MA250 Assignment I 22.01.22

1a.) $v(u, x, t) = \frac{ax}{(t+2)^2}$

$$\xi'(t) = \frac{a\,\xi(t)}{(t+2)^2}$$

$$\int \frac{1}{\xi(t)}\, d\xi(t) = \int \frac{a}{(t+2)^2}\, dt$$

$$\ln(\xi(t)) = -a(t+2)^{-1} + C$$

$$\xi(t) = A e^{\frac{-a}{t+2}}$$

$$\xi(0) = x_0$$

$$x_0 = A e^{-\frac{a}{2}}$$

$$x_0 e^{\frac{a}{2}} = A$$

$$\xi(t) = x_0 e^{\frac{a}{2}} e^{-\frac{a}{t+2}}$$

Let $\xi(t) = x$

$$x = x_0 e^{\frac{a}{2}} e^{\frac{-a}{t+2}}$$

$$x_0 = x e^{\frac{-a}{2}} e^{\frac{a}{t+2}}$$

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$U(x,t) = \Phi(x_0)$

$= \Phi\left(x e^{-\frac{a}{2}} e^{\frac{a}{t+2}} \right)$

Now if $x_0 \notin \left(\frac{1}{8}, \frac{7}{8}\right)$ then

$\Phi(x_0) = 0 \quad \Rightarrow \Phi\left(x e^{-\frac{a}{2}} e^{\frac{a}{t+2}}\right) = 0$

Therefore $U(x,t) = 0 \quad \forall t > 0.$

If $x_0 \in \left(\frac{1}{8}, \frac{1}{2}\right)$ then

$\Phi(x_0) = -\frac{8}{3}\left(\frac{1}{2} - x_0\right) + 1$

$= -\frac{8}{3}\left(\frac{1}{2} - x e^{-\frac{a}{2}} e^{\frac{a}{t+2}}\right) + 1$

$U(x,t) = -\frac{8}{3}\left(\frac{1}{2} - x e^{-\frac{a}{2}} e^{\frac{a}{t+2}}\right) + 1$

if $x_0 \in \left(\frac{1}{8}, \frac{1}{2}\right)$

Take $x_0 \in \left[\frac{1}{2}, \frac{7}{8}\right)$ then

by a similar process

$$U(x,t) = -\frac{8}{3}\left(x e^{-\frac{a}{2}} e^{\frac{a}{t+2}} - \frac{1}{2}\right) + 1$$

Hence

$$U(x,t) = \begin{cases} -\frac{8}{3}\left| x e^{-\frac{a}{2}} e^{\frac{a}{t+2}} - \frac{1}{2}\right| + 1 & \text{if } x_0 \in \left(\frac{1}{8}, \frac{7}{8}\right) \\ 0 & \text{if } x_0 \notin \left(\frac{1}{8}, \frac{7}{8}\right) \end{cases}$$

b.) $(x,t) = \left(\frac{5}{2}, 1\right)$

$$\frac{5}{2} e^{-3\log 5} e^{\frac{6\log 5}{3}} = \frac{5}{2} e^{-3\log 5} e^{2\log 5}$$

$$= \frac{5}{2} e^{-\log 5}$$

$$= \frac{1}{2}$$

This implies that

$$U\left(\frac{5}{2}, 1\right) = 1$$

2a.)

$$\partial_t U(x,t) + \frac{d}{dx}\left(q(U(x,t))\right) = 0$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$

$q(u) = u(1-u)(1-\beta u)$

$= u(\beta u^2 - \beta u - u + 1)$

$= \beta u^3 - \beta u^2 - u^2 + u$

$\frac{\partial}{\partial x} q(u)$

$= 3\beta u^2 \frac{\partial u}{\partial x} - 2\beta u \frac{\partial u}{\partial x} - 2u\frac{\partial u}{\partial x} + \frac{\partial u}{dx}$

$= \partial_x u \,(3\beta u^2 - 2\beta u - 2u + 1)$

Thus

$\partial_t U + (3\beta u^2 - 2\beta u - 2u + 1)\partial_x U = 0$

for $(x,t) \in \mathbb{R} \times (0,\infty)$

b.) $\beta = 0$ then

$$\partial_t u + \partial_x u(1-2u) = 0$$

where $V(u,x,t) = 1-2u$

$$\xi'(t) = 1 - 2u(\xi(t), t)$$

$$\frac{d}{dt}\left(u(\xi(t), t)\right)$$

$$= \partial_x u(\xi(t),t)\,\xi'(t) + \partial_t u(\xi(t),t)$$

$$= \partial_x u(\xi(t),t)\left(1-2u(\xi(t),t)\right) + \partial_t u'(\xi(t), t$$

$$= 0$$

$$\xi'(t) = 1 - 2\Phi(x_0)$$

$$\xi(t) = t - 2t\,\Phi(x_0)$$

$$x = t - 2t\left(1 - \frac{1}{3}x_0\right) + x_0$$

$$x = t - 2t + \frac{2}{3}tx_0 + x_0$$

$$x = \frac{2}{3}tx_0 - t + x_0$$

$$3x = 2tx_0 - 3t + 3x_0$$

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$$\frac{3}{2}x + \frac{3}{2}t = tx_0 + \frac{3}{2}x_0$$

$$3x + 3t = 2tx_0 + 3x_0$$

$$3x + 3t = x_0(2t+3)$$

$$x_0 = \frac{3x+3t}{2t+3}$$

$$U(x,t) = 1 - \frac{1}{3}\left(\frac{3x+3t}{2t+3}\right)$$

$$= 1 - \frac{x+t}{2t+3}$$

$$= \frac{2t+3-(x+t)}{2t+3}$$

$$= \frac{t+3-x}{2t+3}$$

$$U\left(\frac{5}{2}, 1\right) = \frac{1+3-\frac{5}{2}}{5}$$

$$= \frac{4-\frac{5}{2}}{5}$$

$$= \frac{3}{10}$$

$U(\frac{5}{2}, 1) = \frac{3}{10}$

(c.) Assume a characteristic does exist over $(\frac{5}{2}, 1)$ for $\beta = \frac{3}{4}$

For our characteristic curve $\xi(t)$, we get that

$\xi'(t) = \frac{9}{4}u^2 - \frac{6}{4}u - 2u + 1$

Take $x_0 \leq 2$ then $U(\xi(t), t) = \Phi(x_0) = 1$

$\xi'(t) = \frac{9}{4} - \frac{6}{4} - 2 + 1$

$= \frac{3}{4} - 2 + 1$

$= -\frac{1}{4}$

$\xi(t) = -\frac{1}{4}t + x_0$

Subbing in $x = \frac{5}{2}$ and $t = 1$ then

$x_0 = \frac{11}{2}$ which is greater than 2

Hence, $\Phi\left(\frac{11}{2}\right) \neq 1$ therefore

a characteristic does not exist for $\left(\frac{5}{2}, 1\right)$ when $x_0 \leq 2$.

If $x_0 > 2$ then $\Phi(x_0) = 0$ then

$\xi'(t) = 1$

Hence,

$x = x_0 + t \Rightarrow x_0 = x - t$

Now, taking $x = \frac{5}{2}$ and $t = 1$, we get

$x_0 = \frac{3}{2}$

then $\Phi(x_0) = 1$ which is a contradiction as we assumed $x > 2$ then $\Phi(x_0) = 0$

Therefore, there is no characteristic through the point $\left(\frac{5}{2}, 1\right)$ that starts in any point $x_0 \in \mathbb{R}$ at time $t = 0$

3b.) N = 135