)\right]_0^\infty$$

$\bar{X}_n \quad n \to \infty$

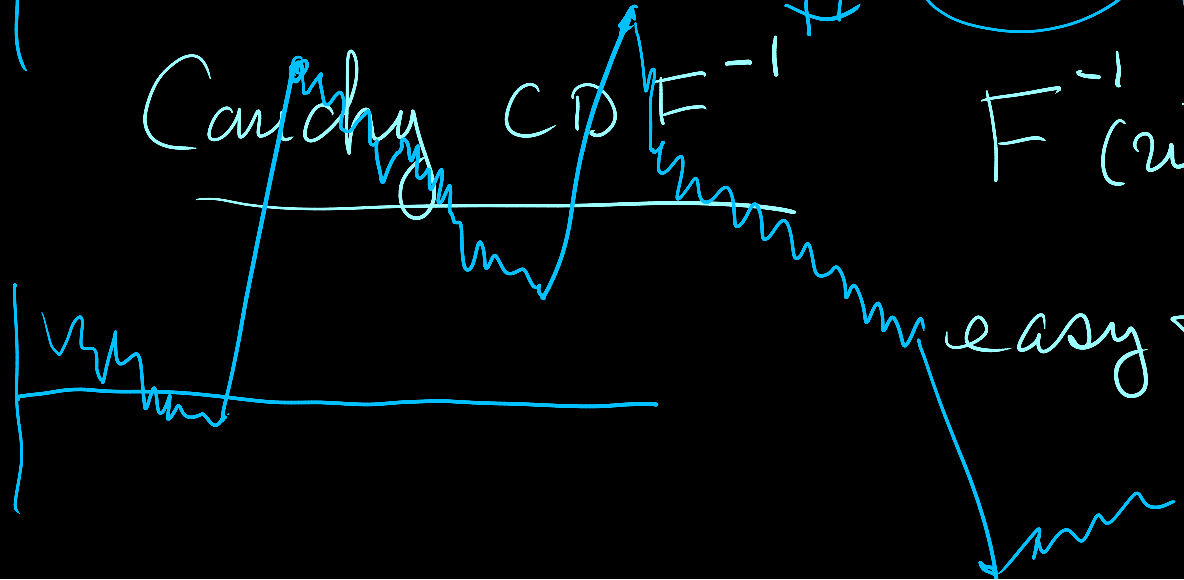


$E(X)$

$$= \frac{2}{\pi} \times \infty \; ! = \infty$$

Cauchy CDF^{-1}

$$F^{-1}(u) = \tan^{-1}\left(\pi u - \frac{\pi}{2}\right)$$



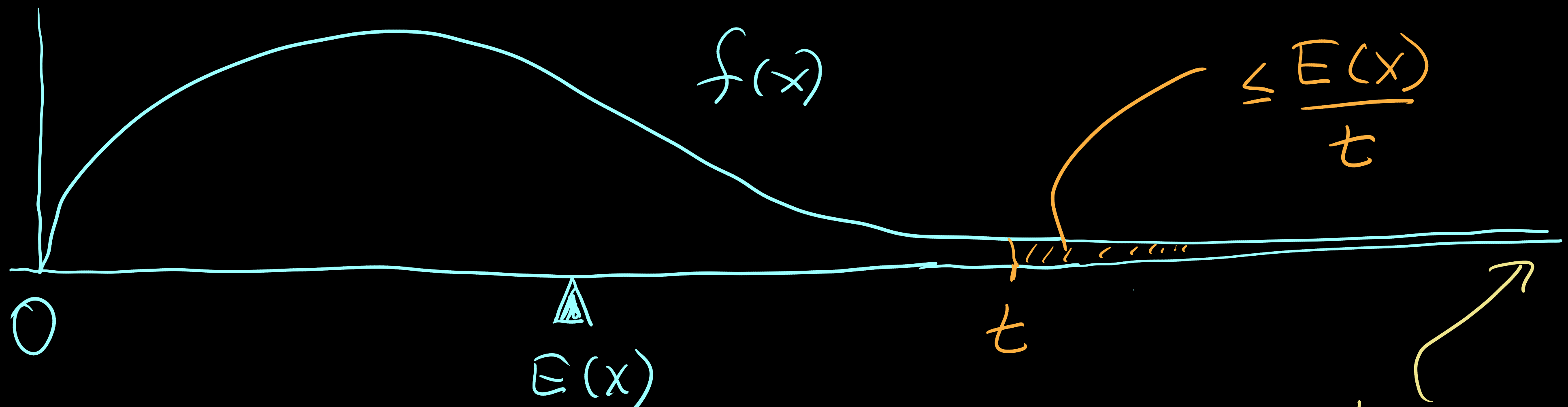
easy to simulate.

Markov Inequality:

If 1) X is a RV with $Pr(X \geq 0) = 1$

2) $E(X) < \infty$

Then for any $t > 0$ $\quad P(X \geq t) \leq \dfrac{E(X)}{t}$



Note E(X) very affected by tail

Proof: $E(X) = \int_0^\infty x f(x)\, dx$

$$= \int_0^t x f(x)\, dx + \int_t^\infty x f(x)\, dx$$

$$\geq \int_t^\infty x f(x)\, dx$$

$$\geq \int_t^\infty t f(x)\, dx$$

$$= t P(X \geq t)$$

need this for discrete case