Chapter 3 - Part 2

3.5 Conditional distributions

Recall:

Multiplication Rule: $P(A \cap B) = P(A)P(B|A)$
$= P(B)P(A|B)$

Conditional Probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ if $P(B) > 0$

$P(B|A) = \dfrac{P(A \cap B)}{P(A)}$ if $P(A) > 0$

Discrete Case:

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

Conditional probabilities from joint probabilities tables

X = # of patients tested in 1 st hour

Y = # positive

Joint table

$p(x,y)$		Y = 0	1	2
X	0	.1	0	0
	1	.2	.1	0
	2	.2	.2	.2

$p(x,y)$		y = 0	1	2	$p_X(x)$
x	0	.1	0	0	.1
	1	.2	.1	0	.3
	2	.2	.2	.2	.6
	$p_Y(y)$	.5	.3	.2	1 ← check

$$P_{Y|X}(y|x) = \frac{p(x,y)}{P_X(x)}$$

$P(x,y)$

		Y: 0	1	2	$P_X(x)$
X	0	.1	0	0	.1
	1	.2	.1	0	.3
	2	.2	.2	.2	.6
	$P_Y(y)$	.5	.3	.2	1

	0	1	2		
0	1	0	0	1	$\div .1$
1	$2/3$	$1/3$	0	1	$\div .3$
2	$1/3$	$1/3$	$1/3$	1	$\div .6$

$P(x,y)$

		Y			
		0	1	2	$P_X(x)$
X	0	.1	0	0	.1
	1	.2	.1	0	.3
	2	.2	.2	.2	.6
	$P_Y(y)$	.5	.3	.2	1

Do: $P_{X|Y}(x|y)$

Example with formula:

Let X be # of car accidents in North York in 1 week

Y " " " " " " " " " " " involving a pedestrian or bicycle

Suppose $X \sim \text{Poisson}(\lambda)$

and for each car accident the probability that it involved a pedestrian or bicycle is p

Note: $E(X) = \lambda$

If $W \sim \text{exponential}(\lambda)$, $E(W) = 1/\lambda$

There a reason!!

What is the distribution of Y ?

$P(Y = y)$??

What do we know?

$$P_X(x) = \frac{1}{x!}\lambda^x e^{-\lambda} \qquad x = 0, 1, \ldots$$

Given $X = x$, each accident has prob. p of involving a bicycle or pedestrian.

So $P_{Y|X}(y|x) \sim \text{Binomial}(x, p)$

$$= \binom{x}{y} p^y (1-p)^{x-y} \qquad y = 0, 1, \ldots, x$$

So $p(x,y) = p_X(x)\, p_{Y|X}(y|x)$

$$= \frac{1}{\cancel{x!}}\lambda^x e^{-\lambda} \times \frac{\cancel{x!}}{y!(x-y)!} p^y (1-p)^{x-y}$$

$0 \le y \le x$

$$p_Y(y) = \sum_{x=y}^{\infty} p(x,y)$$

$$= \frac{p^y}{(1-p)^y}\frac{1}{y!}e^{-\lambda}\sum_{x=y}^{\infty}\frac{\lambda^x}{(x-y)!}(1-p)^x$$

looks like $\sum_{k=0}^{\infty}\frac{\theta^k}{k!} = ?$

$$= \frac{p^y}{(1-p)^y}\frac{1}{y!}e^{-\lambda}\lambda^y(1-p)^y\sum_{x=y}^{\infty}\frac{\lambda^{x-y}}{(x-y)!}(1-p)^{x-y}$$

$$= (\lambda p)^y\frac{1}{y!}e^{-\lambda}\underbrace{\sum_{k=0}^{\infty}\frac{(\lambda(1-p))^k}{k!}}_{e^{\lambda(1-p)}}$$

$$= (\lambda p)^y\frac{1}{y!}e^{-\lambda}\times e^{\lambda(1-p)}$$

$$= (\lambda p)^y \frac{1}{y!} e^{-\lambda p} \qquad y = 0, 1, \ldots$$

So $Y \sim \text{Poisson}(\lambda p)$