Chapter 3 Joint distributions

2 or more random variables

E.G. • Height & Weight

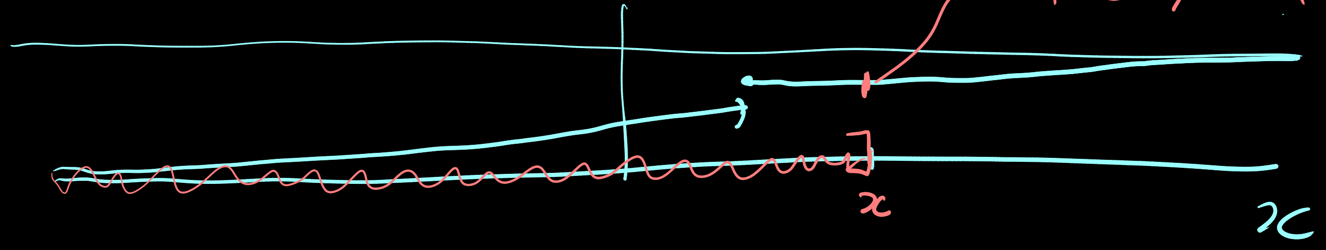
• Dose of a drug & time to recovery

• Gender, Age, Education, Income

CDF One RV: $F(x) = P(X \leq x)$

$$F : \mathbb{R} \to [0,1]$$

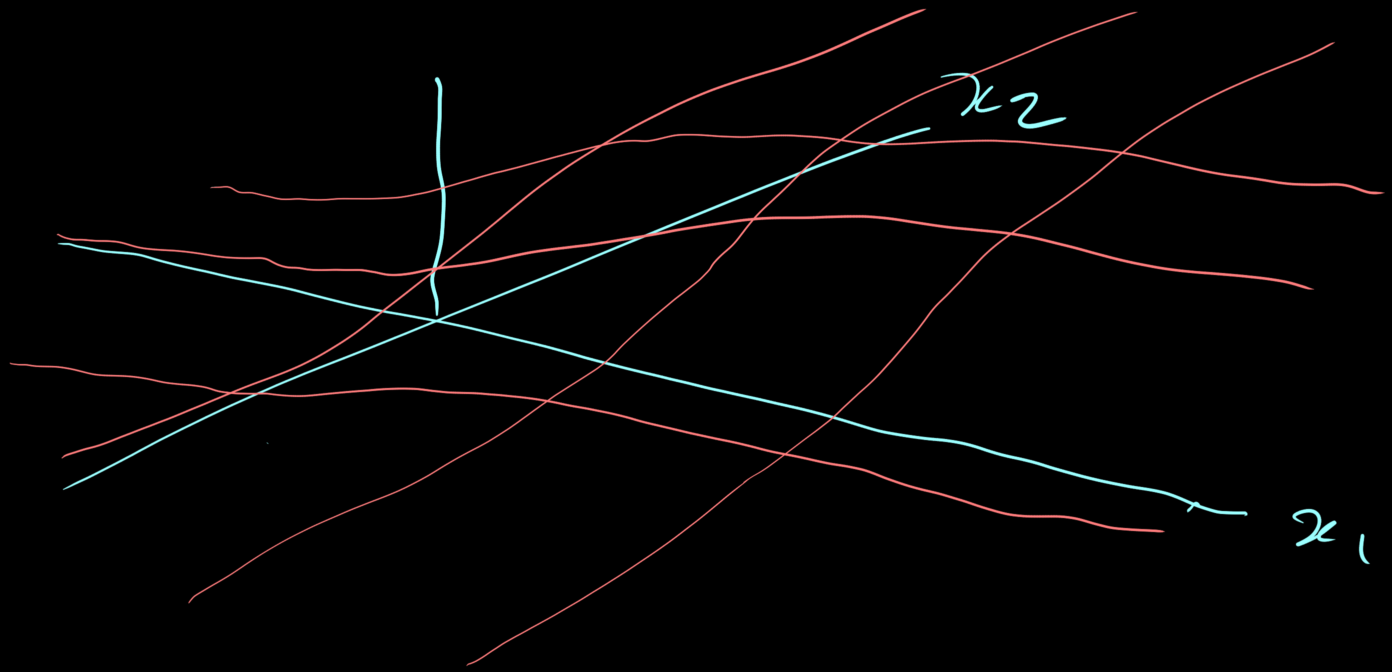
$F(x) = P((-\infty, x])$



CDF: Two R.V.s X_1, X_2

$$F(x_1, x_2) = P(X_1 \le x_2 \text{ and } X_2 \le x_2)$$

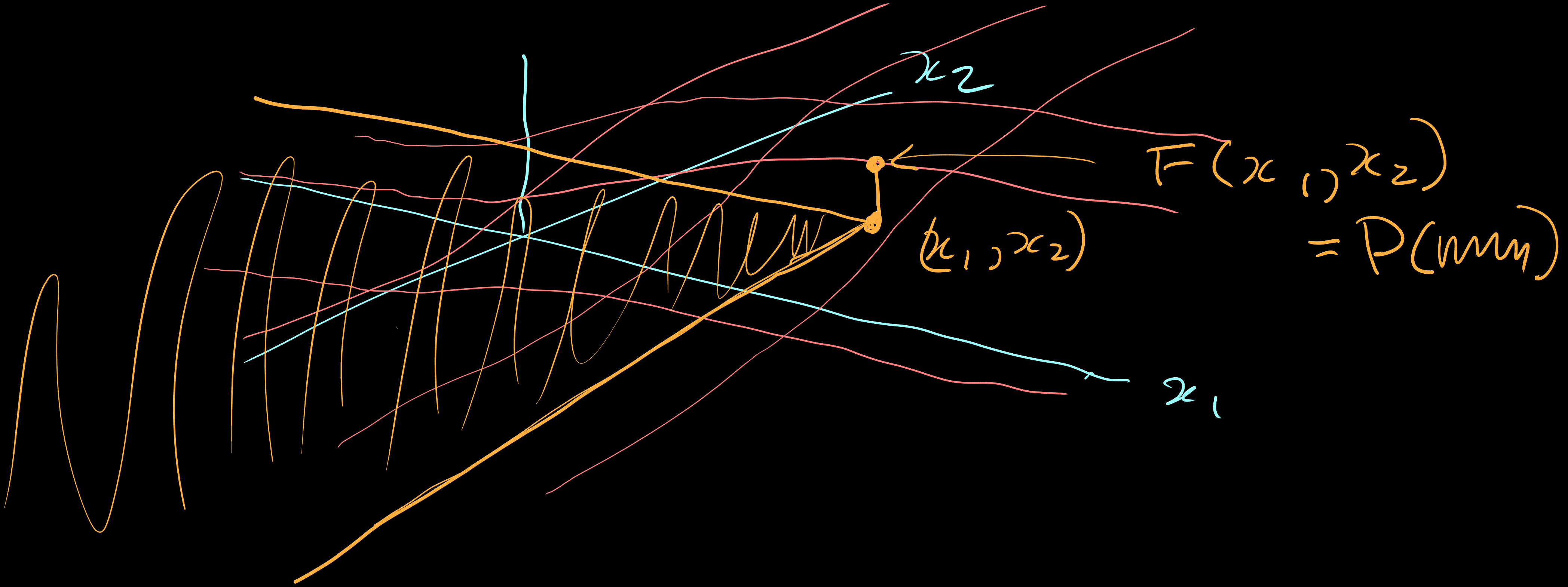
$$F : \mathbb{R}^2 \to [0,1]$$



CDF: Two R.V.s X_1, X_2

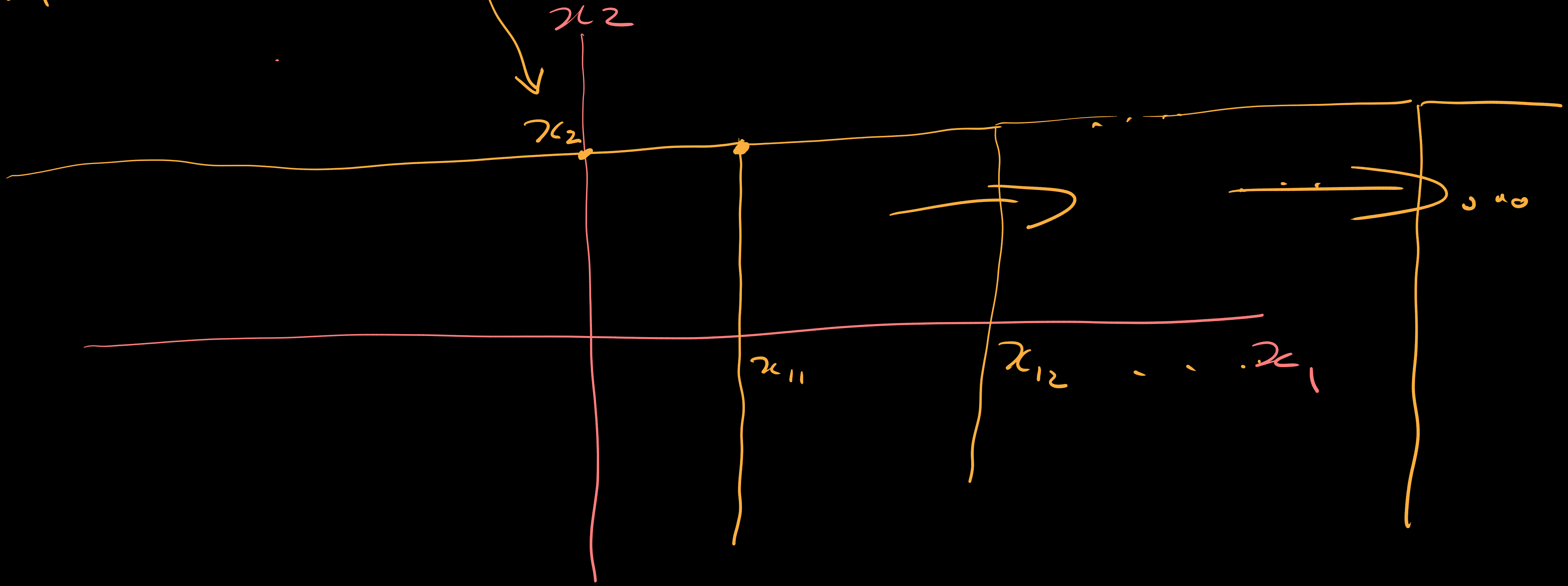
$$F(x_1, x_2) = P(X_1 \le x_2 \text{ and } X_2 \le x_2)$$

$$F : \mathbb{R}^2 \to [0, 1]$$



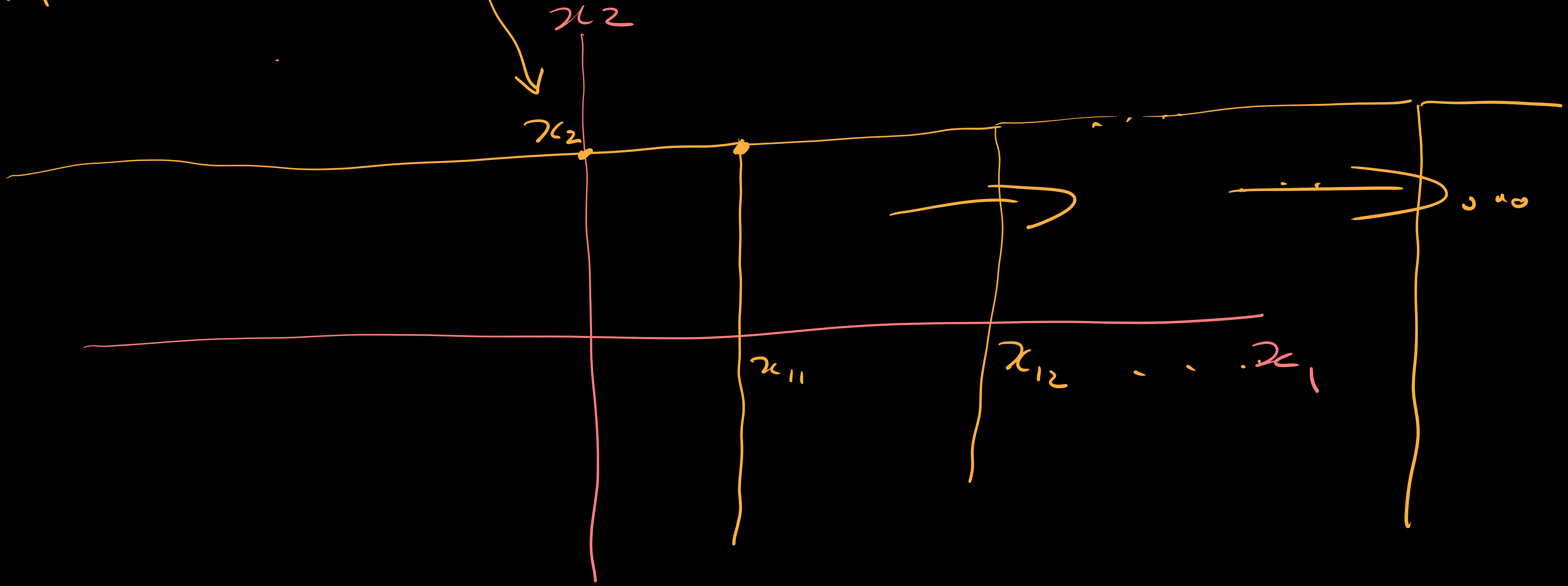
If you fix x_2 and let $x_1 \to \infty$

$$\lim_{x_1 \to \infty} F(x_1, x_2) = \underline{\hspace{6cm}}$$



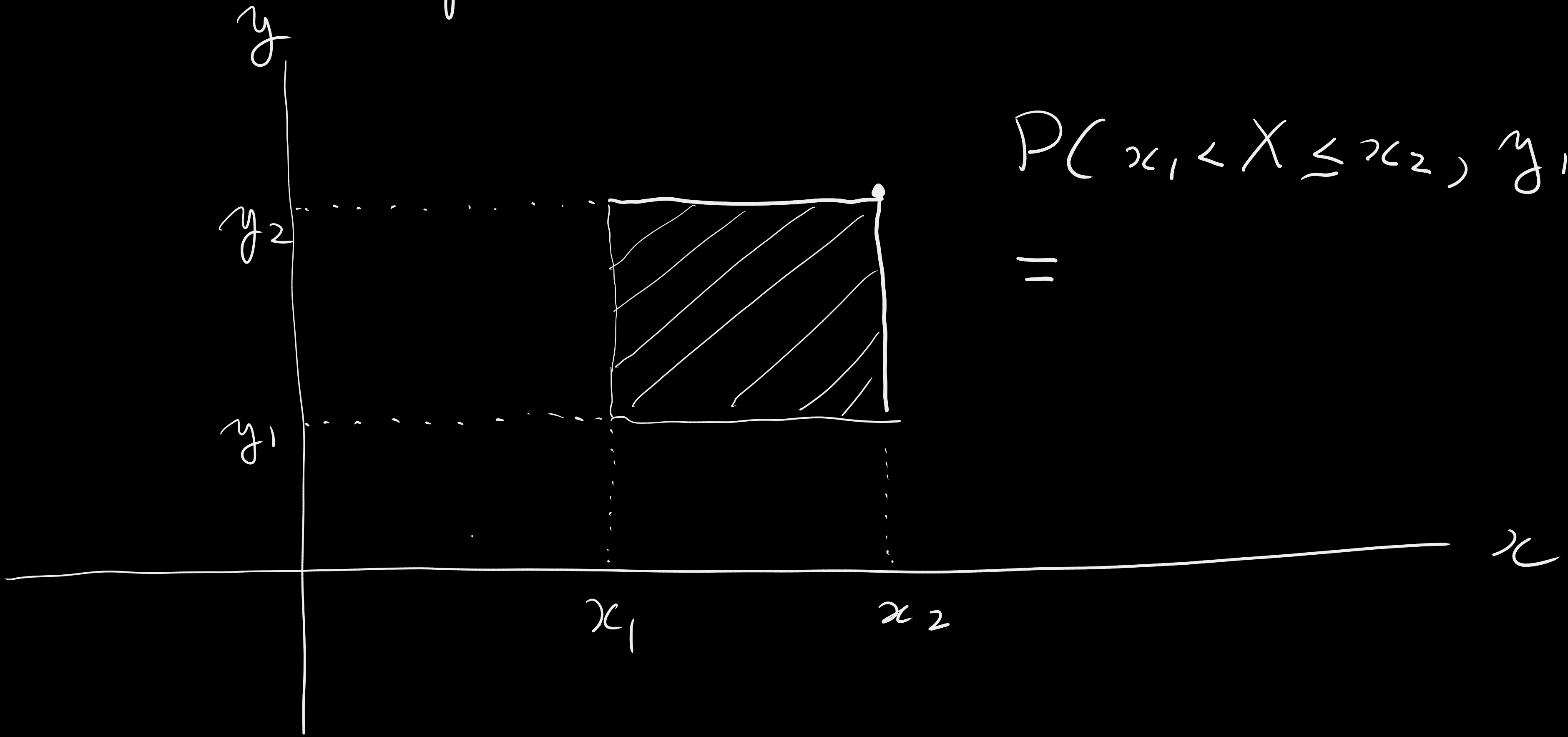
If you fix x_2 and let $x_1 \to \infty$

$$\lim_{x_1 \to \infty} F(x_1, x_2) = F_2(x_2)$$



Similarly $\lim_{x_2 \to \infty} F(x_1, x_2) = F(x_1)$

Probability of a rectangle:

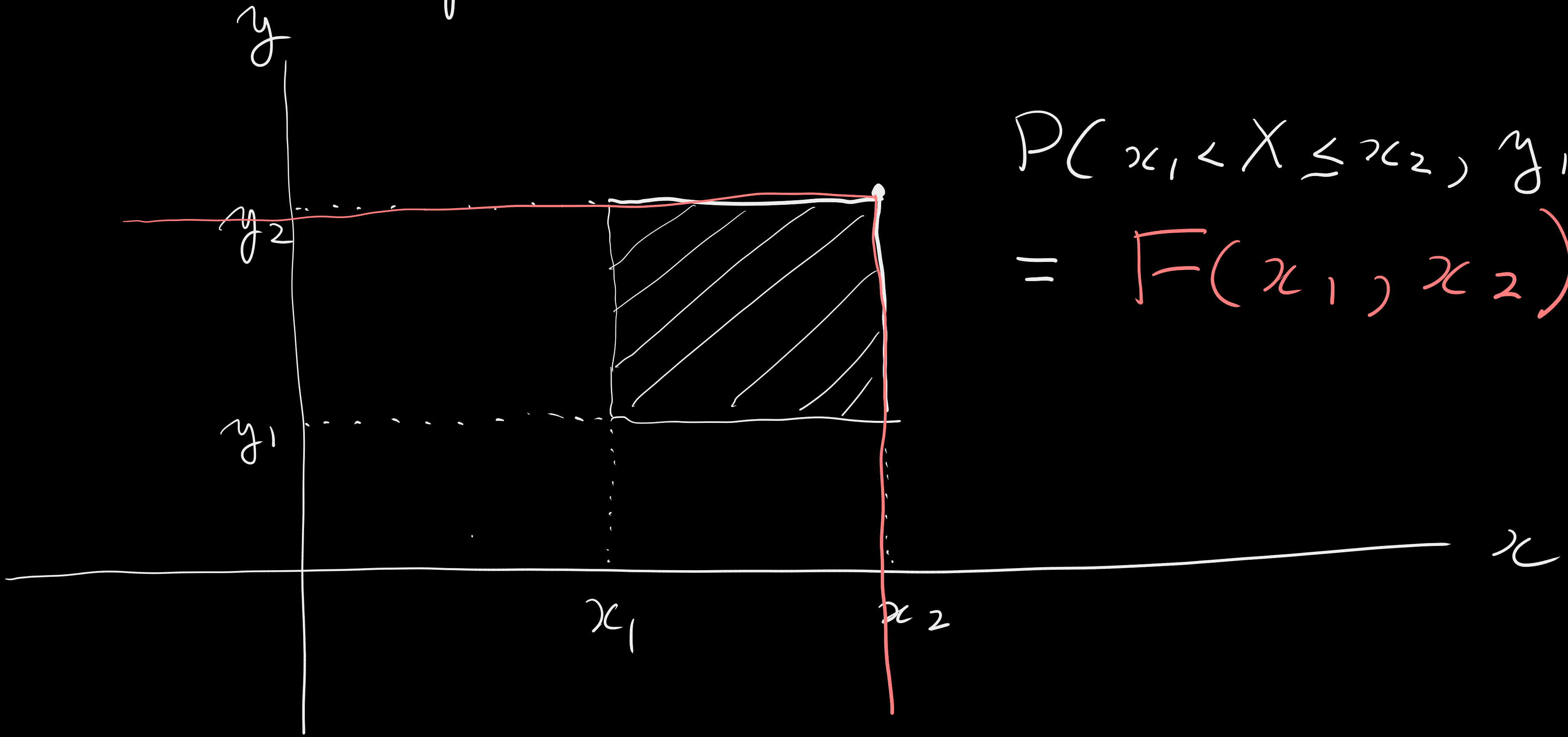


$P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$

$=$

Similarly $\lim_{x_2 \to \infty} F(x_1, x_2) = F(x_1)$

Probability of a rectangle:

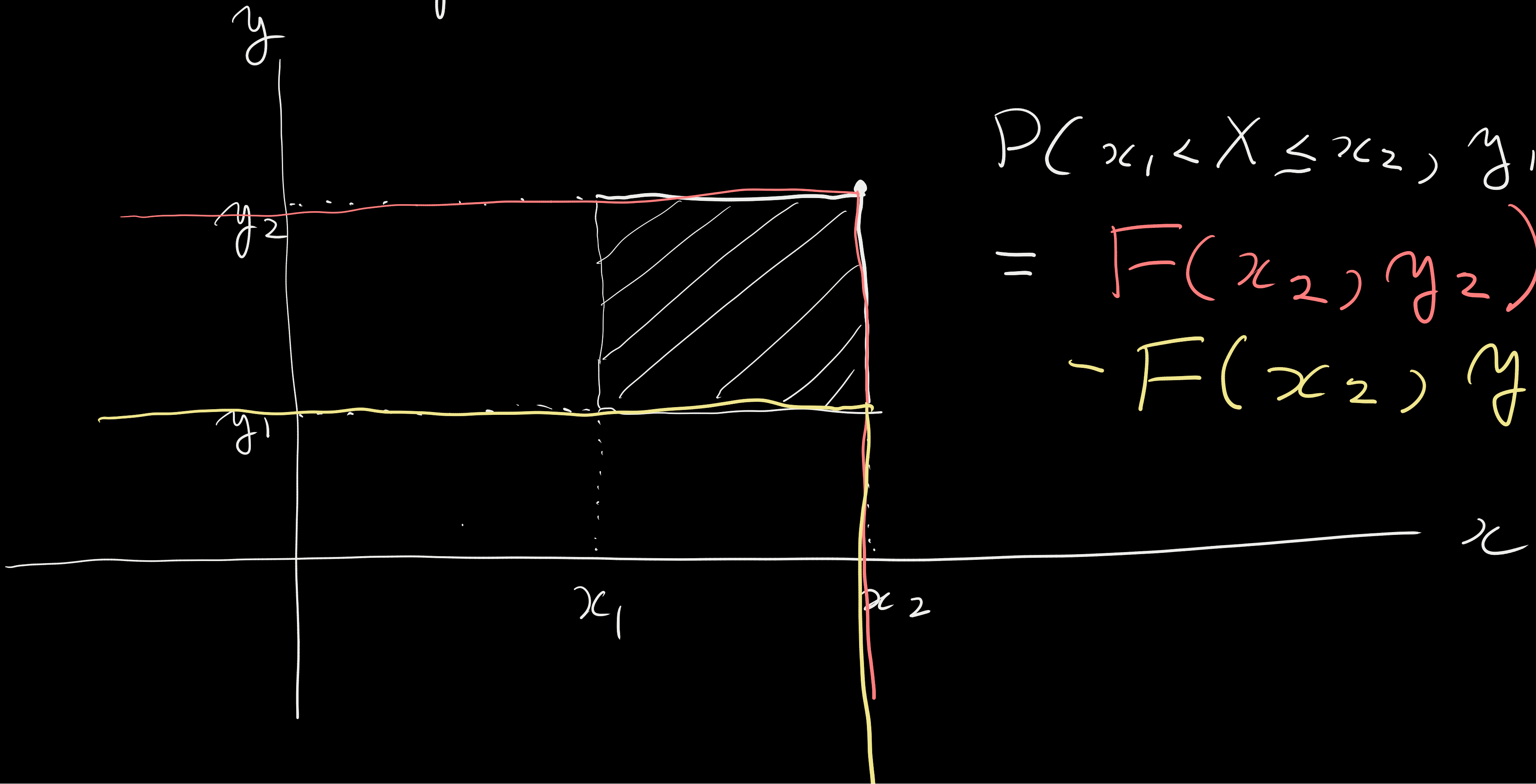


$P(x_1 < X \le x_2,\ y_1 < Y \le y_2)$

$= F(x_1, x_2)$

Similarly $\lim_{x_2 \to \infty} F(x_1, x_2) = F(x_1)$

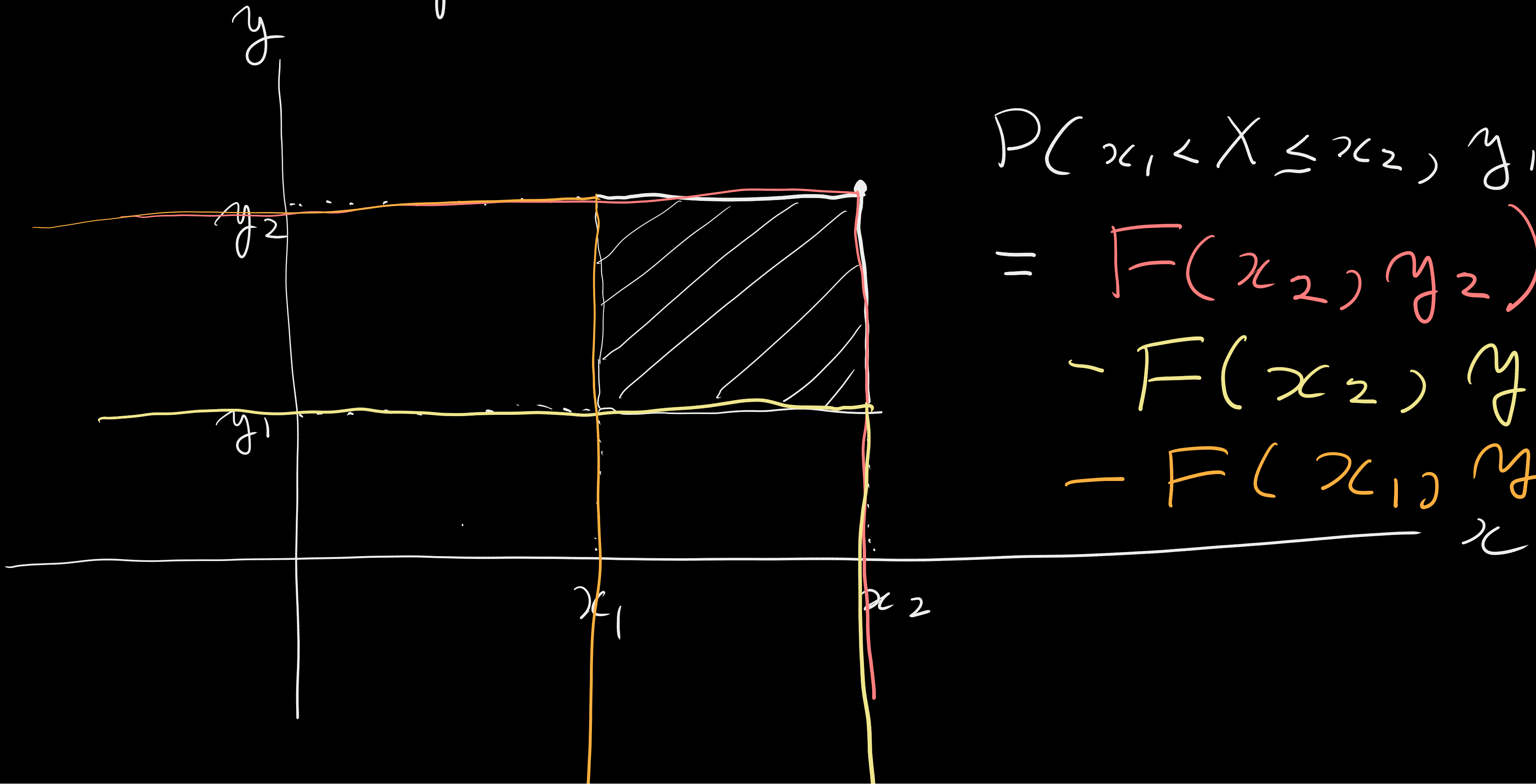
Probability of a rectangle:



$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$$
$$= F(x_2, y_2)$$
$$- F(x_2, y_1)$$

Similarly $\lim_{x_2 \to \infty} F(x_1, x_2) = F(x_1)$

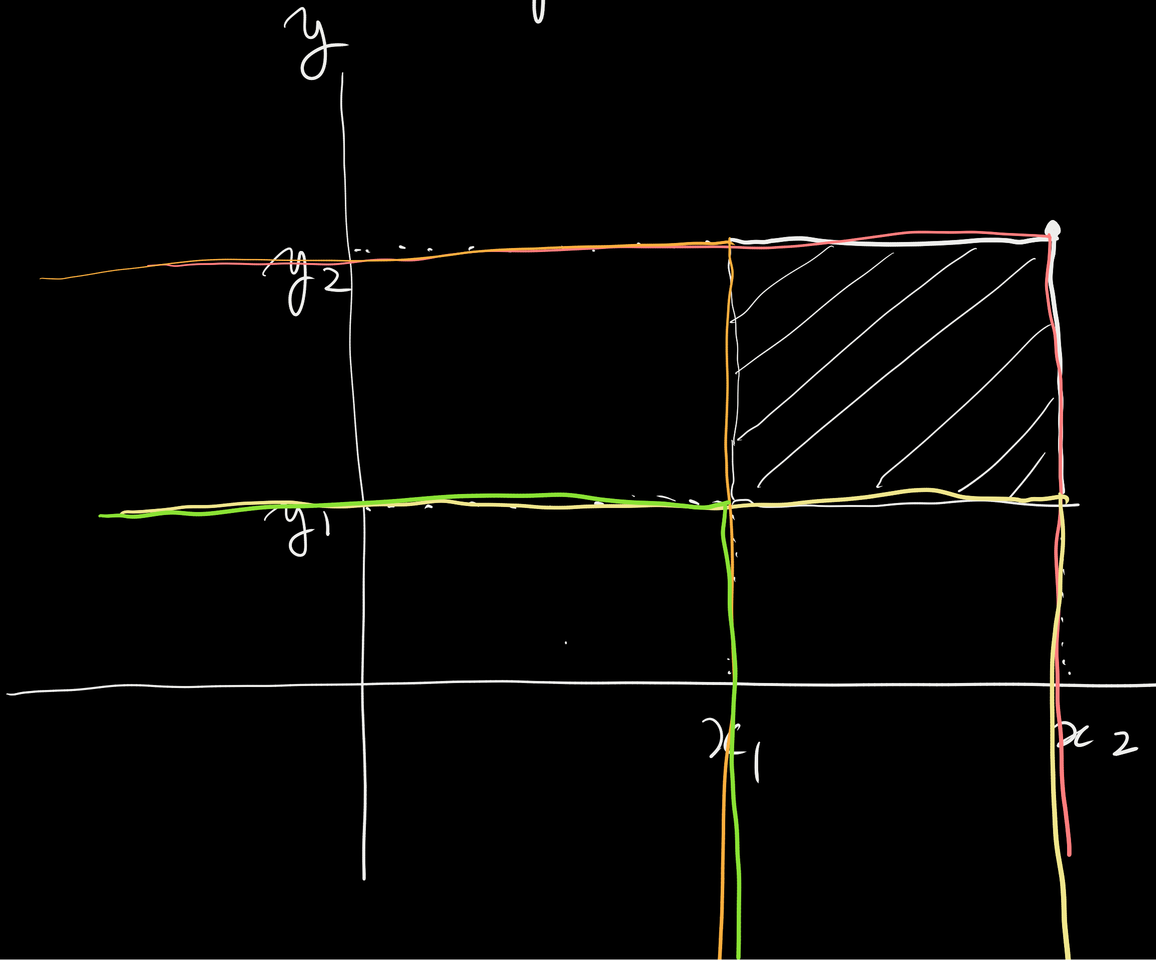
Probability of a rectangle:



$$P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2)$$
$$= F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2)$$

Similarly $\lim_{x_2 \to \infty} F(x_1, x_2) = F(x_1)$

Probability of a rectangle:



$$P(x_1 < X \le x_2, y_1 < Y \le y_2)$$

$$= F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$$

Recap :

$$P(x_1 < X \le x_2, y_1 < Y \le y_2)$$

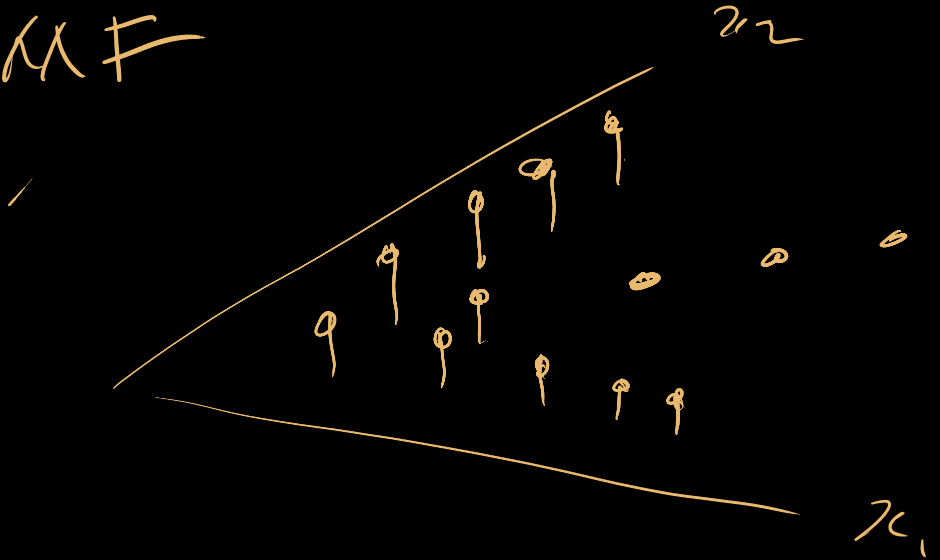
$$= F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$$

Discrete random variables

Joint PMF $P(X_1 = x_1, X_2 = x_2, X_3 = x_3)$

$$= p(x_1, x_2, x_3)$$

Joint PMF



If $\text{supp}(X_1, X_2)$ is finite, we can use a table.

e.g. Toss a coin: $X = \#$ of Heads & a die, $Y = \#$ rolled.

X \ Y	1	2	3	4	5	6	
0	1/12	1/12	1/12	1/12	1/12	1/12	1/2
1	1/12	1/12	1/12	1/12	1/12	1/12	1/2
	1/6	1/6	1/6	1/6	1/6	1/6	1

Joint PMF

marginal PMF for X

$P(\Omega)$

marginal for Y

In general: Joint PMF

$$p(x_1, x_2, x_3, x_4)$$

To get marginal PMF just sum over the variables you don't want:

$$p(x_1, x_3) = \sum_{\substack{\text{all } x_2\text{'s} \\ \text{all } x_4\text{'s}}} p(x_1, x_2, x_3, x_4)$$

Example: Multinomial distribution

Consider $X \sim \text{Binomial}(n, p)$

$P(H) = p \quad P(T) = 1 - p = q$

Toss n times $\quad$ H H H T H H H T H H $\quad (n = 10)$

8 Hs and 2 Ts

So $X = 8$

But we could also record # of Ts, 2
and think of this as a joint distribution
for $X_1 = \#$ of Hs, and $X_2 = \#$ of Ts

Here $(X_1, X_2) = (8, 2)$

Of course $X_1 + X_2 = n$

Here's the joint distribution for $n = \cancel{3}\ 2$

X_1 (Heads) \ X_2 (Tails)	0	1	2
0	0	0	1/4
1	0	1/2	0
2	1/4	0	0

Tosses	P	X_1	X_2
T T	1/4	0	2
T H	1/4	1	1
H T	1/4	1	1
T T	1/4	2	0

$$(X_1, X_2) \sim \text{Multinomial}(\cancel{3}\ 2, p = (1/2, 1/2))$$

This generalized to any number of categories.

E.G. Take a sample of n students and record eye color: Black, Hazel, Blue, Gray

Counts: X_1 X_2 X_3 X_4

$$(X_1, X_1, X_2, X_4) \sim \text{Multinomial}\left(n, (p_1, p_2, p_3, p_4)\right)$$

proportions of B, H, B, G in population

Or. $\underset{\sim}{X} \sim \text{Multinomial}(n, \underset{\sim}{p})$

PMF:

$$p(x_1, \dots, x_n) = \binom{n}{x_1, x_2 \dots x_n} p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}$$

$$\text{if } x_i \geq 0, \ \sum x_i = n,$$

$$p_i > 0, \ \sum p_i = 1$$

Let X_1, X_2, X_3 be Mulinomial $(n, (p_1, p_2, p_3))$

Then $X_1 \sim$ Binomial(n, p_1)

Proof: $P(X_1 = x_1) = \sum_{\substack{x_2 \\ x_3 = n - x_1 - x_2}} \binom{n}{x_1\ x_2\ x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3}$

pulling out factors that don't depend on summands

$$= \frac{n!}{x_1!} p_1^{x_1} \sum_{x_2 + x_3 = n - x_1} \frac{1}{x_2!\, x_3!} p_2^{x_2} p_3^{x_3}$$

Greatest tricks in math: $\begin{cases} \text{multiply} \times 1 \\ \text{add } 0 \end{cases}$

What can we do here?

$= \ldots$

Continuous random variables

$\iint$ instead of Σ

(X, Y) have a bivariate continuous distribution

if $\quad P((X,Y) \in A) = \iint_A f(x,y)\, dx\, dy$

for some function f such that

1) $f \geq 0$ on $\mathbb{R}^2$

2) $\iint f(x,y)\, dx\, dy = 1$

CDF: $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, dv\, du$

$$= \int_{-\infty}^{x} \left\{ \int_{-\infty}^{y} f(u, v)\, dv \right\} du$$

PDF from CDF

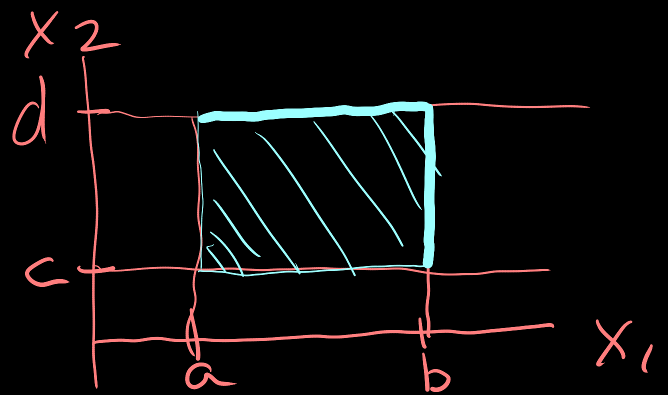
$$f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2)$$

Partial derivatives

Probability of a rectangle (easier)

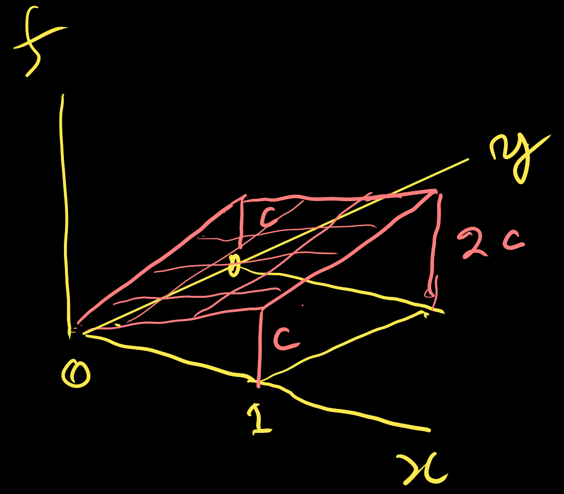
$$P(a < X_1 \le b,\ c < X_2 \le d)$$

$$= \int_a^b \int_c^d f(x_1, x_2)\, dx_2\, dx_1$$



Simple example : $f(x,y) = c(x+y) \quad x,y \in (0,1)$

1) Find c to make f a density



$$\iint f(x,y)\,dx\,dy = \int_0^1\int_0^1 (x+y)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right|_0^1 dx$$

$$= \int_0^1 \left(x+\frac{1}{2}\right) dx = \left[\frac{x^2}{2} + \frac{1}{2}x\right|_0^1$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

So $c = 1$

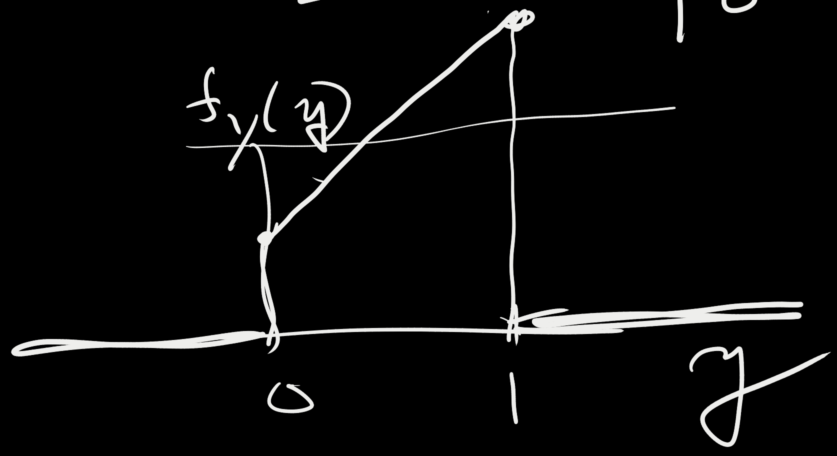
Marginal distribution of Y

Integrate out x :

$$f_Y(y) = \int f(x,y)\,dx$$

$$= \int_0^1 x + y \, dx = \left[\frac{x^2}{2} + xy\right|_0^1$$

$$= \frac{1}{2} + y$$



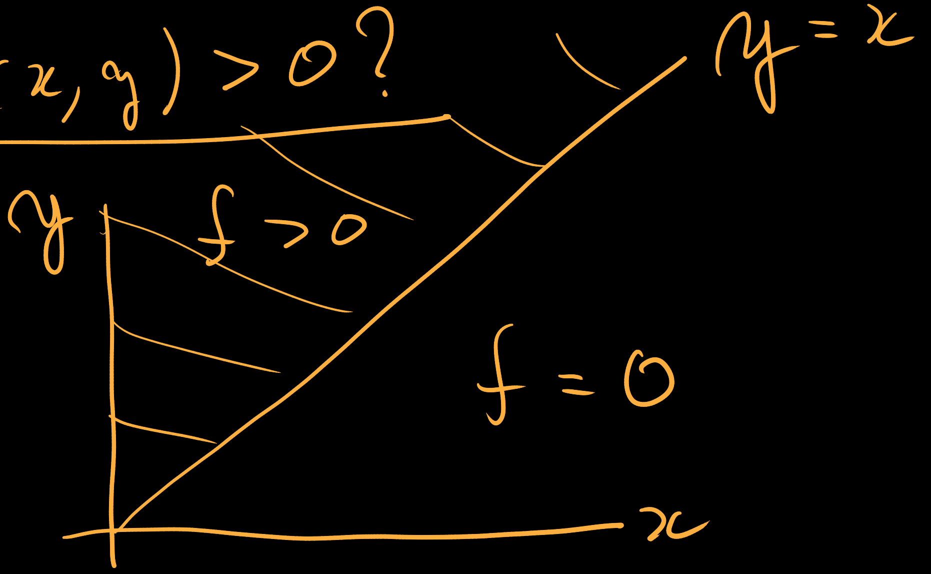
So $P(Y \le 1/2) = \int_0^{1/2} f_Y(y)\, dy$

$$= \int_0^{1/2} \tfrac{1}{2} + y \, dy = \frac{3}{8}$$

Harder example : p.79 Example D

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \le x \le y \\ 0 & \text{otherwise} \end{cases} \qquad \lambda > 0$$

Where is $f(x,y) > 0$?



What does $f(x,y)$ look like?

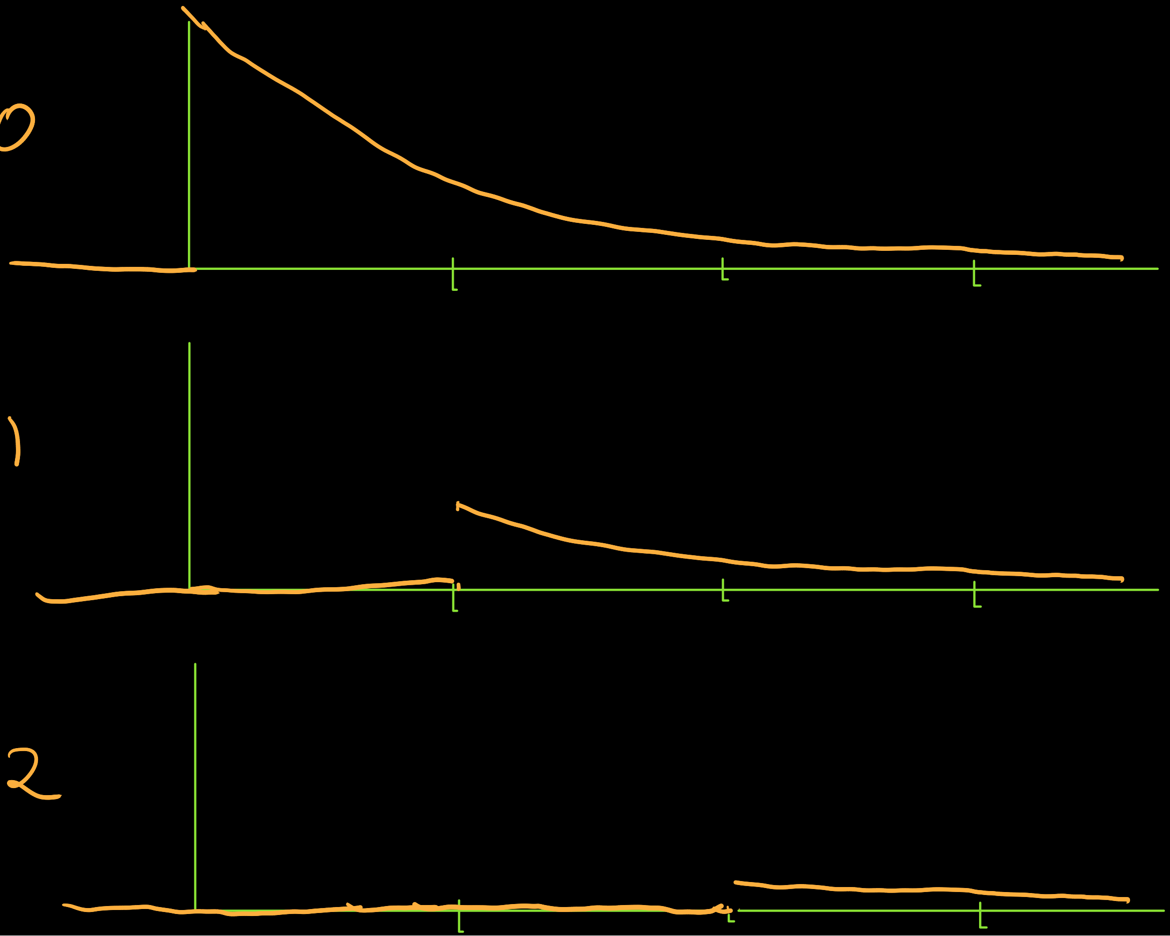
Fix parameter $\lambda = 1$

e^{-y} for $y \geq x$

$x = 0$

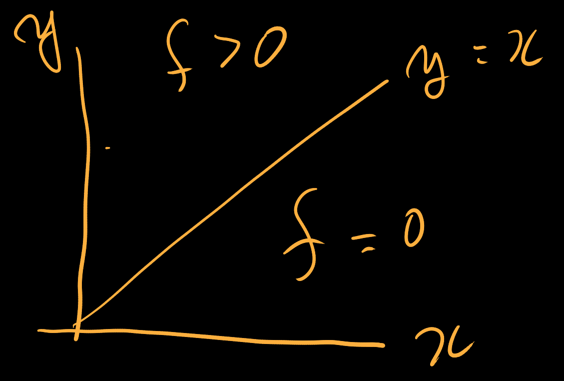
$x = 1$

$x = 2$



Marginal distribution of X

Integrate out Y



$$\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{?}^{?} \lambda^2 e^{-\lambda y}\,dy$$

$$0 < x < y < \infty$$

$$f_X(x) = \int_{x}^{\infty} \lambda^2 e^{-\lambda y}\,dy = \left[\frac{\lambda^2 e^{-\lambda y}}{-\lambda}\right]_{x}^{\infty} = 0 - \left(-\lambda e^{-\lambda x}\right)$$

$$= \lambda e^{-\lambda x}, \quad y > 0$$

So X is exponential

$\underline{Y?}$

$$f_Y(y) = \int f(x,y)\,dx$$

$$= \int_0^y \lambda^2 e^{-\lambda y}\,dx$$

$$= \Big[x \lambda^2 e^{-\lambda y} \Big]_0^y$$

$$= y^1 \lambda^2 e^{-\lambda y} = \frac{y^{2-1} \lambda^2 e^{-\lambda y}}{\Gamma(2)}$$

$$= \frac{y^{\alpha-1} \lambda^{\alpha} e^{-\lambda y}}{\Gamma(\alpha)}, \quad \alpha = 2$$

So $Y \sim$ Gamma with shape parameter $\alpha = 2$ and scale parameter λ

We will return to copulas on p. 78
and the bivariate normal ~~on~~ on p. 81
after talking about independence section 3.4

3.4 Independent random variables

Def: Random variables $X_1, \ldots, X_n$ are <u>independent</u> iff joint CDF factors into product of marginal CDF:

$$F(x_1, x_2, \ldots, x_n) = F_1(x_1) F_2(x_2) \cdots F_n(x_n)$$

$\forall x_1, \ldots, x_n$.

Facts ① If X_1, X_2 are independent RVs then all events involving only X_1 are independent of events involving only X_2.

i.e. $P(X_1 \in A, X_2 \in B)$

$$= P(X_1 \in A)\, P(X_2 \in B)$$

(2) Events for any function g_1 of X_1 and g_2 of X_2 are independent:

$$P(g_1(X_1) \in A,\ g_2(X_2) \in B)$$

$$= P(g_1(X_1) \in A) \times P(g_2(X_2) \in B)$$

(3) Discrete case : $X_1, \cdots, X_n$ are independent

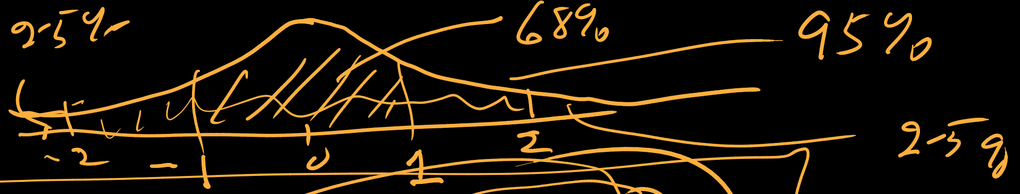
iff $p(x_1, x_2, \cdots, x_n) = p_1(x_1) \cdots p(x_n)$

(4) Continuous case : $X_1, \cdots, X_n$ are independent

iff $f(x_1, \cdots, x_n)$ can be expressed as $= f_1(x_1) \cdots f_n(x_n)$

Back to p. 81 : Bivariate normal

Univariate $N(\mu, \sigma^2)$



Standard Normal

$Z \sim N(0, 1)$: $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$

$X \sim N(\mu, \sigma^2)$

$X = \mu + \sigma Z$ $Z = (X - \mu)/\sigma$

$$f_X(x) = f_Z((x-\mu)/\sigma) \left| \frac{dz}{dx} \right| = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \times \frac{1}{\sigma}$$

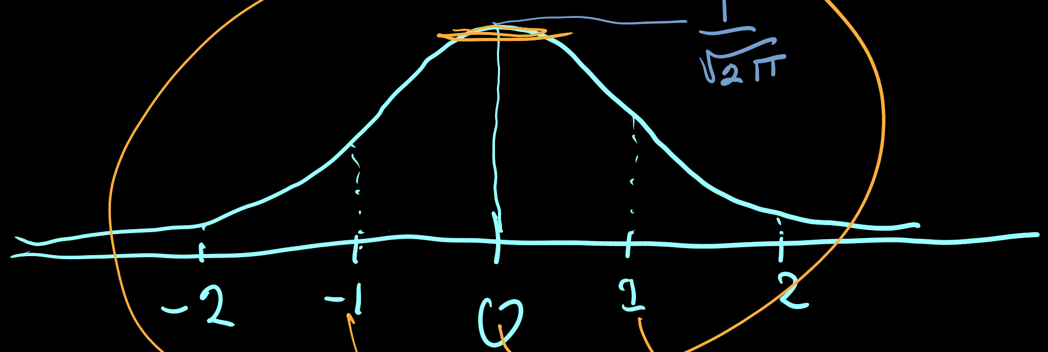
$$= \boxed{\frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}}$$

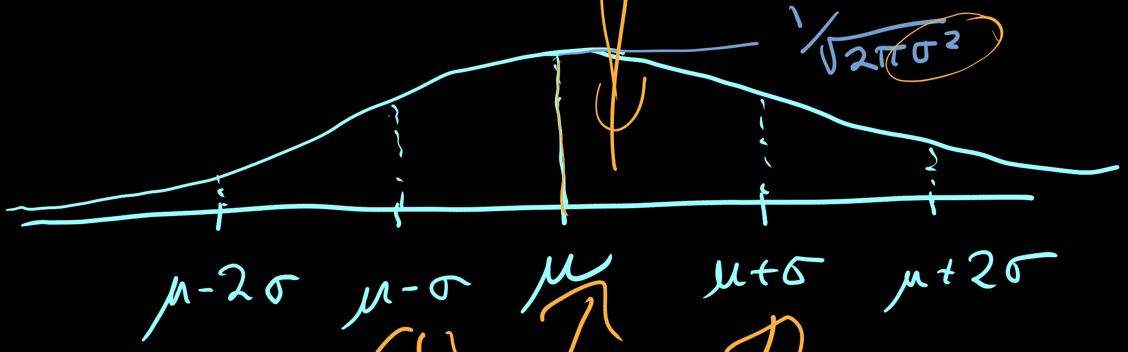
$$\text{OR} \quad = \boxed{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}}$$

Standard Normal

$Z \longrightarrow N(\mu, \sigma^2)$

$X = \mu + \sigma Z$





Multivariate normal in stages

Bivariate standard normal

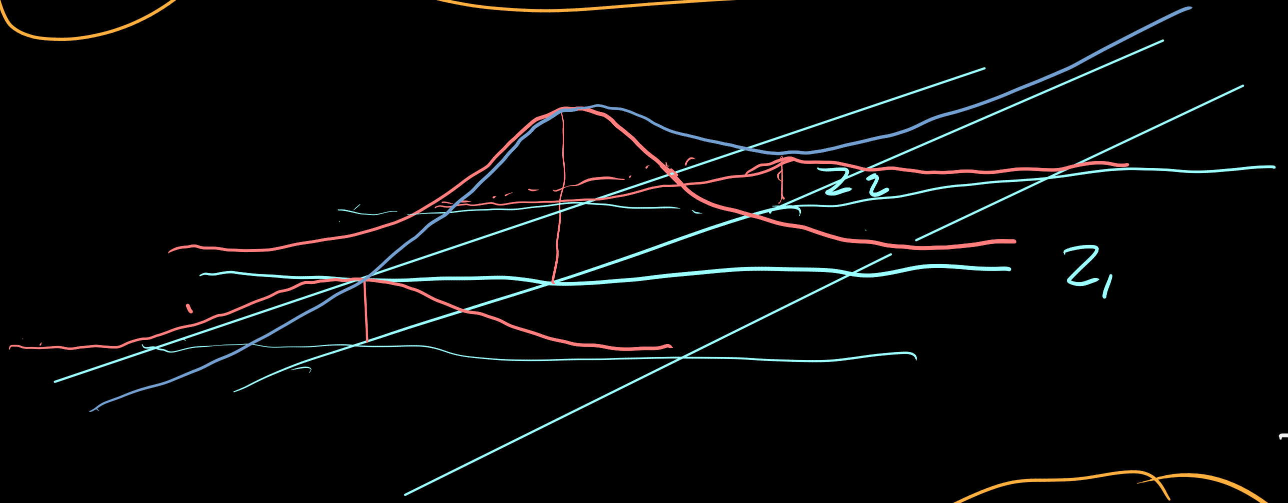
2 independent normals (Z_1, Z_2)

$$(Z_1, Z_2) \sim N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)$$

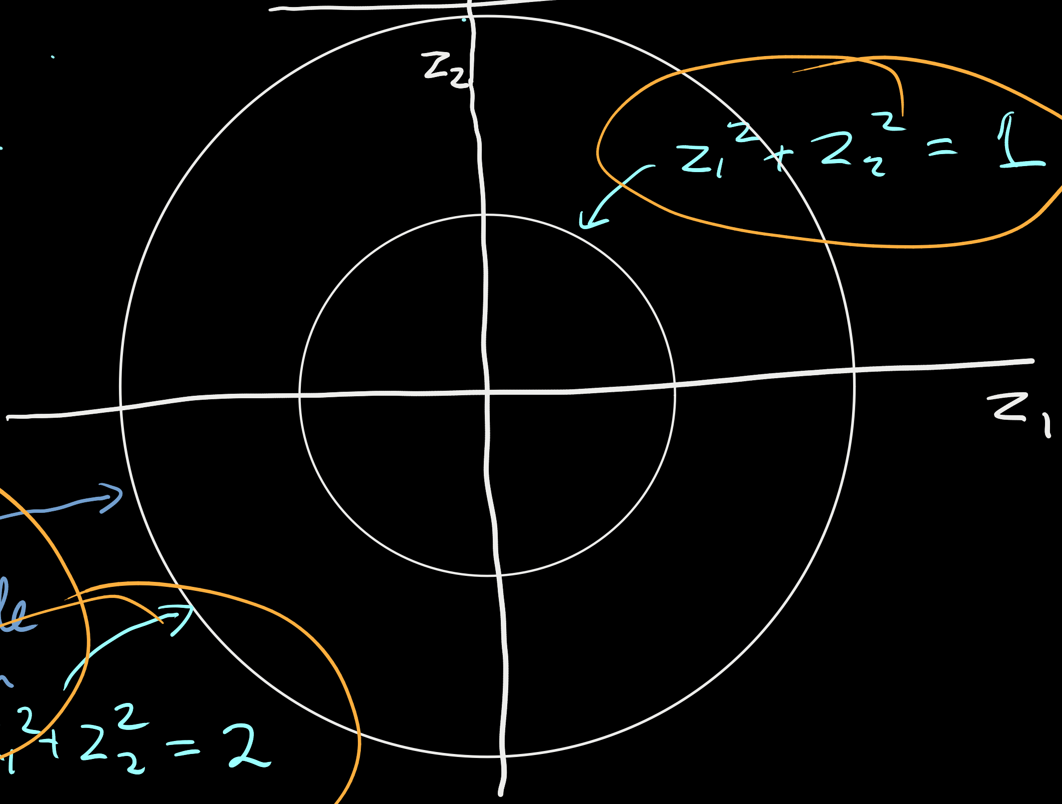
$$f(z_1, z_2) = f(z_1)\,f(z_2)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_1^2} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_2^2}$$

$$\text{dim} = \underbrace{\frac{1}{(2\pi)^{2/2}} \exp\left\{-\frac{1}{2}\left(z_1^2 + z_2^2\right)\right\}}_{K}$$



Contour lines



$z_1^2 + z_2^2 = 1$

lines of equal altitude on a mountain

$z_1^2 + z_2^2 = 2$

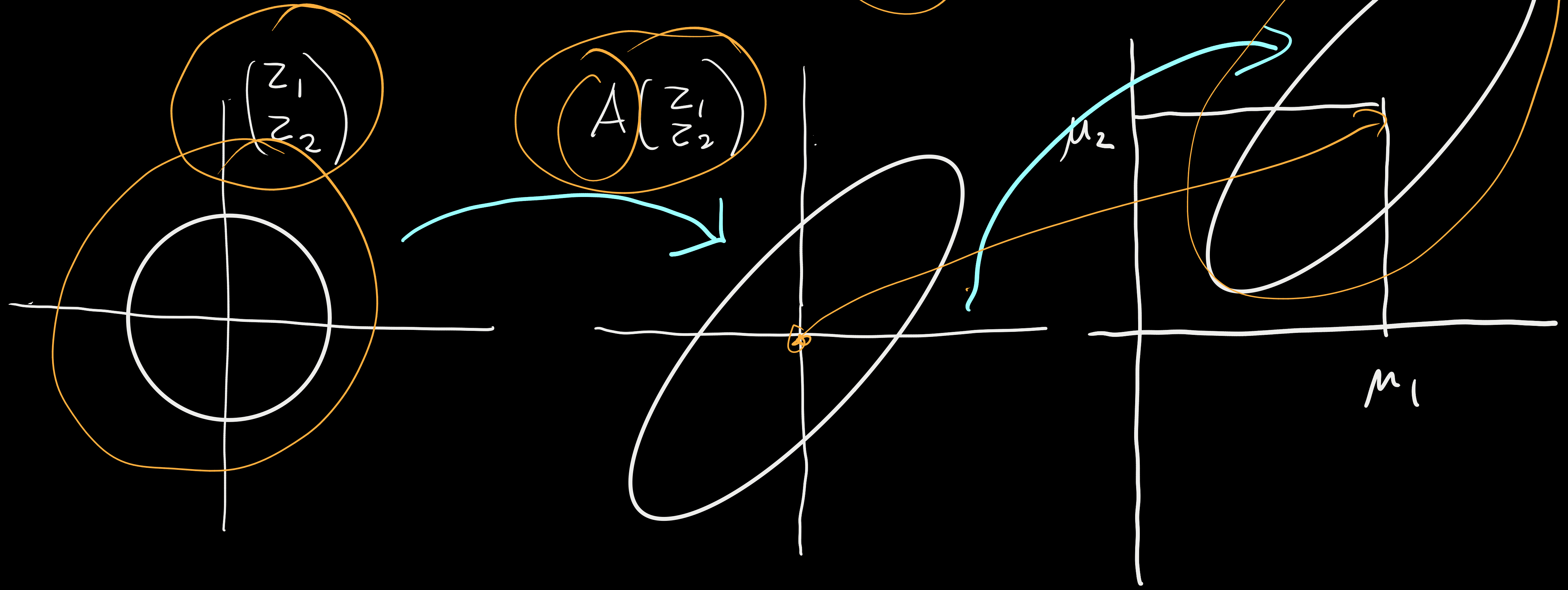
$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ as a ~~linear~~ affine transformation of $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

$$\underset{\sim}{X} = \underset{\sim}{\mu} + A \underset{\sim}{Z}$$

... takes us to the general bivariate normal in the text book:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$



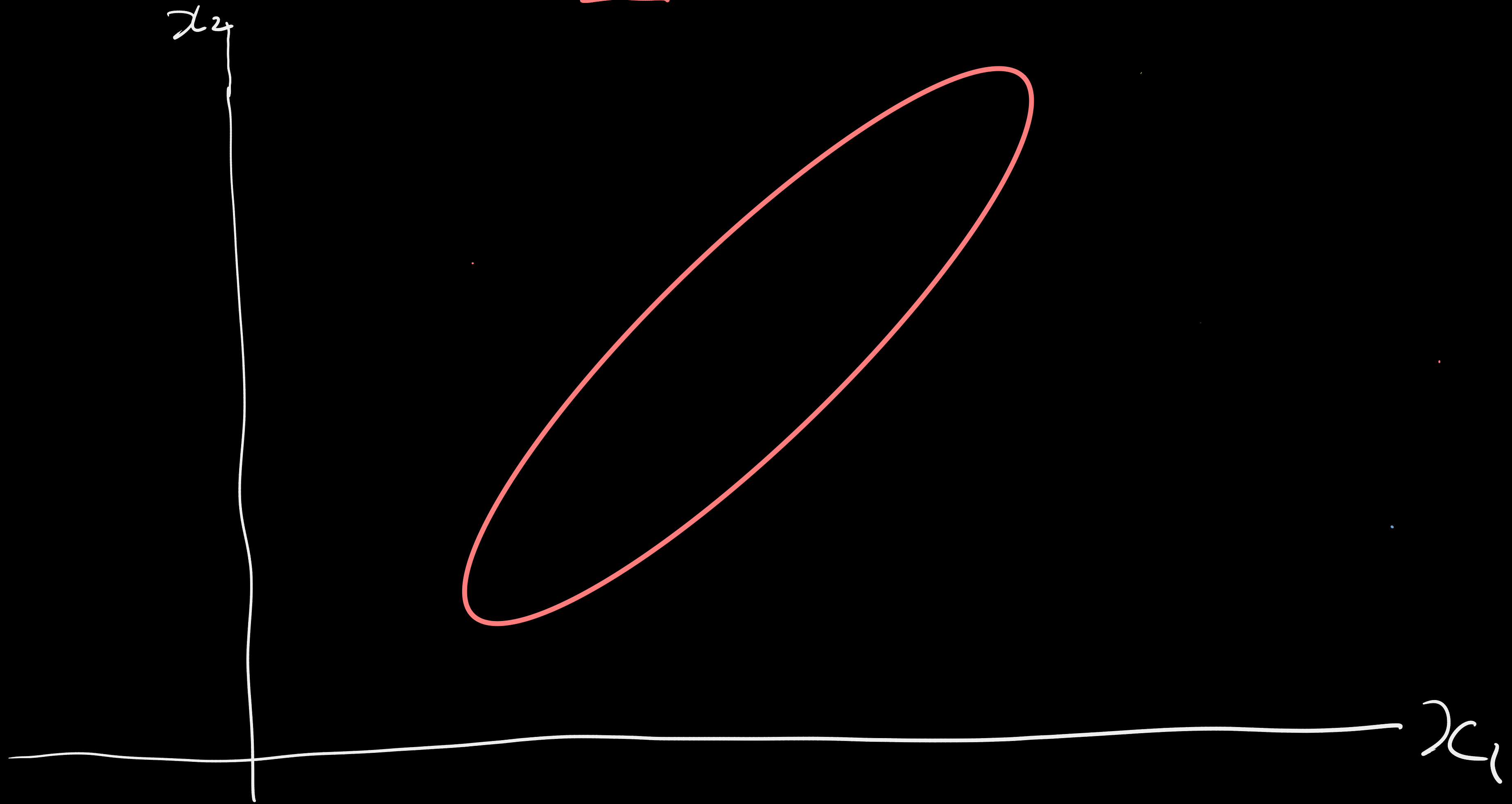
$$f(x_1, x_2) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times$$

correlation ρ ρ

$$\exp\left\{-\frac{1}{2}\left(\left[\frac{1}{1-\rho^2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right)\right\}$$

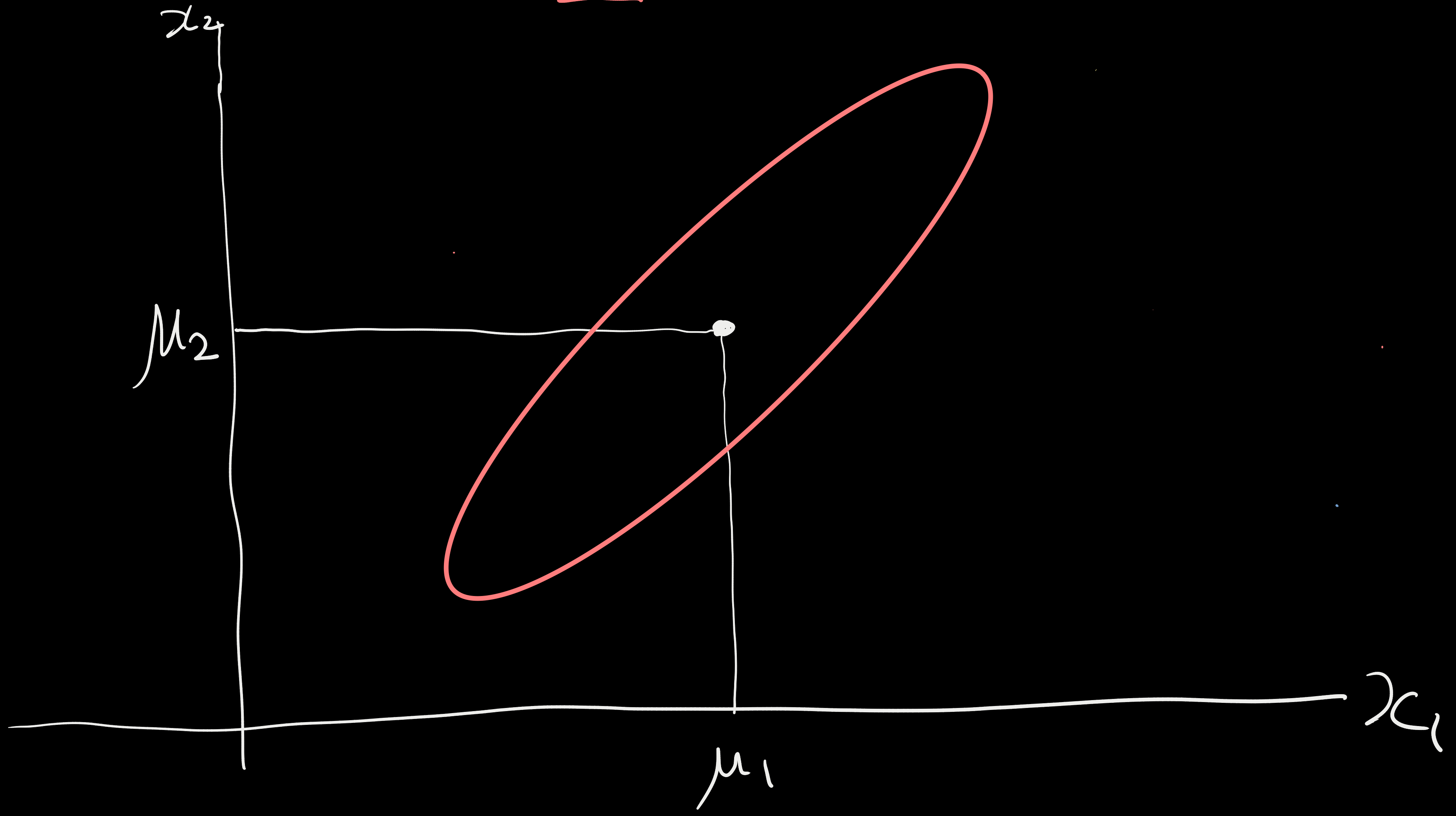
Contours

$$\left[\ \cdots\ \right] = 1$$



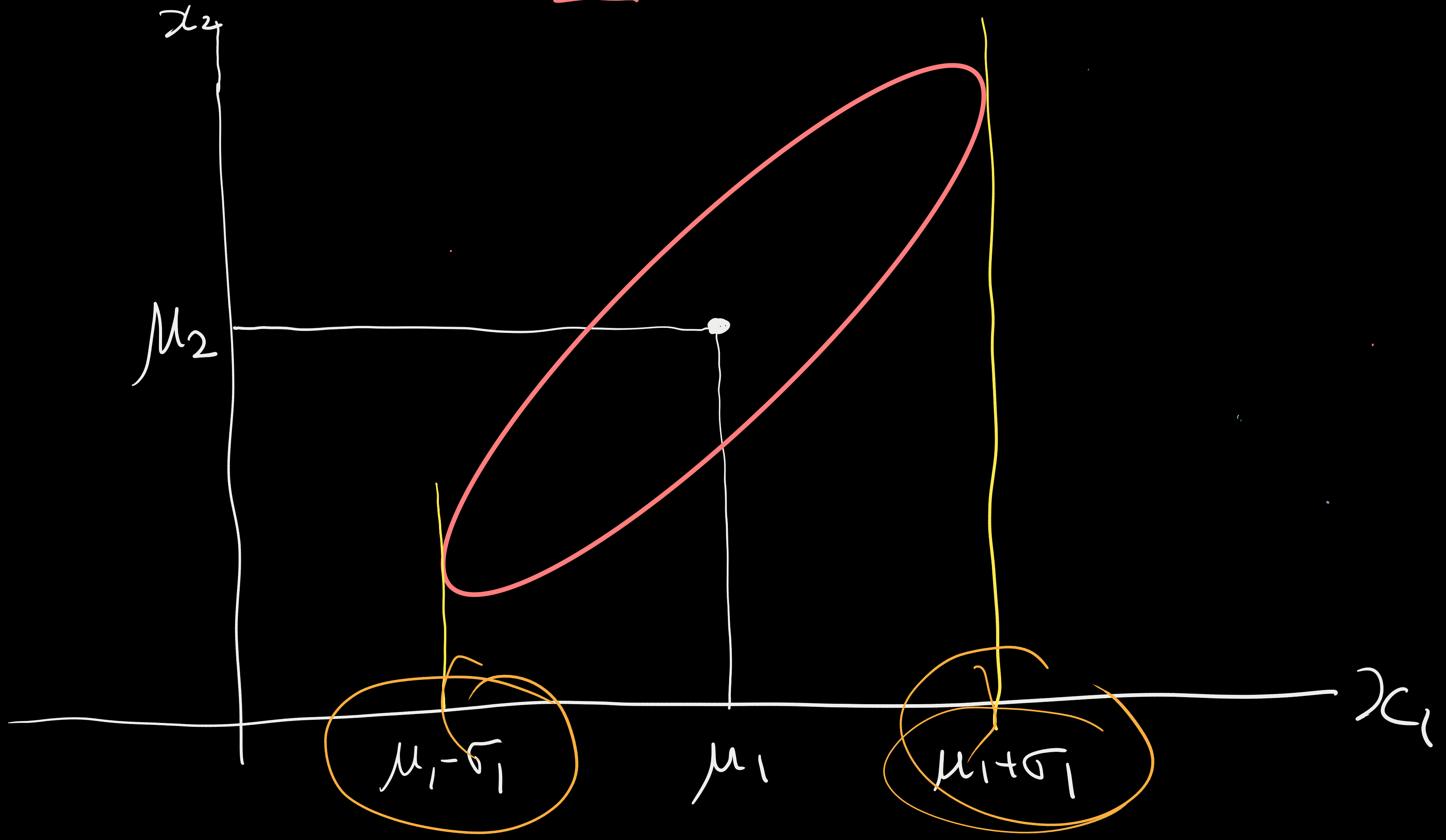
Contours

$$[\cdot\ \cdot\ \cdot] = 1$$



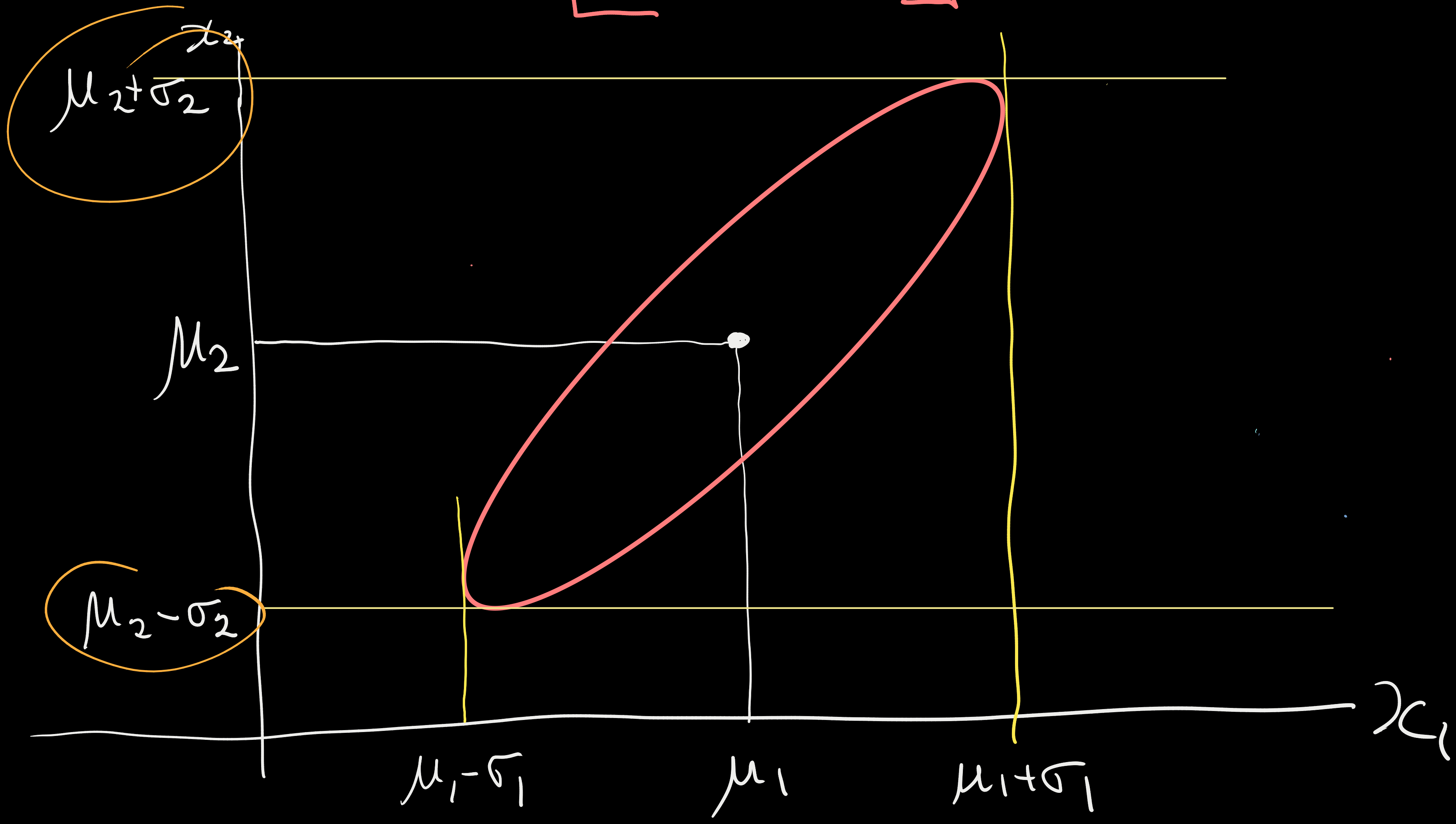
Contours

$$\left[\cdot \cdot \cdot \right] = 1$$



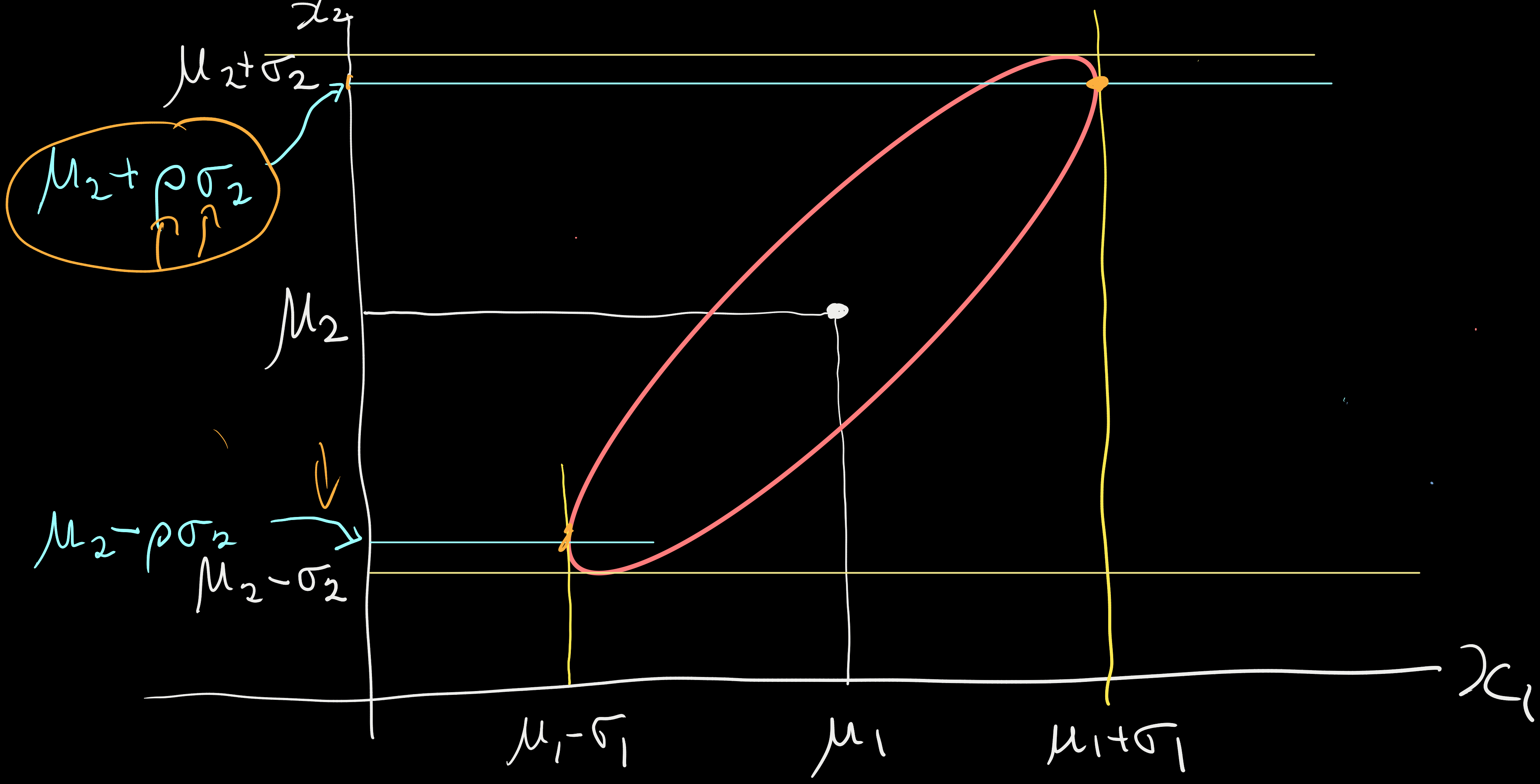
Contours

$$\begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} = 1$$



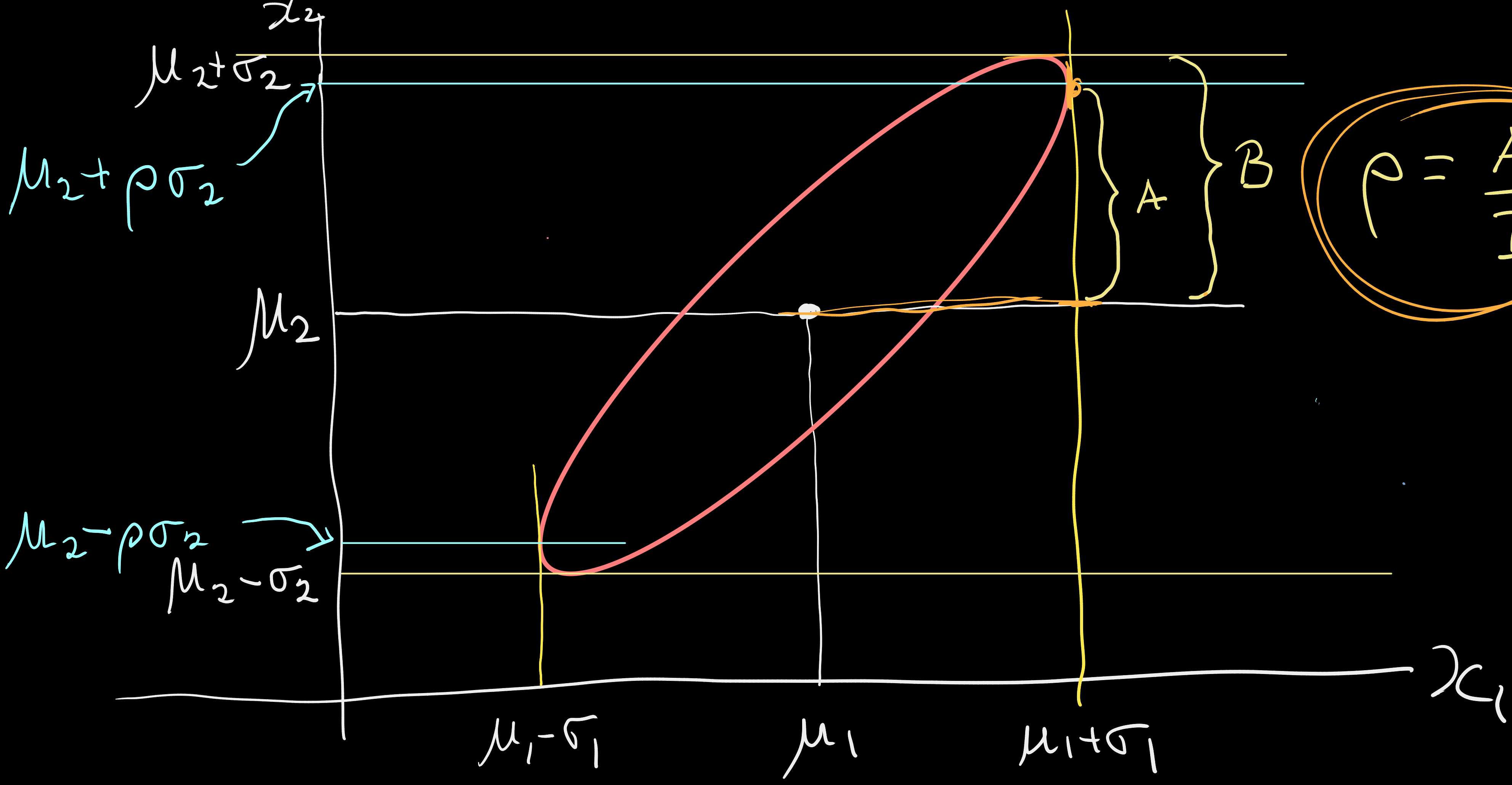
Contours

$$\left[\cdots \right] = 1$$



Contours

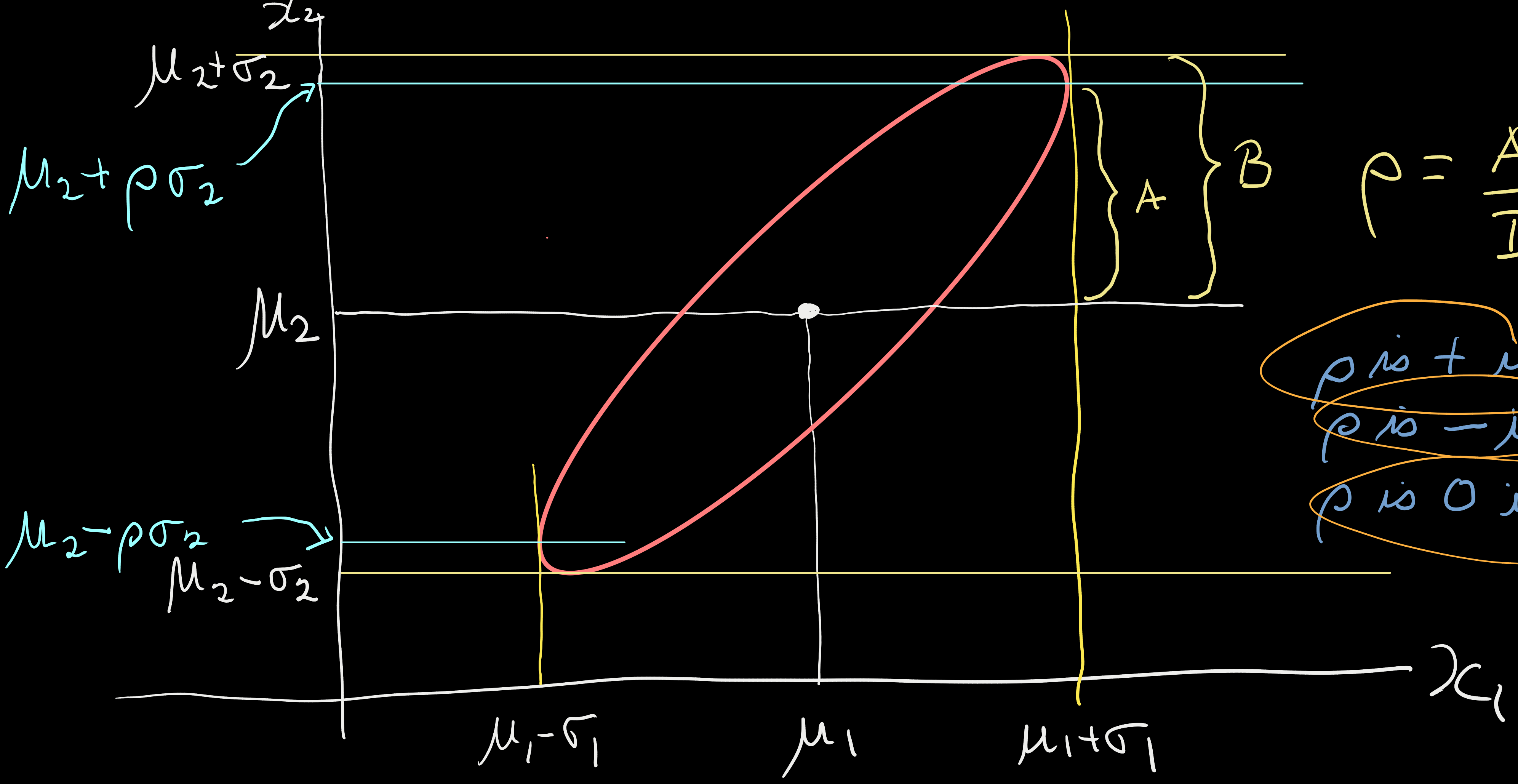
$$[\;\cdot\;\cdot\;\cdot\;] = 1$$



$$\rho = \frac{A}{B}$$

Contours

$$[\ \cdot\ \cdot\ \cdot\ ] = 1$$



$$\rho = \frac{A}{B}$$

ρ is $+$ if

ρ is $-$ if

ρ is 0 if

Back to p 78 : Copula

CDF for X_1, X_2 on $[0,1] \times [0,1]$

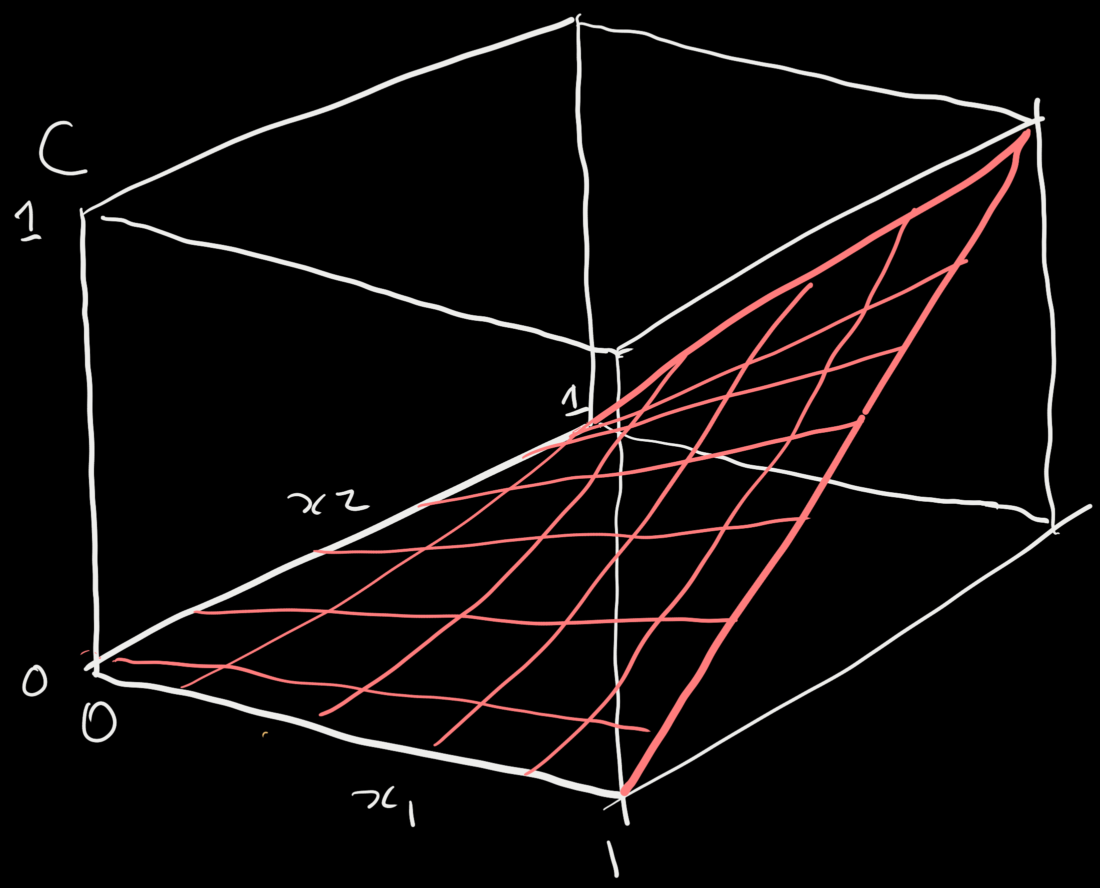
so marginals for X_1, X_2 are $U(0,1)$

$$C(x_1, x_2)$$

Note $C(x_1, \infty) = C(x_1, 1) = x_1$

if $0 < x_1 < 1$

same for $C(\infty, x_1)$



C
1
1
x2
0
0
x1
1

Quiz

A joint distribution where the support for y depends on x (and vice-versa).

E.G.
$$f(x,y) = \begin{cases} c(x+y) & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find c to make f a density

b) Find $f_Y(y)$

Note: When conditions are expressed

a $0 < x < y < 1$ it's easier to split marginal-conditional

$$\begin{array}{|c|} \hline 0 < y < 1 \\ 0 < x < y \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|} \hline 0 < x < 1 \\ x < y < 1 \\ \hline \end{array}$$

$$\int_0^1 \left[\int_0^y \bigcirc \, dx \right] dy$$

$$\int_0^1 \left[\int_x^1 \bigcirc \, dy \right] dx$$