3.7 Extremes and order statistics (continuous)

Let $X_1, \ldots, X_n$ be i.i.d with cdf F_X & pdf f_X.

Let $U = \max(X_i)$ $\quad V = \min(X_i)$

$X_{(1)} < X_{(2)} < X_{(3)} < \ldots < X_{(n-1)} < X_{(n)}$

smallest $\qquad\qquad$ largest

CDF of U: $= X_{(n)}$

$$F_U(u) = \Pr(U \leq u)$$

$$= \Pr\left((X_1 \leq u) \cap (X_2 \leq u) \cap \ldots \cap (X_n \leq u)\right)$$

$$= \Pr(X_1 \leq u) \times \ldots \times \Pr(X_n \leq u)$$

Why?

$$= \underbrace{F_X(u) \times \circ \circ \circ \times F_X(u)}_{n \text{ times}}$$

$$= F_X^n(u)$$

$$f_U(u) = \frac{d}{du} F_U(u)$$

$$= \frac{d}{du} F_X^n(u)$$

$$= n F_X^{n-1}(u) \times \frac{d F_X(u)}{du}$$

$$= \boxed{n F_X^{n-1}(u) f_X(u)}$$

$X_{(1)}$

$(X_{(1)}, X_{(n)})$

$(X_{(3)}, X_{(9)})$

$$F_Y(v) = \Pr(X_1 \le v \cup X_2 \le v \cup \dots \cup X_n \le v)$$

$$= 1 - \Pr(X_1 > v \cap X_2 > v \cap \dots \cap X_n > v)$$

$$= 1 - \underbrace{(1-F(v))(1-F(v))\cdots(1-F(v))}_{n \text{ time}}$$

$$= 1 - (1-F(v)^{n}$$

$$f_V(v) = \frac{dF_V(v)}{dv} = -n(1-F(v)^{n-1}(-f(v))$$

$$= n f(v)(1-F(v))^{n-1}$$

See examples in text

Here a general formula for the joint density of "order statistic" from a sample of n with cdf F and pdf f:
"iid"

This is for ordered indices between 1 and n

e.g. If $n = 10$, indices could be

1 , 3 , 9

smallest — 3rd smallest — 2nd from largest

Notation $\left(X_{(1)}, X_{(3)}, X_{(9)}\right)$

Or just the smallest & largest $(X_{(1)}, X_{(10)})$

$n=10$ $\quad f(x_{(3)}, x_{(6)})$

Positions:

1 2 3 4 5 6 7 8 9 10

We want 2 (3) 1 2 (6) 1 4

$$f(x_{(3)}, x_{(6)}) = \binom{10}{2\ 1\ 2\ 1\ 4} \times$$

$$\frac{10!}{2!\,1!\,2!\,1!\,4!}$$

$$F(x_{(3)})^2 f(x_{(3)})^1 \left(F(x_{(6)})-F(x_{(3)})\right)^2 f(x_{(6)})^1 \left(1-F(x_{(6)})\right)^4 ,$$

$x_{(3)} < x_{(6)}$

$F(-\infty)$ $\quad F(\infty)$

where $\left(\underbrace{\begin{matrix} & & 10 & & \\ 2 & 1 & 2 & 1 & 4 \end{matrix}}_{\text{These must add up to 10}} \right) = \dfrac{10!}{2!\,1!\,2!\,1!\,4!}$

For example $n = 3$

$$f(x_{(1)}, x_{(2)}, x_{(3)}) = \underbrace{\begin{pmatrix} & 3 & \\ 1 & 1 & 1 \end{pmatrix}}_{\frac{3!}{1!\,1!\,1!} = 3!} f(x_{(1)})\, f(x_{(2)})\, f(x_{(3)})$$

$$x_{(1)} < x_{(2)} < x_{(3)}$$

$n = 4 \qquad x_{(1)},\ x_{(3)}$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ x_{(1)} & \underbrace{}_{1} & x_{(3)} & \underbrace{}_{1} \\ 1 & & 1 & \end{array}$$

$$\binom{4}{1\ 1\ 1\ 1} f(x_{(1)})^1 \left[F(x_3) - F(x_{(1)})\right]^1 \times$$

$$f(x_{(3)})^1 \left(1 - F(x_{(3)})\right)^1$$