Parabella

Solution

We know that

$A = (a, a^2)$,

$B = (b, b^2)$,

$C = (c, c^2)$.

We can calculate the gradient of a straight line between two points (x_1, y_1) and (x_2, y_2) as $\frac{y_1-y_2}{x_1-x_2}$.

So the gradient of line AB is equal to $\frac{a^2-b^2}{a-b} = a + b$.

Also, the gradient of line OC is equal to $\frac{0-c^2}{0-c} = c$.

However, we know from the question that these two lines are parallel. Two lines are parallel if and only if they have the same gradient, which means that we must have $a + b = c$.

Now, suppose that we have two points $D = (d, d^2)$ and $E = (e, e^2)$ on the parabola making another line parallel to OC.

The gradient of line DE is equal to $\frac{d^2-e^2}{d-e} = d + e$. As DE is parallel to OC, we also have $d + e = c$.

We can calculate the midpoint of a straight line between two points (x_1, y_1) and (x_2, y_2) as $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

So the midpoint of line OC is

$$\left(\frac{0+c}{2}, \frac{0+c^2}{2}\right) = \left(\frac{c}{2}, \frac{c^2}{2}\right).$$

The midpoint of line AB is

$$\left(\frac{a+b}{2}, \frac{a^2+b^2}{2}\right) = \left(\frac{c}{2}, \frac{a^2+b^2}{2}\right).$$

The midpoint of line DE is

$$\left(\frac{d+e}{2}, \frac{d^2+e^2}{2}\right) = \left(\frac{c}{2}, \frac{d^2+e^2}{2}\right).$$

All three of these points have an x-coordinate of $\frac{c}{2}$. This means that the line $x = \frac{c}{2}$ passes through the midpoints of the lines OC, AB and DE.

Relevance

G2 What is the connection between algebra and geometry, and how can we exploit it?

