MA250 Assignment 2 02.02.22

1.)

$$f(y) = \int_{y^2}^{e^y} (\sin(z) + 4y)\, dz$$

$$a(y) = y^2 \Rightarrow a'(y) = 2y$$

$$b(y) = e^y \Rightarrow b'(y) = e^y$$

$$g(y,z) = \sin(z) + 4y \Rightarrow \partial_y\, g(y,z) = 4$$

$$f'(y) = e^y(\sin(e^y) + 4y) - 2y(\sin(y^2) + 4y) + \int_{y^2}^{e^y} 4\, dz$$

$$f'(y) = e^y \sin(e^y) + 4e^y y - 2y\sin(y^2) - 8y^2 + 4e^y - 4y^2$$

$$f'(y) = e^y \sin(e^y) + 4e^y y - 2y\sin(y^2) + 4e^y - 12y^2$$

2.)

$$H(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(y,s)\, dy\, ds$$

$$g(x,t,s) = \int_{x-c(t-s)}^{x+c(t-s)} h(y,s)\, dy$$

$$\partial_t H = g(x,t,t) + \int_0^t \partial_t g\, ds$$

$$\partial_t H = g(x,t,t) + c \int_0^t h(x+c(t-s),s) - h(x-c(t-s),s)\, ds$$

$$g(x,t,t) = \int_{x-c(t-t)}^{x+c(t-t)} h(y,s)\, dy$$

$$g(x,t,t) = 0$$

$$\partial_t H = c \int_0^t h(x+c(t-s),s) + h(x-c(t-s),s)\, ds$$

Let

$$z(x,t,s) = h(x+c(t-s),s) + h(x-c(t-s),s)$$

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$$\partial_t H = c\int_0^t z(x,t,s)\,ds$$

$$\partial_{tt} H = c\left(z(x,t,t) + \int_0^t \partial_t z(x,t,s)\,ds\right)$$

We know that

$$z(x,t,t) = h(x,t) + h(x,t)$$

$$= 2h(x,t)$$

Moreover

$$\partial_t z(x,t,s) = \partial_t\Big(h(x+c(t-s),s) + h(x-c(t-s),s)\Big)$$

$$= c\,\partial_x h(x+c(t-s),s) - c\,\partial_x h(x-c(t-s),s)$$

$$= c\,\partial_x\big(h(x+c(t-s),s) - h(x-c(t-s),s)\big)$$

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Thus

$$\partial_t H = c\int_0^t h(x+c(t-s),s) + h(x-c(t-s),s)\,ds$$

$$\partial_{tt} H = 2c\,h(x,t)$$

$$+\, c^2 \int_0^t \partial_x \left(h(x+c(t-s),s) - h(x-c(t-s),s) \right) ds$$

*, We can replace ∂t with ∂x as t is a variable of the first argument hence taking the derivative along the first argument of h is equivalent of taking ∂x.

3.)

$$H(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h(y,s)\, dy\, ds$$

Let

$$g(x,t,s) = \int_{x-c(t-s)}^{x+c(t-s)} h(y,s)\, dy$$

$$\partial x\, g = h(x+c(t-s),s) - h(x-c(t-s),s)$$

Then by Leibniz Rule

$$\partial x\, H = \int_0^t h(x+c(t-s),s) - h(x-c(t-s),s)\, ds$$

Doing Leibniz rule again

$$\partial xx\, H = \int_0^t \partial x(h(x+c(t-s),s) - h(x-c(t-s),s))\, ds$$

Therefore if

$$\partial_{tt} H = 2c\,h(x,t)$$

$$+ c^2 \int_0^t \partial_x \big(h(x+c(t-s),s) - h(x-c(t-s),s)\big)\,ds$$

$$\partial_{xx} H = \int_0^t \partial_x \big(h(x+c(t-s),s) - h(x-c(t-s),s)\big)\,ds$$

Then

$$\partial_{tt} H - c^2 \partial_{xx} H = 2ch$$

4.) Let

$$U(x,t) = \tilde{U}(x,t) + \frac{1}{2c} H(x,t)$$

Then consider

$$\partial_{tt} U - c^2 \partial_{xx} U$$

$$= \partial_{tt}\left(\tilde{U}(x,t) + \frac{1}{2c} H(x,t) \right)$$

$$- c^2 \partial_{xx}\left(\tilde{U}(x,t) + \frac{1}{2c} H(x,t) \right)$$

Then by the properties of the derivative, one can rearrange to get that

$$\partial_{tt} U - c^2 \partial_{xx} U$$

$$= \partial_{tt} \tilde{U} - c^2 \partial_{xx} \tilde{U} + \frac{1}{2c}\left(\partial_{tt} H - c^2 \partial_{xx} H \right)$$

We know that

$$\partial_{tt} \tilde{U} - c^2 \partial_{xx} \tilde{U} = 0$$

$$\partial_{tt} H - c^2 \partial_{xx} H = 2c\,h(x,t)$$

Hence

$$\partial_{tt} U - c^2 \partial_{xx} U$$

$$= \frac{1}{2c}\left(2c\, h(x,t) \right)$$

$$= h(x,t)$$

Thus $U(x,t) = \tilde{U}(x,t) + \frac{1}{2c} H(x,t)$

is a solution to inhomogenous Cauchy problem

$\partial_{tt} U - c^2 \partial_{xx} U = h(x,t)$ for $(x,t) \in \mathbb{R} \times (0,\infty)$

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5.) Suppose there exists two unique solution to (1). Call these solutions U_1 and U_2. Then consider

$$W = U_1 - U_2$$

Then

$$\partial_{tt} W - c^2 \partial_{xx} W$$

$$= \partial_{tt}(U_1 - U_2) - c^2 \partial_{xx}(U_1 - U_2)$$

$$= (\partial_{tt} U_1 - c^2 \partial_{xx} U_1) - (\partial_{tt} U_2 - c^2 \partial_{xx} U_2)$$

$$= h(x,t) - h(x,t)$$

$$= 0$$

Thus W satisfies the following properties

$$\partial_{tt} W - c^2 \partial_{xx} W = 0 \quad (x,t) \in (0,\infty) \times \mathbb{R}$$

$$W(x,0) = 0 \quad \forall x \in \mathbb{R}$$

$$\partial_t W(x,0) = 0 \quad \forall x \in \mathbb{R}$$

Since it satisfies the following properties, we can use Lemma 3.5 to imply that the total energy $E_{WE}(t) = C$ for $C > 0$ $\forall t \in (0,\infty)$ (i.e total energy is constant the system).

Now consider

$$E_{WE}(0) = \frac{1}{2}\int_{\mathbb{R}} (\partial_t W(x,0))^2 + c^2(\partial_x W(x,0))^2$$

$$= \frac{1}{2}\int_{\mathbb{R}} 0 + 0 \, dx$$

$$= 0$$

Therefore, $E_{WE}(t) = 0 \ \forall t \geq 0$ due to the previous conclusions.

From the total energy being 0 at all times, we get the implication that

$$\partial_t W(x,t) = 0 \quad \text{and} \quad \partial_x W(x,t) = 0$$

$$\forall (x,t) \in \mathbb{R} \times (0,\infty)$$

This implies that $W(x,t)$ is constant $\forall (x,t) \in \mathbb{R} \times (0,\infty)$. Since $W(x,0) = 0$ then

$$W(x,t) = 0 \quad \forall (x,t) \in \mathbb{R} \times (0,\infty)$$

This implies that $U_1 = U_2$ and that there exists only a single unique solution to (1).

6.) $\Phi(x) = \cos(x)$, $V(x) = 1$ and $h(x,t) = e^{-t}$

Let $\tilde{U}(x,t)$ be the solution to the homog. case

$$\tilde{U}(x,t) = \frac{1}{2}\left(\cos(x+ct) + \cos(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} 1 \, dr$$

$$\tilde{U}(x,t) = \frac{1}{2}\left(\cos(x+ct) + \cos(x-ct)\right) + t$$

$$H(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{-t} \, dy \, ds$$

$$= \int_0^t 2c(t-s)e^{-t} \, ds$$

$$= \int_0^t 2cte^{-t} - 2cse^{-t} \, ds$$

$$= 2ct^2e^{-t} - \left[cs^2e^{-t} \right]_0^t$$

$$= ct^2e^{-t}$$

Therefore the full solution the Cauchy problem is given by

$$U(x,t) = \frac{1}{2}\left(\cos(x+ct) - \cos(x-ct)\right) + t + \frac{1}{2}t^2e^{-t}$$

$$U(x,t) = \frac{1}{2}\left(2\sin(x)\sin(-ct)\right) + t + \frac{1}{2}t^2e^{-t}$$

$$U(x,t) = -\sin(x)\sin(ct) + t + \frac{1}{2}t^2e^{-t}$$