Chapter 4 Expectations of functions of RVs

Example: Toss a die and win X^2 if X is # of spots.

$$E(X^2) = \sum_x x^2 p_X(x) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \dots + 6^2 \cdot \frac{1}{6}$$

$$= 15\tfrac{1}{6}$$

Note $E(X^2) \neq (E(X))^2$

$$(E(X))^2 = \left(1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6}\right)^2$$

$$= (3.5)^2 = 12\tfrac{1}{4}$$

In general: $$E(g(X)) = \sum_x g(x)\, p_X(x)$$

or

$$= \int g(x)\, f_X(x)\, dx$$

Works for function of random vectors:

Let $Y = g(X_1, X_2, \ldots, X_k)$

with pmf $p(x_1, x_2, \ldots, x_k)$

then $$E(Y) = E(g(X_1, \ldots, X_k)) = \sum_{x's} g(x_1, \ldots, x_k)\, p(x_1, \ldots, x_k)$$

or $$E(Y) = \int \left[\int \cdots \int g(x_1, \ldots, x_k) f(x_1, \ldots, x_k)\, dx_k \cdots dx_2 \right] dx_1$$

if the integral (sum) with $|g|$ converges.

---

p. 124 Corollary A

If X & Y are independent, then

$$E(g(X)h(Y)) = E(g(X)) \times E(h(Y))$$

provided ↑ and ↑ exist.

Special case: If X & Y are independent

$$E(XY) = E(X)E(Y)$$

if $\uparrow$ and $\uparrow$ exist.

Beware: 1) not true in general

2) converse not true

Linear combinations of R.V.'s

$$Y = a + b_1 X_1 + b_2 X_2 + b_3 X_3 + \cdots + b_n X_n$$

then $E(Y) = a + b_1 E(X_1) + b_2 E(X_2) + \cdots + b_n E(X_n)$

if $\uparrow$ $\cdots$ exist.

Example A (p. 126)

What is $E(Y)$ if $Y \sim Bin(n, p)$?

$$E(Y) = \sum_{y} \binom{n}{y} y \, p^{y} (1-p)^{n-y}$$

Easier: Use fact that $Y = X_1 + \cdots + X_n$

where X_i's are indep. $Bin(1, p)$

and $E(X_i) = p$

So $E(Y) = \sum_{i=1}^{n} E(X_i) = np$

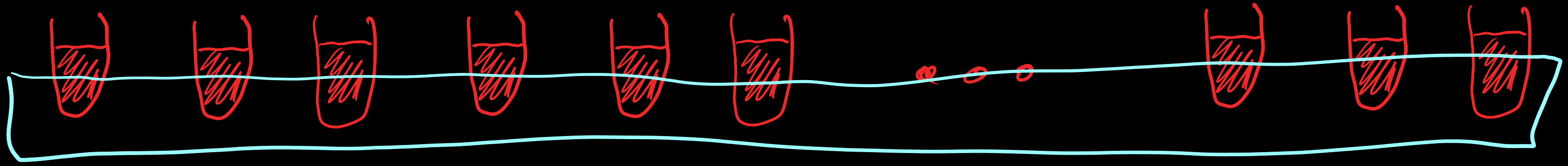
Example C p. 128

Group testing

- n blood samples tested for rare disease

- If you test each individually will need n tests.

Alternative:



take 1/2 of each sample and combine:

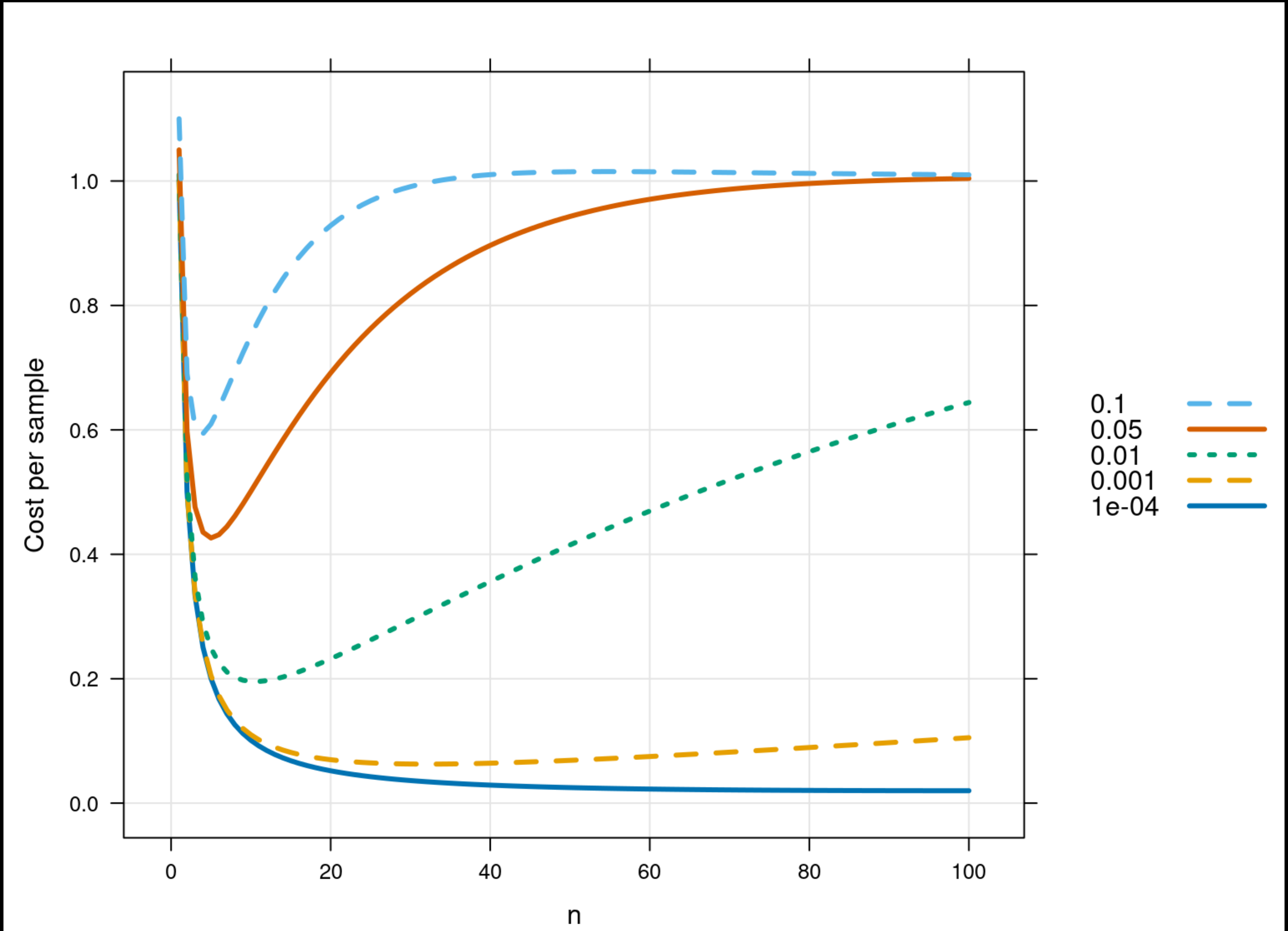
Then test.

If negative - done - all okay

If positive - test n samples.

Let p = probability of a positive.

$$E(\text{Tests}) = 1 \times \underbrace{(1-p)^n}_{\text{pr all negative}} + (n+1)\underbrace{\left(1-(1-p)^n\right)}_{\text{prob. at least 1 positive}}$$



Cost per sample
n
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100
0.1
0.05
0.01
0.001
1e-04

```r
df <- expand.grid(n = 1:100, p = c(.0001,.001, .01, .05, .1))
head(df)
dim(df)
df <- within(df,
             {
               Etests <- 1* (1-p)^n + (n + 1) * (1 - (1 - p)^n)
               Cost_per_sample <- Etests/n
             }
             )

library(latticeExtra)
trellis.par.set(superpose.line = list(lwd=3, lty = 1:3))
xyplot(Cost_per_sample ~ n, df,
       groups = p, type = 'l',
       ylab = "Cost per sample",
       auto.key = list(reverse.rows = T)) +
  layer_(panel.grid(h=-1, v = -1))
```

Example D: Illustrates how

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

does not require independence

DNA Sequences: formed from 4 letters A C T G

If random & each letter with = prob.

$$\underbrace{A\,T\,C\,A\,A\,T\,C\,G\,A\,G\,T \cdots \quad T\,A\,A}$$

- Suppose length = N
- and each letter has $p = 1/4$

How many ATGC do you expect?

Let I_n = event ATGC starts at position n

$$P(I_n) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{256}$$

$$= E(I_n)$$

if $I_n = \begin{cases} 1 & \text{if ATGC starts at position } n \\ 0 & \text{otherwise} \end{cases}$

$$E(\#\text{ of sequences}) = E\left(\sum_{n=1}^{N-3} I_n\right) = \sum_{n=1}^{N-3} E(I_n)$$

$$= (N-3) \times \frac{1}{256}$$

Variance and Standard Deviation

Mean = "location parameter" ← one of many e.g. median, min, max

Next we need a "spread" parameter

Variance: Average squared distance from the mean

$$\mathrm{Var}(X) = E\left[(X-\mu_X)^2\right]$$ if E exists. in squared units

$$SD(X) = \sqrt{\mathrm{Var}(X)}$$ in original units

Facts about variance.

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

also a useful way to calculate Var(X)

Proof $E\left[(X-\mu_X)^2\right]$

$$= E(X^2 - 2\mu_X X + \mu_X^2)$$

$$= E(X^2) - 2\mu_X E(X) + \mu_X^2$$

$$= E(X^2) - 2E(X)E(X) + (E(X))^2$$

$$= E(X^2) - (E(X))^2$$

Corollary: $E(X^2) = E(X)^2$ iff $\text{Var}(X) = 0$

Fact: $\text{Var}(X) = 0$ iff X is a constant

i.e. $P(X = c) = 1$

Fact: If $\text{Var}(X) < \infty$ (i.e. exists)

and $Y = a + bX$

then $\text{Var}(Y) = b^2\,\text{Var}(X)$

Chebyshev's Inequality: Prob & spread

Let X have mean μ and variance σ^2

Then for any $t > 0$:

$$P(|X-\mu| > t) \leq \sigma^2/t^2$$

Proof: Use Markov's inequality on $Y = (X-\mu)^2$

Different form: If $\sigma > 0$, let $t = k\sigma$ for $k > 0$. Then

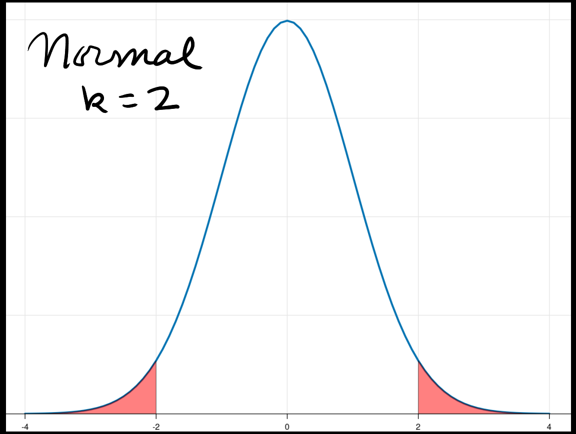
$$P(|X-\mu| > k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = 1/k^2$$

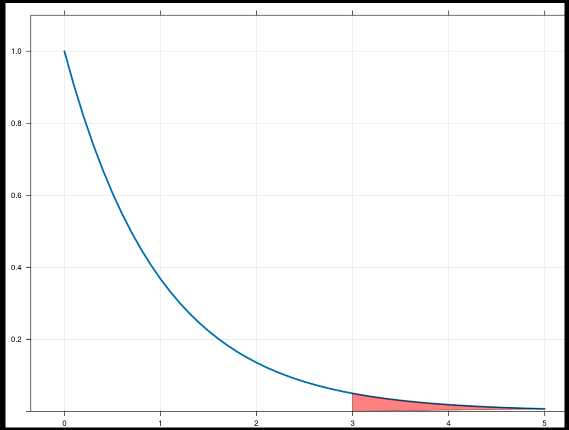
For $k = 1$ this says $P(|X-\mu| \geq \sigma) \leq 1/1^2 = 1$!?

Only useful for $k > 1$; e.g. $P(|X-\mu| \geq 2\sigma) \leq \frac{1}{2^2} = \frac{1}{4}$

Let's compare with some actual values.

	$k = 2$	$k = 3$	$k = 4$
Chebyshev	$\leq 1/4 = 0.25$	$\leq 1/9 = 0.1111\ldots$	$\leq \frac{1}{16} = 0.0625$
Actuals			
Normal	0.0455	0.0027	0.0000633
Exponential	0.0498	0.0183	0.00674
Poisson	0.0803	0.0190	0.00366





Fact (Corollary A, p. 134)

$$\mathrm{Var}(X) = 0 \quad \text{iff} \quad P(X=\mu) = 1$$
$$\text{iff} \quad P(X \text{ is constant}) = 1$$

Proof: Example of Chebyshev's Theorem.

$$P(|X-\mu| > \varepsilon) \le \sigma^2/\varepsilon^2 = 0$$

for every $\varepsilon > 0$.

So $P(|X-\mu| = 0) = 1$

i.e. $P(X=\mu) = 1$

Investment Portfolios (p. 134)

Two investments: R_1: $E(R_1) = \mu_1 = 0.10$

$\sigma_1 = 0.075$

R_2: $E(R_2) = \mu_2 = 0.03$

$\sigma_2 = 0$

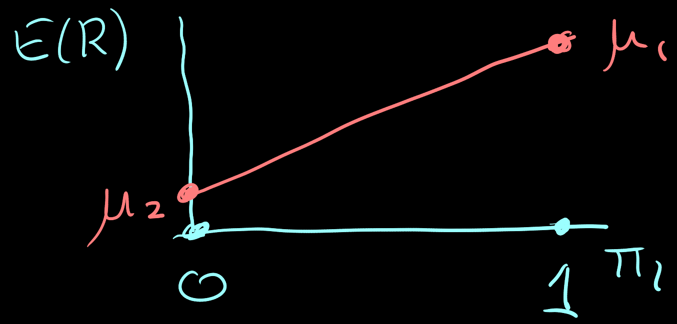
Same cost so investor can choose to invest π_1 in R_1 and $\pi_2 = (1-\pi_1)$ in R_2

Return is a R.V.:

$$R = \pi_1 R_1 + (1-\pi_1) R_2$$

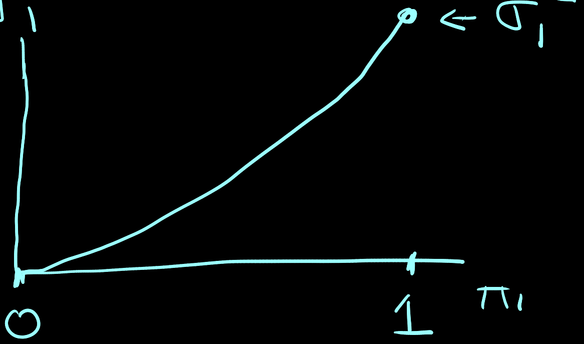
$$E(R) = \pi_1 \mu_1 + (1-\pi_1)\mu_2$$

Expected return



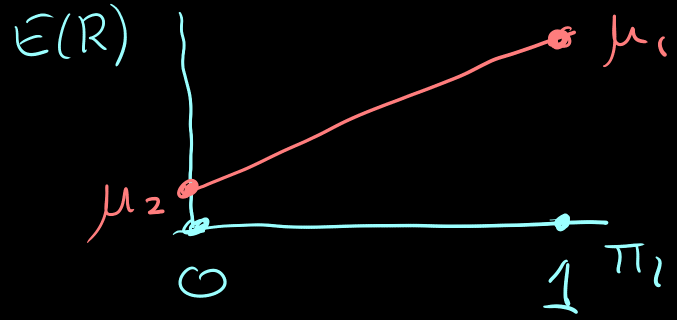
$$Var(R) = \pi_1^2 \sigma_1^2$$

Risk



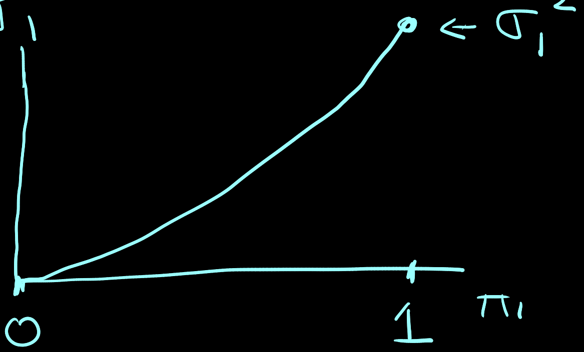
$$E(R) = \pi_1 \mu_1 + (1-\pi_1)\mu_2$$

Expected return



$$Var(R) = \pi_1^2 \sigma_1^2$$

Risk



need to balance expected return against risk.

Notes:

- When $\sigma_2 > 0$ we will also need to consider covariance
- Variance is <u>not</u> the whole story as far as risk is concerned.

Read 4.2.1 Measurement error

Suppose we want to measure some "fixed" but "unknown" quantity.

Call it "x_0."

Measurement $X = \underbrace{x_0}_{\text{fixed unknown}} + \underbrace{\beta}_{\text{systematic error}} + \underbrace{\varepsilon}_{\text{random error}}$

systematic error – same each time

random error varies with each measurement.

β = "bias"

ε = random component: $Var(\varepsilon) = \sigma^2$

$E(\varepsilon) = 0$

Then $\mathrm{Var}(X) = \mathrm{Var}(\underbrace{x_0 + \beta}_{\text{constant}} + \varepsilon)$

$$= \mathrm{Var}(\varepsilon) = \sigma^2$$

But $E(\text{Error}^2) = MSE$

$$= E\left[(X - x_0)^2\right]$$

$$= E\left[(\beta + \varepsilon)^2\right]$$

$$= E\left[\beta^2 + 2\beta\varepsilon + \varepsilon^2\right]$$

$$= \beta^2 + 2\beta \underbrace{E(\varepsilon)}_{=0} + E(\varepsilon^2)$$

$$= \mathrm{Var}(X) + (E(X))^2 = \mathrm{Var}(X) = \sigma^2$$

"MSE" $= \beta^2 + \sigma^2 = \text{Bias}^2 + \sigma^2$

4.3 Covariance & Correlation

$$\mathrm{Cov}(X, Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right]$$

"with" if E exists. "times" is correct here

Fact: $\mathrm{Cov}(X,Y) = E\left[XY - \mu_X Y - \mu_Y X + \mu_X \mu_X\right]$

$$= E(XY) - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y$$

$$= E(XY) - E(X)E(Y)$$

never say Covariance of X ~~times~~ Y BUT always X "with" Y

Note: $\mathrm{Cov}(X, X) \stackrel{?}{=}$

Facts

$$\mathrm{Cov}(a+X, b+Y) = \mathrm{Cov}(X, Y)$$

$$\mathrm{Cov}(aX, bY) = ab\,\mathrm{Cov}(X, Y)$$

$$\mathrm{Cov}(X, Y+Z) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)$$

$$\mathrm{Cov}(aW + bX, cY + dZ)$$

$$= ac\,\mathrm{Cov}(W, Y) + ad\,\mathrm{Cov}(W, Z)$$

$$+ bc\,\mathrm{Cov}(X, Y) + bd\,\mathrm{Cov}(X, Z)$$

Summary:

Linearity of Expectation:

If $Y = a + b_1 X_1 + \cdots + b_n X_n$

then

$$E(Y) = a + b_1 E(X_1) + \cdots + b_n E(X_n)$$

In other words

If $Y = a + \sum_{i=1}^{n} X_i$

then $E(Y) = a + \sum_{i=1}^{n} E(X_i)$

Bilinearity of Covariance

$$\mathrm{Cov}\left(X, a + \sum_{i=1}^{n} b_i Y_i\right)$$

$$= \sum_{i=1}^{n} b_i \mathrm{Cov}(X, Y_i)$$

$$\mathrm{Cov}\left(a + \sum_{i=1}^{n} b_i X_i, Y\right)$$

$$= \sum_{i=1}^{n} b_i \mathrm{Cov}(X_i, Y)$$

$$\mathrm{Cov}\left(a+\sum_{i=1}^{n} b_i X_i \,,\, c+\sum_{j=1}^{m} d_j Y_j\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{b_i d_j \,\mathrm{Cov}(X_i, Y_j)}_{\text{this should look like matrix multiplication}}$$

$$= \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} \times$$

$$\begin{bmatrix} \mathrm{Cov}(X_1, Y_1) & \mathrm{Cov}(X_1, Y_2) & \cdots & \mathrm{Cov}(X_1, Y_m) \\ \mathrm{Cov}(X_2, Y_1) & \mathrm{Cov}(X_2, Y_2) & & \vdots \\ \vdots & \vdots & & \vdots \\ \mathrm{Cov}(X_n, Y_1) & \mathrm{Cov}(X_n, Y_2) & \cdots & \mathrm{Cov}(X_n, Y_m) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

$$\underset{\text{vector}}{(1 \times n)} \times \underset{\text{matrix}}{(n \times m)} \times \underset{\text{vector}}{(m \times 1)} = \underset{1\times 1 \text{ scalar}}{[\quad]}$$

Matrix formula :

Let $\underset{\sim}{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$, $C = \begin{bmatrix} Cov(X_i, Y_j) \end{bmatrix}$, $\underset{\sim}{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}$

$n \times m$ matrix

$$Cov(\underset{\sim}{b}^T \underset{\sim}{X}, \underset{\sim}{d}^T \underset{\sim}{Y}) = \underset{\sim}{b}^T C \underset{\sim}{d}$$

$\underset{\sim}{b}$ and $\underset{\sim}{d}$ could be matrices

$$\mathrm{Cov}(B\underset{\sim}{X}, D\underset{\sim}{Y}) = B\,\mathrm{Cov}(\underset{\sim}{X}, \underset{\sim}{Y})\,D^T$$

Fact:

If X & Y are independent

then $\mathrm{Cov}(X, Y) = 0$

Proof: Recall: if X, Y are independent then $E(g(X)h(Y))$

$= E(g(X))E(h(Y))$

Let $g(X) = X - \mu_X$, $h(Y) = Y - \mu_Y$

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

$$= E(X - \mu_X) \times E(Y - \mu_Y)$$

$$= [\mu_X - \mu_X] \times [\mu_Y - \mu_Y]$$

$$= 0$$

Important to remember:

If (X,Y) are bivariate normal

then X and Y are independent

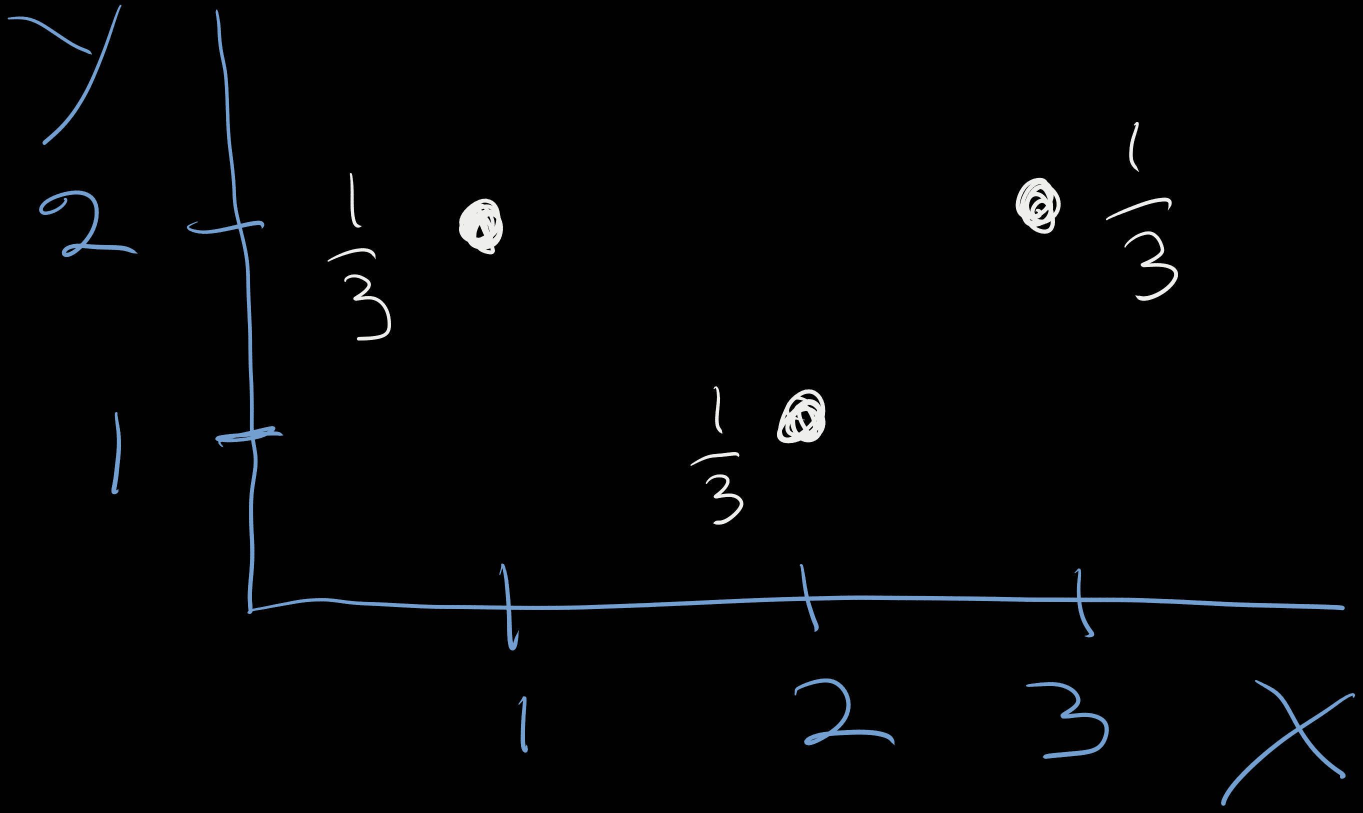
iff $Cov(X,Y)=0$

In general: X, Y independent

implies $Cov(X,Y)=0$

BUT $Cov(X,Y)=0$ DOES NOT IMPLY INDEPENDENCE

Simple counter example:



IMPORTANT RESULT

If $X_1, \dots, X_n$ are R.V.

with $\mathrm{Cov}(X_i, X_j) = 0, i \neq j$

then

$$\mathrm{Var}(X_1 + \dots + X_n)$$
$$= \mathrm{Var}(X_1) + \dots + \mathrm{Var}(X_n)$$

<u>Corollary</u>: If $X_1, \ldots, X_n$ are independent, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

NOTE: NOT TRUE IN GENERAL

Recall:

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

whether X_i's are independent or not

Example B (p. 140)

$X \sim \text{Binomial}(n, p)$

What is $\text{Var}(X)$?

$X = X_1 + \cdots + X_n$

where X_i's are independent Bernoulli(p)

$$E(X_i) = p \cdot 1 + (1-p) \cdot 0 = p$$

$$E(X_i^2) = p \cdot 1^2 + (1-p) \cdot 0^2 =$$

$$\mathrm{Var}(X_i) = E(X^2) - (E(X))^2$$

$$= p - p^2$$

$$= p(1-p)$$

$$= pq \text{ where } q = 1-p$$

So $\mathrm{Var}(X) = \sum_{i=1}^{n} (\mathrm{Var}(X_i)) = npq$

Correlation:

"Pure" measure of covariance

X = height in cm

Y = weight in kg

then $\mathrm{Cov}(X, Y)$ is in "cm × kg"

If we change X to $X' = X/2.54$

so X' is in inches, then

$$\text{Cov}(X', Y) = \text{Cov}(X, Y)/2.54$$

in units of "in × kg"

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{SD(X) \times SD(Y)}$$

$$= \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X) \times \mathrm{Var}(Y)}}$$

$$= \rho_{XY} = \rho$$

Theorem: $|\rho| \le 1$ i.e. $-1 \le \rho \le 1$

and $\rho = \pm 1$ iff $P(Y = a + bX) = 1$ for some $b \ne 0$

See proof p. 143 = Cauchy-Schwarz Theorem

Fact: ρ_{XY} stays the same if X and/or Y are rescaled or translated.

e.g. inches to feet; $X' = X/12$; $^\circ C \to {}^\circ F$ $X' = \frac{9}{5}X + 32$