A Diophantine equation

Solution

Remember that we write $x \mid y$ to mean that x divides y.

We say that two whole numbers are coprime if their highest common factor is 1.

We require $a^2 = 2b^3$, so $2 \mid a^2$. Since 2 is prime, we must have $2 \mid a$. Similarly, $a^2 = 3c^5$, so $3 \mid a^2$ and because 3 is prime we must have $3 \mid a$.

Therefore, let $a = 2^{\alpha_1}3^{\alpha_2}a_1$ where α_1 and α_2 are positive integers and a_1 is a positive integer coprime to 2 and 3.

When we substitute this representation of a into the given equation, we obtain

$$2^{2\alpha_1}3^{2\alpha_2}a_1^2 = 2b^3 = 3c^5.$$

Now we see that $2^{2\alpha_1} \mid 2b^3$, and because α_1 is a positive integer there are at least two factors of 2 in $2^{2\alpha_1}$. Therefore b^3 must be divisible by 2, and because 2 is prime, $2 \mid b$. Also, $3^{2\alpha_2} \mid 2b^3$, and because 3 is prime we have $3 \mid b$.

Similarly $2 \mid c$ and $3 \mid c$, so let

$$b = 2^{\beta_1}3^{\beta_2}b_1$$

and

$$c = 2^{\gamma_1}3^{\gamma_2}c_1$$

where β_1, β_2, γ_1 and γ_2 are positive integers, and b_1 and c_1 are positive integers coprime to 2 and 3.

When we substitute these representations of b and c—along with the representation of a we used earlier—into the given equation, we obtain

$$2^{2\alpha_1}3^{2\alpha_2}a_1^2 = 2^{1+3\beta_1}3^{3\beta_2}b_1^3 = 2^{5\gamma_1}3^{1+5\gamma_2}c_1^5.$$

We are using the uniqueness of prime factorisation here.

Since 3, a_1, b_1 and c_1 are coprime to 2 we must equate the powers of 2, and similarly we must equate the powers of 3.

So we must have

$$2\alpha_1 = 1 + 3\beta_1 = 5\gamma_1$$

and

$$2\alpha_2 = 3\beta_2 = 1 + 5\alpha_2.$$

Since $2\alpha_1 = 5\gamma_1$, we see that $5 \mid 2\alpha_1$ and because 2 and 5 are coprime we have $5 \mid \alpha_1$. So let $\alpha_1 = 5k$ where k is a positive integer. Then $\gamma_1 = 2k$ and $1 + 3\beta_1 = 10k$. So

$$\beta_1 = \frac{10k - 1}{3} = 3k + \frac{k-1}{3}.$$

But β_1 must be a positive integer, so we must have $\frac{k-1}{3} = n$ where n is a non-negative integer (notice that we can have $n = 0$).

Then $k = 3n + 1$, and

$$\alpha_1 = 5(3n+1),$$

$$\beta_1 = 10n + 3$$

and

$$\gamma_1 = 2(3n+1).$$

Using a similar process for $2\alpha_2 = 3\beta_2 = 1 + 5\alpha_2$ shows that

$$\alpha_2 = 3(5m+1),$$

$$\beta_2 = 2(5m+1)$$

and

$$\gamma_2 = 6m + 1$$

for some non-negative integer m.

Therefore all solutions are of the form

$$a = 2^{5(3n+1)}3^{3(5m+1)}a_1,$$

$$b = 2^{10n+3}3^{2(5m+1)}b_1$$

and

$$c = 2^{2(3n+1)}3^{6m+1}c_1$$

where n and m are non-negative integers, and a_1, c_1 and b_1 are positive integers coprime to 2 and 3.

To find the smallest possible solution we need to choose $a_1 = b_1 = c_1 = 1$ and $n = m = 0$.

Then

$$a = 2^5 3^3,$$

$$b = 2^3 3^2$$

and

$$c = 2^2 3.$$

What can we say about a_1, b_1 and c_1 more generally?

Putting our representations of a, b and c into the given equation and cancelling all factors of 2 and 3 shows that

$$a_1^2 = b_1^3 = c_1^5.$$

If a has a prime factor p then we can see immediately that p divides b and c. Similarly all prime factors of b and c are prime factors of a, b and c. So the prime factorisations of a, b and c use exactly the same primes.

Let α, β and γ be the greatest positive integers such that $p^\alpha \mid a$, $p^\beta \mid b$ and $p^\gamma \mid c$. For example, $p^\alpha \mid a$ but $p^{\alpha+1}$ does not divide a.

Then equating powers of p in the given equation shows that

$$2\alpha = 3\beta = 5\gamma.$$

This has solution

$$\alpha = 15r,$$

$$\beta = 10r$$

and

$$\gamma = 6r$$

where r is a positive integer.

Therefore we can let

$$a_1 = d^{15},$$

$$b_1 = d^{10}$$

and

$$c_1 = d^6$$

where d is any positive integer coprime to 2 and 3.

This gives the general solution

$$a = 2^{5(3n+1)}3^{3(5m+1)}d^{15},$$

$$b = 2^{10n+3}3^{2(5m+1)}d^{10}$$

and

$$c = 2^{2(3n+1)}3^{6m+1}d^6$$

where n and m are any non-negative integers and d is any positive integer coprime to 2 and 3.

Notice this can be rearranged to give

$$a = 2^5 3^3 (2^n 3^m d)^{15},$$

$$b = 2^3 3^2 (2^n 3^m d)^{10}$$

and

$$c = 2^2 3 (2^n 3^m d)^6$$

so it is easy to see where our smallest solution came from.

Relevance

NA3 What are highest common factors and why do they matter?

