Factorial fun

Solution

(a)

The highest power of 5 dividing $20!$ We have $20! = 20 \times 19 \times 18 \times \cdots \times 3 \times 2 \times 1$.

Since 5 is a prime number, we can just add the highest power of 5 dividing each of the numbers 1, 2, 3, …, 20. There are only four numbers in this range (5, 10, 15, 20) that are divisible by 5^1, and none of them is divisible by 5^2. So the highest power of 5 dividing $20!$ is 5^4.

Factors of $20!$ The first thing we need to do is find the prime factorisation of $20!$. As $20! = 20 \times 19 \times 18 \times \cdots \times 3 \times 2 \times 1$, we can can do this by finding the prime factorisation of each of the numbers 2, 3, …, 20 (ignoring 1 since it won't contribute any primes) and multiplying them together.

Number	Prime factorisation
2	2
3	3
4	2^2
5	5
6	2×3
7	7
8	2^3
9	3^2
10	2×5
11	11
12	$2^2 \times 3$
13	13
14	2×7
15	3×5
16	2^4
17	17
18	2×3^2
19	19
20	$2^2 \times 5$

Multiplying the prime factorisations in the right-hand column together and simplifying, we get

$20! = 2^{18} \times 3^8 \times 5^4 \times 7^2 \times 11 \times 13 \times 17 \times 19.$

Now we can use this to calculate the number of divisors of $20!$. Each divisor will have a unique prime factorisation, which must be 'contained' within the prime factorisation of $20!$. Let m be a divisor of $20!$. Then there are nineteen possible values for the highest power of 2 dividing m ($0, 1, \dots, 18$). Similarly, there are nine possible values for the highest power of 3 dividing m ($0, 1, \dots, 8$). Continuing in this way for all the prime factors of $20!$, we can calculate that there are $19 \times 9 \times 5 \times 3 \times 2 \times 2 \times 2 \times 2 \times 2 = 41040$ divisors of $20!$.

(b)

The highest power of k that divides $500!$, where $k^t < 500 < k^{t+1}$ We can see that $\left[\frac{500}{k}\right]$ is the number of multiples of k that are less than or equal to 500. For example, if k goes into 500 "seven and a bit" times, this means that $k, 2k, \dots, 7k$ are less than 500 but $8k$ is greater than 500.

Similarly, $\left[\frac{500}{k^2}\right]$ is the number of multiples of k^2 that are less than or equal to 500, and so on.

Now, we can work out the highest power of k that divides $500!$ by considering the number of multiples of $k, k^2, \dots, k^t$ less than 500.

We need to count all the multiples of k. We need to count the multiples of k^2 twice, since they contribute 2 to the exponent, but we have already counted them once in the multiples of k so we need to count them just once more. Similarly, we need to count the multiples of k^3 three times in total, but we have already counted them twice (once in the multiples of k and once in the multiples of k^2), so we need to count them just once more. And so on for the remaining powers. So the highest power of k that divides $500!$ has exponent

$$\left[\frac{500}{k}\right] + \left[\frac{500}{k^2}\right] + \cdots + \left[\frac{500}{k^t}\right]$$

(c)

Factors of $n!$ We can generalise the above result beyond the case of $500!$. The highest power of k that divides $n!$, where $k^t < n < k^{t+1}$, is equal to

$$\left[\frac{n}{k}\right] + \left[\frac{n}{k^2}\right] + \cdots + \left[\frac{n}{k^t}\right],$$

by exactly the same reasoning as in (b).

We can use this information to find the prime factorisation of $n!$. Let p be the largest prime with $p \leq n$. Also, for any number m, let t_m be the integer such that $m^{t_m} \leq n < m^{t_m+1}$.

Using the information from part (b), we can then calculate the prime factorisation of $n!$: we have

$$n! = 2^{\left(\left[\frac{n}{2}\right]+\left[\frac{n}{2^2}\right]+\cdots+\left[\frac{n}{2^{t_2}}\right]\right)} \times 3^{\left(\left[\frac{n}{3}\right]+\left[\frac{n}{3^3}\right]+\cdots+\left[\frac{n}{3^{t_3}}\right]\right)} \times \cdots \times p^{\left(\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^{t_p}}\right]\right)}.$$

Then we can use the same reasoning as in part (a) to calculate the number of factors of $n!$: we get

$$\left(1 + \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \cdots + \left[\frac{n}{2^{t_2}}\right]\right) \times \left(1 + \left[\frac{n}{3}\right] + \left[\frac{n}{3^2}\right] + \cdots + \left[\frac{n}{3^{t_3}}\right]\right) \times \cdots$$
$$\times \left(1 + \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^{t_p}}\right]\right).$$

We can check that this expression works for the example of $20!$:

$$\left(1+\left[\frac{20}{2}\right]+\left[\frac{20}{4}\right]+\left[\frac{20}{8}\right]+\left[\frac{20}{16}\right]\right)\times\left(1+\left[\frac{20}{3}\right]+\left[\frac{20}{9}\right]\right)\times\left(1+\left[\frac{20}{5}\right]\right)\times\left(1+\left[\frac{20}{7}\right]\right)\times$$
$$\left(1+\left[\frac{20}{11}\right]\right)\times\left(1+\left[\frac{20}{13}\right]\right)\times\left(1+\left[\frac{20}{17}\right]\right)\times\left(1+\left[\frac{20}{19}\right]\right)$$
$$=(10+5+2+1+1)\times(6+2+1)\times(4+1)\times(2+1)\times(1+1)\times(1+1)\times(1+1)\times(1+1)$$
$$=19\times 9\times 5\times 3\times 2\times 2\times 2\times 2=41040.$$

This is the same answer as in part (a), suggesting that this formula works!

Relevance

NA3 What are highest common factors and why do they matter?