6

Batch State Estimation

A state without the means of some change is without the means of its conservation. Burke, Edmund

THE previous chapter allows estimation of the states in the model of a dynamical system using *sequential measurements*. We found that the sequential estimation results of §1.3 and the probability concepts introduced in Chapter 2, developed for estimation of *algebraic* systems, remain valid for estimation of *dynamical* systems upon making the appropriate new interpretations of the matrices involved in the estimation algorithms. Specifically, taking a measurement at the current time and an estimate of the state at the previous time with knowledge of its error properties, the methods of Chapter 5 are used to produce a state estimate of the dynamic system at the current time. In this chapter the results of the previous chapter are extended to batch state estimation. The disadvantage of batch estimation methods is they cannot be implemented in real time; however, they have the advantage of providing state estimates with a lower error-covariance than sequential methods. This may be extremely helpful when accuracy is an issue, but real-time application is not required.

The batch methods shown in this chapter are also known as *smoothers*, since they typically are used to "smooth" out the effects of measurement noise. Basically, smoothers are used to estimate the state quantities using measurements made before and after a certain time t. To accomplish this task, two filters are usually used (see Figure 6.1): a forward-time filter and a backward-time filter.[1] Three types of smoothers are usually defined:

1. *Fixed-Interval Smoothing*. This smoother uses the entire batch of measurements over a fixed interval to estimate all the states in the interval. The times 0 and T are fixed and t varies from time 0 to T in this formulation. Since the entire batch of measurements are used to produce an estimate, this smoother provides the best possible estimate over the interval.

2. *Fixed-Point Smoothing*. This smoother estimates the state at a specific fixed point in time t, given a batch of measurements up to the current time T. This smoother is often used to estimate the state at only one time point in the interval.

3. *Fixed-Lag Smoothing*. This smoother estimates the state at a fixed time interval that lags the time of the current measurement at time T. This smoother is often used to refine the optimal forward filter estimate.

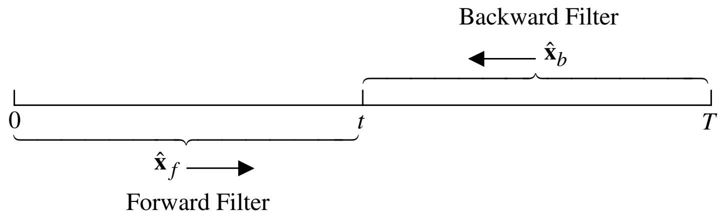

Figure 6.1: Forward-Time and Backward-Time Filtering

The fixed-point and fixed-lag smoothers are batch processes only in the sense that they require measurements up to the current time. The derivation of all of these smoothers can be given from the Kalman filter. In fact, all smoothers use the Kalman filter for forward-time filtering.

The history of smoothing actually predates the Kalman filter. Wiener[2] solved the original fixed-lag smoothing problem in the 1940s, but he only considered the stationary case where the smoother assumes that the entire past history of the input is available for weighting in its estimate.[3] The first practical smoothing algorithms are attributed to Bryson and Frazier,[4] as well as Rauch, Tung, and Striebel (RTS).[5] In particular, the RTS smoothing algorithm has maintained its popularity since the initial paper, and is most likely the most widely used algorithm for smoothing to date.

6.1 Fixed-Interval Smoothing

As mentioned previously, fixed-interval smoothing uses the entire batch of measurements over a fixed interval to estimate all the states in the interval. Fraser and Potter[6] have shown that this smoother can be derived from a combination of two Kalman filters, one of which works forward over the data and the other of which works backward over the fixed interval. Together these two filters use all the available information to provide optimal estimates. Earlier work[4,5] gives the smoother estimate as a correction to the Kalman filter estimate for the same point, and others[7,8] do not have the appearance of a correction to the Kalman filter estimate. All are mathematically equivalent, but the required computations are different for each approach.[9]

6.1.1 Discrete-Time Formulation

We begin our introduction of fixed-interval smoothing by considering discrete-time models and measurements, where the true system is modelled by eqn. (5.27):

$$\boxed{\begin{aligned}\mathbf{x}_{k+1} &= \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k \\ \tilde{\mathbf{y}}_k &= H_k \mathbf{x}_k + \mathbf{v}_k\end{aligned}} \quad \begin{aligned}&(6.1a)\\&(6.1b)\end{aligned}$$

where $\mathbf{w}_k \sim N(\mathbf{0}, Q_k)$ and $\mathbf{v}_k \sim N(\mathbf{0}, R_k)$. The optimal smoother is given by a combination of the estimates of two filters: one, denoted by $\hat{\mathbf{x}}_{fk}$, is given from a filter that runs from the beginning of the data interval to time t, and the other, denoted by $\hat{\mathbf{x}}_{bk}$, that works backward from the end of the time interval. The first step of the optimal smoother involves using the forward Kalman filter summarized in Table 5.1:

forward filter

$$\hat{\mathbf{x}}^-_{fk+1} = \Phi_k \hat{\mathbf{x}}^+_{fk} + \Gamma_k \mathbf{u}_k \tag{6.2a}$$

$$P^-_{fk+1} = \Phi_k P^+_{fk} \Phi^T_k + \Upsilon_k Q_k \Upsilon^T_k \tag{6.2b}$$

$$\hat{\mathbf{x}}^+_{fk} = \hat{\mathbf{x}}^-_{fk} + K_{fk}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}^-_{fk}] \tag{6.2c}$$

$$P^+_{fk} = [I - K_{fk} H_k] P^-_{fk} \tag{6.2d}$$

$$K_{fk} = P^-_{fk} H^T_k [H_k P^-_{fk} H^T_k + R_k]^{-1} \tag{6.2e}$$

The basic Kalman filter structure incorporates a measurement update at time t_k to give $\hat{\mathbf{x}}^+_{fk}$. To derive the backward filter we solve eqn. (6.1a) for $\mathbf{x}_k$, which gives

$$\mathbf{x}_k = \Phi^{-1}_k \mathbf{x}_{k+1} - \Phi^{-1}_k \Gamma_k \mathbf{u}_k - \Phi^{-1}_k \Upsilon_k \mathbf{w}_k \tag{6.3}$$

Clearly, the inverse of Φ must exist, meaning that the state matrix has no zero eigenvalues, but we shall see that the final form of the backward filter does not depend on this condition. The backward estimate is provided by the backward-running filter just *before* the measurement at time t_k.[10] Hence, the backward-time state propagation, denoted by $\hat{\mathbf{x}}^-_{bk}$, is given by

$$\hat{\mathbf{x}}^-_{bk} = \Phi^{-1}_k \hat{\mathbf{x}}^+_{bk+1} - \Phi^{-1}_k \Gamma_k \mathbf{u}_k \tag{6.4}$$

Comparing eqn. (6.4) with eqn. (6.2a) indicates that the backward filter time update and propagation roles are reversed from the forward filter, which is due to the measurement at time t_k going backward in time.

We seek a smoothed estimate that is a function of $\hat{\mathbf{x}}^+_{fk}$ and $\hat{\mathbf{x}}^-_{bk}$. Specifically, using methods similar to the methods of §2.1.2, we seek an optimal estimate that is a linear combination of the forward and backward estimates, given by

$$\hat{\mathbf{x}}_k = M_k \hat{\mathbf{x}}^+_{fk} + N_k \hat{\mathbf{x}}^-_{bk} \tag{6.5}$$

Next, following the error state definitions in §5.3.1, with $\tilde{\mathbf{x}}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k$, $\tilde{\mathbf{x}}^+_{fk} = \hat{\mathbf{x}}^+_{fk} - \mathbf{x}_k$, and $\tilde{\mathbf{x}}^-_{bk} = \hat{\mathbf{x}}^-_{bk} - \mathbf{x}_k$, leads to

$$\tilde{\mathbf{x}}_k = [M_k + N_k - I]\mathbf{x}_k + M_k \tilde{\mathbf{x}}^+_{fk} + N_k \tilde{\mathbf{x}}^-_{bk} \tag{6.6}$$

Clearly an unbiased state estimate (see §2.2) requires

$$N_k = I - M_k \tag{6.7}$$

Therefore, substituting eqn. (6.7) into eqn. (6.5) yields

$$\hat{\mathbf{x}}_k = M_k \hat{\mathbf{x}}_{fk}^{+} + [I - M_k]\hat{\mathbf{x}}_{bk}^{-} \tag{6.8}$$

We now define the following covariance expressions:

$$P_k \equiv E\left\{\tilde{\mathbf{x}}_k \tilde{\mathbf{x}}_k^T\right\} \tag{6.9a}$$

$$P_{fk}^{+} \equiv E\left\{\tilde{\mathbf{x}}_{fk}^{+} \tilde{\mathbf{x}}_{fk}^{+T}\right\} \tag{6.9b}$$

$$P_{bk}^{-} \equiv E\left\{\tilde{\mathbf{x}}_{bk}^{-} \tilde{\mathbf{x}}_{bk}^{-T}\right\} \tag{6.9c}$$

where P_k is the smoother error covariance, P_{fk}^{+} is the forward-filter error covariance, and P_{bk}^{-} is the backward-filter error covariance. Since the forward and backward processes are uncorrelated, then from eqns. (6.6) and (6.9), the smoother covariance can be written as

$$P_k = M_k P_{fk}^{+} M_k^T + [I - M_k] P_{bk}^{-} [I - M_k]^T \tag{6.10}$$

The optimal expression for M_k is given by minimizing the trace of P_k. The necessary conditions, i.e., differentiating with respect to M_k, lead to

$$0 = 2M_k P_{fk}^{+} - 2[I - M_k] P_{bk}^{-} \tag{6.11}$$

Solving eqn. (6.11) for M_k gives

$$M_k = P_{bk}^{-}[P_{fk}^{+} + P_{bk}^{-}]^{-1} \tag{6.12}$$

Also, $I - M_k$ is given by

$$\begin{aligned} I - M_k &= [P_{fk}^{+} + P_{bk}^{-}][P_{fk}^{+} + P_{bk}^{-}]^{-1} - P_{bk}^{-}[P_{fk}^{+} + P_{bk}^{-}]^{-1} \\ &= P_{fk}^{+}[P_{fk}^{+} + P_{bk}^{-}]^{-1} \end{aligned} \tag{6.13}$$

Substituting eqns. (6.12) and (6.13) into eqn. (6.10) and performing some algebraic manipulations (which are left as an exercise for the reader) yields

$$\boxed{P_k = \left[(P_{fk}^{+})^{-1} + (P_{bk}^{-})^{-1}\right]^{-1}} \tag{6.14}$$

Let us consider the physical connotation of eqn. (6.14). For scalar systems eqn. (6.14) reduces down to

$$p_k = \frac{p_{fk}^{+}\, p_{bk}^{-}}{p_{fk}^{+} + p_{bk}^{-}} \tag{6.15}$$

Equation (6.15) clearly shows that $p_k \le p_{fk}^+$ and $p_k \le p_{bk}^-$, which indicates that the smoother error covariance is always less than or equal to either the forward or backward covariance. Therefore, the smoother estimate is always better than either filter alone. This analysis can easily be expanded to higher-order systems.

Equation (6.14) involves matrix inverses of both P_{fk}^+ and P_{bk}^-. The inverse of P_{fk}^+ can be avoided though. We first define the following quantities: $\mathcal{P}_{bk}^- \equiv (P_{bk}^-)^{-1}$ and $\mathcal{P}_{fk}^+ \equiv (P_{fk}^+)^{-1}$. Then, using the matrix inversion lemma in eqn. (1.70) with $A = \mathcal{P}_{fk}^+$, $B = \mathcal{P}_{bk}^-$, and $C = D = I$ leads to

$$\boxed{P_k = P_{fk}^+ - P_{fk}^+ \mathcal{P}_{bk}^- [I + P_{fk}^+ \mathcal{P}_{bk}^-]^{-1} P_{fk}^+} \tag{6.16}$$

Note that eqn. (6.16) requires only one matrix inverse. Equation (6.16) can be further expanded into a symmetric form:

$$P_k = [I - W_k \mathcal{P}_{bk}^-] P_{fk}^+ [I - W_k \mathcal{P}_{bk}^-]^T + W_k \mathcal{P}_{bk}^- W_k^T \tag{6.17}$$

where

$$W_k = P_{fk}^+ [I + P_{fk}^+ \mathcal{P}_{bk}^-]^{-T} \tag{6.18}$$

Equation (6.17) is the sum of two positive definite matrices, which is equivalent to Joseph's stabilized version shown by eqn. (5.39), and provides a more robust approach in terms of numerical stability.

Substituting eqns. (6.12) and (6.13) into eqn. (6.8) and using eqn. (6.14) leads to

$$\hat{\mathbf{x}}_k = P_k \left[(P_{fk}^+)^{-1} \hat{\mathbf{x}}_{fk}^+ + (P_{bk}^-)^{-1} \hat{\mathbf{x}}_{bk}^- \right] \tag{6.19}$$

Equation (6.19) shows the optimal weighting of the forward and backward state estimates to produce the smoothed estimate. Equation (6.19) is also known as *Millman's theorem*,[11] which is also an exact analog to maximum likelihood of a scalar with independent measurements (see exercise 2.7 in Chapter 2 with $\rho = 0$). Equation (6.19) also involves matrix inverses of both P_{fk}^+ and P_{bk}^-. The inverse of P_{fk}^+ can be avoided by substituting eqn. (6.14) into eqn. (6.19) and factoring, which yields

$$\hat{\mathbf{x}}_k = [I + P_{fk}^+ \mathcal{P}_{bk}^-]^{-1} \hat{\mathbf{x}}_{fk}^+ + P_k \mathcal{P}_{bk}^- \hat{\mathbf{x}}_{bk}^- \tag{6.20}$$

Using the matrix inversion lemma in eqn. (1.70) with $A = I$, $B = P_{fk}^+ \mathcal{P}_{bk}^-$, and $C = D = I$ leads to

$$\boxed{\hat{\mathbf{x}}_k = [I - K_k] \hat{\mathbf{x}}_{fk}^+ + P_k \mathcal{P}_{bk}^- \hat{\mathbf{x}}_{bk}^-} \tag{6.21}$$

where the smoother gain is defined by

$$\boxed{K_k \equiv P_{fk}^+ \mathcal{P}_{bk}^- [I + P_{fk}^+ \mathcal{P}_{bk}^-]^{-1}} \tag{6.22}$$

Equation (6.21) gives the desired form for the smoothed state estimate using the combined forward and backward state estimates.

With the definitions of $\mathcal{P}_{bk}^-$ and $\mathcal{P}_{bk}^+$, the inverse of the backward update covariance follows directly from the information filter of §5.3.3, given by eqn. (5.73):

$$\mathcal{P}_{bk}^+ = \mathcal{P}_{bk}^- + H_k^T R_k^{-1} H_k \tag{6.23}$$

To derive a backward recursion for $\mathcal{P}_{bk}^-$ we first subtract eqn. (6.3) from eqn. (6.4), and use the error definitions $\tilde{\mathbf{x}}_{bk}^- = \hat{\mathbf{x}}_{bk}^- - \mathbf{x}_k$ and $\tilde{\mathbf{x}}_{bk}^+ = \hat{\mathbf{x}}_{bk}^+ - \mathbf{x}_k$ to give

$$\tilde{\mathbf{x}}_{bk}^- = \Phi_k^{-1}\tilde{\mathbf{x}}_{bk+1}^+ + \Phi_k^{-1}\Upsilon_k \mathbf{w}_k \tag{6.24}$$

Since $\tilde{\mathbf{x}}_{bk}^+$ and $\mathbf{w}_k$ are uncorrelated, then applying the definition in eqn. (6.9c) with eqn. (6.24) leads to the following backward covariance propagation:

$$P_{bk}^- = \Phi_k^{-1}[P_{bk+1}^+ + \Upsilon_k Q_k \Upsilon_k^T]\Phi_k^{-T} \tag{6.25}$$

The inverse of eqn. (6.25) gives the desired result; however, straightforward implementation of this scheme requires computing P_{bk+1}^+, which is given by the inverse of eqn. (6.23). To overcome this undesired aspect of the smoother covariance the matrix inversion lemma in eqn. (1.70) is again used with $A = P_{bk+1}^+$, $B = \Upsilon_k$, $C = Q_k$ and $D = \Upsilon_k^T$, which leads to

$$\boxed{\mathcal{P}_{bk}^- = \Phi_k^T[I - K_{bk}\Upsilon_k^T]\mathcal{P}_{bk+1}^+\Phi_k} \tag{6.26}$$

where the gain K_{bk} is defined as

$$\boxed{K_{bk} = \mathcal{P}_{bk+1}^+\Upsilon_k[\Upsilon_k^T\mathcal{P}_{bk+1}^+\Upsilon_k + Q_k^{-1}]^{-1}} \tag{6.27}$$

Equation (6.27) involves the inverse of Q_k. However, Fraser[8] showed that only those states that are controllable by the process noise driving the system are smoothable (this will be clearly shown in §6.4.1 using the duality between control and estimation). Therefore, in practice Q_k must have an inverse, otherwise this controllable condition is violated. Another form of eqn. (6.27) is given by (which is left as an exercise for the reader):

$$K_{bk} = \Phi_k^{-T}\mathcal{P}_{bk}^-\Phi_k^{-1}\Upsilon_k Q_k \tag{6.28}$$

Equation (6.26) can be further expanded into a symmetric form (which is again left as an exercise for the reader):

$$\mathcal{P}_{bk}^- = \Phi_k^T[I - K_{bk}\Upsilon_k^T]\mathcal{P}_{bk+1}^+[I - K_{bk}\Upsilon_k^T]^T\Phi_k + \Phi_k^T K_{bk} Q_k^{-1} K_{bk}^T \Phi_k \tag{6.29}$$

Equation (6.29) is the sum of two positive definite matrices, which provides a more robust approach in terms of numerical stability.

Before we can continue with the backward filter update, we must first discuss boundary conditions. The forward filter is implemented using the same initial conditions as given in Table 5.1, with state and covariance initial conditions of $\hat{\mathbf{x}}_{f0}$ and

P_{f0}, respectively, which can be applied to either the updated or propagation state estimate (depending on whether or not a measurement occurs at the initial time). Let t_N denote the terminal time. Since at time $t_k = t_N$ the smoother estimate must be the same as the forward Kalman filter, this clearly requires that $\hat{\mathbf{x}}_N = \hat{\mathbf{x}}_{fN}^+$ and $P_N = P_{fN}^+$. From eqn. (6.14) the covariance condition at the terminal time can only be satisfied when $(P_{bN}^-)^{-1} \equiv \mathcal{P}_{bN}^- = 0$. However, the backward terminal state boundary condition, $\hat{\mathbf{x}}_{bN}$, is yet unknown for the following backward measurement update:

$$\hat{\mathbf{x}}_{bk}^+ = \hat{\mathbf{x}}_{bk}^- + K_{bk}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{bk}^-] \tag{6.30}$$

To overcome this difficulty consider the alternative state update form that is given by eqn. (5.53), rewritten as

$$\hat{\mathbf{x}}_{bk}^+ = P_{bk}^+[\mathcal{P}_{bk}^- \hat{\mathbf{x}}_{bk}^- + H_k^T R_k^{-1} \tilde{\mathbf{y}}_k] \tag{6.31}$$

where the definition of $\mathcal{P}_{bk}^-$ has been used. Left multiplying both sides of eqn. (6.31) by the inverse of P_{bk}^+, and using the definition of $\mathcal{P}_{bk}^+$, gives

$$\mathcal{P}_{bk}^+ \hat{\mathbf{x}}_{bk}^+ = \mathcal{P}_{bk}^- \hat{\mathbf{x}}_{bk}^- + H_k^T R_k^{-1} \tilde{\mathbf{y}}_k \tag{6.32}$$

Define the following new variables:

$$\hat{\chi}_{bk}^+ \equiv \mathcal{P}_{bk}^+ \hat{\mathbf{x}}_{bk}^+ \tag{6.33a}$$
$$\hat{\chi}_{bk}^- \equiv \mathcal{P}_{bk}^- \hat{\mathbf{x}}_{bk}^- \tag{6.33b}$$

Using the definitions in eqn. (6.33), then eqn. (6.32) can be rewritten as

$$\boxed{\hat{\chi}_{bk}^+ = \hat{\chi}_{bk}^- + H_k^T R_k^{-1} \tilde{\mathbf{y}}_k} \tag{6.34}$$

Since $\mathcal{P}_{bN}^- = 0$ then from eqn. (6.33b) we have $\hat{\chi}_{bN}^- = \mathbf{0}$, which is valid for any value of $\hat{\mathbf{x}}_{bN}^-$. The backward update is given by eqn. (6.34). A backward propagation must now be derived. Substituting eqn. (6.30) into eqn. (6.33b), and using the definition in eqn. (6.33a) yields

$$\hat{\chi}_{bk}^- = \mathcal{P}_{bk}^- \Phi_k^{-1} \left[(\mathcal{P}_{bk+1}^+)^{-1} \hat{\chi}_{bk+1}^+ - \Gamma_k \mathbf{u}_k \right] \tag{6.35}$$

Substituting eqn. (6.26) into eqn. (6.35) gives the desired form:

$$\boxed{\hat{\chi}_{bk}^- = \Phi_k^T [I - K_{bk} \Upsilon_k^T][\hat{\chi}_{bk+1}^+ - \mathcal{P}_{bk+1}^+ \Gamma_k \mathbf{u}_k]} \tag{6.36}$$

Equations (6.23), (6.26), (6.27), (6.34), and (6.36) define the backward filter.

A summary of the discrete-time fixed-interval smoother is given in Table 6.1. First, the basic discrete-time Kalman filter is executed forward in time on the data set using eqn. (6.2). Then, the backward filter is run with the gain given by eqn. (6.27). In order to avoid undesirable matrix inversions, the backward updates are implemented

Table 6.1: Discrete-Time Fixed-Interval Smoother

Model	$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k, \quad \mathbf{w}_k \sim N(\mathbf{0}, Q_k)$ $\tilde{\mathbf{y}}_k = H_k \mathbf{x}_k + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{0}, R_k)$
Forward Initialize	$\hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\, \tilde{\mathbf{x}}_f^T(t_0)\}$
Gain	$K_{fk} = P_{fk}^- H_k^T [H_k P_{fk}^- H_k^T + R_k]^{-1}$
Forward Update	$\hat{\mathbf{x}}_{fk}^+ = \hat{\mathbf{x}}_{fk}^- + K_{fk}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}^-]$ $P_{fk}^+ = [I - K_{fk} H_k] P_{fk}^-$
Forward Propagation	$\hat{\mathbf{x}}_{fk+1}^- = \Phi_k \hat{\mathbf{x}}_{fk}^+ + \Gamma_k \mathbf{u}_k$ $P_{fk+1}^- = \Phi_k P_{fk}^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$
Backward Initialize	$\hat{\chi}_{bN}^- = \mathbf{0}$ $\mathcal{P}_{bN}^- = 0$
Gain	$K_{bk} = \mathcal{P}_{bk+1}^+ \Upsilon_k [\Upsilon_k^T \mathcal{P}_{bk+1}^+ \Upsilon_k + Q_k^{-1}]^{-1}$
Backward Update	$\hat{\chi}_{bk}^+ = \hat{\chi}_{bk}^- + H_k^T R_k^{-1} \tilde{\mathbf{y}}_k$ $\mathcal{P}_{bk}^+ = \mathcal{P}_{bk}^- + H_k^T R_k^{-1} H_k$
Backward Propagation	$\hat{\chi}_{bk}^- = \Phi_k^T [I - K_{bk} \Upsilon_k^T][\hat{\chi}_{bk+1}^+ - \mathcal{P}_{bk+1}^+ \Gamma_k \mathbf{u}_k]$ $\mathcal{P}_{bk}^- = \Phi_k^T [I - K_{bk} \Upsilon_k^T] \mathcal{P}_{bk+1}^+ \Phi_k$
Gain	$K_k = P_{fk}^+ \mathcal{P}_{bk}^- [I + P_{fk}^+ \mathcal{P}_{bk}^-]^{-1}$
Covariance	$P_k = [I - K_k] P_{fk}^+$
Estimate	$\hat{\mathbf{x}}_k = [I - K_k] \hat{\mathbf{x}}_{fk}^+ + P_k \hat{\chi}_{bk}^-$

using eqns. (6.23) and (6.34), and the backward propagations are implemented using eqns. (6.26) [or using eqn. (6.29) if numerical stability is of concern] and (6.36). The forward and backward covariances and estimates must be stored in order to evaluate the smoother covariance and estimate. The optimal smoother covariance is computed using eqn. (6.16) [or using eqn. (6.17) if numerical stability is of concern]. Finally, the optimal smoother estimate is computed using eqn. (6.21).

6.1.1.1 Steady-State Fixed-Interval Smoother

If the system matrices and covariance are time-invariant, then a steady-state (i.e., constant gain) smoother can be used, which significantly reduces the computational burden. The steady-state forward filter has been derived in §5.3.4. The only issue for the backward filter is the steady-state Riccati equation for $\mathcal{P}_b^-$. At steady-state from eqn. (6.26) we have

$$\mathcal{P}_b^- = \Phi^T \mathcal{P}_b^+ \Phi - \Phi^T \mathcal{P}_b^+ \Upsilon \left[\Upsilon^T \mathcal{P}_b^+ \Upsilon + Q^{-1} \right]^{-1} \Upsilon^T \mathcal{P}_b^+ \Phi \tag{6.37}$$

Using eqn. (6.23) in eqn. (6.37) yields

$$\mathcal{P}_b^+ = \Phi^T \mathcal{P}_b^+ \Phi - \Phi^T \mathcal{P}_b^+ \Upsilon \left[\Upsilon^T \mathcal{P}_b^+ \Upsilon + Q^{-1} \right]^{-1} \Upsilon^T \mathcal{P}_b^+ \Phi + H^T R^{-1} H \tag{6.38}$$

Comparing eqn. (6.38) to the Riccati (covariance) equation in Table 5.2 and using a similar transformation as eqn. (5.81) yields the following Hamiltonian matrix:

$$\boxed{\mathcal{H} \equiv \begin{bmatrix} \Phi^{-1} & \Phi^{-1} \Upsilon Q \Upsilon^T \\ H^T R^{-1} H \Phi^{-1} & \Phi^T + H^T R^{-1} H \Phi^{-1} \Upsilon Q \Upsilon^T \end{bmatrix}} \tag{6.39}$$

An eigenvalue/eigenvector decomposition of eqn. (6.39) gives

$$\mathcal{H} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}^{-1} \tag{6.40}$$

where Λ is a diagonal matrix of the n eigenvalues outside of the unit circle, and W_{11}, W_{21}, W_{12}, and W_{22} are block elements of the eigenvector matrix. From the derivations of §5.3.4 the steady-state value for $\mathcal{P}_b^+$ is given by

$$\boxed{\mathcal{P}_b^+ = W_{21} W_{11}^{-1}} \tag{6.41}$$

which requires an inverse of an $n \times n$ matrix. To determine the steady-state value for $\mathcal{P}_b^-$ we simply use eqn. (6.23), with

$$\boxed{\mathcal{P}_b^- = \mathcal{P}_b^+ - H^T R^{-1} H} \tag{6.42}$$

The smoother covariance and estimate can now be computed using the steady-state values for P_f^+ and $\mathcal{P}_b^-$. Note that the steady-state value for P in Table 5.2 gives P_f^-, but P_f^+ can be calculated by using eqn. (5.44).

6.1.1.2 RTS Fixed-Interval Smoother

Several other forms of the fixed-interval smoother exist. One of the most convenient forms is given by Rauch, Tung, and Striebel (RTS),[5] who combine the backward filter and smoother into one single backward recursion. Our first task is to determine a recursive expression for the smoother covariance that is independent of the backward covariance. To accomplish this task eqn. (6.16) is rewritten as

$$P_k = P_{fk}^+ - P_{fk}^+[P_{fk}^+ + P_{bk}^-]^{-1} P_{fk}^+ \tag{6.43}$$

We now concentrate our attention on the matrix inverse expression in eqn. (6.43). Substituting eqn. (6.25) into this matrix inversion expression and factoring out Φ_k on both sides yields

$$[P_{fk}^+ + P_{bk}^-]^{-1} = \Phi_k^T [\Phi_k P_{fk}^+ \Phi_k^T + P_{bk+1}^+ + \Upsilon_k Q_k \Upsilon_k^T]^{-1} \Phi_k \tag{6.44}$$

Using eqn. (6.2b) in eqn. (6.44) gives

$$[P_{fk}^+ + P_{bk}^-]^{-1} = \Phi_k^T [P_{fk+1}^- + P_{bk+1}^+]^{-1} \Phi_k \tag{6.45}$$

A more convenient form for P_{bk+1}^+ is required. Solving eqn. (5.73) for $H_k^T R_k^{-1} H_k$, and substituting the resultant into eqn. (6.23) yields

$$P_{bk}^+ = [\mathcal{P}_{bk}^- + \mathcal{P}_{fk}^+ - \mathcal{P}_{fk}^-]^{-1} \tag{6.46}$$

Using eqn. (6.14) in eqn. (6.46) yields

$$P_{bk}^+ = [P_k^{-1} - \mathcal{P}_{fk}^-]^{-1} \tag{6.47}$$

Taking one time-step ahead of eqn. (6.47), and substituting the resulting expression into eqn. (6.45) gives

$$[P_{fk}^+ + P_{bk}^-]^{-1} = \Phi_k^T \left\{ P_{fk+1}^- + [P_{k+1}^{-1} - \mathcal{P}_{fk+1}^-]^{-1} \right\}^{-1} \Phi_k \tag{6.48}$$

Factoring $\mathcal{P}_{fk+1}^-$ yields

$$[P_{fk}^+ + P_{bk}^-]^{-1} = \Phi_k^T \mathcal{P}_{fk+1}^- \left\{ \mathcal{P}_{fk+1}^- + \mathcal{P}_{fk+1}^- [P_{k+1}^{-1} - \mathcal{P}_{fk+1}^-]^{-1} \mathcal{P}_{fk+1}^- \right\}^{-1} \mathcal{P}_{fk+1}^- \Phi_k \tag{6.49}$$

Then, using the matrix inversion lemma in eqn. (1.70) with $A = P_{fk+1}^-$, $B = D = I$ and $C = -P_{k+1}$ leads to

$$[P_{fk}^+ + P_{bk}^-]^{-1} = \Phi_k^T \mathcal{P}_{fk+1}^- [P_{fk+1}^- - P_{k+1}] \mathcal{P}_{fk+1}^- \Phi_k \tag{6.50}$$

Substituting eqn. (6.50) into eqn. (6.43) yields

$$\boxed{P_k = P_{fk}^+ - \mathcal{K}_k [P_{fk+1}^- - P_{k+1}] \mathcal{K}_k^T} \tag{6.51}$$

where the gain matrix $\mathcal{K}_k$ is defined as

$$\boxed{\mathcal{K}_k \equiv P_{fk}^+ \Phi_k^T (P_{fk+1}^-)^{-1}} \tag{6.52}$$

Note that eqn. (6.51) is no longer a function of the backward covariance P_{bk}^+ or P_{bk}^-. Therefore, the smoother covariance can be solved directly from knowledge of the forward covariance alone, which provides a very computationally efficient algorithm.

The RTS smoother state-estimate equation is given by

$$\boxed{\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk}^+ + \mathcal{K}_k[\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}^-]} \tag{6.53}$$

The proof of this form begins by comparing this eqn. (6.53) to eqn. (6.21). From this comparison we need to prove that the following relationship is true:

$$-K_k\hat{\mathbf{x}}_{fk}^+ + P_k\hat{\chi}_{bk}^- = \mathcal{K}_k[\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}^-] \tag{6.54}$$

Substituting eqns. (6.22), (6.51), and (6.52) into eqn. (6.54), and simplifying gives

$$\begin{aligned}&-\mathcal{P}_{bk}^-[I + P_{fk}^+\mathcal{P}_{bk}^-]^{-1}\hat{\mathbf{x}}_{fk}^+ + \hat{\chi}_{bk}^- - \Phi_k^T\mathcal{P}_{fk+1}^-[P_{fk+1}^- - P_{k+1}]\mathcal{P}_{fk+1}^-\Phi_k P_{fk}^+\hat{\chi}_{bk}^-\\ &= \Phi_k^T\mathcal{P}_{fk+1}^-[\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}^-]\end{aligned} \tag{6.55}$$

We will return to eqn. (6.55), but for the time being let's concentrate on determining a more useful expression for $\hat{\mathbf{x}}_{k+1}$, which will be used to help simplify eqn. (6.55). Taking one time-step ahead of eqn. (6.19) gives

$$\hat{\mathbf{x}}_{k+1} = P_{k+1}\mathcal{P}_{fk+1}^+\hat{\mathbf{x}}_{fk+1}^+ + P_{k+1}\hat{\chi}_{bk+1}^- \tag{6.56}$$

Taking one time-step ahead of eqn. (6.34) and solving for $\hat{\chi}_{bk+1}^-$ gives

$$\hat{\chi}_{bk+1}^- = \hat{\chi}_{bk+1}^+ - H_{k+1}^T R_{k+1}^{-1}\tilde{\mathbf{y}}_{k+1} \tag{6.57}$$

Taking one time-step ahead of eqn. (5.30b), with the gain given by eqn. (5.47), and substituting the resultant and eqn. (6.57) into eqn. (6.56) yields

$$\hat{\mathbf{x}}_{k+1} = P_{k+1}\left[\mathcal{P}_{fk+1}^+ - H_{k+1}^T R_{k+1}^{-1} H_{k+1}\right]\hat{\mathbf{x}}_{fk+1}^- + P_{k+1}\hat{\chi}_{bk+1}^+ \tag{6.58}$$

Using one time-step ahead of eqn. (5.73) in eqn. (6.58) now gives a simpler form:

$$\hat{\mathbf{x}}_{k+1} = P_{k+1}\mathcal{P}_{fk+1}^-\hat{\mathbf{x}}_{fk+1}^- + P_{k+1}\hat{\chi}_{bk+1}^+ \tag{6.59}$$

Subtracting $\hat{\mathbf{x}}_{fk+1}^-$ from both sides of eqn. (6.59) and factoring out $\mathcal{P}_{fk+1}^-$ yields

$$\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}^- = [P_{k+1} - P_{fk+1}^-]\mathcal{P}_{fk+1}^-\hat{\mathbf{x}}_{fk+1}^- + P_{k+1}\hat{\chi}_{bk+1}^+ \tag{6.60}$$

Next, rewrite the forward-time prediction, given by eqn. (5.30a), as

$$\hat{\mathbf{x}}_{fk}^+ = \Phi_k^{-1}\hat{\mathbf{x}}_{fk+1}^- - \Phi_k^{-1}\Gamma_k\mathbf{u}_k \tag{6.61}$$

Substituting eqns. (6.60) and (6.61) into eqn. (6.55), and multiplying in $\mathcal{P}_{bk}^{-}$ yields

$$\begin{aligned}
&-[P_{bk}^{-}+P_{fk}^{+}]^{-1}\Phi_k^{-1}\hat{\mathbf{x}}_{fk+1}^{-}+[P_{bk}^{-}+P_{fk}^{+}]^{-1}\Phi_k^{-1}\Gamma_k\mathbf{u}_k\\
&+\hat{\chi}_{bk}^{-}-\Phi_k^T\mathcal{P}_{fk+1}^{-}[P_{fk+1}^{-}-P_{k+1}]\mathcal{P}_{fk+1}^{-}\Phi_k P_{fk}^{+}\hat{\chi}_{bk}^{-}\\
&=\Phi_k^T\mathcal{P}_{fk+1}^{-}[P_{k+1}-P_{fk+1}^{-}]\mathcal{P}_{fk+1}^{-}\hat{\mathbf{x}}_{fk+1}^{-}+\Phi_k^T\mathcal{P}_{fk+1}^{-}P_{k+1}\hat{\chi}_{bk+1}^{+}
\end{aligned}\tag{6.62}$$

Using eqn. (6.50) in eqn. (6.62), and simplifying yields

$$[P_{bk}^{-}+P_{fk}^{+}]^{-1}\Phi_k^{-1}\Gamma_k\mathbf{u}_k+\hat{\chi}_{bk}^{-}-[P_{bk}^{-}+P_{fk}^{+}]^{-1}P_{fk}^{+}\hat{\chi}_{bk}^{-}=\Phi_k^T\mathcal{P}_{fk+1}^{-}P_{k+1}\hat{\chi}_{bk+1}^{+}\tag{6.63}$$

Using eqn. (6.45) in eqn. (6.63), and left multiplying both sides of the resulting equation by $[P_{fk+1}^{-}+P_{bk+1}^{+}]\Phi_k^{-T}$ yields

$$\Gamma_k\mathbf{u}_k+\left([P_{fk+1}^{-}+P_{bk+1}^{+}]\Phi_k^{-T}-\Phi_k P_{fk}^{+}\right)\hat{\chi}_{bk}^{-}=[P_{fk+1}^{-}+P_{bk+1}^{+}]\mathcal{P}_{fk+1}^{-}P_{k+1}\hat{\chi}_{bk+1}^{+}\tag{6.64}$$

Next, rewrite the forward-time covariance prediction, given by eqn. (5.35), as

$$P_{fk}^{+}=\Phi_k^{-1}P_{fk+1}^{-}\Phi_k^{-T}-\Phi_k^{-1}\Upsilon_k Q_k\Upsilon_k^T\Phi_k^{-T}\tag{6.65}$$

Substituting eqn. (6.65) into eqn. (6.64), left multiplying both sides of the resulting equation by $\mathcal{P}_{bk+1}^{+}$, using eqn. (6.47) with one time-step ahead and solving for $\hat{\chi}_{bk}^{-}$ yields

$$\hat{\chi}_{bk}^{-}=\Phi_k^T[I+\mathcal{P}_{bk+1}^{+}\Upsilon_k Q_k\Upsilon_k^T]^{-1}[\hat{\chi}_{bk+1}^{+}-\mathcal{P}_{bk+1}^{+}\Gamma_k\mathbf{u}_k]\tag{6.66}$$

Finally, using the matrix inversion lemma in eqn. (1.70) with $A=I$, $B=\mathcal{P}_{bk+1}^{+}\Upsilon_k$, $C=Q_k$, and $D=\Upsilon_k^T$ gives the same form as eqn. (6.36), which completes the proof.

A summary of the RTS smoother is given in Table 6.2. As before, the forward Kalman filter is executed using the measurements until time T. Storing the propagated and updated state estimates from the forward filter, the smoothed estimate is then determined by executing eqn. (6.53) backward in time. In order to determine the RTS smoothed estimate, the forward filter covariance update and propagation, as well as the state matrix, do not need to be stored. This is due to the fact that the gain in eqn. (6.52) can be computed during the forward filter process and stored to be used in the smoother estimate equation. One of the extraordinary results of the smoother state estimate is the fact that the smoother state in eqn. (6.53) does not involve the smoother covariance P_k! Therefore, eqn. (6.51) is only used to derive the smoother covariance, which may be required for analysis purposes, but is not used to find the optimal smoother state estimate. For all these reasons the RTS smoother is more widely used in practice over the formulation given in Table 6.1. Note, in §6.4.1 we will derive the RTS smoother from optimal control theory, which shows the duality between control and estimation.

6.1.1.3 Stability

The backward state matrix in the RTS smoother defines the stability of the system, which is given by $P_{fk}^{+}\Phi_k^T\mathcal{P}_{fk+1}^{-}$. Note that the smoother state estimate in eqn. (6.53)

Table 6.2: Discrete-Time RTS Smoother

Model	$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k, \quad \mathbf{w}_k \sim N(\mathbf{0}, Q_k)$ $\tilde{\mathbf{y}}_k = H_k \mathbf{x}_k + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{0}, R_k)$
Forward Initialize	$\hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\,\tilde{\mathbf{x}}_f^T(t_0)\}$
Gain	$K_{fk} = P_{fk}^- H_k^T [H_k P_{fk}^- H_k^T + R_k]^{-1}$
Forward Update	$\hat{\mathbf{x}}_{fk}^+ = \hat{\mathbf{x}}_{fk}^- + K_{fk}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}^-]$ $P_{fk}^+ = [I - K_{fk} H_k] P_{fk}^-$
Forward Propagation	$\hat{\mathbf{x}}_{fk+1}^- = \Phi_k \hat{\mathbf{x}}_{fk}^+ + \Gamma_k \mathbf{u}_k$ $P_{fk+1}^- = \Phi_k P_{fk}^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$
Smoother Initialize	$\hat{\mathbf{x}}_N = \hat{\mathbf{x}}_{fN}^+$ $P_N = P_{fN}^+$
Gain	$\mathcal{K}_k \equiv P_{fk}^+ \Phi_k^T (P_{fk+1}^-)^{-1}$
Covariance	$P_k = P_{fk}^+ - \mathcal{K}_k [P_{fk+1}^- - P_{k+1}] \mathcal{K}_k^T$
Estimate	$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk}^+ + \mathcal{K}_k [\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}^-]$

is a backward recursion, which is stable if and only if all the eigenvalues of the state matrix are within the unit circle. The reader should not be confused by the fact that eqn. (6.53) is executed backward in time. All discrete-time recursions, whether executed forward or backward in time, must have state matrix eigenvalues within the unit circle to be stable. Considering only the homogeneous part of eqn. (6.53), the RTS smoother is stable if the following recursion is stable:

$$\hat{\mathbf{x}}_k = P_{fk}^+ \Phi_k^T \mathcal{P}_{fk+1}^- \hat{\mathbf{x}}_{k+1} \tag{6.67}$$

The smoother stability can be proved by using Lyapunov's direct method, which is discussed for discrete-time systems in §3.6. For the discrete-time RTS smoother we consider the following candidate Lyapunov function:

$$V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_{k+1}^T \mathcal{P}_{fk+1}^+ \hat{\mathbf{x}}_{k+1} \tag{6.68}$$

The increment of $V(\hat{\mathbf{x}})$ is given by

$$\Delta V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_{k+1}^T \mathcal{P}_{fk+1}^+ \hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_k^T \mathcal{P}_{fk}^+ \hat{\mathbf{x}}_k \tag{6.69}$$

Substituting eqn. (6.67) into eqn. (6.69) gives

$$\Delta V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_{k+1}^T \left[\mathcal{P}_{fk+1}^{-} \Phi_k P_{fk}^{+} \Phi_k^T \mathcal{P}_{fk+1}^{-} - \mathcal{P}_{fk+1}^{+}\right] \hat{\mathbf{x}}_{k+1} \tag{6.70}$$

Substituting eqn. (6.65) into eqn. (6.70) gives

$$\Delta V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_{k+1}^T \left[\mathcal{P}_{fk+1}^{-} - \mathcal{P}_{fk+1}^{-} \Upsilon_k Q_k \Upsilon_k^T \mathcal{P}_{fk+1}^{-} - \mathcal{P}_{fk+1}^{+}\right] \hat{\mathbf{x}}_{k+1} \tag{6.71}$$

Taking one time-step ahead of the expression in eqn. (5.73) and substituting the resultant into eqn. (6.71) leads to

$$\Delta V(\hat{\mathbf{x}}) = -\hat{\mathbf{x}}_{k+1}^T \left[H_{k+1}^T R_{k+1}^{-1} H_{k+1} + \mathcal{P}_{fk+1}^{-} \Upsilon_k Q_k \Upsilon_k^T \mathcal{P}_{fk+1}^{-}\right] \hat{\mathbf{x}}_{k+1} \tag{6.72}$$

Clearly if R_{k+1} is positive definite and Q_k is at least positive semi-definite, then the Lyapunov condition is satisfied and the discrete-time RTS smoother is stable.

Example 6.1: In this example the model used in example 5.3 is used to demonstrate the power of the fixed-point smoother. For this simulation we are interested in investigating the covariance of the smoother. Therefore, both the forward-time updated and propagated covariance must be stored. The smoothed state estimates and covariance are computed using the RTS formulation. A plot of the smoother attitude-angle error and 3σ bounds is shown in Figure 6.2. Comparing the smoother 3σ bounds with the ones shown in Figure 5.3 indicates that the smoother clearly provides better estimates than the Kalman filter alone. Note that the steady-state covariance can be used for this system with little loss in accuracy. Using the methods of §5.3.4 the steady-state value for the steady-state forward-time propagated covariance, P_f^-, can be computed by solving the algebraic Riccati equation in Table 5.2. Then, the steady-state forward-time updated covariance, P_f^+, can be computed from eqn. (5.44). Finally, the steady-state smoother covariance can be computed by solving the following Lyapunov equation:

$$P = K\,P\,K^T + \left(P_f^+ - K\,P_f^-\,K^T\right)$$

with

$$K = P_f^+ \Phi^T \mathcal{P}_f^-$$

Performing these calculations give a 3σ attitude bound of 4.9216 μrad, which is verified by Figure 6.2. A more dramatic result for the advantages of using the smoother is shown for the bias estimate, given by the bottom plot of Figure 6.3 (the top plot shows the Kalman filter estimate). Clearly, the smoother estimate is far superior than the Kalman filter estimate, which can be very useful for calibration purposes.

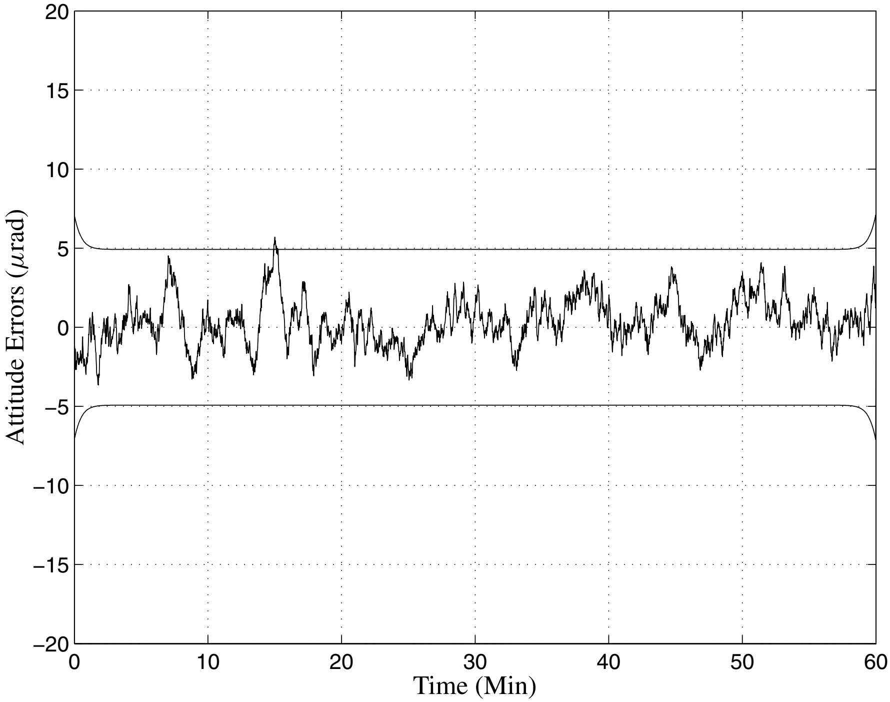

Figure 6.2: Smoother Attitude Error and Bounds

6.1.2 Continuous-Time Formulation

The true system for the continuous-time models and measurements is given by eqn. (5.117):

$$\boxed{\begin{aligned}\frac{d}{dt}\mathbf{x}(t) &= F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t) & \text{(6.73a)}\\ \tilde{\mathbf{y}}(t) &= H(t)\,\mathbf{x}(t) + \mathbf{v}(t) & \text{(6.73b)}\end{aligned}}$$

where $\mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ and $\mathbf{v}(t) \sim N(\mathbf{0}, R(t))$. The optimal smoother is again given by a combination of the estimates of two filters: one, denoted by $\hat{\mathbf{x}}_f(t)$, is given from a filter that runs from the beginning of the data interval to time t, and the other, denoted by $\hat{\mathbf{x}}_b(t)$, that works backward from the end of the time interval. These two filters follow the continuous-time form of the Kalman filter, given in §5.4.1:

forward filter

$$\frac{d}{dt}\hat{\mathbf{x}}_f(t) = F(t)\,\hat{\mathbf{x}}_f(t) + B(t)\,\mathbf{u}(t) + K_f(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_f(t)] \tag{6.74a}$$

$$K_f(t) = P_f(t)\,H^T(t)\,R^{-1}(t) \tag{6.74b}$$

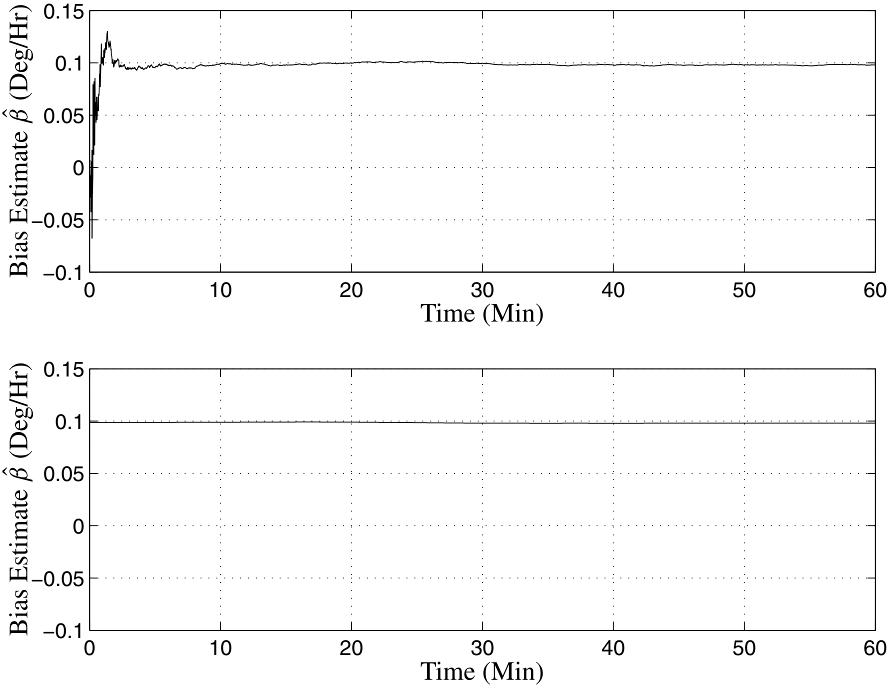

Figure 6.3: Kalman Filter and Smoother Gyro Bias Estimates

$$\begin{aligned}\frac{d}{dt}P_f(t) &= F(t)\,P_f(t) + P_f(t)\,F^T(t)\\ &\quad - P_f(t)\,H^T(t)\,R^{-1}(t)\,H(t)\,P_f(t) + G(t)\,Q(t)\,G^T(t)\end{aligned} \tag{6.74c}$$

backward filter

$$\frac{d}{dt}\hat{\mathbf{x}}_b(t) = F(t)\,\hat{\mathbf{x}}_b(t) + B(t)\,\mathbf{u}(t) + K_b(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_b(t)] \tag{6.75a}$$

$$K_b(t) = P_b(t)\,H^T(t)\,R^{-1}(t) \tag{6.75b}$$

$$\begin{aligned}\frac{d}{dt}P_b(t) &= F(t)\,P_b(t) + P_b(t)\,F^T(t)\\ &\quad - P_b(t)\,H^T(t)\,R^{-1}(t)\,H(t)\,P_b(t) + G(t)\,Q(t)\,G^T(t)\end{aligned} \tag{6.75c}$$

Equation (6.75) must be integrated backward in time. In order to express this integration in a more convenient form, it is convenient to set $\tau = T - t$,[1] where T is the terminal time of the data interval. Since $d\mathbf{x}/dt = -d\mathbf{x}/d\tau$, writing eqn. (6.73a) in terms of τ gives

$$\frac{d}{d\tau}\mathbf{x}(t) = -F(t)\,\mathbf{x}(t) - B(t)\,\mathbf{u}(t) - G(t)\,\mathbf{w}(t) \tag{6.76}$$

Therefore the backward filter equations can be written in terms of τ by replacing $F(t)$ with $-F(t)$, $B(t)$ with $-B(t)$, and $G(t)$ with $-G(t)$, which leads to *backward filter*

$$\frac{d}{d\tau}\hat{\mathbf{x}}_b(t) = -F(t)\,\hat{\mathbf{x}}_b(t) - B(t)\,\mathbf{u}(t) + K_b(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_b(t)] \tag{6.77a}$$

$$K_b(t) = P_b(t)\,H^T(t)\,R^{-1}(t) \tag{6.77b}$$

$$\begin{aligned}\frac{d}{d\tau}P_b(t) &= -F(t)\,P_b(t) - P_b(t)\,F^T(t)\\ &\quad - P_b(t)\,H^T(t)\,R^{-1}(t)\,H(t)\,P_b(t) + G(t)\,Q(t)\,G^T(t)\end{aligned} \tag{6.77c}$$

Therefore, from this point forward whenever $d/d\tau$ is used, this will denote a backward differentiation. We should note that if $F(t)$ is stable going forward in time, then $-F(t)$ is stable going backward in time.

The continuous-time smoother combination of the forward and backward state estimates follows exactly from the discrete-time equivalent of §6.1.1. The continuous-time equivalent of eqn. (6.14) is simply given by

$$\boxed{P(t) = \left[P_f^{-1}(t) + P_b^{-1}(t)\right]^{-1}} \tag{6.78}$$

Also, the continuous-time equivalent of eqn. (6.19) is simply given by

$$\boxed{\hat{\mathbf{x}}(t) = P(t)\left[P_f^{-1}(t)\,\hat{\mathbf{x}}_f(t) + P_b^{-1}(t)\,\hat{\mathbf{x}}_b(t)\right]} \tag{6.79}$$

Equations (6.74), (6.77), (6.78), and (6.79) summarize the basic equations for the smoother. We must now define the boundary conditions. Since at time $t = T$ the smoother estimate must be the same as the forward Kalman filter, this clearly requires that $\hat{\mathbf{x}}(T) = \hat{\mathbf{x}}_f(T)$ and $P(T) = P_f(T)$. From eqn. (6.78) the covariance condition at the terminal time can only be satisfied when $P_b^{-1}(T) = 0$. Therefore, $P_b(t)$ is not finite at the terminal time. To overcome this difficulty consider taking the time derivative of $P_b^{-1}(t)\,P_b(t) = I$, which gives

$$\left[\frac{d}{d\tau}P_b^{-1}(t)\right]P_b(t) + P_b^{-1}(t)\left[\frac{d}{d\tau}P_b(t)\right] = 0 \tag{6.80}$$

Rearranging eqn. (6.80) yields

$$\left[\frac{d}{d\tau}P_b^{-1}(t)\right] = -P_b^{-1}(t)\left[\frac{d}{d\tau}P_b(t)\right]P_b^{-1}(t) \tag{6.81}$$

Substituting (6.77c) into eqn. (6.81) yields

$$\begin{aligned}\frac{d}{d\tau}P_b^{-1}(t) &= P_b^{-1}(t)\,F(t) + F^T(t)\,P_b^{-1}(t)\\ &\quad - P_b^{-1}(t)\,G(t)\,Q(t)\,G^T(t)\,P_b^{-1}(t) + H^T(t)\,R^{-1}(t)\,H(t)\end{aligned} \tag{6.82}$$

which can be integrated backward in time with the appropriate boundary condition of $P_b^{-1}(T) = 0$.

Even with the matrix inverse expression for $P_b^{-1}(t)$, eqn. (6.78) still requires the calculation of two matrix inverses, which is generally not desirable. To overcome this aspect of the smoother covariance, the matrix inversion lemma in eqn. (1.70) is used with $A = P_f^{-1}(t)$, $B = D = I$, and $C = P_b^{-1}(t)$, which leads to

$$\boxed{P(t) = P_f(t) - P_f(t)\, P_b^{-1}(t)[I + P_f(t)\, P_b^{-1}(t)]^{-1} P_f(t)} \tag{6.83}$$

Note that eqn. (6.83), in conjunction with eqn. (6.82), requires only one matrix inverse. Equation (6.83) can be further expanded into a symmetric form:

$$P(t) = \left[I - W(t)\, P_b^{-1}(t)\right] P_f(t)[I - W(t)\, P_b^{-1}(t)]^T + W(t)\, P_b^{-1}(t) W^T(t) \tag{6.84}$$

where

$$W(t) = P_f(t)\,[I + P_f(t)\, P_b^{-1}(t)]^{-T} \tag{6.85}$$

As with the discrete symmetric form, eqn. (6.84) is the sum of two positive definite matrices, which provides a more robust approach in terms of numerical stability.

As previously mentioned, the boundary condition for the smoother state is $\hat{\mathbf{x}}(T) = \hat{\mathbf{x}}_f(T)$, but the boundary condition for $\hat{\mathbf{x}}_b(T)$ is still unknown. This difficulty may be overcome by defining a new variable:

$$\hat{\chi}_b(t) \equiv P_b^{-1}(t)\, \hat{\mathbf{x}}_b(t) \tag{6.86}$$

where $\hat{\chi}_b(T) = \mathbf{0}$ since $P_b^{-1}(T) = 0$ and $\hat{\mathbf{x}}_b(T)$ is finite. Differentiating eqn. (6.86) with respect to time and substituting eqns. (6.77a) and (6.82) into the resulting expression yields

$$\boxed{\begin{aligned} \frac{d}{d\tau}\hat{\chi}_b(t) = \left[F(t) - G(t)\, Q(t)\, G^T(t)\, P_b^{-1}(t)\right]^T \hat{\chi}_b(t) \\ - P_b^{-1}(t)\, B(t)\, \mathbf{u}(t) + H^T(t)\, R^{-1}(t)\, \tilde{\mathbf{y}}(t) \end{aligned}} \tag{6.87}$$

The continuous-time equivalent of eqn. (6.21) is now given by

$$\boxed{\hat{\mathbf{x}}(t) = [I - K(t)]\hat{\mathbf{x}}_f(t) + P(t)\, \hat{\chi}_b(t)} \tag{6.88}$$

where the continuous smoother gain is defined by

$$K(t) \equiv P_f(t)\, P_b^{-1}(t)[I + P_f(t)\, P_b^{-1}(t)]^{-1} \tag{6.89}$$

Note that the definition of $\hat{\chi}_b(t)$ has been used in eqn. (6.88).

A summary of the continuous-time fixed-interval smoother is given in Table 6.3. First, the basic continuous-time Kalman filter is executed forward in time on the data set using eqn. (6.74). Then, the backward filter is run using eqns. (6.82) and (6.87), which avoids undesirable matrix inversions. The forward and backward covariances and estimates must be stored in order to evaluate the smoother covariance and estimate. The optimal smoother covariance is computed using eqn. (6.83), or using eqn. (6.84) if numerical stability is of concern. Finally, the optimal smoother estimate is computed using eqn. (6.88).

Table 6.3: Continuous-Time Fixed-Interval Smoother

Model	$\frac{d}{dt}\mathbf{x}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t),\ \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ $\tilde{\mathbf{y}}(t) = H(t)\,\mathbf{x}(t) + \mathbf{v}(t),\ \mathbf{v}(t) \sim N(\mathbf{0}, R(t))$
Forward Covariance	$\frac{d}{dt} P_f(t) = F(t)\,P_f(t) + P_f(t)\,F^T(t)$ $-P_f(t)\,H^T(t)\,R^{-1}(t) H(t)\,P_f(t)$ $+G(t)\,Q(t)\,G^T(t),$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\,\tilde{\mathbf{x}}_f^T(t_0)\}$
Forward Filter	$\frac{d}{dt}\hat{\mathbf{x}}_f(t) = F(t)\,\hat{\mathbf{x}}_f(t) + B(t)\,\mathbf{u}(t)$ $+P_f(t)\,H^T(t)\,R^{-1}(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_f(t)], \quad \hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$
Backward Covariance	$\frac{d}{d\tau} P_b^{-1}(t) = P_b^{-1}(t)\,F(t) + F^T(t)\,P_b^{-1}(t)$ $-P_b^{-1}(t)\,G(t)\,Q(t)\,G^T(t)\,P_b^{-1}(t)$ $+H^T(t)\,R^{-1}(t)\,H(t), \quad P_b^{-1}(T) = 0$
Backward Filter	$\frac{d}{d\tau}\hat{\chi}_b(t) = \left[F(t) - G(t)\,Q(t)\,G^T(t)\,P_b^{-1}(t)\right]^T \hat{\chi}_b(t)$ $-P_b^{-1}(t)\,B(t)\,\mathbf{u}(t) + H^T(t)\,R^{-1}(t)\,\tilde{\mathbf{y}}(t), \quad \hat{\chi}_b(T) = 0$
Gain	$K(t) = P_f(t)\,P_b^{-1}(t)\left[I + P_f(t)\,P_b^{-1}(t)\right]^{-1}$
Covariance	$P(t) = [I - K(t)]P_f(t)$
Estimate	$\hat{\mathbf{x}}(t) = [I - K(t)]\hat{\mathbf{x}}_f(t) + P(t)\,\hat{\chi}_b(t)$

6.1.2.1 Steady-State Fixed-Interval Smoother

If the system matrices and covariance are time-invariant, then a steady-state (i.e., constant gain) smoother can be used, which significantly reduces the computational burden. The steady-state forward filter has been derived in §5.4.4. The only issue for the backward filter is the steady-state Riccati equation, given by

$$P_b^{-1} F + F^T P_b^{-1} - P_b^{-1} G\,Q\,G^T P_b^{-1} + H^T R^{-1} H = 0 \tag{6.90}$$

Comparing eqn. (6.90) to the Riccati (covariance) equation in Table 5.5 and using a similar transformation as eqn. (5.163) yields the following Hamiltonian matrix:

$$\boxed{\mathcal{H} \equiv \begin{bmatrix} -F & G\,Q\,G^T \\ H^T R^{-1} H & F^T \end{bmatrix}} \tag{6.91}$$

An eigenvalue/eigenvector decomposition of eqn. (6.91) gives

$$\mathcal{H} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}^{-1} \tag{6.92}$$

where Λ is a diagonal matrix of the n eigenvalues in the right half-plane, and W_{11}, W_{21}, W_{12}, and W_{22} are block elements of the eigenvector matrix. From the derivations of §5.4.4 the steady-state value for P_b^{-1} is given by

$$\boxed{P_b^{-1} = W_{21}\,W_{11}^{-1}} \tag{6.93}$$

which requires an inverse of an $n \times n$ matrix. Also, the nonlinear (extended) version of the smoother is straightforward, replacing the space matrices with their equivalent Jacobian matrices evaluated at the current estimate. These equations will be summarized in §6.1.3.

6.1.2.2 RTS Fixed-Interval Smoother

As with the discrete-time smoother shown in §6.1.1 an RTS form can also be derived for the continuous-time smoother, which combines the backward filter and smoother into one single backward recursion. Taking the derivative of $P^{-1}(t) = P_f^{-1}(t) + P_b^{-1}(t)$, and using eqn. (6.81) for the derivative of $P_f^{-1}(t)$ leads to

$$\frac{d}{d\tau} P^{-1}(t) = -P_f^{-1}(t) \left[\frac{d}{d\tau} P_f(t) \right] P_f^{-1}(t) + \frac{d}{d\tau} P_b^{-1}(t) \tag{6.94}$$

Next, using $dP_f/dt = -dP_f/d\tau$ gives

$$\frac{d}{d\tau} P^{-1}(t) = P_f^{-1}(t) \left[\frac{d}{dt} P_f(t) \right] P_f^{-1}(t) + \frac{d}{d\tau} P_b^{-1}(t) \tag{6.95}$$

Substituting eqns. (6.74c) and (6.82) into eqn. (6.95) gives

$$\begin{aligned} \frac{d}{d\tau} P^{-1}(t) = {} & P_f^{-1}(t)\,F(t) + F^T(t)\,P_f^{-1}(t) + P_f^{-1}(t)\,G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t) \\ & + P_b^{-1}(t)\,F(t) + F^T(t)\,P_b^{-1}(t) - P_b^{-1}(t)\,G(t)\,Q(t)\,G^T(t)\,P_b^{-1}(t) \end{aligned} \tag{6.96}$$

Using $P^{-1}(t) = P_f^{-1}(t) + P_b^{-1}(t)$, then eqn. (6.96) can be rewritten as

$$\begin{aligned}\frac{d}{d\tau}P^{-1}(t) &= P^{-1}(t)\,F(t) + F^T(t)\,P^{-1}(t) + P_f^{-1}(t)\,G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)\\ &\quad -\left[P^{-1}(t) - P_f^{-1}(t)\right]G(t)\,Q(t)\,G^T(t)\left[P^{-1}(t) - P_f^{-1}(t)\right]\end{aligned} \tag{6.97}$$

Substituting the following relation into eqn. (6.97):

$$\left[\frac{d}{d\tau}P^{-1}(t)\right] = P^{-1}(t)\left[\frac{d}{dt}P(t)\right]P^{-1}(t) \tag{6.98}$$

and then multiplying both sides of the resulting expression by $P(t)$ yields

$$\boxed{\begin{aligned}\frac{d}{dt}P(t) &= \left[F(t) + G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)\right]P(t)\\ &\quad + P(t)\left[F(t) + G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)\right]^T - G(t)\,Q(t)\,G^T(t)\end{aligned}} \tag{6.99}$$

Since $P_b^{-1}(T) = 0$, then eqn. (6.99) is integrated backward in time with the boundary condition $P(T) = P_f(T)$. This form clearly has significant computational advantages over integrating the backward filter covariance and using eqn. (6.83). Similar to eqn. (6.83) only one matrix inverse is required in eqn. (6.99); however, the smoother covariance is calculated directly without the need to first calculate the backward filter covariance. Also, at steady-state eqn. (6.99) reduces down to an algebraic Lyapunov equation, which is a linear equation.

To derive an expression for the smoother state estimate, we begin with eqn. (6.79), which can be rewritten as

$$P^{-1}(t)\hat{\mathbf{x}}(t) = P_f^{-1}(t)\,\hat{\mathbf{x}}_f(t) + \hat{\chi}_b(t) \tag{6.100}$$

Taking the time derivative of eqn. (6.100), and using eqn. (6.81) for the derivative of $P^{-1}(t)$ and $P_f^{-1}(t)$ leads to

$$\begin{aligned}P^{-1}(t)\left[\frac{d}{dt}\hat{\mathbf{x}}(t)\right] &= P^{-1}(t)\left[\frac{d}{dt}P(t)\right]P^{-1}(t)\,\hat{\mathbf{x}}(t) + P_f^{-1}(t)\left[\frac{d}{dt}\hat{\mathbf{x}}_f(t)\right]\\ &\quad - P_f^{-1}(t)\left[\frac{d}{dt}P_f(t)\right]P_f^{-1}(t)\,\hat{\mathbf{x}}_f(t) + \frac{d}{dt}\hat{\chi}_b(t)\end{aligned} \tag{6.101}$$

Substituting the relations in eqns. (6.74) and (6.87) with $d\hat{\chi}_b/d\tau = -d\hat{\chi}_b/dt$, and (6.99) into eqn. (6.101), and after considerable algebra manipulations (which are left as an exercise for the reader), yields

$$\boxed{\frac{d}{dt}\hat{\mathbf{x}}(t) = F(t)\,\hat{\mathbf{x}}(t) + B(t)\,\mathbf{u}(t) + G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)\left[\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)\right]} \tag{6.102}$$

Table 6.4: Continuous-Time RTS Smoother

Model	$\frac{d}{dt}\mathbf{x}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t),\ \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ $\tilde{\mathbf{y}}(t) = H(t)\,\mathbf{x}(t) + \mathbf{v}(t),\ \mathbf{v}(t) \sim N(\mathbf{0}, R(t))$
Forward Covariance	$\frac{d}{dt}P_f(t) = F(t)\,P_f(t) + P_f(t)\,F^T(t)$ $-P_f(t)\,H^T(t)\,R^{-1}(t)H(t)\,P_f(t)$ $+G(t)\,Q(t)\,G^T(t),$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\,\tilde{\mathbf{x}}_f^T(t_0)\}$
Forward Filter	$\frac{d}{dt}\hat{\mathbf{x}}_f(t) = F(t)\,\hat{\mathbf{x}}_f(t) + B(t)\,\mathbf{u}(t)$ $+P_f(t)\,H^T(t)\,R^{-1}(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_f(t)], \quad \hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$
Smoother Covariance	$\frac{d}{d\tau}P(t) = -[F(t) + G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)]P(t)$ $-P(t)[F(t) + G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)]^T$ $+G(t)\,Q(t)\,G^T(t), \quad P(T) = P_f(T)$
Smoother Estimate	$\frac{d}{d\tau}\hat{\mathbf{x}}(t) = -F(t)\,\hat{\mathbf{x}}(t) - B(t)\,\mathbf{u}(t)$ $-G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)\left[\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)\right], \quad \hat{\mathbf{x}}(T) = \hat{\mathbf{x}}_f(T)$

Equation (6.102) is integrated backward in time with the boundary condition $\hat{\mathbf{x}}(T) = \hat{\mathbf{x}}_f(T)$.

A summary of the RTS smoother is given in Table 6.4. As before, the forward Kalman filter is executed using the measurements until time T. Storing the estimated states from the forward filter, the smoothed estimate is then determined by integrating eqn. (6.102) backward in time. Similar to the discrete-time RTS smoother, eqn. (6.99) is only used to derive the smoother covariance, which is not used to find the optimal smoother state estimate. Also, eqn. (6.102) does not involve the measurement directly, but still uses the forward filter state estimate. For all these reasons the RTS smoother is more widely used in practice over the formulation given in Table 6.3.

6.1.2.3 Stability

The backward state matrix in the RTS smoother defines the stability of the system, which is given by $[F(t) + G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)]$. A backward integration is stable if all the eigenvalues lie in the right-hand plane. This can be re-evaluated using the negative of the RTS smoother state-matrix, so that its eigenvalues must lie in the left-hand plane for stability. Then the backward smoother stability can be evaluated by investigating the dynamics of the following system:

$$\frac{d}{d\tau}\hat{\mathbf{x}}(t) = -[F(t) + G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)]\hat{\mathbf{x}}(t) \tag{6.103}$$

The smoother stability can be proved by using Lyapunov's direct method, which is discussed for continuous-time systems in §3.6. For the continuous-time RTS smoother we consider the following candidate Lyapunov function:

$$V[\hat{\mathbf{x}}(t)] = \hat{\mathbf{x}}^T(t)\,P_f^{-1}(t)\,\hat{\mathbf{x}}(t) \tag{6.104}$$

Taking a time derivative of eqn. (6.104) gives

$$\begin{aligned}\frac{d}{d\tau}V[\hat{\mathbf{x}}(t)] = &\left[\frac{d}{d\tau}\hat{\mathbf{x}}(t)\right]^T P_f^{-1}(t)\,\hat{\mathbf{x}}(t) + \hat{\mathbf{x}}^T(t)\left[\frac{d}{d\tau}P_f^{-1}(t)\right]\hat{\mathbf{x}}(t)\\ &+ \hat{\mathbf{x}}^T(t)\,P_f^{-1}(t)\left[\frac{d}{d\tau}\hat{\mathbf{x}}(t)\right]\end{aligned} \tag{6.105}$$

Using eqn. (6.81) for $P_f^{-1}(t)$ with $dP_f^{-1}/dt = -dP_f^{-1}/d\tau$, and substituting the resulting expression and eqn. (6.103) into eqn. (6.105) leads to

$$\frac{d}{d\tau}V[\hat{\mathbf{x}}(t)] = -\hat{\mathbf{x}}^T(t)\left[H^T(t)\,R^{-1}(t)H(t) + P_f^{-1}(t)\,G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)\right]\hat{\mathbf{x}}(t) \tag{6.106}$$

Clearly if $R(t)$ is positive definite and $Q(t)$ is at least positive semi-definite, then the Lyapunov condition is satisfied and the continuous-time RTS smoother is stable.

Example 6.2: We consider the simple first-order system shown in example 5.4, where the truth model is given by

$$\begin{aligned}\dot{x}(t) &= f\,x(t) + w(t)\\ y(t) &= x(t) + v(t)\end{aligned}$$

where f is a constant, and the variances of $w(t)$ and $v(t)$ are given by q and r, respectively. In the current example the steady-state smoother covariance is investigated. From eqn. (6.99) this value can be determined by solving the following linear differential equation:

$$\frac{d}{dt}p(t) = 2[f + q\,p_f^{-1}(t)]\,p(t) - q$$

where $p_f(t)$ is defined in example 5.4. Since q is a constant, then the steady-state value for $p(t)$ is simply given by

$$\lim_{t\to\infty} p(t) \equiv p = \frac{q}{2\left(f + q\,p_f^{-1}\right)}$$

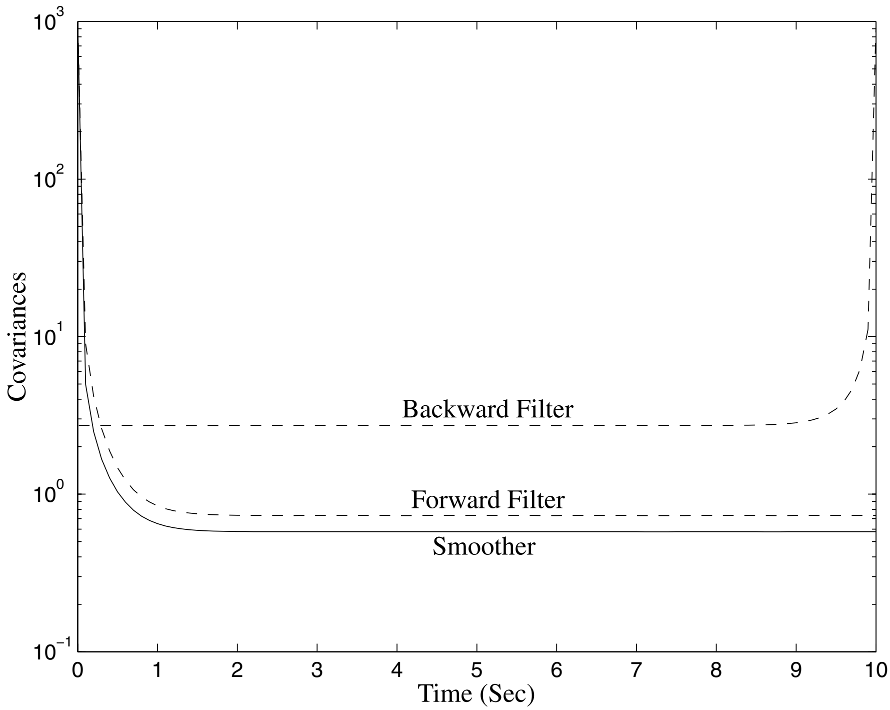

Figure 6.4: Forward Filter, Backward Filter, and Smoother Covariances

Substituting $p_f^{-1} = r^{-1}/(a+f)$, where $a \equiv \sqrt{f^2 + r^{-1}q}$, into the above expression, and after some algebraic manipulations yields

$$p = \frac{q}{2a}$$

From eqn. (6.82) the steady-state backward filter covariance (defined by p_b) can be determined by solving the following quadratic equation:

$$q\, p_b^{-2} - 2\, f\, p_b^{-1} - r^{-1} = 0$$

Taking the positive root yields

$$p_b = \frac{q}{a+f}$$

This can also be easily verified from $p^{-1} = p_f^{-1} + p_b^{-1}$. An interesting aspect of the backward filter covariance is that it is zero when $q = 0$, so that the smoother covariance is equivalent to the forward filter covariance. Hence, for this case the smoother offers no improvements over the forward filter, which is fully proved by Fraser.[8] For all other positive values of q it can be shown that $p \leq p_f$ and $p \leq p_b$, which is left as an exercise for the reader. Consider the following values: $f = -1$,

$q = 2$, and $r = 1$, with an initial condition of $p_f(t_0) = 1,000$. Plots of the forward filter, backward filter, and smoother covariances given by integrating eqns. (6.74c), (6.82), and (6.99), respectively, are shown in Figure 6.4. The analytical steady-state values are given by: $p_f = (\sqrt{3}-1)/1 = 0.7321$, $p_b = 2/(\sqrt{3}-1) = 2.7321$, and $p = 1/\sqrt{3} = 0.5774$, which all agree with the plots in Figure 6.4. An interesting case occurs when $f = 0$, which gives $p_f = p_b = \sqrt{r\,q}$ and $p = \sqrt{r\,q}/2$. From eqn. (6.79) the smoother state estimate for this case is given by

$$\hat{x}(t) = \frac{1}{2}\left[\hat{x}_f(t) + \hat{x}_b(t)\right]$$

Therefore, using the steady-state smoother the optimal estimate of $x(t)$ is the average of the forward and backward filter estimates. This simple example clearly shows the power of the fixed-interval smoother to provide better estimates (i.e., estimates with lower error-covariances) than the standard Kalman filter alone.

6.1.3 Nonlinear Smoothing

In this section the fixed-interval smoothing algorithms derived previously are extended for nonlinear systems. Most modern-day nonlinear applications involve systems with discrete-time measurements and continuous-time models. The first step in the nonlinear smoother involves applying the extended Kalman filter shown in Table 5.9. In order to perform the backward-time integration and measurement updates, straightforward application of the methods in §6.1.2 cannot be applied directly to nonlinear systems. This is due to the fact that we must linearize the backward-time filter about the forward-time filter estimate, not the backward-time filter estimate! Hence, the linearized Kalman filter form shown in §5.6 will be used to derive the backward-time smoother, where the nominal (*a priori*) estimate is given by the forward-time extended Kalman filter. A more formal treatment of nonlinear smoothing is given in Ref. [12].

The derivation of the nonlinear smoother can be shown by using the same procedure leading to the forward/backward filters shown previously. However, we will only show the RTS version of this smoother, since it has clear advantages over the two filter solution, which is given in Ref. [1]. A rigorous proof of the nonlinear RTS smoother is possible using similar methods shown to derive the Kalman filter in §5.5. A detailed derivation for the linear case is given by Bierman.[13] We will prove the nonlinear smoother using variational calculus in §6.4.1.3.

The actual implementation of the RTS nonlinear smoother state estimate is fairly simple. Note that the extended Kalman filter in Table 5.9 provides continuous-time estimates. Therefore, the nonlinear version of eqn. (6.102) can be used directly to determine the smoother state estimate. First, we linearize $\mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t), t)$ about $\hat{\mathbf{x}}_f(t)$.

Table 6.5: Continuous-Discrete Nonlinear RTS Smoother

Model	$\frac{d}{d\tau}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t),\,\mathbf{u}(t),\,t) + G(t)\,\mathbf{w}(t),\ \mathbf{w}(t) \sim N(\mathbf{0},\,Q(t))$ $\tilde{\mathbf{y}}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k,\ \mathbf{v}_k \sim N(\mathbf{0},\,R_k)$
Forward Initialize	$\hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$ $P_{f0} = E\left\{\tilde{\mathbf{x}}_f(t_0)\,\tilde{\mathbf{x}}_f^T(t_0)\right\}$
Forward Gain	$K_{fk} = P_{fk}^- H_k^T(\hat{\mathbf{x}}_{fk}^-)[H_k(\hat{\mathbf{x}}_{fk}^-)P_{fk}^- H_k^T(\hat{\mathbf{x}}_{fk}^-) + R_k]^{-1}$ $H_k(\hat{\mathbf{x}}_{fk}^-) \equiv \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right\|_{\hat{\mathbf{x}}_{fk}^-}$
Forward Update	$\hat{\mathbf{x}}_{fk}^+ = \hat{\mathbf{x}}_{fk}^- + K_{fk}[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}_{fk}^-)]$ $P_{fk}^+ = [I - K_{fk}H_k(\hat{\mathbf{x}}_{fk}^-)]P_{fk}^-$
Forward Propagation	$\frac{d}{dt}\hat{\mathbf{x}}_f(t) = \mathbf{f}(\hat{\mathbf{x}}_f(t),\,\mathbf{u}(t),\,t)$ $\frac{d}{dt}P_f(t) = F(\hat{\mathbf{x}}_f(t),\,t)\,P_f(t) + P_f(t)\,F^T(\hat{\mathbf{x}}_f(t),\,t)$ $+G(t)\,Q(t)\,G^T(t)$ $F(\hat{\mathbf{x}}_f(t),\,t) \equiv \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right\|_{\hat{\mathbf{x}}_f(t)}$
Gain	$K(t) \equiv G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)$
Smoother Covariance	$\frac{d}{d\tau}P(t) = -[F(\hat{\mathbf{x}}_f(t),\,t) + K(t)]P(t)$ $-P(t)[F(\hat{\mathbf{x}}_f(t),\,t) + K(t)]^T$ $+G(t)\,Q(t)\,G^T(t), \quad P(T) = P_f(T)$
Smoother Estimate	$\frac{d}{d\tau}\hat{\mathbf{x}}(t) = -\left[F(\hat{\mathbf{x}}_f(t),\,t) + K(t)\right]\left[\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)\right]$ $-\mathbf{f}(\hat{\mathbf{x}}_f(t),\,\mathbf{u}(t),\,t), \quad \hat{\mathbf{x}}(T) = \hat{\mathbf{x}}_f(T)$

Then, using $d\mathbf{x}/dt = -d\mathbf{x}/d\tau$ to denote the backward-time integration leads to

$$\boxed{\frac{d}{d\tau}\hat{\mathbf{x}}(t) = -\left[F(\hat{\mathbf{x}}_f(t),\,t) + K(t)\right]\left[\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)\right] - \mathbf{f}(\hat{\mathbf{x}}_f(t),\,\mathbf{u}(t),\,t)} \tag{6.107}$$

where

$$\boxed{K(t) \equiv G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)} \tag{6.108}$$

and

$$F(\hat{\mathbf{x}}_f(t), t) \equiv \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}_f(t)} \tag{6.109}$$

Equation (6.107) must be integrated backward in time with a boundary condition of $\hat{\mathbf{x}}(T) = \hat{\mathbf{x}}_f(T)$. Note that eqn. (6.107) is a linear equation in $\hat{\mathbf{x}}(t)$, which allows us to use linear integration methods. Also, the smoother covariance follows the following equation:

$$\boxed{\begin{aligned} \frac{d}{d\tau} P(t) = &- \left[F(\hat{\mathbf{x}}_f(t), t) + K(t) \right] P(t) - P(t) \left[F(\hat{\mathbf{x}}_f(t), t) + K(t) \right]^T \\ &+ G(t)\, Q(t)\, G^T(t) \end{aligned}} \tag{6.110}$$

Equation (6.110) must also be integrated backward in time with a boundary condition of $P(T) = P_f(T)$.

A summary of continuous-discrete nonlinear RTS smoother is given in Table 6.5. First, the extended Kalman filter is executed forward in time on the data set. Then, eqn. (6.107) is integrated backward in time using the stored forward-filter state estimate and covariance. The smoother state and covariance are clearly a function of the inverse of the forward-time covariance. One method to overcome this inverse is to use the information matrix version of the Kalman filter, shown in §5.3.3. Another smoother form that does not require the inverse of the covariance matrix is presented by Bierman.[13] This form uses an "adjoint variable," $\boldsymbol{\lambda}(t)$, to derive the smoother equation. We will derive this form directly using variational calculus in §6.4.1.3. The propagation equations are given by

$$\boxed{\frac{d}{d\tau} \boldsymbol{\lambda}(t) = F^T(\hat{\mathbf{x}}(t), t)\, \boldsymbol{\lambda}(t)} \tag{6.111a}$$

$$\boxed{\frac{d}{d\tau} \Lambda(t) = F^T(\hat{\mathbf{x}}_f(t), t)\, \Lambda(t) + \Lambda(t)\, F(\hat{\mathbf{x}}_f(t), t)} \tag{6.111b}$$

where $\Lambda(t)$ is the covariance of $\boldsymbol{\lambda}(t)$. The backward updates are given by

$$\boxed{\boldsymbol{\lambda}_k^- = \left[I - H_k^T(\hat{\mathbf{x}}_{fk}^-)\, K_{fk}^T \right] \boldsymbol{\lambda}_k^+ - H_k^T(\hat{\mathbf{x}}_{fk}^-)\, D_{fk}^{-1} \left[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_{fk}^-) \right]} \tag{6.112a}$$

$$\boxed{\begin{aligned} \Lambda_k^- = &\left[I - K_{fk}\, H_k(\hat{\mathbf{x}}_{fk}^-) \right]^T \Lambda_k^+ \left[I - K_{fk}\, H_k(\hat{\mathbf{x}}_{fk}^-) \right] \\ &+ H_k^T(\hat{\mathbf{x}}_{fk}^-)\, D_{fk}^{-1} H_k(\hat{\mathbf{x}}_{fk}^-) \end{aligned}} \tag{6.112b}$$

where

$$D_{fk} \equiv H_k(\hat{\mathbf{x}}_{fk}^-)\, P_{fk}^- H_k^T(\hat{\mathbf{x}}_{fk}^-) + R_k \tag{6.113}$$

Note that in this formulation $\boldsymbol{\lambda}_k^-$ is used to denote the backward update just before the measurement is processed. If $T \equiv t_N$ is an observation time, then the boundary conditions are given by

$$\boldsymbol{\lambda}_N^- = -H_N^T(\hat{\mathbf{x}}_{fN}^-)\,D_{fN}^{-1}\left[\tilde{\mathbf{y}}_N - \mathbf{h}_N(\hat{\mathbf{x}}_{fN}^-)\right] \tag{6.114a}$$

$$\Lambda_N^- = H_T N^T(\hat{\mathbf{x}}_{fN}^-)\,D_{fN}^{-1} H_N(\hat{\mathbf{x}}_{fN}^-) \tag{6.114b}$$

If T is not an observation time, then $\boldsymbol{\lambda}$ and Λ simply have boundary conditions of zero. Finally, the smoother state and covariance can be constructed via

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk}^{\pm} - P_{fk}^{\pm}\boldsymbol{\lambda}_k^{\pm} \tag{6.115a}$$

$$P_k = P_{fk}^{\pm} - P_{fk}^{\pm}\Lambda_k^{\pm}P_{fk}^{\pm} \tag{6.115b}$$

where the propagated or updated variables yield the same result. The matrix D_{fk}^{-1} is used directly in the forward-time Kalman filter, which can be stored directly. Therefore, an extra inverse is not required by this alternative approach.

Example 6.3: In this example the model used in example 5.5 is used to demonstrate the power of the RTS nonlinear smoother using continuous-time models with discrete-time measurements. The smoother given in Table 6.5 is used to determine optimal state estimates. The parameters used for this simulation are identical to the parameters given in example 5.5. First, the forward-time extended Kalman filter is executed using the measured data. Then, the smoother state estimate and covariance are determined by integrating eqns. (6.107) and (6.110) backward in time.

A plot of the results is shown in Figure 6.5. Clearly, the smoother covariance and estimate errors are much smaller than the forward-time estimates. Although the smoother estimates cannot be given in real time, these estimates can often provide very useful information. For example, in Ref. [14] a nonlinear smoother algorithm has been used to show uncontrolled motions ("nutation") of a spacecraft, which are not visible in the forward-time estimates. This information may be used to redesign a controller or filter if these nutations lead to unacceptable pointing errors.

6.2 Fixed-Point Smoothing

In this section the fixed-point smoothing algorithm is shown for both discrete-time and continuous-time models. Meditch[15] provides an excellent example on the usefulness of fixed-point smoothing, which we will summarize here. Suppose that a spacecraft is tracked by a ground-based radar, and we implement an orbit determination algorithm using an extended Kalman filter to determine a state estimate of $\mathbf{x}_N$ at some time t_N. The estimate is derived from the measurements $\tilde{\mathbf{y}}_k$, where

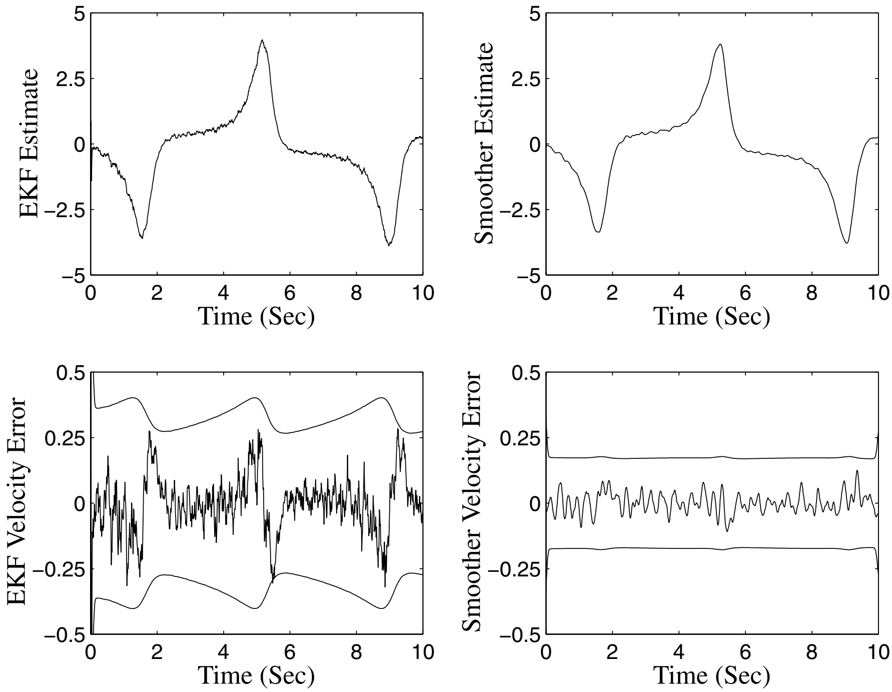

Figure 6.5: Nonlinear RTS Results for Van der Pol's Equation

$k = 1, 2, \ldots, N$, and is denoted by $\hat{\mathbf{x}}_{N|N}$. Suppose now that additional orbital data becomes available, say after an orbital burn, and that we wish to estimate the state at later times. Thus we seek to determine $\hat{\mathbf{x}}_{N|N+1}$, $\hat{\mathbf{x}}_{N|N+2}$, etc., taking the estimate at time t_N into account. Using notation from §2.6, for some fixed N we wish to determine the following quantity:

$$\hat{\mathbf{x}}_{k|N} \equiv E\left\{\hat{\mathbf{x}}_k|[\tilde{\mathbf{y}}_1, \tilde{\mathbf{y}}_2 \ldots, \tilde{\mathbf{y}}_N]\right\} \tag{6.116}$$

for $N > k$, where the notation $k|N$ denotes the smoothed estimate at time t_k, given measurements up to time t_N.

6.2.1 Discrete-Time Formulation

To derive the necessary relations for the discrete-time fixed-point smoother, we start with the measurement residual equation given by eqn. (5.280):

$$\begin{aligned} \upsilon_{fk} &= \tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}^- \\ &= -H_k \tilde{\mathbf{x}}_{fk}^- + \mathbf{v}_k \end{aligned} \tag{6.117}$$

where $\tilde{\mathbf{x}}_{fk}^{-} \equiv \hat{\mathbf{x}}_{fk}^{-} - \mathbf{x}_k$. Meditch[15] shows that the single-stage optimal smoothing relation follows the following equation:

$$\hat{\mathbf{x}}_{k|k+1} = \hat{\mathbf{x}}_{k|k} + P_k^{x\upsilon}(P_{k+1}^{\upsilon\upsilon})^{-1}\boldsymbol{\upsilon}_{fk+1} \tag{6.118}$$

where

$$P_k^{x\upsilon} = E\left\{\mathbf{x}_k \boldsymbol{\upsilon}_{fk+1}^T\right\} \tag{6.119a}$$

$$P_{k+1}^{\upsilon\upsilon} = E\left\{\boldsymbol{\upsilon}_{fk+1} \boldsymbol{\upsilon}_{fk+1}^T\right\} \tag{6.119b}$$

Note the similarities between eqn. (6.118) and eqn. (5.279a). Substituting the one time-step ahead of eqn. (6.117) into eqn. (6.119a) and using the fact that $\mathbf{v}_{k+1}$ has zero mean leads to

$$P_k^{x\upsilon} = -E\left\{\mathbf{x}_k \tilde{\mathbf{x}}_{fk+1}^{-T}\right\} H_{k+1}^T \tag{6.120}$$

Substituting eqn. (5.33) into eqn. (6.120) and using the fact that $\mathbf{w}_k$ has zero mean leads to

$$P_k^{x\upsilon} = -E\left\{\mathbf{x}_k \tilde{\mathbf{x}}_{fk}^{+T}\right\} \Phi_k^T H_{k+1}^T \tag{6.121}$$

Substituting the relationship $\mathbf{x}_k = \hat{\mathbf{x}}_{fk}^{+} - \tilde{\mathbf{x}}_{fk}^{+}$ into eqn. (6.121) and using the orthogonality principle given by eqn. (5.109) yields

$$P_k^{x\upsilon} = P_{fk}^{+} \Phi_k^T H_{k+1}^T \tag{6.122}$$

The covariance of the innovations process can easily be derived using eqn. (6.117), which is given by

$$P_{k+1}^{\upsilon\upsilon} = H_{k+1} P_{fk+1}^{-} H_{k+1}^T + R_{k+1} \tag{6.123}$$

Substituting eqn. (5.30a) into the one time-step ahead of eqn. (6.117), and then substituting the resultant together with eqns. (6.122) and (6.123) into eqn. (6.118) yields

$$\hat{\mathbf{x}}_{k|k+1} = \hat{\mathbf{x}}_{k|k} + \mathcal{M}_{k|k+1}\left[\tilde{\mathbf{y}}_{k+1} - H_{k+1}\Phi_k \hat{\mathbf{x}}_{fk}^{+} - H_{k+1}\Gamma_k \mathbf{u}_k\right] \tag{6.124}$$

where

$$\mathcal{M}_{k|k+1} \equiv P_k^{x\upsilon}(P_{k+1}^{\upsilon\upsilon})^{-1} = P_{fk}^{+}\Phi_k^T H_{k+1}^T \left[H_{k+1} P_{fk+1}^{-} H_{k+1}^T + R_{k+1}\right]^{-1} \tag{6.125}$$

The expression in eqn. (6.124) can be rewritten by using the definition of the forward gain, K_{fk} in Table 6.2, which gives

$$H_{k+1}^T \left[H_{k+1} P_{fk+1}^{-} H_{k+1}^T + R_{k+1}\right]^{-1} = (P_{fk+1}^{-})^{-1} K_{fk+1} \tag{6.126}$$

Also, from eqns. (5.30a) and (5.30b) we have

$$K_{fk+1}\left[\tilde{\mathbf{y}}_{k+1} - H_{k+1}\Phi_k \hat{\mathbf{x}}_{fk}^{+} - H_{k+1}\Gamma_k \mathbf{u}_k\right] = \hat{\mathbf{x}}_{fk+1}^{+} - \hat{\mathbf{x}}_{fk+1}^{-} \tag{6.127}$$

Therefore, eqn. (6.124) can be rewritten as

$$\hat{\mathbf{x}}_{k|k+1} = \hat{\mathbf{x}}_{k|k} + \mathcal{K}_k[\hat{\mathbf{x}}^+_{fk+1} - \hat{\mathbf{x}}^-_{fk+1}] \tag{6.128}$$

where

$$\mathcal{K}_k \equiv P^+_{fk}\Phi^T_k(P^-_{fk+1})^{-1} \tag{6.129}$$

Note that the gain in eqn. (6.129) is the same exact gain used in the discrete-time RTS smoother given in Table 6.2. In fact the RTS smoother can be derived directly from eqn. (6.128) (which is left as an exercise for the reader).

We now develop an expression for the double-stage optimal smoother relationship. This relationship can be derived from the double-stage version of eqn. (6.118):

$$\hat{\mathbf{x}}_{k|k+2} = \hat{\mathbf{x}}_{k|k+1} + P^{x\upsilon}_{k+1}(P^{\upsilon\upsilon}_{k+2})^{-1}\boldsymbol{\upsilon}_{fk+2} \tag{6.130}$$

where

$$P^{x\upsilon}_{k+1} = E\left\{\mathbf{x}_k\boldsymbol{\upsilon}^T_{fk+2}\right\} \tag{6.131a}$$

$$P^{\upsilon\upsilon}_{k+2} = E\left\{\boldsymbol{\upsilon}_{fk+2}\boldsymbol{\upsilon}^T_{fk+2}\right\} \tag{6.131b}$$

Implementing the same procedure that has been used to derive eqn. (6.121), then $P^{x\upsilon}_{k+1}$ is easily shown to be given by

$$P^{x\upsilon}_{k+1} = -E\left\{\mathbf{x}_k\tilde{\mathbf{x}}^{T+}_{fk+1}\right\}\Phi^T_{k+1}H_{k+2} \tag{6.132}$$

Substituting the one time-step ahead of eqn. (5.37) into eqn. (6.132) and using the fact that $\mathbf{v}_{k+1}$ has zero mean leads to

$$P^{x\upsilon}_{k+1} = -E\left\{\mathbf{x}_k\tilde{\mathbf{x}}^{-T}_{fk+1}[I - K_{fk+1}H_{k+1}]^T\right\}\Phi^T_{k+1}H_{k+2} \tag{6.133}$$

Using eqns. (6.120) and (6.122) in eqn. (6.133) yields

$$P^{x\upsilon}_{k+1} = P^+_{fk}\Phi^T_k[I - K_{fk+1}H_{k+1}]^T\Phi^T_{k+1}H_{k+2} \tag{6.134}$$

Using one time-step ahead of eqn. (6.123) with eqn. (6.134) yields

$$\begin{aligned}\mathcal{M}_{k|k+2} \equiv P^{x\upsilon}_{k+1}(P^{\upsilon\upsilon}_{k+2})^{-1} = P^+_{fk}\Phi^T_k&[I - K_{fk+1}H_{k+1}]^T\Phi^T_{k+1}H_{k+2}\\ &\times\left[H_{k+2}P^-_{fk+2}H^T_{k+2} + R_{k+2}\right]^{-1}\end{aligned} \tag{6.135}$$

Note that comparing eqns. (6.125) and (6.135) indicates $\mathcal{M}_{k|k+2}$ is not simply the one time-step ahead of $\mathcal{M}_{k|k+1}$. From eqn. (5.44) we have

$$[I - K_{fk+1}H_{k+1}] = P^+_{fk+1}(P^-_{fk+1})^{-1} \tag{6.136}$$

Substituting eqn. (6.136) and the one time-step ahead of eqn. (6.126) into eqn. (6.135) yields

$$\mathcal{M}_{k|k+2} = \mathcal{K}_k\mathcal{K}_{k+1}K_{fk+2} \tag{6.137}$$

where $\mathcal{K}_{k+1}$ is clearly the one time-step ahead of $\mathcal{K}_k$. Hence, the double-stage optimal smoother follows the following equation:

$$\hat{\mathbf{x}}_{k|k+2} = \hat{\mathbf{x}}_{k|k+1} + \mathcal{K}_k \mathcal{K}_{k+1}[\hat{\mathbf{x}}^+_{f\,k+2} - \hat{\mathbf{x}}^-_{f\,k+2}] \tag{6.138}$$

By induction the discrete-time fixed-point optimal smoother equation follows

$$\boxed{\hat{\mathbf{x}}_{k|N} = \hat{\mathbf{x}}_{k|N-1} + \mathcal{B}_N[\hat{\mathbf{x}}^+_{f\,N} - \hat{\mathbf{x}}^-_{f\,N}]} \tag{6.139}$$

where

$$\boxed{\mathcal{B}_N = \prod_{i=k}^{N-1} \mathcal{K}_i = \mathcal{B}_{N-1}\mathcal{K}_{N-1}} \tag{6.140}$$

and

$$\boxed{\mathcal{K}_i = P^+_{f\,i}\Phi_i^T (P^-_{f\,i+1})^{-1}} \tag{6.141}$$

with boundary condition given by $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}^+_{f\,k}$.

The covariance of the discrete-time fixed-point smoother can be derived from

$$P_{k|N} \equiv E\left\{\hat{\mathbf{x}}_{k|N}\hat{\mathbf{x}}^T_{k|N}\right\} \tag{6.142}$$

First, the following error state is defined:

$$\tilde{\mathbf{x}}_{k|N} = \hat{\mathbf{x}}_{k|N} - \mathbf{x}_k \tag{6.143}$$

Substituting eqn. (6.139) into eqn. (6.143) yields

$$\tilde{\mathbf{x}}_{k|N} = \tilde{\mathbf{x}}_{k|N-1} - \mathcal{B}_N[\hat{\mathbf{x}}^+_{f\,N} - \hat{\mathbf{x}}^-_{f\,N}] \tag{6.144}$$

which is rearranged into

$$\tilde{\mathbf{x}}_{k|N} - \mathcal{B}_N\hat{\mathbf{x}}^+_{f\,N} = \tilde{\mathbf{x}}_{k|N-1} + \mathcal{B}_N\hat{\mathbf{x}}^-_{f\,N} \tag{6.145}$$

Since the terms $\tilde{\mathbf{x}}_{k|N-1}$, $\hat{\mathbf{x}}^+_{f\,N}$, and $\hat{\mathbf{x}}^-_{f\,N}$ are all uncorrelated, the covariance is simply given by

$$P_{k|N} = P_{k|N-1} + \mathcal{B}_N[P^{\hat{x}\hat{x}-}_{f\,N} - P^{\hat{x}\hat{x}+}_{f\,N}]\mathcal{B}^T_N \tag{6.146}$$

where

$$P^{\hat{x}\hat{x}-}_{f\,N} = E\left\{\hat{\mathbf{x}}^-_{f\,N}\hat{\mathbf{x}}^{-T}_{f\,N}\right\} \tag{6.147a}$$

$$P^{\hat{x}\hat{x}+}_{f\,N} = E\left\{\hat{\mathbf{x}}^+_{f\,N}\hat{\mathbf{x}}^{+T}_{f\,N}\right\} \tag{6.147b}$$

Next the following relationship is used (the proof is left as an exercise for the reader):

$$P^{\hat{x}\hat{x}-}_{f\,N} - P^{\hat{x}\hat{x}+}_{f\,N} = P^+_{f\,N} - P^-_{f\,N} \tag{6.148}$$

Table 6.6: Discrete-Time Fixed-Point Smoother

Model	$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k, \quad \mathbf{w}_k \sim N(\mathbf{0}, Q_k)$ $\tilde{\mathbf{y}}_k = H_k \mathbf{x}_k + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{0}, R_k)$
Forward Initialize	$\hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\, \tilde{\mathbf{x}}_f^T(t_0)\}$
Gain	$K_{fk} = P_{fk}^- H_k^T [H_k P_{fk}^- H_k^T + R_k]^{-1}$
Forward Update	$\hat{\mathbf{x}}_{fk}^+ = \hat{\mathbf{x}}_{fk}^- + K_{fk}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}^-]$ $P_{fk}^+ = [I - K_{fk} H_k] P_{fk}^-$
Forward Propagation	$\hat{\mathbf{x}}_{fk+1}^- = \Phi_k \hat{\mathbf{x}}_{fk}^+ + \Gamma_k \mathbf{u}_k$ $P_{fk+1}^- = \Phi_k P_{fk}^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$
Smoother Initialize	$\hat{\mathbf{x}}_{k\|k} = \hat{\mathbf{x}}_{fk}^+$ $P_{k\|k} = P_{fk}^+$
Gain	$\mathcal{B}_N = \prod_{i=k}^{N-1} \mathcal{K}_i, \quad \mathcal{K}_i = P_{fi}^+ \Phi_i^T (P_{fi+1}^-)^{-1}$
Covariance	$P_{k\|N} = P_{k\|N-1} + \mathcal{B}_N [P_{fN}^+ - P_{fN}^-] \mathcal{B}_N^T$
Estimate	$\hat{\mathbf{x}}_{k\|N} = \hat{\mathbf{x}}_{k\|N-1} + \mathcal{B}_N [\hat{\mathbf{x}}_{fN}^+ - \hat{\mathbf{x}}_{fN}^-]$

Substituting eqn. (6.148) into eqn. (6.146) gives

$$\boxed{P_{k|N} = P_{k|N-1} + \mathcal{B}_N [P_{fN}^+ - P_{fN}^-] \mathcal{B}_N^T} \tag{6.149}$$

with boundary condition of $P_{k|k} = P_{fk}^+$. A summary of the discrete-time fixed-point smoother is given in Table 6.6. As with the discrete-time RTS smoother the fixed-point smoother begins by implementing the standard forward-time Kalman filter. The smoother state at a fixed point is simply given by using eqn. (6.139). The smoother covariance at the desired point is computed using eqn. (6.149). Once again the smoother does not require the computation of the covariance to determine the estimate, analogous to the RTS smoother.

Table 6.7: Continuous-Time Fixed-Point Smoother

Model	$\frac{d}{dt}\mathbf{x}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t),\ \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ $\tilde{\mathbf{y}}(t) = H(t)\,\mathbf{x}(t) + \mathbf{v}(t),\ \mathbf{v}(t) \sim N(\mathbf{0}, R(t))$
Forward Covariance	$\frac{d}{dt}P_f(t) = F(t)\,P_f(t) + P_f(t)\,F^T(t)$ $-P_f(t)\,H^T(t)\,R^{-1}(t)H(t)\,P_f(t)$ $+G(t)\,Q(t)\,G^T(t),$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\,\tilde{\mathbf{x}}_f^T(t_0)\}$
Forward Filter	$\frac{d}{dt}\hat{\mathbf{x}}_f(t) = F(t)\,\hat{\mathbf{x}}_f(t) + B(t)\,\mathbf{u}(t)$ $+P_f(t)\,H^T(t)\,R^{-1}(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_f(t)], \quad \hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$
Transition Matrix	$\frac{d}{dT}\Phi(t,T) = -\Phi(t,T)[F(T) + G(T)\,Q(T)\,G^T(T)\,P_f^{-1}(T)],$ $\Phi(t,t) = I$
Smoother Covariance	$\frac{d}{dT}P(t\|T) = -\Phi(t,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)\,P_f(T)\,\Phi^T(t,T),$ $P(t\|t) = P_f(t)$
Smoother Estimate	$\frac{d}{dT}\hat{\mathbf{x}}(t\|T) = \Phi(t,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)[\tilde{\mathbf{y}}(T) - H(T)\,\hat{\mathbf{x}}_f(T)],$ $\hat{\mathbf{x}}(t\|t) = \hat{\mathbf{x}}_f(t)$

6.2.2 Continuous-Time Formulation

The continuous-time fixed-point smoother can be derived from the discrete-time version. A simpler way involves rewriting eqn. (6.102) in terms as the smoother estimate at time t given a state estimate at time T:[1]

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{x}}(t|T) &= [F(t) + G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)]\,\hat{\mathbf{x}}(t|T) + B(t)\,\mathbf{u}(t)\\ &\quad - G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)\,\hat{\mathbf{x}}_f(t)\end{aligned} \tag{6.150}$$

where $T \geq t$ and $\hat{\mathbf{x}}(t|t) = \hat{\mathbf{x}}_f(t)$. The solution of eqn. (6.150) is given by using the methods described in §3.1.3, which is given by

$$\begin{aligned}\hat{\mathbf{x}}(t|T) &= \Phi(t,T)\,\hat{\mathbf{x}}_f(T) + \int_T^t \Phi(t,\tau)\,B(\tau)\,\mathbf{u}(\tau)\,d\tau\\ &\quad - \int_T^t \Phi(t,\tau)\,G(\tau)\,Q(\tau)\,G^T(\tau)\,P_f^{-1}(\tau)\,\hat{\mathbf{x}}_f(\tau)\,d\tau\end{aligned} \tag{6.151}$$

where $\Phi(t,T)$ is the state transition matrix of $F(t)+G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)$, which clearly must obey

$$\frac{d}{dt}\Phi(t,T) = [F(t)+G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)]\,\Phi(t,T), \quad \Phi(t,t) = I \tag{6.152}$$

For the fixed-point smoother we consider the case where t is fixed and allow T to vary. Therefore, in order to derive an expression for the fixed-point smoother estimate, eqn. (6.151) must be differentiated with respect to T, which yields

$$\begin{aligned}\frac{d}{dT}\hat{\mathbf{x}}(t|T) = &\frac{d\Phi(t,T)}{dT}\,\hat{\mathbf{x}}_f(T) + \Phi(t,T)\,\frac{d\hat{\mathbf{x}}_f(T)}{dT} - \Phi(t,T)\,B(T)\,\mathbf{u}(T)\\ &+\Phi(t,T)\,G(T)\,Q(T)\,G^T(T)\,P_f^{-1}(T)\,\hat{\mathbf{x}}_f(T)\end{aligned} \tag{6.153}$$

The expression for the derivative of $\Phi(t,T)$ in eqn. (6.153) is given by differentiating $\Phi(t,T)\,\Phi(T,t) = I$ with respect to t, which yields

$$\begin{aligned}\frac{d\Phi(T,t)}{dt} &= -\Phi^{-1}(t,T)\,\frac{d\Phi(t,T)}{dt}\,\Phi(T,t)\\ &= -\Phi(T,t)\,\frac{d\Phi(t,T)}{dt}\,\Phi^{-1}(t,T)\end{aligned} \tag{6.154}$$

Substituting eqn. (6.152) into eqn. (6.154) yields (after some notational changes)

$$\boxed{\frac{d}{dT}\Phi(t,T) = -\Phi(t,T)[F(T)+G(T)\,Q(T)\,G^T(T)\,P_f^{-1}(T)], \quad \Phi(t,t) = I} \tag{6.155}$$

Hence, substituting the forward-time state filter equation from Table 6.3 and the expression in eqn. (6.155) into eqn. (6.153) leads to

$$\boxed{\frac{d}{dT}\hat{\mathbf{x}}(t|T) = \Phi(t,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)[\tilde{\mathbf{y}}(T) - H(T)\,\hat{\mathbf{x}}_f(T)]} \tag{6.156}$$

Applying the same concepts leading toward eqn. (6.156) to the covariance yields (which is left as an exercise for the reader)

$$\boxed{\frac{d}{dT}P(t|T) = -\Phi(t,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)\,P_f(T)\,\Phi^T(t,T)} \tag{6.157}$$

with $P(t|t) = P_f(t)$.

A summary of the continuous-time fixed-point smoother is given in Table 6.7. As with the discrete-time RTS smoother the fixed-point smoother begins by implementing the standard forward-time Kalman filter. The smoother state at a fixed point is simply given by using eqns. (6.155) and (6.156). The smoother covariance at the desired point is computed using eqn. (6.157), which is not required to determine the state estimate.

Example 6.4: We again consider the simple first-order system shown in example 6.2. Assuming that the forward-pass covariance has reached a steady-state value, given by p_f, the state transition matrix using eqn. (6.155) reduces down to

$$\frac{d\phi(t,T)}{dT} = -\beta\,\phi(t,T), \quad \phi(t,t) = 1$$

where $\beta \equiv (f + q/p_f)$. Note that t is fixed and $T \geq t$. The solution for $\phi(t,T)$ is given by

$$\phi(t,T) = e^{-\beta(T-t)}$$

Then, the smoother covariance using eqn. (6.157) reduces down to

$$\frac{dp(t|T)}{dT} = -\frac{p_f^2}{r} e^{-2\beta(T-t)}$$

The solution for $p(t|T)$ can be shown to be given by (which is left as an exercise for the reader)

$$p(t|T) = p_f\, e^{-2\beta(T-t)} + \frac{q}{2a}\left[1 - e^{-2\beta(T-t)}\right]$$

where $a \equiv \sqrt{f^2 + r^{-1}q}$. Consider when the point of interest is far enough in the past, so that $p(t|T)$ is at steady-state (e.g., after four times the time constant, i.e., when $T - t \geq 2/\beta$).[1] Then, the fixed-point smoother covariance at steady-state is given by $q/(2a)$, which is equivalent to the smoother steady-state covariance given in example 6.2. The smoother state estimate using eqn. (6.156) follows the following differential equation:

$$\frac{d\hat{x}(t|T)}{dT} = \frac{p_f}{r} e^{-\beta(T-t)}[\tilde{y}(T) - \hat{x}_f(T)], \quad \hat{x}(t|t) = \hat{x}_f(t)$$

This differential equation is integrated forward in time from time t until the present time T.

6.3 Fixed-Lag Smoothing

In this section the fixed-lag smoothing algorithm is shown for both discrete-time and continuous-time models. This smoother can be used for estimating the state where a lag is allowable between the current measurement and the estimate. Thus, the fixed-lag smoother is used to determine $\hat{\mathbf{x}}_{k|k+N}$ that is intuitively "better" than $\hat{\mathbf{x}}_{k|k}$, which is obtained through a Kalman filter. The fixed-lag estimate is defined by

$$\hat{\mathbf{x}}_{k|k+N} \equiv E\left\{\hat{\mathbf{x}}_k|[\tilde{\mathbf{y}}_1, \tilde{\mathbf{y}}_2 \ldots, \tilde{\mathbf{y}}_k, \tilde{\mathbf{y}}_{k+1}, \ldots, \tilde{\mathbf{y}}_{k+N}]\right\} \tag{6.158}$$

Thus, the point of time at which we seek the state estimate that lags the most recent measurement time by a fixed interval of time N, so that $t_{k+N} - t_k = \text{constant} > 0$.[15]

6.3.1 Discrete-Time Formulation

To derive the necessary relations for the discrete-time fixed-lag smoother, we start by rewriting the fixed-point smoother given by eqn. (6.53) as

$$\hat{\mathbf{x}}_{k|N} = \hat{\mathbf{x}}^+_{fk} + \mathcal{K}_k[\hat{\mathbf{x}}_{k+1|N} - \hat{\mathbf{x}}^-_{fk+1}] \tag{6.159}$$

where the notation in eqn. (6.158) has been used. Assuming that $\mathcal{K}_k^{-1}$ exists, then eqn. (6.159) can be solved for $\hat{\mathbf{x}}_{k+1|N}$, giving

$$\hat{\mathbf{x}}_{k+1|N} = \hat{\mathbf{x}}^-_{fk+1} + \mathcal{K}_k^{-1}[\hat{\mathbf{x}}_{k|N} - \hat{\mathbf{x}}^+_{fk}] \tag{6.160}$$

Substituting the relation for $\hat{\mathbf{x}}^-_{fk+1}$ in Table 6.2 into eqn. (6.160) gives

$$\hat{\mathbf{x}}_{k+1|N} = \Phi_k \hat{\mathbf{x}}^+_{fk} + \Gamma_k \mathbf{u}_k + \mathcal{K}_k^{-1}[\hat{\mathbf{x}}_{k|N} - \hat{\mathbf{x}}^+_{fk}] \tag{6.161}$$

Adding and subtracting $\Phi_k \hat{\mathbf{x}}_{k|N}$ from the right-hand side of eqn. (6.161) yields

$$\hat{\mathbf{x}}_{k+1|N} = \Phi_k \hat{\mathbf{x}}_{k|N} + \Gamma_k \mathbf{u}_k + [\mathcal{K}_k^{-1} - \Phi_k][\hat{\mathbf{x}}_{k|N} - \hat{\mathbf{x}}^+_{fk}] \tag{6.162}$$

Let us concentrate our attention on $\mathcal{K}_k^{-1} - \Phi_k$. From eqn. (6.52) we have

$$U_k \equiv \mathcal{K}_k^{-1} - \Phi_k = P^-_{fk+1} \Phi_k^{-T} (P^+_{fk})^{-1} - \Phi_k \tag{6.163}$$

Substituting the relation for P^-_{fk+1} in Table 6.2 into eqn. (6.163) gives

$$U_k = \Upsilon_k Q_k \Upsilon_k^T \Phi_k^{-T} (P^+_{fk})^{-1} \tag{6.164}$$

Therefore, substituting eqn. (6.164) into eqn. (6.162) gives

$$\hat{\mathbf{x}}_{k+1|N} = \Phi_k \hat{\mathbf{x}}_{k|N} + \Gamma_k \mathbf{u}_k + U_k[\hat{\mathbf{x}}_{k|N} - \hat{\mathbf{x}}^+_{fk}] \tag{6.165}$$

We now allow the right endpoint of the interval to be variable by replacing N by $k+N$, which gives

$$\hat{\mathbf{x}}_{k+1|k+N} = \Phi_k \hat{\mathbf{x}}_{k|k+N} + \Gamma_k \mathbf{u}_k + U_k[\hat{\mathbf{x}}_{k|k+N} - \hat{\mathbf{x}}^+_{fk}] \tag{6.166}$$

Equation (6.166) will be used to compute the fixed-lag state estimate.

From the results of §6.2.1, replacing k by $k+1$ and N by $k+1+N$ in eqn. (6.139) and using the measurement residual form from eqn. (6.118) yields

$$\hat{\mathbf{x}}_{k+1|k+1+N} = \hat{\mathbf{x}}_{k+1|k+N} + \mathcal{M}_{k+1|k+1+N} \boldsymbol{\upsilon}_{fk+1+N} \tag{6.167}$$

where

$$\mathcal{M}_{k+1|k+1+N} = \mathcal{B}_{k+1+N} K_{fk+1+N} \tag{6.168}$$

with

$$\mathcal{B}_{k+1+N} = \prod_{i=k+1}^{k+N} \mathcal{K}_i \tag{6.169}$$

Substituting eqn. (6.166) into eqn. (6.168) gives

$$\begin{aligned}\hat{\mathbf{x}}_{k+1|k+1+N} = \Phi_k \hat{\mathbf{x}}_{k|k+N} + \Gamma_k \mathbf{u}_k + U_k[\hat{\mathbf{x}}_{k|k+N} - \hat{\mathbf{x}}_{fk}^{+}] \\ + \mathcal{M}_{k+1|k+1+N} \boldsymbol{\upsilon}_{fk+1+N}\end{aligned} \tag{6.170}$$

Using the definition of the residual $\boldsymbol{\upsilon}_{fk+1+N}$ from eqn. (6.117) and the forward-time state propagation in Table 6.2 in eqn. (6.170) leads to

$$\boxed{\begin{aligned}\hat{\mathbf{x}}_{k+1|k+1+N} = \Phi_k \hat{\mathbf{x}}_{k|k+N} + \Gamma_k \mathbf{u}_k \\ + \Upsilon_k Q_k \Upsilon_k^T \Phi_k^{-T} (P_{fk}^{+})^{-1} [\hat{\mathbf{x}}_{k|k+N} - \hat{\mathbf{x}}_{fk}^{+}] \\ + \mathcal{B}_{k+1+N} K_{fk+1+N} \{\tilde{\mathbf{y}}_{k+1+N} \\ - H_{k+1+N} \Phi_{k+N} [\hat{\mathbf{x}}_{fk+N}^{+} + \Gamma_{k+N} \mathbf{u}_{k+N}]\}\end{aligned}} \tag{6.171}$$

where the initial condition for eqn. (6.171) is given by $\hat{\mathbf{x}}_{0|N}$. This initial condition is obtained from the optimal fixed-point smoother starting with $\hat{\mathbf{x}}_{0|0}$, processing measurements to obtain $\hat{\mathbf{x}}_{0|N}$. Then the Kalman filter is employed, where its gain, covariance, and state estimate are used in the fixed-lag smoother.

The fixed-lag smoother covariance is derived by first rewriting eqn. (6.51) as

$$P_{k+1|N} = P_{fk+1}^{-} - \mathcal{K}_k^{-1} [P_{fk}^{+} - P_{k|N}] \mathcal{K}_k^{-T} \tag{6.172}$$

Replacing N by $k+N$ in eqn. (6.172) gives

$$P_{k+1|k+N} = P_{fk+1}^{-} - \mathcal{K}_k^{-1} [P_{fk}^{+} - P_{k|k+N}] \mathcal{K}_k^{-T} \tag{6.173}$$

Next, using eqn. (5.44) we can write

$$P_{fN}^{+} - P_{fN}^{-} = -K_{fN} H_N P_{fN}^{-} \tag{6.174}$$

Substituting eqn. (6.174) into eqn. (6.149), and replacing k by $k+1$ and N by $k+1+N$ leads to

$$P_{k+1|k+1+N} = P_{k+1|k+N} - \mathcal{B}_{k+1+N} K_{fk+1+N} H_{k+1+N} P_{fk+1+N}^{-} \mathcal{B}_{k+1+N}^{T} \tag{6.175}$$

Substituting eqn. (6.173) into eqn. (6.175) yields

$$\boxed{\begin{aligned}P_{k+1|k+1+N} = P_{fk+1}^{-} - \mathcal{K}_k^{-1} [P_{fk}^{+} - P_{k|k+N}] \mathcal{K}_k^{-T} \\ - \mathcal{B}_{k+1+N} K_{fk+1+N} H_{k+1+N} P_{fk+1+N}^{-} \mathcal{B}_{k+1+N}^{T}\end{aligned}} \tag{6.176}$$

where the initial condition for eqn. (6.176) is given by $P_{0|N}$, which is given by the optimal fixed-point covariance.

Table 6.8: Discrete-Time Fixed-Lag Smoother

Model	$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k, \quad \mathbf{w}_k \sim N(\mathbf{0}, Q_k)$ $\tilde{\mathbf{y}}_k = H_k \mathbf{x}_k + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{0}, R_k)$
Forward Initialize	$\hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\,\tilde{\mathbf{x}}_f^T(t_0)\}$
Gain	$K_{fk} = P_{fk}^- H_k^T [H_k P_{fk}^- H_k^T + R_k]^{-1}$
Forward Update	$\hat{\mathbf{x}}_{fk}^+ = \hat{\mathbf{x}}_{fk}^- + K_{fk}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}^-]$ $P_{fk}^+ = [I - K_{fk} H_k] P_{fk}^-$
Forward Propagation	$\hat{\mathbf{x}}_{fk+1}^- = \Phi_k \hat{\mathbf{x}}_{fk}^+ + \Gamma_k \mathbf{u}_k$ $P_{fk+1}^- = \Phi_k P_{fk}^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$
Smoother Initialize	$\hat{\mathbf{x}}_{0\|N}$ from fixed-point smoother $P_{0\|N}$ from fixed-point smoother
Gain	$\mathcal{B}_{k+1+N} = \prod_{i=k+1}^{k+N} \mathcal{K}_i, \quad \mathcal{K}_i = P_{fi}^+ \Phi_i^T (P_{fi+1}^-)^{-1}$
Covariance	$P_{k+1\|k+1+N} = P_{fk+1}^- - \mathcal{K}_k^{-1}[P_{fk}^+ - P_{k\|k+N}]\mathcal{K}_k^{-T}$ $-\mathcal{B}_{k+1+N} K_{fk+1+N} H_{k+1+N} P_{fk+1+N}^- \mathcal{B}_{k+1+N}^T$
Estimate	$\hat{\mathbf{x}}_{k+1\|k+1+N} = \Phi_k \hat{\mathbf{x}}_{k\|k+N} + \Gamma_k \mathbf{u}_k$ $+\Upsilon_k Q_k \Upsilon_k^T \Phi_k^{-T} (P_{fk}^+)^{-1} [\hat{\mathbf{x}}_{k\|k+N} - \hat{\mathbf{x}}_{fk}^+]$ $+\mathcal{B}_{k+1+N} K_{fk+1+N} \{\tilde{\mathbf{y}}_{k+1+N}$ $-H_{k+1+N} \Phi_{k+N} [\hat{\mathbf{x}}_{fk+N}^+ + \Gamma_{k+N} \mathbf{u}_{k+N}]\}$

A summary of the discrete-time fixed-lag smoother is given in Table 6.8. Equation (6.171) incorporates two correction terms. One is applied to the residual between the optimal fixed-lag smoother estimate, $\hat{\mathbf{x}}_{k|k+N}$, and the optimal Kalman filter estimate, $\hat{\mathbf{x}}_{fk}^+$, at time t_k. The other is applied to the measurement residual directly. The first correction reflects the residual back to the fixed-lag estimate. When no process noise is present (i.e., when $Q_k = 0$), this term has no effect on the fixed-lag smoother estimate, which intuitively makes sense. The second correction comes after a "waiting period"[15] where the fixed-lag smoother is dormant over the interval $[0, N]$. Then the fixed-lag smoother depends on the Kalman filter, which leads to the measurement residual in the fixed-lag smoother estimate. Finally, we should note that the fixed-lag smoother can be actually implemented in real time once it has been initialized. However, we still consider this "filter" to be a batch smoother since the sought estimate is not provided in real time, which is derived from future data points.

6.3.2 Continuous-Time Formulation

The continuous-time fixed-lag smoother can be derived using the same methods to derive the continuous-time fixed-point smoother in §6.2.2.[1] Suppose that we seek a smoother solution that lags the most recent measurement by a constant time delay Δ. Replacing t with $T-\Delta$ in eqn. (6.150) gives

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{x}}(T-\Delta|T) = &[F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]\\&\times\hat{\mathbf{x}}(T-\Delta|T)+B(T-\Delta)\,\mathbf{u}(T-\Delta)\\&-G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\,P_f^{-1}(T-\Delta)\,\hat{\mathbf{x}}_f(T-\Delta)\end{aligned}\tag{6.177}$$

The solution of eqn. (6.177) is given by

$$\begin{aligned}\hat{\mathbf{x}}(T-\Delta|T) = &\Psi(T-\Delta,T)\,\hat{\mathbf{x}}_f(T)+\int_T^{T-\Delta}\Psi(T-\Delta,\tau)\,B(\tau)\,\mathbf{u}(\tau)\,d\tau\\&-\int_T^{T-\Delta}\Psi(T-\Delta,\tau)\,G(\tau)\,Q(\tau)\,G^T(\tau)\,P_f^{-1}(\tau)\,\hat{\mathbf{x}}_f(\tau)\,d\tau\end{aligned}\tag{6.178}$$

where $\Psi(T-\Delta,T)$ is the state transition matrix, which clearly must obey

$$\begin{aligned}\frac{d}{dt}\Psi(T-\Delta,T) = &[F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]\\&\times\Psi(T-\Delta,T),\quad \Psi(t,t)=I\end{aligned}\tag{6.179}$$

Note, the matrix $\Phi(t,T)$ from §6.2.2 and the matrix $\Psi(T-\Delta,T)$ are related by

$$\Psi(T-\Delta,T)=\Phi(T-\Delta,t)\,\Phi(t,T)\tag{6.180}$$

with $\Phi(0,T)=\Psi(0,T)$. Differentiating eqn. (6.180) with respect to T and using eqn. (6.152) yields

$$\boxed{\begin{aligned}\frac{d}{dT}\Psi(T-\Delta,T) = &[F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta)\\&\times G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]\Psi(T-\Delta,T)\\&-\Psi(T-\Delta,T)[F(T)+G(T)\,Q(T)\,G^T(T)\,P_f^{-1}(T)]\end{aligned}}\tag{6.181}$$

Table 6.9: Continuous-Time Fixed-Lag Smoother

Model	$\frac{d}{dt}\mathbf{x}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t),\ \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ $\tilde{\mathbf{y}}(t) = H(t)\,\mathbf{x}(t) + \mathbf{v}(t),\ \mathbf{v}(t) \sim N(\mathbf{0}, R(t))$
Forward Covariance	$\frac{d}{dt}P_f(t) = F(t)\,P_f(t) + P_f(t)\,F^T(t)$ $-P_f(t)\,H^T(t)\,R^{-1}(t)H(t)\,P_f(t)$ $+G(t)\,Q(t)\,G^T(t),$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\,\tilde{\mathbf{x}}_f^T(t_0)\}$
Forward Filter	$\frac{d}{dt}\hat{\mathbf{x}}_f(t) = F(t)\,\hat{\mathbf{x}}_f(t) + B(t)\,\mathbf{u}(t)$ $+P_f(t)\,H^T(t)\,R^{-1}(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_f(t)], \quad \hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$
Smoother Initialize	$\hat{\mathbf{x}}(0\|\Delta)$ from fixed-point smoother $P(0\|\Delta)$ from fixed-point smoother
Transition Matrix	$\frac{d}{dT}\Psi(T-\Delta,T) = [F(T-\Delta) + G(T-\Delta)\,Q(T-\Delta)$ $\times G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]\Psi(T-\Delta,T)$ $-\Psi(T-\Delta,T)[F(T) + G(T)\,Q(T)\,G^T(T)\,P_f^{-1}(T)],$ $\Psi(0,\Delta) = \Phi(0,\Delta)$
Smoother Covariance	$\frac{d}{dT}P(T-\Delta\|T) = [F(T-\Delta) + G(T-\Delta)\,Q(T-\Delta)$ $\times G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]P(T-\Delta\|T) + P(T-\Delta\|T)$ $\times[F(T-\Delta) + G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]^T$ $-\Psi(T-\Delta,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)\,P_f(T)\,\Psi(T-\Delta,T)$ $-G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)$
Smoother Estimate	$\frac{d}{dT}\hat{\mathbf{x}}(T-\Delta\|T) = [F(T-\Delta) + G(T-\Delta)\,Q(T-\Delta)$ $\times G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]\hat{\mathbf{x}}(T-\Delta\|T)$ $-G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\,P_f^{-1}(T-\Delta)\,\hat{\mathbf{x}}_f(T-\Delta)$ $+\Psi(T-\Delta,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)[\tilde{\mathbf{y}}(T) - H(T)\,\hat{\mathbf{x}}_f(T)]$

Taking the derivative of eqn. (6.178) with respect to T, and substituting the forward-time state filter equation from Table 6.3 and eqn. (6.181) into the resulting equation leads to (the details are left as an exercise for the reader)

$$\boxed{\begin{aligned}\frac{d}{dT}\hat{\mathbf{x}}(T-\Delta|T) = {} & [F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta) \\ & \times G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]\hat{\mathbf{x}}(T-\Delta|T) \\ & - G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\,P_f^{-1}(T-\Delta)\,\hat{\mathbf{x}}_f(T-\Delta) \\ & + \Psi(T-\Delta,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)[\tilde{\mathbf{y}}(T)-H(T)\,\hat{\mathbf{x}}_f(T)]\end{aligned}} \tag{6.182}$$

where the initial condition for eqn. (6.182) is given by $\hat{\mathbf{x}}(0|\Delta)$. This initial condition is obtained from the optimal fixed-point smoother starting with $\hat{\mathbf{x}}(0|0)$, processing measurements to obtain $\hat{\mathbf{x}}(0|\Delta)$. The covariance can be shown to be given by (the details are left as an exercise for the reader)

$$\boxed{\begin{aligned}\frac{d}{dT}P(T-\Delta|T) = {} & [F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta) \\ & \times G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]P(T-\Delta|T)+P(T-\Delta|T) \\ & \times [F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]^T \\ & - \Psi(T-\Delta,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)\,P_f(T)\,\Psi(T-\Delta,T) \\ & - G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\end{aligned}} \tag{6.183}$$

with initial condition given by $P(0|\Delta)$ from the optimal fixed-point smoother covariance. A summary of the continuous-time fixed-lag smoother is given in Table 6.9. The initial conditions and smoother implementation follow exactly like the discrete-time fixed-lag smoother, but the continuous-time equations are integrated in order to provide the state estimate.

Example 6.5: We again consider the simple first-order system shown in example 6.2. Assuming that the forward-pass covariance has reached a steady-state value, given by p_f, since $p(T-\Delta) = p(T) = p_f$ the state transition matrix using eqn. (6.181) reduces down to

$$\frac{d\psi(T-\Delta,T)}{dT} = 0$$

Note that Δ is fixed and $T \geq \Delta$. The solution for $\psi(t,T)$ is given by using the initial condition from the fixed-point smoother state transition matrix in example 6.4, which gives

$$\psi(T-\Delta,T) = e^{-\beta\Delta}$$

where $\beta \equiv (f + q/p_f)$. Then, the smoother covariance using eqn. (6.183) reduces down to

$$\frac{dp(T-\Delta|T)}{dT} = 2\beta\, p(T-\Delta|T) - \left[r^{-1}p_f^2 e^{-2\beta\Delta} + q\right]$$

Using the initial condition $p(0|\Delta)$, the solution for $p(T-\Delta|T)$ is given by

$$p(T-\Delta|T) = p(0|\Delta)\, e^{2\beta(T-\Delta)} + \frac{r^{-1} p_f^2 e^{-2\beta\,\Delta} + q}{2\beta}\left[1 - e^{2\beta(T-\Delta)}\right]$$

where $p(0|\Delta)$ is evaluated from example 6.4, which leads to

$$p(0|\Delta) = p_f\, e^{-2\beta\,\Delta} + \frac{q}{2a}\left[1 - e^{-2\beta\,\Delta}\right]$$

where $a \equiv \sqrt{f^2 + r^{-1} q}$. Then, the solution for $p(T-\Delta|T)$ can be shown to be given by $p(T-\Delta|T) = p(0|\Delta)$ (which is left as an exercise for the reader). Note, this only occurs since $p_f(t)$ is at steady-state. Consider when Δ is sufficiently large so that the exponential terms decay to near-zero (e.g., after four times the time constant, i.e., when $\Delta \geq 2/\beta$). Then, the fixed-lag smoother covariance at steady-state is given by $q/(2a)$, which is equivalent to the smoother steady-state covariance given in example 6.2. This intuitively makes sense since the accuracy of the fixed-lag smoother should be equivalent to the fixed-interval smoother at steady-state.

6.4 Advanced Topics

In this section we will show some advanced topics used in smoothers. As in previous chapters we encourage the interested reader to pursue these topics further in the references provided. These topics include the duality between estimation and control, and new derivations of the fixed-interval smoothers based on the innovations process.

6.4.1 Estimation/Control Duality

One of the most fascinating aspects of the fixed-interval RTS smoother is that it can be completely derived from optimal control theory. This mathematical *duality* between estimation and control arises from solving the two-point-boundary-value-problem (TPBVP) associated with optimal control theory.[9, 16] In this section we assume that the reader is familiar with the variational approach, which transforms the minimization problem into a TPBVP. More details on the variational approach can be found in Chapter 8. We will derive the discrete-time and continuous-time cases here, as well as a new derivation of the nonlinear RTS smoother with continuous-time models and discrete-time measurements.

6.4.1.1 Discrete-Time Formulation

Consider minimizing the following discrete-time loss function:

$$\begin{aligned} J(\mathbf{w}_k) = &\frac{1}{2}\sum_{k=1}^{N}[\tilde{\mathbf{y}}_k - H_k\mathbf{x}_k]^T R_k^{-1}[\tilde{\mathbf{y}}_k - H_k\mathbf{x}_k] + \mathbf{w}_k^T Q_k^{-1}\mathbf{w}_k \\ &+ \frac{1}{2}[\hat{\mathbf{x}}_{f0} - \mathbf{x}_0]^T P_{f0}^{-1}[\hat{\mathbf{x}}_{f0} - \mathbf{x}_0] \end{aligned} \tag{6.184}$$

subject to the dynamic constraint

$$\mathbf{x}_{k+1} = \Phi_k\mathbf{x}_k + \Gamma_k\mathbf{u}_k + \Upsilon_k\mathbf{w}_k \tag{6.185}$$

Note that $\hat{\mathbf{x}}_{f0}$ is the *a priori* estimate of $\mathbf{x}_0$, with error-covariance P_{f0}, and $J(\mathbf{w}_k)$ is the negative log-likelihood function.[9] Also, we treat $\mathbf{u}_k$ as a deterministic input. Finally, the inverse of Q_k must exist in order to achieve controllability in this minimization problem, which is also discussed in §6.1.1. Let us denote the best estimate of $\mathbf{x}$ as $\hat{\mathbf{x}}$. Then, the minimization of eqn. (6.184) yields the following TPBVP (see §8.4):[16, 17]

$$\hat{\mathbf{x}}_{k+1} = \Phi_k\hat{\mathbf{x}}_k + \Gamma_k\mathbf{u}_k + \Upsilon_k\mathbf{w}_k \tag{6.186a}$$

$$\boldsymbol{\lambda}_k = \Phi_k^T\boldsymbol{\lambda}_{k+1} + H_k^T R_k^{-1} H_k\hat{\mathbf{x}}_k - H_k^T R_k^{-1}\tilde{\mathbf{y}}_k \tag{6.186b}$$

$$\mathbf{w}_k = -Q_k\Upsilon_k^T\boldsymbol{\lambda}_{k+1} \tag{6.186c}$$

where $\boldsymbol{\lambda}_k$ is known as the *costate vector*, which arises from using a Lagrange multiplier for the equality constraint in eqn. (6.185). The boundary conditions are given by

$$\boldsymbol{\lambda}_N = \mathbf{0} \tag{6.187a}$$

$$\boldsymbol{\lambda}_0 = P_{f0}^{-1}[\hat{\mathbf{x}}_{f0} - \hat{\mathbf{x}}_0] \tag{6.187b}$$

Substituting eqn. (6.186c) into eqn. (6.186a) gives the following TPBVP:

$$\boxed{\begin{aligned} \hat{\mathbf{x}}_{k+1} &= \Phi_k\hat{\mathbf{x}}_k + \Gamma_k\mathbf{u}_k - \Upsilon_k Q_k\Upsilon_k^T\boldsymbol{\lambda}_{k+1} && (6.188a) \\ \boldsymbol{\lambda}_k &= \Phi_k^T\boldsymbol{\lambda}_{k+1} + H_k^T R_k^{-1} H_k\hat{\mathbf{x}}_k - H_k^T R_k^{-1}\tilde{\mathbf{y}}_k && (6.188b) \end{aligned}}$$

Equation (6.188) will be used to derive the discrete-time RTS smoother solution.

In order to decouple the state and costate vectors in eqn. (6.188) we use the following inhomogeneous Riccati transformation:

$$\boxed{\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk} - P_{fk}\boldsymbol{\lambda}_k} \tag{6.189}$$

where P_{fk} is an $n\times n$ matrix and $\hat{\mathbf{x}}_{fk}$ is the inhomogeneous vector. We will show in the subsequent derivation that $\hat{\mathbf{x}}_{fk}$ is indeed the forward-time Kalman filter state estimate and $\hat{\mathbf{x}}_k$ is the smoother state estimate. Comparing eqn. (6.189) with eqn. (6.187) indicates that in order for $\boldsymbol{\lambda}_N = \mathbf{0}$ to be satisfied, then $\hat{\mathbf{x}}_N = \hat{\mathbf{x}}_{fN}$. Substituting eqn. (6.189) into eqn. (6.188b), collecting terms and factoring out P_{fk}^{-1} yields

$$\boldsymbol{\lambda}_k = P_{fk}^{-1} Z_k [\Phi_k^T \boldsymbol{\lambda}_{k+1} + H_k^T R_k^{-1} H_k \hat{\mathbf{x}}_{fk} - H_k^T R_k^{-1} \tilde{\mathbf{y}}_k] \tag{6.190}$$

where

$$Z_k \equiv [P_{fk}^{-1} + H_k^T R_k^{-1} H_k]^{-1} \tag{6.191}$$

Taking one time-step ahead of eqn. (6.189) gives

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{fk+1} - P_{fk+1} \boldsymbol{\lambda}_{k+1} \tag{6.192}$$

Substituting eqns. (6.189) and (6.192) into eqn. (6.188a) and rearranging gives

$$[P_{fk+1} - \Upsilon_k Q_k \Upsilon_k^T] \boldsymbol{\lambda}_{k+1} - \Phi_k P_{fk} \boldsymbol{\lambda}_k - \hat{\mathbf{x}}_{fk+1} + \Phi_k \hat{\mathbf{x}}_{fk} + \Gamma_k \mathbf{u}_k = \mathbf{0} \tag{6.193}$$

Substituting eqn. (6.190) into eqn. (6.193) and collecting terms yields

$$\begin{aligned} &[P_{fk+1} - \Phi_k Z_k \Phi_k^T - \Upsilon_k Q_k \Upsilon_k^T] \boldsymbol{\lambda}_{k+1} \\ &+ \Phi_k \hat{\mathbf{x}}_{fk} + \Gamma_k \mathbf{u}_k + \Phi_k Z_k H_k^T R_k^{-1} [\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}] - \hat{\mathbf{x}}_{fk+1} = \mathbf{0} \end{aligned} \tag{6.194}$$

Avoiding the trivial solution of $\boldsymbol{\lambda}_{k+1} = \mathbf{0}$ gives the following two equations:

$$P_{fk+1} = \Phi_k Z_k \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T \tag{6.195a}$$

$$\hat{\mathbf{x}}_{fk+1} = \Phi_k \hat{\mathbf{x}}_{fk} + \Gamma_k \mathbf{u}_k + \Phi_k Z_k H_k^T R_k^{-1} [\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}] \tag{6.195b}$$

We now prove the following identity:

$$Z_k H_k^T R_k^{-1} = K_{fk} \equiv P_{fk} H_k^T [H_k P_{fk} H_k^T + R_k]^{-1} \tag{6.196}$$

Using the matrix inversion lemma in eqn. (1.70) with $A = \mathcal{P}_{fk}^{-1}$, $B = H_k^T$, $C = R_k^{-1}$, and $D = H_k$ leads to the following form for eqn. (6.196):

$$\left\{ P_{fk} - P_{fk} H_k^T [H_k P_{fk} H_k^T + R_k]^{-1} H_k P_{fk} \right\} H_k^T R_k^{-1} = K_{fk} \tag{6.197}$$

Next, using the definition of the forward-time gain K_{fk} and right-multiplying both sides of eqn. (6.197) by R_k leads to

$$P_{fk} H_k^T - K_{fk} H_k P_{fk} H_k^T = K_{fk} R_k \tag{6.198}$$

Collecting terms reduces eqn. (6.198) to

$$P_{fk} H_k^T = K_{fk} [H_k P_{fk} H_k^T + R_k] \tag{6.199}$$

Finally, using the definition of the gain K_{fk} proves the identity. Therefore, we can write eqn. (6.195) as

$$P_{fk+1} = \Phi_k P_{fk} \Phi_k^T - \Phi_k P_{fk} H_k^T [H_k P_{fk} H_k^T + R_k]^{-1} H_k P_{fk} \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T \tag{6.200a}$$

$$\hat{\mathbf{x}}_{fk+1} = \Phi_k \hat{\mathbf{x}}_{fk} + \Gamma_k \mathbf{u}_k + \Phi_k K_{fk} [\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}] \tag{6.200b}$$

Equation (6.200) constitutes the forward-time Kalman filter covariance and state estimate with $P_{fk} \equiv P_{fk}^-$ and $\hat{\mathbf{x}}_{fk} \equiv \hat{\mathbf{x}}_{fk}^-$.

We now need an expression for the state estimate $\hat{\mathbf{x}}_k$. Solving eqns. (6.189) and (6.190) for $\boldsymbol{\lambda}_k$ and $\boldsymbol{\lambda}_{k+1}$, respectively, and substituting the resulting expressions into eqn. (6.186b) gives

$$P_{fk}^{-1}[\hat{\mathbf{x}}_{fk} - \hat{\mathbf{x}}_k] = \Phi_k^T P_{fk+1}^{-1}[\hat{\mathbf{x}}_{fk+1} - \hat{\mathbf{x}}_{k+1}] + H_k^T R_k^{-1} H_k \hat{\mathbf{x}}_k - H_k^T R_k^{-1} \tilde{\mathbf{y}}_k \tag{6.201}$$

Solving eqn. (6.201) for $\hat{\mathbf{x}}_k$ yields

$$\hat{\mathbf{x}}_k = L_k \hat{\mathbf{x}}_{fk} + L_k P_{fk} H_k^T R_k^{-1} \tilde{\mathbf{y}}_k + L_k P_{fk} \Phi_k^T P_{fk+1}^{-1}[\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}] \tag{6.202}$$

where

$$L_k \equiv [I + P_{fk} H_k^T R_k^{-1} H_k]^{-1} = [P_{fk}^{-1} + H_k^T R_k^{-1} H_k]^{-1} P_{fk}^{-1} \tag{6.203}$$

Now, consider the following identities (which are left as an exercise for the reader):

$$L_k P_{fk} = P_{fk}^+ \tag{6.204a}$$

$$L_k = I - K_{fk} H_k \tag{6.204b}$$

Substituting eqn. (6.204) into eqn. (6.202) yields

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk} - K_{fk} H_k \hat{\mathbf{x}}_{fk} + P_{fk}^+ H_k^T R_k^{-1} \tilde{\mathbf{y}}_k + P_{fk}^+ \Phi_k^T P_{fk+1}^{-1}[\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}] \tag{6.205}$$

Finally, using the definitions of the gain K_{fk} from eqn. (5.47), and $P_{fk+1} \equiv P_{fk+1}^-$ and $\hat{\mathbf{x}}_{fk} \equiv \hat{\mathbf{x}}_{fk}^-$ leads to

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk}^+ + \mathcal{K}_k[\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}^-] \tag{6.206}$$

where

$$\mathcal{K}_k \equiv P_{fk}^+ \Phi_k^T (P_{fk+1}^-)^{-1} \tag{6.207}$$

Equation (6.206) is exactly the discrete-time RTS smoother.

6.4.1.2 Continuous-Time Formulation

The continuous-time formulation is much easier to derive than the discrete-time system. Consider minimizing the following continuous-time loss function:

$$\begin{aligned} J[\mathbf{w}(t)] = \frac{1}{2}\int_{t_0}^{t_N} & \Big\{[\tilde{\mathbf{y}}(t) - H(t)\mathbf{x}(t)]^T R^{-1}(t)[\tilde{\mathbf{y}}(t) - H(t)\mathbf{x}(t)] \\ & + \mathbf{w}^T(t) Q^{-1}(t)\mathbf{w}(t)\Big\}\, dt \\ & + \frac{1}{2}[\hat{\mathbf{x}}_f(t_0) - \mathbf{x}(t_0)]^T P_f^{-1}(t_0)[\hat{\mathbf{x}}_f(t_0) - \mathbf{x}(t_0)] \end{aligned} \tag{6.208}$$

subject to the dynamic constraint

$$\frac{d}{dt}\mathbf{x}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t) \tag{6.209}$$

Note that for the continuous-time case the loss function in eqn. (6.208) becomes infinite if the measurement and process noises are represented by white noise. However, since white noise can be formulated as a limiting case of nonwhite noise, the derivation of the final results can be achieved by avoiding stochastic calculus.[9] Let us again denote the best estimate of $\mathbf{x}$ as $\hat{\mathbf{x}}$. Then, the minimization of eqn. (6.208) yields the following TPBVP (see §8.2):[16, 17]

$$\frac{d}{dt}\hat{\mathbf{x}}(t) = F(t)\,\hat{\mathbf{x}}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t) \tag{6.210a}$$

$$\frac{d}{dt}\boldsymbol{\lambda}(t) = -F^T(t)\,\boldsymbol{\lambda}(t) - H^T(t)\,R^{-1}(t)\,H(t)\,\hat{\mathbf{x}}(t) + H^T(t)\,R^{-1}(t)\,\tilde{\mathbf{y}}(t) \tag{6.210b}$$

$$\mathbf{w}(t) = -Q(t)\,G^T(t)\,\boldsymbol{\lambda}(t) \tag{6.210c}$$

The boundary conditions are given by

$$\boldsymbol{\lambda}(T) = \mathbf{0} \tag{6.211a}$$

$$\boldsymbol{\lambda}(t_0) = P_f^{-1}(t_0)\,[\hat{\mathbf{x}}_f(t_0) - \hat{\mathbf{x}}(t_0)] \tag{6.211b}$$

Substituting eqn. (6.210c) into eqn. (6.210a) gives the following TPBVP:

$$\boxed{\begin{aligned} \frac{d}{dt}\hat{\mathbf{x}}(t) &= F(t)\,\hat{\mathbf{x}}(t) + B(t)\,\mathbf{u}(t) - G(t)\,Q(t)\,G^T(t)\boldsymbol{\lambda}(t) && (6.212\text{a}) \\ \frac{d}{dt}\boldsymbol{\lambda}(t) &= -F^T(t)\,\boldsymbol{\lambda}(t) - H^T(t)\,R^{-1}(t)\,H(t)\,\hat{\mathbf{x}}(t) \\ &\quad + H^T(t)\,R^{-1}(t)\,\tilde{\mathbf{y}}(t) && (6.212\text{b}) \end{aligned}}$$

Equation (6.212) will be used to derive the continuous-time RTS smoother solution.

As with the discrete-time case we consider the following inhomogeneous Riccati transformation:

$$\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_f(t) - P_f(t)\,\boldsymbol{\lambda}(t) \tag{6.213}$$

Comparing eqn. (6.213) with eqn. (6.211) indicates that in order for $\boldsymbol{\lambda}(T) = \mathbf{0}$ to be satisfied, then $\hat{\mathbf{x}}(T) = \hat{\mathbf{x}}_f(T)$. Taking the time-derivative of eqn. (6.213) gives

$$\frac{d}{dt}\hat{\mathbf{x}}(t) = \frac{d}{dt}\hat{\mathbf{x}}_f(t) - \left[\frac{d}{dt}P_f(t)\right]\boldsymbol{\lambda}(t) - P_f(t)\left[\frac{d}{dt}\boldsymbol{\lambda}(t)\right] \tag{6.214}$$

Substituting eqn. (6.212) into eqn. (6.214) gives

$$\begin{aligned} &F(t)\,\hat{\mathbf{x}}(t) + B(t)\,\mathbf{u}(t) - G(t)\,Q(t)\,G^T(t)\boldsymbol{\lambda}(t) \\ &- \frac{d}{dt}\hat{\mathbf{x}}_f(t) + \left[\frac{d}{dt}P_f(t)\right]\boldsymbol{\lambda}(t) - P_f(t)\,F^T(t)\,\boldsymbol{\lambda}(t) \\ &- P_f(t)\,H^T(t)\,R^{-1}(t)\,H(t)\,\hat{\mathbf{x}}(t) + P_f(t)\,H^T(t)\,R^{-1}(t)\,\tilde{\mathbf{y}}(t) = \mathbf{0} \end{aligned} \tag{6.215}$$

Substituting eqn. (6.213) into eqn. (6.215), and collecting terms gives

$$\begin{aligned}&\left[\frac{d}{dt}P_f(t) - F(t)\,P_f(t) - P_f(t)\,F^T(t) + P_f(t)\,H^T(t)\,R^{-1}(t)\,H(t)\,P_f(t)\right.\\&\left.- G(t)\,Q(t)\,G^T(t)\right]\boldsymbol{\lambda}(t) + F(t)\,\hat{\mathbf{x}}_f(t) + B(t)\,\mathbf{u}(t)\\&+ P_f(t)\,H^T(t)\,R^{-1}(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_f(t)] - \frac{d}{dt}\hat{\mathbf{x}}_f(t) = \mathbf{0}\end{aligned} \tag{6.216}$$

Avoiding the trivial solution of $\boldsymbol{\lambda}(t) = \mathbf{0}$ gives the following two equations:

$$\begin{aligned}\frac{d}{dt}P_f(t) = F(t)\,P_f(t) + P_f(t)\,F^T(t) - P_f(t)\,H^T(t)\,R^{-1}(t)\,H(t)\,P_f(t)\\+ G(t)\,Q(t)\,G^T(t)\end{aligned} \tag{6.217a}$$

$$\frac{d}{dt}\hat{\mathbf{x}}_f(t) = F(t)\,\hat{\mathbf{x}}_f(t) + B(t)\,\mathbf{u}(t) + K_f(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_f(t)] \tag{6.217b}$$

where

$$K_f(t) \equiv P_f(t)\,H^T(t)\,R^{-1}(t) \tag{6.218}$$

Equation (6.217) constitutes the forward-time Kalman filter covariance and state estimate. The smoother equation is easily given by solving eqn. (6.213) for $\boldsymbol{\lambda}(t)$ and substituting the resulting expression into eqn. (6.212a), which yields

$$\frac{d}{dt}\hat{\mathbf{x}}(t) = F(t)\,\hat{\mathbf{x}}(t) + B(t)\,\mathbf{u}(t) + G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)\left[\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)\right] \tag{6.219}$$

Equation (6.219) is exactly the continuous-time RTS smoother.

6.4.1.3 Nonlinear Formulation

In this section the results of §6.1.3 will fully be derived. The literature for nonlinear smoothing involving continuous-time models and discrete-time measurements is sparse though. An algorithm is presented in Ref. [1] without proof or reference. This algorithm relies upon the computation and use of the discrete-time model state-transition matrix as well as the discrete-time process noise covariance. From a practical point of view this approach may become unstable if the measurement frequency is not within Nyquist's limit. McReynolds[9] fills in many of the gaps in the derivations of early linear fixed-interval smoothers. In this current section McReynolds' results are extended for nonlinear continuous-discrete time systems. Consider minimizing the following mixed continuous-discrete loss function:

$$\begin{aligned}J = &\frac{1}{2}\sum_{k=1}^{N}\left[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\mathbf{x}_k)\right]^T R_k^{-1}\left[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\mathbf{x}_k)\right]\\&+ \frac{1}{2}\int_{t_0}^{t_N}\mathbf{w}^T(t)\,Q^{-1}(t)\,\mathbf{w}(t)\,dt\\&+ \frac{1}{2}\left[\hat{\mathbf{x}}_f(t_0) - \mathbf{x}(t_0)\right]^T P_f(t_0)^{-1}\left[\hat{\mathbf{x}}_f(t_0) - \mathbf{x}(t_0)\right]\end{aligned} \tag{6.220}$$

subject to the dynamic constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\, \mathbf{u}(t),\, t) + G(t)\,\mathbf{w}(t) \tag{6.221a}$$

$$\tilde{\mathbf{y}}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k \tag{6.221b}$$

Let us again denote the best estimate of $\mathbf{x}$ as $\hat{\mathbf{x}}$. The optimal control theory for continuous-time loss functions including discrete state penalty terms has been studied by Geering.[18] Using this theory the minimization of eqn. (6.220) yields the following TPBVP:[19]

$$\boxed{\begin{aligned} \frac{d}{dt}\hat{\mathbf{x}}(t) &= \mathbf{f}(\hat{\mathbf{x}}(t),\, \mathbf{u}(t),\, t) - G(t)\,Q(t)\,G^T(t)\boldsymbol{\lambda}(t) && (6.222\text{a})\\ \dot{\boldsymbol{\lambda}}(t) &= -F^T(\hat{\mathbf{x}}(t),\, t)\,\boldsymbol{\lambda}(t) && (6.222\text{b})\\ \boldsymbol{\lambda}_k^+ &= \boldsymbol{\lambda}_k^- + H_k^T(\hat{\mathbf{x}}_k)\,R_k^{-1}\left[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_k)\right] && (6.222\text{c}) \end{aligned}}$$

where

$$F(\hat{\mathbf{x}}(t),\, t) \equiv \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}(t)}, \quad H_k(\hat{\mathbf{x}}_k) \equiv \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}_k} \tag{6.223}$$

The boundary conditions are given by

$$\boldsymbol{\lambda}(T) = \mathbf{0} \tag{6.224a}$$

$$\boldsymbol{\lambda}(t_0) = P_f^{-1}(t_0)[\hat{\mathbf{x}}_f(t_0) - \hat{\mathbf{x}}(t_0)] \tag{6.224b}$$

Note that discrete-joints are present in the costate vector at the measurement times, but these discontinuities do not directly appear in the state vector estimate equation.

In order to decouple the state and costate vectors in eqn. (6.222) we use the inhomogeneous Riccati transformation given by eqns. (6.213) and (6.214). Our first step in the derivation of the smoother equation is to linearize $\mathbf{f}(\hat{\mathbf{x}}(t),\, \mathbf{u}(t),\, t)$ about $\hat{\mathbf{x}}_f(t)$, which yields

$$\begin{aligned} \frac{d}{dt}\hat{\mathbf{x}}(t) = \mathbf{f}(\hat{\mathbf{x}}_f(t),\, \mathbf{u}(t),\, t) + F(\hat{\mathbf{x}}_f(t),\, t)\left[\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)\right]\\ - G(t)\,Q(t)\,G^T(t)\,\boldsymbol{\lambda}(t) \end{aligned} \tag{6.225}$$

Substituting eqn. (6.213) into eqn. (6.225), and then substituting the resulting expression and eqn. (6.222b) into eqn. (6.214) yields

$$\begin{aligned} &\left[\frac{d}{dt}P_f(t) - F(\hat{\mathbf{x}}_f(t),\, t)\,P_f(t) - P_f(t)\,F^T(\hat{\mathbf{x}}(t),\, t) - G(t)\,Q(t)\,G^T(t)\right]\boldsymbol{\lambda}(t)\\ &+\left[\mathbf{f}(\hat{\mathbf{x}}_f(t),\, \mathbf{u}(t),\, t) - \frac{d}{dt}\hat{\mathbf{x}}_f(t)\right] = \mathbf{0} \end{aligned} \tag{6.226}$$

Avoiding the trivial solution of $\boldsymbol{\lambda}(t) = \mathbf{0}$ for all time leads to the following two equations:

$$\frac{d}{dt}\hat{\mathbf{x}}_f(t) = \mathbf{f}(\hat{\mathbf{x}}_f(t), \mathbf{u}(t), t) \tag{6.227a}$$

$$\frac{d}{dt}P_f(t) = F(\hat{\mathbf{x}}_f(t), t)\, P_f(t) + P_f(t)\, F^T(\hat{\mathbf{x}}_f(t), t) + G(t)\, Q(t)\, G^T(t) \tag{6.227b}$$

Equation (6.227) represents the Kalman filter propagation, where $\hat{\mathbf{x}}_f(t)$ denotes the forward-time estimate and $P_f(t)$ denotes the forward-time covariance. Note that $F^T(\hat{\mathbf{x}}(t), t)$ has been replaced by $F^T(\hat{\mathbf{x}}_f(t), t)$ in eqn. (6.227b). This substitution results in second-order error effects, which are neglected in the linearization assumption.

We now investigate the costate update equation. Solving eqn. (6.189) for $\boldsymbol{\lambda}_k$ and substituting the resulting expression into eqn. (6.222c) gives

$$\mathcal{P}_{fk}^{+}[\hat{\mathbf{x}}_{fk}^{+} - \hat{\mathbf{x}}_k] = \mathcal{P}_{fk}^{-}[\hat{\mathbf{x}}_{fk}^{-} - \hat{\mathbf{x}}_k] + H_k^T(\hat{\mathbf{x}}_k)\, R_k^{-1}[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_k)] \tag{6.228}$$

where $\mathcal{P}_{fk}^{+}$ is the matrix inverse of P_{fk}^{+}, and $\mathcal{P}_{fk}^{-}$ is the matrix inverse of P_{fk}^{-}. Note that the smoother state $\hat{\mathbf{x}}_k$ does not contain discontinuities at the measurement points, but its derivative is discontinuous due to the update in the costate. Linearizing $\mathbf{h}_k(\hat{\mathbf{x}}_k)$ about $\hat{\mathbf{x}}_{fk}^{-}$ gives

$$\mathbf{h}_k(\hat{\mathbf{x}}_k) = \mathbf{h}_k(\hat{\mathbf{x}}_{fk}^{-}) + H_k(\hat{\mathbf{x}}_{fk}^{-})[\hat{\mathbf{x}}_k - \hat{\mathbf{x}}_{fk}^{-}] \tag{6.229}$$

Substituting eqn. (6.229) into eqn. (6.228), and replacing $H_k^T(\hat{\mathbf{x}}_k)$ by $H_k^T(\hat{\mathbf{x}}_{fk}^{-})$, which again leads to second-order errors that are neglected, yields

$$\begin{aligned}
&[\mathcal{P}_{fk}^{-} - \mathcal{P}_{fk}^{+} + H_k^T(\hat{\mathbf{x}}_{fk}^{-})\, R_k^{-1} H_k(\hat{\mathbf{x}}_{fk}^{-})]\hat{\mathbf{x}}_k \\
&+ \mathcal{P}_{fk}^{+}\hat{\mathbf{x}}_{fk}^{+} - \mathcal{P}_{fk}^{-}\hat{\mathbf{x}}_{fk}^{-} - H_k^T(\hat{\mathbf{x}}_{fk}^{-}) R_k^{-1}[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_{fk}^{-}) + H_k(\hat{\mathbf{x}}_{fk}^{-})\hat{\mathbf{x}}_{fk}^{-}] = \mathbf{0}
\end{aligned} \tag{6.230}$$

Avoiding the trivial solution of $\hat{\mathbf{x}}_k = \mathbf{0}$ for all time leads to the following two equations:

$$\mathcal{P}_{fk}^{-} = \mathcal{P}_{fk}^{+} - H_k^T(\hat{\mathbf{x}}_{fk}^{-})\, R_k^{-1} H_k(\hat{\mathbf{x}}_{fk}^{-}) \tag{6.231a}$$

$$\mathcal{P}_{fk}^{+}\hat{\mathbf{x}}_{fk}^{+} = \mathcal{P}_{fk}^{-}\hat{\mathbf{x}}_{fk}^{-} + H_k^T(\hat{\mathbf{x}}_{fk}^{-}) R_k^{-1}[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_{fk}^{-}) + H_k(\hat{\mathbf{x}}_{fk}^{-})\hat{\mathbf{x}}_{fk}^{-}] \tag{6.231b}$$

Note that eqn. (6.231a) is the information form of the covariance update shown in §5.3.3. Substituting eqn. (6.231a) into eqn. (6.231b) leads to

$$\hat{\mathbf{x}}_{fk}^{+} = \hat{\mathbf{x}}_{fk}^{-} + K_{fk}[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_{fk}^{-})] \tag{6.232}$$

where the Kalman gain K_{fk} is given by

$$K_{fk} = P_{fk}^{+} H_k^T(\hat{\mathbf{x}}_{fk}^{-})\, R_k^{-1} \equiv V_{fk} D_{fk}^{-1} \tag{6.233}$$

with

$$V_{fk} \equiv P_{fk}^- H_k^T(\hat{\mathbf{x}}_{fk}^-) \tag{6.234a}$$

$$D_{fk} \equiv H_k(\hat{\mathbf{x}}_{fk}^-)\, V_{fk} + R_k \tag{6.234b}$$

Equation (6.232) gives the forward-time Kalman filter update, which is used to update the filter propagation in eqn. (6.227a). The covariance update is easily derived using the matrix inversion lemma on eqn. (6.231a), which yields

$$P_{fk}^+ = [I - K_{fk} H_k(\hat{\mathbf{x}}_{fk}^-)] P_{fk}^- \tag{6.235}$$

Equations (6.227), (6.232), (6.233), and (6.235) constitute the standard extended Kalman filter equations.

Since no jump discontinuities exist in the smoother state estimate equation, the smoother estimate can simply be found by solving eqn. (6.213) for $\boldsymbol{\lambda}(t)$ and substituting the resulting expression into eqn. (6.225), which yields

$$\frac{d}{dt}\hat{\mathbf{x}}(t) = [F(\hat{\mathbf{x}}_f(t),\, t) + K(t)][\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)] + \mathbf{f}(\hat{\mathbf{x}}_f(t),\, \mathbf{u}(t),\, t) \tag{6.236}$$

where

$$K(t) \equiv G(t)\, Q(t)\, G^T(t)\, P_f^{-1}(t) \tag{6.237}$$

Equation (6.236) must be integrated backward in time with a boundary condition of $\hat{\mathbf{x}}(T) = \hat{\mathbf{x}}_f(T)$, which satisfies eqn. (6.224a). Note that eqn. (6.222a) can be used instead of eqn. (6.236), but the advantage of using eqn. (6.236) is that a linear integration can be implemented.

The smoother state estimate shown in eqn. (6.107) requires an inversion of the propagated forward-time Kalman filter covariance. This can be overcome by using the information matrix version of the Kalman filter of §5.3.3, which directly involves the inverse of the covariance matrix. Another approach that avoids this matrix inversion involves using the costate equation directly to derive the smoother state.[13] This approach can easily be extended for the nonlinear case. Substituting eqn. (6.229) into eqn. (6.222c) gives

$$\boldsymbol{\lambda}_k^+ = \boldsymbol{\lambda}_k^- + H_k^T(\hat{\mathbf{x}}_k)\, R_k^{-1}\{\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_{fk}^-) - H_k(\hat{\mathbf{x}}_{fk}^-)[\hat{\mathbf{x}}_k - \hat{\mathbf{x}}_{fk}^-]\} \tag{6.238}$$

Solving eqn. (6.189) for $\boldsymbol{\lambda}_k$ and substituting the resulting expression into eqn. (6.238), and once again replacing $H_k^T(\hat{\mathbf{x}}_k)$ by $H_k^T(\hat{\mathbf{x}}_{fk}^-)$ yields

$$\boldsymbol{\lambda}_k^+ = [I + H_k^T(\hat{\mathbf{x}}_{fk}^-)\, R_k^{-1} H_k(\hat{\mathbf{x}}_{fk}^-)\, P_{fk}^-]\boldsymbol{\lambda}_k^- + H_k^T(\hat{\mathbf{x}}_{fk}^-)\, R_k^{-1}[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_{fk}^-)] \tag{6.239}$$

Solving eqn. (6.239) for $\boldsymbol{\lambda}_k^-$ and using the matrix inversion lemma leads to

$$\boldsymbol{\lambda}_k^- = [I - H_k^T(\hat{\mathbf{x}}_{fk}^-)\, K_{fk}^T]\boldsymbol{\lambda}_k^+ - H_k^T(\hat{\mathbf{x}}_{fk}^-)\, D_{fk}^{-1}[\tilde{\mathbf{y}}_k - \mathbf{h}_k(\hat{\mathbf{x}}_{fk}^-)] \tag{6.240}$$

which is used to update the backward integration of eqn. (6.222b). The covariance of the costate follows (which is left as an exercise for the reader)

$$\frac{d}{dt}\Lambda(t) = -F^T(\hat{\mathbf{x}}_f(t),t)\,\Lambda(t) - \Lambda(t)\,F(\hat{\mathbf{x}}_f(t),t) \tag{6.241a}$$

$$\Lambda_k^- = [I - K_{fk}\,H_k(\hat{\mathbf{x}}_{fk}^-)]^T\,\Lambda_k^+[I - K_{fk}\,H_k(\hat{\mathbf{x}}_{fk}^-)] + H_k^T(\hat{\mathbf{x}}_{fk}^-)\,D_{fk}^{-1}H_k(\hat{\mathbf{x}}_{fk}^-) \tag{6.241b}$$

The boundary conditions are given by

$$\boldsymbol{\lambda}_N^- = -H_N^T(\hat{\mathbf{x}}_{fN}^-)\,D_{fN}^{-1}[\tilde{\mathbf{y}}_N - \mathbf{h}_k(\hat{\mathbf{x}}_{fN}^-)]\delta_{t_n,N} \tag{6.242a}$$

$$\Lambda_N^- = H_N^T(\hat{\mathbf{x}}_{fN}^-)\,D_{fN}^{-1}H_N(\hat{\mathbf{x}}_{fN}^-)\,\delta_{t_n,N} \tag{6.242b}$$

where $\delta_{t_n,N}$ is the Kronecker symbol (if N is not an observation time, then $\boldsymbol{\lambda}$ and Λ have end boundary conditions of zero). Finally, the smoother state and covariance can be constructed via

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk}^{\pm} - P_{fk}^{\pm}\boldsymbol{\lambda}_k^{\pm} \tag{6.243a}$$

$$P_k = P_{fk}^{\pm} - P_{fk}^{\pm}\Lambda_k^{\pm}P_{fk}^{\pm} \tag{6.243b}$$

where the propagated or updated variables yield the same result. The nonlinear algorithm derived in this section does not require the computation of the discrete-time model state-transition matrix nor the discrete-time process noise covariance, which has clear advantages over the algorithm presented in Ref. [1]. Also, when linear models are used the smoothing solution reduces to the classical smoothing algorithms shown in Refs. [9] and [13]. For example, in the linear case the costate vector integration in eqn. (6.222b) with jump discontinuities given by eqn. (6.240) is equivalent to the adjoint filter variable given by Bierman.[13]

6.4.2 Innovations Process

In §6.4.1 the RTS smoother has been derived from optimal control theory. From this theory the costate vector (adjoint variable) is seen to be directly related to the process noise vector, shown by eqns. (6.186c) and (6.210c). Although this provides a nice mathematical representation of the smoothing problem, the physical meaning of the adjoint variable is somewhat unclear from this framework. In this section a different derivation of the TPBVP is shown, which helps to provide some physical meaning to the adjoint variable. This derivation is based upon the innovations process, which can be used to derive the Kalman filter. Here we will use the innovations process to directly derive the TPBVP associated with the RTS smoother. More details on this approach can be found in Refs. [20] and [21].

6.4.2.1 Discrete-Time Formulation

For the discrete-time case, we begin the derivation of the smoother by considering the following innovations process:

$$\begin{aligned}\mathbf{e}_{fk} &\equiv \tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_{fk} \\ &= -H_k \tilde{\mathbf{x}}_{fk} + \mathbf{v}_k\end{aligned} \tag{6.244}$$

where $\tilde{\mathbf{x}}_{fk} \equiv \hat{\mathbf{x}}_{fk} - \mathbf{x}_k$ and $\mathbf{v}_k$ is the measurement noise. The covariance of the error in eqn. (6.244) is given by eqn. (5.245), so that

$$\boxed{E\left\{\mathbf{e}_{fk}\mathbf{e}_{fk}^T\right\} = H_k P_{fk} H_k^T + R_k \equiv \mathcal{R}_{fk}} \tag{6.245}$$

To derive the smoother state estimate we use the following general formula for state estimation given the innovations process:[21]

$$\boxed{\hat{\mathbf{x}}_k = \sum_{i=0}^{N} E\left\{\mathbf{x}_k\, \mathbf{e}_{fi}^T\right\} \mathcal{R}_{fi}^{-1} \mathbf{e}_{fi}} \tag{6.246}$$

This relation can be directly derived from the orthogonality of the innovations, which is closely related to the projection property in least squares estimation (see §1.6.4). Setting $N = k-1$ in eqn. (6.246) gives the state estimate $\hat{\mathbf{x}}_{fk}$. This implies that the summation for $i \geq k$ can be broken up as

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk} + \sum_{i=k}^{N} E\left\{\mathbf{x}_k\, \mathbf{e}_{fi}^T\right\} \mathcal{R}_{fi}^{-1} \mathbf{e}_{fi} \tag{6.247}$$

We now concentrate our attention to the expectation in eqn. (6.247). Substituting eqn. (6.244) into the expectation in eqn. (6.247) gives

$$E\left\{\mathbf{x}_{fk}\, \mathbf{e}_{fi}^T\right\} = -E\left\{\mathbf{x}_k\, \tilde{\mathbf{x}}_{fi}^T\right\} H_i^T + E\left\{\mathbf{x}_k \mathbf{v}_i^T\right\} \tag{6.248}$$

Substituting $\mathbf{x}_k = \hat{\mathbf{x}}_{fk} - \tilde{\mathbf{x}}_{fk}$ into eqn. (6.248) gives

$$E\left\{\mathbf{x}_{fk}\, \mathbf{e}_{fi}^T\right\} = E\left\{\tilde{\mathbf{x}}_{fk}\, \tilde{\mathbf{x}}_{fi}^T\right\} H_i^T - E\left\{\hat{\mathbf{x}}_{fk}\, \tilde{\mathbf{x}}_{fi}^T\right\} H_i^T + E\left\{\mathbf{x}_k \mathbf{v}_i^T\right\} \tag{6.249}$$

Since the state estimate is orthogonal to its error (see §5.3.6) and since the measurement noise is uncorrelated with the true state, then eqn. (6.249) reduces down to

$$E\left\{\mathbf{x}_{fk}\, \mathbf{e}_{fi}^T\right\} = E\left\{\tilde{\mathbf{x}}_{fk}\, \tilde{\mathbf{x}}_{fi}^T\right\} H_i^T \equiv P_{fk,i}\, H_i^T \tag{6.250}$$

where $P_{fk,i} \equiv E\left\{\tilde{\mathbf{x}}_{fk}\, \tilde{\mathbf{x}}_{fi}^T\right\}$. Therefore, substituting eqn. (6.250) into eqn. (6.247) gives

$$\boxed{\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk} + \sum_{i=k}^{N} P_{fk,i}\, H_i^T \mathcal{R}_{fi}^{-1} \mathbf{e}_{fi}} \tag{6.251}$$

Note that $P_{fk,i}$ gives the correlation between the error states at different times. If $i=k$ then $P_{fk,i}$ is exactly the forward-time Kalman filter error-covariance.

The smoother error covariance can be derived by first subtracting $\mathbf{x}_k$ from both sides of eqn. (6.251), which leads to

$$\tilde{\mathbf{x}}_k = \tilde{\mathbf{x}}_{fk} + \sum_{i=k}^{N} P_{fk,i}\, H_i^T \mathcal{R}_{fi}^{-1} \mathbf{e}_{fi} \tag{6.252}$$

Substituting eqn. (6.244) into eqn. (6.252) and performing the covariance operation $P_k \equiv E\left\{\tilde{\mathbf{x}}_k\, \tilde{\mathbf{x}}_k^T\right\}$ yields

$$\boxed{P_k = P_{fk} - \sum_{i=k}^{N} P_{fk,i}\, H_i^T \mathcal{R}_{fi}^{-1} H_i\, P_{fk,i}^T} \tag{6.253}$$

Note the sign difference between eqns. (6.252) and (6.253).

Our next step involves determining a relationship between $P_{fk,i}$ and P_{fk}. Substituting eqn. (5.37) into eqn. (5.33) gives the forward-time state error:

$$\tilde{\mathbf{x}}_{fk+1} = Z_{fk}\tilde{\mathbf{x}}_{fk} + \mathbf{b}_{fk} \tag{6.254}$$

where

$$Z_{fk} \equiv \Phi_k[I - K_{fk}H_k] \tag{6.255a}$$
$$\mathbf{b}_{fk} \equiv \Phi_k K_{fk}\mathbf{v}_k - \Upsilon_k \mathbf{w}_k \tag{6.255b}$$

Taking one time-step ahead of eqn. (6.254) gives

$$\begin{aligned}\tilde{\mathbf{x}}_{fk+2} &= Z_{fk+1}\tilde{\mathbf{x}}_{fk+1} + \mathbf{b}_{k+1} \\ &= Z_{fk+1}Z_{fk}\tilde{\mathbf{x}}_{fk} + Z_{fk+1}\mathbf{b}_{fk} + \mathbf{b}_{fk+1}\end{aligned} \tag{6.256}$$

where eqn. (6.254) has been used. Taking more time-steps ahead leads to the following relationship for $i \geq k$:

$$\tilde{\mathbf{x}}_{fi} = \mathcal{Z}_{fi,k}\tilde{\mathbf{x}}_{fk} + \sum_{j=k}^{i-1} \mathcal{Z}_{fi,j+1}\mathbf{b}_{fj} \tag{6.257}$$

where

$$\mathcal{Z}_{fi,k} = \begin{cases} Z_{fi-1}Z_{fi-2}\cdots Z_{fk} & \text{for } i > k \\ I & \text{for } i = k \end{cases} \tag{6.258}$$

Then the relationship between $P_{fk,i}$ and P_{fk} is simply given by

$$P_{fk,i} \equiv E\left\{\tilde{\mathbf{x}}_{fk}\,\tilde{\mathbf{x}}_{fi}^T\right\} = P_{fk}\mathcal{Z}_{fi,k}^T \tag{6.259}$$

where eqn. (6.259) is valid for $i \geq k$. Substituting eqn. (6.259) into eqn. (6.251) gives

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk} - P_{fk}\boldsymbol{\lambda}_k \tag{6.260}$$

where

$$\boxed{\boldsymbol{\lambda}_k \equiv -\sum_{i=k}^{N} \mathcal{Z}_{fi,k}^T H_i^T \mathcal{R}_{fi}^{-1} \mathbf{e}_{fi}} \tag{6.261}$$

This result clearly shows the relationship between the adjoint variable $\boldsymbol{\lambda}_k$ and the forward-time residual. Comparing this result with eqn. (6.186c) shows an interesting relationship between the process noise and the innovations process in the adjoint variable.

Using the definition of $\mathcal{Z}_{fi,k}$ in eqn. (6.258) immediately implies that $\boldsymbol{\lambda}_k$ can be given by the following backward recursion:

$$\boldsymbol{\lambda}_k = Z_{fk}^T \boldsymbol{\lambda}_{k+1} - H_k^T \mathcal{R}_{fk}^{-1} \mathbf{e}_{fk}, \quad \boldsymbol{\lambda}_N = \mathbf{0} \tag{6.262}$$

Substituting eqn. (6.244) into eqn. (6.262), and using the definitions of Z_{fk} from eqn. (6.255a) and $\mathcal{R}_{fk}$ from eqn. (6.245) gives

$$\boxed{\boldsymbol{\lambda}_k = [I - K_{fk} H_k]^T \Phi_k^T \boldsymbol{\lambda}_{k+1} + H_k^T [H_k P_{fk} H_k^T + R_k]^{-1} [H_k \hat{\mathbf{x}}_{fk} - \tilde{\mathbf{y}}_k]} \tag{6.263}$$

Next, solving eqn. (6.260) for $\hat{\mathbf{x}}_{fk}$ and substituting the resulting expression into eqn. (6.263) yields

$$\begin{aligned} \boldsymbol{\lambda}_k &= [I - K_{fk} H_k]^T \Phi_k^T \boldsymbol{\lambda}_{k+1} \\ &\quad + H_k^T [H_k P_{fk} H_k^T + R_k]^{-1} [H_k \hat{\mathbf{x}}_k - \tilde{\mathbf{y}}_k + H_k P_{fk} \boldsymbol{\lambda}_k] \end{aligned} \tag{6.264}$$

Finally, collecting terms and solving eqn. (6.264) for $\boldsymbol{\lambda}_k$ leads to

$$\begin{aligned} \boldsymbol{\lambda}_k = [I - W_{fk}]^{-1} \Big\{ &[I - K_{fk} H_k]^T \Phi_k^T \boldsymbol{\lambda}_{k+1} \\ &+ H_k^T [H_k P_{fk} H_k^T + R_k]^{-1} [H_k \hat{\mathbf{x}}_k - \tilde{\mathbf{y}}_k] \Big\} \end{aligned} \tag{6.265}$$

where

$$W_{fk} \equiv H_k^T [H_k P_{fk} H_k^T + R_k]^{-1} H_k P_{fk} \tag{6.266}$$

At first glance eqn. (6.188b) and eqn. (6.265) do not appear to be equivalent. However, upon further inspection the following identities can be proven (which are left as an exercise for the reader):

$$[I - W_{fk}]^{-1} [I - K_{fk} H_k]^T = I \tag{6.267a}$$

$$[I - W_{fk}]^{-1} H_k^T [H_k P_{fk} H_k^T + R_k]^{-1} = H_k^T R_k \tag{6.267b}$$

Hence, eqn. (6.188b) and eqn. (6.265) are indeed equivalent. The state equation can be derived by taking one time-step ahead of eqn. (6.260) and substituting the forward-time Kalman filter equations from eqn. (5.54), which leads to

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} &= \Phi_k \hat{\mathbf{x}}_{fk} + \Gamma_k \mathbf{u}_k + \Phi_k K_{fk} [\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}] \\ &\quad - [\Phi_k P_{fk} \Phi_k^T - \Phi_k K_k H_k P_{fk} \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T] \boldsymbol{\lambda}_{k+1} \end{aligned} \tag{6.268}$$

Now, solving eqn. (6.260) for $\hat{\mathbf{x}}_{fk}$ and substituting the resulting expression into eqn. (6.268) yields

$$\begin{aligned}\hat{\mathbf{x}}_{k+1} &= \Phi_k\hat{\mathbf{x}}_k + \Gamma_k\mathbf{u}_k - \Upsilon_k Q_k \Upsilon_k^T \boldsymbol{\lambda}_{k+1} + \Phi_k P_{fk}\boldsymbol{\lambda}_k \\ &\quad - \Phi_k K_{fk}[H_k\hat{\mathbf{x}}_{fk} - \tilde{\mathbf{y}}_k] - [\Phi_k P_{fk}\Phi_k^T - \Phi_k K_k H_k P_{fk}\Phi_k^T]\boldsymbol{\lambda}_{k+1}\end{aligned} \tag{6.269}$$

Substituting eqn. (6.263) into eqn. (6.269) simply produces the state equation in eqn. (6.188a).

The innovations process leads to some important conclusions. For example, as previously mentioned, the innovations process can be used to directly derive the Kalman filter, where the filter is used to *whiten* the innovations process (see Refs. [11] and [21] for more details). The derivations provided in this section give a very important result, since they show yet another approach to derive the RTS smoother. In fact, a form of the RTS smoother has been derived more directly than the lengthy algebraic process shown in §6.1.1. This form involves using eqn. (6.263) to solve for the adjoint variable directly from the forward-time Kalman filter quantities. Then, the smoothed estimate can be found directly from eqn. (6.260), which can be implemented in conjunction with the adjoint variable calculation.

6.4.2.2 Continuous-Time Formulation

For the continuous-time case, we begin the derivation of the smoother by considering the following innovations process:

$$\begin{aligned}\mathbf{e}_f(t) &\equiv \tilde{\mathbf{y}}(t) - \hat{\mathbf{y}}_f(t) \\ &= -H(t)\,\tilde{\mathbf{x}}_f(t) + \mathbf{v}(t)\end{aligned} \tag{6.270}$$

where $\tilde{\mathbf{x}}_f(t) = \hat{\mathbf{x}}_f(t) - \mathbf{x}(t)$ and $\mathbf{v}(t)$ is the measurement noise. One's first natural instinct is to assume that the innovations covariance is just the continuous-time version of eqn. (6.245). However, a rigorous derivation of the continuous-time covariance for the innovations process is far more complicated than the discrete-time case. It can be shown that this covariance obeys[21]

$$\boxed{E\left\{\mathbf{e}_f(t)\,\mathbf{e}_f^T(\tau)\right\} = R(t)\,\delta(t-\tau)} \tag{6.271}$$

Equation (6.271) can be proven in many ways, i.e., using the orthogonality conditions or by working with white noise replaced with a Wiener process and using martingale theory.[21] Also, since the innovations process is uncorrelated between different times, then the expression in eqn. (6.271) is valid for all time.

The continuous-time version of eqn. (6.246) is given by

$$\boxed{\hat{\mathbf{x}}(t) = \int_0^T E\left\{\mathbf{x}(t)\,\mathbf{e}_f^T(\tau)\right\} R^{-1}(\tau)\,\mathbf{e}_f(\tau)\,d\tau} \tag{6.272}$$

As with the discrete-time case, eqn. (6.272) can be broken up into two parts, given by

$$\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_f(t) + \int_t^T E\left\{\mathbf{x}(t)\,\mathbf{e}_f^T(\tau)\right\} R^{-1}(\tau)\,\mathbf{e}_f(\tau)\,d\tau \tag{6.273}$$

for $0 \le t \le T$. We now concentrate our attention on the expectation in eqn. (6.273). Substituting eqn. (6.270) into the expectation in eqn. (6.273) gives

$$E\left\{\mathbf{x}(t)\,\mathbf{e}_f^T(\tau)\right\} = -E\left\{\mathbf{x}(t)\,\tilde{\mathbf{x}}_f^T(\tau)\right\} H^T(\tau) + E\left\{\mathbf{x}(t)\,\mathbf{v}^T(\tau)\right\} \tag{6.274}$$

Substituting $\mathbf{x}(t) = \hat{\mathbf{x}}_f(t) - \tilde{\mathbf{x}}_f(t)$ into eqn. (6.274) gives

$$\begin{aligned} E\left\{\mathbf{x}(t)\,\mathbf{e}_f^T(\tau)\right\} = E\left\{\tilde{\mathbf{x}}_f(t)\,\tilde{\mathbf{x}}_f^T(\tau)\right\} H^T(\tau) - E\left\{\hat{\mathbf{x}}_f(t)\,\tilde{\mathbf{x}}_f^T(\tau)\right\} H^T(\tau) \\ + E\left\{\mathbf{x}(t)\,\mathbf{v}^T(\tau)\right\} \end{aligned} \tag{6.275}$$

Since the state estimate is orthogonal to its error and since the measurement noise is uncorrelated with the true state, then eqn. (6.275) reduces down to

$$E\left\{\mathbf{x}(t)\,\mathbf{e}_f^T(\tau)\right\} = E\left\{\tilde{\mathbf{x}}_f(t)\,\tilde{\mathbf{x}}_f^T(\tau)\right\} H^T(\tau) \equiv P_f(t,\,\tau)\,H^T(\tau) \tag{6.276}$$

where $P_f(t,\,\tau) \equiv E\left\{\tilde{\mathbf{x}}_f(t)\,\tilde{\mathbf{x}}_f^T(\tau)\right\}$. Substituting eqn. (6.276) into eqn. (6.273) gives

$$\boxed{\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_f(t) + \int_t^T P_f(t,\,\tau)\,H^T(\tau)\,R^{-1}(\tau)\,\mathbf{e}_f(\tau)\,d\tau} \tag{6.277}$$

Note that $P_f(t,\,\tau)$ gives the correlation between the error states at different times. If $\tau = t$ then $P_f(t,\,\tau)$ is exactly the forward-time Kalman filter error-covariance. The smoother error covariance can be shown to be given by (which is left as an exercise for the reader)

$$\boxed{P(t) = P_f(t) - \int_t^T P_f(t,\,\tau)\,H^T(\tau)\,R^{-1}(\tau)\,H(\tau)\,P_f(t,\,\tau)\,d\tau} \tag{6.278}$$

As with the discrete-time case, note the sign difference between eqns. (6.277) and (6.278).

Our next step involves determining a relationship between $P_f(t,\,\tau)$ and $P_f(t)$. From eqn. (5.120) we can write

$$\frac{d}{d\tau}\tilde{\mathbf{x}}_f(\tau) = E_f(\tau)\,\tilde{\mathbf{x}}_f(\tau) + \mathbf{z}_f(\tau) \tag{6.279}$$

where

$$E_f(\tau) = F(\tau) - K_f(\tau)\,H(\tau) \tag{6.280}$$

$$\mathbf{z}_f(\tau) = -G(\tau)\,\mathbf{w}(\tau) + K_f(\tau)\,\mathbf{v}(\tau) \tag{6.281}$$

The solution for $\tilde{\mathbf{x}}_f(\tau)$ in eqn. (6.279) is given by

$$\tilde{\mathbf{x}}_f(\tau) = \Psi(\tau,\,t)\,\tilde{\mathbf{x}}_f(t) + \int_t^\tau \Phi(\tau,\,\eta)\,\mathbf{z}_f(\eta)\,d\eta \tag{6.282}$$

where $\Psi(\tau, t)$ is the state transition matrix of $E_f(\tau)$, which follows

$$\frac{d}{d\tau}\Psi(\tau, t) = [F(\tau) - K_f(\tau)\, H(\tau)]\Psi(\tau, t), \quad \Psi(\tau, \tau) = I \tag{6.283}$$

Substituting eqn. (6.283) into $P_f(t, \tau) = E\left\{\tilde{\mathbf{x}}_f(t)\,\tilde{\mathbf{x}}_f^T(\tau)\right\}$ leads to

$$P_f(t, \tau) = E\left\{\tilde{\mathbf{x}}_f(t)\,\tilde{\mathbf{x}}_f^T(\tau)\Psi^T(\tau, t)\right\} = P_f(t)\,\Psi^T(\tau, t) \tag{6.284}$$

where the fact that $\tilde{\mathbf{x}}_f(t)$ is uncorrelated to $\mathbf{z}_f(\tau)$ has been used to yield eqn. (6.284). Substituting eqn. (6.284) into eqn. (6.277) gives

$$\boxed{\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_f(t) - P_f(t)\,\boldsymbol{\lambda}(t)} \tag{6.285}$$

where

$$\boxed{\begin{aligned}\boldsymbol{\lambda}(t) &\equiv -\int_t^T \Psi^T(\tau, t)\, H^T(\tau)\, R^{-1}(\tau)\, \mathbf{e}_f(\tau)\, d\tau \\ &= \int_T^t \Psi^T(\tau, t)\, H^T(\tau)\, R^{-1}(\tau)\, \mathbf{e}_f(\tau)\, d\tau\end{aligned}} \tag{6.286}$$

The physical interpretation of the continuous-time adjoint variable is analogous to the discrete-time case, which is an intuitively pleasing result. Taking the time derivative of eqn. (6.286) leads to

$$\frac{d}{dt}\boldsymbol{\lambda}(t) = \int_T^t \left[\frac{d}{dt}\Psi^T(\tau, t)\right] H^T(\tau)\, R^{-1}(\tau)\, \mathbf{e}_f(\tau)\, d\tau + H^T(t)\, R^{-1}(t)\, \mathbf{e}_f(t) \tag{6.287}$$

Using the result shown in exercise 3.1 as well as the definition of $\boldsymbol{\lambda}$ in eqn. (6.286), with the definitions of $\mathbf{e}_f(t)$ in eqn. (6.270) and $E(t)$ in eqn. (6.280), leads to

$$\boxed{\begin{aligned}\frac{d}{dt}\boldsymbol{\lambda}(t) = &-[F(t) - K_f(t)\, H(t)]^T\boldsymbol{\lambda}(t) \\ &- H^T(t)\, R^{-1}(t) H(t)\,\hat{\mathbf{x}}_f(t) + H^T(t)\, R^{-1}(t)\,\tilde{\mathbf{y}}(t)\end{aligned}} \tag{6.288}$$

Solving eqn. (6.285) for $\boldsymbol{\lambda}(t)$ and substituting the resulting expression into the differential equation of eqn. (6.288) gives

$$\begin{aligned}\frac{d}{dt}\boldsymbol{\lambda}(t) = &-F^T(t)\,\boldsymbol{\lambda}(t) + H^T(t)\, K_f^T(t)\, P_f^{-1}(t)[\hat{\mathbf{x}}_f(t) - \hat{\mathbf{x}}(t)] \\ &- H^T(t)\, R^{-1}(t) H(t)\,\hat{\mathbf{x}}_f(t) + H^T(t)\, R^{-1}(t)\,\tilde{\mathbf{y}}(t)\end{aligned} \tag{6.289}$$

Substituting eqn. (6.218) into eqn. (6.289) leads exactly to eqn. (6.212b). Also, the differential equation for $\hat{\mathbf{x}}(t)$ follows directly from the steps leading to eqn. (6.219). Equation (6.288) can be used to solve for the adjoint variable directly from the forward-time Kalman filter quantities. The results in the section validate the associated TPBVP shown in eqn. (6.212) using the innovations process, which leads to the continuous-time RTS smoother.

6.5 Summary

In this chapter several smoothing algorithms have been presented that are based on using a batch set of measurement data. The advantage of using a smoother has been clearly shown by the fact that its associated error-covariance is always less than (or equal to) the Kalman filter error covariance. This indicates that better estimates can be achieved by using the optimal smoother; however, a significant disadvantage of a smoother is that a real-time estimate is not possible. The fixed-interval smoother of §6.1 is particularly useful though for many applications, such as sensor bias calculations and parameter estimation.

Intrinsic in all smoothing algorithms presented in this chapter is the Kalman filter. The fixed-interval smoother can conceptually be divided into two separate filters: a forward-time Kalman filter and a backward time recursion. For the fixed-interval smoother the backward-time recursion has been derived two different ways. One uses a backward-time Kalman filter-type implementation, where the smoother estimate is given by an optimally-derived combination of both filters. The other uses a direct computation of the smoother estimate without the need of combining the forward-time and backward-time estimates. Each approach is equivalent to one another from a theoretical point of view. However, depending on the particular situation, one approach may provide a computational advantage of another. A comparison of the computational requirements in the various smoother equation approaches is given by McReynolds.[9]

Several theoretical aspects of the optimal smoother are given in this chapter. For example, a formal proof of the stability of the RTS smoother has been provided using a Lyapunov stability analysis. Fairly complete derivations of the fixed-point and fixed-lag smoothers have also been provided so that the reader can better understand the intricacies of the properties of these smoothers. One of the most interesting aspects of smoothing is the dual relationship with optimal control, which has been presented in §6.4.1. At first glance one might not realize this relationship, but after closer examination we realize than *any* estimation problem can be rewritten as a control problem. The results of §6.4.2 further strengthen this statement. Several references have been provided in this chapter, and the reader is strongly encouraged to further study smoothing approaches in the open literature.

A summary of the key formulas presented in this chapter is given below. All variables with the subscript f denote the forward-time Kalman filter.

- Fixed-Interval Smoother (Discrete-Time)

$$K_{bk} = \mathcal{P}^+_{bk+1}\Upsilon_k[\Upsilon_k^T\mathcal{P}^+_{bk+1}\Upsilon_k + Q_k^{-1}]^{-1}$$

$$\hat{\chi}^+_{bk} = \hat{\chi}^-_{bk} + H_k^T R_k^{-1}\tilde{\mathbf{y}}_k$$
$$\mathcal{P}^+_{bk} = \mathcal{P}^-_{bk} + H_k^T R_k^{-1} H_k$$

$$\hat{\chi}_{bk}^{-} = \Phi_k^T [I - K_{bk} \Upsilon_k^T][\hat{\chi}_{bk+1}^{+} - \mathcal{P}_{bk+1}^{+} \Gamma_k \mathbf{u}_k]$$
$$\mathcal{P}_{bk}^{-} = \Phi_k^T [I - K_{bk} \Upsilon_k^T] \mathcal{P}_{bk+1}^{+} \Phi_k$$

$$K_k = P_{fk}^{+} \mathcal{P}_{bk}^{-} [I + P_{fk}^{+}]$$

$$\hat{\mathbf{x}}_k = [I - K_k] \hat{\mathbf{x}}_{fk}^{+} + P_k \hat{\chi}_{bk}^{-}$$
$$P_k = [I - K_k] P_{fk}^{+}$$

- RTS Smoother (Discrete-Time)

$$\mathcal{K}_k \equiv P_{fk}^{+} \Phi_k^T (P_{fk+1}^{-})^{-1}$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk}^{+} + \mathcal{K}_k [\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}^{-}]$$
$$P_k = P_{fk}^{+} - \mathcal{K}_k [P_{fk+1}^{-} - P_{k+1}] \mathcal{K}_k^T$$

- Fixed-Interval Smoother (Continuous-Time)

$$\begin{aligned}\frac{d}{d\tau} P_b^{-1}(t) &= P_b^{-1}(t)\, F(t) + F^T(t)\, P_b^{-1}(t) \\ &\quad - P_b^{-1}(t)\, G(t)\, Q(t)\, G^T(t)\, P_b^{-1}(t) + H^T(t)\, R^{-1}(t)\, H(t)\end{aligned}$$

$$\begin{aligned}\frac{d}{d\tau} \hat{\chi}_b(t) &= \left[F(t) - G(t)\, Q(t)\, G^T(t)\, P_b^{-1}(t) \right]^T \hat{\chi}_b(t) \\ &\quad - P_b^{-1}(t)\, B(t)\, \mathbf{u}(t) + H^T(t)\, R^{-1}(t)\, \tilde{\mathbf{y}}(t)\end{aligned}$$

$$K(t) = P_f(t)\, P_b^{-1}(t) \left[I + P_f(t)\, P_b^{-1}(t) \right]^{-1}$$

$$\hat{\mathbf{x}}(t) = [I - K(t)] \hat{\mathbf{x}}_f(t) + P(t)\, \hat{\chi}_b(t)$$
$$P(t) = [I - K(t)] P_f(t)$$

- RTS Smoother (Continuous-Time)

$$\frac{d}{d\tau} \hat{\mathbf{x}}(t) = -F(t)\, \hat{\mathbf{x}}(t) - B(t)\, \mathbf{u}(t) - G(t)\, Q(t)\, G^T(t)\, P_f^{-1}(t) \left[\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t) \right]$$

$$\begin{aligned}\frac{d}{d\tau} P(t) &= -[F(t) + G(t)\, Q(t)\, G^T(t)\, P_f^{-1}(t)] P(t) \\ &\quad - P(t) [F(t) + G(t)\, Q(t)\, G^T(t)\, P_f^{-1}(t)]^T + G(t)\, Q(t)\, G^T(t)\end{aligned}$$

- Nonlinear RTS Smoother

$$K(t) \equiv G(t)\,Q(t)\,G^T(t)\,P_f^{-1}(t)$$

$$\begin{aligned}
\frac{d}{d\tau}\hat{\mathbf{x}}(t) &= -\left[F(\hat{\mathbf{x}}_f(t),\,t)+K(t)\right]\left[\hat{\mathbf{x}}(t)-\hat{\mathbf{x}}_f(t)\right]-\mathbf{f}(\hat{\mathbf{x}}_f(t),\,\mathbf{u}(t),\,t)\\
\frac{d}{d\tau}P(t) &= -[F(\hat{\mathbf{x}}_f(t),\,t)+K(t)]P(t)-P(t)[F(\hat{\mathbf{x}}_f(t),\,t)+K(t)]^T\\
&\quad +G(t)\,Q(t)\,G^T(t)
\end{aligned}$$

- Fixed-Point Smoother (Discrete-Time)

$$\mathcal{B}_N = \prod_{i=k}^{N-1}\mathcal{K}_i$$
$$\mathcal{K}_i = P_{f\,i}^{+}\Phi_i^T(P_{f\,i+1}^{-})^{-1}$$

$$\begin{aligned}
\hat{\mathbf{x}}_{k|N} &= \hat{\mathbf{x}}_{k|N-1}+\mathcal{B}_N[\hat{\mathbf{x}}_{f\,N}^{+}-\hat{\mathbf{x}}_{f\,N}^{-}]\\
P_{k|N} &= P_{k|N-1}+\mathcal{B}_N[P_{f\,N}^{+}-P_{f\,N}^{-}]\mathcal{B}_N^T
\end{aligned}$$

- Fixed-Point Smoother (Continuous-Time)

$$\frac{d}{dT}\Phi(t,T) = -\Phi(t,T)[F(T)+G(T)\,Q(T)\,G^T(T)\,P_f^{-1}(T)],$$

$$\begin{aligned}
\frac{d}{dT}\hat{\mathbf{x}}(t|T) &= \Phi(t,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)[\tilde{\mathbf{y}}(T)-H(T)\,\hat{\mathbf{x}}_f(T)]\\
\frac{d}{dT}P(t|T) &= -\Phi(t,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)\,P_f(T)\,\Phi^T(t,T)
\end{aligned}$$

- Fixed-Lag Smoother (Discrete-Time)

$$\mathcal{B}_{k+1+N} = \prod_{i=k+1}^{k+N}\mathcal{K}_i$$
$$\mathcal{K}_i = P_{f\,i}^{+}\Phi_i^T(P_{f\,i+1}^{-})^{-1}$$

$$\begin{aligned}
\hat{\mathbf{x}}_{k+1|k+1+N} &= \Phi_k\hat{\mathbf{x}}_{k|k+N}+\Gamma_k\mathbf{u}_k\\
&\quad +\Upsilon_k Q_k\Upsilon_k^T\Phi_k^{-T}(P_{f\,k}^{+})^{-1}[\hat{\mathbf{x}}_{k|k+N}-\hat{\mathbf{x}}_{f\,k}^{+}]\\
&\quad +\mathcal{B}_{k+1+N}K_{f\,k+1+N}\{\tilde{\mathbf{y}}_{k+1+N}\\
&\quad -H_{k+1+N}\Phi_{k+N}[\hat{\mathbf{x}}_{f\,k+N}^{+}+\Gamma_{k+N}\mathbf{u}_{k+N}]\}\\
P_{k+1|k+1+N} &= P_{f\,k+1}^{-}-\mathcal{K}_k^{-1}[P_{f\,k}^{+}-P_{k|k+N}]\mathcal{K}_k^{-T}\\
&\quad -\mathcal{B}_{k+1+N}K_{f\,k+1+N}H_{k+1+N}P_{f\,k+1+N}^{-}\mathcal{B}_{k+1+N}^T
\end{aligned}$$

- Fixed-Lag Smoother (Continuous-Time)

$$\begin{aligned}\frac{d}{dT}\Psi(T-\Delta,T) &= [F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta)\\ &\quad\times G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]\Psi(T-\Delta,T)\\ &\quad-\Psi(T-\Delta,T)[F(T)+G(T)\,Q(T)\,G^T(T)\,P_f^{-1}(T)]\end{aligned}$$

$$\begin{aligned}\frac{d}{dT}\hat{\mathbf{x}}(T-\Delta|T) &= [F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta)\\ &\quad\times G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]\hat{\mathbf{x}}(T-\Delta|T)\\ &\quad-G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\,P_f^{-1}(T-\Delta)\,\hat{\mathbf{x}}_f(T-\Delta)\\ &\quad+\Psi(T-\Delta,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)[\tilde{\mathbf{y}}(T)-H(T)\,\hat{\mathbf{x}}_f(T)]\end{aligned}$$

$$\begin{aligned}\frac{d}{dT}P(T-\Delta|T) &= [F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta)\\ &\quad\times G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]P(T-\Delta|T)+P(T-\Delta|T)\\ &\quad\times[F(T-\Delta)+G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\,P_f^{-1}(T-\Delta)]^T\\ &\quad-\Psi(T-\Delta,T)\,P_f(T)\,H^T(T)\,R^{-1}(T)\,P_f(T)\,\Psi(T-\Delta,T)\\ &\quad-G(T-\Delta)\,Q(T-\Delta)\,G^T(T-\Delta)\end{aligned}$$

- Innovations Process (Discrete-Time)

$$\boldsymbol{\lambda}_k \equiv -\sum_{i=k}^{N}\mathcal{Z}_{f\,i,k}^T H_i^T \mathcal{R}_{f\,i}^{-1}\mathbf{e}_{f\,i}$$

$$\mathcal{Z}_{f\,i,k} = \begin{cases} Z_{f\,i-1}Z_{f\,i-2}\cdots Z_{f\,k} & \text{for } i>k\\ I & \text{for } i=k\end{cases}$$

$$Z_{f\,k} \equiv \Phi_k[I-K_{f\,k}H_k]$$

$$\mathcal{R}_{f\,k} \equiv H_k P_{f\,k} H_k^T + R_k$$

$$\mathbf{e}_{f\,k} \equiv \tilde{\mathbf{y}}_k - H_k\hat{\mathbf{x}}_{f\,k}$$

$$\boldsymbol{\lambda}_k = Z_{f\,k}^T\boldsymbol{\lambda}_{k+1} - H_k^T\mathcal{R}_{f\,k}^{-1}\mathbf{e}_{f\,k}, \quad \boldsymbol{\lambda}_N = \mathbf{0}$$

- Innovations Process (Continuous-Time)

$$\boldsymbol{\lambda}(t) \equiv -\int_t^T \Psi^T(\tau,t)\,H^T(\tau)\,R^{-1}(\tau)\,\mathbf{e}_f(\tau)\,d\tau$$

$$\frac{d}{d\tau}\Psi(\tau,t) = [F(\tau)-K_f(\tau)\,H(\tau)]\Psi(\tau,t), \quad \Psi(\tau,\tau)=I$$

$$\mathbf{e}_f(t) \equiv \tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}_f(t)$$

$$\frac{d}{dt}\boldsymbol{\lambda}(t) = -[F(t) + K_f(t)\,H(t)]^T\,\boldsymbol{\lambda}(t) \\ - H^T(t)\,R^{-1}(t)\,H(t)\,\hat{\mathbf{x}}_f(t) + H^T(t)\,R^{-1}(t)\,\tilde{\mathbf{y}}(t)$$

Exercises

6.1 After substituting eqns. (6.12) and (6.13) into eqn. (6.10) prove that the expression in eqn. (6.14) is valid.

6.2 Prove that the backward gain expressions given in eqns. (6.27) and (6.28) are equivalent to each other. Also, prove that the backward inverse covariance expressions given in eqns. (6.26) and (6.29) are equivalent to each other.

6.3 Write a general program that solves the discrete-time algebraic Riccati equation using the eigenvalue/eigenvector decomposition algorithm of the Hamiltonian matrix given by eqn. (6.39). Compare the steady-state values computed from your program to the values computed by the backward propagation in eqn. (6.29). Pick any order system with various values for Φ, H, Q, Υ, and R to test your program.

6.4 Reproduce the results of example 6.1 using your own simulation. Also, instead of using the RTS smoother form, use the two-filter algorithm shown in Table 6.1. Do you obtain the same results as the RTS smoother? Compute the steady-state values for P_{fk}^+ and P_{bk}^- using the eigenvalue/eigenvector decompositions of eqns. (5.89) and (6.39). Next, from these values compute the steady-state value for the smoother covariance P_k. Compare the 3σ attitude bound from this approach with the solution given in example 6.1.

6.5 Use the discrete-time fixed-interval smoother to provide smoothed estimates for the system described in problems 5.12 and 5.13.

6.6 Show that the solution for the optimal smoother estimate given by eqn. (6.79) can be derived by minimizing the following loss function:

$$J[\hat{\mathbf{x}}(t)] = [\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)]^T\,P_f^{-1}(t)[\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_f(t)] \\ + [\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_b(t)]^T\,P_b^{-1}(t)[\hat{\mathbf{x}}(t) - \hat{\mathbf{x}}_b(t)]$$

What are the physical connotations of this result?

6.7 Write a general program that solves the continuous-time algebraic Riccati equation using the eigenvalue/eigenvector decomposition algorithm of the Hamiltonian matrix given by eqn. (6.91). Compare the steady-state values computed from your program to the values computed by the backward propagation in eqn. (6.29). Pick any order system with various values for F, H, Q, G, and R to test your program.

6.8 After substituting the relations given in eqns. (6.74) and (6.87) with $d\hat{\chi}_b/d\tau = -d\hat{\chi}_b/dt$, and (6.99) into eqn. (6.101), prove that the expression given in eqn. (6.102) is valid.

6.9 What changes need to be made (if any) to the RTS smoother equations if the process noise and measurement noise are correlated? Discuss both the discrete-time and continuous-time cases.

6.10 ♣ Using the approach outlined in §5.4.2, beginning with the discrete-time fixed-interval smoother shown in Table 6.1, derive the continuous-time version shown in Table 6.3. Also, perform the same derivation for the RTS version of the smoother.

6.11 In example 6.2 show that at steady-state the smoother variance p is always less than half the forward-time filter variance p_f. Also, show $p \leq p_b$.

6.12 The nonlinear RTS smoother shown in Table 6.5 is also valid for linear systems with continuous-time models and discrete-time measurements. Use the smoother to provide smoothed estimates for the system described in exercise 5.29.

6.13 Use the nonlinear RTS smoother to provide smoothed estimates for the system described in exercise 5.33.

6.14 Use the nonlinear RTS smoother to provide a smoothed estimate for the damping coefficient described in the parameter identification problem shown in example 5.6.

6.15 ♣ Fully derive the expression shown for the single-stage optimal smoother in eqn. (6.118).

6.16 Derive the discrete-time RTS smoother directly from eqn. (6.128).

6.17 Prove the expression shown in eqn. (6.148).

6.18 Use the fixed-point discrete-time shown in Table 6.6 to find a fixed-point smoother estimate at some time reference for the system described in example 5.3.

6.19 ♣ Using the approach outlined in §5.4.2, beginning with the discrete-time fixed-point smoother shown in Table 6.6, derive the continuous-time version shown in Table 6.7.

6.20 Prove the covariance expression shown in eqn. (6.157) using similar steps outlined to obtain eqn. (6.156).

6.21 Prove that the solution for $p(t|T)$ given in example 6.4 satisfies its differential equation.

6.22 Use the fixed-lag discrete-time shown in Table 6.8 to find a fixed-lag smoother estimate for the system described in example 5.3. Choose any constant lag in your simulation.

6.23 ♣ Using the approach outlined in §5.4.2, beginning with the discrete-time fixed-lag smoother shown in Table 6.8, derive the continuous-time version shown in Table 6.9.

6.24 After taking the derivative of eqn. (6.178) with respect to T, and substituting the forward-time state filter equation from Table 6.3 and eqn. (6.181) into the resulting equation, prove that the expression in eqn. (6.182) is valid.

6.25 Starting with the fixed-lag estimate in eqn. (6.182) derive the covariance expression given in eqn. (6.183).

6.26 In example 6.5 verify that the fixed-lag smoother variance solution is given $p(T-\Delta|T) = p(0|\Delta)$.

6.27 Prove the identities given in eqn. (6.204).

6.28 Starting with the costate differential equation shown in eqn. (6.222b) and update shown in eqn. (6.222c), prove that the covariance of the costate is given by eqn. (6.241).

6.29 ♣ Using the approach outlined in §5.4.2, beginning with the discrete-time TPBVP shown in eqn. (6.188), derive the continuous-time version shown eqn. (6.212).

6.30 ♣ In the nonlinear formulation of §6.4.1 the quantity $\hat{\mathbf{x}}(t)$ has been replaced by $\hat{\mathbf{x}}_f(t)$ in a number of cases, e.g., in eqn. (6.227b). Prove that this substitution leads to second-order errors that can be ignored in the linearization assumption.

6.31 Prove the identities shown in eqn. (6.267).

6.32 ♣ The general linear least-mean-square estimator for $\mathbf{x}$, given a set of N measurements, can be represented by

$$\hat{\mathbf{x}} = \sum_{k=0}^{N} E\left\{\mathbf{x}\,\mathbf{e}_k^T\right\} ||\mathbf{e}_k||^{-2}\mathbf{e}_k$$

where $\mathbf{e}_k \equiv \tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k$. Prove this relationship using the orthogonality of the innovations process.

6.33 ♣ Derive the forward-time discrete-time Kalman filter beginning with the following basic formula for state estimation:

$$\hat{\mathbf{x}}_{f\,k+1} = \sum_{i=0}^{N} E\left\{\mathbf{x}_{k+1}\,\mathbf{e}_{f\,i}^T\right\} \mathcal{R}_{f\,i}^{-1}\mathbf{e}_{f\,i} \tag{6.290}$$

where $\mathbf{e}_{fi} \equiv \tilde{\mathbf{y}}_i - H_i \hat{\mathbf{x}}_{fi}$, and $\mathcal{R}_{fi}$ is the covariance of $\mathbf{e}_{fi}$.

6.34 Starting with the state estimate given in eqn. (6.277), prove that the smoother error covariance is given by eqn. (6.278). How can eqn. (6.278) be used to verify that the smoother error covariance is always less than or equal to the forward-time error covariance?

6.35 ♣ Using the results of §5.7.5, derive error equations for the continuous-time fixed-interval, fixed-point, and fixed-lag smoothers.

6.36 Intrinsic in all smoothing algorithms derived in this chapter is the forward-time Kalman filter. However, a better approach may involve using the Unscented filter shown in §5.7.6 as the forward-time filter. Using the model of a vertically falling body in example 5.9, compare the performance of the RTS nonlinear smoother using the forward-time Kalman filter versus the Unscented filter.

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