3.5.2 Continuous Conditional Densities

Given a joint density $f_{XY}(x,y)$

We can get $f_X(x) = \int f_{XY}(x,y)\,dy$

Conditional density for Y given $X=x$

$$f_{Y|X}(y \mid x) = \begin{cases} \dfrac{f_{XY}(x,y)}{f_X(x)} & \text{if } f_X(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Note: If X is continuous then $P(X=x)=0$
so we are conditioning on a set of probability 0

But that's okay because $f_{Y|X}(y|x)$
is NOT the probability of $Y=y$ given $X=x$,
It's just a density that gives you
conditional probabilities when you integrate.

The Law of Total Probability

$$f_X(x) = \int \boxed{f(x,y)}\, dy$$

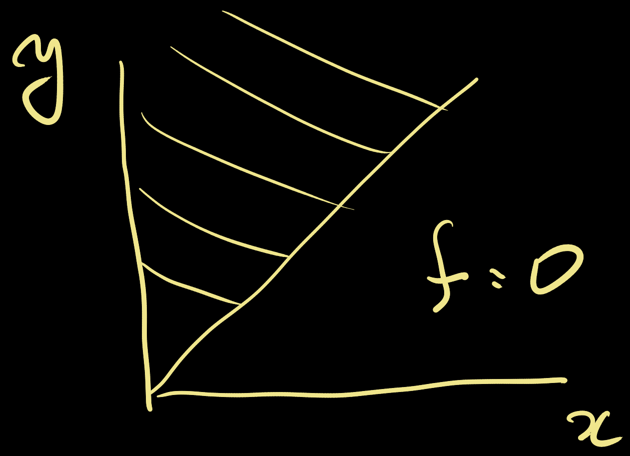
$$= \int \boxed{\underline{f(x|y)}\,\underline{f_Y(y)}}\, dy$$

$$= f(x|y) \text{ averaged over } Y$$

Marginal Probability = Mean Conditional probability

"Example D"

$$f_{XY}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$



Note: How to integrate f

- Choose 1 variable for marginal bound
- Other for conditional

$0 < x < y$

y-first

$$\boxed{\begin{gathered} 0 < y < \infty \\ 0 < x < y \end{gathered}}$$

outer integral

inner

$$\int_0^\infty \left[\int_0^y f(x,y)\,dx \right] dy$$

x-first

$$\boxed{\begin{gathered} 0 < x < \infty \\ x < y < \infty \end{gathered}}$$

$$\int_0^\infty \left[\int_x^\infty f(x,y)\,dy \right] dx$$

Define: outer then inner

Evaluate: inner then outer

$$f_{XY}(x,y)=\begin{cases}\lambda^2 e^{-\lambda y} & 0<x<y\\ 0 & \text{otherwise}\end{cases}$$

Find $f_{X|Y}(x|y)$

Step 1 : Find marginal for conditioning variable, Y.

$$f_Y(y)=\int_0^y \lambda^2 e^{-\lambda y}\,dx=\Big[x\lambda^2 e^{-\lambda y}\Big|_0^y$$

$$=y\lambda^2 e^{-\lambda y}\quad,\quad y>0$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\lambda^2 e^{-\lambda y}}{y\lambda^2 e^{-\lambda y}}, \quad 0 < x < y$$

$$= \frac{1}{y} \qquad 0 < x < y$$

$$\therefore\ x \mid y \sim U\left(0, \frac{1}{y}\right)$$