Assignment 3 MA2SO 19.02.22

1a.)

$$S_n(f)(x) = \int_{-\pi}^{\pi} K_n(x-z) f(z)\, dz$$

Take $\theta = x - z$, then

$$S_n(f)(x) = \int_{\pi}^{-\pi} K_n(\theta) f(x-\theta)\, d\theta$$

$$= \int_{\pi}^{0} K_n(\theta) f(x-\theta)\, d\theta + \int_{0}^{-\pi} K_n(\theta) f(x-\theta)\, d\theta$$

Note that by symmetry along $\theta = 0$ for a fixed x then one can say

$$\int_{\pi}^{0} K_n(\theta) f(x-\theta)\, d\theta$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} K_n(\theta) f(x - |\theta|)\, d\theta$$

$$\int_{0}^{-\pi} K_n(\theta) f(x-\theta)\, d\theta$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} K_n(\theta) f(x+|\theta|)\, d\theta \quad *$$

To be clear, one can write the torme intergrands in the form of $*$ due to the symmetry along $\theta = 0$ for a fixed 0. with $\pm|\theta|$ enabling to take the integral with the same value both from the left and the right. Hence, where these exist.

Therefore.

$$S_n(f)(x) = \frac{1}{2} S_n^{+}(f)(x) + \frac{1}{2} S_n^{-}(f)(x)$$

b.) Consider

$$f(x\pm) - S_n^{\pm}(f)(x)$$

$$= \int_{-\pi}^{\pi} K_n(\theta)\,d\theta\,\left(f(x\pm) - S^{\pm}(f)(x)\right)$$

$$= \int_{-\pi}^{\pi} K_n(\theta)\left(f(x\pm) - f(x \pm |\theta|)\right)d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(n\theta)\cos\left(\frac{\theta}{2}\right)\frac{\left(f(x\pm) - f(x\pm|\theta|)\right)}{\sin\left(\frac{\theta}{2}\right)}\,d\theta$$

$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(n\theta)\left(f(x\pm) - f(x\pm|\theta|)\right)d\theta$$

Since Riemann-Lebesgue lemma holds for piecewise continuous functions then

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} \cos(n\theta)\left(f(x\pm) - f(x\pm|\theta|)\right)d\theta = 0$$

as $\theta \mapsto \left(f(x\pm) - f(x\pm|\theta|)\right)$ is a piecewise continuous function.

Now consider

$$\lim_{\theta \to 0} \frac{f(x_\pm) - f(x \pm |\theta|)}{\sin\left(\frac{\theta}{2}\right)}$$

Then, since we have $|\theta|$, consider the one hand limit

$$\lim_{\theta \to 0^+} \frac{f(x_\pm) - f(x \pm \theta)}{\sin\left(\frac{\theta}{2}\right)} \quad *$$

Then by symmetry along x, the limit from the left, i.e $\theta \to 0^-$, will be equivalent to the limit from the right.

Eventhough, f is piecewise continuous, by considering the domain of f as punctured, we can still apply L'Hôpital's rule to $*$ to get

$$\lim_{\theta \to 0^+} \frac{f(x_\pm) - f(x \pm \theta)}{\sin\left(\frac{\theta}{2}\right)} = \mp 2 f(x)$$

Since all function involved a 2π periodic then for all $\theta = 2\pi m$ for $m \in \mathbb{Z}$, piecewise continuity is preserved and that $*$ is not unbounded. Therefore

$$\theta \mapsto \cos\left(\frac{\theta}{2}\right) \frac{f(x_\pm) - f(x \pm |\theta|)}{\sin\left(\frac{\theta}{2}\right)}$$

can be extended to a piecewise continuous function where $\theta = 0$

Finally, one can apply the Riemann-Lebesgue lemma for piecewise functions to state that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(n\theta)\cos\left(\frac{\theta}{2}\right)\frac{f(x\pm)-f(x\pm|\theta|)}{\sin\left(\frac{\theta}{2}\right)} \to 0$$

as $n \to \infty$

Therefore, as $n \to \infty$,

$$f(x\pm) - S_n^{\pm}(f)(x) \to 0$$

as $n \to \infty$. Hence,

$$S_n^{\pm}(f)(x) \to f(x\pm)$$

Then

$$S_n(f)(x) \to \frac{1}{2}\left(f(x+) + f(x-)\right)$$

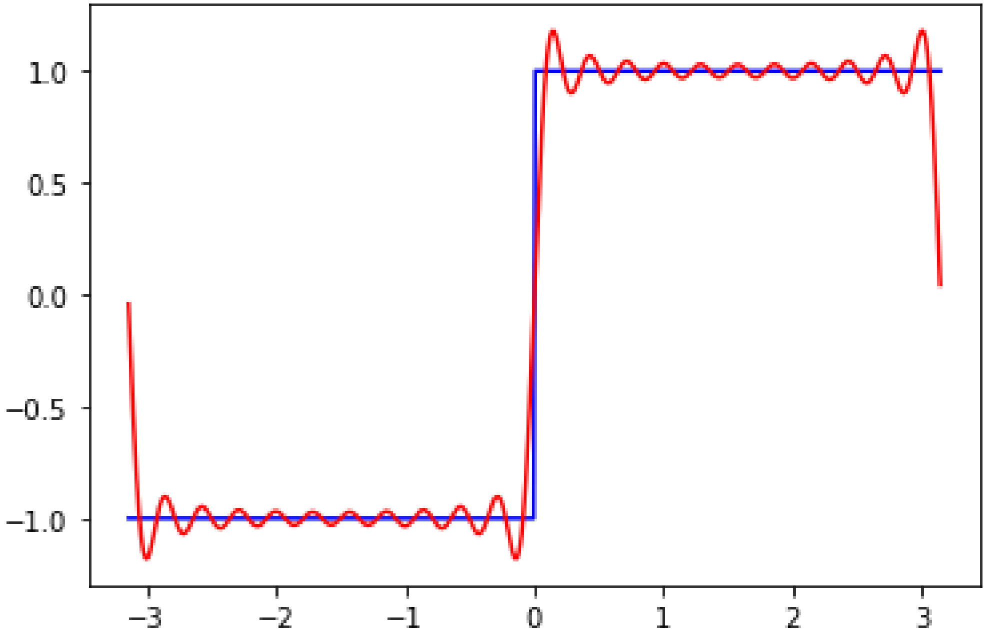
as $n \to \infty$

2a.) Since

$$\phi(x) = 2\sin(3x) + 0.7\sin(11x)$$

Then the maximum frequency any component wave (i.e. $2\sin(3x)$ and $0.7\sin(11x)$) can get is 11 due to $\phi(x)$ being the sum of $2\sin(3x)$ and $0.7\sin(11x)$. Hence $n=11$ as all frequencies greater than 11 do not matter to ϕ.

b.) $S_n(\phi)(0) = 0$


1.0
0.5
0.0
−0.5
−1.0
−3
−2
−1
0
1
2
3

```python
def phifct(x):
    if x>0:
        return 1
    if x<0:
        return -1
    return 0
```