Chapter 4.5 Moment-generating functions

Distributions for random variables can be represented in many ways. We've seen some:

CDF: $F(x) = P(X \leq x)$

pmf: $p(x) = P(X = x)$

pdf: $f(x)$: $P(X \in A) = \int_A f(x)\,dx$

We know that $E(X)$ does not determine a distribution unless restricting to a special family: e.g. Poisson or exponential.

The moment-generating function (MGF) is another way that works for many distributions.

Suppose a function has a Taylor series expansion around 0, (Maclaurin series)

$$f(t) = a_0 + \frac{a_1}{1!} t + \frac{a_2}{2!} t^2 + a_3 \frac{t^3}{3!} + a_4 \frac{t^4}{4!} + a_5 \frac{t^5}{5!} + \cdots + a_n \frac{t^n}{n!} + \cdots$$

and the power series converges in an open interval around 0.

Then

1) $\frac{d^n}{dt^n} f(0) = a_n$ and has derivatives of all orders.

2) So the derivatives of f at 0 determine the Taylor series and thus determine the entire function.

Defn: The $\underbrace{r\text{th}}_{\text{central}}$ moment of X is $E(X^r)$ if it exists.

$E[(X-E(X))^r]$

Note: $Var(X)$ is the second central moment.

$E[(X-\mu_X)^3]$ is the "skewness"

sometimes standardized as $E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^3\right]$

MGF for r.v. X:

a function of t defined as:

$$M(t) = E(e^{tX}) = \begin{cases} \int e^{tx} f(x)\, dx \\ \sum_x e^{tx} p(x) \end{cases}$$

provided $|m(t)| < \infty$ in an interval $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$.

$$M'(t) = \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{tx} \right) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x e^{tx} f(x)\, dx$$

So $M'(0) = \int_{-\infty}^{\infty} x e^{0x} f(x)\, dx = E(X)$

$$M''(t) = \int_{-\infty}^{\infty} x^2 e^{tx} f(x)\, dx$$

$$M''(0) = E(X^2)$$

$\vdots$

$$M^{(k)}(0) = E(X^k)$$

Two theorems:

If $M_X(t)$ exists in an open interval about 0

then

1) $E(X^r) = M^{(r)}(0)$

2) The distribution of X is uniquely determined by M

So the MGF joins the CDF, pmf and pdf as ways of characterizing distributions.

Note: All distribution for RVs have a (unique) CDF.

Only some have a pdf, pmf or mgf.

- Both continuous and discrete distribution can have mgfs.

Some mgf's

Poisson: $p(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$ $\quad x = 0, 1, \ldots; \lambda > 0$

$$M(t) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!} e^{-\lambda}$$

$\sum x^r \frac{\lambda^x}{x!} e^{-\lambda}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t-1)}$$ Converges for all t

Differentiating to find moments is much easier than summing infinite series!

$$M'(t) = e^{\lambda(e^t-1)} \times \lambda e^t$$

$$M'(0) = e^{\lambda(e^0-1)} \times \lambda e^0 = e^0 \times \lambda e^0 = \lambda$$

$$M''(t) = \underbrace{e^{\lambda(e^t-1)} \times \lambda e^t}_{\lambda} \times \underbrace{\lambda e^t}_{\lambda} + \underbrace{e^{\lambda(e^t-1)}}_{1} \underbrace{\lambda e^t}_{\lambda}$$

$$M''(0) = \lambda \quad \lambda + 1 \quad \lambda = \lambda^2 + \lambda$$

So $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$$= (\lambda^2 + \lambda) - \lambda^2$$

$$= \lambda$$

What happens if we add 2 independent Poisson(λ): Y_1 and Y_2

Finding the pdf/pmf of a sum was tough. Let's see the MGF.

$$E\left(e^{t(Y_1+Y_2)}\right) = E\left(e^{tY_1}\right)E\left(e^{tY_2}\right) \text{ by independence}$$

$$= e^{\lambda(e^t-1)}\, e^{\lambda(e^t-1)}$$

$$M_{Y_1+Y_2}(t) = e^{2\lambda(e^t-1)}$$

Recognize this ?

It's the MGF for a Poisson (2λ)

$\therefore$ by uniqueness of MGF

$Y_1 + Y_2 \sim \text{Poisson}(2\lambda)$

How about $Y_1 + Y_2 + \dots Y_n$

where Y_i's are independent

Poisson $\lambda_1, \lambda_2, \dots \lambda_n$?

$Y_1 + Y_2 + \dots Y_n \sim \text{Poisson}(\sum \lambda_i)$

Facts about MGFs

If the mgf exists in an interval around 0 then

1) $M^{(r)}(0) = E(X^r)$

2) If $X_1, X_2, \ldots, X_n$ are independent with mgfs $M_1(t), \ldots, M_n(t)$, then

$$M_{X_1+\cdots X_n}(t) = \prod_{i=1}^{n} M_i(t)$$

3) ... with the same mgf $M(t)$ then

$$M_{X_1+\cdots X_n}(t) = [M(t)]^n$$

4) If $Y = a + bX$ then $M_Y(t) = e^{at} M_X(bt)$

Some MGFs :

Exponential (λ)

$$M_{\lambda}(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x} \, dx$$

$$= \frac{\lambda}{\lambda - t} \int_0^{\infty} (\lambda - t) \, e^{-(\lambda - t)x} dx$$

$$= \frac{\lambda}{\lambda - t} \quad \text{if } \lambda - t > 0$$

i.e. $t < \lambda$

$$\text{Gamma}(\alpha, \lambda) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$$

shape

So sum of n independent Exponential(λ)'s

$$\text{Gamma}(n, \lambda)$$

Normal$(0,1)$ $\quad e^{t^2/2}$

Normal mean μ and variance σ^2

$Y = \mu + \sigma Z$ where Z is $N(0,1)$

$$M_Y(t) = E(e^{tY}) = E(e^{t(\mu + \sigma Z)})$$

$$= E(e^{t\mu + t\sigma Z})$$

$$= e^{t\mu} E(e^{t\sigma Z})$$

$$= e^{t\mu} e^{(t\sigma)^2/2}$$

$$= e^{t\mu + t^2\sigma^2/2}$$

Does it work?

$$M_Y'(t) = e^{t\mu + \frac{t^2}{2}\sigma^2}(\mu + t\sigma^2)$$

$$M_Y''(t) = e^{t\mu + \frac{t^2}{2}\sigma^2}(\mu + t\sigma^2)(\mu + t\sigma^2)$$

$$+ e^{t\mu + \frac{t^2}{2}\sigma^2}\sigma^2$$

$$= e^{t\mu + \frac{t^2}{2}\sigma^2}\left\{(\mu + t\sigma^2)^2 + \sigma^2\right\}$$

So $E(Y) = M_Y'(0) = e^0(\mu + 0\cdot\sigma^2) = \mu$

$E(Y^2) = M_Y''(0) = e^0\{\mu^2 + \sigma^2\} = \mu^2 + \sigma^2$

$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$

$= \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

Sums of independent Normals

$Y_1 \sim N(\mu_1, \sigma_1^2)$, $Y_2 \sim N(\mu_2, \sigma_2^2)$

$$E\left(e^{t(Y_1+Y_2)}\right) = e^{t\mu_1 + \frac{t^2}{2}\sigma_1^2 + t\mu_2 + \frac{t^2}{2}\sigma_2^2}$$

$$= \exp\left\{t(\mu_1+\mu_2) + \frac{t^2}{2}(\sigma_1^2+\sigma_2^2)\right\}$$

$$\therefore Y_1 + Y_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2) \quad !!!$$