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1 General approach

The program in the form of a dynamic library is designed to solve the Fokker-Plank equation in form:

$$\frac{\partial f}{\partial t} = -\mu u \frac{\partial f}{\partial z} + u \frac{(1-\mu^2)}{2} \frac{d \ln B}{dz} \frac{\partial f}{\partial \mu} - \frac{ej}{m_e \sigma} \frac{(1-\mu^2)}{u} \frac{\partial f}{\partial \mu} - \mu u \frac{ej}{\sigma} \frac{\partial f}{\partial E} + \\ + \frac{m_e c^4}{\lambda_0} \frac{\partial}{\partial E} \left(\frac{f}{u} \right) + \frac{c^4}{\lambda_0 u^3 \left(E/m_e c^2 + 1 \right)^2} \frac{\partial}{\partial \mu} \left[(1-\mu)^2 \frac{\partial f}{\partial \mu} \right] + \\ + \frac{u}{2\lambda_{turb}} \frac{\partial}{\partial \mu} \left[(1-\mu)^2 \frac{\partial f}{\partial \mu} \right] + S(t, E, \mu, z), \quad (1)$$

where

$$j = -e \int\limits_{E_{min}}^{\infty} \int\limits_{-1}^{1} \mu u f(t, E, \mu, z) d\mu dE \quad (2)$$

- is the current density of nonthermal electrons along the loop; S - function of injection of nonthermal electrons into the loop, $m^{-3} MeV^{-1} s^{-1}$; $u = c\sqrt{1-(E/m_e c^2+1)^{-2}}$ - electron velocity module; c - is light speed; E - kinetic energy of a nonthermal electron; e, m_e - electron charge modulus and electron mass; $\lambda_0 = \frac{1}{4\pi r_0^2 \Lambda(z) n(z)}$; $\Lambda(z)$ - Coulomb logarithm; $n(z)$ - concentration of thermal plasma electrons along the loop; r_0 - classical electron radius; $f(t, E, \mu, z)$ - is the distribution function of nonthermal electrons, $m^{-3} MeV^{-1}$; μ - is the cosine angle between electron velocity and the direction of the magnetic field (the cosine of the pitch angle); z - coordinate along the magnetic loop; t - time; σ - conductivity of fully ionized plasma along the magnetic field, $Omh^{-1} m^{-1}$; $B(z)$ - projection of the magnetic field for the magnetic loop direction; E_{min} - electron energy threshold, electrons with energies above E_{min} are considered non-thermal; λ_{turb} - mean free path in turbulence model of diffusion, for example we can use the empirical model where the mean free path can be written as

$$\lambda_{turb}(E) = \lambda_{turb0} \left(\frac{E_{turb0}}{E} \right)^a, \quad (3)$$

where λ_{turb0}, a and E_{turb0} - parameters of the empirical turbulence model.

For a numerical solution, we write equations (1), (2) in the useful form:

$$\frac{\partial F}{\partial \tau} = a_1 \frac{\partial F}{\partial x} + (a_2 + a_{J2} J) \frac{\partial F}{\partial \mu} + (a_3 + a_{J3} J) \frac{\partial F}{\partial \varepsilon} + a_4 \frac{\partial^2 F}{\partial \mu^2} + a_5 F + \widetilde{S}(\varepsilon, \mu, x, \tau), \quad (4)$$

where

$$J = - \int\limits_{\varepsilon_{min}}^{\infty} \int\limits_{-1}^{1} \mu \beta F d\mu d\varepsilon, \quad (5)$$

$a_1 = -\frac{ct_{max}}{z_{max}}\beta\mu$; $a_2 = \frac{ct_{max}(1-\mu^2)\beta}{2z_{max}}\frac{d\ln B}{dx} - \frac{2ct_{max}\mu}{\lambda_0\beta^3(\varepsilon+1)^2} - \frac{ct_{max}}{\lambda turb}\beta\mu$; $a_{J2} = -\frac{e^2 n_0 t_{max}}{m_e\sigma}\frac{(1-\mu^2)}{\beta}$; $a_3 = \frac{ct_{max}}{\lambda_0\beta}$; $a_{J3} = -\frac{e^2 n_0 t_{max}}{m_e\sigma}\beta\mu$; $a_4 = \frac{ct_{max}}{\lambda_0}\frac{(1-\mu^2)}{\beta^3(\varepsilon+1)^2} + \frac{ct_{max}}{2\lambda_{turb}}\beta(1-\mu^2)$; $a_5 = -\frac{ct_{max}}{\lambda_0}\frac{1}{(\varepsilon^2+2\varepsilon)^{1.5}}$; $\beta = u/c$; $F(\tau,\varepsilon,\mu,x) = f\frac{m_e c^2}{n_0 e}10^{-6}$ – dimensionless distribution function of nonthermal electrons; $J = j/ecn_0$; $\varepsilon = E\frac{e}{m_e c^2}10^6$; $\widetilde{S} = S\frac{m_e c^2 t_{max}}{n_0 e}10^{-6}$; $x = z/z_{max}$; $\tau = t/t_{max}$; z_{max} – length of magnetic loop; t_{max} – time of calculation.

2 Boundary conditions

The problem is solved in the area: $E_{min} \le E < \infty$; $-1 \le \mu \le 1$; $-\infty \le z \le \infty$; $0 \le t \le t_{max}$. Initial conditions $f(t = 0, E, \mu, z)$ and $j(t = 0, z)$ are chosen by the user. The boundary conditions of the problem are chosen as follows:

$$f(t, E = \infty, \mu, z) = 0; \tag{6}$$

$$\partial^2 f(t, E, \mu = \pm 1, z)/\partial\mu^2 = 0; \tag{7}$$

$$f(t, E, \mu, z = \pm\infty) = 0; \tag{8}$$

$$j(t, z = \pm\infty) = 0. \tag{9}$$

The area $0 \le z \le z_{max}$ corresponds to the magnetic loop. The area z $<$ 0 and z $>$ z_{max} corresponds to the corona-chromosphere transition layer, the background plasma concentration there rapidly increases to a value of the order n_∞ .

3 Function: "create_solver"

The function is create solver. Output argument for "create_solver" function is void. Input arguments:

- general_param = [NE, Nmu, Nz, Nz_loop_param, switch_reverse_current, switch_turb] — 5-element array (type long):
  - parameters NE, Nmu, Nz set the size of the distribution function in terms of energy, pitch angle and coordinate, respectively;
  - parameter Nz_loop_param set the size of array "loop_param" – which defined distributions of parameters along loop;
  - switch_reverse_current — switch (1/0), turn on/off reverse current in equation (1);
  - switch_turb — switch (1/0), turn on/off turbulent diffusion in equation (1).
- grid_param = [E_{min}, z_{max}, t_{max}, dt_0, dt_{max}, γ, r_{gridE}, r_{gridz}], 8-element array (type double):
  - E_{min} – lectron energy threshold, electrons with energies above E_{min} are considered non-thermal (in MeV);
  - z_{max} – is length of magnetic loop (in m);
  - t_{max} – is end time of calculation (in s); the start time of the calculation is 0;
  - dt_0 – is initial time step (time step in t = 0), in s;

- dt_{max} – is the maximum possible time step, in s;
- γ – is the parameter which use for control time step value (dimensionless). If $\frac{\max|f_{new} - f_{old}|}{\max|f_{old}|} \leq \gamma$, then go to the next time step. If the previous and current time steps are not equal, then we don't decrease the step, if the previous and current steps are equal, then we increase the step by the factor of 1.5. If $\frac{\max|f_{new} - f_{old}|}{\max|f_{old}|}$ $> \gamma$, then decrease the time step by a factor of 1.5 times and try to calculate again;
- r_{gridE} (in MeV), r_{gridz} (in m) – is coefficients in non-uniform grids for E and for z. The non-uniform grid E define as $E = r_{gridE} \cdot \ln(1 - k_E)$, where $k_E = [k_{Emin}, 1]$, $k_{Emin} = 1 - \exp(-E_{min}/r_{gridE})$ - is uniform grid. The non-uniform grid z define as $z{=}{\pm}r_{gridz} \cdot \ln(1 \pm k_z)$, where $k_z = [0, 1]$ - is uniform grid.

- inf_param=$[n_\infty, log_\infty, \sigma_\infty]$, 3-element array (type double), $n_\infty, log_\infty, \sigma_\infty$ sets the value of the background plasma density (in m^{-3}), Coulomb logarithm and conductivity (in $Omh^{-1}m^{-1}$), respectively, in the area z $<$ 0 and z $>$ z_{max} (outside the loop).

- loop_param — 2D array with dimension 5× Nz_loop_param (type double), defines distributions of parameters along loop:
  - loop_param[0, *] — coordinate values z, in m;
  - loop_param[1, *] — magnetic field B, in G;
  - loop_param[2, *] — plasma density n, in m^{-3};
  - loop_param[3, *] — Coulomb logarithm Λ, dimensionless;
  - loop_param[4, *] — conductivity of plasma σ, in $Omh^{-1}m^{-1}$.

- turb_param — array with dimension NE+1 (type double), determines the energy dependence of the mean free path in turbulent diffusion $\lambda_{turb}(E)$, where $E = -r_{gridE} \cdot \ln(1 - k_E)$, $k_E = [k_{Emin}, 1]$, $k_{Emin} = 1 - \exp(-E_{min}/r_{gridE})$; $\lambda_{turb}(E)$ in m.

NOTE to do a calculation with a constant time step You must to assume $dt_0 = dt_{max}$ and gamma $>$ 1.

4 Function: "delete_solver"

The function is delete solver. Output argument for "delete_solver" function is void. Input arguments is none.

5 Function: "get_grids"

In order to set the initial distribution function, the initial distribution of reverse current and function of injection of non-thermal electrons you must to have the grids on distance, cos pitch angle and energy. The function "get_grids" give the grids on distance, cos pitch angle and energy, and also give the distribution of density thermal plasma and distribution of $\frac{d \ln B}{dz}$.

Output argument for "get_grids" function is void. Input arguments:

- E – array (type double) with dimension NE+1, grid on energy, MeV;

- μ – array (type double) with dimension Nmu+1, grid on cos pitch angle;

- z – array (type double) with dimension 2Nz+1, grid on coordinate, m;
- dlnB – array (type double) with dimension 2Nz+1, the distribution of $\frac{d\ln B}{dz}$ along z;
- n_loop – array (type double) with dimension 2Nz+1, the distribution of density thermal plasma along z, m^{-3}.

6 Function: "solve"

Function "solve" do the time step from some t_1 to $t_2 = t_1 + \Delta t$.

Output argument for "solve" function is void. Input arguments:

- t_2 – pointer to variable, type double, time value after call function "solve", s;
- S – function of injection $S(E, \mu, z)$ of non-thermal electron, 3D array with dimension (NE+1)×(Nmu+1)×(2Nz+1) type double. You must to define the function of injection $S(E, \mu, z)$ of non-thermal electron for each time step;
- j – reverse current of non-thermal electrons $j(z)$, 2D array with dimension 2Nz+1. Before the function "solve" called, this array contains the initial value of the reverse current at the moment t_1, after the function called, this array contains the calculated value of the reverse current at the moment t_2;
- f – distribution function of non-thermal electrons $f(E, \mu, z)$, 3D array with dimension (NE+1)×(Nmu+1)×(2Nz+1) type double. Before the function "solve" called, this array contains the initial value of the distribution function at the moment t_1, after the function called, this array contains the calculated value of the distribution function at the moment t_2;
- n_{fast} – density of non-thermal electrons $n_{fast}(z) = \int\limits_{E_{min}}^{\infty} \int\limits_{-1}^{1} f(E, \mu, z) d\mu dE$, 1D array with dimension 2Nz+1.

7 Method convergence test

To check the convergence of the solution, we set the specific form of the solution of system (4),(5):

$$F_{an}(\varepsilon, \mu, x, \tau) = F_\varepsilon(\varepsilon) F_\mu(\mu) F_x(x) F_\tau(\tau). \tag{10}$$

Then, use the (10) we find the injection function from system (4) and (5) and use this injection function for numerical solution. Result of numerical solution must be same to equation (10). Consider a solution like:

$$F_\varepsilon(\varepsilon) = \varepsilon_{min}^5 / \varepsilon^5; \tag{11}$$

$$F_\mu(\mu) = 1/10 - \mu^3/10; \tag{12}$$

$$F_x(x) = \sin(\pi x); \tag{13}$$

$$F_\tau(\tau) = \begin{cases} \tau & \tau \le \tau_0, \\ \tau_0 \exp\left(-2(\tau - \tau_0)^2\right) & \tau > \tau_0. \end{cases} \tag{14}$$

The parameters of problem is: $B(z) = \exp((z/z_{max} - 0.5)^2 4 \cdot \ln 5)$; $\tau_0 = 0.3$; $E_{min} = 3 \cdot 10^{-2}$ MeV; $z_{max} = 6 \cdot 10^7$ m; t_{max}=3 s; $r_{gridz} = 0.25 \cdot z_{max}$; $r_{gridE} = 3.0 \cdot E_{min}$.

From (4) we find the injection function:

$$\widetilde{S} = \frac{\partial F_{an}}{\partial \tau} - a_1 \frac{\partial F_{an}}{\partial x} - (a_2 + a_{J2} J_{an}) \frac{\partial F_{an}}{\partial \mu} - (a_3 + a_{J3} J_{an}) \frac{\partial F_{an}}{\partial \varepsilon} - a_4 \frac{\partial^2 F_{an}}{\partial \mu^2} - a_5 F_{an}, \qquad (15)$$

where

$$J_{an} = -\frac{F_x F_\tau \varepsilon_{min}^5}{25} \int\limits_{\varepsilon_{min}}^{\infty} \frac{\beta}{\varepsilon^5} d\varepsilon. \qquad (16)$$

For $E_{min} = 3 \cdot 10^{-2}$ MeV, the integral $\int\limits_{\varepsilon_{min}}^{\infty} \frac{\beta}{\varepsilon^5} d\varepsilon = 7822.691892717618$.

Let's do two calculations (the program contain in file "convergence.py"): with the number of calculate intervals $\mathrm{NE}_1 = 30$, $\mathrm{Nmu}_1 = 30$, $\mathrm{Nz}_1 = 30$, $\mathrm{Nt}_1 = 40$, and with twice as many number intervals: $\mathrm{NE}_1 = 60$, $\mathrm{Nmu}_1 = 60$, $\mathrm{Nz}_1 = 60$, $\mathrm{Nt}_1 = 80$, in this case we expect the error to decrease by the factor of 2.

The fig. 1 shows the calculation result of the distribution function f for the values $E = 0.03$ MeV; μ = -0.93 at which the maximum error $R_1 = \dfrac{\max|f_{an} - f_1|}{\max|f_{an}|} = 0.154$ is observed, here f_1 – numerical solution for number of intervals NE_1, Nmu_1, Nz_1, Nt_1.

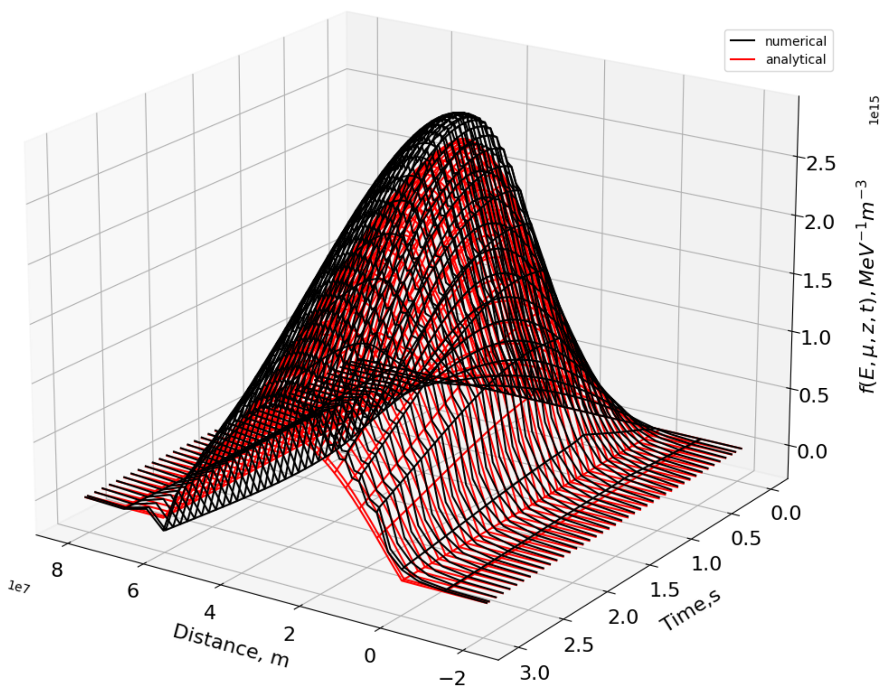

Figure 1: Result of calculations for injection function (4), for the values E = 0.03 MeV; μ = -0.93, number of calculate intervals: $\mathrm{NE}_1 = 30$, $\mathrm{Nmu}_1 = 30$, $\mathrm{Nz}_1 = 30$, $\mathrm{Nt}_1 = 40$.

The fig. 2 shows the calculation result of the distribution function f for the number of intervals: $\mathrm{NE}_2 = 60$, $\mathrm{Nmu}_2 = 60$, $\mathrm{Nz}_2 = 60$, $\mathrm{Nt}_2 = 80$. For value $E = 0.03$ MeV; μ = -0.97 we observe the maximum error $R_2 = \dfrac{\max|f_{an} - f_2|}{\max|f_{an}|} = 0.0754$, where f_2 – numerical solution for number of intervals NE_2, Nmu_2, Nz_2, Nt_2.

We can to calculated effective order of accuracy of the numerical method $p = \ln(R_1/R_2)/\ln 2 \approx 1$, this value is equal to the theoretical 1 order of accuracy, it means the method is converged.

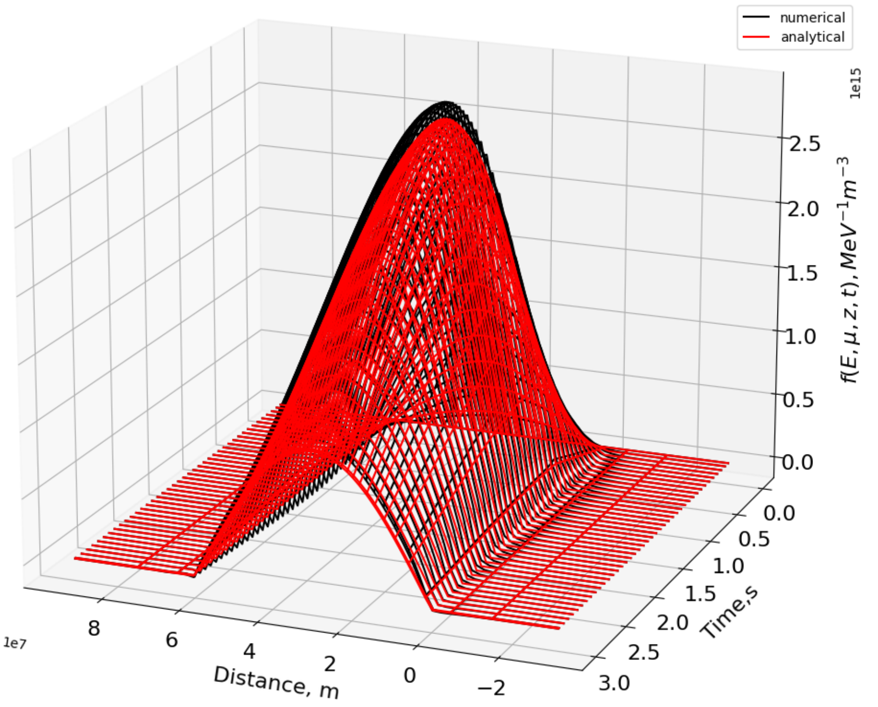

Figure 2: Result of calculations for injection function (4), for value E = 0.03 MeV; μ = -0.97, number of calculate intervals:
$\mathrm{NE}_2 = 60$, $\mathrm{Nmu}_2 = 60$, $\mathrm{Nz}_2 = 60$, $\mathrm{Nt}_2 = 80$.

8 Turbulent diffusion on/off

Let's do two calculations for cases: when turbulent diffusion is present and when there is no (the program contain in file "turbulent_ diffusion_on_off.py"). We define the initial distribution function in form:

$$f(t=0, E, \mu, z) = 10.0 \cdot \exp(-500(\mu - 0.7)^2)(E_{min}/E)^5 \exp\left(-\frac{(z/z_{max} - 0.5)^2}{z_0^2}\right) \cdot \frac{e \cdot 10^6}{m_e c^2}, \quad (17)$$

where $z_0 = 3 \cdot 10^6 / z_{max}$. Function of injection non-thermal electron take $S(t, E, \mu, z) = 0$; we define mean free path for turbulent diffusion in form:

$$\lambda_{turb}(E) = \lambda_{turb0}(E_{min}/E)^{a_{turb}},$$

where $\lambda_{turb0} = 10^6$ m; $a_{turb} = 1$; other parameters: $B(z) = \exp((z/z_{max} - 0.5)^2 4 \cdot \ln 5)$; $E_{min} = 3 \cdot 10^{-2}$ MeV; $z_{max} = 6 \cdot 10^7$ m; t_{max}=1 s; $r_{gridz} = 0.25 \cdot z_{max}$; $r_{gridE} = 5.0 \cdot E_{min}$; NE = 40, Nmu = 40, Nz = 50, $dt_0 = t_{max} \cdot 10^{-4}$; $dt_{max} = t_{max} \cdot 0.02$; $\gamma = 0.1$.

The fig. 3 shows the calculation result of the density non-thermal electron $n(t, z) = \int\limits_{E_{min}}^{\infty} \int\limits_{-1}^{1} f(t, E, \mu, z) d\mu dE$ for case when there is no turbulent diffusion. It can be seen from the figure that the initial beam of electron moves to the end of the loop and is reflected from the magnetic mirror. The fig. 4 shows the calculation result of the density non-thermal electron $n(z)$ for case when there is turbulent diffusion. In this case we see that the initial beam of electrons propagates randomly and the beam width increases according to the law $z \sim \sqrt{t}$.

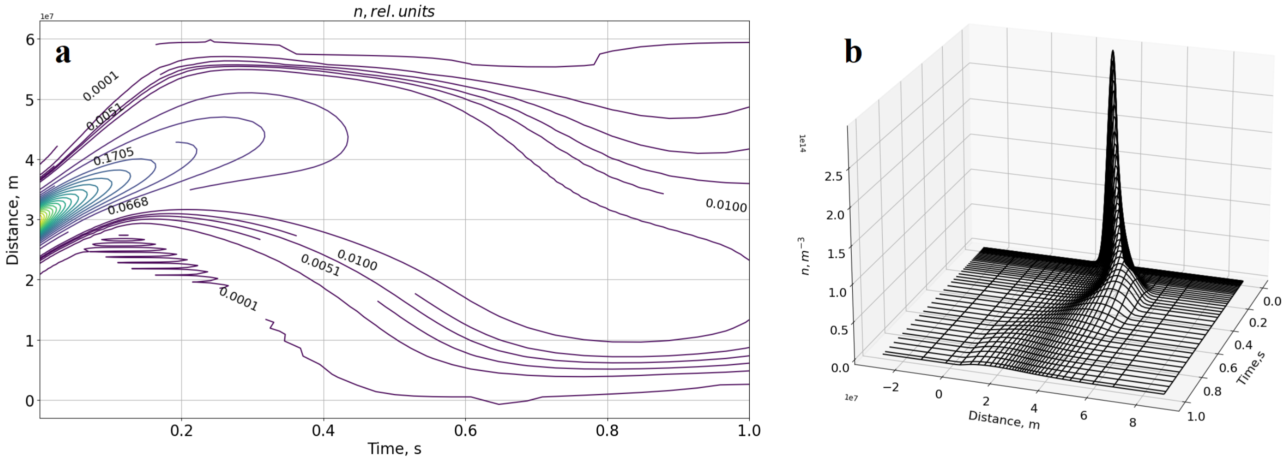

Figure 3: The result of calculations for the case when there is no turbulent diffusion.
a – contour plot; b – 3D graph.

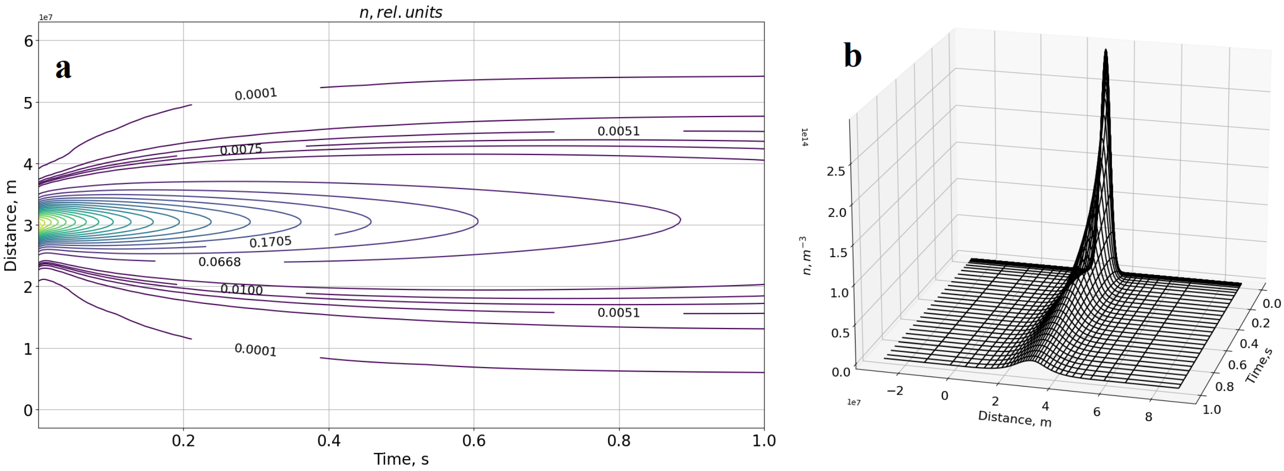

Figure 4: The result of calculations for the case when there is turbulent diffusion.
a – contour plot; b – 3D graph.

9 Comparison with reduced equation

In case when turbulent diffusion is present, we can to simplify the system (1), (2):

$$\frac{\partial N}{\partial t} = \frac{u\lambda_{turb}}{3}\frac{\partial^2 N}{\partial z^2} + \frac{m_e c^4}{\lambda_0}\frac{\partial}{\partial E}\left(\frac{N}{u}\right), \tag{18}$$

in this case the density of non-thermal electron defined as $n(t,z) = \int\limits_{E_{min}}^{\infty} N(t,E,z)dE$.

Now we compare the two numerical solutions: the solution for (18) and solution for system (1), (2), this solutions must be close in case when the turbulent diffusion is present. The parameters of the problem are the same as in section 8, except max time step: in this problem we take $dt_{max} = t_{max} \cdot 0.005$. Also when solving equation (18) we must take the initial value of the distribution function in form:

$$N(t=0,E,z) = \left(\frac{E_{min}}{E}\right)^5 \exp\left(-\frac{(z/z_{max}-0.5)^2}{z_0^2}\right) \cdot \int\limits_{-1}^{1} \exp(-500(\mu-0.7)^2)d\mu \cdot \frac{e\cdot 10^7}{m_e c^2}. \tag{19}$$

In figures 5 and 6 shown the result of calculations (the program contain in file "compare_with_reduced.py") for two equations. The differences between the solutions at t < 0.1 s are due to the heterogeneous of the initial distribution function, in the rest of the region the solutions conform well. Thus, in the presence of turbulent diffusion, we can to calculate the distribution function of nonthermal electrons by the simplified equation (18).

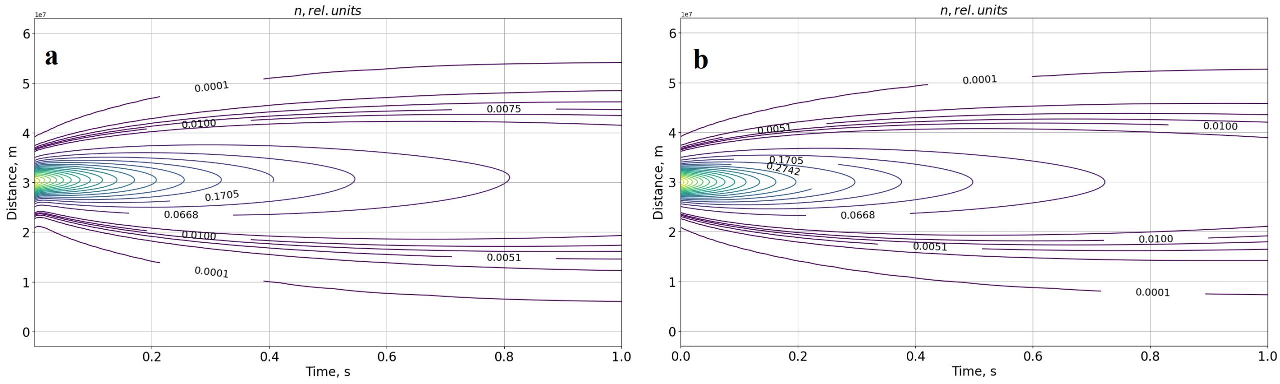

Figure 5: Comparison of two solutions:
a – solution for system (1) and (2); b – solution for (18).

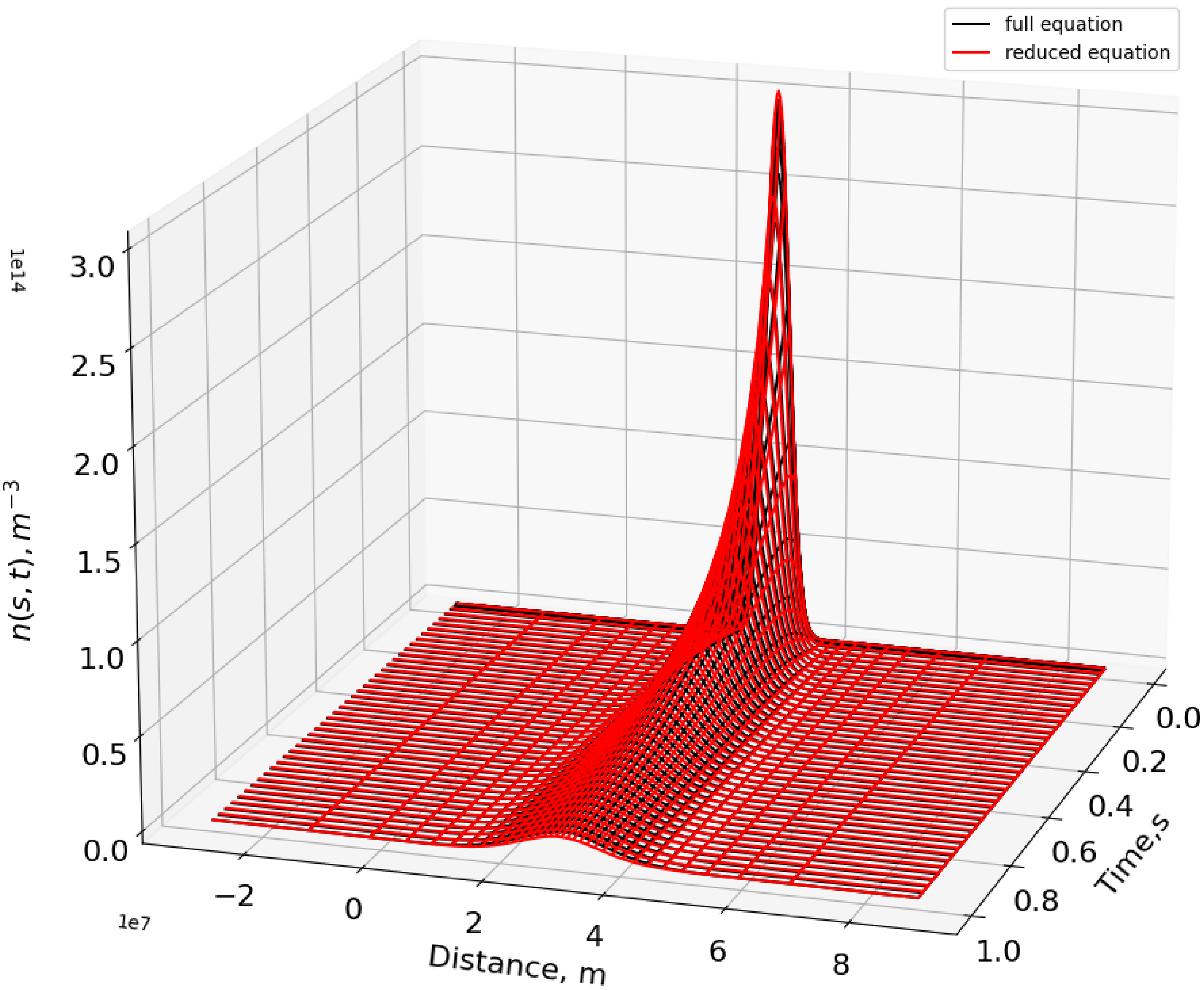

Figure 6: Comparison of solution for system (1), (2) and solution for (18), 3D graph.