The equation of a circle

G2_RT3

What do we mean by a *circle*?

It's the set of all points at some fixed distance from some fixed point.

For example, we might talk about the circle of radius 3 with centre $(-1,2)$—this is the set of all points that are at distance 3 from $(-1,2)$.

Now we're going to try to express this set of points as a set of solutions to an equation, in much the same way that we can describe a line as the set of solutions to an equation. For example, the line with gradient $\frac{1}{2}$ passing through $(0,1)$ is precisely the set of all points (x,y) such that $2y = x + 2$.

When is the point (x,y) at distance 3 from $(-1,2)$?

Here's a useful picture.

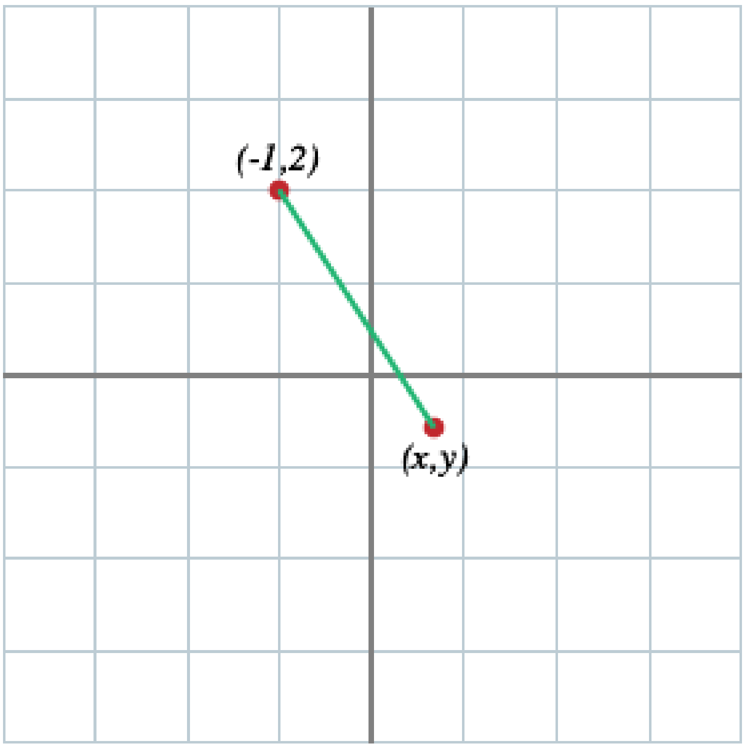


Figure G2_RT3.1: When is (x,y) at distance 3 from $(-1,2)$?

The coordinate system gives us a very natural way to get a helpful right-angled triangle from this.

If the point (x,y) were placed differently relative to $(-1,2)$, then we might get different triangles.

Exercise Find all the possible diagrams, and the side lengths of the resulting triangles.

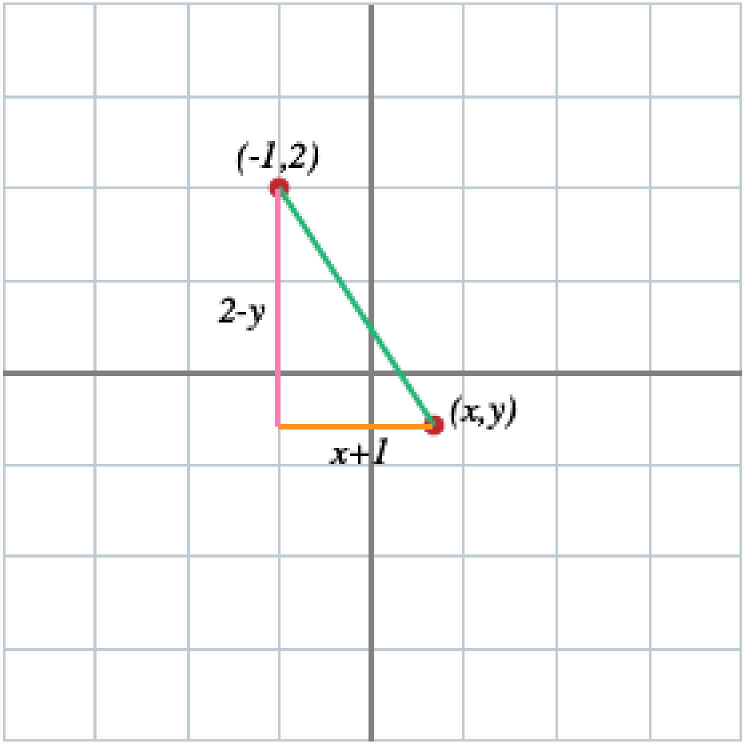
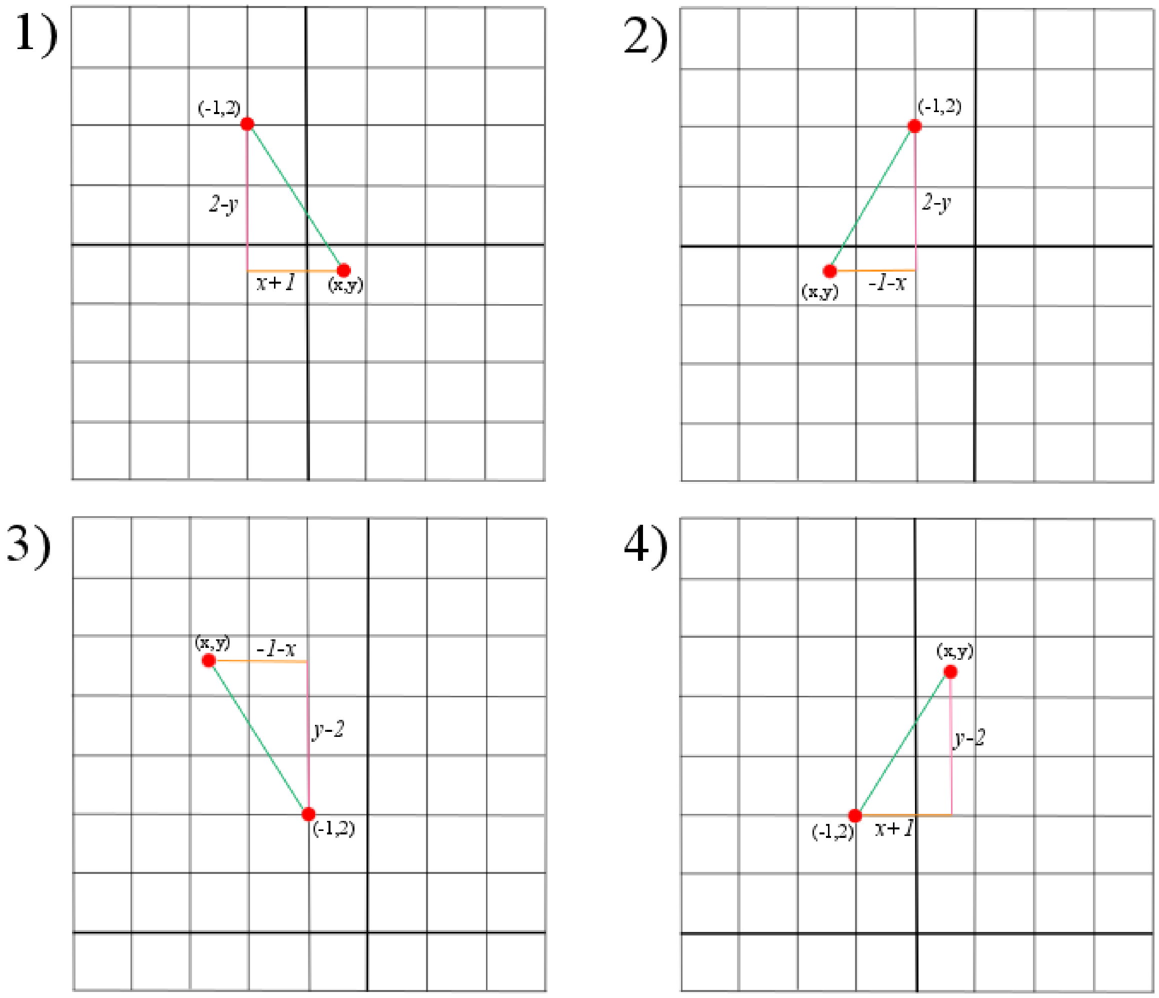


Figure G2_RT3.2: When is (x, y) at distance 3 from $(-1, 2)$?



Now Pythagoras's theorem helps us to find the distance between the points (x, y) and $(-1, 2)$: it's

$$\sqrt{(x+1)^2 + (2-y)^2}.$$

Exercise Find the corresponding expressions for the distances in the other possible diagrams.

The expressions for the distance between $(-1, 2)$ and (x, y) in each of the above diagrams are:
1) $\sqrt{(x+1)^2 + (2-y)^2}$,
2) $\sqrt{(-1-x)^2 + (2-y)^2}$,
3) $\sqrt{(-1-x)^2 + (y-2)^2}$,
4) $\sqrt{(x+1)^2 + (y-2)^2}$.
Using the fact that $(-a)^2 = a^2$ for any real number a, we can see that these expressions are all equivalent!

So (x, y) lies on the circle of radius 3 with centre $(-1, 2)$ if, and only if,

$$\sqrt{(x+1)^2 + (2-y)^2} = 3.$$

This, in turn, is satisfied if, and only if,

$$(x+1)^2 + (2-y)^2 = 9.$$

(In one direction this is clear: by squaring both sides of the first equation we obtain the second. In the other direction, we can use the fact that $\sqrt{(x+1)^2 + (2-y)^2}$ must be positive to justify taking the positive root on both sides.)

So the equation of the circle of radius 3 with centre $(-1, 2)$ is

$$(x+1)^2 + (2-y)^2 = 9.$$

Exercise Expand out the brackets and experiment with different ways of writing this equation. Which way(s) do you find most convenient, and why?

Exercise Pick a radius and a centre, and find the equation of the corresponding circle, with an explanation. Repeat until you feel confident.

Relevance

E2 How is the solution of equations related to problems in geometry?

G2 What is the connection between algebra and geometry, and how can we exploit it?

