

Assignment 3 MA250

19.02.22

1a.)

$$S_n(f)(x) = \int_{-\pi}^{\pi} k_n(x-z) f(z) dz$$

Take $\theta = x - z$, then

$$S_n(f)(x) = \int_{\pi}^{0} k_n(\theta) f(x-\theta) d\theta$$

$$= \int_{\pi}^{0} k_n(\theta) f(x-\theta) d\theta + \int_{0}^{-\pi} k_n(\theta) f(x-\theta) d\theta$$

Note that by symmetry along $\theta = 0$ for a fixed x , then one can say

$$\int_{\pi}^{0} k_n(\theta) f(x-\theta) d\theta$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} k_n(\theta) f(x-|\theta|) d\theta$$

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$$\begin{aligned}
 & \int_0^{-\pi} K_n(\theta) f(x-\theta) d\theta \\
 &= \frac{1}{2} \int_{-\pi}^{\pi} K_n(\theta) f(x+|\theta|) d\theta *
 \end{aligned}$$

To be clear, one can write the terms integrands in the form of * due to the symmetry along $\theta = 0$ for a fixed θ with $|\theta|$ enabling to take the integral with same value both from the left and the right. Hence, where there exist.

Therefore,

$$S_n(f)(x) = \frac{1}{2} S_n^+(f)(x) + \frac{1}{2} S_n^-(f)(x)$$

b.) Consider

$$f(x_{\pm}) - S_n^{\pm}(f)(x)$$

$$= \int_{-\pi}^{\pi} k_n(\theta) d\theta (f(x_{\pm}) - S_n^{\pm}(f)(x))$$

$$= \int_{-\pi}^{\pi} k_n(\theta) (f(x_{\pm}) - f(x \pm |\theta|)) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(n\theta) \cos\left(\frac{\theta}{2}\right) (f(x_{\pm}) - f(x \pm |\theta|)) d\theta$$

$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\theta) (f(x_{\pm}) - f(x \pm |\theta|)) d\theta$$

Since Riemann-Lebesgue lemma holds for piecewise continuous functions then

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} \cos(n\theta) (f(x_{\pm}) - f(x \pm |\theta|)) d\theta = 0$$

as $\theta \mapsto (f(x_{\pm}) - f(x \pm |\theta|))$ is a piecewise continuous function.

Now consider

$$\lim_{\theta \rightarrow 0} \frac{f(x \pm) - f(x \pm |\theta|)}{\sin(\frac{\theta}{2})}$$

Then, since we have $|\theta|$, consider the one hand limit

$$\lim_{\theta \rightarrow 0^+} \frac{f(x \pm) - f(x \pm \theta)}{\sin(\frac{\theta}{2})} *$$

Then by symmetry along x , the limit from the left, i.e. $\theta \rightarrow 0^-$, will be equivalent to the limit from the right.

Even though, f is piecewise continuous, by considering the domain of f as punctured, we can still apply L'Hospital's rule to $*$ to get

$$\lim_{\theta \rightarrow 0^+} \frac{f(x \pm) - f(x \pm \theta)}{\sin(\frac{\theta}{2})} = \mp 2f'(x)$$

Since all function involved a 2π periodic then for all $\theta = 2\pi m$ for $m \in \mathbb{Z}$, piecewise continuity is preserved and that $*$ is not unbounded. Therefore

$$\theta \mapsto \cos\left(\frac{\theta}{2}\right) \frac{f(x \pm) - f(x \pm |\theta|)}{\sin\left(\frac{\theta}{2}\right)}$$

can be extended to a piecewise continuous function where $\theta = 0$

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Finally, one can apply the Riemann-Lebesgue lemma for piecewise functions to state that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(n\theta) \cos\left(\frac{\theta}{2}\right) \frac{f(x_+) - f(x_-)}{\sin\left(\frac{\theta}{2}\right)} d\theta \rightarrow 0$$

as $n \rightarrow \infty$

Therefore, as $n \rightarrow \infty$,

$$f(x_+) - S_n(f)(x) \xrightarrow{+} 0$$

as $n \rightarrow \infty$. Hence,

$$S_n^+(f)(x) \rightarrow f(x_+)$$

Then

$$S_n(f)(x) \rightarrow \frac{1}{2} (f(x_+) + f(x_-))$$

as $n \rightarrow \infty$

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2a.) Since

$$\phi(x) = 2\sin(3x) + 0.7\sin(11x)$$

Then the maximum frequency any component wave (i.e. $2\sin(3x)$ and $0.7\sin(11x)$) can get is 11 due to $\phi(x)$ being the sum of $2\sin(3x)$ and $0.7\sin(11x)$. Hence $n=11$ as all frequencies greater than 11 do not matter to ϕ .

b.) $S_n(\phi)(0) = 0$