BEE 506 **Project: Electronic Filter Design** **Winter 2022**

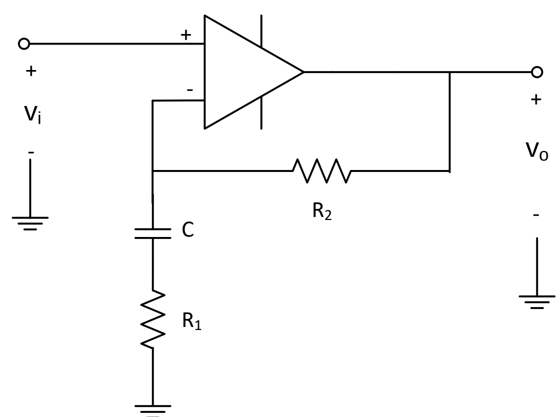
Due midnight on Friday, 3/11/2022 through Canvas.

Please work on this project on your own without help from an outside source.

Organize your project work in a Word file format. Show details of your calculations (handwritten is ok). Explain your work in detail, label and present the simulation models and plots in a way that they would be easily understood. Include your MATLAB and LTspice model and results in your submission through Canvas.

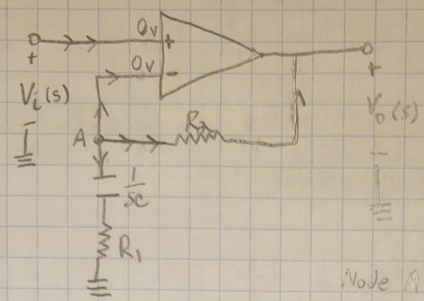
In this project you will derive the transfer function of an active filter circuit and characterize the filter behavior. Additionally, you will design the filter for given specifications and verify your design with LTspice and MATLAB.

1. In the active circuit shown below the op amp can be considered to be ideal. Analyze the circuit to derive the expression for the transfer function $H(s)=V_o(s)/V_i(s)$. <u>Hint</u>: In an ideal op amp the voltages at the inputs are equal, V+ = V-, and there are no currents going into the input terminals. You can apply nodal analysis at the common node between R_2 and C (op amp negative input).



Show that the transfer function is equal to the following (equation 1):

$$H(s) = \frac{V_o(s)}{V_i\,(s)} = \frac{R_1 + R_2}{R_1} \frac{s + \dfrac{1}{(R_1 + R_2)C}}{s + \dfrac{1}{R_1 C}}$$



Node $A = V_i(s)$

$$\frac{V_i(s)-V_0(s)}{R_2}+\frac{V_i(s)-0}{R_1+\frac{1}{sc}}=0$$

$$\frac{1}{R_2}\cdot\left(V_i(s)-V_0(s)\right)+\frac{1}{R_1+\frac{1}{sc}}\left(V_i(s)\right)=0$$

$$\frac{V_i(s)}{R_2}-\frac{V_0(s)}{R_2}+\left(V_i(s)\right)\frac{1}{R_1+\frac{1}{sc}}=0$$

$$\frac{V_i(s)}{R_2}+V_i(s)\left(\frac{1}{R_1+\frac{1}{sc}}\right)=\frac{V_o(s)}{R_2}$$

$$V_i(s)\left[\frac{1}{R_2}+\frac{1}{R_1+\frac{1}{sc}}\right]=\frac{V_o(s)}{R_2}$$

$$R_2\left[\frac{1}{R_2}+\frac{1}{R_1+\frac{1}{sc}}\right]=\frac{V_o(s)}{V_i(s)}$$

$$1+\frac{R_2}{R_1+\frac{1}{sc}}=\frac{V_o(s)}{V_i(s)}$$

$$\frac{R_1+\frac{1}{sc}}{R_1+\frac{1}{sc}}+\frac{R_2}{R_1+\frac{1}{sc}}=\frac{V_o(s)}{V_i(s)} \Rightarrow \frac{V_o(s)}{V_i(s)}=\frac{R_1+R_2+\frac{1}{sc}}{R_1+\frac{1}{sc}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_1 + R_2 + \frac{1}{sC}}{R_1 + \frac{1}{sC}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{R_1 sC + R_2 sC + 1}{sC}}{\frac{R_1 sC + 1}{sC}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_1 sC + R_2 sC + 1}{sC} \cdot \frac{sC}{R_1 sC + 1}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_1 sC + R_2 sC + 1}{R_1 sC + 1} \Rightarrow \frac{sC(R_1 + R_2) + 1}{R_1 sC + 1}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\left[\frac{sC(R_1+R_2)}{C(R_1+R_2)} + \frac{1}{C(R_1+R_2)}\right] C(R_1+R_2)}{R_1 sC + 1}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\left[s + \frac{1}{C(R_1+R_2)}\right] C(R_1+R_2)}{R_1 sC + 1} \Rightarrow \left[\frac{R_1 sC}{R_1 C} + \frac{1}{R_1 C}\right] R_1 C$$

$$= \left[s + \frac{1}{R_1 C}\right] R_1 C$$

$$\frac{V_o(s)}{V_i(s)} = \frac{C(R_1+R_2) \cdot \left[s + \frac{1}{C(R_1+R_2)}\right]}{\left[s + \frac{1}{R_1 C}\right] R_1 C} \Rightarrow \frac{(R_1+R_2)\left[s + \frac{1}{C(R_1+R_2)}\right]}{R_1 \left[s + \frac{1}{R_1 C}\right]}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{R_1 + R_2}{R_1} \cdot \frac{s + \frac{1}{C(R_1+R_2)}}{s + \frac{1}{R_1 C}}}$$

2. Choose numerical values for the resistors (in kΩ range) and the capacitor (in microfarad range). Choose $R_2 > R_1$. Substitute the values of the resistors and the capacitor in the equation above. Input the transfer function into MATLAB and generate its Bode plot. What type of filter does the circuit represent?

   This circuit represents a high pass filter.

$R_1 = 1\,k\Omega \qquad R_2 = 4\,k\Omega \qquad C = 1\,\mu F$

$$H(s) = \frac{R_1 + R_2}{R_1} \cdot \frac{s + \frac{1}{C(R_1+R_2)}}{s + \frac{1}{R_1 C}}$$

$$= \frac{(1\times10^3 + 4\times10^3)}{1\times10^3} \cdot \frac{s + \frac{1}{1\times10^{-6}(1\times10^3 + 4\times10^3)}}{s + \frac{1}{(1\times10^3)(1\times10^{-6})}}$$

$$\boxed{H(s) = 5 \cdot \frac{s + 200}{s + 1000}}$$

Command Window

```
>> num = [5 1000];
>> den = [1 1000];
>> H = tf(num, den)

H =

  5 s + 1000
  ----------
   s + 1000

Continuous-time transfer function.
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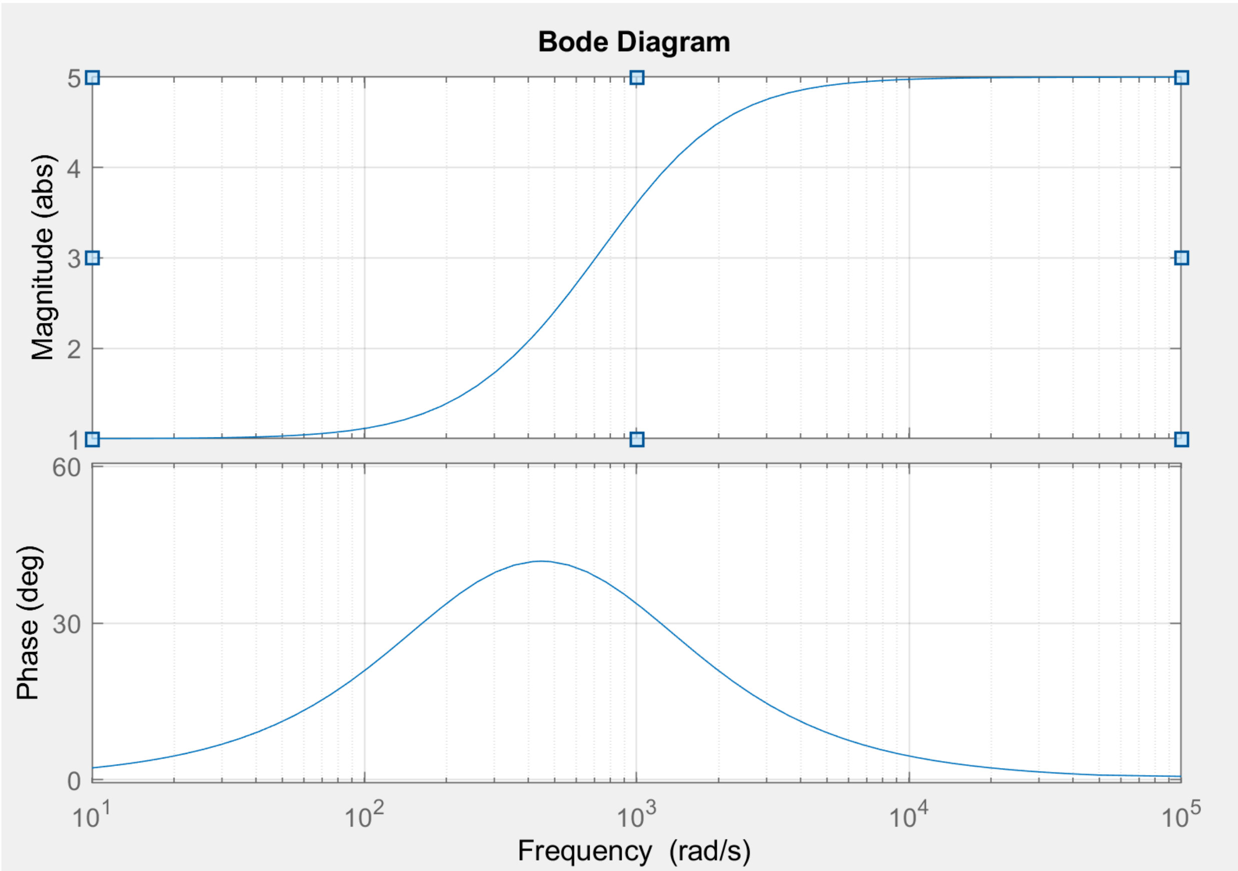


Bode Diagram
Magnitude (abs)
Phase (deg)
Frequency (rad/s)
5
4
3
2
1
60
30
0
10^1
10^2
10^3
10^4
10^5

3. Substitute s=jω in equation 1 above to obtain H(jω). Find the expression for the magnitude function of H(jω), |H(jω)|, in terms of R_1, R_2, C and ω. Evaluate |H(jω)| at ω=0 and at ω=infinity. Verify that these results agree with the numerical results from MATLAB.

After evaluating |H(jω)| at ω=0 and at ω=infinity I was able to verify that my results matched with my numerical results from MATLAB.

$$H(j\omega) = \frac{R_1+R_2}{R_1} \cdot \frac{j\omega + \frac{1}{C(R_1+R_2)}}{j\omega + \frac{1}{R_1 C}}$$

$$\frac{1}{C(R_1+R_2)} + j\omega\left(\frac{C(R_1+R_2)}{C(R_1+R_2)}\right)$$

$$\text{Num} = \frac{1 + (R_1+R_2)j\omega C}{C(R_1+R_2)}$$

$$\text{Den} = \frac{1}{R_1 C} + j\omega\left(\frac{R_1 C}{R_1 C}\right) = \frac{1 + R_1 C (j\omega)}{(R_1 C)}$$

$$= \frac{1 + (R_1+R_2)j\omega C}{\cancel{C(R_1+R_2)}} \cdot \frac{\cancel{R_1 C}}{1 + R_1 C (j\omega)} \cdot \frac{\cancel{R_1+R_2}}{\cancel{R_1}}$$

$$\boxed{H(j\omega) = \frac{1 + (R_1+R_2)j\omega C}{1 + R_1 j\omega C}}$$

$$H(j\omega) = \frac{1 + j\omega C(R_1 + R_2)}{1 + j\omega C R_1}$$

$$\| H(j\omega) \| = \frac{\sqrt{1^2 + (j\omega C)^2 (R_1 + R_2)^2}}{\sqrt{1^2 + (j\omega C)^2 (R_1)^2}}$$

$$\| H(j\omega) \| = \frac{\sqrt{1 + (\omega C)^2 (R_1 + R_2)^2}}{\sqrt{1 + (\omega C)^2 (R_1)^2}}$$

$$\boxed{\| H(j\omega) \| \Big|_{\omega = 0} = \frac{\sqrt{1 + 0}}{\sqrt{1 + 0}} = 1}$$

$$\| H(j\omega) \| = \frac{\sqrt{\omega^2 \left[\frac{1 + \omega^2 C^2 (R_1 + R_2)^2}{\omega^2} \right]}}{\sqrt{\omega^2 \left[\frac{1 + \omega^2 C^2 (R_1)^2}{\omega^2} \right]}} = \frac{\sqrt{\omega^2 \left[\frac{1}{\omega^2} + C^2 (R_1 + R_2)^2 \right.}}{\sqrt{\omega^2 \left[\frac{1}{\omega^2} + C^2 R_1^2 \right.}}$$

$$= \frac{\omega \sqrt{\frac{1}{\omega^2} + C^2 (R_1 + R_2)^2}}{\omega \sqrt{\frac{1}{\omega^2} + C^2 R_1^2}} = \frac{\sqrt{\frac{1}{\omega^2} + C^2 (R_1 + R_2)^2}}{\sqrt{\frac{1}{\omega^2} + C^2 R_1^2}}$$

$$\| H(j\omega) \| \Big|_{\omega \to \infty} = \frac{\sqrt{0 + C^2 (R_1 + R_2)^2}}{\sqrt{0 + C^2 R_1^2}} = \boxed{\frac{R_1 + R_2}{R_1}}$$

$$\| H(j\omega) \| \Big|_{\omega \to \infty} \frac{R_1 + R_2}{R_1} \Rightarrow \frac{1000 + 4000}{1000} \boxed{= 5}$$

4. The filter cutoff frequency is the frequency where the magnitude is equal to its maximum value divided by sqrt(2) (or 0.7071 $|H(j\omega)|_{max}$). Show that for $R_2 > R_1$ the expression for the filter cutoff frequency ω_c is equal to the following:

$$w_c = \frac{1}{C}\sqrt{\frac{1}{R_1^2} - \frac{2}{(R_1+R_2)^2}}$$

#4

$$\left[\frac{R_1+R_2}{\sqrt{2}R_1}\right]^2 = \left[\frac{\sqrt{1+\omega_c^2(R_1+R_2)^2C^2}}{\sqrt{1+\omega_c^2R_1^2C^2}}\right]^2$$

$$\frac{(R_1+R_2)^2}{2R_1^2} = \frac{1+\omega_c^2(R_1+R_2)^2C^2}{1+\omega_c^2R_1^2C^2}$$

$$(R_1+R_2)^2\left[1+\omega_c^2R_1^2C^2\right] = 2\cdot R_1^2\left[1+\omega_c^2(R_1+R_2)^2C^2\right]$$

$$(R_1+R_2)^2 + (R_1+R_2)^2\omega_c^2R_1^2C^2 = 2R_1^2 + \omega_c^2(R_1+R_2)^2C^2\,2R_1^2$$

LHS:

$$\left[R_1^2+R_2^2+2R_1R_2\right] + R_1^4\omega_c^2c^2 + \omega_c^2R_1^2R_2^2C^2 + 2R_1^3C^2\omega_c^2R_2$$

$$\left[R_1^2+R_2^2+2R_1R_2\right] + (\omega_c^2R_1^2C^2)\left[R_1^2+R_2^2+2R_1R_2\right]$$

LHS: $(\omega_c^2R_1^2C^2)\left[R_1^2+R_2^2+2R_1R_2\right] + R_1^2+R_2^2+2R_1R_2$

RHS: $2R_1^2 + \left[R_1^2+R_2^2+2R_1R_2\right]\omega_c^2C^2\,2R_1^2$

$$2R_1^2 + \omega_c^2C^2\,2R_1^4 + \omega_c^2C^2\,2R_1^2R_2^2 + 4R_1^3R_2\,\omega_c^2C^2$$

$$2R_1^2 + (2R_1^2\omega_c^2C^2)\left[R_1^2+R_2^2+2R_1R_2\right]$$

$$(\omega_c^2R_1^2C^2)\left[R_1^2+R_2^2+2R_1R_2\right] + \left[R_1^2+R_2^2+2R_1R_2\right] = (2R_1^2\omega_c^2C^2)\left[R_1^2+R_2^2+2R_1R_2\right] + 2R_1^2$$

$$(R_1^2+R_2^2+2R_1R_2)\left[\omega_c^2R_1^2C^2+1\right] = (2R_1^2\omega_c^2C^2)(R_1^2+R_2^2+2R_1R_2) + 2R_1^2$$

$$\cancel{(R_1^2+R_2^2+2R_1R_2)\left[\omega_c^2R_1^2C^2+1\right]}$$

4 continued

$$(R_1^2 + R_2^2 + 2R_1R_2)\left[\omega_c^2 R_1^2 C^2 + 1\right] = (2R_1^2\omega_c^2C^2)(R_1^2+R_2^2+2R_1R_2) + 2R_1^2$$

$$\left[\omega_c^2 R_1^2 C^2 + 1\right] = 2R_1^2\omega_c^2C^2 + \frac{2R_1^2}{R_1^2+R_2^2+2R_1R_2}$$

$$\omega_c^2 R_1^2 C^2 + 1 = R_1^2\left[2\omega_c^2C^2 + \frac{2}{R_1^2+R_2^2+2R_1R_2}\right]$$

$$\frac{\omega_c^2 R_1^2 C^2 + 1}{R_1^2} = 2\omega_c^2C^2 + \frac{2}{R_1^2+R_2^2+2R_1R_2}$$

$$\frac{\omega_c^2 \cancel{R_1^2} C^2}{\cancel{R_1^2}} + \frac{1}{R_1^2} = 2\omega_c^2C^2 + \frac{2}{R_1^2+R_2^2+2R_1R_2}$$

$$\omega_c^2C^2 + \frac{1}{R_1^2} = 2\omega_c^2C^2 + \frac{2}{R_1^2+R_2^2+2R_1R_2}$$

$$\frac{1}{R_1^2} - \frac{2}{(R_1+R_2)^2} = 2\omega_c^2C^2 - \omega_c^2C^2$$

$$\frac{1}{R_1^2} - \frac{2}{(R_1+R_2)^2} = \omega_c^2C^2$$

$$\sqrt{\omega_c^2C^2} = \sqrt{\frac{1}{R_1^2} - \frac{2}{(R_1+R_2)^2}}$$

$$\omega_c C = \sqrt{\frac{1}{R_1^2} - \frac{2}{(R_1+R_2)^2}}$$

$$\boxed{\omega_c = \frac{1}{C}\sqrt{\frac{1}{R_1^2} - \frac{2}{(R_1+R_2)^2}}}$$

5. Design the filter circuit for a cutoff frequency of 2000 Hz and a gain of 10 (or $V_o/V_i=10$) in its passband. This step involves choosing values for R_1, R_2 and C to meet the given specifications.

5

$$\frac{V_o}{V_i} = \frac{R_1+R_2}{R_1} \Rightarrow \frac{R_1+R_2}{R_1} = 10$$

Let $R_1 = 10\ k\Omega$

$$\frac{10^4+R_2}{10^4} = 10$$

$$10^4 + R_2 = 10 \cdot 10^4$$

$$10^4 + R_2 = 10^5$$

$$R_2 = 10^5 - 10^4$$

$$R_2 = 90{,}000\ \Omega$$

$$12566.37 = \frac{1}{C}\sqrt{\frac{1}{(10000)^2} - \frac{2}{(10000+90000)^2}}$$

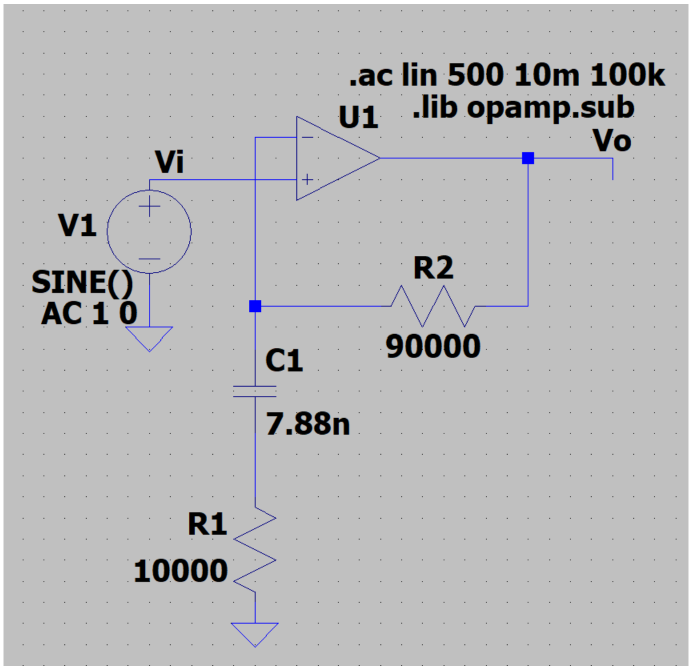
$$12566.37 = \frac{1}{C}\left[9.8995\times10^{-5}\right]$$

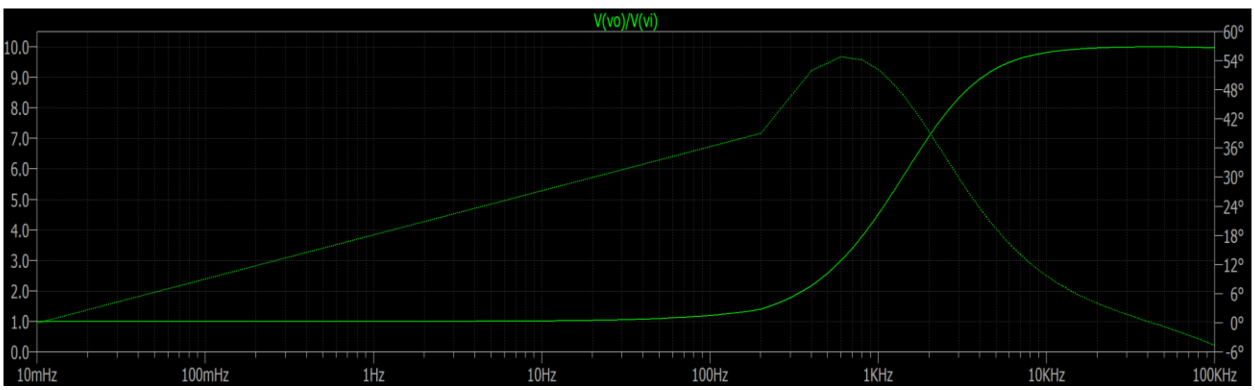
$$\frac{12566.37}{9.8995\times10^{-5}} = \frac{1}{C}$$

$$\frac{1}{C} = 126939512.5$$

$$C = \frac{1}{126939512.5} \Rightarrow \boxed{\begin{aligned} R_1 &= 10\ k\Omega \\ R_2 &= 90\ k\Omega \\ C &= 7.88\times10^{-9} \end{aligned}}$$

6. Verify your filter design with MATLAB and LTspice. Include your model and Bode plot with your submission.





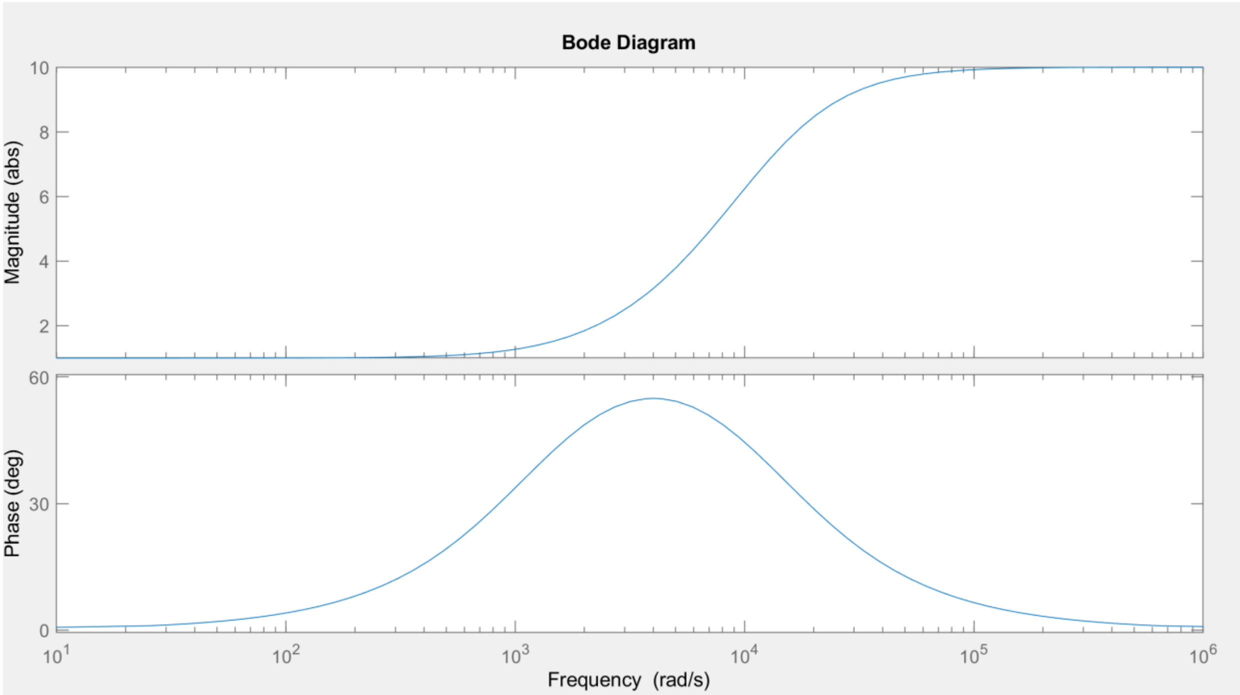

```
>> num = [10 12690.35533];
>> den = [1 12690.35533];
>> H = tf(num, den)

H =

  10 s + 1.269e04
  ---------------
   s + 1.269e04

Continuous-time transfer function.

>> bode(H)
>>
```



Bode Diagram
Magnitude (abs)
Phase (deg)
Frequency (rad/s)
10
8
6
4
2
60
30
0
10^1
10^2
10^3
10^4
10^5
10^6