3.5 Functions of random vectors

Sums: Given a joint distribution for (X, Y) find the distribution $Z = X + Y$

Example

$Y \backslash X$	0	1	2
2	0	.2	.2
1	.1	.1	.1
0	.1	.2	0

What are possible values for $Z = X + Y$

$$P(Z=0) = \sum_{x,y \,:\, x+y=0} P(x,y) = \sum_{x,\, y=0-x} P(x,y)$$

$$= \sum_{x} p(x, 0-x)$$

$$P(Z=1) = \sum_{x} p(x, 1-x)$$

$$P(Z=2) = \sum_{x} p(x, 2-x)$$

$$P(Z=3) = \dots$$

$$P(Z=4) = \dots$$

Y \ X	0	1	2
2	0	.2	.2
1	.1	.1	.1
0	.1	.2	0

General formula:

Of $Z = X + Y$

Discrete case

$$P_Z(z) = \sum_x p(x, z-x)$$

$$\overset{\text{or}}{=} \sum_y p(z-y, y)$$

Continuous: $f_z(z) = \int_{-\infty}^{-\infty} f(x, z-x)\, dx$

$$= \int_{-\infty}^{\infty} f(z-y, y)\, dy$$

If X & Y are independent: $f(x,y) = f_X(x) f_Y(y)$

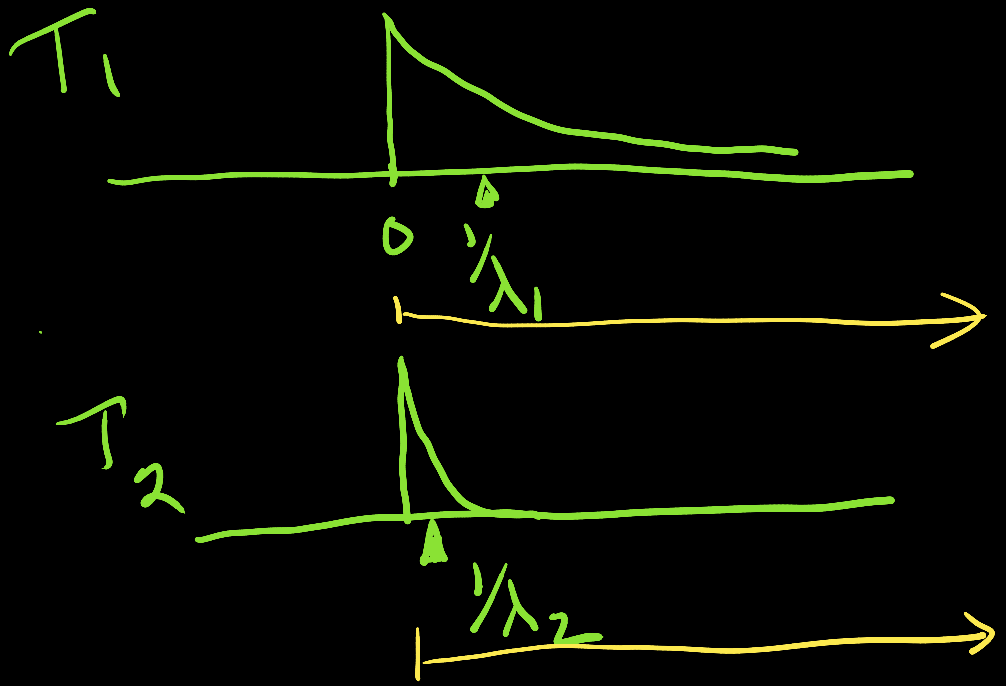
and

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

"Convolution of f_X & f_Y"

Example A: Computer has lifetime $T_1 \sim \text{Exponential}(\lambda_1)$
Automatic backup $T_2 \sim \text{Exponential}(\lambda_2)$



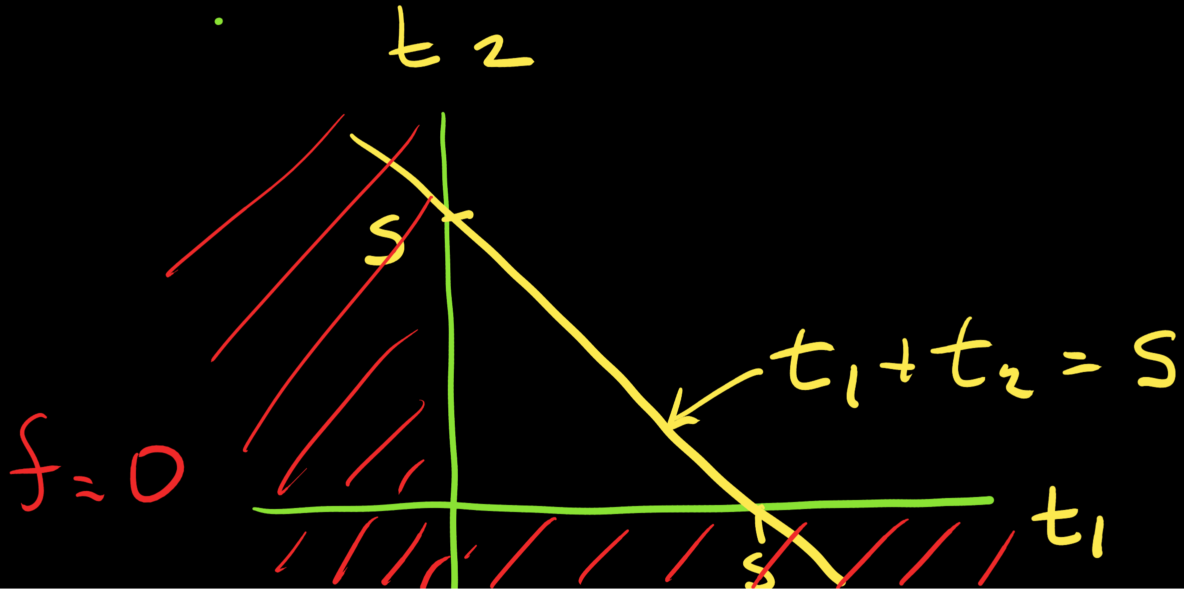
$$f_1(t_1) = \lambda_1 e^{-\lambda_1 t_1}, \quad t_1 > 0$$

$$f_2(t_2) = \lambda_2 e^{-\lambda_2 t_2}, \quad t_2 > 0$$

$S = T_1 + T_2$; T_1, T_2 independent

$$f_S(s) = \int_{-\infty}^{\infty} f_1(t_1) f_2(s-t_1)\, dt_1$$

$$= \int_{?}^{?} \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 (s-t_1)}\, dt_1$$



So

$? < t_1 < ?$

$$= \int_{?=0}^{?=s} \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 (s-t_1)} \, dt_1$$

"Simple" if $\lambda_1 = \lambda_2 = \lambda$ i.e. Same expected lifetime.

$$f_S(s) = \int_0^s \underbrace{\lambda e^{-\lambda t} \lambda e^{-\lambda(s-t)}}_{\lambda^2 e^{-\lambda s}} \, dt$$

$$= \int_0^s \lambda^2 e^{-\lambda s} \, dt$$

$$= \left[t\lambda^2 e^{-\lambda s} \right|_0^s$$

$$= s\lambda^2 e^{-\lambda s}, \quad 0 < s$$

$\sim \text{Gamma}(2, \lambda)$ i.e. $\frac{1}{\Gamma(\alpha)} s^{\alpha-1} \lambda^\alpha e^{-\lambda s}$,

$0 < s, \alpha = 2$

General Case: Using Jacobians

Transformation $\mathbb{R}^1 \to \mathbb{R}^1$

of $Y = h(X)$

h is differentiable and monotone

"strictly" increasing on an interval I

or " decreasing " " " ".

$$f_Y(y) = f_X(g^{-1}(y)) \times \left|\frac{dx}{dy}\right|$$

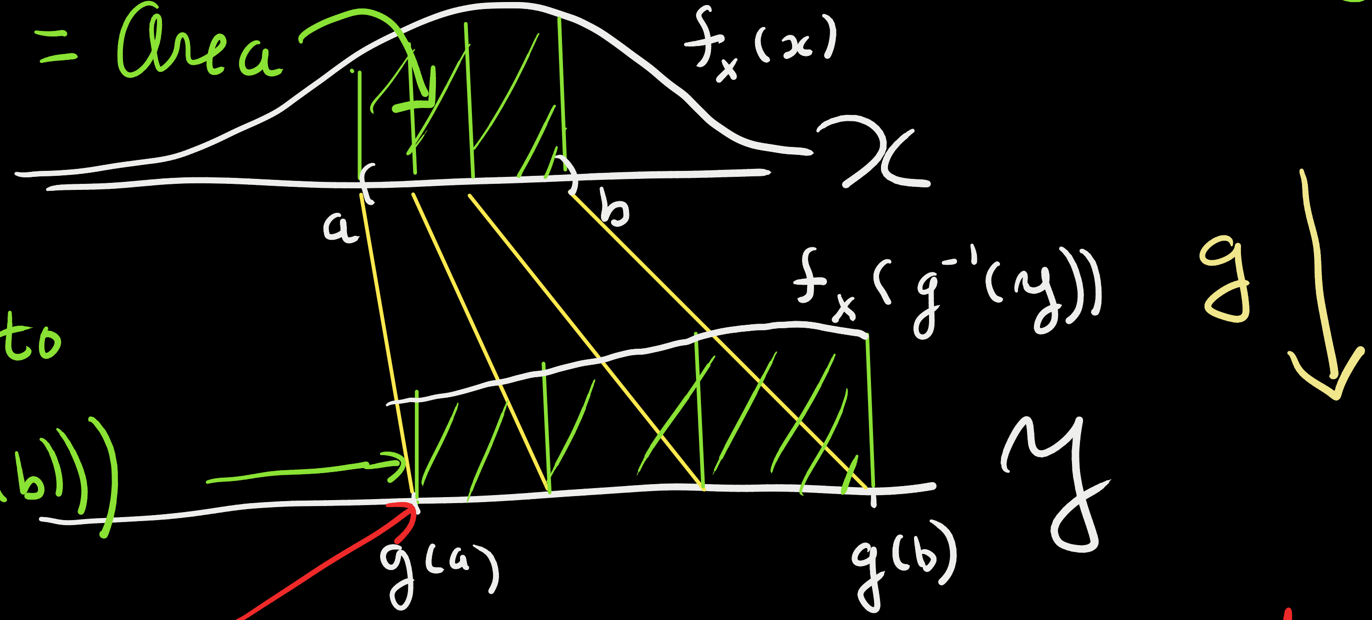
$|\ldots|$ so it works whether increasing or decreasing

$P(X \in (a,b)) = \text{Area}$

$\shortparallel$

needs to be equal to

$P(Y \in (g(a), g(b)))$



But this area is too big because $g(x)$ stretches the base

What to do?

Undo the effect of stretching by dividing by $\left|\frac{dy}{dx}\right|$ i.e. the amount of stretching

Now $\left|\frac{dy}{dx}\right| = \left|\frac{dx}{dy}\right|^{-1}$

So we can multiply by $\left|\frac{dx}{dx}\right|$

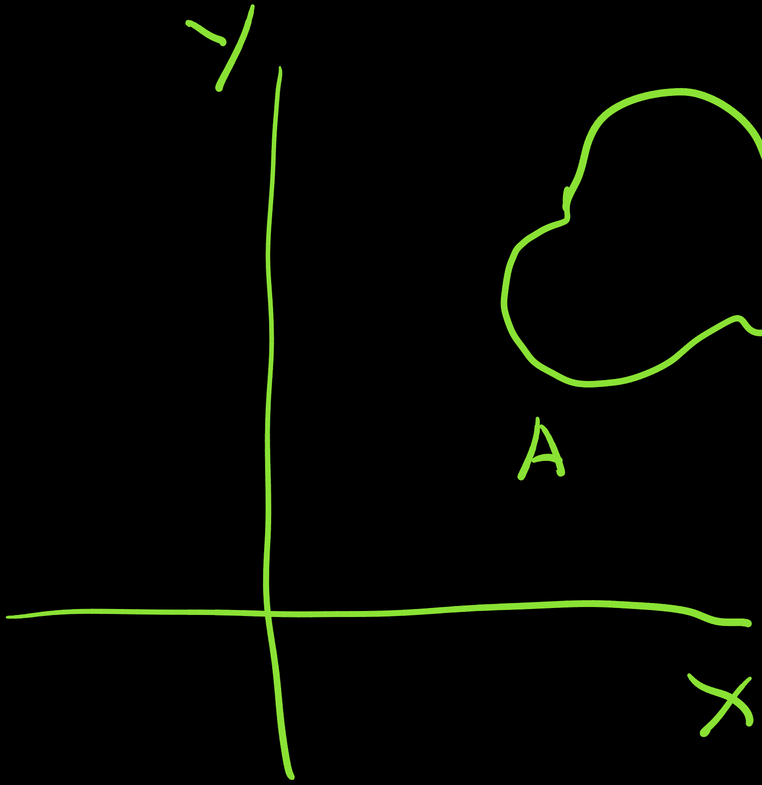
Formula:

$$f_Y(y) = f_X(g^{-1}(y)) \times \left|\frac{dx}{dy}\right|$$

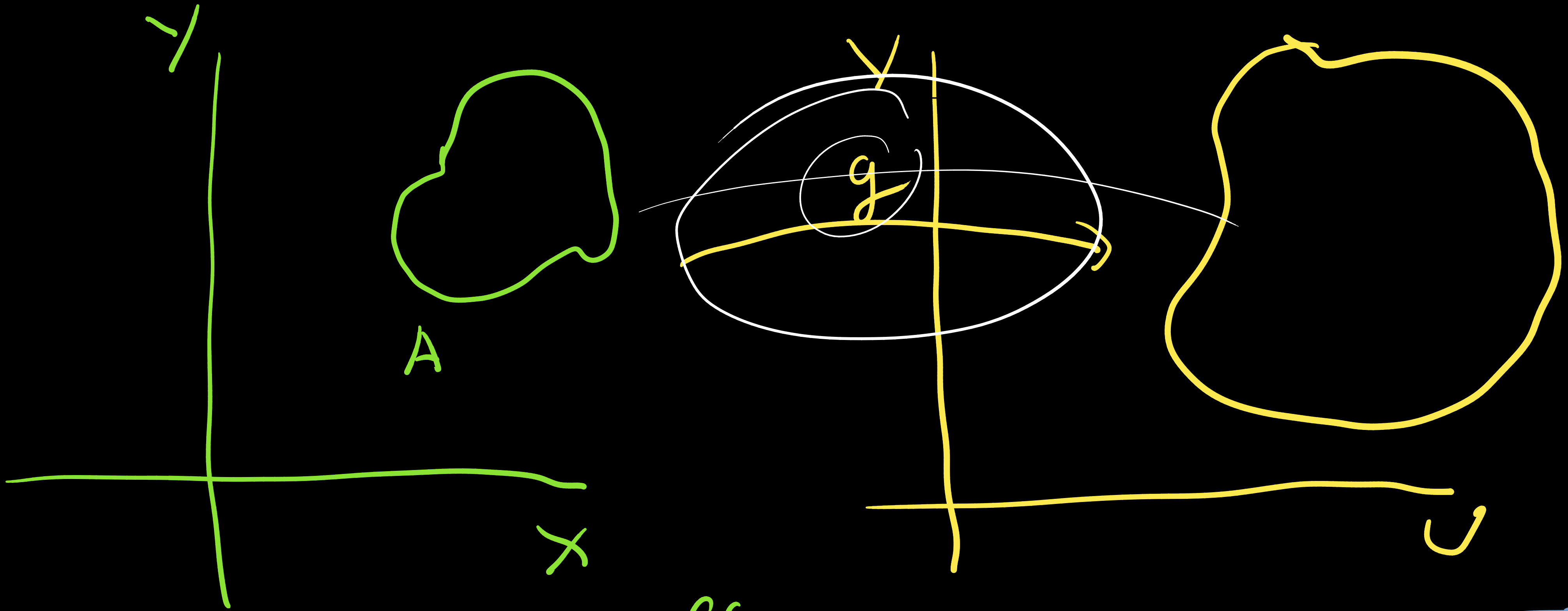
on $g(I)$

$$= f_X(g^{-1}(y)) \times \left|\frac{d}{dy} g^{-1}(y)\right|$$

How does this work in $\mathbb{R}^2$?



$$P((X,Y) \in A) = \iint_A f_{XY}(x,y)\,dx\,dy$$



$$P\{(X,Y)\in A\} = \iint_A f_{XY}(x,y)\,dx\,dy$$

$$P\{(U,V)\in g(A)\} = \iint f_{XY}\left[x^{-1}(u,v),\, y^{-1}(u,v)\right] du\,dv$$

too big if g stretches area

stretching by g

need to undo stretching g(A)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} = & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$= \iint_{g(A)} \frac{f_{XY}(g^{-1}(u,v))}{\underbrace{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|}_{\text{stretching by } g}} \, du\, dy$$

$$\iint_{g(A)} f_{XY}(g^{-1}(u,v)) \underbrace{\left\| J_{g^{-1}}(u,v) \right\|}_{\text{shrinking by } g^{-1}} \, du\, dv$$

So: $f_{UV}(u,v) = f_{XY}(g^{-1}(u,v)) \left\| \underbrace{J_{g^{-1}}(u,v)}_{\text{Jacobian matrix}} \right\|$

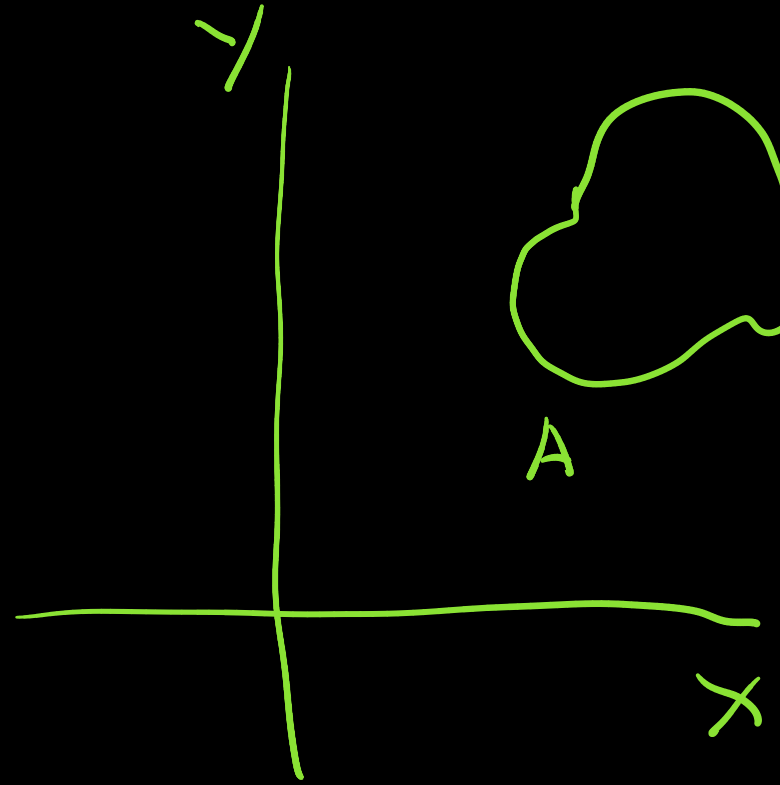
absolute value of Jacobian determinant

$\underbrace{\text{Jacobian matrix}}$ Jacobian determinant

$$(u, v) \leftarrow (x, y) \qquad (u, v) \rightarrow (x, y)$$

$$\left|\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}\right| = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right|^{-1}$$

What is $|J|$: Determinant of Jacobian Matrix



How much does g $\left(\begin{array}{c}\text{stretch}\\ \text{or}\\ \text{shrink}\end{array}\right)$ area?

$$J_g(x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$\underbrace{\left|\overbrace{J_g(x,y)}^{\text{determinant}}\right|}_{\text{absolute value}} = \underbrace{\left|\overbrace{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}^{\text{determinant}}\right|}_{\text{absolute value.}}$$

$$f_{X_1, X_2}(x_1, x_2)$$

$$\boxed{\begin{aligned} y_1 &= g_1(x_1, x_2) \\ y_2 &= g_2(x_1, x_2) \end{aligned}}$$

$$\begin{aligned} x_1 &= h_1(y_1, y_2) \\ x_2 &= h_2(y_1, y_2) \end{aligned}$$

1) $f_{X_1, X_2}(x_1, x_2)$ substitute (y_1, y_2)

2) X by $\left\| \dfrac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right\|$

or $\cancel{\frac{\overline{x}}{p}}$ by $\left\{ \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right| \right\}$

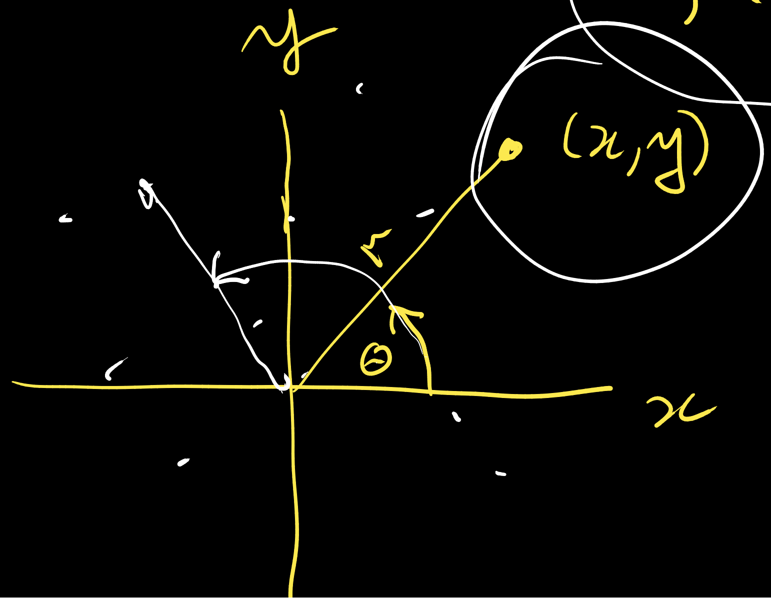
Example: Polar coordinates

Let (X,Y) have a bivariate standard normal distribution:

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$-\infty < x < \infty$

$-\infty < y < \infty$



Let g be transformation to polar co-ordinates.

$$r = \sqrt{x^2+y^2}$$

$$\theta = \tan2^{-1}(y, x)$$

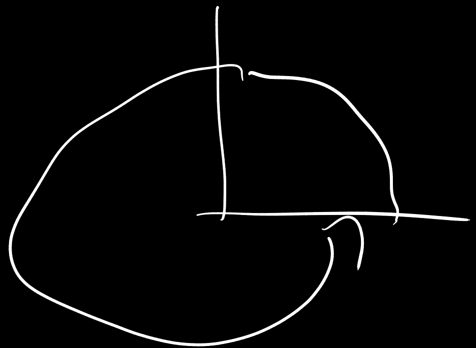
The arc-tangent of two arguments `atan2(y, x)` returns the angle between the x-axis and the vector from the origin to (x, y), i.e., for positive arguments `atan2(y, x) == atan(y/x)`.

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \\ \frac{\pi}{2}\,\mathrm{sgn}(y) & \text{if } x = 0,\ y \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$$

Inverse easier

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$r \geq 0, \quad 0 \leq \theta < 2\pi$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = g^{-1}\begin{pmatrix} r \\ \theta \end{pmatrix}$$

$$J_{g^{-1}}(r,\theta) = \begin{bmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{bmatrix}$$

$$\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| \text{ w.r.t.}$$

$$= \begin{bmatrix} \cos\theta & r \times (-\sin\theta) \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= \left|\begin{bmatrix} \cos\theta & r(-\sin\theta) \\ \sin\theta & r\cos\theta \end{bmatrix}\right|$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r \qquad r > 0$$

$$x = r\cos\theta$$

$$|J_{g^{-1}}| = r\cos^2\theta - r(-\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

$$f(x,y) = \frac{1}{2\pi} e^{-1/2(x^2+y^2)}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}r^2}$$

$$f(r,\theta) = \boxed{} \times \left\| \frac{\partial(x,y)}{\partial(r,\theta)} \right\|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}r^2} \times r$$

$\| J_{g^{-1}} \| = |r| = r$ since $r > 0$ anyways.

$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$$f_{R,\Theta}(r,\theta) = \left(\frac{1}{2\pi} e^{-\frac{1}{2}r^2} \right) \times r \,, \quad \underline{\begin{matrix} r > 0 \\ 0 < \theta < 2\pi \end{matrix}}$$

See where $\frac{1}{2\pi}$ came from!

Now let $Z = R^2$

If one variable stays the same you don't need to work out the Jacobian

$$\begin{pmatrix} z \\ \theta \end{pmatrix} = g\begin{pmatrix} R \\ \theta \end{pmatrix} = \begin{pmatrix} R^2 \\ \theta \end{pmatrix}$$

Note: Even if this is not 0 we still have $J_g = \frac{dz}{dr}$

$$J_g = \begin{bmatrix} \frac{dz}{dr} & \frac{dz}{d\theta} \\ \frac{d\theta}{dr} & \frac{d\theta}{d\theta} \end{bmatrix} = \begin{bmatrix} 2R & 0 \\ 0 & 1 \end{bmatrix} = 2R$$

$$f_{Z,\theta}(z,\theta) = \left(\text{Substitute } z,\theta \text{ in } f_{R,\theta}\right) \times \left|\frac{dr}{dz}\right|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}r^2} \times r \times \left|\frac{dr}{dz}\right|$$

$Z = R^2$

$\theta = \theta$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}z} \times \sqrt{z} \times \frac{1}{2} z^{-1/2}$$

$R = \sqrt{Z}$

$\frac{dR}{dz} = \frac{1}{2} Z^{-1/2}$

$$= \frac{1}{2\pi} \frac{1}{2} e^{-1/2\, z}, \quad z > 0,\ 0 < \theta < 2\pi$$

$$= \underbrace{f_\Theta(\theta)}_{U(0,2\pi)} \times \underbrace{f_Z(z)}_{\text{Exponential}(1/2)}$$

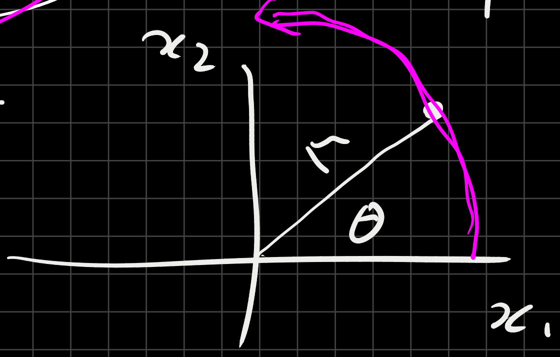
Independent.

Note: This gives us a way of generating random standard normals in pairs <u>even though</u> it is very difficult to generate them one at a time.

HOW?

Example for Wednesday Quiz

Let X_1, X_2 be independent with Exponential (λ) distribution.



Find the distribution of R and Θ where

$$\boxed{\begin{aligned} X_1 &= R\cos\Theta \\ X_2 &= R\sin\Theta \end{aligned}}$$

$$\boxed{f(x_1, x_2) = \lambda^2 e^{-\lambda x_1} e^{-\lambda x_2},} \quad x_1, x_2 > 0$$

Find $f_{R,\Theta}(r, \theta)$

How: 1) Substitute r, θ in $f(x_1, x_2)$

and 2) <u>multiply by:</u>

$$\| J_{g^{-1}} \| = \left\| \frac{\partial(x_1, x_2)}{\partial(r, \theta)} \right\| = \left\| \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} \right\|$$

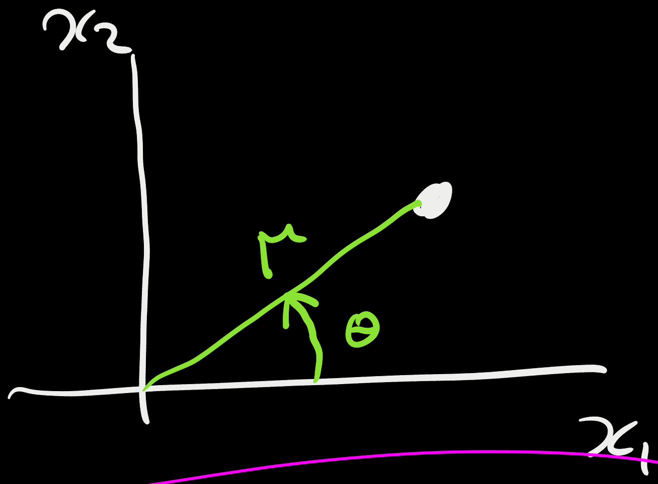
OR

<u>divide by:</u>

$$\| J_g \| = \left\| \frac{\partial(r, \theta)}{\partial(x_1, x_2)} \right\| = \left\| \begin{vmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{vmatrix} \right\|$$

$$f(x_1, x_2) = \lambda^2 e^{-\lambda x_1} e^{-\lambda x_2}, \quad x_1, x_2 > 0$$

$$x_1 = r\cos\theta$$
$$x_2 = r\sin\theta$$



Find $f_{R,\Theta}(r, \theta)$

$$\frac{\partial(x_1, x_2)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

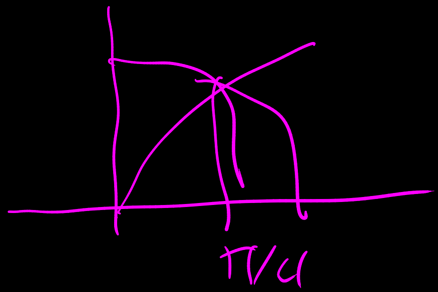
$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$f(x_1, x_2) = \lambda^2 e^{-\lambda x_1} e^{-\lambda x_2}, \quad x_1, x_2 > 0$$

$$x_1 = r\cos\theta$$
$$x_2 = r\sin\theta$$

$$f_{R,\theta}(r, \theta) = \lambda^2 e^{-\lambda r\cos\theta} e^{-\lambda r\sin\theta} \times r$$
$$= r\lambda^2 e^{-\lambda r(\cos\theta + \sin\theta)}$$

$$r > 0, \quad 0 < \theta < \frac{\pi}{2}$$



$$\frac{\sin\left(\alpha + \frac{\pi}{4}\right)}{\cos(\pi/4)} = \sin(\alpha)\cos\left(\frac{\pi}{4}\right) + \cos(\alpha)\sin\left(\frac{\pi}{4}\right)$$

2) divide by

$$\left\| J_g \right\| = \left\| \frac{\partial(r, \theta)}{\partial(x_1, x_2)} \right\| = \left\| \begin{matrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{matrix} \right\|$$