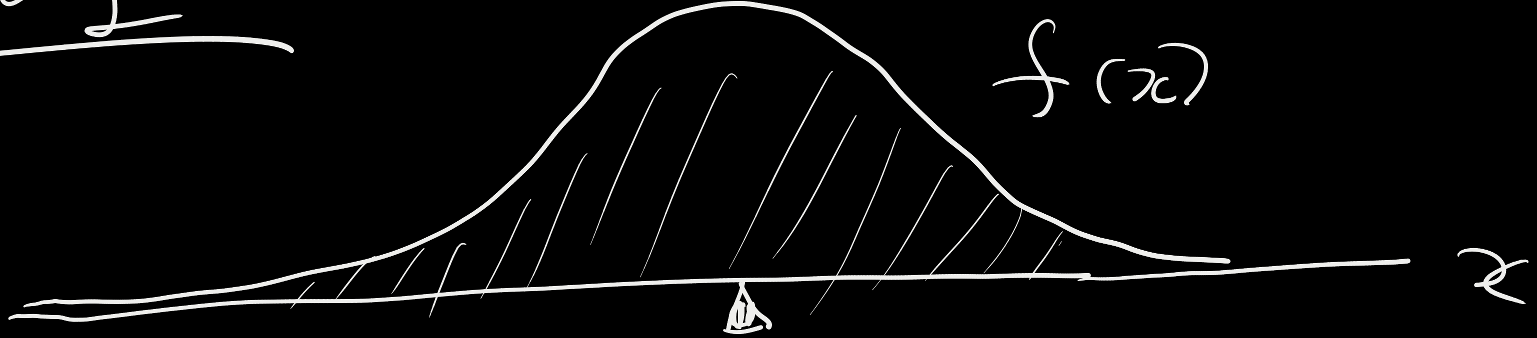
Chapter 4 Expected Value

Geometrically

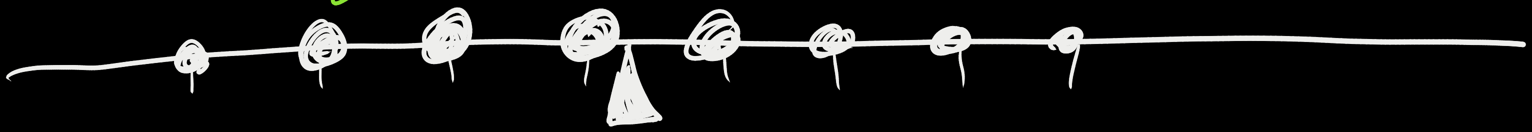
$X \sim f$



$E(X)$ is point of balance

$X \sim p$

weights $= p(x)$



$E(X)$

Defn $E(X) = \sum x\, p(x)$ provided $\sum |x| p(x) < \infty$
otherwise "not defined"

$$E(X) = \int x\, f(x)\, dx \quad \text{provided } \int |x| f(x)\, dx < \infty$$

Examples:

Exponential (λ)

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0,\ \lambda > 0$$

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x}\, dx$$

$$= \Gamma(2)\frac{1}{\lambda}\int \underbrace{\frac{x^{2-1}\lambda^{2}e^{-\lambda x}}{\Gamma(2)}}_{\text{Gamma}(2,\lambda)} dx \qquad \alpha = 2$$

Use Gamma distribution

$$\int_0^\infty \frac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x}\,dx = 1$$

$$= \frac{1}{\lambda}$$

Poisson:

$$E(X) = \sum_{k=0}^{\infty} \underbrace{\frac{k\lambda^{k}}{k!}e^{-\lambda}}_{\text{first term is } 0}$$

$$= e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k}}{(k-1)!}$$

We know $\sum_{j=0}^{\infty}\frac{\lambda^{j}}{j!} = e^{\lambda}$

$$= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

but $\bigcirc = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$

$$= e^{-\lambda} \lambda e^{\lambda}$$

$$= \lambda$$

St. Petersburg Paradox

How to make a sure \$1

Find someone willing to make fair bets on the toss of a coin.

i.e. if coin is H they give me \$N

" " " T I give them \$N

Let X = amount I win.

$$E(X) = \frac{1}{2}N + \frac{1}{2}(-N) = 0$$

"Fair bet"

My strategy:

- Bet \$1
- If I win stop
- If I lose bet double the amount
- Keep doubling until I win

If my first win is at the kth bet

my winnings are:

$$\boxed{-1-2-4 \cdots -2^j - \cdots - 2^{k-1}} + 2^k$$

$= \$1$

I am sure to eventually win $1

next - first

Recall for geometric series

$$\sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a-1} \quad \text{if } a \neq 1$$

ratio - 1

- works for all finite n
- For "$n = \infty$" need $|a| < 1$

Let W = amount won

It looks like $E(W) = 1$

not much, but sure!

Formally

$$E(W) = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} + \cdots +$$

$$= 1\left(\frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$= 1$$

But let X be amount won on <u>first winning bet</u>:

$$E(X) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \times \frac{1}{8} + 8 \cdot \frac{1}{16} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots$$

$$= \infty$$

Let L be amount lost until first winning bet.

$$L = X - 1$$

$$E(L) = \infty.$$

$$\boxed{\infty - \infty = 1 \quad ?}$$

<u>Problem:</u> You need ∞ capital to play this game

- If you have finite capital you might go bankrupt before winning
- Suppose you only have enough money to lose K times, then

$$P\left(W = -\sum_{i=1}^{K} 2^{i-1}\right) = 1 - \sum_{i=1}^{k} \left(\frac{1}{2}\right)^{i}$$

$$P\left(W = -\frac{2^k - 1}{2-1}\right) = 1 - \frac{\left(\left(\frac{1}{2}\right)^{k+1} - \frac{1}{2}\right)}{\frac{1}{2} - 1}$$

$$P\left\{W = -\left(2^k - 1\right)\right\} = 1 - \left(1 - \left(\frac{1}{2}\right)^k\right) = \frac{1}{2^k}$$

$$E(W) = -\left(2^k - 1\right) \times \frac{1}{2^k} + 1\left(1 - \frac{1}{2^k}\right)$$

$$= -1 + \frac{1}{2^k} + 1 - \frac{1}{2^k}$$

$$= 0$$

So a "fair" bet with a high probability of winning small at the cost of a small probability of losing big.

<u>Scam strategy:</u> Play the game with IOU's.

Gamma

$$E(X) = \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$$

$$= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \int_0^\infty \underbrace{\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x}\, dx}_{\text{Gamma}(\alpha+1) \text{ density}}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{1}{\lambda} = \frac{\alpha}{\lambda}$$ since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

Cauchy: $$f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right), \quad -\infty < x < +\infty$$

But $$E(|X|) = \int_{-\infty}^{\infty} |x| \frac{1}{\pi} \frac{1}{1+x^2}\, dx$$

$$= 2\int_0^\infty \frac{1}{\pi}\frac{x}{1+x^2}\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\log(x^2+1)\right]_0^\infty$$

$$= \infty \quad !$$

<u>Cauchy CDF^{-1}</u> $\quad F^{-1}(u) = \tan^{-1}\left(\pi u - \frac{\pi}{2}\right)$

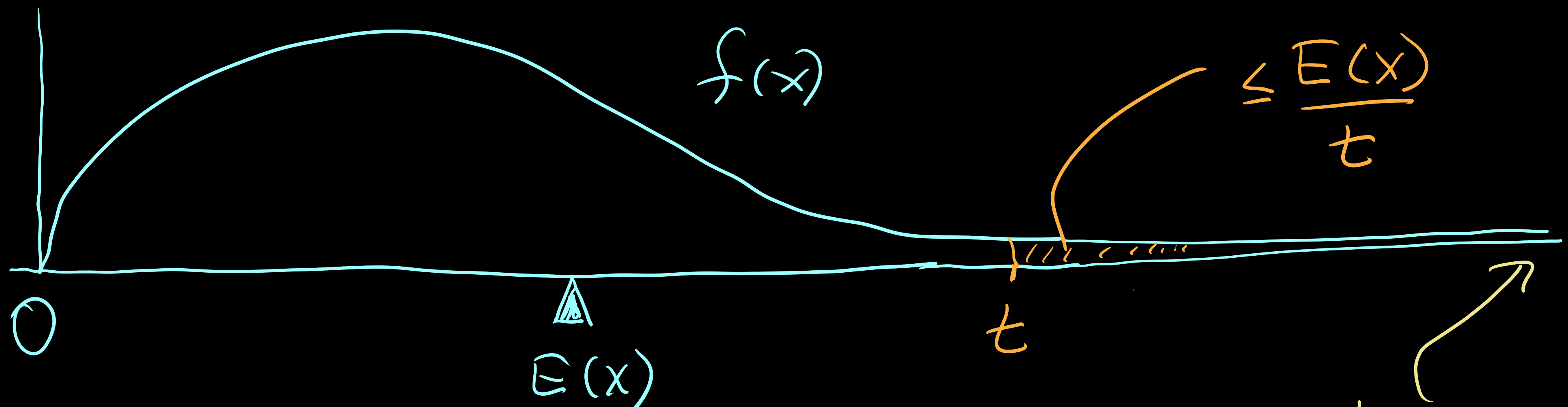
easy to simulate.

Markov Inequality:

If 1) X is a RV with $\Pr(X \geq 0) = 1$

2) $E(X) < \infty$

Then for any $t > 0$ $\quad P(X \geq t) \leq \dfrac{E(X)}{t}$



Note $E(X)$ very affected by tail

Proof: $E(X) = \int_0^\infty x f(x)\,dx$

$$= \int_0^t x f(x)\,dx + \int_t^\infty x f(x)\,dx$$

$$\geq \int_t^\infty x f(x)\,dx$$

$$\geq \int_t^\infty t f(x)\,dx$$

$$= t\,P(X \geq t)$$

need this for discrete case

Expectation of a geometric R.V.

keep tossing until H

$P(H) = p$

$X = \#$ of tosses

$$f_X(x) = p(1-p)^{x-1}, \quad x = 1, 2, \ldots$$

$$E(X) = \sum_{x=1}^{\infty} x\, p(1-p)^{x-1} = p \sum_{x=1}^{\infty} x\, q^{x-1}, \quad q = 1-p$$

$$= p \sum_{x=1}^{\infty} \frac{d}{dq} q^{x}$$

$$= p \frac{d}{dq} \sum_{x=1}^{\infty} q^{x}$$

$$= p \frac{d}{dq} \left(\frac{q^{\infty} - q}{q - 1} \right)$$

$$= p \frac{d}{dq} \left(\frac{q}{1-q} \right)$$

$$= p \left(\frac{1 \times (1-q) - (-1) q}{(1-q)^2} \right)$$

$$= p \left(\frac{1}{1-q} \right)^2 = \frac{p}{p^2} = \frac{1}{p}$$

Uses idea that we can $\frac{d}{dx}$ & $\sum$ commute

$\frac{d}{dx}$ & $\int$

When? Works most of the time
but there are pathological
counterexamples.