Chapter 4 Expected Value

Geometrically

$X \sim f$

$f(x)$

x
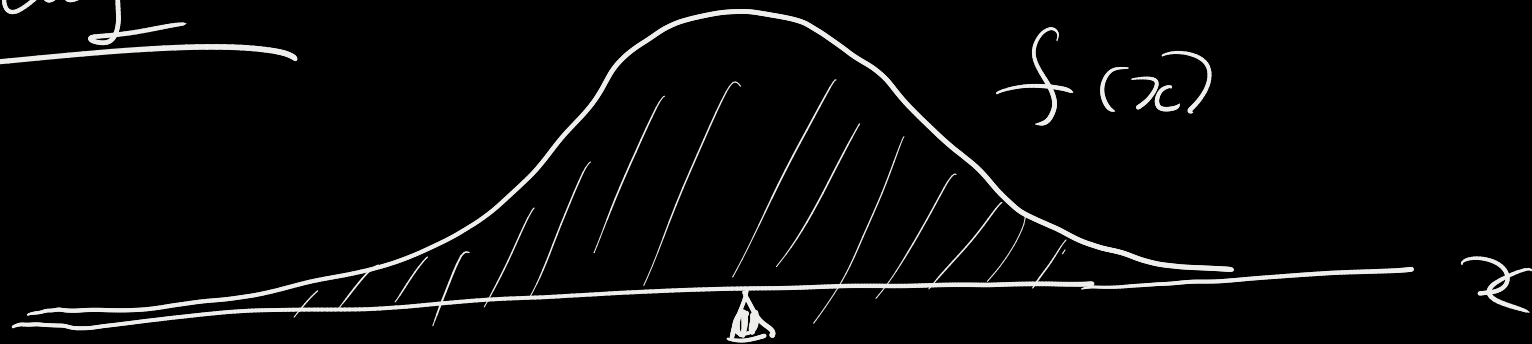
E(X) is point of balance

$X \sim p$

weights = P(x)

E(X)
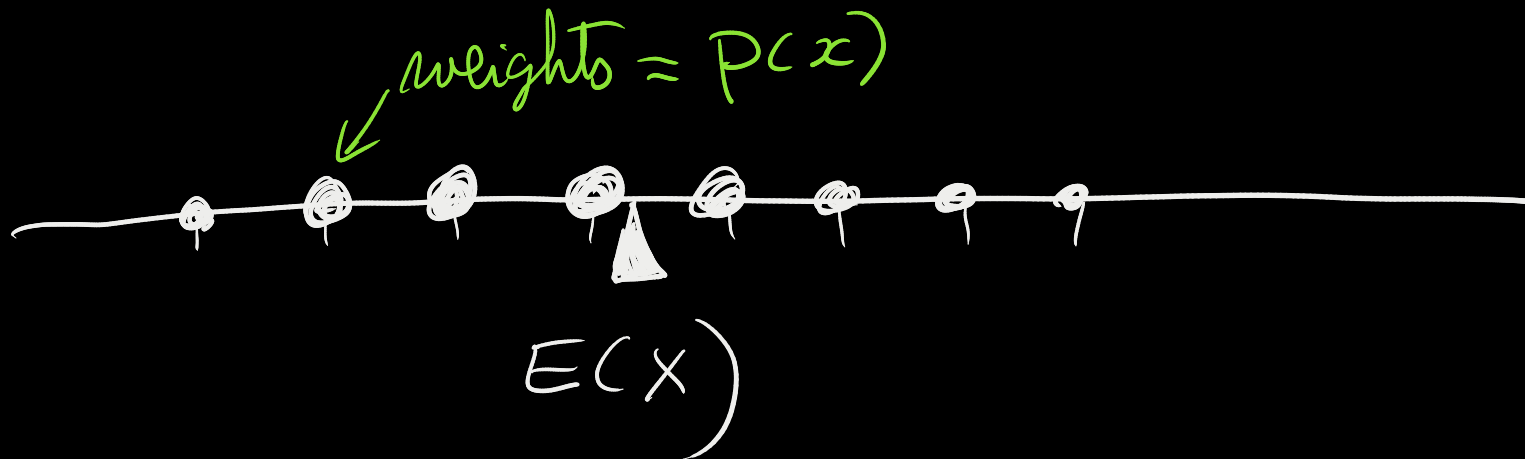
Defn $\qquad E(X) = \sum x \, p(x)$ provided $\sum |x| p(x) < \infty$

otherwise "not defined"

$$E(X) = \int x \, f(x) \, dx \quad \text{provided} \quad \int |x| f(x) \, dx < \infty$$

Examples :

Exponential (λ)

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0, \; \lambda > 0$$

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} \, dx$$

$$= \Gamma(2) \frac{1}{\lambda} \int \frac{x^{2-1} \overset{\alpha=2}{\lambda^2} e^{-\lambda x}}{\Gamma(2)} dx$$

$$\underbrace{\qquad}_{\text{gamma}(2, \lambda)}$$

$$= \frac{1}{\lambda}$$

Use gamma distribution

$$\int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x} dx$$

$$= 1$$

Poisson: $E(x) = \sum_{k=0}^\infty \frac{k \lambda^k}{k!} e^{-\lambda}$

first term is 0

$$= e^{-\lambda} \sum_{k=1}^\infty \frac{\lambda^k}{(k-1)!}$$

We know $\sum_{j=0}^\infty \frac{\lambda^j}{j!} = e^{\lambda}$

$$= e^{-\lambda} \lambda \left(\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right) \qquad \text{but } e = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \lambda \, e^{\lambda}$$

$$= \lambda$$

St. Petersburg Paradox

How to make a sure \$1

Find someone willing to make fair bets on the toss of a coin.

i.e. if coin is H   they give me $N

" " " T   & give them $N

Let X = amount I win.

$$E(X) = \frac{1}{2} N + \frac{1}{2} (-N) = 0$$

"Fair bet"

My strategy:

- Bet $1

- If I win stop

- If I lose bet double the amount

- Keep doubling until I win

If my first win is at the kth bet.
my winnings are:

$$\boxed{-1 - 2 - 4 \ldots - 2^j - \ldots - 2^{k-1}} + 2^k$$

$$= \$1$$

I am sure to eventually win $\$1$

Recall for Geometric series

$$\sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a - 1} \quad \text{if } a \neq 1$$

(next - first)

(ratio - 1)

— works for all finite n
— For "$n = \infty$" need $|a| < 1$

Let W = amount won

It looks like $E(W) = \underline{1}$

not much, but sure!

Formally:

$$E(W) = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} + \cdots +$$

$$= 1 \left(\frac{1}{2} + \frac{1}{4} + \cdots \right)$$

$$= \underline{1}$$

But let X be amount won on first winning bet:

$$E(X) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \times \frac{1}{8} + 8 \cdot \frac{1}{16} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots$$

$$= \infty$$

Let L be amount lost until first winning bet.

$$L = X - 1$$

$$E(L) = \infty.$$

$$\boxed{\infty - \infty = 1 \quad ?}$$

Problem: You need ∞ capital to play this game

— If you have finite capital
you might go bankrupt before winning

— Suppose you only have enough money
to lose k times, then

$$P\left(W = -\sum_{i=1}^{k} 2^{i-1}\right) = 1 - \sum_{i=1}^{k}\left(\frac{1}{2}\right)^{i}$$

$$P\left(W = -\frac{2^k - 1}{2-1}\right) = 1 - \frac{\left(\left(\frac{1}{2}\right)^{k+1} - \frac{1}{2}\right)}{\frac{1}{2} - 1}$$

$$P\left\{W = -(2^k - 1)\right\} = 1 - \left(1 - \left(\frac{1}{2}\right)^k\right) = \frac{1}{2^k}$$

$$E(W) = -(2^k - 1) \times \frac{1}{2^k} + 1\left(1 - \frac{1}{2^k}\right)$$

$$= -1 + \frac{1}{2^k} + 1 - \frac{1}{2^k}$$

$$= 0$$

So a "fair" bet with a
high probability of winning small
at the cost of a small probability
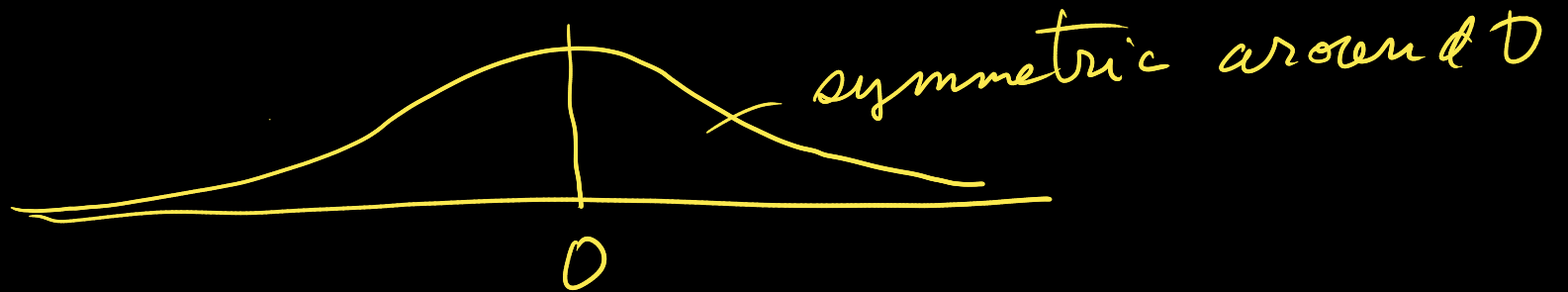of losing big.

Scam strategy : Play the game
with I O U's.

Gamma

$$E(X) = \int_0^\infty x \, \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \, dx$$

$$= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha} e^{-\lambda x} \, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \int_0^\infty \underbrace{\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x}}_{\text{Gamma}(\alpha+1) \text{ density}} \, dx$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{1}{\lambda} = \frac{\alpha}{\lambda} \quad \text{since } \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

Cauchy:
$$f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right), \quad -\infty < x < +\infty$$

symmetric around 0

But $E(|X|) = \displaystyle\int_{-\infty}^{\infty} |x| \frac{1}{\pi} \frac{1}{1+x^2} \, dx$

$$= 2 \int_0^\infty \frac{1}{\pi} \frac{x}{1+x^2} \, dx$$

$$= \frac{2}{\pi} \left[\frac{1}{2} \log(x^2+1) \right]\Bigg|_0^\infty$$

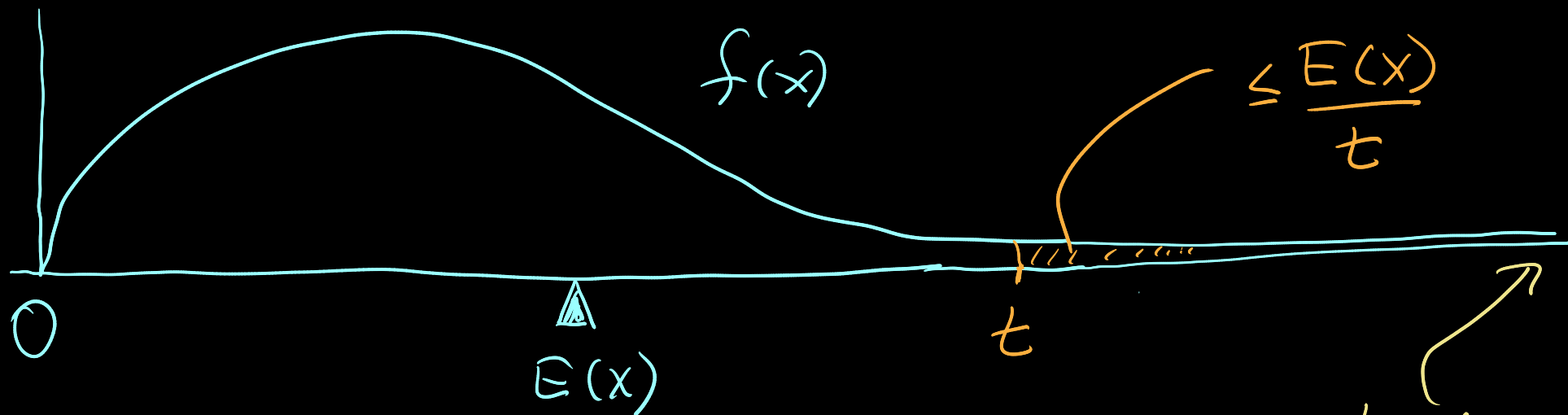$$= \infty \quad !$$

Cauchy CDF^{-1}

$$F^{-1}(u) = \tan^{-1}\left(\pi u - \frac{\pi}{2} \right)$$

easy to simulate.

Markov Inequality:

If 1) X is a RV with $\Pr(X \geq 0) = 1$

2) $E(X) < \infty$

Then for any $t > 0$ $\quad P(X \geq t) \leq \dfrac{E(X)}{t}$

$f(x)$

$\leq \dfrac{E(X)}{t}$

O

$E(X)$

t

Note $E(X)$ very affected by tail

Proof : $E(X) = \int_0^\infty x f(x) \, dx$

$$= \int_0^t x f(x) \, dx + \int_t^\infty x f(x) \, dx$$

$$\geq \int_t^\infty x f(x) \, dx$$

$$\geq \int_t^\infty t \, f(x) \, dx \qquad \text{need this for discrete case}$$

$$= t \, P(X \geq t)$$