Dr Björn Stinner Term 2, 2021/2022

MA250 Introduction to PDEs
Assignment 3

Please submit your answers to all questions via Moodle by

Friday 25 February 2022 (week 7), 12 noon (UK time).

Some but possibly not all questions will be marked for credit.

---

1. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function such that f and its derivative f' are piecewise continuous. More precisely, the limits
$$f(x_\pm) := \lim_{\delta \searrow 0} f(x \pm \delta)$$
exist for all points $x \in \mathbb{R}$ and coincide except for a finite number of points on $(-\pi, \pi]$, and analogously for
$$f'(x_\pm) := \lim_{\delta \searrow 0} f'(x \pm \delta).$$
You may assume in the following that the *Riemann-Lebesgue* lemma holds true for piecewise continuous functions.

   (a) Show that the trigonometric polynomials with the Fourier coefficients of f satisfy
$$S_n(f)(x) = \frac{1}{2} S_n^+(f)(x) + \frac{1}{2} S_n^-(f)(x) \quad \text{where}$$
$$S_n^\pm(f)(x) := \int_{-\pi}^{\pi} K_n(\theta) f(x \pm |\theta|) d\theta,$$
which features the Dirichlet kernel
$$K_n(\theta) = \frac{1}{2\pi} \sum_{|k| \le n} e^{ik\theta} = \frac{\sin((n + \frac{1}{2})\theta)}{2\pi \, \sin(\frac{1}{2}\theta)}.$$
Hint: Starting with a suitable integral formula for $S_n(f)(x)$, split the integral $\int_{-\pi}^{\pi} \dots = \int_{-\pi}^{0} \dots + \int_{0}^{\pi} \dots$ and use suitable extensions of the integrand.

   (b) Show that $S_n^\pm(f)(x) \to f(x_\pm)$ as $n \to \infty$ and conclude that
$$S_n(f)(x) \to \frac{1}{2}\big(f(x_+) + f(x_-)\big).$$

2. Retrieve one of the files (depending on what you prefer to work with)

   `FSapprox.mlx` (Matlab) or `FSapprox.py` (Python).

   Each file contains an implementation of the function
$$\phi(x) = 2\sin(3x) + 0.7\sin(11x).$$
   This function is sampled at $N \in \mathbb{N}$ equispaced points in $[-\pi, \pi]$ and plotted (blue).
   Then a discrete (fast) Fourier transform (`fft`) is applied to the sample, and absolute values of the amplitues are plotted (black).
   After, for a given number $n \in \mathbb{N}$, $n < N$, the amplitudes are set to zero for all frequencies bigger than n. The manipulated amplitudes are plotted again (green).
   Finally, the cut-off amplitudes are transformed back with an inverse discrete (fast) Fourier transform (`ifft`). The resulting (sample of a) function is plotted in red, together with the original function sample (blue).

(a) State with some justification which minimal frequency n is required to exactly recover the above function ϕ.
(By changing n in the code you can check that this minimal n indeed is sufficient, i.e., in the last plot the blue and the red curve coincide; no need to report on your checks in your submission.)

(b) Replace now the function ϕ with

$$\phi(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Submit your implementation of this function (just this function, no other code).
Using the result from the previous question, state the analytical value of $S_n(\phi)(x)$ in $x = 0$.
(Check for yourself that this is also true for the computational values for several values of n; no need to report on these checks in your submission.)
Provide a picture of the last plot with the function ϕ (blue) and the approximating Fourier series (red) for $n = 21$.

Remark: In the last plot, you should observe that the Fourier series 'overshoots' close to the jump point, and the amount by which it overshoots is rather independent of n. This observation is known as Gibbs phenomenon. *Appendix D in the lecture notes contains more detail.*