MA250 Assignment 4 8.03.22

1a.) Let $d_1, d_2 > 0$ where

$$D = \begin{pmatrix} -d_1 & 0 \\ 0 & -d_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

Then $\text{trace}(D) = -d_1 - d_2$ and $\det(D) = d_1 d_2 > 0$.

Since $\text{trace}(D) < 0$ and $\det(D) > 0$ the the system is stable and will converges to $(0,0)^T$ as $t \to \infty$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -d_1 & 0 \\ 0 & -d_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$y_1' = -d_1 y_1 \implies y_1 = Ae^{-d_1 t}$$
$$y_2 = -d_2 y_2 \implies y_2 = Be^{-d_2 t}$$

Therefore if $y(0) = (2, -3)^T$ then

$$y(t) = \begin{pmatrix} 2e^{-d_1 t} \\ -3e^{-d_2 t} \end{pmatrix}$$

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b) Let $B = \begin{pmatrix} 2 & -4 \\ 4 & -5 \end{pmatrix}$ then

$\text{trace}(B) = -3$

$\det(B) = -10 - -16$

$\det(B) = 6$

Since $\text{trace}(B) < 0$ and $\det(B) > 0$ then the system is stable

(c)

$B + D = \begin{pmatrix} 2 & -4 \\ 4 & -5 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -15 \end{pmatrix}$

$= \begin{pmatrix} 1 & -4 \\ 4 & -20 \end{pmatrix}$

Let $A = B + D$ then

$\text{trace}(A) = -19$ and $\det(A) = -4$

Since $\text{trace}(A) = -19$ and $\det(A) = -4$ then one of the eigenvalues has a positive real part. Therefore, the system is unstable

2a.) If $\bar{u}$ is a stationary point then

$$v(\bar{u}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus

$$rb\left(1 + \frac{u_1^2}{u_2}\right) - \mu u_1 = 0$$

$$1 + \frac{u_1^2}{u_2} = \frac{\mu u_1}{rb}$$

Now, if $u_1^2 = \frac{\alpha u_2}{rb}$ then

$$1 + \frac{1}{u_2}\left(\frac{\alpha u_2}{rb}\right) = \frac{\mu u_1}{rb}$$

$$1 + \frac{\alpha}{rb} = \frac{\mu u_1}{rb}$$

$$\frac{rb}{\mu}\left(1 + \frac{\alpha}{rb}\right) = u_1$$

$$u_1 = \frac{rb + \alpha}{\mu}$$

$$u_2 = \frac{rb}{\alpha} \frac{rb^2}{\mu^2}\left(1 + \frac{\alpha}{rb}\right)^2$$

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$$u_1 = \frac{rb}{\mu}\left(1 + \frac{\alpha}{rb}\right)$$

$$u_2 = \frac{rb^3}{\alpha \mu^2}\left(1 + \frac{\alpha}{rb}\right)^2$$

Now, where $\bar{u} = (\bar{u}_1, \bar{u}_2)$ is a stationary point

Now

$$\partial_{u_1}\left(rb\left(1 + \frac{u_1^2}{u_2}\right) - \mu u_1\right)$$

$$= \frac{2u_1}{u_2} - \mu$$

$$\partial_{u_1} v_2 = 2rbu_1$$

$$\partial_{u_2} v_1 = -\frac{u_1^2}{u_2^2}$$

$$\partial_{u_2} v_2 = -\alpha$$

Therefore

$$B = \begin{pmatrix} \frac{2\bar{u}_1}{\bar{u}_2} - \mu & -\frac{\bar{u}_1^2}{\bar{u}_2^2} \\ 2rb\bar{u}_1 & -\alpha \end{pmatrix}$$

$$B_{11} = \frac{2rb}{\mu} \frac{\alpha \mu^2}{rb^3} \left(1 + \frac{\alpha}{rb}\right)^{-1}$$

$$B_{11} = \frac{2\alpha\mu}{rb^2} \left(1 + \frac{\alpha}{rb}\right)^{-1} - \mu$$

$$B_{21} = 2rb \left(\frac{rb}{\mu}\left(1 + \frac{\alpha}{rb}\right)\right)$$

$$B_{21} = \frac{2rb^2}{\mu} \left(1 + \frac{\alpha}{rb}\right)$$

$$B_{12} = -\frac{rb^2}{\mu^2} \frac{\alpha^2\mu^4}{rb^6} \left(1 + \frac{\alpha}{rb}\right)^{-2}$$

$$B_{12} = -\frac{\alpha^2\mu^2}{rb^4} \left(1 + \frac{\alpha}{rb}\right)^{-2}$$

$$B = \begin{pmatrix} \frac{2\alpha\mu}{rb^2}\left(1+\frac{\alpha}{rb}\right)^{-1} - \mu & -\frac{\alpha^2\mu^2}{rb^4}\left(1+\frac{\alpha}{rb}\right)^{-2} \\ \frac{2rb^2}{\mu}\left(1+\frac{\alpha}{rb}\right) & -\alpha \end{pmatrix}$$

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b.)

$$\operatorname{trace} B = \frac{2\alpha\mu}{rb^2}\left(1+\frac{\alpha}{rb}\right)^{-1} - \alpha - \mu$$

$$= \frac{2\alpha\mu}{rb(rb+\alpha)} - \alpha - \mu$$

$$\det B = -\alpha\left(\frac{2\alpha\mu}{rb^2}\left(1+\frac{\alpha}{rb}\right)^{-1} - \mu\right)$$

$$+ \frac{\alpha^2\mu^2}{rb^4}\,\frac{2rb^2}{\mu}\left(1+\frac{\alpha}{rb}\right)^{-1}$$

$$= -\frac{2\alpha^2\mu}{rb(rb+\alpha)} + \alpha\mu + \frac{2\alpha^2\mu}{rb(rb+\alpha)}$$

$$= \frac{2\alpha^2\mu - 2\alpha^2\mu}{rb(rb+\alpha)} + \alpha\mu$$

$$\operatorname{trace} B = \frac{2\alpha\mu}{rb(rb+\alpha)} - \alpha - \mu$$

$$\det B = \alpha\mu$$

Take $(\alpha, rb, \mu) = (1, 1, 1)$ Then

$$\text{trace } B = \frac{2}{2} - 1 - 1$$

$$\text{trace } B = -1$$

$$\det B = \frac{2-2}{2} + 1$$

$$\det B = 1$$

Since trace $B < 0$ and det $B > 0$ therefore the system is stable

3a.) Let $A = D_u + B$

$$\operatorname{trace} A = \frac{2\alpha\mu}{rb(rb+\alpha)} - \alpha - \mu - d_1|u|^2 - d_2|u|^2$$

$$\det A = \left(\frac{2\alpha\mu}{rb^2}\left(1+\frac{\alpha}{rb}\right)^{-1} - \mu - d_1|u|^2\right)$$

$$\left(-\alpha - d_2|u|^2\right)$$

$$+ \frac{2\alpha^2\mu}{rb(rb+\alpha)}$$

$$\det A = \left(\frac{2\alpha\mu}{rb(rb+\alpha)} - \mu - d_1|u|^2\right)$$

$$\left(-\alpha - d_2|u|^2\right)$$

$$+ \frac{2\alpha^2\mu}{rb(rb+\alpha)}$$

b.) $6 \le k \le 124$ (Programmed in Computer)

4b.) $(u_1, u_2) = (50, 50)$