$n^5 - n$

Solution

The general term of this sequence is $n^5 - n$. We can factorise this expression as follows, using the expression for the difference of two squares:

$$n^5 - n = n(n^4 - 1)$$

$$= n(n^2 - 1)(n^2 + 1)$$

$$= n(n-1)(n+1)(n^2 + 1).$$

This expression is divisible by 2. That's because n and $n+1$ are two consecutive integers, so one of them must be even and the other odd. Then, as $n^5 - n$ is divisible by both n and $n+1$, it has at least one even factor and must therefore be even (the product of an even integer and any other integer is always even).

This expression is also divisible by 3. That's because $n-1$, n and $n+1$ are three consecutive integers, so one of them must be a multiple of 3. We can see this by considering the remainder left upon dividing n by 3: the only possible values are 0, 1, and 2. If it is 0, then n is a multiple of 3. If it is 3, then $n-1$ is a multiple of 3. If it is 2, then $n+1$ is a multiple of 3.

Similarly to above, since $n^5 - n$ is divisible by $n-1$, n, and $n+1$, it must have a factor which is a multiple of 3, and therefore must itself be divisible by 3.

This expression is also divisible by 5, although this is slightly tricker to show than in the previous two parts. Firstly, we consider the remainder left when we divide n by 5. This can take the values 0, 1, 2, 3, and 4. We'll consider the five cases separately.

Case 1 If this remainder is 0, then n itself is divisible by 5, and then so is $n^5 - n$, since it is divisible by n.

Case 2 If this remainder is 1, then $n-1$ is divisible by 5, and then so is $n^5 - n$, as it is divisible by $n-1$.

Case 3 If this remainder is 2, then n is 2 greater than a multiple of 5. That is, we can write $n = 5k+2$ for some integer k. Then

$$\begin{aligned} n^2+1 &= (5k+2)^2+1 \\ &= 25k^2+20k+4+1 \\ &= 25k^2+20k+5 \\ &= 5(5k^2+4k+1). \end{aligned}$$

As k is an integer, $5k^2+4k+1$ is also an integer, and so n^2+1 is a multiple of 5. Then so is n^5-5, as it is divisible by n^2+1.

Case 4 Similarly, if this remainder is 3, then we can write $n = 5m+3$, for some integer m. Then

$$\begin{aligned} n^2+1 &= (5m+3)^2+1 \\ &= 25m^2+30m+10 \\ &= 5(5m^2+6m+2). \end{aligned}$$

So again, n^2+1 is a multiple of 5, meaning that n^5-n is too.

Case 5 If the remainder is 4, then $n+1$ is divisible by 5, and then so is n^5-n, as it is divisible by $n+1$.

We have shown that, for all n, n^5-n is divisible by 2, 3, and 5. This means that every term in the sequence is divisible by the lowest common multiple of 2, 3 and 5. In this case this is simply their product, 30, as they have no common prime factors.

So 30 divides every number in the sequence. This means that the largest integer which divides every term in the sequence must be at least 30.

Now, look at the second term in the sequence: 2^5-2. This is equal to 30, which obviously is not divisible by any integers greater than itself. So, 30 is the largest integer which divides every term in the sequence.

Relevance

A2 What interesting things can we do with squares and square roots?

NA3 What are highest common factors and why do they matter?

