1.2 Sample Spaces

Probability theory is concerned with situations in which the outcomes occur randomly. Generically, such situations are called *experiments*, and the set of all possible outcomes is the **sample space** corresponding to an experiment. The sample space is denoted by Ω, and an element of Ω is denoted by ω. The following are some examples.

EXAMPLE **A** Driving to work, a commuter passes through a sequence of three intersections with traffic lights. At each light, she either stops, s, or continues, c. The sample space is the set of all possible outcomes:

$$\Omega = \{ccc, ccs, css, csc, sss, ssc, scc, scs\}$$

where csc, for example, denotes the outcome that the commuter continues through the first light, stops at the second light, and continues through the third light. ■

EXAMPLE **B** The number of jobs in a print queue of a mainframe computer may be modeled as random. Here the sample space can be taken as

$$\Omega = \{0, 1, 2, 3, \ldots\}$$

that is, all the nonnegative integers. In practice, there is probably an upper limit, N, on how large the print queue can be, so instead the sample space might be defined as

$$\Omega = \{0, 1, 2, \ldots, N\}$$ ■

EXAMPLE **C** Earthquakes exhibit very erratic behavior, which is sometimes modeled as random. For example, the length of time between successive earthquakes in a particular region that are greater in magnitude than a given threshold may be regarded as an experiment. Here Ω is the set of all nonnegative real numbers:

$$\Omega = \{t \mid t \geq 0\}$$ ■

We are often interested in particular subsets of Ω, which in probability language are called **events.** In Example A, the event that the commuter stops at the first light is the subset of Ω denoted by

$$A = \{sss, ssc, scc, scs\}$$

Events : Subsets of the sample space
measurable

Union (OR) : $A \cup B$ ← Why not "A and B"

Because "or" refers to the elements of A and B

"$x \in A \cup B$" if

$x \in A$ OR $x \in B$ ... or both.

Intersection (AND) : $A \cap B$

Complement (NOT) A^c

Empty set : $\emptyset$

Disjoint : A & B disjoint if $A \cap B = \emptyset$

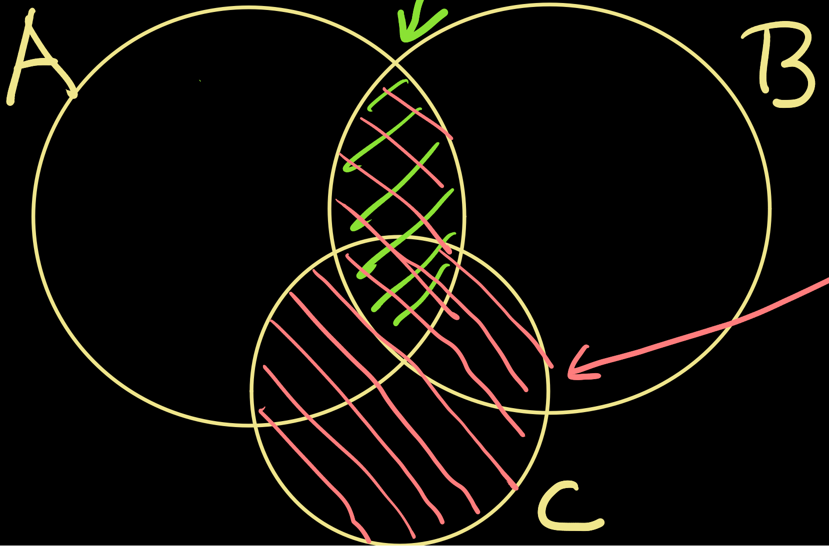
Commutative Laws: $A \cup B = B \cup A$

$A \cap B = B \cap A$

Associative Laws: $(A \cup B) \cup C = A \cup (B \cup C)$

Distributive Laws: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

$\underbrace{(\overbrace{A \cap B}) \cup C} = (A \cup C) \cap (B \cup C)$



Definition of probability measure:

P is a function : Subsets of $\Omega \to [0,1]$

measurable

P is a probability measure if

1) $P(\Omega) = 1$

2) $A \subset \Omega$ then $P(A) \geq 0$

3a) If $A_1 \cap A_2 = \emptyset$ then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

3b) If $A_1, A_2, \dots$ are mutually disjoint the

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

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Axioms of σ-additive probability

"normed" 1) $P(\Omega) = 1$

"positive" 2) $A \subset \Omega$ then $P(A) \geq 0$

"additive" 3a) If $A_1 \cap A_2 = \emptyset$ then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

"σ-additive" 3b) If $A_1, A_2, \ldots$ are mutually disjoint the

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Can prove:

$$P(A^c) = 1 - P(A)$$

$$P(\emptyset) = 0$$

$$A \subset B \Rightarrow P(A) \leq P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C)$$

$$P(A \cup B \cup C \cup D) = ?$$

$+3 - 3 + 1$

Counting methods:

When Ω is finite and all elements have the same probability, then

$$P(A) = \frac{\#(A)}{\#(\Omega)} = \frac{\#\text{ of ways } A \text{ can occur}}{\#(\Omega)}$$

EXAMPLE B *Simpson's Paradox*

A black urn contains 5 red and 6 green balls, and a white urn contains 3 red and 4 green balls. You are allowed to choose an urn and then choose a ball at random from the urn. If you choose a red ball, you get a prize. Which urn should you choose to draw from? If you draw from the black urn, the probability of choosing a red ball is $\frac{5}{11} = .455$ (the number of ways you can draw a red ball divided by the total number of outcomes). If you choose to draw from the white urn, the probability of choosing

... skipping for now — will be back!

SP has MUCH larger implications in the right context.

Multiplication Principle

If one experiment has m outcomes

and

for each outcome a second experiment has n outcomes,

then there are $m \times n$ outcomes altogether.

If experiment 1 has n_1 outcomes, and

for each, experiment 2 has n_2 outcomes, and

for each combination of prior expts,

experiment 3 has n_3 outcomes

...

for each combination of prior expts,

experiment p has n_p outcomes

Then there are $n_1 \times n_2 \times n_3 \times \cdots n_p$ outcomes altogether.

Permutations and Combinations

n distinct objects : $n!$ possible orderings.

Choosing r objects from n :

\# of ways : n^r with replacement

$n \times (n-1) \times \cdots (n-(r-1))$ without replacement