Stability and Performance Analysis of Randomized Blended Control Under Novel Disturbance

Paper number 4171: Proofs

1 Notation and Preliminaries

1.1 System Model

We define a discrete-time dual switching linear system Ψ, a class of switched linear systems with two (either stochastic or deterministic) switching variables, as follows:

Definition 1 (State-Space model Ψ) *A discrete-time dual switching linear system Ψ is described by*

$$\boldsymbol{x}(k+1) = A_{\sigma(k)}^{\gamma(k)}\boldsymbol{x}(k) + B_{\sigma(k)}^{\gamma(k)}\boldsymbol{u}(k) \quad (1)$$

$$\boldsymbol{y}(k+1) = C_{\sigma(k)}^{\gamma(k)}\boldsymbol{x}(k) \quad (2)$$

where k is the discrete time index, $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state, $\boldsymbol{u}(k) \in \mathbb{R}^m$ is a control input, $\boldsymbol{y}(k) \in \mathbb{R}^p$ is the performance output, $\sigma(k)$ and is a time homogeneous Markov process taking values in the set $\Upsilon = \{\upsilon_1, \upsilon_2, \cdots \upsilon_V\}$, with transition probability matrix Λ, $\gamma(k)$ is a switching signal taking values in a finite set $\mathcal{U} = \{\boldsymbol{u}_1, \omega_2, \cdots \boldsymbol{u}_M\}$, with transition probability matrix Π. More precisely, the entry $\lambda_{ij} \geq 0$ of Λ represents the probability of a transition from mode υ_i to mode υ_j, namely $\lambda_{ij} = P\{\sigma(k+1) = \upsilon_j | \sigma(k) = \upsilon_i\}$. Λ is a right-stochastic matrix.

A matrix, e.g., $A_{\sigma(k)}^{\gamma(k)}\boldsymbol{x}(k)$, defines the dynamics for Ψ at time k given system mode $\sigma(k)$ and control mode $\gamma(k)$. If we define $\rho(k)$ as the process probability distribution at time k, then its evolution is governed by the difference equation $\rho(k+1)' = \rho(k)'\Lambda$, where $\rho(0) = \rho_0$. If we further assume that Λ and Π are irreducible and aperiodic Markov matrices, then the Markov process admits a unique stationary (strictly positive) probability distribution; for example, the distribution $\bar{\rho}$ satisfies $\bar{\rho}' = \bar{\rho}'\Lambda$. The state dynamics of the overall system Ψ is characterized by a set of tuples (A_i^j, B_i^j, C_i^j), $i \in \Upsilon$, $j \in \mathcal{U}$.

Symbol	Key
$\boldsymbol{x}(k)$	state vector
$\boldsymbol{u}(k)$	control vector
$\boldsymbol{y}(k)$	observation vector
$\sigma(k)$	process switching function
Λ	process probability matrix
$\boldsymbol{\upsilon}(k)$	process mode vector
$\boldsymbol{\alpha}(k)$	process parameter vector
$\gamma(k)$	control switching function
Π	control probability matrix
$\boldsymbol{\varphi}(k)$	control mode blend distribution
$\tilde{\mathcal{T}}_{1:T}$	reference task over time $1:T$
$\xi_{1:T}$	loss function over time $1:T$

Table 1: Notation

2 Theoretical Results

This section summarises our theoretical results. We summarise our results as follows:

1. *Stability*: given a system Ψ subject to stochastic disturbance, we show that there exists system matrices A and B such that Ψ converges to a stable state, which we define in terms of mean-square stability (MSS).
2. *Stabilizability*: given a system Ψ subject to stochastic disturbance, there exists a switching function γ that generates a control sequence $\{\gamma\}_{k\in\mathbb{N}}$ that guarantees MSS.
3. *Performance given known modes Υ*: there exists a randomized blended control sequence that converges to the performance of a reference controller.
4. *Performance given unknown modes Υ'*: there exists a randomized blended control sequence that converges to the performance of a reference controller if the worst-case loss is bounded for any unknown mode $\breve{\upsilon} \in \Upsilon'$.

2.1 Stability Properties

Assessing stability of system Ψ given switching due to system disturbance process $\{\sigma(k)\}_{k\in\mathbb{N}}$ is a robust stability problem. The standard approach defines stochastic stability, in terms of mean-square stability (MSS), which states that the expectation of the system state norm asymptotically converges to zero.

Definition 2 (Mean Square Stability) *A system Ψ exhibits mean square stability (MSS) if for any initial condition $(\boldsymbol{x}_0, \upsilon_0)$, and any switching sequence $(\sigma_k)_{k\in\mathbb{N}}$,*

$$\lim_{k\to\infty} E(\| \boldsymbol{x}(k) \|^2) \to 0. \quad (3)$$

Using this definition, we can state the conditions for stability in terms of the system matrix $A_{\sigma(k)}$, independent of the control switching process.

Lemma 1 (Mean Square Stability) *A system Ψ exhibits mean square stability (MSS) if $\exists$ a system matrix $A_{\sigma(k)}$ such that for any initial condition $(\boldsymbol{x}_0, \upsilon_0)$, and any switching sequence $\{\sigma(k)\}_{k\in\mathbb{N}}$, equation 3 holds.*

Proof: We begin be writing the transition probability for the Markov chain $\{\sigma(k)\}_{k\in\mathbb{N}}$ as $p_{ij} = Pr(\upsilon_{k+1} = j|\upsilon_k = i)$. We can further define a symmetric matrix $M_{(l,i)} \in \mathbb{R}^{n\times n}$ such that $l \in \mathcal{U}$ and $i \in \Upsilon$.

For any initial condition $(\boldsymbol{x}_0, \upsilon_0)$ and any switching sequence p_{ij}, equation 3 holds if the following two properties are satisfied $\forall l, m \in \mathcal{U}$ [Etienne *et al.*, 2019]:

$$\begin{aligned} M_{(l,i)} &> 0 \\ \sum_{j\in\Upsilon} p_{ij} A^T_{(l,i)} M_{(m,j)} A^T_{(l,i)} - M_{(l,i)} &< 0 \end{aligned}$$

□

[Costa *et al.*, 2006] also prove this Lemma, using the matrix M based on possible system trajectories.

2.2 Stabilizability Properties

Assessing stability of system Ψ given switching due to control inputs $\{\gamma(k)\}_{k\in\mathbb{N}}$ is a stabilization problem and requires assessing the properties of a switching control law.

Lemma 2 (Mean Square Stabilizability [Etienne *et al.*, 2019]) *A system Ψ (Definition 1) is mean square stabilizable if there exists a switching control law $\{\gamma(k)\}_{k\in\mathbb{N}}$ defined as a time homogeneous Markov chain such that, for any initial condition $(\boldsymbol{x}_0, \gamma_0)$, equation 3 holds.*

Proof: We can prove this property by rewriting the state equation of Definition 1 such that $\boldsymbol{u}(k) = K^{\gamma(k)}_{\sigma(k)}\boldsymbol{x}(k)$, and defining the matrix $\tilde{A}^{\gamma(k)}_{\sigma(k)} = A^{\gamma(k)}_{\sigma(k)} + B^{\gamma(k)}_{\sigma(k)} K^{\gamma(k)}_{\sigma(k)}$. This enables us to rewrite the state equation of Definition 1 as $\boldsymbol{x}(k+1) = \left[A^{\gamma(k)}_{\sigma(k)} + B^{\gamma(k)}_{\sigma(k)} K^{\gamma(k)}_{\sigma(k)}\right]\boldsymbol{x}(k) = \tilde{A}^{\gamma(k)}_{\sigma(k)}\boldsymbol{x}(k)$.

We can prove our Lemma using properties analogous to the properties in Lemma 1, where we replace A in Lemma 1 with $\tilde{A}$. □

2.3 Performance Computation

This section provides a theoretical basis for assessing the performance of randomised switching.

Distance Metrics for Randomised Switching

Because we allow randomised controllers, we must be able to compute properties of stochastic controllers. For this purpose, we will use the Wasserstein distance [Panaretos and Zemel, 2019], which defines a metric on distributions that enables us to analyse the convergence of a time-varying distribution of a system Ψ with respect to a reference distribution. The Wasserstein metric is a natural way to compare the probability distributions of two variables Y and Z, where one variable is derived from the other by small, non-uniform perturbations.

Definition 3 *The r-Wasserstein distance between probability measures μ and ν on $\mathbb{R}^d$ is defined as*

$$W_r(\mu, \nu) := \left(\inf_{X\sim\mu,\, Y\sim\nu} E \parallel X - Y \parallel^r\right)^{1/r}, \quad r \geq 1 \quad (4)$$

where the infimum is taken over all pairs of d-dimensional random vectors X and Y marginally distributed as μ and ν, respectively.[1]

In the following, we want to compare the expected loss of the randomized methods against the reference. For a loss function $\xi(\boldsymbol{u}(k), \boldsymbol{\alpha}(k)) \in [0, 1]$ and a stochastic blend distribution over $\boldsymbol{\varphi}(k)$, the expected loss is $\mathscr{L}(k) \triangleq E\left[\xi\left(\boldsymbol{u}(k), \boldsymbol{\alpha}(k)\right)\right] = \xi\left(\boldsymbol{u}(k), \boldsymbol{\alpha}(k)\right)\varphi(k)$. Since the reference control is deterministic, we can define it using a Dirac delta distribution, and use Lemma 3 for performance comparisons between reference and randomized controllers.

Lemma 3 *[Lee* et al.*, 2015] If we fix the Dirac distribution as a reference measure, then distributional convergence in the Wasserstein W_2 metric is a necessary and sufficient measure for MS convergence.*

Proof: Consider a sequence of N-dimensional joint PDFs $\{\rho(x)^{\infty}_{j=1}\}$, which converges to $\delta(x)$ in distribution, i.e., $\lim_{j\to\infty} \mathcal{W}(\rho_j(x), \delta(x)) = 0$. From (2), we have

$$\begin{aligned} \mathcal{W}^2(\rho_j(x), \delta(x)) &= \inf_{\varsigma\in\mathcal{P}_2(\rho_j(x),\delta(x))} E\left[\parallel X_j - 0 \parallel^2_{\ell_2(\mathbb{R}^n)}\right] \\ &= E\left[\parallel X_j \parallel^2_{\ell_2(\mathbb{R}^n)}\right] \end{aligned}$$

where the random variable $X_j \sim \rho_j(x)$. The last equality follows from the fact that $\mathcal{P}_2(\rho_j(x), \delta(x)) = \{\rho(x)_j\}\forall j$, thus the infimum is obviated. From (3),

$$\lim_{j\to\infty} \mathcal{W}(\rho_j(x), \delta(x)) = 0 \;\Rightarrow\; \lim_{j\to\infty} E\left[\parallel X_j \parallel^2_{\ell_2(\mathbb{R}^n)}\right] = 0.$$

Hence, distributional convergence to $\delta(x))$ implies MS convergence. Conversely, MS convergence distributional convergence, is well-known (Grimmett & Stirzaker, 1992) and unlike the other direction, holds for an arbitrary reference measure. □

Expected Loss for Randomised Switching

Assume that we must complete a task that requires Ψ to execute control sequence $\boldsymbol{u}_{1:T}$. In the following, we assume that we have a reference controller that executes the ideal control sequence, characterized by $\{\boldsymbol{\varphi}\}_{1:T}$. Expressing relative loss in terms of a difference of control distributions, as in Lemma 4, enables us to show convergence of randomized and reference controllers.

Lemma 4 *The relative loss $\Delta(\tilde{\varphi}_{1:T}, \varphi_{1:T})$ for two randomised switching sequences, $\varphi_{1:T}$ and $\tilde{\varphi}_{1:T}$, can be expressed such that*

$$\Delta(\tilde{\varphi}_{1:T}, \varphi_{1:T}) \propto W_2(\tilde{\varphi}_{1:T}, \varphi_{1:T}). \quad (5)$$

Proof Sketch: We can express relative expected loss as

$$\begin{aligned} \Delta(\tilde{\boldsymbol{u}}_{1:T}, \boldsymbol{u}_{1:T}) &= E \parallel \xi^{\tilde{\varphi}}_{1:T} - \xi^{\varphi}_{1:T} \parallel \\ &= E \parallel \tilde{\boldsymbol{u}}_{1:T}\tilde{\varphi}_{1:T} - \boldsymbol{u}_{1:T}\varphi_{1:T} \parallel \\ &= E \parallel \boldsymbol{u}_{1:T}\left(\tilde{\varphi}_{1:T} - \varphi_{1:T}\right) \parallel \\ &= \left[E \parallel \boldsymbol{u} \parallel\right]\left[E \parallel \tilde{\boldsymbol{u}}_{1:T}\tilde{\varphi}_{1:T} - \boldsymbol{u}_{1:T}\varphi_{1:T} \parallel\right] \\ &\propto E \parallel \tilde{\varphi}_{1:T} - \varphi_{1:T} \parallel \\ &\propto W_2(\tilde{\varphi}_{1:T},\, \varphi_{1:T}). \end{aligned}$$

[1] This is an obviously non-empty set, since one can always construct independent random variables with prescribed marginals.

We will use Lemmas 3 and 4 to show MS convergence of the loss of randomized and reference controllers.

2.4 Performance: Known Mode Set Υ

This section shows results for relative performance of reference and randomized controllers when the system mode switching covers a set Υ of known system modes.

When the system modes are known and appropriate controllers are tuned for these modes, we can show that the loss of a randomized control system converges (in expectation) to that of a reference control system. To do this, we first can show that the relative loss for randomised control is equivalent to the distance between the reference and randomised distribution sequences.

Lemma 5 *For a task $\mathcal{T}_{1:T}$ and randomised and reference distribution sequences, $\{\tilde{\boldsymbol{\varphi}}(k)\}_{k\in\mathbb{N}}$ and $\{\boldsymbol{\varphi}(k)\}_{k\in\mathbb{N}}$ respectively, the relative loss is given by $W_2(\tilde{\varphi}_{1:T}, \varphi_{1:T}) = 0$ as $k \to \infty$.*

Proof: From Lemma 4, we can express the relative loss for $\{\tilde{\boldsymbol{\varphi}}(k)\}_{k\in\mathbb{N}}$ and $\{\boldsymbol{\varphi}(k)\}_{k\in\mathbb{N}}$ as $W_2(\tilde{\varphi}_{1:T}, \varphi_{1:T})$. We can view the reference controller as a Dirac delta distribution at each $k \in \mathbb{N}$, since its value is a single control signal. Since we are computing the W_2-distance between a Dirac delta distribution and a distribution, we know (by Lemma 3), that $\lim k \to \infty W_2(\tilde{\varphi}_{1:k}, \varphi_{1:k}) = 0$. □

We use the next lemma to show the necessary and sufficient conditions for mean square convergence (MSC) given a randomised controller.

Lemma 6 *Given a system Ψ and system switching sequence $\{\sigma(k)\}_{k\in\mathbb{N}}$ with switching probability $\pi(k)$, Ψ is MS-stable if and only if the matrix $A^j_{\sigma(i)}$ at time step i (given controller switching mode i) satisfies*

$$\lim_{k\to\infty} \left[\prod_{i=k}^{1} \left(\sum_{j=1}^{m} \pi_j(i)(A^j_{\sigma(i)} \otimes A^j_{\sigma(i)}), \right) \right] \to 0.$$

Proof: We start by using a result (Theorem 1) from [Lee *et al.*, 2015], which provides the following identity for W^2:

$$W^2(k) = [I_n]^T \Gamma(k) [\hat{\mu}(0)\hat{\mu}(0)^T + \hat{\Sigma(0)}, \qquad (6)$$

where

$$\Gamma(k) = \left[\prod_{i=k}^{1} \left(\sum_{j=1}^{m} \pi_j(i)(A^j_{\sigma(i)} \otimes A^j_{\sigma(i)}) \right) \right]$$

, I_n is the $n \times n$ identity matrix, and $\hat{\mu}(0)$ and $\hat{\Sigma(0)}$ are the mean and the covariance of the distribution of the pdf $\pi(k)$.

We now must justify the necessary and sufficient properties of our claim.

$\Rightarrow$: If

$$\lim_{k\to\infty} \Gamma(k) = 0,$$

this implies that $W^2(k) \to 0$ as $k \to \infty$, which in turn implies that $W \to 0$.

$\Leftarrow$: We prove this part of the claim by contradiction. Suppose that

$$\lim_{k\to\infty} \Gamma(k) \neq 0.$$

Then W never reaches 0 (by equation 6), which then implies that MS stability does not hold. Hence we have a contradiction. □

These two previous lemmas enable us to show the mean square convergence (MSC) of randomised control.

Lemma 7 *$\exists$ a system Ψ (defined by Lemma 6) and randomised controller sequence $\{\boldsymbol{\varphi}(k)\}_{k\in\mathbb{N}}$ such that mean square convergence (MSC) is guaranteed for Υ known.*

Proof: This Lemma follows from Lemmas 6 and 7. If we assume that a reference controller guarantees convergence, then Lemma 6 states that the randomized controller is MSC, and Lemma 7 shows that the system Ψ must be definable in terms of the matrix A. □

2.5 Performance: Unknown Mode Set

In the case when the system modes $\breve{\upsilon} \in \Upsilon$ may be unknown, we must introduce a notion of bounded loss to provide reasonable guarantees for any controller. By this we ensure that, for any controller, the instantaneous loss for any unknown system mode $\breve{\upsilon} \in \Upsilon$ does not cause total failure of the system Ψ. We thus add an extra task constraint to Definition **??**, i.e., a bound ϵ on the instantaneous loss that corresponds to a system crash, e.g., due to a fault for which no recovery action is possible.

Definition 4 *Bounded instantaneous loss is a loss such that: $\xi(\boldsymbol{u}(k), \boldsymbol{\alpha}(k)) < \epsilon$ for $k = 1, ..., T$ and $\epsilon \in [0, 1]$.*

Given this additional constraint, we can prove the following result using Lemma 7:

Lemma 8 (Unknown Mode MSC) *$\exists$ a system Ψ and randomised controller sequence $\boldsymbol{\varphi}_{1:T}$ such that Ψ converges to the performance of a reference controller if the worst-case loss is bounded for any unknown mode $\breve{\upsilon} \in \Upsilon'$.*

Proof: We start by assuming a bound on worst-case loss; this guarantees that the disturbance will not cause even a reference controller to be unable to stabilize the system. If the reference controller guarantees stabilizability, then by Lemma 6 the randomized controller will also guarantee stabilizability. Further, by Lemma 7 we know there must exist a system (defined in terms of matrix A) such that MSC is guaranteed. □

References

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