# There's always one, isn't there?

## Solution

**Claim** If the two numbers have a highest common factor greater than $1$, then dividing one by the other cannot give a remainder of $1$.

**Proof** Let two numbers $a$ and $b$ (with $a > b$) have a hcf of $d$, which is larger than $1$. We may assume that $a$ is not divisible by $b$ (otherwise it is not an interesting question). Then when we divide $a$ by $b$, we have $a = qb + r$ for some integers $q$ and $r$ with $1 \leq r < b$. Rearranging this gives

$$r = a - qb.$$

But now the RHS is divisible by $d$, so $r$ must also be divisible by $d$. Hence the remainder is some multiple of $d$, and so must be greater than $1$.

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**Claim** If instead the hcf of the two numbers is $1$ (i.e. they are *coprime*), then there will always be a remainder of 1.

**Proof** If $a$ and $b$ (with $a > b$) are coprime, then their lowest common multiple is $ab$.

The remainders of the numbers in the question, upon division by $b$, could be written $r_1, r_2, \ldots, r_{b-1}$. We want to show that no two of these remainders are the same. If we suppose that two of them **are** the same, then we can try to force a contradiction.

Suppose that $r_k$ and $r_l$ are the same, with $k < l$. Call this value of the remainder $R$. Then, for some $m$ and $n$, we have

$$ka = nb + R$$

and

$$la = mb + R$$

for some integers $m$ and $n$.

Now, since $R$ is the same in both cases, we can take away one from the other to get

$$la - ka = mb - nb$$

$$\iff (l - k)a = (m - n)b.$$

This means that $(l - k)a$ is divisible by $b$. However, $l - k$ is an integer value somewhere between $1$ and $b - 2$. We already know that $ab$ is the smallest multiple of $a$ that is divisible by $b$, so there is no possible value of $l - k$ that will make $(l - k)a$ divisible by $b$. This is a contradiction!

The conclusion is that no two of these remainders can be the same—they must all be different.

Since there are $b - 1$ remainders, which have to take one of the values $1, 2, ..., b - 1$, it follows that one of the remainders will be $1$, which is what we wanted to prove.

## Relevance

NA3 What are highest common factors and why do they matter?

 