3.7 Extremes and order statistics

Let $X_1, \ldots, X_n$ be iid with cdf F_X & pdf f_X.

Let $U = \max(X_i)$ $\quad V = \min(X_i)$

CDF of U:

$$F_U(u) = \Pr(U \le u)$$
$$= \Pr(X_1 \le u \cap X_2 \le u \cap \ldots \cap X_n \le u)$$
$$= \Pr(X_1 \le u) \times \ldots \times \Pr(X_n \le u)$$ Why?

$$= \underbrace{F_X(u) \times \circ \circ \circ \times F_X(u)}_{n \text{ times}}$$

$$= F_X^n(u)$$

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \frac{d}{du} F_X^n(u)$$

$$= n F_X^{n-1}(u) \times \frac{d}{du} F_X(u)$$

$$= n F_X^{n-1}(u) f_X(u)$$

$$F_V(v) = \Pr(\underline{X_1 \le v} \cup \underline{X_2 \le v} \cup \dots \cup \underline{X_n \le v})$$

$$= 1 - \Pr(X_1 > v \cap X_2 > v \cap \dots \cap X_n > v)$$

$$= 1 - \underbrace{(1-F(v))(1-F(v))\cdots(1-F(v))}_{n \text{ time}}$$

$$= 1 - (1-F(v))^n$$

$$f_V(v) = \frac{dF_V(v)}{dv} = -n(1-F(v))^{n-1}(-f(v))$$

$$= n f(v)(1-F(v))^{n-1}$$

See examples in text

Here a general formula for the joint density of "order statistic" from a sample of n with cdf F and pdf f:
"iid"

This is for ordered indices between 1 and n

e.g. If $n=10$, indices could be

1, 3, 9

smallest

3rd smallest

2nd from largest

Notation $(X_{(1)}, X_{(3)}, X_{(9)})$

Or just the smallest & largest $(X_{(1)}, X_{(10)})$

Positions:

1 2 3 4 5 6 7 8 9 10

We want $\underbrace{\qquad}_{2}$ $\underbrace{(3)}_{1}$ $\underbrace{\qquad}_{2}$ $\underbrace{(6)}_{1}$ $\underbrace{\qquad\qquad}_{4}$

2 1

$$f(x_{(3)}, x_{(6)}) = \binom{10}{2\ 1\ 2\ 1\ 4} \times$$

$$F(x_{(3)})^2 f(x_{(3)})^1 (F(x_{(6)}) - F(x_{(3)}))^2 f(x_{(6)})^1 (1 - F(x_{(6)}))^4,$$

$$x_{(3)} < x_{(6)}$$

where $\begin{pmatrix} 10 \\ \underbrace{2 \quad 1 \quad 2 \quad 1 \quad 4}_{\text{These must add up to } 10} \end{pmatrix} = \dfrac{10!}{2!\,1!\,2!\,1!\,4!}$

For example $n = 3$

$$f(x_{(1)}, x_{(2)}, x_{(3)}) = \begin{pmatrix} 3 \\ 1 \quad 1 \quad 1 \end{pmatrix} f(x_{(1)}) f(x_{(2)}) f(x_{(3)})$$

$$x_{(1)} < x_{(2)} < x_{(3)}$$

$$\frac{3!}{1!\,1!\,1!} = 3!$$

$n = 4 \qquad x_{(1)},\ x_{(3)}$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ x_{(1)} & \underbrace{\quad}_{1} & x_{(3)} & \underbrace{\quad}_{1} \\ 1 & & 1 & \end{array}$$

$$\binom{4}{1\ 1\ 1\ 1} f(x_{(1)})^1 \left[F(x_3) - F(x_{(1)})\right]^1 \times$$

$$f(x_{(3)})^1 \left(1 - F(x_{(3)})\right)^1$$