8

Optimal Control and Estimation Theory

Technology makes it possible for people to gain control over everything, except over technology. Tudor, John

THE optimal estimation foundations and applications of Chapters 2 through 7 are rooted in probability theory. Although the optimal algorithms derived in these chapters can be implemented solely for estimation and filtering applications, they are oftentimes used in control applications as well. For example, the Kalman filter is typically used to provide optimal estimates of state variables that are implemented in a control algorithm to guide a dynamic system along a desired trajectory. A practical scenario of this concept involves using the α-β filter to provide optimal position and rate estimates from position measurements only, which are required for a proportional-derivative controller. If the rate estimates are adequate then a rate hardware sensor may not be needed, which may produce significant cost savings.

The overall pointing error of a dynamic system inherently encompasses both estimation *and* control errors, which can occur from either hardware or algorithmic inaccuracies (or even both). Estimation errors typically arise from measurement errors (hardware), but may include errors associated with tuning parameters (algorithmic), as discussed in 7.4.1. Control errors typically arise from actuation constraints (hardware), as well as modelling errors (algorithmic). Estimation errors can be quantified using probability theory, but control errors usually cannot. When considering the overall pointing error one must keep in mind a dynamic system can only be controlled to within the accuracy of the estimation algorithm, which exemplifies the need for optimal estimation theory discussed in this book.

It seems natural to assume that control theory and estimation theory are two vastly different notions. However, as surmised in §6.4.1.3, the relationship between control and estimation is not a vague facet at all. In particular, §6.4.1.3 shows a derivation of fixed-interval smoother directly from optimal control theory, which proves the existence of a duality between control and estimation. The present chapter serves to provide the necessary foundations and tools of optimal control theory, which can be used to control a dynamic system to a desired point, and to follow a derived trajectory. Also, this theory can be used to fully comprehend the duality between control and estimation.

We begin by showing the most fundamental foundation in optimal control theory, called the *calculus of variations*. Then, Pontryagin's necessary conditions are presented, which can be used for non-smooth control inputs. The linear quadratic regulator is next shown, which provides an algorithm for an optimal controller of a system by minimizing a quadratic loss function using full state knowledge. We follow this theory with the linear quadratic-Gaussian controller, which incorporates the Kalman filter for state estimation. Finally, an example involving spacecraft attitude control is shown to demonstrate the practical aspects of the combined control and estimation theory.

8.1 Calculus of Variations

Modern optimal control theory has its roots in the calculus of variations, a subject placed upon the solid foundations during the 1800s by the monumental works of Lagrange, Hamilton, and Jacobi. Variational calculus was motivated directly by the apparent existence of minimum principles and other variational laws (e.g., Hamilton's principle) in analytical dynamics. In this section we develop the fundamental concepts of calculus of variations and optimal control in a fashion that encompasses a very large class of dynamic systems.

A fundamental class of variational problems seeks an optimum space-time path $\mathbf{x}(t)$ that minimizes (or maximizes) the following loss function:

$$\boxed{J \equiv J(\mathbf{x}(t), t_0, t_f) = \int_{t_0}^{t_f} \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)\, dt} \tag{8.1}$$

with $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}^T$. Without loss in generality, we assume our task is to minimize eqn. (8.1). It is evident that a simple change of sign converts a maximization problem to a minimization problem.

To obtain the most fundamental classical results, we restrict initial attention to ϑ and $\mathbf{x}$ of class C_2 (smooth, continuous functions having two continuous derivatives with respect to all arguments). Let $\mathbf{x}(t)$, t_0, and t_f represent the unknown path, and start and stop times, respectively, for which J of eqn. (8.1) has a local minimum value. Let an arbitrary neighboring, generally suboptimal, path be denoted by $\bar{\mathbf{x}}(t)$, with neighboring terminal times $\bar{t}_0$ and $\bar{t}_f$. We restrict the *varied path* $\bar{\mathbf{x}}(t)$ to be of class C_2 and to be near $\mathbf{x}(t)$ in the sense that the path variation

$$\delta\mathbf{x}(t) = \bar{\mathbf{x}}(t) - \mathbf{x}(t) \tag{8.2}$$

is of differential size for $\bar{t}_0 \le t \le \bar{t}_f$. We can consider $\bar{\mathbf{x}}(t)$ and $\dot{\bar{\mathbf{x}}}(t)$ to be generated by small arbitrary variations $\delta\mathbf{x}(t)$ of class C_2 as

$$\bar{\mathbf{x}}(t) = \mathbf{x}(t) + \delta\mathbf{x}(t) \tag{8.3a}$$

$$\dot{\bar{\mathbf{x}}}(t) = \dot{\mathbf{x}}(t) + \delta\dot{\mathbf{x}}(t) \tag{8.3b}$$

Clearly $\delta\dot{\mathbf{x}}(t) = \dot{\bar{\mathbf{x}}}(t) - \dot{\mathbf{x}}(t)$ is continuous, since both $\mathbf{x}(t)$ and $\bar{\mathbf{x}}(t)$ are continuous.

Along the varied path $\bar{\mathbf{x}}(t)$ initiating at time $\bar{t}_0 = t_0 + \delta t_0$ and terminating at $\bar{t}_f = t_f + \delta t_f$, the loss function of eqn. (8.1) has neighboring value

$$\bar{J} \equiv J(\bar{\mathbf{x}}(t), \bar{t}_0, \bar{t}_f) = \int_{\bar{t}_0}^{\bar{t}_f} \vartheta(\mathbf{x}(t) + \boldsymbol{\delta}\mathbf{x}(t), \dot{\mathbf{x}}(t) + \boldsymbol{\delta}\dot{\mathbf{x}}(t), t)\, dt \tag{8.4}$$

We define, for the case of finite $\boldsymbol{\delta}\mathbf{x}(t)$, the *finite variation* of J by differencing eqns. (8.4) and (8.1) as

$$\begin{aligned} \Delta J \equiv \bar{J} - J = &\int_{\bar{t}_0}^{\bar{t}_f} \vartheta(\mathbf{x}(t) + \boldsymbol{\delta}\mathbf{x}(t), \dot{\mathbf{x}}(t) + \boldsymbol{\delta}\dot{\mathbf{x}}(t), t)\, dt \\ &- \int_{t_0}^{t_f} \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)\, dt \end{aligned} \tag{8.5}$$

We restrict our attention to infinitesimal variations $\boldsymbol{\delta}\mathbf{x}(t_f)$ and δt_f only, since the initial state, $\mathbf{x}(t_0)$, and t_0 are usually defined *a priori*. Therefore, eqn. (8.5) reduces down to

$$\begin{aligned} \Delta J = &\int_{t_0}^{t_f} [\vartheta(\mathbf{x}(t) + \boldsymbol{\delta}\mathbf{x}(t), \dot{\mathbf{x}}(t) + \boldsymbol{\delta}\dot{\mathbf{x}}(t), t) - \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)]\, dt \\ &+ \int_{t_f}^{t_f + \delta t_f} \vartheta(\bar{\mathbf{x}}(t), \dot{\bar{\mathbf{x}}}(t), t)\, dt \end{aligned} \tag{8.6}$$

where $\bar{\mathbf{x}}(t) = \mathbf{x}(t) + \boldsymbol{\delta}\mathbf{x}(t)$ and its derivative have been used in eqn. (8.6). Now define the differential *first variation* δJ as the linear part of ΔJ. We find δJ by expanding the first integral of eqn. (8.6) in a Taylor series in $\boldsymbol{\delta}\mathbf{x}(t)$, $\boldsymbol{\delta}\dot{\mathbf{x}}(t)$, and δt_f to be

$$\begin{aligned} \delta J = &\int_{t_0}^{t_f} \left[\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \mathbf{x}^T(t)} \boldsymbol{\delta}\mathbf{x}(t) + \frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)} \boldsymbol{\delta}\dot{\mathbf{x}}(t) \right] dt \\ &+ \vartheta(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f)\, \delta t_f \end{aligned} \tag{8.7}$$

where $\partial\vartheta/\partial\mathbf{x}^T(t)$ and $\partial\vartheta/\partial\dot{\mathbf{x}}^T(t)$ denote row vectors. The second term on the right-hand side of eqn. (8.7) is derived by expanding $\vartheta(\bar{\mathbf{x}}(t_f), \dot{\bar{\mathbf{x}}}(t_f), t_f)$ in a Taylor series as follows

$$\begin{aligned} \vartheta(\bar{\mathbf{x}}(t_f), \dot{\bar{\mathbf{x}}}(t_f), t_f) = &\; \vartheta(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f) \\ &+ \left. \frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \mathbf{x}^T(t)} \right|_{t_f} \boldsymbol{\delta}\mathbf{x}(t_f) \\ &+ \left. \frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)} \right|_{t_f} \boldsymbol{\delta}\dot{\mathbf{x}}(t_f) \end{aligned} \tag{8.8}$$

Substituting eqn. (8.8) into (8.6) yields eqn. (8.7) since $\boldsymbol{\delta}\mathbf{x}(t_f)\,\delta t_f$ and $\boldsymbol{\delta}\dot{\mathbf{x}}(t_f)\,\delta t_f$ represent higher-order terms, which vanish in the first variation.

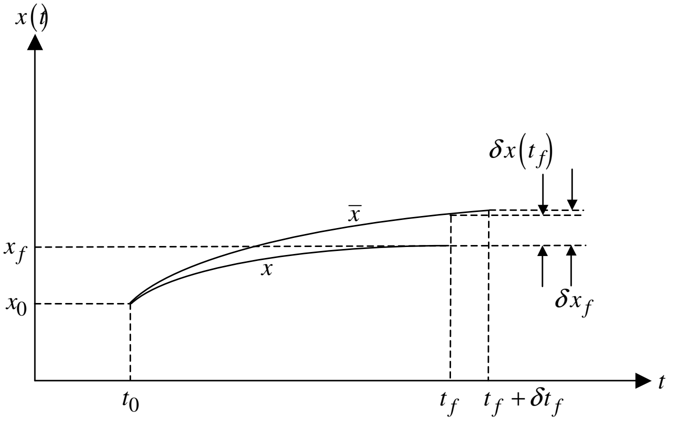

Figure 8.1: An Extremal and an Arbitrary Neighboring Path

In preparation for making arguments on the arbitrariness of $\boldsymbol{\delta}\mathbf{x}(t)$ and δt_f, we seek to eliminate the $\boldsymbol{\delta}\dot{\mathbf{x}}(t)$ term in eqn. (8.7). This is accomplished by using the integration by parts:

$$\int_{t_0}^{t_f} \frac{\partial \vartheta}{\partial \dot{\mathbf{x}}^T(t)} \boldsymbol{\delta}\dot{\mathbf{x}}(t)\, dt = \left. \frac{\partial \vartheta}{\partial \dot{\mathbf{x}}^T(t)} \boldsymbol{\delta}\mathbf{x}(t) \right|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{dt}\left[\frac{\partial \vartheta}{\partial \dot{\mathbf{x}}^T(t)}\right] \boldsymbol{\delta}\mathbf{x}(t)\, dt \tag{8.9}$$

Using eqn. (8.9) to replace the second term in the integrand of eqn. (8.7) yields

$$\begin{aligned} \delta J = &\int_{t_0}^{t_f} \left\{ \frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \mathbf{x}^T(t)} - \frac{d}{dt}\left[\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)}\right]\right\} \boldsymbol{\delta}\mathbf{x}(t)\, dt \\ &+ \left. \frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)} \right|_{t_f} \boldsymbol{\delta}\mathbf{x}(t_f) + \vartheta(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f)\, \delta t_f = 0 \end{aligned} \tag{8.10}$$

Note $\delta t_0 = 0$ since $\mathbf{x}(t_0)$ is assumed to be known. Equation (8.10) is set to zero as a *necessary condition* for J to have a minimum, i.e., we require δJ to vanish for all admissible variations $\boldsymbol{\delta}\mathbf{x}(t)$ and δt_f. As a result the trajectories $\mathbf{x}(t)$ and terminal time t_f satisfying eqn. (8.10) yield a *stationary* value for $J(\mathbf{x}(t), t_0, t_f)$. If both t_f and $\mathbf{x}(t_f)$ are free, a relationship between them still exists. A scalar version of this relationship is demonstrated in Figure 8.1,[1] where δx_f is the difference between the ordinates at the end points. The first-order multidimensional approximation for this relationship is given by

$$\boldsymbol{\delta}\mathbf{x}(t_f) = \boldsymbol{\delta}\mathbf{x}_f - \dot{\mathbf{x}}(t_f)\, \delta t_f \tag{8.11}$$

Substituting eqn. (8.11) into eqn. (8.10) gives

$$
\begin{aligned}
\delta J = & \int_{t_0}^{t_f} \left\{ \frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \mathbf{x}^T(t)} - \frac{d}{dt}\left[\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)}\right]\right\} \boldsymbol{\delta}\mathbf{x}(t)\, dt \\
& + \left.\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)}\right|_{t_f} \boldsymbol{\delta}\mathbf{x}_f \\
& + \left[\vartheta(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f) - \left.\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)}\right|_{t_f} \dot{\mathbf{x}}(t_f)\right] \delta t_f = 0
\end{aligned}
\tag{8.12}
$$

Since $\boldsymbol{\delta}\mathbf{x}(t)$ can assume an infinity of functional values, irrespective of the boundary conditions, we see that the integrand of the first term of eqn. (8.12) must vanish identically. Furthermore, since the boundary variations are generally independent of $\boldsymbol{\delta}\mathbf{x}(t)$, the boundary terms must also vanish independently. Thus eqn. (8.12) leads immediately to the *Euler-Lagrange necessary conditions*:

Euler-Lagrange Equations

$$
\boxed{\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \mathbf{x}(t)} - \frac{d}{dt}\left[\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}(t)}\right] = \mathbf{0}}
\tag{8.13}
$$

Transversality Conditions

$$
\left.\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)}\right|_{t_f} \boldsymbol{\delta}\mathbf{x}_f = 0
\tag{8.14a}
$$

$$
\left[\vartheta(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f) - \left.\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)}\right|_{t_f} \dot{\mathbf{x}}(t_f)\right] \delta t_f = 0
\tag{8.14b}
$$

For example, if the initial and final times are fixed constants, and if the initial and final states are fully prescribed as $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$, then the admissible path variations $\boldsymbol{\delta}\mathbf{x}(t)$ must vanish at t_0 and t_f, and δt_0 and δt_f must vanish as well. Thus for the *fixed time and fixed end point problem*, we find that the transversality conditions of eqn. (8.14) are trivially satisfied and the necessary conditions reduce to the Euler-Lagrange equations of eqn. (8.13) subject to the $2n$ boundary conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$.

For more general boundary condition specifications, the transversality conditions provide replacement or "natural" boundary conditions for terminal variables not constrained to prescribed values. In the simplest such case, a single variable may be totally "free." For example, if the final time t_f is not constrained (and unknown) and $\mathbf{x}(t_f)$ is specified, we must admit δt_f as nonzero and arbitrary. As a result, it is apparent by inspection of the transversality condition on eqn. (8.14b) that the unknown "free" final time is implicitly determined from the generally nonlinear *stopping con-*

dition

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{8.15a}$$

$$\mathbf{x}(t_f) = \mathbf{x}_f \tag{8.15b}$$

$$\vartheta(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f) - \left. \frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)} \right|_{t_f} \dot{\mathbf{x}}(t_f) = 0 \tag{8.15c}$$

If, on the other hand, t_f and $\mathbf{x}(t_f)$ are free and independent, the stopping conditions are given by

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{8.16a}$$

$$\left. \frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}(t)} \right|_{t_f} = \mathbf{0} \tag{8.16b}$$

$$\vartheta(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f) = 0 \tag{8.16c}$$

In §8.2 we will subsequently consider the more general case that the terminal states and time are frequently constrained to lie in a generally nonlinear constraint manifold of the form given by

$$\boldsymbol{\psi}(\mathbf{x}(t_f), t_f) = \mathbf{0} \tag{8.17}$$

where the ψ_j are a set of independent functions of the class C_2.

Notice, in any event, that typically n boundary conditions (i.e., specified boundary conditions and transversality replacement boundary conditions) will be available at time t_0, while the remaining conditions are associated with time t_f. Thus, the terminal boundary conditions on eqn. (8.13) are split, and as a result we have a *two-point-boundary-value-problem* (TPBVP). Equation (8.13) generally provides n second-order nonlinear, stiff differential equations that can usually be solved for the second derivatives in the functional form

$$\ddot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) \tag{8.18}$$

Typically, numerical methods are required to solve eqn. (8.18), even if we have an *initial-value problem* in which $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$ are fully prescribed.[2, 3] Nonlinear TPBVPs are inherently more difficult to solve than nonlinear initial-value problems. In general, iterative numerical methods must be employed in some fashion to solve TPBVPs, where convergence is usually difficult to guarantee *a priori*.

Given a solution, $\mathbf{x}(t)$, of the Euler-Lagrange equations in eqn. (8.18) satisfying the appropriate terminal boundary conditions in eqn. (8.14) and/or $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$, we have a *stationary trajectory*. If this stationary trajectory in fact minimizes (or maximizes) J, we have a local *extremal trajectory*. Analogous to minima-maxima theory in ordinary calculus, a curvature test is required to establish sufficiency for a local minimum (or maximum). Functional curvature of $J[\mathbf{x}(t) + \boldsymbol{\delta}\mathbf{x}(t)]$ is tested using the *second variation*.[4] Since formal sufficiency tests and the second variation play a relatively restricted role in practical applications, we elect not to treat these concepts here. Fortunately, a resourceful analyst can often achieve a high degree of confidence that a candidate trajectory is at least a local minimum, even if a formal sufficiency test proves intractable.

8.2 Optimization with Differential Equation Constraints

We now turn our attention to development of the fundamental results needed for optimal control of nonlinear systems. Suppose we have a system whose behavior is described by solving ordinary differential equations. It is usually possible to arrange the system of differential equations in the standard first-order form

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)} \tag{8.19}$$

The $u_i(t)$ are p control functions of class C_2 that are to be chosen to maneuver the system described by eqn. (8.19) from the prescribed initial state

$$\boxed{\mathbf{x}(t_0) = \mathbf{x}_0, \quad t_0 \text{ fixed}} \tag{8.20}$$

to a generally unspecified final time t_f and final state $\mathbf{x}(t_f)$ satisfying a nonlinear *manifold* system of q algebraic equations of the form given by

$$\boxed{\boldsymbol{\psi}(\mathbf{x}(t_f), t_f) = \mathbf{0}} \tag{8.21}$$

The loss function to be minimized has the form given by

$$\boxed{J = \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \vartheta(\mathbf{x}(t), \mathbf{u}(t), t)\, dt} \tag{8.22}$$

Introducing the two vector of Lagrange multipliers[4, 5] $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\alpha}$ of dimension $n \times 1$ and $q \times 1$, respectively, we form the *augmented functional*

$$\begin{aligned} J &= \phi(\mathbf{x}(t_f), t_f) + \boldsymbol{\alpha}^T \boldsymbol{\psi}(\mathbf{x}(t_f), t_f) \\ &\quad + \int_{t_0}^{t_f} \left\{ \vartheta(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}^T(t)[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)] \right\} dt \end{aligned} \tag{8.23}$$

Considering the neighboring trajectory associated with the variations $\bar{\mathbf{x}}(t) = \mathbf{x}(t) + \boldsymbol{\delta}\mathbf{x}(t)$, $\bar{\mathbf{u}}(t) = \mathbf{u}(t) + \boldsymbol{\delta}\mathbf{u}(t)$, $\bar{t}_f = t_f + \delta t_f$, we find from the linear part of $\Delta J = \bar{J} - J$ that the first variation of J is

$$\begin{aligned} \delta J &= \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial \mathbf{x}(t)} + \dot{\boldsymbol{\lambda}}(t) \right]^T \boldsymbol{\delta}\mathbf{x}(t)\, dt \\ &\quad + \int_{t_0}^{t_f} [\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)]^T \boldsymbol{\delta\lambda}(t)\, dt + \int_{t_0}^{t_f} \frac{\partial H}{\partial \mathbf{u}^T(t)} \boldsymbol{\delta}\mathbf{u}(t)\, dt \\ &\quad + \left[H + \frac{\partial \Phi(\mathbf{x}(t), t)}{\partial t} \right]\Bigg|_{t_f} \delta t_f + \left[\frac{\partial \Phi(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)} - \boldsymbol{\lambda}(t) \right]^T \Bigg|_{t_f} \boldsymbol{\delta}\mathbf{x}(t_f) = 0 \end{aligned} \tag{8.24}$$

where the auxiliary definition of the *Hamiltonian* is

$$\boxed{H \equiv \vartheta(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}^T(t)\,\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)} \tag{8.25}$$

and the augmented terminal function

$$\boxed{\Phi(\mathbf{x}(t_f), t_f) \equiv \phi(\mathbf{x}(t_f), t_f) + \boldsymbol{\alpha}^T \boldsymbol{\psi}(\mathbf{x}(t_f), t_f)} \tag{8.26}$$

It follows, by inspection of the variational statement of eqn. (8.24), that the following necessary conditions hold:

$$\dot{\mathbf{x}}(t) = \frac{\partial H}{\partial \boldsymbol{\lambda}(t)} \equiv \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{8.27a}$$

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial H}{\partial \mathbf{x}(t)} \equiv -\frac{\partial \vartheta(\mathbf{x}(t), \mathbf{u}(t), t)}{\partial \mathbf{x}(t)} - \left[\frac{\partial \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)}{\partial \mathbf{x}(t)}\right]^T \boldsymbol{\lambda}(t) \tag{8.27b}$$

$$\frac{\partial H}{\partial \mathbf{u}(t)} = \mathbf{0} \tag{8.27c}$$

$$\left[\frac{\partial \Phi(\mathbf{x}(t), t)}{\partial t} + H\right]\Bigg|_{t_f} \delta t_f = 0 \tag{8.27d}$$

$$\left[\frac{\partial \Phi(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)} - \boldsymbol{\lambda}(t)\right]^T\Bigg|_{t_f} \boldsymbol{\delta}\mathbf{x}(t_f) = 0 \tag{8.27e}$$

and, of course, the boundary conditions of eqns. (8.20) and (8.21). If the final time is fixed, then $\delta t_f = 0$ and eqn. (8.27d) becomes trivially satisfied. If none of the $\mathbf{x}(t_f)$ are directly specified and the final time is free, conditions of eqns. (8.27d) and (8.27e) provide the transversality conditions

$$\left[\frac{\partial \phi(\mathbf{x}(t), t)}{\partial t} + \boldsymbol{\alpha}^T \frac{\partial \boldsymbol{\psi}(\mathbf{x}(t), t)}{\partial t} + H\right]\Bigg|_{t_f} = 0 \tag{8.28a}$$

$$\boldsymbol{\lambda}(t_f) = \left\{\frac{\partial \phi(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)} + \left[\frac{\partial \boldsymbol{\psi}(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)}\right]^T \boldsymbol{\alpha}\right\}\Bigg|_{t_f} \tag{8.28b}$$

Equation (8.28a) is the "stopping condition" used to implicitly determine the optimal final time. Notice eqn. (8.28b) determines a *final* boundary condition on the costate $\boldsymbol{\lambda}(t_f)$, which must be considered simultaneously with eqn. (8.21) to determine $\boldsymbol{\alpha}$, whereas eqn. (8.20) provides the *initial* condition on the state $\mathbf{x}(t_0)$. Thus the boundary conditions on eqns. (8.27a) and (8.27b) are *split* and we generally have a TPBVP. The algebraic equation provided by eqn. (8.27c) is usually simple enough to solve for $\mathbf{u}(t)$ as a function of $\mathbf{x}(t)$ and $\boldsymbol{\lambda}(t)$, and thereby eliminate $\mathbf{u}(t)$ from eqns. (8.27a) and (8.27b).

8.3 Pontryagin's Optimal Control Necessary Conditions

In many control applications, the above formulation suffers a serious shortcoming; the requirement (limitation!) that the admissible controls $\mathbf{u}(t)$ be smooth functions with two continuous derivatives immediately precludes on/off controls and the (often necessary) imposition of inequality bounds on the control input's magnitude and its derivatives. Several important generalizations of optimal control formulations have made it possible to routinely solve problems with inequality constraints on both the control and state variables.[1, 5]

If we allow admissible controls which are bounded and only piecewise continuous (in lieu of restricting them to belong to class C_2), the necessary conditions generalize in such a way that the only change from the conditions in eqn. (8.27) is the replacement of eqn. (8.27c) by Pontryagin's Principle:[6] *The optimal control* $\mathbf{u}(t)$ *is determined at each instant to render the Hamiltonian a minimum over all admissible control functions*. For example, Pontryagin's Principle requires for controls of class C_2 that eqn. (8.27c) is true *and* $\partial^2 H/\partial \mathbf{u}^2(t)$ must be positive definite. Thus Pontryagin's Principle is consistent with the developments of §8.2, but with the additional constraint that $\partial^2 H/\partial \mathbf{u}^2(t)$ be positive definite.

The most significant utility of Pontryagin's Principle, however, lies in finding optimal controls when the admissible controls *do not* belong to class C_2. For example, suppose we have an optimal maneuver problem of the form given by

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t) + \mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f} \tag{8.29}$$

The loss function to be minimized is given by

$$\boxed{J = \frac{1}{2}\int_{t_0}^{t_f} \mathbf{x}^T(t)\, Q\, \mathbf{x}(t)\, dt} \tag{8.30}$$

where Q is an $n \times n$ positive definite or positive semi-definite matrix. The Hamiltonian for this system is given by

$$\boxed{H = \frac{1}{2}\mathbf{x}^T(t)\, Q\, \mathbf{x}(t) + \boldsymbol{\lambda}^T(t)[\mathbf{f}(\mathbf{x}(t), t) + \mathbf{u}(t)]} \tag{8.31}$$

If $\mathbf{u}(t)$ is of class C_2 then the solution for the optimal control input simply follows the conditions given in eqn. (8.27). However, we are also given that the admissible control inputs must satisfy the constraints

$$\boxed{|u_j(t)| \le u_{\max_j}, \quad j = 1, 2, \ldots, p} \tag{8.32}$$

The necessary conditions of eqns. (8.27a) and (8.27b) are still valid, which give

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t),\, t) + \mathbf{u}(t) \\ \dot{\boldsymbol{\lambda}}(t) &= -\left[\frac{\partial \mathbf{f}(\mathbf{x}(t),\, t)}{\partial \mathbf{x}(t)}\right]^T \boldsymbol{\lambda}(t) - \mathcal{Q}\,\mathbf{x}(t)\end{aligned}} \qquad \begin{aligned}&(8.33\text{a})\\ &(8.33\text{b})\end{aligned}$$

and Pontryagin's Principle requires the Hamiltonian of eqn. (8.31) to be minimized with respect to $\mathbf{u}(t)$ over all admissible control inputs satisfying eqn. (8.32). Since the Hamiltonian contains $\mathbf{u}(t)$ linearly, we know that the extreme of H with respect to $\mathbf{u}(t)$ must lie on the boundary of the region defined by eqn. (8.32). Thus we find that the $\lambda_i(t)$ are *switching functions* for the element $u_i(t)$ of the control input vector $\mathbf{u}(t)$:

$$\boxed{\mathbf{u}(t) = \begin{bmatrix} s_1 u_{\max_1} \\ s_2 u_{\max_2} \\ \vdots \\ s_p u_{\max_p} \end{bmatrix}} \tag{8.34}$$

where

$$\boxed{s_i = \text{sign}[\lambda_i(t)]} \tag{8.35}$$

Equation (8.34) is not valid, however, for the unusual event that one or more of the elements of $\boldsymbol{\lambda}(t)$ vanishes identically for a finite time interval. This latter case of problems is known as *singular* optimal control problems.[3] While the singular optimal control problem is of significant theoretical and some practical interest, we elect not to treat this subject formally here.

Example 8.1: In this example we consider the case of a rigid body constrained to rotate about a fixed axis, where the equation of motion is given by the single axis version of eqn. (3.180):

$$\ddot{\theta}(t) = \frac{1}{J} L(t) \equiv u(t)$$

where $\dot{\theta} \equiv \omega$ from eqn. (3.180) and J is the inertia (see §3.7.2). Suppose we seek a $u(t)$ of class C_2 that maneuvers the body frame from the prescribed initial conditions

$$\begin{aligned}\theta(t_0) &= \theta_0 \\ \dot{\theta}(t_0) &= \dot{\theta}_0\end{aligned}$$

to the desired final conditions

$$\begin{aligned}\theta(t_f) &= \theta_f \\ \dot{\theta}(t_f) &= \dot{\theta}_f\end{aligned}$$

The loss function to be minimized is given by

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)\, dt$$

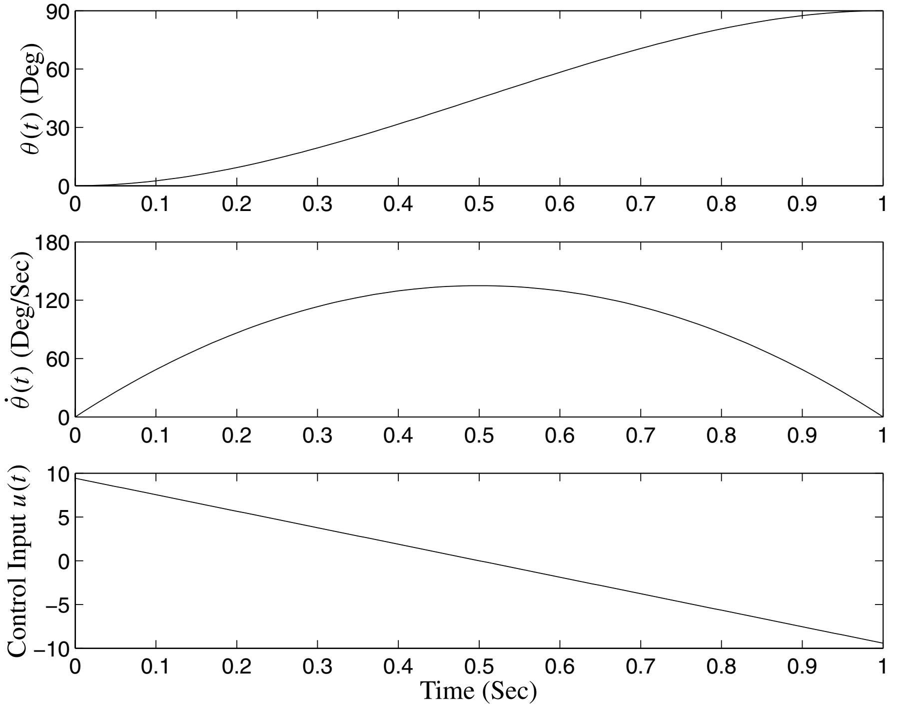

Figure 8.2: Optimal Rest-to-Rest Maneuver for $\ddot{\theta}(t) = u(t)$

where this J is not to be confused with the inertia. We restrict attention to the case that $t_0 = 0$ and $t_f = T$ are fixed. Two methods are considered to derive the optimal maneuver. First we note that direct substitution of the dynamics equation into the loss function yields an equation of the form given by

$$\vartheta(\theta, \dot{\theta}, \ddot{\theta}, t) = \frac{1}{2}\ddot{\theta}^2(t)$$

This form is not identical to the form presented in eqn. (8.1); however, the extension of the Euler-Lagrange equations to higher-order derivatives is straightforward (which is left as an exercise for the reader). For this specific case the Euler-Lagrange equation is given as

$$\frac{d^4\theta(t)}{dt^4} = 0$$

This equation is trivially integrated to obtain the cubic polynomial

$$\theta(t) = a_1 + a_2 t + a_3 t^2 + a_4 t^3$$

as the extremal trajectory.

The four integration constants can be determined as a function of the boundary conditions and the maneuver time T by simply enforcing the boundary conditions on the cubic polynomial equation and its time derivative. The solution of the resulting four algebraic equations gives

$$
\begin{aligned}
a_1 &= \theta_0 \\
a_2 &= \dot{\theta}_0 \\
a_3 &= \frac{3(\theta_f - \theta_0)}{T^2} - \frac{2\dot{\theta}_0 + \dot{\theta}_f}{T} \\
a_4 &= -\frac{2(\theta_f - \theta_0)}{T^3} + \frac{\dot{\theta}_0 + \dot{\theta}_f}{T^2}
\end{aligned}
$$

Furthermore, it is obvious that taking a second time derivative of the cubic polynomial gives the optimal control torque, which is a linear function of time:

$$u(t) = 2a_3 + 6a_4 t$$

As a specific example, consider the following numerical values with $t_0 = 0$ and $T = 1$:

$$
\begin{aligned}
\theta(0) &= 0, \quad \dot{\theta}(0) = 0 \\
\theta(1) &= \pi/2, \quad \dot{\theta}(1) = 0
\end{aligned}
$$

These boundary conditions will yield a *rest-to-rest* maneuver. Using these conditions gives the following control torque:

$$u(t) = \ddot{\theta}(t) = 3\pi(1 - 2t)$$

Also, the maneuver angle, $\theta(t)$, and angular velocity, $\dot{\theta}(t)$, are given by

$$
\begin{aligned}
\theta(t) &= 3\pi\left(t^2/2 - t^3/3\right) \\
\dot{\theta}(t) &= 3\pi\left(t - t^2\right)
\end{aligned}
$$

A plot of the maneuver angle, angular velocity, and control torque is shown in Figure 8.2. Clearly, the initial and final boundary conditions are satisfied with this control torque.

Notice, since we admitted only controls of class C_2, we were able to use the generalized version of Euler-Lagrange's equations in lieu of the Pontryagin-form necessary conditions of §8.2. The constraint in this simple example is enforced by simply substituting it into the loss function directly. In the approach of §8.2, we enforce the differential equation constraints by using the Lagrange multiplier rule. To illustrate the equivalence in the present transparent example, we resolve for the optimal maneuvering using the approach and notations of §8.2.

Before we proceed, it is necessary to convert the dynamics equations $\ddot{\theta}(t) = u(t)$ to the first-order form of eqn. (8.19). This is accomplished by using the change of variables introduced in eqn. (3.3). For the present example the following state variables are introduced:

$$x_1(t) = \theta(t)$$
$$x_2(t) = \dot{\theta}(t)$$

Then the desired equivalent first-order equations follow as

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

The Hamiltonian described in eqn. (8.25) is given by

$$H = \frac{1}{2}u^2(t) + \lambda_1(t)\,x_2(t) + \lambda_2(t)\,u(t)$$

The necessary conditions for the optimal maneuver then follow from eqns. (8.27a) to (8.27c) as

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$
$$\dot{\lambda}_1(t) = 0$$
$$\dot{\lambda}_2(t) = -\lambda_1(t)$$
$$u(t) + \lambda_2(t) = 0$$

The solutions for the costate variables $\lambda_1(t)$ and $\lambda_2(t)$ follow as

$$\lambda_1(t) = b_1 = \text{constant}$$
$$\lambda_2(t) = -b_1 t + b_2$$

Also, the control input follows $u(t) = -\lambda_2(t)$:

$$u(t) = b_1 t - b_2$$

Having $u(t)$ then $x_1(t)$ and $x_2(t)$ are trivially solved to be

$$x_1(t) \equiv \theta(t) = b_4 + b_3 t - b_2 t^2/2 + b_1 t^3/6$$
$$x_2(t) \equiv \dot{\theta}(t) = b_3 - b_2 t + b_1 t^2/2$$

This solution is identical to the previous solution using the Euler-Lagrange approach, with the obvious relationship of the integration constants $b_4 = a_1$, $b_3 = a_2$, $b_2 = -2a_3$, and $b_1 = 6a_4$. For the case of one constraint, i.e., one state variable, and controls of class C_2, it appears that the multiplier rule slightly increased the algebra. For the cases in which constraints can be eliminated by direct substitution and for controls of class C_2 this pattern is typical. However, such ideal circumstances represent the minority of applications. Implicit, nonlinear constraints, nonlinear differential

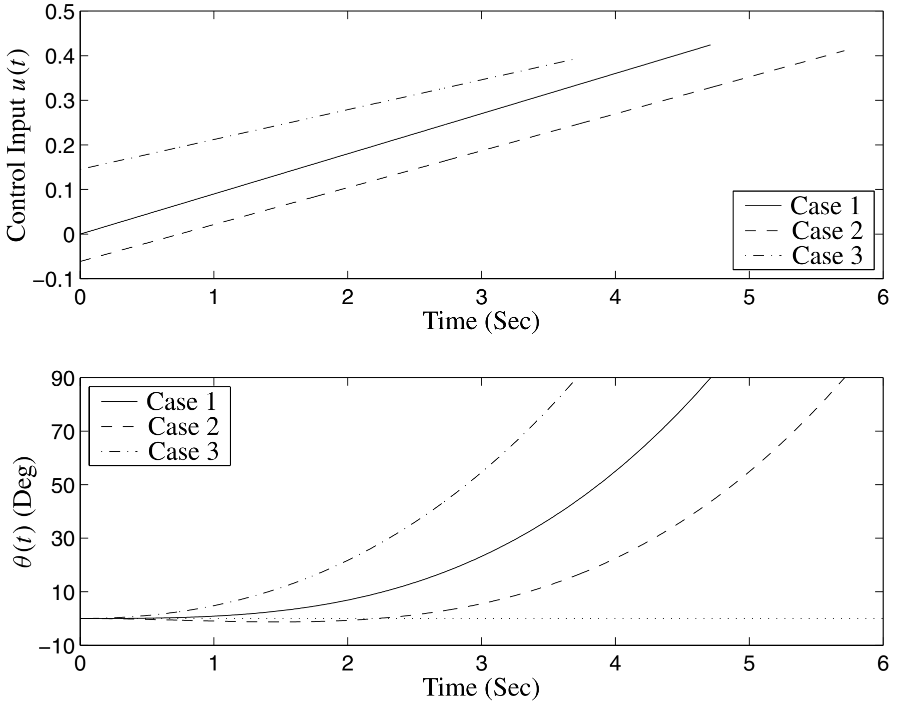

Figure 8.3: Spinup Maneuver: Effect of Final Time Variation

equations, and discontinuous controls abound in modern-day applications. For these cases, the introduction of Lagrange multipliers and the use of Pontryagin-form necessary conditions have been found to be advantageous.

For the case that the final time T is free, we have from eqn. (8.27d) the stopping condition $H(T) = 0$, which leads to

$$H(T) = -\frac{2}{T^4}(a\,T^2 + b\,T + c) = 0$$

where

$$\begin{aligned} a &= \dot\theta_0^2 + \dot\theta_0\dot\theta_f + \dot\theta_f^2 \\ b &= 6(\theta_0 - \theta_f)(\dot\theta_0 + \dot\theta_f) \\ c &= 9(\theta_f - \theta_0)^2 \end{aligned}$$

Thus, there are three final times for which $H(T) = 0$:

$$T_1^* = \infty, \quad T_{2,3}^* = \frac{3(\theta_f - \theta_0)\left[\dot\theta_0 + \dot\theta_f \pm \sqrt{\dot\theta_0\dot\theta_f}\right]}{\dot\theta_0^2 + \dot\theta_0\dot\theta_f + \dot\theta_f^2}$$

The value of $T_1^* = \infty$ corresponds to the global optimal free time, whereas T_2^* and T_3^*, when real, are local maxima or minima of J, at finite times; these have some significance in practical applications. It is obvious by inspection of the final time conditions that for the rest-to-rest case ($\dot{\theta}_0 = \dot{\theta}_f = 0$) the only zero of $H(T)$ is $T = \infty$. Thus, the optimum rest-to-rest maneuvers are carried out very slowly. Furthermore, consider the special cases of maneuvers for which $\dot{\theta}_0 = 0$, which cause the discriminant in the solution for $T_{2,3}^*$ to vanish, and we have a double root:

$$T^* = T_2^* = T_3^* = \frac{3(\theta_f - \theta_0)}{\dot{\theta}_f}$$

This causes an inflection at $J(T)$. For $\theta_f = \pi/2$, $\theta_0 = 0$ and $\dot{\theta}_f = 1$, i.e., a spinup maneuver, we show in Figure 8.3 trajectories for the following three cases:

Case 1: $T = T^* = 3\pi/2 = 4.7124$
Case 2: $T = T^* - 1 = 3.7124$ ($T < T^*$)
Case 3: $T = T^* + 1 = 5.7124$ ($T > T^*$)

From Figure 8.3, it is evident that fixing the final time greater than T^* has the undesirable consequence that θ initially counter rotates (e.g., Case 3). The performance, as measured by J, is actually slightly less for Case 3 than for Case 1. This example illustrates that counterintuitive and undesirable results sometimes stem from "optimal" control developments.

If *both* initial and final rates ($\dot{\theta}_0$ and $\dot{\theta}_f$) are zero, the inflection of J disappears, and the only zero of $H(T)$ occurs at $T = \infty$. The global minimum of J is zero and is approached as the maneuver time approaches infinity. The optimal control, angular velocity, and angle of rotation profiles (for this rest-to-rest class of maneuvers) are all completely analogous to the maneuver shown in Figure 8.2.

We should note that the *open-loop* approaches for the solution of optimal control problems shown in §8.1 and §8.2 are not generally robust to parametric variations, unlike *feedback control* methods. This is easily illustrated by multiplying the control torque $u(t)$ in example 8.1 by some scalar, which simulates an error in the inertia J, and using this control input with the identical boundary conditions shown in the example. This will yield suboptimal results for various scalar multiplication factors (which is left as an exercise for the reader to investigate).

8.4 Discrete-Time Control

The importance of discrete-time systems, described in §3.5, is well known with the reliance on digital computers, which are used to process sampled-data systems for estimation and control purposes. As discussed in §8.3, the Lagrange multiplier approach with the use of Pontryagin-form necessary conditions is better suited for modern-day problems. Hence, we only present this approach for the optimal control theory involving discrete-time systems. A more thorough treatise involving the discrete-time Euler-Lagrange equations and associated transversality conditions can be found in Refs. [2] and [3]. Consider finding a control sequence $\mathbf{u}_0, \ldots, \mathbf{u}_{N-1}$ and final time t_f that minimizes the following loss function:

$$\boxed{J = \phi(\mathbf{x}_N, t_f) + \sum_{k=0}^{N-1} \vartheta_k(\mathbf{x}_k, \mathbf{u}_k, k)} \tag{8.36}$$

subject to the constraints

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, k) \tag{8.37a}$$

$$\psi(\mathbf{x}_N, t_f) = \mathbf{0} \tag{8.37b}$$

with $t_f = N\Delta t$, where N is the total number of steps and Δt is the time-step. As in §8.2 we assume that the initial state and time are fixed and known, so that $\mathbf{x}(t_0) = \mathbf{x}_0$ and t_0 is fixed. The augmented functional is formed by introducing two Lagrange multipliers, $\boldsymbol{\lambda}_{k+1}$ and $\boldsymbol{\alpha}$, of dimension $n \times 1$ and $q \times 1$, respectively:

$$\begin{aligned} J &= \phi(\mathbf{x}_N, t_f) + \boldsymbol{\alpha}^T \psi(\mathbf{x}_N, t_f) \\ &+ \sum_{k=0}^{N-1} \vartheta_k(\mathbf{x}_k, \mathbf{u}_k, k) + \boldsymbol{\lambda}_{k+1}^T \left[\mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, k) - \mathbf{x}_{k+1}\right] + \boldsymbol{\lambda}_0^T [\mathbf{x}_0 - \mathbf{x}(t_0)] \end{aligned} \tag{8.38}$$

As with the continuous-time development we introduce the following Hamiltonian and augmented terminal function:

$$H_k \equiv \vartheta_k(\mathbf{x}_k, \mathbf{u}_k, k) + \boldsymbol{\lambda}_{k+1}^T \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, k) \tag{8.39a}$$

$$\Phi(\mathbf{x}_N, t_f) \equiv \phi(\mathbf{x}_N, t_f) + \boldsymbol{\alpha}^T \psi(\mathbf{x}_N, t_f) \tag{8.39b}$$

Changing indices of summation on the last term in eqn. (8.38) yields[3, 5]

$$J = \Phi(\mathbf{x}_N, t_f) - \boldsymbol{\lambda}_N^T \mathbf{x}_N + \sum_{k=0}^{N-1} \left[H_k - \boldsymbol{\lambda}_k^T \mathbf{x}_k\right] + \boldsymbol{\lambda}_0^T \mathbf{x}_0 \tag{8.40}$$

Similar to the steps leading to eqn. (8.27), taking the first variation of eqn. (8.40) leads to the following conditions:

$$\boxed{\begin{aligned} \mathbf{x}_{k+1} &= \frac{\partial H_k}{\partial \boldsymbol{\lambda}_{k+1}} \equiv \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, k) && (8.41a)\\ \boldsymbol{\lambda}_k &= \frac{\partial H_k}{\partial \mathbf{x}_k} \equiv \frac{\partial \vartheta_k(\mathbf{x}_k, \mathbf{u}_k, k)}{\partial \mathbf{x}_k} + \left[\frac{\partial \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, k)}{\partial \mathbf{x}_k}\right]^T \boldsymbol{\lambda}_{k+1} && (8.41b)\\ &\frac{\partial H_k}{\partial \mathbf{u}_k} = \mathbf{0} && (8.41c)\\ &\left[\frac{\partial \Phi(\mathbf{x}_k, t_f)}{\partial \Delta t} + \sum_{k=0}^{N-1} \frac{\partial H_k}{\partial \Delta t}\right] \delta \Delta t = 0 && (8.41d)\\ &\left[\frac{\partial \Phi(\mathbf{x}_k, t_f)}{\partial \mathbf{x}_k} - \boldsymbol{\lambda}_k\right]^T \Bigg|_N \boldsymbol{\delta}\mathbf{x}_N = 0 && (8.41e) \end{aligned}}$$

and, of course, the boundary conditions of $\mathbf{x}(t_0) = \mathbf{x}_0$ and eqn. (8.37b). If none of the $\mathbf{x}_N$ are directly specified and the final time is free, conditions of eqns. (8.41d) and (8.41e) provide the transversality conditions

$$\boxed{\begin{aligned} &\frac{\partial \Phi(\mathbf{x}_k, t_f)}{\partial \Delta t} + \sum_{k=0}^{N-1} \frac{\partial H_k}{\partial \Delta t} = 0 && (8.42a)\\ &\boldsymbol{\lambda}_N = \left\{\frac{\partial \phi(\mathbf{x}_k, t_f)}{\partial \mathbf{x}_k} + \left[\frac{\partial \psi(\mathbf{x}_k, t_f)}{\partial \mathbf{x}_k}\right]^T \boldsymbol{\alpha}\right\}\Bigg|_N && (8.42b) \end{aligned}}$$

As with the continuous-time formulation, eqn. (8.42a) is the stopping condition used to implicitly determine the optimal final time through the determination of the optimal time step Δt.

8.5 Linear Regulator Problems

The formulations of the foregoing developments naturally lead to *open-loop* optimal controls that are designed to calculate an optimal trajectory from a prescribed initial state to a prescribed final state. Such controls can be pre-computed, under the assumption of perfectly known initial conditions. However, upon application of open-loop controls to a real system, even small modelling errors and initial state errors result in usually unacceptable divergence of the actual system's behavior from the optimal trajectory. In many cases *perturbation feedback controls* need to be superimposed (à la "guidance" in rocket flight path control) to continually correct for model errors and other disturbances.

In some cases, we will see that it is possible to formulate optimal controls so that they can be calculated directly in a *terminal controller* feedback form:

$$\mathbf{u}(t) = \mathbf{f}[\mathbf{x}(t) - \mathbf{x}(t_f), t_f - t] \tag{8.43}$$

in which the optimal control is a function of instantaneous displacement from the desired final state and the "time-to-go" $\tau = t_f - t$. Such controls are of enormous practical impact, since we are, in essence, continuously re-initializing the control calculations with current best estimates of $\mathbf{x}(t)$, from a Kalman filter for example, which can be updated continuously based upon measurements (and thereby counteract the accumulation of ever-present errors due to an erroneous model and other disturbances). In this section we develop one such case for linear time-invariant models belonging to the class of linear regulator problems.

8.5.1 Continuous-Time Formulation

In this section the continuous-time linear quadratic regulator (LQR) problem is solved using Bellman's *Principle of Optimality*[7] and directly from the Hamiltonian formulation of §8.2. If we initiate at an arbitrary start point $[\mathbf{x}(t), t]$, the cost-to-go for an arbitrary control $\mathbf{u}(t)$ is given by

$$J = \phi(\mathbf{x}(t_f), t_f) + \int_t^{t_f} \vartheta(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau)\, d\tau \tag{8.44}$$

Note that unlike eqn. (8.22), the integration is over the interval t to t_f. We are concerned only with trajectories that satisfy the differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{8.45}$$

and satisfy the terminal constraints

$$\boldsymbol{\psi}(\mathbf{x}(t_f), t_f) = \mathbf{0} \tag{8.46}$$

In §8.2 we developed the necessary conditions for minimizing eqn. (8.44) subject to $\mathbf{x}(t)$ being on a trajectory of eqn. (8.45) satisfying the prescribed boundary conditions. The principle of optimality is concerned with the instantaneous time-to-go $t_f - t$ rather than the fixed $t_f - t_0$ interval. The principle of optimality states that J must be a minimum over every subinterval of the time Δt, satisfying $t_f \geq t + \Delta t \geq t_0$, along an optimal trajectory. Having stated this principle, it seems obviously true that we do not concern ourselves with a formal proof. Clearly, if an optimal control had been employed everywhere *except* during the interval from t to $t + \Delta t$ the only way to minimize J of eqn. (8.44) is to choose $\mathbf{u}(t)$ to minimize J over the interval Δt in question.

The optimal control is implicitly defined by the requirement that it yields the minimum cost-to-go which we denote by

$$J^*(\mathbf{x}(t), t) = \min_{\mathbf{u}(t)} \left\{ \phi(\mathbf{x}(t_f), t_f) + \int_t^{t_f} \vartheta(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau)\, d\tau \right\} \tag{8.47}$$

Notice that $J = J(\mathbf{x}(t), \mathbf{u}(t), t)$ in eqn. (8.44), along with a non-optimal trajectory, but $J^* = J^*(\mathbf{x}(t), t)$ upon carrying out the minimization of eqn. (8.44) over all admissible controls $\mathbf{u}(t)$.

In order to develop an important partial differential equation, we now investigate eqn. (8.47) locally. Suppose optimal control is used everywhere on the interval (t, t_f) *except* during the initial Δt where a non-optimal $\mathbf{u}(t)$ is employed. For Δt sufficiently small, the system will be displaced from $[\mathbf{x}(t), t]$ to a neighboring point $[\mathbf{x}(t)+\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\,\Delta t, t+\Delta t]$. Now suppose from these perturbed initial conditions an optimal control is employed; it is apparent that the perturbed cost-to-go is

$$\tilde{J}^*(\mathbf{x}(t), t) = J^*[\mathbf{x}(t)+\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\,\Delta t, t+\Delta t] + \vartheta(\mathbf{x}(t), \mathbf{u}(t), t)\,\Delta t \tag{8.48}$$

Since $\mathbf{u}(t)$ over the interval Δt is generally non-optimal it is clear that

$$\tilde{J}^*(\mathbf{x}(t), t) \geq J^*(\mathbf{x}(t), t) \tag{8.49}$$

The equality holds only if we choose $\mathbf{u}(t)$ to minimize eqn. (8.48). Thus

$$J^*(\mathbf{x}(t), t) = \min_{\mathbf{u}(t)} \left\{ J^*[\mathbf{x}(t)+\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)\,\Delta t, t+\Delta t] + \vartheta(\mathbf{x}(t), \mathbf{u}(t), t)\,\Delta t \right\} \tag{8.50}$$

Upon expanding in Taylor's series and taking the limit as $\Delta t \to 0$,[1] eqn. (8.50) leads directly to the partial differential equation

$$\frac{\partial J^*(\mathbf{x}(t), t)}{\partial t} + \min_{\mathbf{u}(t)} \left\{ \vartheta(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{\partial J^*(\mathbf{x}(t), \mathbf{u}(t), t)}{\partial \mathbf{x}^T(t)} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \right\} = 0 \tag{8.51}$$

Comparison of eqn. (8.51) with eqn. (8.25) reveals that eqn. (8.51) can be written as the *Hamilton-Jacobi-Bellman* (HJB) equation:

$$\frac{\partial J^*(\mathbf{x}(t), t)}{\partial t} + \min_{\mathbf{u}(t)} \left\{ H\left(\mathbf{x}(t), \frac{\partial J^*(\mathbf{x}(t), \mathbf{u}(t), t)}{\partial \mathbf{x}(t)}, \mathbf{u}(t), t \right) \right\} = 0 \tag{8.52}$$

where the costate is defined by

$$\boldsymbol{\lambda}(t) = \frac{\partial J^*(\mathbf{x}(t), \mathbf{u}(t), t)}{\partial \mathbf{x}(t)} \tag{8.53}$$

The significance of finding a globally valid analytical solution of the HJB equation for $J^* = J^*(\mathbf{x}(t), t)$ is that the solution of the Lagrange multiplier $\boldsymbol{\lambda}(t)$ is reduced to taking the gradient of J^*. This immediately allows determination of the corresponding optimal control from Pontryagin's Principle, *in feedback form*.

Unfortunately obtaining such global analytical solutions of the HJB equation can only be accomplished for special cases. The most important special case for which the HJB equation is solvable is the *linear quadratic regulator* for which we seek to minimize

$$J = \frac{1}{2}\mathbf{x}^T(t_f) S_f \mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)\,dt \tag{8.54}$$

where S_f, $\mathcal{Q}(t)$ and $\mathcal{R}(t)$ are symmetric, non-negative weight matrices, subject to the constraint

$$\boxed{\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0} \tag{8.55}$$

The HJB equation of eqn. (8.52) for this case becomes

$$\begin{aligned} \frac{\partial J^*}{\partial t} + \min_{\mathbf{u}(t)} \Bigg\{ \frac{1}{2} \left[\mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t) \right] \\ + \frac{\partial J^*}{\partial \mathbf{x}^T(t)} \left[F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) \right] \Bigg\} = 0 \end{aligned} \tag{8.56}$$

Carrying out the minimization over $\mathbf{u}(t)$ of eqn. (8.56) yields

$$\mathbf{u}(t) = -\mathcal{R}^{-1}(t)\,B^T(t)\,\frac{\partial J^*}{\partial \mathbf{x}(t)} \tag{8.57}$$

Thus the HJB equation of eqn. (8.56) becomes

$$\begin{aligned} \frac{\partial J^*}{\partial t} + \frac{1}{2}\frac{\partial J^*}{\partial \mathbf{x}^T(t)} F(t)\,\mathbf{x}(t) + \frac{1}{2}\mathbf{x}^T(t)\,F^T(t)\frac{\partial J^*}{\partial \mathbf{x}(t)} \\ + \frac{1}{2}\mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t) - \frac{1}{2}\frac{\partial J^*}{\partial \mathbf{x}^T(t)} B(t)\,\mathcal{R}^{-1}(t)\,B^T(t)\frac{\partial J^*}{\partial \mathbf{x}(t)} = 0 \end{aligned} \tag{8.58}$$

It can be verified by direct substitution (which is left as an exercise for the reader) that the general solution of the HJB equation of eqn. (8.58) is the quadratic form

$$J^*(\mathbf{x}(t), t) = \frac{1}{2}\mathbf{x}^T(t)\,S(t)\,\mathbf{x}(t) \tag{8.59a}$$

$$\frac{\partial J^*}{\partial \mathbf{x}(t)} = S(t)\,\mathbf{x}(t) \tag{8.59b}$$

$$\frac{\partial J^*}{\partial t} = \frac{1}{2}\mathbf{x}^T(t)\,\dot{S}(t)\,\mathbf{x}(t) \tag{8.59c}$$

where $S(t)$ is a positive definite matrix satisfying the *matrix Riccati equation*

$$\boxed{\dot{S}(t) = -S(t)\,F(t) - F^T(t)\,S(t) + S(t)\,B(t)\,\mathcal{R}^{-1}(t)\,B^T(t)\,S(t) - \mathcal{Q}(t)} \tag{8.60}$$

with the terminal boundary condition

$$S(t_f) = S_f \tag{8.61}$$

Since we gave eqns. (8.53) and (8.57), the optimal control is thus obtained globally in the *time-varying linear feedback* form

$$\boxed{\mathbf{u}(t) = -L(t)\,\mathbf{x}(t)} \tag{8.62}$$

where the *optimal gain matrix* is

$$\boxed{L(t) = \mathcal{R}^{-1}(t)\,B^T(t)\,S(t)} \tag{8.63}$$

Table 8.1: Continuous-Time Linear Quadratic Regulator

Model	$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$
Gain	$L(t) = \mathcal{R}^{-1}(t)\,B^T(t)\,S(t)$
Riccati Equation	$\dot{S}(t) = -S(t)\,F(t) - F^T(t)\,S(t)$ $+S(t)\,B(t)\,\mathcal{R}^{-1}(t)\,B^T(t)\,S(t) - \mathcal{Q}(t), \quad S(t_f) = S_f$
Control Input	$\mathbf{u}(t) = -L(t)\,\mathbf{x}(t)$

Note the similarity between the formulation presented here and the continuous-time Kalman filter in Table 5.4, which leads to the duality results of §6.4.1. A summary of the continuous-time LQR is shown in Table 8.1. Once the gain matrices $\mathcal{R}(t)$ and $\mathcal{Q}(t)$ are chosen, the matrix Riccati solution in eqn. (8.60) is integrated backward in time with boundary conditions given by eqn. (8.61). Storing the entire matrix $S(t)$ over all time, the gain matrix in eqn. (8.63) is then calculated. Finally, eqn. (8.55) is integrated forward in time with the known initial state condition.

The stability of the LQR controller can be proved by using Lyapunov's direct method, which is discussed for continuous-time systems in §3.6. The closed-loop dynamics are given by substituting eqn. (8.62) into eqn. (8.55), which leads to

$$\dot{\mathbf{x}}(t) = \left[F(t) - B(t)\,\mathcal{R}^{-1}(t)\,B^T(t)\,S(t)\right]\mathbf{x}(t) \tag{8.64}$$

We consider the following candidate Lyapunov function:

$$V[\mathbf{x}(t)] = \mathbf{x}^T(t)\,S(t)\,\mathbf{x}(t) \tag{8.65}$$

Taking the time derivative of eqn. (8.65) yields

$$\dot{V}[\mathbf{x}(t)] = \dot{\mathbf{x}}^T(t)\,S(t)\,\mathbf{x}(t) + \mathbf{x}^T(t)\,S(t)\,\dot{\mathbf{x}}(t) + \mathbf{x}^T(t)\,\dot{S}(t)\,\mathbf{x}(t) \tag{8.66}$$

Substituting eqns. (8.60) and (8.64) into eqn. (8.66), and simplifying leads to

$$\dot{V}[\mathbf{x}(t)] = -\mathbf{x}^T(t)\left[S(t)\,B(t)\,\mathcal{R}^{-1}(t)\,B^T(t)\,S(t) + \mathcal{Q}(t)\right]\mathbf{x}(t) \tag{8.67}$$

Clearly if $\mathcal{R}(t)$ is positive definite and $\mathcal{Q}(t)$ is at least positive semi-definite then the Lyapunov condition is satisfied and LQR controller is stable.

In order to implement the control input given by eqn. (8.62), we first must integrate eqn. (8.60) *backward* in time and store matrix $S(t)$ at all times. For the case that all system and weight matrices are constant, and $t_f \to \infty$ in eqn. (8.54), it can be shown (for a controllable system[5, 8]) that $S(t)$ approaches the constant positive semi-definite solution of the algebraic Riccati equation (ARE) given by

$$S\,F + F^T S - S\,B\,\mathcal{R}^{-1} B^T S + \mathcal{Q} = 0 \tag{8.68}$$

Thus, eqns. (8.62) and (8.63) provide a constant gain feedback control that can be implemented in *real time*. The solution of the ARE in eqn. (8.68) can be found by employing the methods of §5.4.4. First, we define the following Hamiltonian matrix:

$$\mathcal{H} \equiv \begin{bmatrix} F & -B\mathcal{R}^{-1}B^T \\ -\mathcal{Q} & -F^T \end{bmatrix} \tag{8.69}$$

The eigenvalues of $\mathcal{H}$ can be arranged in a diagonal matrix given by

$$\mathcal{H}_\Lambda = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} \tag{8.70}$$

where Λ is a diagonal matrix of the n eigenvalues in the right half-plane. Assuming that the eigenvalues are distinct, we can perform a linear state transformation, as shown in §3.1.4, such that

$$\mathcal{H}_\Lambda = W^{-1}\mathcal{H}\,W \tag{8.71}$$

where W is the matrix of eigenvectors, which can be represented in block form as

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{8.72}$$

Going backward in time the stable eigenvalues dominate, which leads to the following solution for S at steady-state:

$$\boxed{S = W_{22}W_{12}^{-1}} \tag{8.73}$$

It is important to note that all states must be observed in order to implement the LQR controller in real time. Unfortunately, this is rarely the case in practice. However, an estimator, such as the Kalman filter, is often employed to provide state estimates for the unmeasured states, which will be discussed in §8.6.

The Riccati solution for the LQR problem can be derived another way. The Hamiltonian of eqn. (8.25) for the minimization problem shown by eqns. (8.54) and (8.55) is given by

$$H = \frac{1}{2}\left[\mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)\right] + \boldsymbol{\lambda}^T(t)\,[F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t)] \tag{8.74}$$

From the necessary conditions of eqn. (8.27) the following equations must be satisfied:

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{8.75a}$$

$$\dot{\boldsymbol{\lambda}}(t) = -F^T(t)\,\boldsymbol{\lambda}(t) - \mathcal{Q}(t)\,\mathbf{x}(t) \tag{8.75b}$$

$$\mathbf{u}(t) = -\mathcal{R}^{-1}(t)\,B^T(t)\,\boldsymbol{\lambda}(t) \tag{8.75c}$$

$$\boldsymbol{\lambda}(t_f) = S_f\,\mathbf{x}(t_f) \tag{8.75d}$$

where eqn. (8.27e) has been used to derive eqn. (8.75d). Suppose we assume that the solution for the costate $\boldsymbol{\lambda}(t)$ follows the form of eqn. (8.75d) for all time, which seems to be a reasonable assumption due to the linearity of the system. Hence, we assume

$$\boldsymbol{\lambda}(t) = S(t)\,\mathbf{x}(t) \tag{8.76}$$

Taking the time derivative of eqn. (8.76) gives

$$\dot{\boldsymbol{\lambda}}(t) = \dot{S}(t)\,\mathbf{x}(t) + S(t)\,\dot{\mathbf{x}}(t) = -F^T(t)\,\boldsymbol{\lambda}(t) - \mathcal{Q}(t)\,\mathbf{x}(t) \tag{8.77}$$

where eqn. (8.75b) has been used in eqn. (8.77). Substituting eqn. (8.75c) into eqn. (8.75a) gives

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) - B(t)\,\mathcal{R}^{-1}(t)\,B^T(t)\,\boldsymbol{\lambda}(t) \tag{8.78}$$

Now, substituting eqn. (8.76) into eqn. (8.78) gives

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) - B(t)\,\mathcal{R}^{-1}(t)\,B^T(t)\,S(t)\,\mathbf{x}(t) \tag{8.79}$$

Finally, substituting eqns. (8.76) and (8.79) into eqn. (8.77) and collecting terms yields

$$\left[\dot{S}(t) + S(t)\,F(t) + F^T(t)\,S(t) - S(t)\,B(t)\,\mathcal{R}^{-1}(t)\,B^T(t)\,S(t) + \mathcal{Q}(t)\right]\mathbf{x}(t) = \mathbf{0} \tag{8.80}$$

Since eqn. (8.80) must hold for all nonzero $\mathbf{x}(t)$, then the term within the brackets pre-multiplying $\mathbf{x}(t)$ must be zero, which leads directly to eqn. (8.60). Also, substituting eqn. (8.76) into eqn. (8.75c) leads directly to eqn. (8.62).

Example 8.2: In this example we wish to apply the LQR approach to asymptotically control the following linear time-invariant system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbf{u}(t)$$

Note that this system is unstable, with eigenvalues given by $\lambda_{12} = 1 \pm j$. The weighting matrices for the control design are chosen to be $\mathcal{R} = 0.1$ and $\mathcal{Q} = I_{2\times 2}$. Since this system is time-invariant we choose to employ the steady-state feedback gain approach, which allows for real-time implementation. Solving the steady-state ARE in eqn. (8.68) and the steady-state gain in eqn. (8.63) gives

$$S = \begin{bmatrix} 1.9645 & 0.1742 \\ 0.1742 & 0.6181 \end{bmatrix}, \quad L = \begin{bmatrix} 1.7417 & 6.1813 \end{bmatrix}$$

The eigenvalues of the closed-loop system, $F - B\,L$, are given by $\lambda_1 = -1.2974$ and $\lambda_2 = -2.8839$, which yield a stable closed-loop response as expected. A plot of the closed-loop response is shown in Figure 8.4. Clearly, the states approach zero. The weighting matrices dictate the characteristics of the closed-loop response. In general

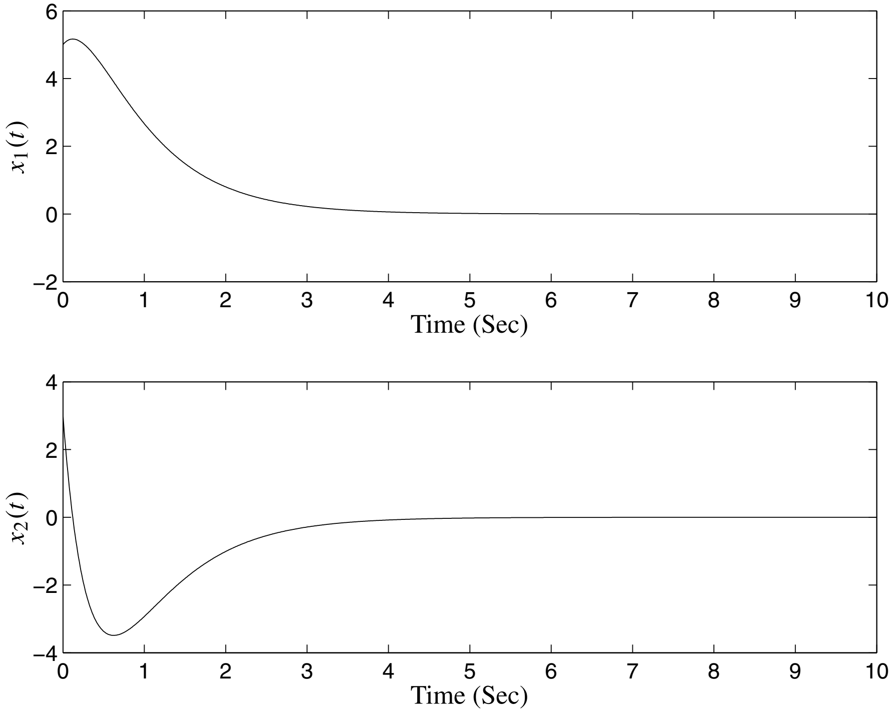

Figure 8.4: Linear Quadratic Regulator Control Example

as $\mathcal{Q}$ is increased, the faster the response time of the closed-loop system, but this comes at the price of a larger control gain. This also occurs as $\mathcal{R}$ is decreased. In a scalar sense it is the ratio of $\mathcal{Q}$ and $\mathcal{R}$ that is important in the final LQR design.

8.5.2 Discrete-Time Formulation

In this section the discrete-time linear quadratic regulator problem is solved using the Hamiltonian formulation of §8.4. The HJB equation can be extended to discrete-time systems, but this is beyond the scope of the present text. Here, we will focus our attentions only on the final discrete-time LQR solution form obtained through a Riccati transformation. Consider the minimization of the following loss function:

$$J = \frac{1}{2}\mathbf{x}_N^T S_f \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}_k^T \mathcal{Q}_k \mathbf{x}_k + \mathbf{u}_k^T \mathcal{R}_k \mathbf{u}_k \tag{8.81}$$

subject to the constraint

$$\boxed{\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k, \quad \mathbf{x}(t_0) = \mathbf{x}_0} \tag{8.82}$$

The Hamiltonian of eqn. (8.39a) for the minimization problem shown by eqns. (8.81) and (8.82) is given by

$$H_k = \frac{1}{2}\left[\mathbf{x}_k^T \mathcal{Q}_k \mathbf{x}_k + \mathbf{u}_k^T \mathcal{R}_k \mathbf{u}_k\right] + \boldsymbol{\lambda}_{k+1}^T \left[\Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k\right] \tag{8.83}$$

From the necessary conditions of eqn. (8.41) the following equations must be satisfied:

$$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{8.84a}$$

$$\boldsymbol{\lambda}_k = \Phi_k^T \boldsymbol{\lambda}_{k+1} + \mathcal{Q}_k \mathbf{x}_k \tag{8.84b}$$

$$\mathbf{u}_k = -\mathcal{R}_k^{-1} \Gamma_k^T \boldsymbol{\lambda}_{k+1} \tag{8.84c}$$

$$\boldsymbol{\lambda}_N = S_f \mathbf{x}_N \tag{8.84d}$$

where eqn. (8.41e) has been used to derive eqn. (8.84d). Suppose we assume that the solution for the costate $\boldsymbol{\lambda}_k$ follows the form of eqn. (8.84d) for all time, which seems to be a reasonable assumption due to the linearity of the system. Hence, we assume

$$\boldsymbol{\lambda}_k = S_k \mathbf{x}_k \tag{8.85}$$

Taking one time-step ahead of eqn. (8.85) gives

$$\boldsymbol{\lambda}_{k+1} = S_{k+1} \mathbf{x}_{k+1} \tag{8.86}$$

Substituting eqns. (8.85) and (8.86) into eqn. (8.84b), and collecting terms yields

$$\Phi_k^T S_{k+1} \mathbf{x}_{k+1} + (\mathcal{Q}_k - S_k)\mathbf{x}_k = \mathbf{0} \tag{8.87}$$

Substituting eqn. (8.84c) into eqn. (8.84a) gives

$$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k - \Gamma_k \mathcal{R}_k^{-1} \Gamma_k^T \boldsymbol{\lambda}_{k+1} \tag{8.88}$$

Now, substituting eqn. (8.86) into eqn. (8.88) gives

$$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k - \Gamma_k \mathcal{R}_k^{-1} \Gamma_k^T S_{k+1} \mathbf{x}_{k+1} \tag{8.89}$$

Solving eqn. (8.89) for $\mathbf{x}_{k+1}$ gives

$$\mathbf{x}_{k+1} = \left[I + \Gamma_k \mathcal{R}_k^{-1} \Gamma_k^T S_{k+1}\right]^{-1} \Phi_k \mathbf{x}_k \tag{8.90}$$

Substituting eqn. (8.90) into eqn. (8.87) and collecting terms yields

$$\left\{\Phi_k^T S_{k+1}\left[I + \Gamma_k \mathcal{R}_k^{-1} \Gamma_k^T S_{k+1}\right]^{-1} \Phi_k + \mathcal{Q}_k - S_k\right\} \mathbf{x}_k = \mathbf{0} \tag{8.91}$$

Since eqn. (8.91) must hold for all nonzero $\mathbf{x}_k$, then the term within the brackets pre-multiplying $\mathbf{x}_k$ must be zero, which leads directly to

$$S_k = \Phi_k^T S_{k+1} \left[I + \Gamma_k \mathcal{R}_k^{-1} \Gamma_k^T S_{k+1} \right]^{-1} \Phi_k + \mathcal{Q}_k \tag{8.92}$$

Since S_{k+1} is assumed to have an inverse, then eqn. (8.92) can be rewritten as

$$S_k = \Phi_k^T \left[S_{k+1}^{-1} + \Gamma_k \mathcal{R}_k^{-1} \Gamma_k^T \right]^{-1} \Phi_k + \mathcal{Q}_k \tag{8.93}$$

Using the matrix inversion lemma in eqn. (1.70) with $A = S_{k+1}^{-1}$, $B = \Gamma_k$, $C = \mathcal{R}_k^{-1}$, and $D = \Gamma_k^T$ gives

$$\boxed{S_k = \Phi_k^T S_{k+1} \Phi_k - \Phi_k^T S_{k+1} \Gamma_k \left[\Gamma_k^T S_{k+1} \Gamma_k + \mathcal{R}_k \right]^{-1} \Gamma_k^T S_{k+1} \Phi_k + \mathcal{Q}_k} \tag{8.94}$$

with terminal boundary condition

$$S_N = S_f \tag{8.95}$$

Equation (8.94) represents the discrete-time matrix Riccati equation, which is propagated backward in time. The discrete-time LQR gain for the time-varying linear feedback form is more complicated than the continuous-time case. We first substitute eqn. (8.86) into eqn. (8.84c) to yield

$$\mathcal{R}_k \mathbf{u}_k = -\Gamma_k^T S_{k+1} \mathbf{x}_{k+1} \tag{8.96}$$

Substituting eqn. (8.82) into eqn. (8.96) and solving the resulting equation for $\mathbf{u}_k$ gives

$$\boxed{\mathbf{u}_k = -L_k \mathbf{x}_k} \tag{8.97}$$

where the *optimal gain matrix* is

$$\boxed{L_k = \left[\Gamma_k^T S_{k+1} \Gamma_k + \mathcal{R}_k \right]^{-1} \Gamma_k^T S_{k+1} \Phi_k} \tag{8.98}$$

Note the similarity between the formulation presented here and the discrete-time Kalman filter in Table 5.1, which leads to the duality results of §6.4.1. A summary of the discrete-time LQR is shown in Table 8.2. Once the gain matrices $\mathcal{R}_k$ and $\mathcal{Q}_k$ are chosen, the matrix Riccati solution in eqn. (8.94) is executed backward in time with a boundary condition given by eqn. (8.95). Storing the entire matrix S_k over all time, the gain matrix in eqn. (8.98) is then calculated. Finally, eqn. (8.82) is executed forward in time with the known initial state condition.

The stability of the discrete-time LQR controller can be proved by using Lyapunov's direct method, which is discussed for discrete-time systems in §3.6. The closed-loop dynamics are given by substituting eqn. (8.97) into eqn. (8.82), which leads to

$$\mathbf{x}_{k+1} = [\Phi_k - \Gamma_k L_k] \mathbf{x}_k \tag{8.99}$$

Table 8.2: Discrete-Time Linear Quadratic Regulator

Model	$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k, \quad \mathbf{x}(t_0) = \mathbf{x}_0$
Gain	$L_k = \left[\Gamma_k^T S_{k+1} \Gamma_k + \mathcal{R}_k\right]^{-1} \Gamma_k^T S_{k+1} \Phi_k$
Riccati Equation	$S_k = \Phi_k^T S_{k+1} \Phi_k + \mathcal{Q}_k$ $-\Phi_k^T S_{k+1} \Gamma_k \left[\Gamma_k^T S_{k+1} \Gamma_k + \mathcal{R}_k\right]^{-1} \Gamma_k^T S_{k+1} \Phi_k, \;\; S_N = S_f$
Control Input	$\mathbf{u}_k = -L_k \mathbf{x}_k$

We consider the following candidate Lyapunov function:

$$V(\mathbf{x}) = \mathbf{x}_k^T S_k \mathbf{x}_k \tag{8.100}$$

The increment of $V(\mathbf{x}_k)$ is given by

$$\Delta V(\mathbf{x}) = \mathbf{x}_{k+1}^T S_{k+1} \mathbf{x}_{k+1} - \mathbf{x}_k^T S_k \mathbf{x}_k \tag{8.101}$$

Using the definition of the gain in eqn. (8.98), the Riccati equation in eqn. (8.94) can be rewritten as

$$S_k = \Phi_k^T S_{k+1} \Phi_k - \Phi_k^T S_{k+1} \Gamma_k L_k + \mathcal{Q}_k \tag{8.102}$$

Equation (8.94) can be rewritten as (which is left as an exercise for the reader)

$$S_k = [\Phi_k - \Gamma_k L_k]^T S_{k+1} [\Phi_k - \Gamma_k L_k] + L_k^T \mathcal{R}_k L_k + \mathcal{Q}_k \tag{8.103}$$

Substituting eqns. (8.99) and (8.103) into eqn. (8.101), and simplifying yields

$$\Delta V(\mathbf{x}) = -\mathbf{x}_k^T \left[L_k^T \mathcal{R}_k L_k + \mathcal{Q}_k\right] \mathbf{x}_k \tag{8.104}$$

Clearly if $\mathcal{R}_k$ is positive definite and $\mathcal{Q}_k$ is at least positive semi-definite then the Lyapunov condition is satisfied and the discrete-time LQR controller is stable.

As with the continuous-time case a steady-state discrete-time LQR can be derived if all weighting and system matrices in the Riccati equation of eqn. (8.94) are constant. This leads to the following discrete-time algebraic Riccati equation:

$$S = \Phi^T S \Phi - \Phi^T S \Gamma \left[\Gamma^T S \Gamma + \mathcal{R}\right]^{-1} \Gamma^T S \Phi + \mathcal{Q} \tag{8.105}$$

In order to solve eqn. (8.105) using the method shown in §5.3.4, we must first derive the discrete-time Hamiltonian matrix. Assuming constant system matrices, then solving eqn. (8.84b) for $\boldsymbol{\lambda}_{k+1}$ gives

$$\boldsymbol{\lambda}_{k+1} = \Phi^{-T} \boldsymbol{\lambda}_k - \Phi^{-T} \mathcal{Q} \mathbf{x}_k \tag{8.106}$$

Substituting eqn. (8.106) into eqn. (8.88) gives

$$\mathbf{x}_{k+1} = \left[\Phi + \Gamma \mathcal{R}^{-1} \Gamma^T \Phi^{-T} \mathcal{Q}\right] \mathbf{x}_k - \Gamma \mathcal{R}^{-1} \Gamma^T \Phi^{-T} \boldsymbol{\lambda}_k \tag{8.107}$$

Combining eqns. (8.106) and (8.107) leads to

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \boldsymbol{\lambda}_{k+1} \end{bmatrix} = \mathcal{H} \begin{bmatrix} \mathbf{x}_k \\ \boldsymbol{\lambda}_k \end{bmatrix} \tag{8.108}$$

where the Hamiltonian matrix is defined by[9]

$$\mathcal{H} \equiv \begin{bmatrix} \Phi + \Gamma \mathcal{R}^{-1} \Gamma^T \Phi^{-T} \mathcal{Q} & -\Gamma \mathcal{R}^{-1} \Gamma^T \Phi^{-T} \\ -\Phi^{-T} \mathcal{Q} & \Phi^{-T} \end{bmatrix} \tag{8.109}$$

The eigenvalues of $\mathcal{H}$ can be arranged in a diagonal matrix given by

$$\mathcal{H}_\Lambda = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \tag{8.110}$$

where Λ is a diagonal matrix of the n eigenvalues outside of the unit circle. Assuming that the eigenvalues are distinct, we can perform a linear state transformation, as shown in §3.1.4, such that

$$\mathcal{H}_\Lambda = W^{-1} \mathcal{H} W \tag{8.111}$$

where W is the matrix of eigenvectors, which can be represented in block form as

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{8.112}$$

Going backward in time the stable eigenvalues dominate, which leads to the following solution for S at steady-state:

$$\boxed{S = W_{22} W_{12}^{-1}} \tag{8.113}$$

Note that the inverse of Φ must exist for a valid solution. This usually poses no problems though, since Φ does not usually have a zero eigenvalue in practice.

8.6 Linear Quadratic-Gaussian Controllers

The LQR feedback control laws of eqns. (8.62) and (8.97) clearly require full state knowledge, which is not always possible or even practical in real-world systems. It seems natural to use the Kalman filter to provide state estimates, which can be used in place of the "true" states in the LQR feedback control law. In actuality this seemingly ad hoc approach turns out to be the optimal approach, which leads to the so-called *linear quadratic-Gaussian* (LQG) controller.[10] In this section combining the LQR feedback control law with the standard estimator form of the Kalman filter is proven to be optimal using the *Separation Theorem*, which is also known as the *Certainty Equivalence Principle*.[11–13] This theorem states that the solution of overall optimal control problem with incomplete state knowledge is given by the solution of two separate sub-problems: 1) the estimation problem used to provide optimal state estimates, which is solved using the Kalman filter, and 2) the control problem using the optimal states estimates, which is derived from the standard LQR results. Another way to show this separation of the overall control design involves the eigenvalue separation property,[14] which states that the eigenvalues of the overall closed-loop system are given by the eigenvalues of the LQR system together with those of the state estimator system.

8.6.1 Continuous-Time Formulation

In the continuous-time LQG problem we assume that the state model is given by eqn. (5.117):

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t) \tag{8.114a}$$

$$\tilde{\mathbf{y}}(t) = H(t)\,\mathbf{x}(t) + \mathbf{v}(t) \tag{8.114b}$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are zero-mean Gaussian noise processes with covariances given by eqn. (5.118). Note that unlike eqn. (8.55), the state model in eqn. (8.114) is random. Therefore, we must take the expected value of the loss function in eqn. (8.54), which leads to the LQG loss function to be minimized:

$$J = E\left\{\int_{t_0}^{t_f} \mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)\,dt\right\} \tag{8.115}$$

Note that the terminal condition is omitted here for brevity since the results of the Separation Theorem extended easily for this case (also the factor of one half is not needed to prove the theorem). There are many ways to prove the Separation Theorem (e.g., see Refs. [2] and [13]), but we choose to use the approach presented in Ref. [14], which is fairly straightforward without requiring rigorous stochastic optimal control theory. Let us first concentrate on the expression $E\left\{\mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t)\right\}$. Adding and subtracting the state estimate $\hat{\mathbf{x}}(t)$ to $\mathbf{x}(t)$ gives

$$E\left\{\mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t)\right\} = E\left\{\left[\hat{\mathbf{x}}(t) - \tilde{\mathbf{x}}(t)\right]^T \mathcal{Q}(t)\left[\hat{\mathbf{x}}(t) - \tilde{\mathbf{x}}(t)\right]\right\} \tag{8.116}$$

where the estimation error is defined as $\tilde{\mathbf{x}}(t) \equiv \hat{\mathbf{x}}(t) - \mathbf{x}(t)$. Expanding eqn. (8.116) and using the trace property $\mathrm{Tr}(A\,\mathbf{z}\mathbf{z}^T) = \mathbf{z}^T A\,\mathbf{z}$ (see Appendix A) leads to

$$\begin{aligned} E\left\{\mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t)\right\} = E\left\{\hat{\mathbf{x}}^T(t)\,\mathcal{Q}(t)\,\hat{\mathbf{x}}(t)\right\} - 2E\left\{\mathrm{Tr}\left[\mathcal{Q}(t)\tilde{\mathbf{x}}(t)\,\hat{\mathbf{x}}^T(t)\right]\right\} \\ + E\left\{\mathrm{Tr}\left[\mathcal{Q}(t)\,\tilde{\mathbf{x}}(t)\,\tilde{\mathbf{x}}^T(t)\right]\right\} \end{aligned} \tag{8.117}$$

The orthogonality principle of the Kalman filter, which is shown for discrete-time systems in §5.3.6 and exercise 5.22, states that the estimation error is orthogonal to the state estimate. This is obviously also true for continuous-time systems, which gives $E\left\{\tilde{\mathbf{x}}(t)\,\hat{\mathbf{x}}^T(t)\right\}=0$. Therefore, eqn. (8.117) reduces down to

$$E\left\{\mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t)\right\}=E\left\{\hat{\mathbf{x}}^T(t)\,\mathcal{Q}(t)\,\hat{\mathbf{x}}(t)\right\}+E\left\{\mathrm{Tr}\left[\mathcal{Q}(t)\,\tilde{\mathbf{x}}(t)\,\tilde{\mathbf{x}}^T(t)\right]\right\} \tag{8.118}$$

Using the definition of the covariance $P(t)$ in eqn. (5.125), eqn. (8.118) can be rewritten as

$$E\left\{\mathbf{x}^T(t)\,\mathcal{Q}(t)\,\mathbf{x}(t)\right\}=E\left\{\hat{\mathbf{x}}^T(t)\,\mathcal{Q}(t)\,\hat{\mathbf{x}}(t)\right\}+\mathrm{Tr}\left[\mathcal{Q}(t)\,P(t)\right] \tag{8.119}$$

Substituting eqn. (8.119) into eqn. (8.115) leads to the following equivalent minimization problem:

$$J=E\left\{\int_{t_0}^{t_f}\hat{\mathbf{x}}^T(t)\,\mathcal{Q}(t)\,\hat{\mathbf{x}}(t)+\mathbf{u}^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)\,dt\right\}+\int_{t_0}^{t_f}\mathrm{Tr}\left[\mathcal{Q}(t)\,P(t)\right]\,dt \tag{8.120}$$

subject to the new dynamic constraint

$$\dot{\hat{\mathbf{x}}}(t)=F(t)\,\hat{\mathbf{x}}(t)+B(t)\,\mathbf{u}(t)+K(t)[\tilde{\mathbf{y}}(t)-H(t)\,\hat{\mathbf{x}}(t)] \tag{8.121}$$

which is the linear continuous estimator for $\mathbf{x}(t)$.

The goal of our overall process is to convert the constrained minimization problem given by eqns. (8.120) and (8.121) into an unconstrained problem (thus avoiding the use of Lagrange multipliers). For the subsequent developments we will need an expression for $W(t)\equiv E\left\{\hat{\mathbf{x}}(t)\,\hat{\mathbf{x}}^T(t)\right\}$. Using the methods of §5.4.1 and the definition of the innovations process in §6.4.2.2, this expression can be shown to follow (which is left as an exercise for the reader)

$$\begin{aligned}\dot{W}(t)&=F(t)\,W(t)+W(t)\,F^T(t)+K(t)\,R(t)\,K^T(t)\\&\quad+E\left\{B(t)\,\mathbf{u}(t)\,\hat{\mathbf{x}}^T(t)+\hat{\mathbf{x}}(t)\,\mathbf{u}^T(t)\,B^T(t)\right\}\end{aligned} \tag{8.122}$$

with $W(t_0)=E\left\{\hat{\mathbf{x}}(t_0)\,\hat{\mathbf{x}}^T(t_0)\right\}$. Also, we need an expression for

$$\frac{d}{dt}[S(t)\,W(t)]=\dot{S}(t)\,W(t)+S(t)\,\dot{W}(t) \tag{8.123}$$

Substituting eqns. (8.60) and (8.122) into eqn. (8.123), taking the trace of the resulting equation, and using the definition of $L(t)$ in eqn. (8.63) leads to

$$\begin{aligned}\mathrm{Tr}\left\{\frac{d}{dt}[S(t)\,W(t)]\right\}&=\mathrm{Tr}\Big[L^T(t)\,\mathcal{R}(t)\,L(t)\,W(t)-\mathcal{Q}(t)\,W(t)\\&\qquad+S(t)\,K(t)\,R(t)\,K^T(t)+2E\left\{\hat{\mathbf{x}}^T(t)\,L^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)\right\}\Big]\end{aligned} \tag{8.124}$$

Using $\mathrm{Tr}[\mathcal{Q}(t)\,W(t)] = \hat{\mathbf{x}}^T(t)\,\mathcal{Q}(t)\,\hat{\mathbf{x}}(t)$ in eqn. (8.124), and solving for the quantity $\hat{\mathbf{x}}^T(t)\,\mathcal{Q}(t)\,\hat{\mathbf{x}}(t)$ yields

$$\begin{aligned}\hat{\mathbf{x}}^T(t)\,\mathcal{Q}(t)\,\hat{\mathbf{x}}(t) &= 2E\left\{\hat{\mathbf{x}}^T(t)\,L^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)\right\}\\ &+\mathrm{Tr}\left\{-\frac{d}{dt}[S(t)\,W(t)] + S(t)\,K(t)\,R(t)\,K^T(t) + L^T(t)\,\mathcal{R}(t)\,L(t)\,W(t)\right\}\end{aligned} \tag{8.125}$$

We now find an expression for $\mathbf{u}^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)$. This is accomplished by first expanding the following expression:

$$\begin{aligned}E\left\{\left[\mathbf{u}(t)+L(t)\,\hat{\mathbf{x}}(t)\right]^T\mathcal{R}(t)\left[\mathbf{u}(t)+L(t)\,\hat{\mathbf{x}}(t)\right]\right\} &= E\left\{\mathbf{u}^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)\right\}\\ +2E\left\{\hat{\mathbf{x}}^T(t)\,L^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)\right\} &+ E\left\{\hat{\mathbf{x}}^T(t)\,L^T(t)\,\mathcal{R}(t)\,L(t)\,\hat{\mathbf{x}}(t)\right\}\end{aligned} \tag{8.126}$$

Using the trace property $\mathrm{Tr}(A\,\mathbf{z}\mathbf{z}^T) = \mathbf{z}^T A\,\mathbf{z}$, and solving eqn. (8.126) for the desired expression, $\mathbf{u}^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)$, yields

$$\begin{aligned}\mathbf{u}^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t) &= E\left\{\left[\mathbf{u}(t)+L(t)\,\hat{\mathbf{x}}(t)\right]^T\mathcal{R}(t)\left[\mathbf{u}(t)+L(t)\,\hat{\mathbf{x}}(t)\right]\right\}\\ &-2E\left\{\hat{\mathbf{x}}^T(t)\,L^T(t)\,\mathcal{R}(t)\,\mathbf{u}(t)\right\} - \mathrm{Tr}\left[L^T(t)\,\mathcal{R}(t)\,L(t)\,W(t)\right]\end{aligned} \tag{8.127}$$

Substituting eqns. (8.125) and (8.127) into eqn. (8.120) leads to

$$\begin{aligned}J = \int_{t_0}^{t_f} E\Big\{&\left[\mathbf{u}(t)+L(t)\,\hat{\mathbf{x}}(t)\right]^T\mathcal{R}(t)\left[\mathbf{u}(t)+L(t)\,\hat{\mathbf{x}}(t)\right] + \mathrm{Tr}[\mathcal{Q}(t)\,P(t)]\\ &+\mathrm{Tr}\left[S(t)\,K(t)\,R(t)\,K^T(t)\right]\Big\}\,dt - \{\mathrm{Tr}[S(t)\,W(t)]\}\big|_{t_0}^{t_f}\end{aligned} \tag{8.128}$$

Minimizing eqn. (8.128) with respect to $\mathbf{u}(t)$ gives

$$\mathbf{u}(t) = -L(t)\,\hat{\mathbf{x}}(t) \tag{8.129}$$

Equation (8.129) is identical to eqn. (8.62) with the exception that the true state $\mathbf{x}(t)$ is replaced with the estimated state $\hat{\mathbf{x}}(t)$! This clearly shows that the optimal solution with partial state information is given by using the linear estimator of the form in eqn. (8.114a). Any estimator with this form is valid; however, the Kalman filter is most widely used in practical applications. This attests to the separation of the estimator design with the control design.

A block diagram of the LQG controller is shown in Figure 8.5. The control input has been generalized in this diagram to be given by

$$\mathbf{u}(t) = -L(t)\,\hat{\mathbf{x}}(t) + \mathbf{u}_{\mathrm{ext}}(t) \tag{8.130}$$

where $\mathbf{u}_{\mathrm{ext}}(t)$ denotes an external input, which may include a term $-L(t)\,\mathbf{x}_d(t)$, where $\mathbf{x}_d(t)$ is some desired state trajectory; or a feedforward term $L_r(t)\,\mathbf{r}(t)$, where

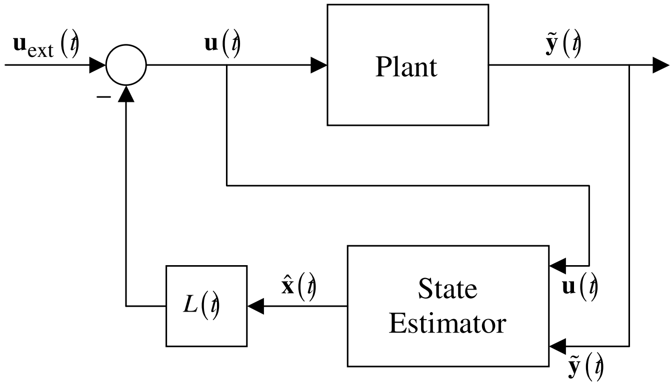

Figure 8.5: The Linear Quadratic-Gaussian Controller

$\mathbf{r}(t)$ is a reference trajectory.[14] Reference [14] also shows other possible arrangements, e.g., where the external input is combined with the output prior to entering the state estimator.

Another, much easier way to show the separation of the estimator and controller involves the investigation of the closed-loop LQG system. We only consider the time-invariant case with constant system matrices to illustrate this concept. Substituting eqn. (8.129) into eqn. (8.114a) gives

$$\dot{\mathbf{x}}(t) = F\,\mathbf{x}(t) - B\,L\,\hat{\mathbf{x}}(t) + G\,\mathbf{w}(t) \tag{8.131}$$

Substituting eqns. (8.114b) and (8.129) into eqn. (8.121) yields

$$\dot{\hat{\mathbf{x}}}(t) = [F - B\,L - K\,H]\hat{\mathbf{x}}(t) + K\,H\mathbf{x}(t) + K\,\mathbf{v}(t) \tag{8.132}$$

Combining eqns. (8.131) and (8.132) leads to

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} F & -B\,L \\ K\,H & F - B\,L - K\,H \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{v}(t) \end{bmatrix} \tag{8.133}$$

Unfortunately the stability of this system is not obvious at first glance. To overcome this difficulty we use the definition of the error state from §5.4.1: $\tilde{\mathbf{x}}(t) \equiv \hat{\mathbf{x}}(t) - \mathbf{x}(t)$. Taking the time derivative of this quantity and substituting eqns. (8.131) and (8.132) into the resulting expression yields

$$\dot{\tilde{\mathbf{x}}}(t) = [F - K\,H]\tilde{\mathbf{x}}(t) + K\,\mathbf{v}(t) - G\,\mathbf{w}(t) \tag{8.134}$$

Combining eqns. (8.131) and (8.134) leads to

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\tilde{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} F - B\,L & -B\,L \\ 0 & F - K\,H \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \tilde{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} G & 0 \\ -G & K \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{v}(t) \end{bmatrix} \tag{8.135}$$

The eigenvalues of the state matrix in eqns. (8.133) and (8.135) can be shown to be equivalent (which is left as an exercise for the reader). The form in eqn. (8.135) is much easier to visualize than the one of eqn. (8.133), since the eigenvalues of the block diagonal structure are given by (see Appendix A)

$$\det(\lambda I - F + B\,L)\,\det(\lambda I - F + K\,H) = 0 \tag{8.136}$$

Equation (8.136) clearly shows that the eigenvalues of the controller and estimator are separate from each other in the overall LQG closed-loop system. This again shows the Separation Principle. The obvious advantage of having a time-invariant system is the application of real-time control/estimation in a feedback system. The optimality of the time-invariant closed-loop system is proven by Tse.[15]

Yet another way to prove the Separation Theorem involves using the *Stochastic Hamilton-Jacobi-Bellman* (SHJB) equation,[2] given by

$$\boxed{\begin{aligned}\frac{\partial J^*(\mathbf{x}(t),\,t)}{\partial t} + \min_{\mathbf{u}(t)}\Bigg\{&\vartheta(\mathbf{x}(t),\,\mathbf{u}(t),\,t) + \frac{\partial J^*(\mathbf{x}(t),\,\mathbf{u}(t),\,t)}{\partial \mathbf{x}^T(t)}\mathbf{f}(\mathbf{x}(t),\,\mathbf{u}(t),\,t)\\ &+ \frac{1}{2}\mathrm{Tr}\left[G(t)\,Q(t)\,G(t)\,\frac{\partial^2 J^*(\mathbf{x}(t),\,t)}{\partial \mathbf{x}^2(t)}\right]\Bigg\} = 0\end{aligned}} \tag{8.137}$$

with terminal condition

$$J^*(\mathbf{x}(t_f),\,t_f) = \phi(\mathbf{x}(t_f),\,t_f) \tag{8.138}$$

The cost-to-go function for the stochastic problem is given by

$$J^*(\mathbf{x}(t),\,t) = \min_{\mathbf{u}(t)} E\left\{\phi(\mathbf{x}(t_f),\,t_f) + \int_t^{t_f} \vartheta(\mathbf{x}(\tau),\,\mathbf{u}(\tau),\,\tau)\,d\tau|\mathbf{x}(t)\right\} \tag{8.139}$$

subject to the dynamic constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\,\mathbf{u}(t),\,t) + G(t)\,\mathbf{w}(t) \tag{8.140}$$

For the LQG problem the control input that satisfies the SHJB equation can be shown to be given by eqn. (8.129).

8.6.2 Discrete-Time Formulation

The results of the previous section can be extended to discrete-time systems. Rather than repeating the steps here, we choose to only show the steps required to prove the Separation Theorem for discrete-time systems (see Refs. [2] and [11] for more details). In the discrete-time LQG problem we assume that the state model is given by eqn. (5.27):

$$\mathbf{x}_{k+1} = \Phi_k\mathbf{x}_k + \Gamma_k\mathbf{u}_k + \Upsilon_k\mathbf{w}_k \tag{8.141a}$$
$$\tilde{\mathbf{y}}_k = H_k\mathbf{x}_k + \mathbf{v}_k \tag{8.141b}$$

where $\mathbf{v}_k$ and $\mathbf{w}_k$ are assumed to be zero-mean Gaussian white-noise processes with covariances given by eqns. (5.28) and (5.29), respectively. The discrete-time version of the loss function in eqn. (8.115) is given by

$$J = E\left\{\sum_{k=0}^{N-1} \mathbf{x}_k^T \mathcal{Q}_k \mathbf{x}_k + \mathbf{u}_k^T \mathcal{R}_k \mathbf{u}_k\right\} \tag{8.142}$$

The discrete-time problem involves finding a control input to minimize eqn. (8.142) given a set of measurements $\mathbf{Y}_{k-1} = [\tilde{\mathbf{y}}_0, \ldots, \tilde{\mathbf{y}}_{k-1}]$. Equation (8.142) can be rewritten as

$$J = E\left\{\sum_{s=0}^{k-1} \mathbf{x}_s^T \mathcal{Q}_s \mathbf{x}_s + \mathbf{u}_s^T \mathcal{R}_s \mathbf{u}_s\right\} + E\left\{\sum_{s=k}^{N-1} \mathbf{x}_s^T \mathcal{Q}_s \mathbf{x}_s + \mathbf{u}_s^T \mathcal{R}_s \mathbf{u}_s\right\} \tag{8.143}$$

Note that the second term in the loss function of eqn. (8.143) depends on $\mathbf{u}_k$. Hence, we seek to minimize the following cost-to-go function:

$$J^* = E\left\{\sum_{s=k}^{N-1} \mathbf{x}_s^T \mathcal{Q}_s \mathbf{x}_s + \mathbf{u}_s^T \mathcal{R}_s \mathbf{u}_s\right\} \tag{8.144}$$

It is more convenient to express eqn. (8.144) in terms of a conditional probability, similar to the approach shown in §2.6. This leads to the following equivalent minimizing function:

$$J^* = E\left[\min_{\mathbf{u}_k} E\left\{\sum_{s=k}^{N-1} \mathbf{x}_s^T \mathcal{Q}_s \mathbf{x}_s + \mathbf{u}_s^T \mathcal{R}_s \mathbf{u}_s \,\middle|\, \mathbf{Y}_{k-1}\right\}\right] \tag{8.145}$$

where the first expectation in eqn. (8.143) denotes the expectation with respect to the distribution $\mathbf{Y}_{k-1}$, and the minimum is taken with respect to all strategies that express $\mathbf{u}_k$ and a function of $\mathbf{Y}_{k-1}$.[11] Repeating the arguments for $k = N-1,\ N-2, \ldots$ leads to

$$\min_{\mathbf{u}_k, \ldots, \mathbf{u}_{N-1}} E\left\{\sum_{s=k}^{N-1} \mathbf{x}_s^T \mathcal{Q}_s \mathbf{x}_s + \mathbf{u}_s^T \mathcal{R}_s \mathbf{u}_s \,\middle|\, \mathbf{Y}_{k-1}\right\} \equiv \tilde{V}_k(\mathbf{Y}_{k-1}) \tag{8.146}$$

Since $\mathbf{x}_k$ and $\mathbf{u}_k$ are not causally affected by $\mathbf{u}_{k+1}, \ldots, \mathbf{u}_{N-1}$, then eqn. (8.146) can be written as[2]

$$\begin{aligned}\tilde{V}_k(\mathbf{Y}_{k-1}) = \min_{\mathbf{u}_k} E\Bigg\{&\mathbf{x}_k^T \mathcal{Q}_k \mathbf{x}_k + \mathbf{u}_k^T \mathcal{R}_k \mathbf{u}_k \\ &+ \min_{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_{N-1}} \left[\sum_{s=k+1}^{N-1} \mathbf{x}_s^T \mathcal{Q}_s \mathbf{x}_s + \mathbf{u}_s^T \mathcal{R}_s \mathbf{u}_s\right] \Bigg|\, \mathbf{Y}_{k-1}\Bigg\}\end{aligned} \tag{8.147}$$

Equation (8.147) is equivalent to

$$\begin{aligned}\tilde{V}_k(\mathbf{Y}_{k-1}) = \min_{\mathbf{u}_k}\Bigg[& E\left\{ \mathbf{x}_k^T \mathcal{Q}_k \mathbf{x}_k + \mathbf{u}_k^T \mathcal{R}_k \mathbf{u}_k \,\middle|\, \mathbf{Y}_{k-1} \right\} \\ & + E\left\{ \min_{\mathbf{u}_{k+1},\ldots,\mathbf{u}_{N-1}} E\left\{ \sum_{s=k+1}^{N-1} \mathbf{x}_s^T \mathcal{Q}_s \mathbf{x}_s + \mathbf{u}_s^T \mathcal{R}_s \mathbf{u}_s \,\middle|\, \mathbf{Y}_k \right\} \,\middle|\, \mathbf{Y}_{k-1} \right\} \Bigg]\end{aligned} \tag{8.148}$$

Finally, using the definition of the conditional expectation (see Appendix B) allows us to notionally simplify eqn. (8.148) to

$$\begin{aligned}\tilde{V}_k(\mathbf{Y}_{k-1}) = \min_{\mathbf{u}_k}\Bigg[& E\left\{ \mathbf{x}_k^T \mathcal{Q}_k \mathbf{x}_k + \mathbf{u}_k^T \mathcal{R}_k \mathbf{u}_k \,\middle|\, \mathbf{Y}_{k-1} \right\} \\ & + E\left\{ E\left\{ \tilde{V}_{k+1}(\mathbf{Y}_k) \,\middle|\, \mathbf{Y}_k \right\} \,\middle|\, \mathbf{Y}_{k-1} \right\} \Bigg]\end{aligned} \tag{8.149}$$

Note that eqn. (8.149) does not include a summation anymore.

In order to prove the Separation Theorem for discrete-time systems, we need to show that for each $k = N,\ N-1,\ \ldots,\ 0$, there exists a function V_k dependent on $\hat{\mathbf{x}}_k$, a matrix S_k, and a scalar s_k such that $\tilde{V}_k(\mathbf{Y}_{k-1}) = V_k(\hat{\mathbf{x}}_k)$. Let us assume that this relationship is of the form given by

$$V(\hat{\mathbf{x}}_k,\, k) = \hat{\mathbf{x}}_k^T S_k \hat{\mathbf{x}}_k + s_k \tag{8.150}$$

We first concentrate our efforts on the expression $E\left\{\mathbf{x}_k^T \mathcal{Q}_k \mathbf{x}_k | \mathbf{Y}_{k-1}\right\}$. This expression can be given directly from the discrete-time version of eqn. (8.119):

$$E\left\{\mathbf{x}_k^T \mathcal{Q}_k \mathbf{x}_k | \mathbf{Y}_{k-1}\right\} = \hat{\mathbf{x}}_k^T \mathcal{Q}_k \hat{\mathbf{x}}_k + \mathrm{Tr}\left(\mathcal{Q}_k P_k\right) \tag{8.151}$$

Starting with the Kalman filter equations of eqn. (5.54), the mean and covariance of $\hat{\mathbf{x}}_{k+1}$ can be shown to be given by (which is left as an exercise for the reader)

$$E\left\{\hat{\mathbf{x}}_{k+1} | \mathbf{Y}_{k-1}\right\} = \Phi_k \hat{\mathbf{x}}_k + \Gamma_k \mathbf{u}_k \tag{8.152a}$$

$$\mathrm{cov}\left\{\hat{\mathbf{x}}_{k+1} | \mathbf{Y}_{k-1}\right\} = \Phi_k K_k \left[H_k P_k H_k^T + R_k\right] K_k^T \Phi_k^T \tag{8.152b}$$

Summing up we find that $V(\hat{\mathbf{x}}_k)$ is given by

$$\begin{aligned}V(\hat{\mathbf{x}}_k) = \min_{\mathbf{u}_k}\Big\{ & \hat{\mathbf{x}}_k^T \mathcal{Q}_k \hat{\mathbf{x}}_k + \mathrm{Tr}\left(\mathcal{Q}_k P_k\right) + \mathbf{u}_k^T \mathcal{R}_k \mathbf{u}_k \\ & + \left[\Phi_k \hat{\mathbf{x}}_k + \Gamma_k \mathbf{u}_k\right]^T S_{k+1} \left[\Phi_k \hat{\mathbf{x}}_k + \Gamma_k \mathbf{u}_k\right] \\ & + \mathrm{Tr}\left(S_{k+1} \Phi_k K_k \left[H_k P_k H_k^T + R_k\right] K_k^T \Phi_k^T\right) + s_{k+1} \Big\}\end{aligned} \tag{8.153}$$

Equation (8.153) is equivalent to

$$\begin{aligned}V(\hat{\mathbf{x}}_k) = \min_{\mathbf{u}_k}\Big\{ & \hat{\mathbf{x}}_k^T \left(\Phi_k^T S_{k+1} \Phi_k + \mathcal{Q}_k - L_k^T \left[\Gamma_k^T S_{k+1} \Gamma_k + \mathcal{R}_k\right] L_k\right) \hat{\mathbf{x}}_k \\ & + \left(\mathbf{u}_k + L_k \hat{\mathbf{x}}_k\right)^T \left[\Gamma_k^T S_{k+1} \Gamma_k + \mathcal{R}_k\right] \left(\mathbf{u}_k + L_k \hat{\mathbf{x}}_k\right) + \mathrm{Tr}\left(\mathcal{Q}_k P_k\right) \\ & + \mathrm{Tr}\left(S_{k+1} \Phi_k K_k \left[H_k P_k H_k^T + R_k\right] K_k^T \Phi_k^T\right) + s_{k+1} \Big\}\end{aligned} \tag{8.154}$$

where L_k is given by eqn. (8.98). Also, comparing eqn. (8.150) to eqn. (8.154) gives

$$s_k = s_{k+1} + \mathrm{Tr}\,(\mathcal{Q}_k P_k) + \mathrm{Tr}\left(S_{k+1}\Phi_k K_k \left[H_k P_k H_k^T + R_k\right] K_k^T \Phi_k^T\right) \tag{8.155}$$

and

$$S_k = \Phi_k^T S_{k+1}\Phi_k + \mathcal{Q}_k - L_k^T \left[\Gamma_k^T S_{k+1}\Gamma_k + \mathcal{R}_k\right] L_k \tag{8.156}$$

Note that eqn. (8.156) is equivalent to eqn. (8.102) with the gain matrix L_k given by eqn. (8.98)!

The optimal $\mathbf{u}_k$ that satisfies eqn. (8.154) is clearly given by

$$\mathbf{u}_k = -L_k \hat{\mathbf{x}}_k \tag{8.157}$$

Equation (8.157) is identical to eqn. (8.97) with the exception that the true state $\mathbf{x}(t)$ is replaced with the estimated state $\hat{\mathbf{x}}(t)$! In order for eqn. (8.157) to truly achieve the minimum of the LQG loss function, the matrix $[\Gamma_k^T S_{k+1}\Gamma_k + \mathcal{R}_k]$ must be positive definite. This condition will obviously always be met since S_{k+1} is always positive definite, which is shown by the form in eqn. (8.103). For autonomous systems, the discrete-time Separation Theorem can be proved using an eigenvalue separation of the controller and estimator (see exercise 8.30), similar to the steps leading to eqn. (8.135).

8.7 Loop Transfer Recovery

As discussed in example 5.5, the Kalman filter estimates are usually derived by "tuning" the process noise covariance matrix until desired estimation characteristics are obtained. The difficulties of this usually "ad hoc" approach are often mitigated through intimate experience of the dynamical system. However, the process is further complicated when we wish to investigate the robustness properties of the combined estimator/controller in the overall LQG design. As discussed in the introduction section of this chapter, the overall pointing error is a function of both the estimation *and* control errors. Problems with LQG designs may arise from two possible undesirable characteristics: 1) poor stability margins and 2) poor performance of the overall LQG dynamics. One might expect that since the Kalman filter and linear regulator have nice properties, then the LQG controller would exhibit nice properties as well. But, Doyle[16] has shown that LQG designs can exhibit poor stability margins, which leads to the first problem in LQG designs. Also, a natural and seemingly logical assumption in the LQG design involves setting the Kalman gain so that the estimator errors have converged well before the controller errors, which should provide well-behaved feedback properties. However, Doyle and Stein[17] show that stability margins can actually be degraded by making the estimator dynamics faster in some cases, which leads to the second problem in LQG designs.

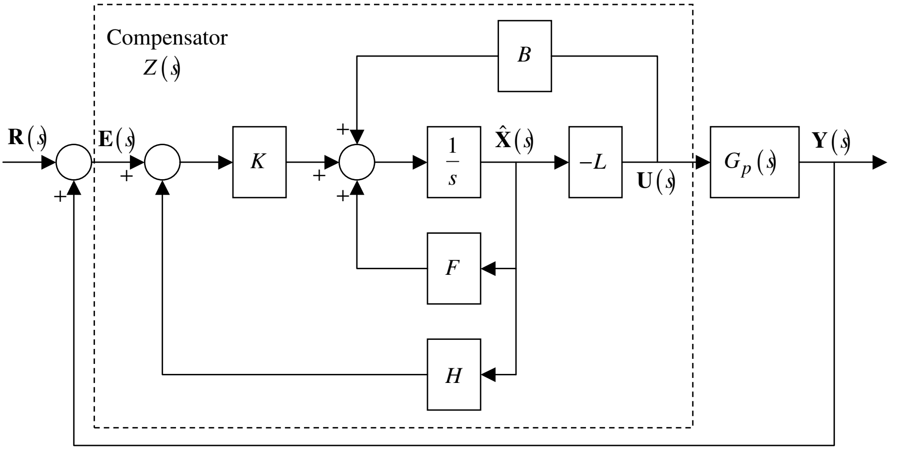

Figure 8.6: The Linear Quadratic-Gaussian Controller with Reference Input

The Loop Transfer Recovery (LTR) approach[17, 18] overcomes these problems by tuning the Kalman filter so that the original (true state) regulator dynamics are "recovered" at the control input. A block diagram of the LQG controller with a reference input is shown in Figure 8.6. Notice that unlike Figure 8.5, we now incorporate unity feedback into the control structure, which provides a more likely plant/controller arrangement for use in practice.[14] Also, this arrangement allows us to compute useful stability/robustness parameters, such as phase and/or gain margins (see Refs. [19]-[24] for more details on these tools).

To help motivate the LTR concept, we begin by considering the closed-loop dynamics of the LQR system. It is assumed that the number of inputs is equal to the number of outputs. Taking the Laplace transform of both sides of eqn. (8.55) leads to

$$\mathbf{X}(s) = (sI - F)^{-1} B\,\mathbf{U}(s) \tag{8.158}$$

Multiplying both sides of eqn. (8.158) by $-L$ and using the definition of the LQR control law in eqn. (8.62) gives

$$\mathbf{U}(s) = -L\,(sI - F)^{-1} B\,\mathbf{U}(s) \tag{8.159}$$

The matrix $-L\,(sI - F)^{-1} B$ represents the desired return ratio. Referring to Figure 8.6, the transfer function from the error signal $\mathbf{E}(s)$ to the state estimate $\hat{\mathbf{X}}(s)$ is given by

$$\hat{\mathbf{X}}(s) = (sI - F + B\,L + K\,H)^{-1} K\,\mathbf{E}(s) \tag{8.160}$$

Taking the Laplace transform of eqn. (8.129) and substituting eqn. (8.160) into the resulting expression gives

$$\mathbf{U}(s) \equiv Z(s)\,\mathbf{E}(s) = -L\,(sI - F + B\,L + K\,H)^{-1} K\,\mathbf{E}(s) \tag{8.161}$$

where $Z(s) \equiv -L\,(s\,I - F + B\,L + K\,H)^{-1}K$ is the LQG compensator matrix. Using the transfer function model of eqn. (3.14), with D assumed to be zero, for the plant $G_p(s)$ gives the following *loop-gain transfer function* matrix:

$$Z(s)\,G_p(s) = -L\,(s\,I - F + B\,L + K\,H)^{-1}K\,H\,(s\,I - F)^{-1}B \tag{8.162}$$

This is the return ratio of the overall LQG system. Our goal is to tune the Kalman filter gain so that $Z(s)\,G_p(s)$ approaches the matrix $-L\,(s\,I - F)^{-1}B$ shown in eqn. (8.159). Define the following quantities:

$$\Phi(s) \equiv (s\,I - F)^{-1} \tag{8.163a}$$

$$\Psi(s) \equiv (s\,I - F + B\,L)^{-1} \tag{8.163b}$$

With these definitions eqn. (8.162) can be rewritten as

$$Z(s)\,G_p(s) = -L\,[\Psi^{-1}(s) + K\,H]^{-1}K\,H\,\Phi(s)\,B \tag{8.164}$$

Equation (8.164) can be shown to be equivalent to (which is left as an exercise for the reader)

$$Z(s)\,G_p(s) = -L\,\Psi(s)\,K[I + H\,\Psi(s)\,K]^{-1}K\,H\,\Phi(s)\,B \tag{8.165}$$

where the matrix inversion lemma of eqn. (1.69) can be used to prove eqn. (8.165).

In the LTR approach it is assumed that the process noise covariance matrix $G\,Q\,G^T$ is replaced by

$$\boxed{G\,Q\,G^T = G\,Q_0G^T + q^2B\,B^T} \tag{8.166}$$

where Q_0 is some initial guess for Q, and q is a real and positive tuning parameter. Using eqn. (8.166), the new algebraic Riccati equation for the Kalman filter covariance, shown in Table 5.5, can be written as

$$\boxed{F\left(\frac{P}{q^2}\right) + \left(\frac{P}{q^2}\right)F^T - q^2\left(\frac{P}{q^2}\right)H^TR^{-1}H\left(\frac{P}{q^2}\right) + \frac{G\,Q_0G^T}{q^2} + B\,B^T = 0} \tag{8.167}$$

Kwakernaak and Sivan[25] show that if the plant has no transmission zeros in the right half-plane, then

$$\lim_{q\to\infty}\frac{P}{q^2} = 0 \tag{8.168}$$

Equation (8.168) indicates that as q increases, the covariance matrix P is increasing more slowly than the process noise covariance (if the stated assumptions hold).[18] Consequently, from eqn. (8.167) we have

$$q^2\left(\frac{P}{q^2}\right)H^TR^{-1}H\left(\frac{P}{q^2}\right) \to B\,B^T \tag{8.169}$$

Since the Kalman gain is given by $K = P\,H^TR^{-1}$, then from eqn. (8.169) we now have

$$K \to q\,B\,R^{-1/2} \text{ as } q \to \infty \tag{8.170}$$

Substituting eqn. (8.170) into eqn. (8.165), with the assumption that $H\,\Psi(s)\,B$ is square (i.e., the number of inputs is equal to the number of outputs) yields

$$Z(s)\,G_p(s) \to -L\,\Psi(s)\,R^{-1/2}\left[H\,\Psi(s)\,B\,R^{-1/2}\right]^{-1} H\,\Phi(s)\,B \text{ as } q \to \infty \tag{8.171}$$

Equation (8.171) can be further simplified since R is a square matrix:

$$Z(s)\,G_p(s) \to -L\,\Psi(s)\,[H\,\Psi(s)\,B]^{-1}\,H\,\Phi(s)\,B \text{ as } q \to \infty \tag{8.172}$$

Now, consider the following identity:[18]

$$\Psi(s) = \Phi(s)\,[I + B\,L\,\Phi(s)]^{-1} \tag{8.173}$$

Substituting eqn. (8.173) into eqn. (8.172), and performing some simple algebraic manipulations yields

$$\begin{aligned}\lim_{q\to\infty} Z(s)\,G_p(s) = &-L\,\Phi(s)\,B\,[I + L\,\Phi(s)\,B]^{-1} \\ &\times \left\{H\,\Phi(s)\,B\,[I + L\,\Phi(s)\,B]^{-1}\right\}^{-1} H\,\Phi(s)\,B\end{aligned} \tag{8.174}$$

Since $H\,\Phi(s)\,B$ is assumed to be a square matrix, then eqn. (8.174) reduces down to

$$\lim_{q\to\infty} Z(s)\,G_p(s) = -L\,\Phi(s)\,B = -L\,(sI - F)^{-1}B \tag{8.175}$$

where the definition of $\Phi(s)$ from eqn. (8.163a) has been used. Hence the desired return ratio in eqn. (8.159) is achieved. It can be shown that the LTR approach drives the filter eigenvalues to the plant's zeros as q is increased.[18] Therefore, in order to maintain stability, the plant must have no transmission zeros in the right half-plane, which is also required by eqn. (8.168).

The design procedure for the LTR approach is as follows. First, design a Kalman filter and LQR control law to meet the desired estimation and control characteristics, treating them as separate design issues. The usual checks in the Kalman filter, trading off performance versus noisy estimates, should be employed to tune the initial process noise covariance. Once the initial estimator and control designs are completed, check the characteristics of the overall LQG system. If the stability margins are poor, then use eqn. (8.166) with some value for q to adjust the Kalman filter gain. Increase q until reasonable stability margins are given. It is imperative to make q only as large as possible because, in general, larger values for the process noise covariance introduce more high frequency noise into the filter state estimates.

Example 8.3: In this example we will show the usefulness of the LTR design procedure. We consider the following continuous-time system:[17]

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 35 \\ -61 \end{bmatrix} w(t) \\ \tilde{y}(t) &= \begin{bmatrix} 2 & 1 \end{bmatrix}\mathbf{x}(t) + v(t)\end{aligned}$$

Table 8.3: Summary of LTR Example Results

q^2	Filter Gain K	Phase Margin	Covariance P
0	$\begin{bmatrix} 30.00 \\ -49.96 \end{bmatrix}$	14.85°	$\begin{bmatrix} 96.23 & -162.46 \\ -162.46 & 274.95 \end{bmatrix}$
100	$\begin{bmatrix} 26.83 \\ -40.21 \end{bmatrix}$	19.39°	$\begin{bmatrix} 139.70 & -252.57 \\ -252.57 & 464.93 \end{bmatrix}$
500	$\begin{bmatrix} 20.38 \\ -17.75 \end{bmatrix}$	32.37°	$\begin{bmatrix} 212.59 & -404.80 \\ -404.80 & 791.85 \end{bmatrix}$
1,000	$\begin{bmatrix} 16.69 \\ -1.93 \end{bmatrix}$	42.50°	$\begin{bmatrix} 244.98 & -473.27 \\ -473.27 & 944.61 \end{bmatrix}$
10,000	$\begin{bmatrix} 6.94 \\ 84.62 \end{bmatrix}$	74.44°	$\begin{bmatrix} 297.68 & -588.43 \\ -588.43 & 1261.47 \end{bmatrix}$

with process noise and measurement noise variances given by $Q_0 = 1$ and $R = 1$, respectively. The resulting Kalman filter gain is given by $K = \begin{bmatrix} 30.00 & -49.96 \end{bmatrix}^T$. The estimator poles are placed at $s = -7.02 \pm 1.95j$ with this gain matrix. Suppose we now design an LQR control law with weighting matrices $\mathcal{R} = 1$ and $\mathcal{Q} = M^T M$, with $M = 4\sqrt{5}\begin{bmatrix} \sqrt{35} & 1 \end{bmatrix}$. Solving the LQR problem with these weighting matrices gives $L = \begin{bmatrix} 50 & 10 \end{bmatrix}$. The closed-loop LQR poles are placed at $s = -7 \pm 2j$ with this gain matrix, which are nearly identical to the estimator poles. The phase margin for the LQR system, which can be derived from the loop $-L(sI - F)^{-1}B$, is 85.94° (in general, the larger the phase margin the better the closed-loop characteristics). This indicates that LQR controller gives a well behaved closed-loop system.

Suppose we now use the Kalman filter in an LQG design with the predetermined Kalman and LQR gain matrices. The phase margin for the LQG system, which can be derived from the loop-gain transfer function matrix in eqn. (8.162), is now only 14.85°. This clearly has decreased the performance of the overall LQG controller design, compared with the original LQR design. Since the estimator poles are nearly identical to the LQR control poles, a natural assumption to make, before ever learning about the LTR approach, might be to place the estimator poles further down the left half-plane. Suppose we use Ackermann's formula, given by eqn. (5.19), to place the estimator's poles at $s = -22 \pm 17.86j$, which gives a gain matrix of $K = \begin{bmatrix} 720 & -1400 \end{bmatrix}^T$. The phase margin for this LQG designed system is now 4.17°, which is even worse than the original design! It fact, the margins go asymptotically to zero for large gains, which is clearly undesirable.

We now employ the LTR design approach, using eqn. (8.166) to recover the desired performance characteristics, with q^2 = 100, 500, 1,000, and 10,000. A summary of the results is shown in Table 8.3. Clearly, as q^2 is increased the phase margin also increases, which provides better closed-loop characteristics. Note when $q^2 = 10,000$ the Kalman gain approaches its limit, given by eqn. (8.170), of $K \rightarrow \begin{bmatrix} 0 & 100 \end{bmatrix}^T$. The improved closed-loop performance comes at a price though, since the filter covariance also increases as expected. The second state corresponds to the rate estimate, and the noise associated with this state substantially increases from the original design of $q^2 = 0$. In practice, hopefully, a satisfactory compromise between closed-loop stability margins and high frequency noise rejection can be found.

8.8 Spacecraft Control Design

In this section an LQG-based control system is designed to optimally orientate a spacecraft along a desired reference trajectory. The control of spacecraft for large angle slewing maneuvers poses a difficult problem. Some of these difficulties include: the highly nonlinear characteristics of the governing equations, control rate and saturation constraints and limits, and incomplete state knowledge due to sensor failure or omission. The control of spacecraft with large angle slews can be accomplished by either open-loop or closed-loop schemes. Open-loop schemes usually require a pre-determined pointing maneuver and are typically determined using optimal control techniques, which involve the solution of a TPBVP (e.g., the time optimal maneuver problem[26]). Also, open-loop schemes are sensitive to spacecraft parameter uncertainties and unexpected disturbances.[27, 28] Closed-loop systems can account for parameter uncertainties and disturbances, and as shown in this chapter, provide a more robust design methodology.

Several spacecraft attitude controllers have been developed that are devoted to the closed-loop design of spacecraft with large angle slews. An exhaustive history of this problem is beyond the present text; a starting reference point for many of these controllers can be found in Refs. [29] and [30]. In fact, a plethora of nonlinear and robust controllers have been developed, each with their own advantages and disadvantages. Our goal in the present text involves first using an LQR approach with *linear* dynamics. Paielli and Bach[31] present an optimal control design that provides linear closed-loop error dynamics for tracking a desired trajectory . However, this approach is singular for ±180° error-rotations about any axis. Schaub et al.[32] derive an optimal controller using the modified Rodrigues parameters[33] (MRPs), which are singular for +360° rotations. By switching between the original and alternative sets of MRPs (known as the *shadow set*), it is possible to achieve a globally nonsingular attitude parameterization for all possible ±360° rotations. An approach using MRPs is beyond the scope of this text, so we only will present the approach of Ref. [31]. Our derivation is slightly different than the one shown in Ref. [31], but the end result is the same. First, recall the kinematics and dynamics equations of motion given in §3.7:

$$\boxed{\begin{aligned}\dot{\mathbf{q}} &= \frac{1}{2}\Xi(\mathbf{q})\boldsymbol{\omega} = \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q} \qquad (8.176a)\\ \dot{\boldsymbol{\omega}} &= -J^{-1}[\boldsymbol{\omega}\times]J\boldsymbol{\omega} + J^{-1}\mathbf{L} \qquad (8.176b)\end{aligned}}$$

where $\mathbf{q}$ is the quaternion, $\boldsymbol{\omega}$ is the angular velocity vector, J is the inertia matrix, and $\mathbf{L}$ is the applied torque. Also, the quantities $\Xi(\mathbf{q})$ and $\Omega(\boldsymbol{\omega})$ are defined by eqns. (3.155a) and (3.162), respectively. Suppose that a desired quaternion, $\mathbf{q}_d$, is given that also follows the following kinematics equation:

$$\dot{\mathbf{q}}_d = \frac{1}{2}\Xi(\mathbf{q}_d)\boldsymbol{\omega}_d \tag{8.177}$$

where $\boldsymbol{\omega}_d$ is the desired angular velocity vector. We now define the following error quaternion:

$$\boldsymbol{\delta}\mathbf{q} = \mathbf{q} \otimes \mathbf{q}_d^{-1} \tag{8.178}$$

with $\boldsymbol{\delta}\mathbf{q} \equiv \begin{bmatrix}\boldsymbol{\delta\varrho}^T & \delta q_4\end{bmatrix}^T$. Also, the quaternion inverse is defined by eqn. (3.169). Using the rules of quaternion multiplication, discussed in §3.7.1, $\boldsymbol{\delta\varrho}$ and δq_4 can be shown to be given by

$$\boldsymbol{\delta\varrho} = \Xi^T(\mathbf{q}_d)\mathbf{q} \tag{8.179a}$$

$$\delta q_4 = \mathbf{q}_d^T\mathbf{q} \tag{8.179b}$$

Note that as $\boldsymbol{\delta\varrho}$ approaches zero, then the actual quaternion approaches the desired quaternion. Let us assume that the closed-loop dynamics are desired to have the following prescribed *linear* form:

$$\boldsymbol{\delta}\ddot{\boldsymbol{\varrho}} + L_2\boldsymbol{\delta}\dot{\boldsymbol{\varrho}} + L_1\boldsymbol{\delta\varrho} = \mathbf{0} \tag{8.180}$$

where L_1 and L_2 are 3×3 gain matrices. These matrices can be determined using an LQR approach:

$$\boldsymbol{\delta}\ddot{\boldsymbol{\varrho}} = \mathbf{u} \tag{8.181}$$

with

$$\mathbf{u} = -L\begin{bmatrix}\boldsymbol{\delta\varrho}\\ \boldsymbol{\delta}\dot{\boldsymbol{\varrho}}\end{bmatrix} \tag{8.182}$$

where $L \equiv \begin{bmatrix}L_1 & L_2\end{bmatrix}$. The state-space formulation of eqn. (8.181) is given by

$$\dot{\mathbf{x}} = \begin{bmatrix}0_{3\times3} & I_{3\times3}\\ 0_{3\times3} & 0_{3\times3}\end{bmatrix}\mathbf{x} + \begin{bmatrix}0_{3\times3}\\ I_{3\times3}\end{bmatrix}\mathbf{u} \tag{8.183}$$

where $\mathbf{x} \equiv \left[\delta\boldsymbol{\varrho}^T \;\; \delta\dot{\boldsymbol{\varrho}}^T\right]^T$. If L_1 and L_2 are assumed to be scalars, then these gains can be directly designed to yield the desired closed-loop dynamics without solving the LQR problem.

Our goal is to find a control torque input, $\mathbf{L}$, that achieves the desired closed-loop dynamics given by eqn. (8.180). Toward this end goal, two time derivatives of eqn. (8.179a) are first taken and then substituted into eqn. (8.180), which yields

$$\Xi^T(\mathbf{q}_d)\ddot{\mathbf{q}} + \left[2\Xi^T(\dot{\mathbf{q}}_d) + L_2\Xi^T(\mathbf{q}_d)\right]\dot{\mathbf{q}} + \left[\Xi^T(\ddot{\mathbf{q}}_d) + L_2\Xi^T(\dot{\mathbf{q}}_d) + L_1\Xi^T(\mathbf{q}_d)\right]\mathbf{q} = \mathbf{0} \tag{8.184}$$

Taking the time derivative of eqn. (8.176a) leads to

$$\begin{aligned}\ddot{\mathbf{q}} &= \frac{1}{2}\Xi(\mathbf{q})\dot{\boldsymbol{\omega}} + \frac{1}{2}\Omega(\boldsymbol{\omega})\dot{\mathbf{q}} \\ &= \frac{1}{2}\Xi(\mathbf{q})\dot{\boldsymbol{\omega}} - \frac{1}{4}(\boldsymbol{\omega}^T\boldsymbol{\omega})\mathbf{q}\end{aligned} \tag{8.185}$$

where the identity $\Omega^2(\boldsymbol{\omega}) = -(\boldsymbol{\omega}^T\boldsymbol{\omega})I_{4\times 4}$ has been used. An identical expression for the desired quaternion is also given:

$$\ddot{\mathbf{q}}_d = \frac{1}{2}\Xi(\mathbf{q}_d)\dot{\boldsymbol{\omega}}_d - \frac{1}{4}(\boldsymbol{\omega}_d^T\boldsymbol{\omega}_d)\mathbf{q}_d \tag{8.186}$$

where $\dot{\boldsymbol{\omega}}_d$ can be derived from a desired dynamics equation, using eqn. (8.176b), or it can be pre-specified from the known desired dynamical motion. Substituting eqn. (8.176b) into eqn. (8.185) gives

$$\ddot{\mathbf{q}} = -\frac{1}{2}\Xi(\mathbf{q})J^{-1}[\boldsymbol{\omega}\times]J\boldsymbol{\omega} - \frac{1}{4}(\boldsymbol{\omega}^T\boldsymbol{\omega})\mathbf{q} + \frac{1}{2}\Xi(\mathbf{q})J^{-1}\mathbf{L} \tag{8.187}$$

Substituting eqns. (8.176a) and (8.187) into eqn. (8.184), and solving for $\mathbf{L}$ yields

$$\boxed{\begin{aligned}\mathbf{L} = [\boldsymbol{\omega}\times]J\boldsymbol{\omega} + 2J\left[\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q})\right]^{-1}&\left\{\frac{1}{4}(\boldsymbol{\omega}^T\boldsymbol{\omega})\Xi^T(\mathbf{q}_d) - \Xi^T(\dot{\mathbf{q}}_d)\Omega(\boldsymbol{\omega})\right. \\ &\left. - \Xi^T(\ddot{\mathbf{q}}_d) - L_1\Xi^T(\mathbf{q}_d) - L_2\left[\frac{1}{2}\Xi^T(\mathbf{q}_d)\Omega(\boldsymbol{\omega}) + \Xi^T(\dot{\mathbf{q}}_d)\right]\right\}\mathbf{q}\end{aligned}} \tag{8.188}$$

Note that the inverse of $\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q})$ always exists as long as $\delta q_4 = \mathbf{q}_d^T\mathbf{q}$ is nonzero. This can easily be shown by the following identities:

$$\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q}) = \delta q_4 I_{3\times 3} + [\delta\boldsymbol{\varrho}\times] \tag{8.189a}$$

$$\left[\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q})\right]^{-1} = \delta q_4 I_{3\times 3} - [\delta\boldsymbol{\varrho}\times] + \frac{\delta\boldsymbol{\varrho}\,\delta\boldsymbol{\varrho}^T}{\delta q_4} \tag{8.189b}$$

From the definition of the scalar part of the quaternion in eqn. (3.153b) and from eqn. (8.179b), δq_4 is zero for $\pm 180°$ rotations in the tracking error, which is analogous to the approach shown in Ref. [31]. Hence, care must be exercised when the tracking errors approach $\pm 180°$. If L_1 and L_2 are scalars, with $L_1 = l_1$ and $L_2 = l_2$, then eqn. (8.188) simplifies to

$$\mathbf{L} = [\boldsymbol{\omega}\times]J\boldsymbol{\omega} + J\left\{\delta A\,\dot{\boldsymbol{\omega}}_d - [\boldsymbol{\omega}\times]\,\delta A\,\boldsymbol{\omega}_d - l_2\boldsymbol{\delta\omega} - 2\left[\frac{4l_1 - (\boldsymbol{\delta\omega}^T\boldsymbol{\delta\omega})}{4\delta q_4}\right]\boldsymbol{\delta\varrho}\right\} \tag{8.190}$$

with

$$\delta A = A(\mathbf{q})A^T(\mathbf{q}_d) \tag{8.191a}$$

$$\boldsymbol{\delta\omega} = \boldsymbol{\omega} - \delta A\,\boldsymbol{\omega}_d \tag{8.191b}$$

where the attitude matrix is defined by eqn. (3.154). Equation (8.190) can be proven using the following identities:

$$\left[\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q})\right]^{-1}\boldsymbol{\delta\varrho} = \frac{\boldsymbol{\delta\varrho}}{\delta q_4} \tag{8.192a}$$

$$2\left[\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q})\right]^{-1}\Xi^T(\dot{\mathbf{q}}_d)\Omega(\boldsymbol{\omega})\mathbf{q} = [\boldsymbol{\omega}\times]\,\delta A\,\boldsymbol{\omega}_d + \frac{\boldsymbol{\omega}^T\delta A\,\boldsymbol{\omega}_d}{\delta q_4}\boldsymbol{\delta\varrho} \tag{8.192b}$$

$$2\left[\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q})\right]^{-1}\Xi^T(\ddot{\mathbf{q}}_d)\mathbf{q} = -\left[\delta A\,\dot{\boldsymbol{\omega}}_d + \frac{\boldsymbol{\omega}_d^T\boldsymbol{\omega}_d}{2\delta q_4}\boldsymbol{\delta\varrho}\right] \tag{8.192c}$$

$$2\left[\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q})\right]^{-1}\Xi^T(\dot{\mathbf{q}}_d)\mathbf{q} = -\delta A\,\boldsymbol{\omega}_d \tag{8.192d}$$

Equation (8.190) is equivalent to the control law given in Ref. [31]. Note that eqn. (8.190) does not explicitly involve $\dot{\mathbf{q}}_d$ and $\ddot{\mathbf{q}}_d$.

The procedure for the spacecraft attitude controller proceeds as follows. First, given a desired quaternion, $\mathbf{q}_d$, angular velocity vector, $\boldsymbol{\omega}_d$, and angular acceleration vectors, $\dot{\boldsymbol{\omega}}_d$, compute the desired quaternion rate and acceleration using eqns. (8.177) and (8.186). Then, design an LQR feedback gain to achieve the desired closed-loop tracking dynamics shown by eqns. (8.180) and (8.183), which gives the matrices L_1 and L_2. Finally, use the control law given by eqn. (8.188), to drive the spacecraft's attitude and angular velocity to the desired trajectories. This controller can be combined with an extended Kalman filter to filter noisy measurements and to estimate gyro biases, as shown in §7.2.1, which leads to an LQG-type control system.

Example 8.4: In this example the control law given by eqn. (8.188) is combined with the EKF of §7.2.1 to maneuver a spacecraft along a desired trajectory. The assumed sensors include "quaternion-out" star trackers (see exercise 7.14) and three-axis gyros. The noise parameters for the gyro measurements are given by $\sigma_u = \sqrt{10}\times 10^{-10}$ rad/sec$^{3/2}$ and $\sigma_v = \sqrt{10}\times 10^{-7}$ rad/sec$^{1/2}$. The initial bias for each axis is given by 0.1 deg/hr. A combined quaternion from two trackers is assumed for the measurement. In order to generate synthetic measurements the following model is used:

$$\tilde{\mathbf{q}} = \begin{bmatrix} 0.5\mathbf{v} \\ 1 \end{bmatrix} \otimes \mathbf{q}$$

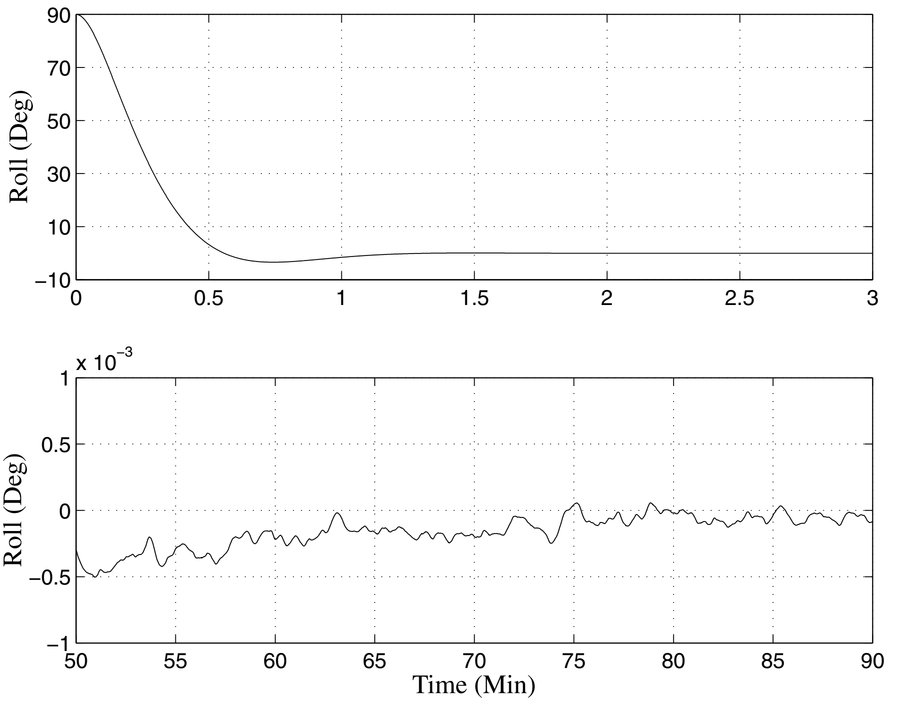

Figure 8.7: Roll Pointing Errors

where $\tilde{\mathbf{q}}$ is the quaternion measurement, $\mathbf{q}$ is the truth, and $\mathbf{v}$ is the measurement noise, which is assumed to be a zero-mean Gaussian noise process with covariance given by $0.001 I_{3\times3}$ deg^2. Note, the measured quaternion is normalized to within first-order, but a brute-force normalization is still taken to ensure a normalized measurement. All quaternion and gyro measurements are sampled at 10 Hz. The initial covariances for the attitude error and gyro drift are taken exactly from example 7.2.

The spacecraft desired motion includes a constant angular velocity vector given by $\boldsymbol{\omega}_d = \begin{bmatrix} 0 & 0.0011 & 0 \end{bmatrix}^T$ rad/sec, which corresponds to an Earth-pointing spacecraft in low-Earth orbit. The actual initial angular velocity of the spacecraft is given by $\boldsymbol{\omega}(t_0) = \mathbf{0}$. The initial desired and actual quaternions are given by

$$\mathbf{q}_d(t_0) = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}, \quad \mathbf{q}(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Equation (8.177) is used to propagate the desired quaternion over time. The space-

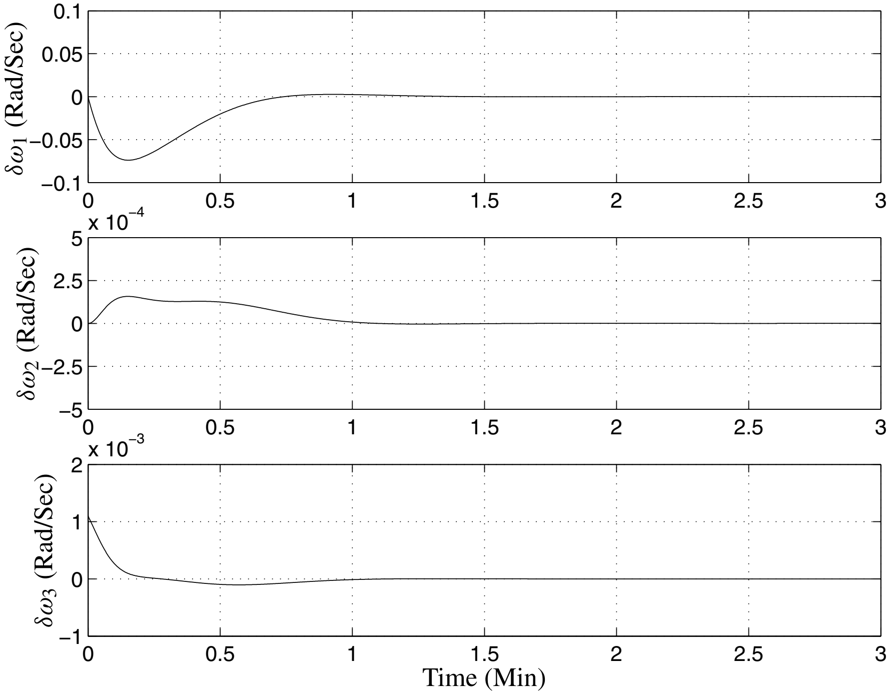

Figure 8.8: Angular Velocity Errors

craft inertia matrix is given by[32]

$$J = \begin{bmatrix} 30 & 10 & 5 \\ 10 & 20 & 3 \\ 5 & 3 & 15 \end{bmatrix} \text{ kg-m}^2$$

An LQR is designed with the model in eqn. (8.183), using the method outlined in §8.5.1. The steady-state Riccati equation in eqn. (8.68) is solved to determine L_1 and L_2. The weighting matrices are given by $\mathcal{Q} = 1 \times 10^{-4} I_{6\times 6}$ and $\mathcal{R} = I_{3\times 3}$. Using these weights gives $L_1 = 0.01 I_{3\times 3}$ and $L_2 = 0.14177 I_{3\times 3}$. The closed-loop natural frequencies and damping ratios (see §3.10) are given by $\omega_n = 0.1$ rad/sec and $\zeta = 0.709$. These gains are used in eqn. (8.188), with the estimated quaternion and angular velocities determined for the EKF (i.e., an LQG-type design), which provides the control torque input into the spacecraft.

A plot of the roll pointing-error trajectory is shown in Figure 8.7. The time required for the damped oscillations to reach and stay within ±2% of the steady-state value is given by $4/(\zeta\omega_n)$.[23] For the LQR design this formula gives a settling time of 56.4175 seconds, which agrees with the result shown in Figure 8.7. This result can also be checked by integrating eqn. (8.180). The bottom plot of Figure 8.7 shows the roll error in finer detail. Clearly, fine pointing can be achieved with this control

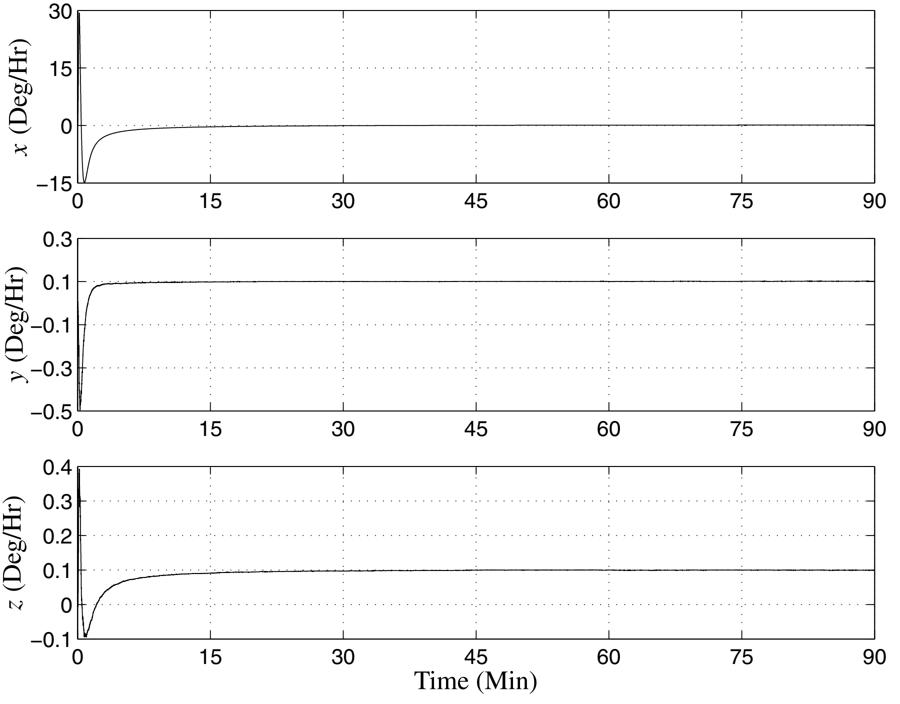

Figure 8.9: Gyro Drift Estimates

law and assumed sensors. A plot of the angular velocity errors is shown in Figure 8.8. Clearly, the desired angular velocity motion is achieved. A plot of the gyro drift estimates using the EKF is shown in Figure 8.9. The x axis has a large response due to the roll maneuver. All axes still converge to the actual bias of 0.1 deg/hr. Note that in a practical setting, the gyro biases are normally allowed to converge before a significant maneuver takes place. Still, this example clearly shows how an EKF can be combined with a control law to achieve effective overall pointing of a very practical system involving large-angle spacecraft maneuvers.

8.9 Summary

This chapter provided only a brief introduction to the theory of optimal control. Several texts and books have been written that provide much more depth in the subject area that can be covered here (e.g., see the references used in this chapter). Optimal control theory has uses well beyond the control of dynamic systems (e.g., optimal path planning for shipping routes), and we encourage the interested reader to pursue other topics where this theory can be used. The main results of this chapter involve the LQR control law and Separation Theorem used in the LQG controller. Although from a practical point of view, the theory used in the Separation Theorem is masked behind the actual control implementation, we believe that the reader will benefit from the derivation and understanding of this elegant theory. Also, the LTR approach of §8.7 is especially useful to recover the originally designed regulator dynamics. A general "rule-of-thumb" is to only use LTR when needed, because increasing the process noise covariance may lead to too much high frequency noise in the output estimates.

A summary of the key formulas presented in this chapter is given below.

- Euler-Lagrange Equations and Transversality Conditions

$$J \equiv J(\mathbf{x}(t), t_0, t_f) = \int_{t_0}^{t_f} \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)\, dt$$

$$\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \mathbf{x}(t)} - \frac{d}{dt}\left[\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}(t)}\right] = \mathbf{0}$$

$$\left.\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)}\right|_{t_f} \boldsymbol{\delta}\mathbf{x}_f = 0$$

$$\left[\vartheta(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f) - \left.\frac{\partial \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)}{\partial \dot{\mathbf{x}}^T(t)}\right|_{t_f} \dot{\mathbf{x}}(t_f)\right] \delta t_f = 0$$

- Optimization with Differential Equation Constraints

$$J = \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \vartheta(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$H \equiv \vartheta(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}^T(t)\, \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$\dot{\mathbf{x}}(t) = \frac{\partial H}{\partial \boldsymbol{\lambda}(t)} \equiv \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial H}{\partial \mathbf{x}(t)} \equiv -\frac{\partial \vartheta(\mathbf{x}(t), \mathbf{u}(t), t)}{\partial \mathbf{x}(t)} - \left[\frac{\partial \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)}{\partial \mathbf{x}(t)}\right]^T \boldsymbol{\lambda}(t)$$

$$\frac{\partial H}{\partial \mathbf{u}(t)} = \mathbf{0}$$

$$\left.\left[\frac{\partial \phi(\mathbf{x}(t), t)}{\partial t} + \boldsymbol{\alpha}^T \frac{\partial \psi(\mathbf{x}(t), t)}{\partial t} + H\right]\right|_{t_f} = 0$$

$$\boldsymbol{\lambda}(t_f) = \left\{ \frac{\partial \phi(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)} + \left[\frac{\partial \boldsymbol{\psi}(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)} \right]^T \boldsymbol{\alpha} \right\} \Bigg|_{t_f}$$

- Pontryagin's Optimal Control Necessary Conditions

$$J = \frac{1}{2} \int_{t_0}^{t_f} \mathbf{x}^T(t)\, \mathcal{Q}\, \mathbf{x}(t)\, dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t) + \mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\boldsymbol{\psi}(\mathbf{x}(t_f), t_f) = \mathbf{0}$$

$$H = \frac{1}{2} \mathbf{x}^T(t)\, \mathcal{Q}\, \mathbf{x}(t) + \boldsymbol{\lambda}^T(t)[\mathbf{f}(\mathbf{x}(t), t) + \mathbf{u}(t)]$$

$$\Phi(\mathbf{x}(t_f), t_f) \equiv \phi(\mathbf{x}(t_f), t_f) + \boldsymbol{\alpha}^T \boldsymbol{\psi}(\mathbf{x}(t_f), t_f)$$

$$|u_j(t)| \le u_{\max_j}, \quad j = 1, 2, \ldots, p$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t) + \mathbf{u}(t)$$

$$\dot{\boldsymbol{\lambda}}(t) = -\left[\frac{\partial \mathbf{f}(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)} \right]^T \boldsymbol{\lambda}(t) - \mathcal{Q}\, \mathbf{x}(t)$$

$$\mathbf{u}(t) = \begin{bmatrix} s_1 u_{\max_1} \\ s_2 u_{\max_2} \\ \vdots \\ s_p u_{\max_p} \end{bmatrix}, \quad s_i = \operatorname{sign}[\lambda_i(t)]$$

- Discrete-Time Control

$$J = \phi(\mathbf{x}_N, t_f) + \sum_{k=0}^{N-1} \vartheta_k(\mathbf{x}_k, \mathbf{u}_k, k)$$

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, k)$$

$$\boldsymbol{\psi}(\mathbf{x}_N, t_f) = \mathbf{0}$$

$$H_k \equiv \vartheta_k(\mathbf{x}_k, \mathbf{u}_k, k) + \boldsymbol{\lambda}_{k+1}^T \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, k)$$

$$\Phi(\mathbf{x}_N, t_f) \equiv \phi(\mathbf{x}_N, t_f) + \boldsymbol{\alpha}^T \boldsymbol{\psi}(\mathbf{x}_N, t_f)$$

$$\mathbf{x}_{k+1} = \frac{\partial H_k}{\partial \boldsymbol{\lambda}_{k+1}} \equiv \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, k)$$

$$\boldsymbol{\lambda}_k = \frac{\partial H_k}{\partial \mathbf{x}_k} \equiv \frac{\partial \vartheta_k(\mathbf{x}_k, \mathbf{u}_k, k)}{\partial \mathbf{x}_k} + \left[\frac{\partial \mathbf{f}_k(\mathbf{x}_k, \mathbf{u}_k, k)}{\partial \mathbf{x}_k}\right]^T \boldsymbol{\lambda}_{k+1}$$

$$\frac{\partial H_k}{\partial \mathbf{u}_k} = \mathbf{0}$$

$$\left[\frac{\partial \Phi(\mathbf{x}_k, t_f)}{\partial \Delta t} + \sum_{k=0}^{N-1} \frac{\partial H_k}{\partial \Delta t}\right] \delta \Delta t = 0$$

$$\frac{\partial \Phi(\mathbf{x}_k, t_f)}{\partial \Delta t} + \sum_{k=0}^{N-1} \frac{\partial H_k}{\partial \Delta t} = 0$$

$$\boldsymbol{\lambda}_N = \left\{\frac{\partial \phi(\mathbf{x}_k, t_f)}{\partial \mathbf{x}_k} + \left[\frac{\partial \psi(\mathbf{x}_k, t_f)}{\partial \mathbf{x}_k}\right]^T \boldsymbol{\alpha}\right\}\Bigg|_N$$

- Linear Quadratic Regulator (Continuous-Time)

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

$$\mathbf{u}(t) = -L(t)\,\mathbf{x}(t)$$

$$\dot{S}(t) = -S(t)\,F(t) - F^T(t)\,S(t) + S(t)\,B(t)\,\mathcal{R}^{-1}(t)\,B^T(t)\,S(t) - \mathcal{Q}(t)$$

$$L(t) = \mathcal{R}^{-1}(t)\,B^T(t)\,S(t)$$

- Linear Quadratic Regulator (Discrete-Time)

$$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

$$\mathbf{u}_k = -L_k \mathbf{x}_k$$

$$S_k = \Phi_k^T S_{k+1} \Phi_k - \Phi_k^T S_{k+1} \Gamma_k \left[\Gamma_k^T S_{k+1} \Gamma_k + \mathcal{R}_k\right]^{-1} \Gamma_k^T S_{k+1} \Phi_k + \mathcal{Q}_k$$

$$L_k = \left[\Gamma_k^T S_{k+1} \Gamma_k + \mathcal{R}_k\right]^{-1} \Gamma_k^T S_{k+1} \Phi_k$$

- Stochastic Hamilton-Jacobi-Bellman Equation

$$\frac{\partial J^*(\mathbf{x}(t), t)}{\partial t} + \min_{\mathbf{u}(t)} \left\{ \vartheta(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{\partial J^*(\mathbf{x}(t), \mathbf{u}(t), t)}{\partial \mathbf{x}^T(t)} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \right.$$
$$\left. + \frac{1}{2} \mathrm{Tr}\left[G(t)\,Q(t)\,G(t) \frac{\partial^2 J^*(\mathbf{x}(t), t)}{\partial \mathbf{x}^2(t)} \right] \right\} = 0$$

- Loop Transfer Recovery

$$G\,Q\,G^T = G\,Q_0 G^T + q^2 B\,B^T$$

$$F\left(\frac{P}{q^2}\right) + \left(\frac{P}{q^2}\right) F^T - q^2 \left(\frac{P}{q^2}\right) H^T R^{-1} H \left(\frac{P}{q^2}\right) + \frac{G\,Q_0 G^T}{q^2} + B\,B^T = 0$$

- Spacecraft Control

$$\dot{\mathbf{q}} = \frac{1}{2}\Xi(\mathbf{q})\boldsymbol{\omega} = \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q}$$
$$\dot{\boldsymbol{\omega}} = -J^{-1}[\boldsymbol{\omega}\times]J\boldsymbol{\omega} + J^{-1}\mathbf{L}$$

$$\mathbf{L} = [\boldsymbol{\omega}\times]J\boldsymbol{\omega} + 2\,J\left[\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q})\right]^{-1}\left\{\frac{1}{4}(\boldsymbol{\omega}^T\boldsymbol{\omega})\Xi^T(\mathbf{q}_d) - \Xi^T(\dot{\mathbf{q}}_d)\Omega(\boldsymbol{\omega})\right.$$
$$\left. - \Xi^T(\ddot{\mathbf{q}}_d) - L_1\,\Xi^T(\mathbf{q}_d) - L_2\left[\frac{1}{2}\Xi^T(\mathbf{q}_d)\Omega(\boldsymbol{\omega}) + \Xi^T(\dot{\mathbf{q}}_d)\right]\right\}\mathbf{q}$$

Exercises

8.1 Suppose that both $\mathbf{x}(t_f)$ and t_f are free, but related by $\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$. Derive the transversality condition to determine the final time for this constraint.

8.2 Consider minimizing the loss function in eqn. (8.22), with $\phi(\mathbf{x}(t_f), t_f) = 0$, equality constraint given by eqn. (8.19), and final time fixed. The continuous-time solution is given by the TPBVP shown in eqn. (8.27), with $\boldsymbol{\lambda}(t_f) = \mathbf{0}$. Using first-order finite difference approximations for the state and costate derivatives, develop simple discrete-time approximations to the TPBVP equations involving a constant sampling interval Δt. An alternative approach to this approximation is given by discretizing the loss function and equality constraint:

$$J = \Delta t\sum_{k=0}^{N-1}\vartheta(\mathbf{x}_k, \mathbf{u}_k, k)$$
$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t\,\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k)$$

Now, with this discretization derive the associated TPVBP using the methods of §8.4. You will see that the equations associated with this TPBVP are not equivalent to the ones obtained by discretizing the continuous-time TBPVP equations. Under what conditions do both sets of equations give nearly identical solutions?

8.3 Take a Taylor series expansion of eqn. (8.50) to prove the expression given in eqn. (8.51).

8.4 The minimum energy required to charge a capacitor for a portable defibrillator using an RC circuit can be achieved by minimizing the following loss function:

$$J = \int_0^1\left[\dot{v}(t) + \frac{1}{RC}v(t)\right]^2 dt, \quad v(0) = 0, \quad v(1) = 400 \text{ volts}$$

where R and C are constants, and $v(t)$ is the voltage. Using the methods of §8.1 show the associated Euler-Lagrange equations for this problem. Find the optimal trajectory for $v(t)$ that minimizes J in closed form. How does the trajectory change for increasing RC?

8.5 Consider the minimization of the following loss function:

$$J = \int_0^1 \left[x^2(t) + \dot{x}^2(t)\right] dt, \quad x(0) = 1, \quad x(1) = 0$$

Using the methods of §8.1 show the associated Euler-Lagrange equations for this problem. Find the optimal trajectory for $x(t)$ that minimizes J in closed form. Show that δJ is zero for all admissible perturbations $\delta x(t)$.

8.6 Consider the minimization of the following loss function:

$$J = \frac{1}{2}\int_0^{t_f} \dot{\mathbf{x}}^T(t)\dot{\mathbf{x}}(t)\, dt + \frac{1}{2}x_1^2(t_f), \quad (t_f \text{ free})$$
$$\mathbf{x}(0) = \begin{bmatrix}0 & 1\end{bmatrix}^T, \quad x_2(t_f) = -t_f$$

where $\mathbf{x} = \begin{bmatrix}x_1 & x_2\end{bmatrix}^T$. Using the variational methods of §8.1 derive the following transversality conditions for this problem:

$$\left.\frac{\partial\phi(\mathbf{x}(t),t)}{\partial\dot{\mathbf{x}}^T(t)}\right|_{t_f}\boldsymbol{\delta}\mathbf{x}(t_f) + \left.\frac{\partial\vartheta(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial\dot{\mathbf{x}}^T(t)}\right|_{t_f}\boldsymbol{\delta}\mathbf{x}(t_f)$$
$$+ \left.\left[\vartheta(\mathbf{x}(t),\dot{\mathbf{x}}(t),t) - \frac{\partial\vartheta(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial\dot{\mathbf{x}}^T(t)}\dot{\mathbf{x}}(t)\right]\right|_{t_f}\delta t_f = 0$$

where $\phi(\mathbf{x}(t_f),t_f) \equiv x_1^2(t_f)/2$ and $\vartheta(\mathbf{x}(t),\dot{\mathbf{x}}(t),t) \equiv \dot{\mathbf{x}}^T(t)\dot{\mathbf{x}}(t)/2$. Using the Euler-Lagrange equations with these transversality conditions, determine the solutions for the optimal $\mathbf{x}(t)$ and final time t_f. If t_f is fixed rather than free, how would the optimal trajectory differ?

8.7 Consider the minimization of the following loss function:

$$J = \frac{1}{2}\int_0^1 \left[x^2(t) + \dot{x}^2(t)\right] dt, \quad x(0) = 1, \quad x(1) = \text{ free}$$

Using the methods of §8.1 show the associated Euler-Lagrange equations and transversality conditions for this problem. Find the optimal trajectory for $x(t)$ that minimizes J in closed form.

8.8 Consider the minimization of the following loss function:

$$J = \frac{1}{2}\int_{-1}^1 \left[u^2(t) + 5x^2(t)\right] dt$$
$$\dot{x}(t) = -2x(t) + u(t), \quad x(0) = 1$$

Using the methods of §8.1 show the associated Euler-Lagrange equations and transversality conditions for this problem (note: a boundary condition is given at $t = 0$, but the integral is given from $t = -1$ to $t = +1$). Find the optimal trajectory for $x(t)$ that minimizes J in closed form.

8.9 Consider the minimization of the following loss function:

$$J = \frac{1}{2}\int_0^{t_f} \left[1+u^2(t)\right] dt, \quad (t_f \text{ free})$$

$$\dot{x}(t) = -a\,x(t) + u(t), \quad x(0) = 1, \quad x(t_f) = 1$$

where a is a positive constant. Using the methods of §8.1 show the associated Euler-Lagrange equations and transversality conditions for this problem. Find the optimal trajectory for $x(t)$ that minimizes J in closed form.

8.10 Consider the minimization of the following loss function:

$$J = \frac{1}{2}\int_0^{t_f} \sqrt{1+\dot{x}^2(t)}\, dt, \quad (t_f \text{ free})$$

$$x(0) = 0, \quad x(t_f) = -5t_f + 15$$

Using the methods of §8.1 and the results from exercise 8.6 show the associated Euler-Lagrange equations and transversality conditions for this problem. Find the optimal trajectory for $x(t)$ and the final time t_f that minimize J in closed form.

8.11 ♣ Consider the following functional:

$$J = \int_{t_0}^{t_f} \vartheta(\mathbf{x}(t), \dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t), t)\, dt + \phi(\mathbf{x}(t_f), \dot{\mathbf{x}}(t_f), t_f)$$

where t_f is fixed. Express δJ in terms of $\delta\mathbf{x}(t)$ and endpoint perturbations to derive the Euler-Lagrange equations and transversality conditions (hint: integrate by parts twice). Note that this is a generalized extension of the first problem, not a first problem.

8.12 Consider the minimization of the following loss function:

$$J = \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \vartheta(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

Suppose that instead of a differential constraint given by eqn. (8.19) we have the general (possibly nonlinear) constraint given by

$$\mathbf{g}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0$$

Using a set of Lagrange multipliers derive the associated Euler-Lagrange equations and transversality conditions for this problem. First assume that t_f is fixed, then allow it to be free.

8.13 Consider the minimization of the following loss function:

$$J = \frac{1}{2}\int_0^{t_f} u^2(t)\, dt + \frac{1}{2}x_1^2(t_f), \quad (t_f \text{ free})$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

$$x_1(0) = 0, \quad x_2(0) = \text{free}$$

$$x_2(t_f) = x_1^2(t_f) - 1$$

where $\mathbf{x} = \left[x_1 \;\; x_2\right]^T$. Using the Hamiltonian approach of §8.2 derive the state and costate equations. Also, specify the appropriate boundary conditions. The optimal input $u(t)$ is a function of what two time functions? How would the answer to the solution of this minimization problem change if the constraint $0 \leq |u(t)| \leq 1$ were added to the problem statement?

8.14 In this exercise you will test the robustness of the open-loop control law developed in example 8.1. First, reproduce the results shown in Figure 8.2. Then, multiply the control torque $u(t)$ in example 8.1 by some scalar, which simulates an error in the inertia J, and use this control input with the identical boundary conditions shown in the example. How do the state and control input trajectories change for various scalar multiplication factors?

8.15 In example 8.1 a rigid body constrained to rotate about a fixed axis is considered with the final time fixed at $t_f = T$. Consider the following boundary conditions: $\theta(0) = \theta_0$, $\dot{\theta}(0) = \dot{\theta}_0$, $\theta(T) = 0$, and $\dot{\theta}(T) = 0$. Also, consider only piecewise continuous controls satisfying $|u(t)| \leq 1$. We seek to minimize the maneuver "time-to-go"

$$J = c\int_0^T dt$$

where c is a positive scale factor whose arbitrary value will be chosen to accomplish a useful normalization of the costate variables. Using the Hamiltonian approach with Pontryagin's Principle of §8.3 derive the state and costate equations. Show that only one sign change (at most) can occur in $u(t)$.

8.16 ♣ Using the equations derived in exercise 8.15, show that if $\lambda_1(t)$ and $\lambda_2(t)$ are solutions to the costate differential equations, then $\alpha\lambda_1(t)$ and $\alpha\lambda_2(t)$ are also solutions for α = an arbitrary positive constant. Deduce that the α-scaling on λ_i dictates a specific c value:

$$c = -[\lambda_1(T)\,x_2(T) + \lambda_2(T)\,u(T)]$$

Since an infinity of linearly scaled costates generate the same control, we take advantage of this truth to scale initial conditions on the λ's so that the initial costates lie on the unit circle

$$\lambda_1^2(0) + \lambda_2^2(0) = 1$$

or, alternatively, we can define the complete family of trajectories by introducing an initial phase γ such that $\lambda_1(0) = \cos\gamma$ and $\lambda_2(0) = \sin\gamma$, where $0 \leq \gamma \leq 360°$. Show that the optimal control is given by $u(t) = -\text{sign}(\sin\gamma - t\cos\gamma)$, and that the switch times, t_s, are related to the γ-values as $t_s = \tan\gamma$. Construct an analytical solution for the $u = +1$ and $u = -1$ trajectories. Show that the control in the second quadrant of a $[x_1(t),\, x_2(t)]$ phase plot switches from positive to negative when the positive torque trajectories intersect the switching curve $x_1(t) = -x_2^2(t)/2$, whereas the control in the fourth quadrant of a $[x_1(t),\, x_2(t)]$ phase plot switches from negative to positive when the initially negative torque trajectories intersect the switching curve $x_1(t) = +x_2^2(t)/2$. Construct a global portrait of the time optimal "bang-bang" trajectories.

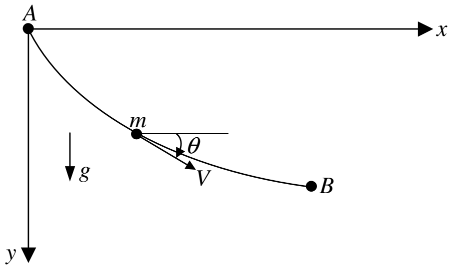

Figure 8.10: The Brachistochrone Problem

8.17 Consider the following second-order dynamical system:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

where $x_1(t)$ is equivalent to $\theta(t)$ from example 8.1. We now wish to develop a control-rate penalty technique that minimizes the loss function

$$J = \frac{1}{2}\int_0^{t_f} \left[w^2 u^2(t) + \dot{u}^2(t) \right] dt$$

where $u(t)$ is assumed to have two continuous derivatives, and w is a positive constant weight. We can easily convert this loss function into a standard form by simply introducing a new "state variable" $x_3(t) = u(t)$ and defining a new control variable $v(t) \equiv \dot{u}(t)$. Thus we seek to minimize

$$J = \frac{1}{2}\int_0^{t_f} \left[w^2 x_3^2(t) + v^2(t) \right] dt$$

subject to

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = x_3(t)$$
$$\dot{x}_3(t) = v(t)$$

Using the Hamiltonian approach of §8.2 derive the state and costate equations. Assume that the state boundary conditions are given by $x_1(0) = \theta_0$, $x_2(0) = \dot{\theta}_0$, $x_1(t_f) = \theta_f$, and $x_2(t_f) = \dot{\theta}_f$. Also, since we require that the control be zero initially and vanish upon completion, we have $u(0) = 0$ and $u(t_f) = 0$. Find analytical expressions for $x_1(t)$ and $u(t)$ that minimize J in closed form. How does the solution change for the special case where $w = 0$?

8.18 Consider the classical *brachistochrone problem* shown in Figure 8.10. Given two points, A and B, in space with A higher than B, but not vertically above B, what shape of wire connecting A to B will have the property that a bead sliding along it under gravity gets from A to B in the shortest time? Using the principle of energy we can write

$$\frac{1}{2}m\,[\dot{x}^2(t)+\dot{y}^2(t)] = mg\,y(t)$$

where m is the mass of the bead and g is the gravity constant. This equation can be written as

$$\frac{dx(t)}{dt}\left[\sqrt{\frac{\dot{y}(t)}{\dot{x}(t)}}+1\right] = \sqrt{2g\,y(t)}$$

We wish to minimize the time taken, so

$$J = \int_0^T dt = c\int_0^1 \left[\frac{1+\dot{y}^2(t)/\dot{x}^2(t)}{y(t)}\right]^{1/2} dx$$

where $c = 1/\sqrt{2g}$. But since $\dot{y}(t)/\dot{x}(t) = dy/dx$, the minimization problem can be stated as

$$J = c\int_0^1 f\left(y(x),\,\frac{dy}{dx}\right)dx$$
$$y(0)=0, \quad y(1)=1$$

where

$$f(y,s) = \left[\frac{1+s^2}{y}\right]^{1/2}$$

The velocity components can be written as[5]

$$\dot{x}(t) = V(y)\cos\theta(t) \tag{8.193}$$
$$\dot{y}(t) = V(y)\sin\theta(t) \tag{8.194}$$

where the velocity is given by $V(y) = \sqrt{V_0^2+2g\,y(t)}$, and V_0 is the initial velocity at point A. Solve the Euler-Lagrange equations to show that the paths for $x(t)$ and $y(t)$ are cycloids, i.e., paths generated by a point on a circle rolling without slipping in a horizontal directions, and that $\dot{\theta}$ is constant.

8.19 Verify by direct substitution that the solution of HJB equation in eqn. (8.58) is indeed given by eqn. (8.59).

8.20 Take the second variation of eqn. (8.54). What are the sufficient conditions on $\mathcal{Q}$, $\mathcal{R}$, and S_f to guarantee a minimum?

8.21 In the LQR loss function of eqn. (8.54) no weighting between the cross-correlation of the state $\mathbf{x}(t)$ and input $\mathbf{u}(t)$ is given. Suppose that we now wish to minimize the following loss function, which includes this cross-weighting:

$$J = \frac{1}{2}\mathbf{x}^T(t_f)S_f\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{x}^T(t)\;\mathbf{u}^T(t)\right]\begin{bmatrix}\mathcal{Q}(t) & \mathcal{N}(t)\\ \mathcal{N}^T(t) & \mathcal{R}(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{bmatrix}dt$$

where $\mathcal{N}(t)$ is the cross-weighting matrix. Using the methods of §8.5.1, derive new LQR results using a Riccati transformation that minimizes this loss function.

8.22 A similar loss function to the one shown in exercise 8.21 can be derived for the discrete-time case:

$$J = \frac{1}{2}\mathbf{x}_N^T S_f \mathbf{x}_N + \sum_{k=0}^{N-1} \begin{bmatrix} \mathbf{x}_k^T & \mathbf{u}_k^T \end{bmatrix} \begin{bmatrix} \mathcal{Q}_k & \mathcal{N}_k \\ \mathcal{N}_k^T & \mathcal{R}_k \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix}$$

Using the methods of §8.5.2, derive new LQR results using a Riccati transformation that minimizes this loss function.

8.23 Consider the minimization of the following discrete-time loss function:[2]

$$J = \frac{1}{2}\sum_{k=0}^{9} u_k^2$$

$$x_{k+1} = x_k + \gamma\, u_k$$

$$x_0 = 1, \quad x_{10} = 0$$

where γ is a constant. Determine a closed-form solution for x_k that minimizes this loss function and meets the desired boundary conditions.

8.24 Prove that the discrete-time Riccati equation in eqn. (8.103) is equivalent to eqn. (8.102).

8.25 Starting with the expression given in eqn. (8.121), prove the expression given in eqn. (8.122) using the methods of §5.4.1 and the definition of the innovations process in §6.4.2.2.

8.26 Prove that the eigenvalues of the system matrices in eqns. (8.133) and (8.135) are equivalent to each other.

8.27 ♣ Using the Stochastic Hamilton-Jacobi-Bellman equation of eqn. (8.137) to prove the Separation Theorem for continuous-time systems.

8.28 Starting with the Kalman filter equations of eqn. (5.54), show that the mean and covariance of $\hat{\mathbf{x}}_{k+1}$ are given by the expressions in eqn. (8.152).

8.29 Substituting the gain L_k, given by eqn. (8.98), show that eqn. (8.154) is equivalent to eqn. (8.153). Also, show that the matrix $[\Gamma_k S_{k+1} \Gamma_k + \mathcal{R}_k]$ is always positive definite.

8.30 Starting with the Kalman filter estimator form in eqn. (5.54a) and truth model in eqn. (8.141), prove the eigenvalue separation of the combined estimator and controller system by showing that the closed-loop LQG dynamics are given by

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \tilde{\mathbf{x}}_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi - \Gamma L & -\Gamma L \\ 0 & \Phi(I - K H) \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \tilde{\mathbf{x}}_k \end{bmatrix} + \begin{bmatrix} \Upsilon & 0 \\ -\Upsilon & \Phi K \end{bmatrix} \begin{bmatrix} \mathbf{w}_k \\ \mathbf{v}_k \end{bmatrix}$$

where $\tilde{\mathbf{x}}_k \equiv \hat{\mathbf{x}}_k - \mathbf{x}_k$, and all system and covariance matrices are assumed to be constants.

8.31 Show that eqn. (8.164) is equivalent to eqn. (8.165) by using the matrix inversion lemma.

8.32 Reproduce the results shown for the LTR system in example 8.3. Create synthetic measurements using various standard deviations for the measurement noise with the linear system described in the example. Using the LTR filter gains test the performance of the overall system executing various simulated runs. At what noise levels in the measurements can you find a satisfactory compromise between closed-loop stability margins and high-frequency noise rejection? Discuss the metrics used to qualify this compromise in your simulations.

8.33 In this exercise you will design an optimal controller involving a terminal guidance system for satellite rendezvous. Although the relative equations of the motion for two spacecraft flying in formation are highly nonlinear, if the spacecraft are close to each other, then a linearized solution works well for short periods. A commonly used set of linearized equations is given by the Clohessy-Wiltshire equations or Hill's equations:[34]

$$\begin{aligned}
\ddot{r}(t) - 2n\,\dot{s}(t) - 3n^2 r(t) &= F_r(t)\\
\ddot{s}(t) + 2n\,\dot{r}(t) &= F_s(t)\\
\ddot{z}(t) + n^2 z(t) &= F_z(t)
\end{aligned}$$

where $r(t)$ is the radial direction, $s(t)$ is the cross-track direction, $z(t)$ is perpendicular to the reference orbit plane, n is the mean motion (see §3.8.2) of the leader spacecraft, and $F_r(t)$, $F_s(t)$, and $F_z(t)$ are control variables. Assuming a low-Earth orbit (with n = 0.0011 rad/sec), design a steady-state LQR controller for this system. For your design assume that the position states in all directions are initially about 1 km with zero velocity errors, and bring the errors to zero within 20 minutes (set $\mathcal{Q}$ to be the identity matrix and adjust $\mathcal{R}$ to meet the design specifications). Use your LQR steady-state control input on the full nonlinear equations of motion, given by

$$\begin{aligned}
\ddot{r}(t) - 2n\,\dot{s}(t) - n^2[a + r(t)][1 - g(t)] &= F_r(t)\\
\ddot{s}(t) + 2n\,\dot{r}(t) - n^2 s(t)[1 - g(t)] &= F_s(t)\\
\ddot{z}(t) + n^2 z(t)\, g(t) &= F_z(t)
\end{aligned}$$

where

$$g(t) \equiv \frac{a^3}{\{[a + r(t)]^2 + s^2(t) + z^2(t)\}^{3/2}}$$

and a is the semimajor axis given by $a = 6,906.4$ km. How well does your linear controller work for other (larger) initial conditions?

8.34 ♣ Prove the identities in eqns. (8.189) and (8.192). Show that the determinant of the matrix $\Xi^T(\mathbf{q}_d)\Xi(\mathbf{q})$, which is used in the control law given by eqn. (8.188), is given by $\mathbf{q}_d^T\mathbf{q}$. Finally, prove that eqn. (8.188) reduces down to eqn. (8.190) when L_1 and L_2 are scalars.

8.35 A spacecraft equipped with reaction wheels[35] can also be used for attitude maneuvering purposes. Although the spacecraft can no longer be considered a rigid body with the internal wheels, Euler's rotational equations of §3.7.2 can still be used to describe the overall system. The equations of motion using reaction wheels can be written as

$$(J-\bar{J})\dot{\boldsymbol{\omega}} = -[\boldsymbol{\omega}\times](J\,\boldsymbol{\omega}+\bar{J}\,\bar{\boldsymbol{\omega}})-\bar{\mathbf{u}}$$
$$\bar{J}(\dot{\bar{\boldsymbol{\omega}}}+\dot{\boldsymbol{\omega}}) = \bar{\mathbf{u}}$$

where J is the inertia of the spacecraft which now includes the wheels, $\bar{J}$ is the inertia of the wheels, $\bar{\boldsymbol{\omega}}$ is the wheel angular velocity vector relative to the spacecraft, and $\bar{\mathbf{u}}$ is the wheel torque vector. Derive a wheel control law that provides the linear error-dynamics given by eqn. (8.180). Consider a wheel inertia matrix given by

$$\bar{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ kg-m}^2$$

Assuming that the wheels initially begin at rest, $\bar{\boldsymbol{\omega}}(t_0)=\mathbf{0}$, use the derived wheel control law to maneuver the spacecraft along a desired trajectory, with the simulation parameters shown in example 8.4. Also, test the robustness of the wheel control law by using a different spacecraft inertia matrix in the assumed model. How robust is this control law to parameter variations?

8.36 Consider the nonlinear equations of motion for a highly maneuverable aircraft given in exercise 4.22. Neglecting higher-order terms we can write the equations of motion in linear form as

$$\begin{bmatrix} \dot{\alpha}(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} -0.88 & 0 & 1 \\ 0 & 0 & 1 \\ -4.208 & 0 & -0.396 \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} -0.22 \\ 0 \\ -20.967 \end{bmatrix} \delta_E(t) \tag{8.195}$$

Design a steady-state LQR controller to bring the states with initial conditions of $\alpha_0 = 1$ deg, $\theta = 10$ deg, and $\dot{\theta}_0 = 0$ deg/sec to zero within 15 to 20 seconds. Use your LQR steady-state control input on the full nonlinear equations of motion. How well does your linear controller work for other (larger) initial conditions?

8.37 Consider using a linear Kalman filter to estimate the states for the system described in exercises 4.22 and 8.36. Design a filter with the linear model shown in exercise 8.36 using measurements of angle of attack, $\alpha(t)$, and pitch angle, $\theta(t)$. Assume standard deviations of the measurement errors to be the same as the ones given in exercise 4.4. Tune the process noise covariance matrix, Q, to yield sufficiently filtered estimates with adequate filter convergence properties. Use the designed estimator in an LQG design to control the aircraft with the gain developed in exercise 8.36. Try various initial conditions in the actual system as well as the Kalman filter to test the overall robustness of your design. Also, use measurements of only the pitch angle and compare the results with those obtained using measurements of both angle of attack and pitch in the Kalman filter.

8.38 Example 4.5 shows mass, stiffness, and damping matrices of a 4-mode system. Convert the continuous-time model in discrete-time using the methods of §3.5 with a sampling interval of 0.1 seconds. Assuming initial conditions of one for the position states and zero for the velocity states, design an LQR controller to bring all states to zero within 10 seconds. Then, use a Kalman filter to estimate all states from position measurements only. Assume that the standard deviation of the measurement noise is given by $\sqrt{1 \times 10^{-5}}$ for all measurements. Add discrete-time process noise into the velocity states only (i.e., assume that the kinematically relationships are exact, as discussed in §7.4.1, so do not add process noise to these states). Assume that the discrete-time standard deviation for the process noise is given by 0.1 for all velocity states. Implement an LQG controller using the Kalman filter estimates in the control law. Try various values for the process noise and measurement noise covariances to generate the true states. Discuss the performance of the overall controller to these variations.

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