Question 2:

Let Y_1 have a Poisson(λ_1) distribution and let Y_2 be independently distributed as a Poisson(λ_2).

Derive the probability mass function of $Y = Y_1 + Y_2$.

$$P_{Y_1}(y_1) = \frac{e^{-\lambda_1}\lambda_1^{y_1}}{y_1!} \qquad P_{Y_2}(y_2) = \frac{e^{-\lambda_2}\lambda_2^{y_2}}{y_2!}$$

$$Y = Y_1 + Y_2$$

$$P_Y(t) = \sum_{y_1=0}^{t} e^{-\lambda_1}\frac{\lambda_1^{y_1}}{y_1!}\, e^{-\lambda_2}\frac{\lambda_2^{t-y_1}}{(t-y_1)!}$$

$$= e^{-\lambda_1-\lambda_2}\frac{(\lambda_1+\lambda_2)^t}{t!} \sum_{y_1=0}^{t} \frac{t!}{y_1!(t-y_1)!}\frac{\lambda_1^{y_1}}{(\lambda_1+\lambda_2)^{y_1}}\frac{\lambda_2^{t-y_1}}{(\lambda_1+\lambda_2)^{t-y_1}}$$

$$= \boxed{e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^t}{t!}} \times 1$$

, sum of binomial prob.

$$\text{Poisson}(\lambda_1+\lambda_2)$$

$$Y_1 \sim \text{Bin}\left(t, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$\text{Bin}(n, p) \cdot \boxed{\left(\frac{n!}{y!\,(n-y)!}\right) p^y (1-p)^{n-y}}$$

$$\sum_{y=0}^{n}$$

$$= 1$$

Question 3:

Let $X_1, X_2, \ldots, X_n$ be independent Exponential(λ) random variables. Derive the CDF and the PDF of $X_{(n)}$ where $X_{(n)} = \max_i X_i$.

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x)$$

$\max_i X_i \le x$ iff

$X_1 \le x$ and $X_2 \le x$ .... and $X_n \le x$

$$P(\max X_i \le x) = P(X_1 \le x) \times P(X_2 \le x) \times \ldots \times P(X_n \le x)$$

$$= \prod_{i=1}^{n} F_{X_i}(x)$$

CDF for Exponential (λ) $\quad 1 - e^{-\lambda x}$ $\quad x > 0$, $\lambda > 0$

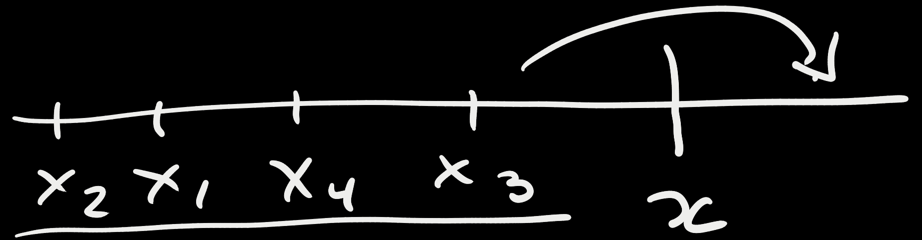
$$P(\max X_i \le x) = \prod_{i=1}^{n} (1-e^{-\lambda x})$$

$$F_{X_{(n)}}(x) = \boxed{(1-e^{-\lambda x})^{n}}$$

PDF $$\frac{d}{dx} F_{X_{(n)}}(x) = n(1-e^{-\lambda x})^{n-1}(-1)\, e^{-\lambda x}(-\lambda)$$

$$= \boxed{n(1-e^{-\lambda x})^{n-1} \lambda e^{-\lambda x}}$$

$$\max_i X_i \le x$$



$\max_i X_i \le x$ iff all X_i's $\le x$

$= X_1 \le x$ and $X_2 \le x \cdots X_n \le x$

$$\Pr(X_1 \le x \cap X_2 \le x \cap \cdots \cap X_n \le x)$$

$$= \underline{\Pr(X_1 \le x)} \times \cdots \times \underline{P(X_n \le x)}$$

$$= (1 - e^{-\lambda x})^n$$

$\underline{\min X_i \le x}$ iff $\underline{\text{at least one } X_i \le x}$

$1 - F(x)$ $(\min X_i > x)$ iff

$X_1 > x \cap X_2 > x \cap \cdots \cap X_n > x$

$$P(\quad) = (P(X_i > x))^n$$

Question 1:

A random rectangle has sides whose lengths are independent random variables, each with a uniform distribution on the interval (0, 2). Find the expected value of the area and the expected value of the perimeter of the square.

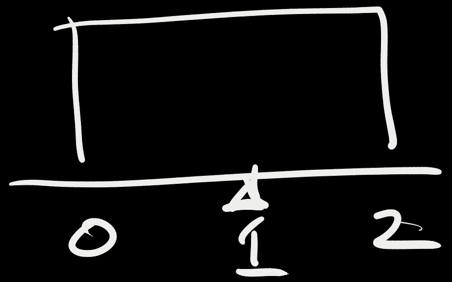
$$X, Y \sim \text{iid } U(0, 2)$$

$$E(XY) = E(X)E(Y)$$

$$E(X) = 1 = E(Y)$$

$$\Rightarrow 1 \cdot 1 = 1.$$

$$E(2X + 2Y) = 4$$



Chapter 4.5 Moment-generating functions

Distributions for random variables can be represented in many ways. We've seen some:

CDF: $F(x) = P(X \le x)$

pmf: $p(x) = P(X = x)$

pdf: $f(x)$: $P(X \in A) = \boxed{\int_A f(x)\,dx}$

We know that $E(X)$ does not determine a distribution unless restricting to a special family: e.g. Poisson or exponential.

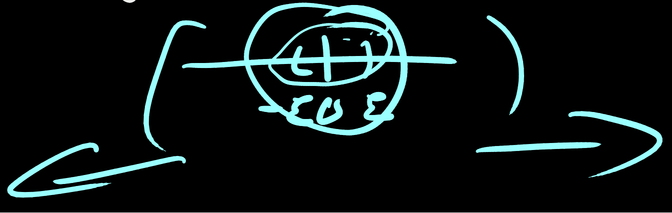
$N(\mu, \sigma^2)$

The moment-generating function (MGF) is another way that works for many distributions.

Suppose a function has a Taylor series expansion around 0, (Maclaurin series)

$$f(t) = a_0 + a_1\frac{t}{1!} + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + a_4\frac{t^4}{4!} + a_5\frac{t^5}{5!} + \cdots + a_n\frac{t^n}{n!} + \cdots$$

and the power series converges in an open interval around 0.



Then

1) $\frac{d^n}{dt^n} f(0) = a_n$ and has derivatives of all orders.

2) So the derivatives of f at 0 determine the Taylor series and thus determine the entire function. in radius of convergence.

Defn: The rth moment of X is $E(X^r)$ if it exists.

central: $E[(X-E(X))^r]$

Note: $\mathrm{Var}(X)$ is the second central moment.

$E[(X-\mu_X)^3]$ is the "skewness"

sometimes standardized as $E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^3\right]$

4 kurtosis

MGF for r.v. X:

$$\boxed{C(t) = E(e^{itX})}$$

a function of t defined as:

$$M(t) = E(e^{tX}) = \begin{cases} \int e^{tx} f(x)\, dx \\ \sum_x e^{tx} p(x) \end{cases}$$

provided $|m(t)| < \infty$ in an interval $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$.

$$M'(t) = \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{tx}\right) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x e^{tx} f(x)\, dx$$

So $M'(0) = \int_{-\infty}^{\infty} x e^{0x} f(x)\, dx = E(X)$

$$M''(t) = \int_{-\infty}^{\infty} x^2 e^{tx} f(x)\, dx$$

$$M''(0) = E(X^2)$$

$\vdots$

$M^{(k)}(0) = E(X^k)$

Two theorems:

If $M_X(t)$ exists in an open interval about 0 then

1) $E(X^r) = M^{(r)}(0)$

2) The distribution of X is uniquely determined by M

So the MGF joins the CDF, pmf and pdf as ways of characterizing distributions.

Note: All distribution for RVs have a (unique) CDF.

Only some have a pdf, pmf or mgf.

– Both continuous and discrete distribution can have mgfs.

Some mgf's

Poisson: $p(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$ $\quad x = 0, 1, \dots; \ \lambda > 0$

$$M(t) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!} e^{-\lambda}$$

$\sum x^r \dfrac{\lambda^x}{x!} e^{-\lambda}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t-1)}$$ Converges for all t

Differentiating to find moments is much easier than summing infinite series!

$$M'(t) = e^{\lambda(e^t-1)} \times \lambda e^t$$

$$M'(0) = e^{\lambda(e^0-1)} \times \lambda e^0 = e^0 \times \lambda e^0 = \lambda$$

$$M''(t) = \underbrace{e^{\lambda(e^t-1)} \times \lambda e^t}_{\lambda} \times \underbrace{\lambda e^t}_{\lambda} + \underbrace{e^{\lambda(e^t-1)}}_{1} \underbrace{\lambda e^t}_{\lambda}$$

$$M''(0) = \lambda \quad \lambda + 1 \quad \lambda = \lambda^2 + \lambda$$

So $Var(X) = E(X^2) - (E(X))^2$

$$= (\lambda^2 + \lambda) - \lambda^2$$

$$= \lambda$$

What happens if we add 2 independent Poisson(λ): Y_1 and Y_2

Finding the pdf/pmf of a sum was tough. Let's see the MGF.

$$E\left(e^{t(Y_1+Y_2)}\right) = E\left(e^{tY_1}\right)E\left(e^{tY_2}\right) \text{ by independence}$$

$$= e^{\lambda(e^t-1)}\, e^{\lambda(e^t-1)}$$

$$M_{Y_1+Y_2}(t) = e^{2\lambda(e^t-1)}$$

Recognize this?

It's the MGF for a Poisson(2λ)

$\therefore$ by uniqueness of MGF

$Y_1+Y_2 \sim \text{Poisson}(2\lambda)$

How about $Y_1 + Y_2 + \dots Y_n$

where Y_i's are independent

Poisson $\lambda_1, \lambda_2, \dots \lambda_n$?

$Y_1 + Y_2 + \dots Y_n \sim \text{Poisson}(\sum \lambda_i)$

Facts about MGFs

If the mgf exists in an interval around 0 then

1) $M^{(r)}(0) = E(X^r)$

2) If $X_1, X_2, \ldots, X_n$ are independent with mgfs $M_1(t), \ldots, M_n(t)$, then

$$M_{X_1 + \ldots X_n}(t) = \prod_{i=1}^{n} M_i(t)$$

3) ... with the same mgf $M(t)$ then

$$M_{X_1 + \ldots X_n}(t) = [M(t)]^n$$

4) If $Y = a + bX$ then $M_Y(t) = e^{at} M_X(bt)$

Some MGFs :

Exponential (λ)

$$M_\lambda(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{-(\lambda-t)x}\, dx$$

$$= \frac{\lambda}{\lambda-t} \int_0^\infty (\lambda-t)\, e^{-(\lambda-t)x}\, dx$$

$$= \frac{\lambda}{\lambda-t} \quad \text{if } \lambda - t > 0$$

i.e. $t < \lambda$

$\text{Gamma}(\alpha, \lambda) = \left(\dfrac{\lambda}{\lambda - t}\right)^{\alpha}$

α: shape

So sum of n independent Exponential(λ)'s

Gamma(n, λ)

Normal$(0,1)$ $\quad e^{t^2/2}$

Normal mean μ and variance σ^2

$Y = \mu + \sigma Z$ where Z is $N(0,1)$

$M_Y(t) = E(e^{tY}) = E(e^{t(\mu + \sigma Z)})$

$$= E\left(e^{t\mu + t\sigma Z}\right)$$

$$= e^{t\mu} E\left(e^{t\sigma Z}\right)$$

$$= e^{t\mu} e^{(t\sigma)^2/2}$$

$$= e^{t\mu + t^2\sigma^2/2}$$

Does it work?

$$M_Y'(t) = e^{t\mu + \frac{t^2}{2}\sigma^2}\left(\mu + t\sigma^2\right)$$

$$M_Y''(t) = e^{t\mu + \frac{t^2}{2}\sigma^2}\left(\mu + t\sigma^2\right)\left(\mu + t\sigma^2\right)$$

$$+ e^{t\mu + \frac{t^2}{2}\sigma^2}\,\sigma^2$$

$$= e^{t\mu + \frac{t^2}{2}\sigma^2}\left\{(\mu + t\sigma^2)^2 + \sigma^2\right\}$$

So $E(Y) = M_Y'(0) = e^0(\mu + 0\cdot\sigma^2) = \mu$

$E(Y^2) = M_Y''(0) = e^0\{\mu^2 + \sigma^2\} = \mu^2 + \sigma^2$

$Var(Y) = E(Y^2) - (E(Y))^2$

$= \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

Sums of <u>independent</u> Normals

$Y_1 \sim N(\mu_1, \sigma_1^2)$, $Y_2 \sim N(\mu_2, \sigma_2^2)$

$$E\left(e^{t(Y_1+Y_2)}\right) = e^{t\mu_1 + \frac{t^2}{2}\sigma_1^2 + t\mu_2 + \frac{t^2}{2}\sigma_2^2}$$

$$= \exp\left\{t(\mu_1+\mu_2) + \frac{t^2}{2}(\sigma_1^2+\sigma_2^2)\right\}$$

$$\therefore Y_1+Y_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2) \quad !!!$$

Example 14 **Compound Poisson Distribution**

Sum of independent RVs where # of RVs summed is an RV.

Example: # of Hs if $\Pr(H) = p$ and # of Tosses is a Poisson (λ)

Let $S = \sum_{i=1}^{N} X_i$, X_i i.i.d ~ Binomial$(1, p)$

"Bernoulli(p)"

N independent Poisson(λ)

$$M_S(t) = E(e^{tS})$$
$$= E_N[E(e^{tS} \mid N)]$$
$$= E_N[E(e^{t(X_1+\dots+X_N)}) \mid N)]$$
$$= E_N[E(e^{tX_1}) \cdots E(e^{tX_N}) \mid N]$$
$$= E_N[(pe^t+q)^N \mid N]$$

$$\boxed{\begin{aligned} &E(e^{tX}) \\ &= e^{t1} \cdot p + e^{t0} \cdot q \\ &= pe^t + q \end{aligned}}$$

$$= \sum_{i=0}^{\infty} (pe^t+q)^i \frac{\lambda^i}{i!} e^{-\lambda}$$

$$= \sum_{i=0}^{\infty} \frac{[\lambda(pe^t+q)]^i}{i!} e^{-\lambda}$$

$$= \exp\{\lambda(pe^t+q)-\lambda\}$$

$$= \exp\{\lambda pe^t - \lambda p\}$$

$$= \exp\{\lambda p(e^t-1)\}$$

Poisson(λ) MGF

$M(t) = \exp\{\lambda(e^t-1)\}$

$$= \sum_{i=0}^{\infty} (pe^t+q)^i \frac{\lambda^i}{i!} e^{-\lambda}$$

$$= \sum_{i=0}^{\infty} \frac{[\lambda(pe^t+q)]^i}{i!} e^{-\lambda}$$

$$= \exp\{\lambda(pe^t+q)-\lambda\}$$

$$= \exp\{\lambda pe^t - \lambda p\}$$

$$= \exp\{\lambda p(e^t-1)\}$$

Poisson(λ) MGF

$M(t) = \exp\{\lambda(e^t-1)\}$

So $M_S(t)$ is MGF of Poisson(λp)

General Principle

If $S = X_1 + \cdots + X_N$

X_i iid with MGF $M_X(t)$

N is random with MGF $M_N(t)$

Then $M_S(t) = E\left(e^{t(X_1 + \cdots + X_n)}\right)$

$$= E_N\left[E\left(e^{t(X_1 + \cdots X_n)} \middle| N\right)\right]$$

$$= E_N\left[E\left(\prod_{i=1}^{N} e^{tX_i} \middle| N\right)\right]$$

$$= E_N\left[E\left(\prod_{i=1}^{N} M_X(t) \mid N\right)\right]$$

$$= E_N\left[M_X(t)^N \mid N\right]$$

$$= E_N\left[M_X(t)^N\right]$$

$$= E_N\left[e^{N \log(M_X(t))}\right]$$

$$= E_N\left[E\left(\prod_{i=1}^{N} M_X(t) \,\middle|\, N\right)\right]$$

$$= E_N\left[M_X(t)^N \,\middle|\, N\right]$$

$$= E_N\left[M_X(t)^N\right]$$

$$= E_N\left[e^{N \log(M_X(t))}\right]$$

$$= M_N\left(\log(M_X(t))\right)$$

Let's see how that would have worked in the last example.

$$X \sim \text{Bernoulli}(p)$$

$$M_X(t) = pe^t + q$$

$$N \sim \text{Poisson}(\lambda)$$

$$M_N(t) = \exp\{\lambda(e^t - 1)\}$$

$$M_S(t) = M_N(\log M_X(t))$$

$$= \exp\{\lambda(e^{\log M_X(t)} - 1)\}$$

$$= \exp\left\{\lambda\left(e^{\log(pe^t+q)} - 1\right)\right\}$$

$$= \exp\left\{\lambda\left(pe^t + q - 1\right)\right\}$$

$$= \exp\left\{\lambda p\left(e^t - 1\right)\right\}$$

MGF of Poisson (λp)

Skip 4.6