Two useful results:

$\boxed{P \text{ is "continuous"}}$

<u>Thm:</u> Let $A_1, A_2, \ldots$ be a monotone increasing sequence of sets (i.e. $A_1 \subset A_2 \subset A_3 \subset \ldots$)

note: includes =

and let $A = \bigcup_{i=1}^{\infty} A_i$

Then $P(A) = \lim_{i \to \infty} P(A_i)$

Proof: Let $D_1 = A_1$, $D_i = A_i - A_{i-1}$ for $i > 1$

Then $A = \bigcup_{i=1}^{\infty} D_i$ and D_i's are disjoint

So $P(A) = \sum_{i=1}^{\infty} P(D_i)$ (σ-additivity)

But $P(A_i) = \sum_{j=1}^{i} P(D_i)$

which is a partial sum of

$\sum_{i=1}^{\infty} P(D_i)$

So $\lim_{j\to\infty} P(A_j) = \lim_{j\to\infty} \sum_{i=1}^{j} P(D_i)$

$= \sum_{i=1}^{\infty} P(D_i)$

$= P(A)$

Q.E.D.

Continuity Theorem Part 2

Let $A_1 \supset A_2 \supset \dots$ be a monotone decreasing sequence of sets.

Let $A = \bigcap_{i=1}^{\infty} A_i$.

Then $\lim_{i \to \infty} P(A_i) = P(A)$

Proof: Consider that $A_1^c \subset A_2^c \subset \dots$ is monotone increasing and apply the previous theorem. [Complete the details]

These theorems are useful to understand the continuity – or lack thereof – of CDFs.

EXERCISES

1) Let $a < 1$, $b > 4$.

a) Find $\bigcup_{i=1}^{\infty} \left(a + \frac{1}{n}, b - \frac{1}{n}\right]$

$\bigcap_{i=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right]$

$\bigcup_{i=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right)$

$\bigcap_{i=1}^{\infty} \left[a - \frac{1}{n}, b + \frac{1}{n}\right)$

b) Show

$$\bigcup_{n=1}^{\infty} \left(-\infty, x - \frac{1}{n}\right] = (-\infty, x)$$

$$\bigcap_{n=1}^{\infty} \left(-\infty, x + \frac{1}{n}\right] = (-\infty, x]$$

c) Recall: If X is a random variable then its CDF, $F : \mathbb{R} \to [0,1]$ is defined as

$$F(x) = P(X \leq x)$$
$$= P(X \in (-\infty, x])$$

Show that F is <u>right continuous</u>

i.e $\lim_{x \to c^+} F(x) = F(c) \quad \forall c \in \mathbb{R}.$

i.e. if $x_1 > x_2 > x_3 \cdots\cdots > c$
and $\lim_{i \to \infty} x_i = c$
then $\lim_{i \to \infty} F(x_i) = F(c)$

Hint: Consider the fact that

$$F(x_i) = P_X((-\infty, x_i]))$$

and apply the continuity theorem.

Fill in the details.