Dr Björn Stinner Term 2, 2021/2022

MA250 Introduction to PDEs
Assignment 2

Please submit your answers to all questions via Moodle by

Friday 4 February 2022 (week 4), 12 noon (UK time).

Some but possibly not all questions will be marked for credit.

In this assignment we consider the *inhomogeneous* Cauchy problem for the wave equation

$$\left.\begin{aligned} \partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) &= h(x,t), && (x,t) \in \mathbb{R}\times(0,\infty), \\ u(x,0) &= \Phi(x), && x \in \mathbb{R}, \\ \partial_t u(x,0) &= V(x), && x \in \mathbb{R}, \end{aligned}\right\} \tag{1}$$

where $h(x,t)$, $\Phi(x)$, and $V(x)$ are given smooth functions and $c > 0$.
You may use in the following that d'Alembert's formula

$$\tilde{u}(x,t) = \frac{1}{2}\big(\Phi(x+ct) + \Phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} V(r)dr \tag{2}$$

yields a solution to the *homogeneous* Cauchy problem (i.e., $h = 0$ in (1)).

1. Use Leibniz' rule to compute the derivative of $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(y) = \int_{y^2}^{e^y} (\sin(z) + 4y)dz.$$

2. Consider the function
$$H(x,t) = \int_0^t \Big(\int_{x-c(t-s)}^{x+c(t-s)} h(y,s)dy\Big)ds.$$
Compute $\partial_t H(x,t)$ and then $\partial_{tt}H(x,t)$.
(*Hint: In Leibniz' rule the integrand may be an integral again, and you can apply it here because all functions are assumed to be smooth.*)

3. Now show that the function H from the previous parts satisfies
$$\partial_{tt}H(x,t) - c^2\partial_{xx}H(x,t) = 2c\,h(x,t).$$

4. Use the result from the previous part and $\tilde{u}$ defined in (2) to show that
$$u(x,t) = \tilde{u}(x,t) + \frac{1}{2c}H(x,t)$$
solves the Cauchy problem (1).

5. Prove that the solution to (1) is unique by considering the difference $w = u^{(1)} - u^{(2)}$ of two solutions in $C^2(\mathbb{R}\times(0,\infty))$ and using an 'energy argument'.

6. Find the solution to (1) for the specific data
$$\Phi(x) = \cos(x), \quad V(x) = 1, \quad h(x,t) = e^{-t}.$$