3

Review of Dynamical Systems

> *All the effects of nature are only the mathematical consequences of a small number of immutable laws. Laplace, Pierre-Simon*

THIS chapter serves to provide a review of the equations and concepts of dynamical systems. These equations will subsequently be used in later chapters to illustrate the importance of estimation for actual applications in dynamical systems. In particular several systems will be reviewed in this chapter; including spacecraft dynamics, orbital mechanics, aircraft flight dynamics, and vibrational systems. A thorough treatise of these subjects is not possible, and only the fundamental equations and concepts will be reviewed here. The interested reader can pursue these subjects in more depth by studying the many references cited in this chapter.

The mathematical models of most physical processes are embodied by one or more differential equations. A large fraction of practical problems is included if we restrict our attention to the case in which the state is the solution of a system of ordinary differential equations (ODEs). The differential equations usually arise quite naturally from application of fundamental principles (e.g., Newton's laws of motion) known to govern the particular dynamical system's behavior. In a significant fraction of the applications, it is possible to obtain explicit algebraic solutions of the system of differential equations; when this is possible, the results of the first two chapters may be immediately employed (e.g., see example 1.1). If simple algebraic analytical solutions of the differential equations cannot be found, one need not (necessarily!) despair, as will be demonstrated in later chapters. We begin the present chapter with an overview of the analytical and numerical methods for solving differential equations.

3.1 Linear System Theory

We first consider linear ODEs, which can be used to describe the behavior of a large class of dynamical systems. A linear system follows the *superposition principle*,[1] which states that a linear combination of inputs produces an output that is the superposition (linear combination) of the outputs if the outputs of each input term were applied separately. Mathematically expressed, a system is linear if the following holds true:

$$\begin{aligned} y = f(a\,x_1 + b\,x_2) &= a f(x_1) + b f(x_2) \\ &= a\,y_1 + b\,y_2 \end{aligned} \tag{3.1}$$

where $y_1 = f(x_1)$ and $y_2 = f(x_2)$, and a and b are constants.

Example 3.1: We wish to investigate the linearity of the following functions:

1. $y = m\,x$
2. $y = x^2$
3. $y = 3\ddot{x} + 4\dot{x}$

The first equation is clearly linear since $y = m(a\,x_1 + b\,x_2) = a\,y_1 + b\,y_2$, with $y_1 = m\,x_1$ and $y_2 = m\,x_2$. The second equation is not linear since $y = (a\,x_1 + b\,x_2)^2 \neq a\,x_1^2 + b\,x_2^2$. The third equation is linear even though it involves a differential equation since $y = 3(a\,\ddot{x}_1 + b\,\ddot{x}_2) + 4(a\,\dot{x}_1 + b\,\dot{x}_2) \equiv a\,y_1 + b\,y_2$. Superposition is a powerful tool for solving linear ODEs since the homogeneous and forced response can be found individually, and then summed to form the entire solution.

3.1.1 The State Space Approach

The state space approach is extremely useful for many reasons, including: the approach reduces an n^{th}-order linear ODE to n first-order ODEs, matrix analysis tools can easily be used, and it provides a convenient representation for multi-input-multi-output (MIMO) systems. We begin this topic by considering a simple single-input-single-output (SISO) n^{th}-order linear ODE, given by

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0\,y = u \tag{3.2}$$

where y is the output variable and u is the input variable. In order to convert the ODE into first-order form, consider the following variable change:

$$\begin{aligned} x_1 &= y \\ x_2 &= \frac{dy}{dt} \\ &\ \ \vdots \\ x_n &= \frac{d^{n-1} y}{dt^{n-1}} \end{aligned} \tag{3.3}$$

This leads to the following equivalent system of n first-order equations:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u
\end{aligned} \tag{3.4}$$

which can be represented in matrix form by

$$\dot{\mathbf{x}}(t) = F\,\mathbf{x} + B\,u(t) \tag{3.5}$$

where the vector $\mathbf{x}$ contains the *state variables*:

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \tag{3.6}$$

and the matrices F and B are given by

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \tag{3.7a}$$

$$B = \begin{bmatrix} 0 \; 0 \; \cdots \; 1 \end{bmatrix}^T \tag{3.7b}$$

The general SISO n^{th}-order linear ODE is given by

$$\begin{aligned}
&\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0\, y \\
&= b_n \frac{d^n u}{dt^n} + b_{n-1}\frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_1 \frac{du}{dt} + b_0\, u
\end{aligned} \tag{3.8}$$

In order to convert the ODE into first-order form we first rewrite eqn. (3.8) into an equivalent form involving two ODEs, given by

$$y = b_n \frac{d^n x}{dt^n} + b_{n-1} d^{n-1} x dt^{n-1} + \cdots + b_1 \frac{dx}{dt} + b_0\, x \tag{3.9a}$$

$$u = \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0\, x \tag{3.9b}$$

where x is an intermediate variable. Now, consider the following variable change:

$$\begin{aligned}
x_1 &= x \\
x_2 &= \frac{dx}{dt} \\
&\vdots \\
x_n &= \frac{d^{n-1} x}{dt^{n-1}}
\end{aligned} \tag{3.10}$$

This leads to the following equivalent system of n first-order equations, given in matrix form by

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= F\,\mathbf{x}(t) + B\,u(t) \\ y(t) &= H\,\mathbf{x}(t) + D\,u(t)\end{aligned}} \qquad \begin{aligned}(3.11\text{a}) \\ (3.11\text{b})\end{aligned}$$

where the matrices F and B are given by eqn. (3.7), and H and D are given by

$$H = \left[(b_0 - b_n\,a_0)\;\;(b_1 - b_n\,a_1)\;\;\cdots\;\;(b_{n-1} - b_n\,a_{n-1})\right] \tag{3.12a}$$

$$D = b_n \tag{3.12b}$$

Clearly, if $b_0 = 1$ and the remaining coefficients $b_i = 0$, $i = 1, 2, \ldots, n$, then the intermediate variable $x = y$, which reduces the general case in eqn. (3.8) to the simple case in eqn. (3.2).

The matrix representation in eqn. (3.11) is the basis for modern estimation and controls. The matrix F is known as the *state matrix* and defines the *stability* of the overall system. The matrix representation is useful for MIMO systems as well since additional inputs can be added simply by using additional columns in the B matrix (likewise, additional outputs can be added by using additional rows in the H matrix). Among developers of computer software for solution of ODEs, there is now a universal adoption of the standardized form of eqn. (3.11). Thus, in theoretical developments whose end products are likely to be implemented on a computer, adherence to this convention is justified on practical grounds. We mention that the particular transformations of eqns. (3.9) and (3.10) represent only one of many linear transformations that brings eqn. (3.8) to the form of eqn. (3.11). Each such transformation, leading to the associated $(F,\ B,\ H,\ D)$, is called a *realization*. Other aspects of the state space representation (such as transmission zeroes, internal and external descriptions, geometric visualization, balanced realizations, etc.), can be found in Refs. [1]-[9].

The MIMO version of the system in eqn. (3.11) can be represented in *transfer function* form by taking the Laplace transform[10] of both sides with zero initial conditions:

$$s\,\mathbf{X}(s) = F\,\mathbf{X}(s) + B\,\mathbf{U}(s) \tag{3.13a}$$

$$\mathbf{Y}(s) = H\,\mathbf{X}(s) + D\,\mathbf{U}(s) \tag{3.13b}$$

where s is the Laplace variable. Solving for $\mathbf{X}(s)$ in eqn. (3.13a) and substituting the resulting expression into eqn. (3.13b) yields

$$\mathbf{Y}(s) = \left\{H\,[sI - F]^{-1}\,B + D\right\}\mathbf{U}(s) \tag{3.14}$$

Since the inverse of $[sI - F]$ is given by its adjoint divided by its determinant, then the determinant of $[sI - F]$ gives the *poles* of the transfer function. Also, the eigenvalues of F are equivalent to the roots of the denominator of the transfer function. The transfer function representation can be useful; however, it becomes impractical for large order systems.

3.1.2 Homogeneous Linear Dynamical Systems

Consider the homogeneous matrix differential equation

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t), \quad \mathbf{x}(t_0) \text{ known} \tag{3.15}$$

The standard approach for solving equations of the form (3.15) is to determine the "fundamental" or "state transition" matrix $\Phi(t, t_0)$ which "maps" the initial state into the current state as

$$\mathbf{x}(t) = \Phi(t, t_0)\,\mathbf{x}(t_0) \tag{3.16}$$

Before developing means for determining $\Phi(t, t_0)$, three important group properties of the transition matrix which follow from inspection of eqn. (3.16) are stated as

$$\Phi(t_0, t_0) = I \tag{3.17a}$$

$$\Phi(t_0, t) = \Phi^{-1}(t, t_0) \tag{3.17b}$$

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0) \tag{3.17c}$$

A differential equation for determining $\Phi(t, t_0)$ can be developed by substituting eqn. (3.16) into the right-hand side of eqn. (3.15) and the derivative of eqn. (3.16) into the left-hand side of eqn. (3.15) to obtain

$$\dot{\Phi}(t, t_0)\,\mathbf{x}(t_0) = F(t)\,\Phi(t, t_0)\,\mathbf{x}(t_0) \tag{3.18}$$

from which we conclude that the transition matrix satisfies the differential equation

$$\dot{\Phi}(t, t_0) = F(t)\,\Phi(t, t_0) \tag{3.19}$$

with the identity matrix in eqn. (3.17a) as the initial condition. Only under ideal circumstances can a practical analytical solution of eqn. (3.19) be obtained; otherwise, numerical techniques must be employed to compute $\Phi(t, t_0)$. We now consider several standard approaches for extracting analytical or approximate solutions for $\Phi(t, t_0)$.

To develop one approach for solving eqn. (3.19), we rewrite it in integral form as

$$\Phi(t, t_0) = I + \int_{t_0}^{t} F(\tau_1)\,\Phi(\tau_1, t_0)\,d\tau_1 \tag{3.20}$$

which is a "matrix Volterra integral equation." We "casually note" that the integrand of eqn. (3.20) contains the left side; so it does not appear that any progress has been made writing eqn. (3.19) in integral form. One "might consider the wisdom" of substituting eqn. (3.20) *into its own integrand*; while this process may appear not only obscene, but futile, it does turn out to be profitable! For $\Phi(\tau_1, t_0)$ in the integrand of eqn. (3.20), we substitute from eqn. (3.20)

$$\Phi(\tau_1, t_0) = I + \int_{t_0}^{\tau_1} F(\tau_2)\,\Phi(\tau_2, t_0)\,d\tau_2 \tag{3.21}$$

to obtain

$$\Phi(t,t_0) = I + \int_{t_0}^{t} F(\tau_1)\, d\tau_1 + \int_{t_0}^{t} F(\tau_1) \int_{t_0}^{\tau_1} F(\tau_2)\, \Phi(\tau_2,t_0)\, d\tau_2\, d\tau_1 \tag{3.22}$$

One can now re-use eqn. (3.20) to write

$$\Phi(\tau_2,t_0) = I + \int_{t_0}^{\tau_2} F(\tau_3)\, \Phi(\tau_3,t_0)\, d\tau_3 \tag{3.23}$$

which, when substituted into the final integrand of eqn. (3.22) yields

$$\begin{aligned}\Phi(t,t_0) = I &+ \int_{t_0}^{t} F(\tau_1)\, d\tau_1 \\ &+ \int_{t_0}^{t} F(\tau_1) \int_{t_0}^{\tau_1} F(\tau_2)\, d\tau_2\, d\tau_1 \\ &+ \int_{t_0}^{t} F(\tau_1) \int_{t_0}^{\tau_1} F(\tau_2) \int_{t_0}^{\tau_2} F(\tau_3)\, d\tau_3\, d\tau_2\, d\tau_1 \\ &+ \cdots\end{aligned} \tag{3.24}$$

This procedure is known as the Peano-Baker Method; as is shown by Ince (1926),[11] uniform and absolute convergence is guaranteed. Whether or not this process is practical depends, of course, upon how difficult the elements of the $F(t)$ are to integrate, and how quickly convergence occurs.

Considering an important special case that F equals a constant matrix; F can be brought from under all integrands of eqn. (3.24), we immediately find

$$\Phi(t,t_0) = I + F(t-t_0) + \frac{1}{2!}F^2(t-t_0)^2 + \cdots + \frac{1}{n!}F^n(t-t_0)^n + \cdots \tag{3.25}$$

which is recognized to be the e^x series with the matrix $F(t-t_0)$ as the argument. For notational compactness, eqn. (3.25) is often written compactly as

$$\Phi(t,t_0) = e^{F(t-t_0)}, \quad \text{for } F = \text{constant} \tag{3.26}$$

Thus, returning to eqn. (3.16), we see that the solution for constant F is

$$\boxed{\mathbf{x}(t) = e^{F(t-t_0)}\mathbf{x}(t_0)} \tag{3.27}$$

Consider the analogy of the matrix differential equation (3.15) with the scalar differential equation

$$\dot{x}(t) = f(t)\,x(t), \quad x(t_0) \text{ known} \tag{3.28}$$

For the special case that f equals a constant, then the solution of eqn. (3.28) is

$$x(t) = x(t_0)e^{f(t-t_0)} \tag{3.29}$$

Thus, except for the constrained order of multiplication, the matrix solution (3.27) of eqn. (3.15) is completely analogous to the scalar solution (3.29) of eqn. (3.28) for constant coefficient matrices.

For the general case that f does not equal a constant, the general solution of eqn. (3.28) is

$$x(t) = x(t_0)e^{\int_{t_0}^{t} f(\tau)\, d\tau} \tag{3.30}$$

One might naturally conjecture that the general solution of eqn. (3.15) is

$$\mathbf{x}(t) = \Phi(t, t_0)\,\mathbf{x}(t_0) = \left[e^{\int_{t_0}^{t} F(\tau)\, d\tau}\right]\mathbf{x}(t_0) \tag{3.31}$$

This conjecture turns out to be false, in general. To see under what conditions eqn. (3.31) *is* a correct solution of eqn. (3.15), note

$$\Phi = e^{\int_{t_0}^{t} F\, d\tau} = I + \left[\int_{t_0}^{t} F\, d\tau\right] + \frac{1}{2!}\left[\int_{t_0}^{t} F\, d\tau\right]^2 + \frac{1}{3!}\left[\int_{t_0}^{t} F\, d\tau\right]^3 + \cdots \tag{3.32}$$

$$\begin{aligned}\dot{\Phi} = 0 + F + \frac{1}{2!}F\left[\int_{t_0}^{t} F\, d\tau\right] + \frac{1}{2!}\left[\int_{t_0}^{t} F\, d\tau\right]F \\ + \frac{1}{3!}F\left[\int_{t_0}^{t} F\, d\tau\right]^2 + \frac{1}{3!}\left[\int_{t_0}^{t} F\, d\tau\right]F\left[\int_{t_0}^{t} F\, d\tau\right] \\ + \frac{1}{3!}\left[\int_{t_0}^{t} F\, d\tau\right]^2 F + \cdots\end{aligned} \tag{3.33}$$

and

$$F\Phi = F + F\left[\int_{t_0}^{t} F\, d\tau\right] + \frac{1}{2!}F\left[\int_{t_0}^{t} F\, d\tau\right]^2 + \frac{1}{3!}F\left[\int_{t_0}^{t} F\, d\tau\right]^3 \tag{3.34}$$

Clearly, for eqns. (3.33) and (3.34) to be equal {which they must if eqn. (3.31) is a solution of eqn. (3.15)} then it is necessary that the following "commutivity property" be satisfied:

$$F(t)\left[\int_{t_0}^{t} F(\tau)\, d\tau\right] = \left[\int_{t_0}^{t} F(\tau)\, d\tau\right]F(t) \tag{3.35}$$

This property defines only a very special class of matrices!

The conclusion is that the analogy between solutions of eqn. (3.16) and its scalar analog is not complete. The Peano-Baker solution (3.24) can be written in shorthand notation as

$$\Phi(t, t_0) = I + \sum_{i=1}^{\infty} l_i(t) \tag{3.36}$$

where the integrals are defined as

$$l_1(t) = \int_{t_0}^{t} F(\tau_1) d\tau_1 \tag{3.37a}$$

$$l_2(t) = \int_{t_0}^{t} F(\tau_1) \int_{t_0}^{\tau_1} F(\tau_2)\, d\tau_2\, d\tau_1 = \int_{t_0}^{t} F(\tau_1) l_1(\tau_1)\, d\tau_1 \tag{3.37b}$$

$$l_3(t) = \int_{t_0}^{t} F(\tau_1) l_2(\tau_1)\, d\tau_1 \tag{3.37c}$$

or

$$l_n(t) = \int_{t_0}^{t} F(\tau_1) l_{n-1}(\tau_1)\, d\tau_1, \quad \text{for } n \geq 2 \tag{3.38}$$

As an alternative to the Peano-Baker solution, consider the Taylor's Series

$$\Phi(t, t_0) = I + \sum_{j=1}^{\infty} \frac{(t-t_0)^j}{j!} \left. \frac{d^j \Phi}{dt^j} \right|_{t=t_0} \tag{3.39}$$

where the necessary partial derivatives are evaluated sequentially from the following equations:

In General	Evaluated Initially
$\frac{d\Phi}{dt} = F\Phi$	$\left.\frac{d\Phi}{dt}\right\vert_{t=t_0} = F(t_0)$
$\frac{d^2\Phi}{dt^2} = \frac{dF}{dt}\Phi + F\frac{d\Phi}{dt}$	$\left.\frac{d^2\Phi}{dt^2}\right\vert_{t=t_0} = \left.\frac{dF}{dt}\right\vert_{t=t_0} + F^2(t_0)$
$\vdots$	$\vdots$

In particular, if F is constant, then

$$\left. \frac{d^j \Phi}{dt^j} \right|_{t=t_0} = F^j \tag{3.40}$$

and eqn. (3.39) becomes

$$\Phi(t, t_0) = I + \sum_{j=1}^{\infty} \frac{(t-t_0)^j}{j!} F^j \equiv e^{F(t-t_0)} \tag{3.41}$$

In practice, if F is not constant, and the Peano-Baker or Taylor's Series prove too cumbersome (due to slow convergence or algebraic difficulties), then one must resort to a numerical solution of eqn. (3.18) or eqn. (3.15).

3.1.3 Forced Linear Dynamical Systems

We now direct our attention to the multi-input inhomogeneous differential equation

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) \tag{3.42}$$

Using Lagrange's method of *variation of parameters*, a solution of eqn. (3.42) having the following form is assumed:

$$\mathbf{x}(t) = \Phi(t,t_0)\,\mathbf{g}(t), \quad \mathbf{g}(t_0) = \mathbf{x}(t_0) \tag{3.43}$$

where $\mathbf{g}(t)$ is an $n \times 1$ vector of unknown functions and $\Phi(t,t_0)$ is the homogeneous transition matrix. Differentiating eqn. (3.43), we obtain

$$\dot{\mathbf{x}}(t) = \Phi(t,t_0)\,\dot{\mathbf{g}}(t) + \dot{\Phi}(t,t_0)\,\mathbf{g}(t) \tag{3.44}$$

which, upon substitution of eqn. (3.18) for $\dot{\Phi}(t,t_0)$, becomes

$$\dot{\mathbf{x}}(t) = \Phi(t,t_0)\,\dot{\mathbf{g}}(t) + F(t)\,\Phi(t,t_0)\,\mathbf{g}(t) \tag{3.45}$$

Substituting eqns. (3.43) and (3.45) into eqn. (3.42) yields

$$\Phi(t,t_0)\,\mathbf{g}(t) + F(t)\,\Phi(t,t_0)\,\dot{\mathbf{g}}(t) = F(t)\,\Phi(t,t_0)\,\mathbf{g}(t) + B(t)\,\mathbf{u}(t) \tag{3.46}$$

Therefore

$$\dot{\mathbf{g}}(t) = \Phi^{-1}(t,t_0)\,B(t)\,\mathbf{u}(t) \tag{3.47}$$

which we integrate to obtain {noting $\mathbf{g}(t_0) = \mathbf{x}(t_0)$}

$$\mathbf{g}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \Phi^{-1}(\tau,t_0)\,B(\tau)\,\mathbf{u}(\tau)\,d\tau \tag{3.48}$$

Therefore, the general solution of eqn. (3.42) is

$$\mathbf{x}(t) = \Phi(t,t_0)\,\mathbf{x}(t_0) + \Phi(t,t_0)\int_{t_0}^{t} \Phi^{-1}(\tau,t_0)\,B(\tau)\,\mathbf{u}(\tau)\,d\tau \tag{3.49}$$

Application of eqn. (3.17b) allows the integrand to be written as

$$\Phi^{-1}(\tau,t_0) = \Phi(t_0,\tau) \tag{3.50}$$

Using eqn. (3.17c) gives

$$\Phi^{-1}(\tau,t_0) = \Phi(t_0,t)\,\Phi(t,\tau) \tag{3.51}$$

or

$$\Phi^{-1}(\tau,t_0) = \Phi^{-1}(t,t_0)\,\Phi(t,\tau) \tag{3.52}$$

which, when substituted into eqn. (3.49) yields

$$\boxed{\mathbf{x}(t) = \Phi(t,t_0)\,\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,\mathbf{u}(\tau)\,d\tau} \tag{3.53}$$

as the final form of the solution of eqn. (3.42) for arbitrary $F(t)$, $B(t)$, and $\mathbf{u}(t)$. Equation (3.53) must typically be solved numerically.

Example 3.2: Consider the motion of a projectile in a constant gravity field. The equations of motion are

$$\begin{aligned}\ddot{x} &= 0\\ \ddot{y} &= 0\\ \ddot{z} &= -g\end{aligned}$$

which integrate immediately to give

$$\begin{aligned}\dot{x} &= \dot{x}_0\\ \dot{y} &= \dot{y}_0\\ \dot{z} &= \dot{z}_0 - g\,(t-t_0)\end{aligned}$$

and

$$\begin{aligned}x &= x_0 + \dot{x}_0\,(t-t_0)\\ y &= y_0 + \dot{y}_0\,(t-t_0)\\ z &= z_0 + \dot{z}_0\,(t-t_0) - 1/2\,g\,(t-t_0)^2\end{aligned}$$

where g is the gravity constant, and $(x_0,\ y_0,\ z_0)$ and $(\dot{x}_0,\ \dot{y}_0,\ \dot{z}_0)$ are the initial positions and velocities, respectively.

Alternatively, we could have employed the variable change

$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = \dot{x},\ x_5 = \dot{y},\ x_6 = \dot{z},\ u = -g$$

so that the following state space form can be written:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\end{bmatrix} = \begin{bmatrix}0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{bmatrix} - \begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 1\end{bmatrix} g$$

Notice, by inspection of the analytical position and velocity solutions that the state transition matrix is

$$\Phi(t,t_0) = \begin{bmatrix}1&0&0&(t-t_0)&0&0\\ 0&1&0&0&(t-t_0)&0\\ 0&0&1&0&0&(t-t_0)\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\end{bmatrix}$$

Therefore, the "forced" solution (including gravity) follows from eqn. (3.53), given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & (t-t_0) & 0 & 0 \\ 0 & 1 & 0 & 0 & (t-t_0) & 0 \\ 0 & 0 & 1 & 0 & 0 & (t-t_0) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \\ x_4(t_0) \\ x_5(t_0) \\ x_6(t_0) \end{bmatrix} - \int_{t_0}^{t} \left[0\ 0\ (\tau - t_0)\ 0\ 0\ 1\right]^T g\, d\tau$$

or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = \begin{bmatrix} x_1(t_0) + x_4(t_0)(t-t_0) \\ x_2(t_0) + x_5(t_0)(t-t_0) \\ x_3(t_0) + x_6(t_0)(t-t_0) - 1/2\, g\, (t-t_0)^2 \\ x_4(t_0) \\ x_5(t_0) \\ x_6(t_0) - g\,(t-t_0) \end{bmatrix}$$

which verify (again) the previous results and demonstrates the equivalence between the preceding results of this section and conventional integration of the differential equations.

3.1.4 Linear State Variable Transformations

The matrix exponential for an arbitrary constant matrix is expensive to compute if one requires a large number of terms in eqn. (3.25). Often, one can carry out a coordinate transformation which "blasts this problem into trivia." Consider the introduction of a new state vector $\mathbf{z}$ which is linearly related to $\mathbf{x}$ via

$$\mathbf{x} = T\,\mathbf{z} \tag{3.54}$$

where T is a constant $n \times n$ matrix. Taking the time derivative of eqn. (3.54) and solving for $\dot{\mathbf{z}}$ yields

$$\dot{\mathbf{z}} = T^{-1}\dot{\mathbf{x}} \tag{3.55}$$

Now, substitution of the $\mathbf{x}$-differential equation (3.15) yields

$$\dot{\mathbf{z}} = T^{-1} F\,\mathbf{x} \tag{3.56}$$

and substitution of eqn. (3.54) then yields the differential equation for $\mathbf{z}$ as

$$\dot{\mathbf{z}} = \Lambda\,\mathbf{z} \tag{3.57}$$

where the new coefficient matrix is given by the similarity transformation

$$\Lambda = T^{-1} F\, T \tag{3.58}$$

Now the unspecified T-matrix can often be judiciously chosen so that Λ is diagonal; more generally, Λ can be brought to a block diagonal form (the "Jordan Canonical Form"). If Λ is in fact diagonal, it is clear that the solution is trivial since eqn. (3.57) can be written as

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \\ \vdots \\ \dot{\mathbf{z}}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_n \end{bmatrix} \tag{3.59}$$

or

$$\dot{z}_i = \lambda_i z_i, \quad i = 1, 2, \ldots, n \tag{3.60}$$

and the solution is simply

$$z_i(t) = z_i(t_0)\, e^{\lambda_i (t-t_0)}, \quad i = 1, 2, \ldots, n \tag{3.61}$$

The solution in eqn. (3.61) can be written in state transition matrix form as

$$\mathbf{z}(t) = \Psi(t, t_0)\, \mathbf{z}(t_0) \tag{3.62}$$

where

$$\Psi(t, t_0) \equiv \begin{bmatrix} e^{\lambda_1 (t-t_0)} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 (t-t_0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n (t-t_0)} \end{bmatrix} \tag{3.63}$$

Now substituting eqn. (3.62) into eqn. (3.54) and using $\mathbf{z}(t_0) = T^{-1}\mathbf{x}(t_0)$ yields

$$\mathbf{x}(t) = T\, \Psi(t, t_0)\, T^{-1} \mathbf{x}(t_0) \tag{3.64}$$

The state transition matrix for $\mathbf{x}$ is then clearly identified as

$$\Phi(t, t_0) \equiv T\, \Psi(t, t_0)\, T^{-1} \tag{3.65}$$

Let us now see how to construct the elements of the T and Λ matrices. We require that the similarity transformation yields a diagonal Λ matrix as

$$\Lambda = T^{-1} F\, T \tag{3.66}$$

or

$$T\, \Lambda = F\, T \tag{3.67}$$

In detail, the equations (3.67) are

$$\begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix} \tag{3.68}$$

Equating the i^{th} column resulting from the matrix product on the left-hand side of eqn. (3.68) to the i^{th} column on the right-hand side yields

$$\lambda_i \begin{bmatrix} t_{1i} \\ t_{2i} \\ \vdots \\ t_{ni} \end{bmatrix} = F \begin{bmatrix} t_{1i} \\ t_{2i} \\ \vdots \\ t_{ni} \end{bmatrix}, \quad i = 1, 2, \ldots, n \tag{3.69}$$

Thus, the conclusion is that the diagonal elements of Λ are the *eigenvalues* of F, and the columns of the required matrix T are the corresponding *eigenvectors* of F. The λ's are the n roots of the characteristic equation

$$\det(\lambda I - F) = 0 \rightarrow \lambda_1, \lambda_2, \ldots, \lambda_n \tag{3.70}$$

Upon determining λ_i's from eqn. (3.70), the t_{ij}'s are determined (to within an arbitrary multiplicative constant for each column) from eqn. (3.69). For the most common case that the n λ's satisfying eqn. (3.70) are distinct, the independent columns of T can always be found to satisfy eqn. (3.70). For the case that eqn. (3.70) has multiple roots, it is not always possible to find independent columns of T from eqn. (3.69) which will guarantee eqn. (3.58) to be diagonal. The difficulties encountered for repeated eigenvalues are not always trivial to resolve; see Ref. [12] for a more detailed treatment of this subject.

We can easily prove that a transformation of state does not alter the transfer function of a system. Taking the Laplace transform of eqn. (3.54) and substituting the resultant into eqn. (3.13) gives

$$s\,\mathbf{Z}(s) = T^{-1} F T\, \mathbf{Z}(s) + T^{-1} B\, \mathbf{U}(s) \tag{3.71a}$$

$$\mathbf{Y}(s) = H T\, \mathbf{Z}(s) + D\, \mathbf{U}(s) \tag{3.71b}$$

The transfer function from $\mathbf{U}(s)$ to $\mathbf{Y}(s)$ is given by

$$\begin{aligned} \mathbf{Y}(s) &= \left\{ HT \left[sI - T^{-1} F T \right]^{-1} T^{-1} B + D \right\} \mathbf{U}(s) \\ &= \left\{ HT \left[T^{-1} (sI - F)\, T \right]^{-1} T^{-1} B + D \right\} \mathbf{U}(s) \\ &= \left\{ HTT^{-1} [sI - F]^{-1}\, TT^{-1} B + D \right\} \mathbf{U}(s) \\ &= \left\{ H\, [sI - F]^{-1}\, B + D \right\} \mathbf{U}(s) \end{aligned} \tag{3.72}$$

Therefore, the overall transfer function is unaffected. Clearly, there are an infinity number of state-space representations that yield identical transfer functions.

3.2 Nonlinear Dynamical Systems

We now consider the circumstance in which the original system of differential equations is nonlinear and can be brought to the standard form

$$\boxed{\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x},\mathbf{u}) \\ \mathbf{y} &= \mathbf{h}(t,\mathbf{x},\mathbf{u})\end{aligned}} \qquad \begin{aligned}(3.73\text{a}) \\ (3.73\text{b})\end{aligned}$$

Some of the nonlinear systems of differential equations encountered in applications can be solved for an exact analytical solution (e.g., as will be demonstrated for the elliptic two-body problem in §3.8.2). Unfortunately, only a minority of these systems have known analytical solutions and no standardized methods exist for finding *exact analytical solutions*. In many cases a reference motion may be known, which is "close" to the actual state history. In these cases the *departure* of the actual state history from a known reference motion may be adequately described by eqn. (3.11). The nominal reference ($\mathbf{x}_N$) trajectory's integration is formally indicated as

$$\mathbf{x}_N(t) = \mathbf{x}_N(t_0) + \int_0^t \mathbf{f}(\tau,\mathbf{x}_N,\mathbf{u}_N)\, d\tau \tag{3.74a}$$

$$\mathbf{y}(t) = \mathbf{h}(t,\mathbf{x}_N,\mathbf{u}_N) \tag{3.74b}$$

Now, we assume that the actual quantities are given by the nominal quantities plus a perturbation:

$$\mathbf{x}(t) = \mathbf{x}_N(t) + \boldsymbol{\delta}\mathbf{x}(t) \tag{3.75a}$$

$$\mathbf{u}(t) = \mathbf{u}_N(t) + \boldsymbol{\delta}\mathbf{u}(t) \tag{3.75b}$$

$$\mathbf{y}(t) = \mathbf{y}_N(t) + \boldsymbol{\delta}\mathbf{y}(t) \tag{3.75c}$$

where $\boldsymbol{\delta}\mathbf{x}(t)$, $\boldsymbol{\delta}\mathbf{u}(t)$, and $\boldsymbol{\delta}\mathbf{y}(t)$ are state, input, and output perturbations, respectively. Results from a first-order Taylor series expansion for $\mathbf{f}(t,\mathbf{x},\mathbf{u})$ and $\mathbf{h}(t,\mathbf{x},\mathbf{u})$ yield

$$\boldsymbol{\delta}\dot{\mathbf{x}}(t) = F(t)\,\boldsymbol{\delta}\mathbf{x}(t) + B(t)\,\boldsymbol{\delta}\mathbf{u}(t) \tag{3.76a}$$

$$\boldsymbol{\delta}\mathbf{y}(t) = H(t)\,\boldsymbol{\delta}\mathbf{x}(t) + D(t)\,\boldsymbol{\delta}\mathbf{u}(t) \tag{3.76b}$$

where

$$F(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_N,\mathbf{u}_N}, \quad B(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{x}_N,\mathbf{u}_N} \tag{3.77a}$$

$$H(t) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}_N,\mathbf{u}_N}, \quad D(t) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_{\mathbf{x}_N,\mathbf{u}_N} \tag{3.77b}$$

Equation (3.76) can be integrated and then employed in eqns. (3.75a) and (3.75c) to approximate trajectories in a sufficiently small neighborhood of eqn. (3.74). Errors arise when the "departure" from the nominal reference trajectory is not small (i.e., when the higher-order expansion terms in Taylor's series are not negligible).

For the *perturbation class* of nonlinear system whose differential equations can be brought to the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}) + \delta\mathbf{f}(t, \mathbf{x}, \mathbf{u}) \tag{3.78}$$

in which the "unperturbed" *generating system*

$$\dot{\mathbf{z}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}) \tag{3.79}$$

has a known analytical solution, and the perturbation $\delta\mathbf{f}(t, \mathbf{x}, \mathbf{u})$ follows

$$||\delta\mathbf{f}(t, \mathbf{x}, \mathbf{u})|| \ll ||\mathbf{f}(t, \mathbf{x}, \mathbf{u})|| \tag{3.80}$$

for all t and $\mathbf{x}$ of interest, numerous methods are available for construction of *approximate* analytical solutions. The interested reader is referred to Refs. [13] and [14] for development of basic perturbation methods which are not developed herein due to space limitations. Let us remark, however, that the perturbation approach suffers from one fundamental drawback; for each specification of the functions $\mathbf{f}$ and $\delta\mathbf{f}$ in eqn. (3.78), lengthy algebraic developments must be carried through to obtain an *approximate* solution. In many cases the practical constraints imposed by "having but one life to give" and the desirability of constructing general-purpose algorithms make the analytical perturbation approach unattractive. On the other hand, general purpose numerical methods exist which are routinely employed to solve a wide variety of highly nonlinear systems of the form (3.73) with excellent, near arbitrary control over precision of the solution (e.g., Runge-Kutta methods).

Estimation theory based upon a linear differential equation of the form (3.76) is seen to be applicable (at least approximately) to a wide class of dynamical systems. In any given application to nonlinear problems, of course, one must realistically face the problems of choosing suitable nominal trajectories to linearize about, and analyzing the effects of errors introduced through the linearization. Many of the available tools for linear systems (such as superposition, Laplace transforms, Bode plots, observability, etc.[1−9]) are not directly applicable to nonlinear systems. Still, the linearized system in eqn. (3.76) can be used to prove local stability and analyze the nonlinear system near an equilibrium point by using Lyapunov's linearization method. Also, Lyapunov's direct method can be used to prove global stability (whether the system is linear or nonlinear) by examining the variation of a single *scalar* function, which is often the total energy of the dynamical system.[15] These concepts are demonstrated in §3.6.

Example 3.3: In this example the linear perturbation technique described previously is used to study the behavior of a highly maneuverable aircraft which exhibits nonlinear behavior. This behavior occurs when the aircraft operates at high angles of attack, in which the lift coefficient cannot be accurately represented as a linear function of angle of attack. Using the coefficients for an F-8 aircraft and normalizing with respect to trim values yields the following nonlinear differential equations for

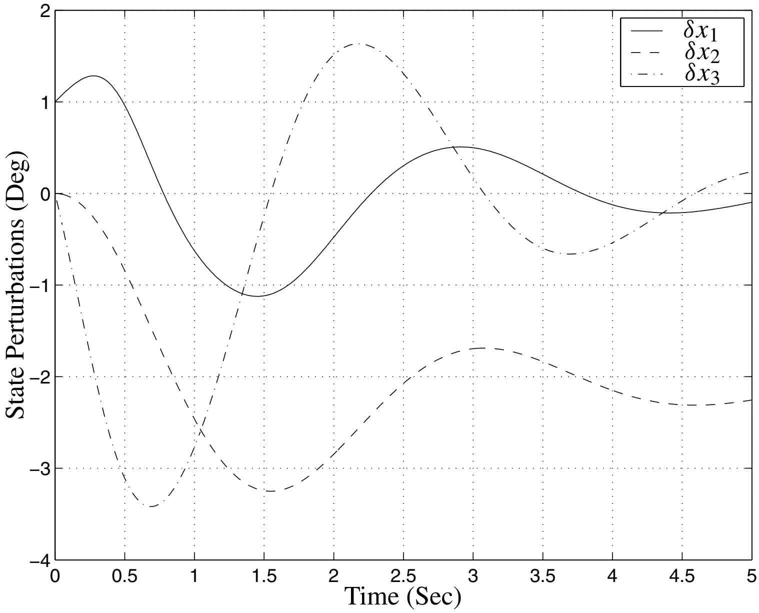

Figure 3.1: State Perturbation Trajectories

the longitudinal motion:[16]

$$\dot{\alpha} = \dot{\theta} - \alpha^2\dot{\theta} - 0.09\alpha\dot{\theta} - 0.88\alpha + 0.47\alpha^2 + 3.85\alpha^3 - 0.02\theta^2$$
$$\ddot{\theta} = -0.396\dot{\theta} - 4.208\alpha - 0.47\alpha^2 - 3.564\alpha^3$$

where α is the angle of attack and θ is the pitch angle (see §3.9). The state vector is chosen as $\mathbf{x} = \left[\alpha\ \theta\ \dot{\theta}\right]^T$. Therefore, the linearized state matrix is

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ 0 & 0 & 1 \\ f_{31} & 0 & f_{33} \end{bmatrix}$$

where

$$f_{11} = -2x_1x_3 - 0.09x_3 - 0.88 + 0.94x_1 + 11.55x_1^2$$
$$f_{12} = -0.04x_2$$
$$f_{13} = 1 - x_1^2 - 0.09x_1$$
$$f_{31} = -4.208 - 0.94x_1 - 10.692x_1^2$$
$$f_{33} = -0.396$$

For the actual system the initial angle of attack is 25 degrees and the pitch and pitch rate are both zero. The nominal state quantities are found by integrating the nonlinear equations with initial conditions given by 24 degrees for the angle of attack and

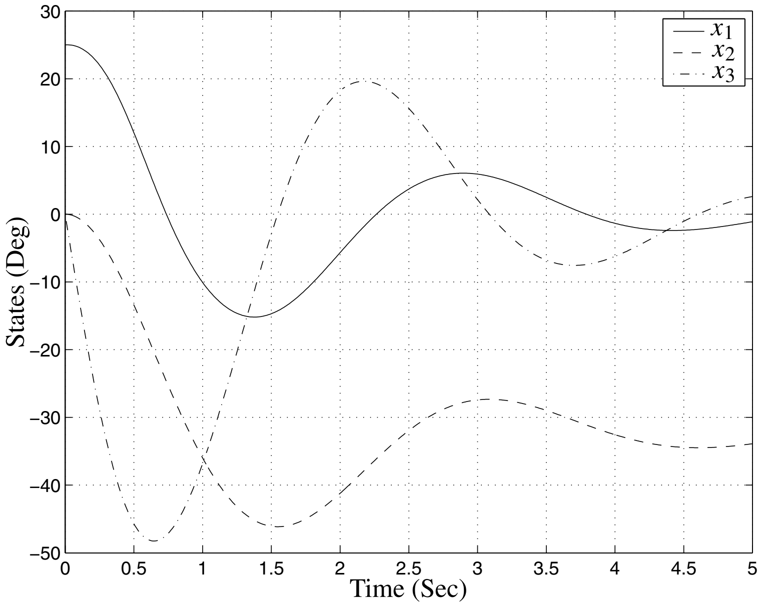

Figure 3.2: State Trajectories

zero for both the pitch and pitch rate. Then the linearized system is integrated with initial conditions given by $\boldsymbol{\delta}\mathbf{x}(t_0) = \left[\pi/180\ 0\ 0\right]^T$. A plot of the state perturbations is shown in Figure 3.1. As shown by this plot the perturbation trajectories are small compared to the large initial condition for the angle of attack. These perturbations are then added to the nominal quantities to form the state trajectories, shown in Figure 3.2. These trajectories closely match the actual state trajectories. Although the nominal trajectory typically involves the integration of the full nonlinear equations, the exercise of performing the linearization still remains useful, as will be demonstrated in the extended Kalman filter of §5.6.

3.3 Parametric Differentiation

Estimation or optimization algorithms are often applied to systems whose state is governed by a system of equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{p}) \tag{3.81}$$

where

$$\mathbf{p} = \begin{bmatrix} p_1 & p_2 & \cdots & p_q \end{bmatrix}^T \tag{3.82}$$

is a set of q *model constants* which appear in the system's differential equations. In many applications, the initial conditions $\mathbf{x}(t_0)$ of eqn. (3.81) will be poorly known, as well as one or more elements of the model parameter vector $\mathbf{p}$. Thus it may be necessary to estimate both $\mathbf{x}(t_0)$ and $\mathbf{p}$ based upon measurements of $\mathbf{x}(t)$ or a function thereof. As will be seen in the applications of Chapter 4, conventional estimation will require the partial derivative matrices

$$\Phi(t,t_0) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(t_0)} \tag{3.83}$$

and

$$\Psi(t,t_0) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{p}} \tag{3.84}$$

We now investigate methods for calculating these derivative matrices.

Equation (3.81) can be written in integral form as

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{f}(\tau,\mathbf{x},\mathbf{p})\, d\tau \tag{3.85}$$

from which it follows

$$\Phi(t,t_0) = I + \int_{t_0}^{t} \frac{\partial \mathbf{f}(\tau,\mathbf{x},\mathbf{p})}{\partial \mathbf{x}(\tau)} \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}(t_0)}\, d\tau \tag{3.86}$$

and

$$\Psi(t,t_0) = \int_{t_0}^{t} \left(\frac{\partial \mathbf{f}(\tau,\mathbf{x},\mathbf{p})}{\partial \mathbf{p}} + \frac{\partial \mathbf{f}(\tau,\mathbf{x},\mathbf{p})}{\partial \mathbf{x}(\tau)} \frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{p}} \right) d\tau \tag{3.87}$$

Taking the time derivative of eqns. (3.86) and (3.87), it follows that the desired derivative matrices satisfy the first-order linear differential equations

$$\dot{\Phi}(t,t_0) = F(t)\Phi(t,t_0), \quad \Phi(t_0,t_0) = I \tag{3.88}$$

and

$$\dot{\Psi}(t,t_0) = F(t)\Psi(t,t_0) + \frac{\partial \mathbf{f}(t,\mathbf{x},\mathbf{p})}{\partial \mathbf{p}}, \quad \Psi(t_0,t_0) = 0 \tag{3.89}$$

where

$$F(t) \equiv \frac{\partial \mathbf{f}(t,\mathbf{x},\mathbf{p})}{\partial \mathbf{x}(t)} \tag{3.90}$$

Observe that $F(t)$ and $\partial \mathbf{f}(t,\mathbf{x},\mathbf{p})/\partial \mathbf{p}$ in eqns. (3.88) and (3.89) depend on $\mathbf{x}(t)$. If numerical methods are required to solve the differential equations (3.81) for $\mathbf{x}(t)$, it is usually convenient to employ the same numerical process to simultaneously integrate eqns. (3.88) and (3.89) to obtain $\Phi(t,t_0)$ and $\Psi(t,t_0)$. Clearly, if the original system can be solved analytically for $\mathbf{x}(t)$, then the partial derivatives can be taken formally and analytical solutions can be determined for $\Phi(t,t_0)$ and $\Psi(t,t_0)$.

As is evident by comparison of eqns. (3.88) and (3.76a), the derivative matrix has the interpretation

$$\boldsymbol{\delta}\mathbf{x}(t) = \Phi(t, t_0)\,\boldsymbol{\delta}\mathbf{x}(t_0) \tag{3.91}$$

where $\boldsymbol{\delta}\mathbf{x}$ are small variations about a reference solution of eqn. (3.81). One important conclusion of the above is that if the original nonlinear system can be solved analytically, then the linear variational equations (3.76a), (3.88), and (3.89) can be solved analytically (i.e., their solution is reduced to a process of formal partial differentiation). This approach will be demonstrated in the orbit determination problem given in §4.3.

The above developments can be derived via a different path that is illuminating. Consider using the following augmented system:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{p}) \tag{3.92a}$$

$$\dot{\mathbf{p}} = \mathbf{0} \tag{3.92b}$$

Equation (3.92) can be rewritten in compact form as

$$\dot{\mathbf{z}} = \mathbf{g}(t, \mathbf{z}) \tag{3.93}$$

where $\mathbf{z} \equiv \begin{bmatrix}\mathbf{x}^T & \mathbf{p}^T\end{bmatrix}^T$ and $\mathbf{g}(t, \mathbf{z}) \equiv \begin{bmatrix}\mathbf{f}^T & \mathbf{0}^T\end{bmatrix}^T$. We now seek the following augmented matrix:

$$\Gamma(t, t_0) \equiv \frac{\partial \mathbf{z}(t)}{\partial \mathbf{z}(t_0)} = \begin{bmatrix} \Phi(t, t_0) & \Psi(t, t_0) \\ 0 & I \end{bmatrix} \tag{3.94}$$

We know the augmented state transition matrix satisfies

$$\dot{\Gamma}(t, t_0) = \frac{\partial \mathbf{g}(t, \mathbf{z})}{\partial \mathbf{z}(t)} \Gamma(t, t_0), \quad \Gamma(t_0, t_0) = I \tag{3.95}$$

where

$$\frac{\partial \mathbf{g}(t, \mathbf{z})}{\partial \mathbf{z}(t)} = \begin{bmatrix} F(t) & \dfrac{\partial \mathbf{f}(t, \mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} \\ & \\ 0 & 0 \end{bmatrix} \tag{3.96}$$

Making use of eqns. (3.94) and (3.96) in eqn. (3.95) immediately verifies eqns. (3.88) and (3.89). Thus augmenting the state vector as in eqns. (3.92) and (3.93) and computing the augmented state transition matrix as in eqns. (3.94) and (3.95) is theoretically equivalent to the sensitivities computed from eqns. (3.88) and (3.89).

3.4 Observability

This section presents one of the most useful concepts in estimation. Observability gives us an indication of the state quantities that can be monitored ("observed") from the measurements. An observable state-space form is given by the observer canonical form:

$$\dot{\mathbf{x}}_o = F_o\,\mathbf{x}_o + B_o\,u \tag{3.97a}$$

$$y_o = H_o\,\mathbf{x}_o + D_o\,u \tag{3.97b}$$

where the matrices F_o, B_o, H_o, and D_o are given by

$$F_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \tag{3.98a}$$

$$B_o = \left[(b_0 - b_n\,a_0)\ (b_1 - b_n\,a_1)\ \cdots\ (b_{n-1} - b_n\,a_{n-1})\right]^T \tag{3.98b}$$

$$H_o = \left[0\ 0\ \cdots\ 1\right] \tag{3.98c}$$

$$D_o = b_n \tag{3.98d}$$

Clearly, since all states are "coupled" together in the F_o matrix, we only need to monitor one state (given as the last state by H_o) to observe *all* states. The matrix F_o is called the *right companion matrix* to the characteristic equation since the coefficients of eqn. (3.9b) appear on the right side of the matrix.

A general single-output system (F, B, H, D) is "fully observable" if it can be converted into observer canonical form. This is achieved via a transformation of state shown in §3.1.4:

$$F_o = T^{-1}F\,T \tag{3.99}$$

where T is a nonsingular constant matrix. To demonstrate the general form for T, we begin by considering the third-order case (the extension to the general case will be clear from this development). Left multiplying both sides of eqn. (3.99) by T gives

$$T\,F_o = F\,T \tag{3.100}$$

For the third-order case let T be partitioned into column vectors so that $T = \left[\mathbf{t}_1\ \mathbf{t}_2\ \mathbf{t}_3\right]$. This leads directly to

$$\left[\mathbf{t}_1\ \mathbf{t}_2\ \mathbf{t}_3\right] \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} = F\left[\mathbf{t}_1\ \mathbf{t}_2\ \mathbf{t}_3\right] \tag{3.101}$$

Next, solving for $\mathbf{t}_2$ and $\mathbf{t}_3$ gives

$$\mathbf{t}_2 = F\,\mathbf{t}_1 \tag{3.102a}$$

$$\mathbf{t}_3 = F\,\mathbf{t}_2 \tag{3.102b}$$

Since $\mathbf{x} = T\,\mathbf{x}_o$, then $H\,T = H_o$, which gives the following three equations:

$$H\,\mathbf{t}_1 = 0 \tag{3.103a}$$

$$H\,\mathbf{t}_2 = 0 \tag{3.103b}$$

$$H\,\mathbf{t}_3 = 1 \tag{3.103c}$$

Substituting eqn. (3.102) into eqn. (3.103) leads to

$$\mathbf{t}_1 = \begin{bmatrix} H \\ H\,F \\ H\,F^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.104}$$

Clearly, the original system can only be transformed into observer canonical form if the matrix inverse in eqn. (3.104) exists. The extension to higher-order systems is given by the following $n \times n$ *observability matrix*:

$$\boxed{\mathcal{O} = \begin{bmatrix} H \\ H\,F \\ H\,F^2 \\ \vdots \\ H\,F^{n-1} \end{bmatrix}} \tag{3.105}$$

For a system to be fully observable, the observability matrix $\mathcal{O}$ must be non-singular. Also, a multi-output system is observable if the rank of the $mn \times n$ matrix (where m is the number of outputs) is equal to n.

Example 3.4: In this simple example we consider only a second-order system, with state matrices given by

$$F = \begin{bmatrix} 0 & 1 \\ -2 & -f_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

where f_{22}, b_{11}, and b_{21} are real numbers. Computing the observability matrix in eqn. (3.105) with $n = 2$ gives

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ -2 & 1 - f_{22} \end{bmatrix}$$

Clearly, the system is observable unless $f_{22} = 3$. Let us compute the transfer function using eqn. (3.14) to gain some physical insight for the case when $f_{22} = 3$:

$$\frac{Y(s)}{U(s)} = \frac{(b_{11} + b_{21})(s+1)}{(s+1)(s+3)}$$

This clearly indicates that a "pole-zero cancellation" has occurred (i.e., one of the roots of the numerator polynomial cancels one of the roots of the denominator polynomial). Therefore, we cannot observe the state associated with $s + 1 = 0$.

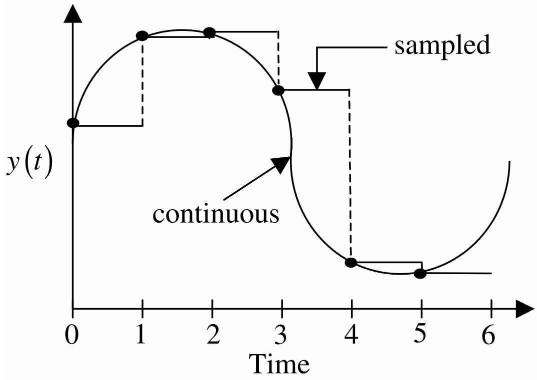

Figure 3.3: Continuous Signal and Sampled Zero-Order Hold

Observability is a powerful tool for state estimation. If a system is not fully observable then all is not lost. A singular value decomposition of the observability matrix can give us insight as to what states are observable. If the observed states are adequate for the dynamical system's requirements (e.g., for control requirements), then a fully observable system may not be necessary. Finally, an extension of observability to nonlinear systems is possible; however, for most nonlinear dynamical systems only local observability can be proven mathematically.[17, 18]

3.5 Discrete-Time Systems

All of the concepts shown in §3.1 extend to discrete-time systems. Discrete-time systems have now become standard in most dynamical applications with the advent of digital computers, which are used to process sampled-data systems for estimation and control purposes. The mechanism that acts on the sensor output and supplies numbers to the digital computer is the analog-to-digital (A/D) converter. Then, the numbers are processed through numerical subroutines and sent to the dynamical system input through the digital-to-analog (D/A) converter. This allows the use of software driven systems to accommodate the estimation/control aspect of a dynamical system, which can be modified simply by uploading new subroutines to the computer.

We shall only consider the most common sampled-type system given by a "zero-order hold" which holds the sampled point to a constant value throughout the interval. Figure 3.3 shows a sampled signal using a zero-order hold. Obviously as the sample interval decreases the sampled signal more closely approximates the continuous signal. Consider the case where time is set to the first sample interval, denoted by Δt, and $F(t)$ and $B(t)$ are constants in eqn. (3.42). Then eqn. (3.53) reduces to

$$\mathbf{x}(\Delta t) = e^{F\Delta t}\mathbf{x}(0) + \left[\int_0^{\Delta t} e^{F(\Delta t-\tau)}\, d\tau\right] B\,\mathbf{u}(0) \tag{3.106}$$

The integral on the right-hand side of eqn. (3.106) can be simplified by defining $\zeta = \Delta t - \tau$, which leads to

$$\int_0^{\Delta t} e^{F(\Delta t-\tau)}\, d\tau = -\int_{\Delta t}^{0} e^{F\zeta}\, d\zeta = \int_0^{\Delta t} e^{F\zeta}\, d\zeta \tag{3.107}$$

Therefore, eqn. (3.106) becomes

$$\mathbf{x}(\Delta t) = \Phi\,\mathbf{x}(0) + \Gamma\,\mathbf{u}(0) \tag{3.108}$$

where

$$\Phi \equiv e^{F\Delta t} \tag{3.109a}$$

$$\Gamma \equiv \left[\int_0^{\Delta t} e^{Ft}\, dt\right] B \tag{3.109b}$$

Expanding (3.108) for $k+1$ samples gives

$$\mathbf{x}[(k+1)\Delta t] = \Phi\,\mathbf{x}(k\,\Delta t) + \Gamma\,\mathbf{u}(k\,\Delta t) \tag{3.110}$$

It is common convention to drop Δt notation from eqn. (3.110) so that the entire discrete state-space representation is given by

$$\boxed{\begin{aligned} \mathbf{x}_{k+1} &= \Phi\,\mathbf{x}_k + \Gamma\,\mathbf{u}_k && (3.111\text{a})\\ \mathbf{y}_k &= H\,\mathbf{x}_k + D\,\mathbf{u}_k && (3.111\text{b}) \end{aligned}}$$

Notice that the output system matrices H and D are unaffected by the conversion to a discrete-time system. The system can be shown to be stable if all eigenvalues of Φ lie within the unit circle.[3]

Example 3.5: In this example we will perform a conversion from the continuous-time domain to the discrete-time domain for a second-order system, given by

$$F = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To compute Φ we will enlist the help of Laplace transforms, with

$$\Phi = e^{F\Delta t} = \left\{\mathcal{L}^{-1}[sI - F]^{-1}\right\}\Big|_{\Delta t} = \left\{\mathcal{L}^{-1}\begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \dfrac{1}{s(s+1)} & \dfrac{1}{s} \end{bmatrix}\right\}\Bigg|_{\Delta t}$$
$$= \begin{bmatrix} e^{-\Delta t} & 0 \\ 1 - e^{-\Delta t} & 1 \end{bmatrix}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. The matrix Γ is computed using eqn. (3.109b):

$$\Gamma = \int_0^{\Delta t} \begin{bmatrix} e^{-t} \\ 1-e^{-t} \end{bmatrix} dt = \begin{bmatrix} 1-e^{-\Delta t} \\ \Delta t + e^{-\Delta t} - 1 \end{bmatrix}$$

If the sampling interval is chosen to be $\Delta t = 0.1$ seconds, then Φ and Γ become

$$\Phi = \begin{bmatrix} 0.9048 & 0 \\ 0.0952 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.0952 \\ 0.0048 \end{bmatrix}$$

Determining analytical expressions for Φ and Γ can be tedious and difficult for large order systems. Fortunately, several numerical approaches exist for computing these matrices.[19] A computationally efficient and accurate approach involves a series expansion:

$$\boxed{\Phi = I + F\Delta t + \frac{1}{2!}F^2\Delta t^2 + \frac{1}{3!}F^3\Delta t^3 + \cdots} \tag{3.112}$$

The matrix Γ is obtained from integration of eqn. (3.112):

$$\boxed{\Gamma = \left[I\Delta t + \frac{1}{2!}F\Delta t^2 + \frac{1}{3!}F^2\Delta t^3 + \cdots \right] B} \tag{3.113}$$

Adequate results can be obtained in most cases using only a few of the terms in the series expansion. For the matrices in example 3.5, using three terms in the series expansion yields

$$\Phi = \begin{bmatrix} 0.9048 & 0 \\ 0.0952 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.0952 \\ 0.0048 \end{bmatrix} \tag{3.114}$$

The series results for Φ and Γ are accurate to within four significant digits. Results vary with sampling interval. As a general rule of thumb, if the sampling interval is below Nyquist's upper limit, then three to four terms in the series expansion gives accurate results.[20]

The concept of observability can be extended to discrete-time systems. The discrete system is observable if there exists a finite k such that knowledge of the outputs to $k-1$ is sufficient to determine the initial state of the system.[21] Expanding eqn. (3.111), for single output with $\mathbf{u}_k = \mathbf{0}$, to $n-1$ points to obtain n equations for the n unknown initial condition gives

$$\begin{aligned} y_0 &= H\mathbf{x}_0 \\ y_1 &= H\mathbf{x}_1 = H\Phi\,\mathbf{x}_0 \\ y_2 &= H\mathbf{x}_2 = H\Phi^2\,\mathbf{x}_0 \\ &\vdots \\ y_{n-1} &= H\mathbf{x}_{n-1} = H\Phi^{n-1}\,\mathbf{x}_0 \end{aligned} \tag{3.115}$$

Solving eqn. (3.115) for $\mathbf{x}_0$ yields

$$\mathbf{x}_0 = \begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \\ \vdots \\ H\Phi^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} \tag{3.116}$$

Clearly, the initial state $\mathbf{x}_0$ can be obtained only if the following observability matrix is nonsingular:

$$\boxed{\mathcal{O}_d = \begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \\ \vdots \\ H\Phi^{n-1} \end{bmatrix}} \tag{3.117}$$

If multiple outputs are given, then for the system to be fully observable $\mathcal{O}_d$ must have rank n.

The main difference in the analysis tools for discrete-time versus continuous-time systems is in the sampling interval. The sampling interval can adversely affect the system's response, but it can also be actually used as another design parameter in a dynamical system to achieve a desired response characteristic. Available tools for discrete-time systems include: z-transforms, bilinear transformations, stability, etc. These concepts are beyond the scope of the present text, since only the required basic fundamentals have been presented. The interested reader can pursue these subjects in more depth by studying the references cited in this section.

3.6 Stability of Linear and Nonlinear Systems

Stability of linear and nonlinear systems is extremely important in both control and estimation algorithms. In estimation the stability of a sequential process is a stringent requirement so that the estimated quantities remain within a bounded region. The general definition of stability begins with Bounded-Input-Bounded-Output (BIBO) stability. Before providing the definition of BIBO stability, we first must describe a *relaxed system*. A system is said to be relaxed at time t_0 if and only if the output $\mathbf{y}_{[t_o,\infty)}$ is solely and uniquely excited by $\mathbf{u}_{[t_o,\infty)}$.[12] For linear systems the relaxed condition follows $\mathbf{y}(t) = H\mathbf{u}_{(-\infty,t_0)} = \mathbf{0}$ for all $t \geq t_0$. A relaxed system is said to be BIBO stable if and only if for any bounded input, the output is bounded.

Let us consider the linear time-invariant model of eqn. (3.14). Since we assume that the input is bounded, we have

$$||\mathbf{u}(t)|| \leq \alpha < \infty \quad \text{for all } t \geq 0 \tag{3.118}$$

where α is a positive constant. The solution for $\mathbf{y}(t)$ assuming a relaxed condition (i.e., $\mathbf{x}(t_0) = \mathbf{0}$) is given by eqn. (3.53):

$$\mathbf{y}(t) = H \int_{t_0}^{t} \Phi(t,\tau)\, B\, \mathbf{u}(\tau)\, d\tau \tag{3.119}$$

Since BIBO stability must be valid for all time, we can allow $t \to \infty$. Next, making use of the convolution integral for $\mathbf{y}(t)$ as $t \to \infty$ gives

$$\mathbf{y}(\infty) = H \int_{t_0}^{\infty} \Phi(\tau)\, B\, \mathbf{u}(t-\tau)\, d\tau \tag{3.120}$$

Taking the norm of both sides of eqn. (3.120) and using eqn. (3.118) yields

$$||\mathbf{y}(\infty)|| \le \alpha \left\| H \int_{t_0}^{\infty} \Phi(\tau)\, B\, d\tau \right\| \tag{3.121}$$

Therefore, the system is bounded if

$$\left\| H \int_{t_0}^{\infty} \Phi(\tau)\, B\, d\tau \right\| < \infty \tag{3.122}$$

which can only be true if

$$\lim_{t\to\infty} ||\Phi(t,t_0)|| = 0 \tag{3.123}$$

From eqn. (3.26) the condition in eqn. (3.123) is satisfied if and only if all the eigenvalues of F have negative real parts.

BIBO stability for nonlinear systems is much more difficult to prove. Fortunately, Lyapunov methods can be applied to show BIBO stability for both nonlinear and linear systems. Two methods for stability were introduced by Lyapunov. The first is given by Lyapunov's linearization method. Before proceeding with this method we must first define an equilibrium point. An equilibrium is defined as a point where the system states remain indefinitely, so that $\dot{\mathbf{x}} = \mathbf{0}$. For linear systems there is usually only one equilibrium point given at $\mathbf{x} = \mathbf{0}$, although there are exceptions (see exercise 3.9). In Lyapunov's linearization method each equilibrium point is considered and evaluated in the linearized model of eqn. (3.76). The equilibrium point is said to be Lyapunov stable if we can select a bound on initial conditions that results in trajectories that remain with a chosen finite limit. Furthermore, the equilibrium point is asymptotically stable if the state also approaches zero as time approaches infinity. Lyapunov's linearization method gives the following stability conditions:[15]

- The equilibrium point is asymptotically stable for the actual nonlinear system if the linearized system is strictly stable, with all eigenvalues of F strictly in the left-hand plane.

- The equilibrium point is unstable for the actual nonlinear system if the linearized system is strictly unstable, with at least one eigenvalue strictly on the right-hand plane.

- Nothing can be concluded if the linearized system is marginally stable, with at least one eigenvalue of F on the imaginary axis and the remainder in the left-hand plane (the equilibrium point may be stable or unstable for the nonlinear system).

Lyapunov's linearization method provides a powerful approach to help qualify the stability of a system if a control (or estimation) scheme is designed to remain within a linear region, but does not give a thorough understanding of the nonlinear system in many cases.

Lyapunov's direct method gives a global stability condition for the general nonlinear system. This concept is closely related to the energy of a system, which is a *scalar* function. The scalar function must in general be continuous and have continuous derivatives with respect to all components of the state vector. Lyapunov showed that if the total energy of a system is dissipated, then the state is confined to a volume bounded by a surface of constant energy, so that the system must eventually settle to an equilibrium point. This concept is valid for both linear and nonlinear systems. Lyapunov stability is given if a chosen scalar function $V(\mathbf{x})$ satisfies the following conditions:

- $V(\mathbf{0}) = 0$
- $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$
- $\dot{V}(\mathbf{x}) \leq 0$

If these conditions are met, then $V(\mathbf{x})$ is a *Lyapunov function*. Furthermore, if $\dot{V}(\mathbf{x}) < 0$ for $\mathbf{x} \neq \mathbf{0}$ then the system is asymptotically stable.

Example 3.6: Consider the following spring-mass-damper system with nonlinear spring and damper components:

$$m\ddot{x} + c\dot{x}|\dot{x}| + k_1 x + k_2 x^3 = 0$$

where m, c, k_1, and k_2 have positive values. The system can be represented in first-order form by defining the following state vector $\mathbf{x} = \begin{bmatrix} x & \dot{x} \end{bmatrix}^T$:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(k_1/m)x_1 - (k_2/m)x_1^3 - (c/m)x_2|x_2|
\end{aligned}$$

The system has only one equilibrium point at $\mathbf{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ that is physically correct (the other one is complex). We wish to investigate the stability of this nonlinear system using Lyapunov's direct method. Intuitively, we choose a candidate Lyapunov function that is given by the total mechanical energy of the system, which is the sum of its kinetic and potential energies:

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \int_0^x (k_1 x + k_2 x^3)\, dx$$

Evaluating this integral yields

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_1 x^2 + \frac{1}{4}k_2 x^4$$

Note that zero energy corresponds to the equilibrium point ($\mathbf{x} = \mathbf{0}$), which satisfies the first condition for a valid Lyapunov function. Also, the second condition, $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$, is clearly satisfied. Taking the time derivative of $V(\mathbf{x})$ gives

$$\dot{V}(\mathbf{x}) = m\ddot{x}\dot{x} + (k_1 x + k_2 x^3)\dot{x}$$

Solving the original system equation for $m\ddot{x}$, and substituting the resulting expression into the equation for $\dot{V}(\mathbf{x})$ yields

$$\dot{V}(\mathbf{x}) = -c|\dot{x}|^3$$

Clearly this expression meets the final condition for a valid Lyapunov function since $\dot{V}(\mathbf{x}) \leq 0$ for all nonzero values of x and $\dot{x}$. Therefore, since $V(\mathbf{x})$ is indeed a Lyapunov function the system is asymptotically stable. This example shows how an "energy-like" function can be used to find a Lyapunov function, since the energy of this system is dissipated by the damper until the mass settles down. More details on Lyapunov methods for stability can be found in Ref. [15].

Lyapunov's global method also is valid for linear time-invariant systems with $\dot{\mathbf{x}} = F\mathbf{x}$. Consider the function $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$, where P is a positive definite symmetric matrix. Clearly, $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$. The time derivative of $V(\mathbf{x})$ is given by

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{x}^T P\dot{\mathbf{x}} \tag{3.124a}$$

$$= \mathbf{x}^T (F^T P + PF)\mathbf{x} \tag{3.124b}$$

Next, define the following *matrix Lyapunov equation*:

$$\boxed{F^T P + PF = -Q} \tag{3.125}$$

If Q is strictly positive definite then the system is asymptotically stable. Lyapunov showed that this condition is true if and only if all eigenvalues of F are strictly in the left-hand plane. The proof begins by using $\mathbf{x}(t) = e^{Ft}\mathbf{x}_0$ and setting

$$P = \int_0^\infty e^{F^T t} Q e^{Ft}\, dt \tag{3.126}$$

where Q is assumed to be strictly positive definite. Then $F^T P + PF$ is given by

$$F^T P + PF = \int_0^\infty \left(F^T e^{F^T t} Q e^{Ft} + e^{F^T t} Q e^{Ft} F\right) dt \tag{3.127}$$

Next, we use the time derivative of e^{Ft}, which is given by

$$\frac{d}{dt}e^{Ft} = F e^{Ft} = e^{Ft} F \tag{3.128}$$

The second equality in eqn. (3.128) is due to the fact that F and e^{Ft} commute (see Appendix A). Then, the quantity within the integral of eqn. (3.127) can be written as

$$\frac{d}{dt}\left(e^{F^T t} Q e^{Ft}\right) = F^T e^{F^T t} Q e^{Ft} + e^{F^T t} Q e^{Ft} F \tag{3.129}$$

Therefore, we have

$$\begin{aligned} F^T P + P F &= \int_0^\infty \frac{d}{dt}\left(e^{F^T t} Q e^{Ft}\right) dt \\ &= e^{F^T t} Q e^{Ft}\Big|_0^\infty \end{aligned} \tag{3.130}$$

If all eigenvalues of F have negative real parts then the integral in eqn. (3.130) is given by

$$e^{F^T t} Q e^{Ft}\Big|_0^\infty = -Q \tag{3.131}$$

which gives the original matrix Lyapunov equation. Since eqn. (3.126) actually shows the existence of a solution P for *any* square matrix Q, then for any Q the solution for P is unique.[15] A simple choice of Q is given by the identity matrix.

Example 3.7: Given the following state matrix:

$$F = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix}$$

we wish to determine the ranges for a and b that yield a stable response using Lyapunov's direct method. Choosing $Q = I$, Lyapunov's matrix equation leads to the following three algebraic equations:

$$\begin{aligned} -a\,p_{11} - b\,p_{12} - a\,p_{11} - b\,p_{12} &= -1 \\ -a\,p_{12} - b\,p_{22} - a\,p_{12} + b\,p_{11} &= 0 \\ -a\,p_{22} + b\,p_{12} - a\,p_{22} + b\,p_{12} &= -1 \end{aligned}$$

where p_{11}, p_{22}, and p_{12} are the elements of the P matrix. The solutions for these elements are straightforward and are given by $p_{11} = p_{22} = 1/2a$ and $p_{12} = 0$, so that

$$P = \begin{bmatrix} \frac{1}{2a} & 0 \\ 0 & \frac{1}{2a} \end{bmatrix}$$

The matrix P is positive definite when $a > 0$, which gives the range for stability of the overall system matrix. This is easily confirmed by computing the eigenvalues of F, which are found from the roots of the following characteristic equation:

$$s^2 + 2a\,s + a^2 + b^2 = 0$$

This again shows that the real parts of s are negative when $a > 0$. Note that b may take any value.

Lyapunov's linearization and direct methods can also be applied to discrete-time systems. A nonlinear discrete system with no forcing input is represented by

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) \tag{3.132}$$

Equilibrium points are determined by allowing $k+1 \to k$. Lypaunov's linearization method involves evaluating the equilibrium points using the following linearized model:

$$\Phi = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \tag{3.133}$$

The stability conditions are exactly the same as in the case for the continuous system. All eigenvalues of Φ must be within the unit circle for the equilibrium point to be stable. If at least one eigenvalue of Φ is on the unit circle then nothing can be concluding for the linearization method. The theory for the discrete-time case of Lyapunov's direct method has been presented by Kalman and Bertram.[22] For Lyapunov's direct method the discrete-time system is stable if the following conditions are satisfied for a chosen scalar function $V(\mathbf{x})$:[3]

- $V(\mathbf{0}) = 0$
- $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$
- $\Delta V(\mathbf{x}) = V[\mathbf{f}(\mathbf{x})] - V(\mathbf{x}) \leq 0$

When these conditions are satisfied then $V(\mathbf{x})$ is a discrete Lyapunov function. Furthermore, if $\Delta V(\mathbf{x}) < 0$ for $\mathbf{x} \neq \mathbf{0}$ then the system is asymptotically stable.

Lyapunov's global method also is valid for linear time-invariant systems with $\mathbf{x}_{k+1} = \Phi \mathbf{x}_k$. Consider the function $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$, where P is a positive definite symmetric matrix. Clearly, $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$. The increment of $V(\mathbf{x})$ is given by

$$\Delta V(\mathbf{x}) = V(\Phi \mathbf{x}) - V(\mathbf{x}) \tag{3.134a}$$

$$= \mathbf{x}^T (\Phi^T P \Phi - P)\mathbf{x} \tag{3.134b}$$

Next, define the following matrix Lyapunov equation:

$$\boxed{\Phi^T P \Phi - P = -Q} \tag{3.135}$$

If Q is strictly positive definite then the system is asymptotically stable. This condition is true if and only if all eigenvalues of Φ are within the unit circle.

We shall now prove that the linear sequential estimator given by eqns. (1.77) to (1.80) is asymptotically stable. For this proof (assuming a bounded input $\mathbf{y}$) we can ignore the measurements and only treat the following recursion:

$$\hat{\mathbf{x}}_{k+1} = [I - K_{k+1} H_{k+1}]\hat{\mathbf{x}}_k \tag{3.136}$$

Next, we consider the following candidate Lyapunov function!

$$V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}^T P^{-1} \hat{\mathbf{x}} \tag{3.137}$$

The increment of $V(\hat{\mathbf{x}})$ is given by

$$\Delta V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_{k+1}^T P_{k+1}^{-1} \hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_k^T P_k^{-1} \hat{\mathbf{x}}_k \tag{3.138}$$

Substituting eqn. (3.136) and the inverse of eqn. (1.80) into eqn. (3.138), and simplifying yields

$$\Delta V(\hat{\mathbf{x}}) = -\hat{\mathbf{x}}_k^T H_{k+1}^T K_{k+1}^T P_k^{-1} \hat{\mathbf{x}}_k \tag{3.139}$$

Finally, substituting the transpose of eqn. (1.79) into eqn. (3.139) gives

$$\Delta V(\hat{\mathbf{x}}) = -\hat{\mathbf{x}}_k^T H_{k+1}^T [H_{k+1} P_k H_{k+1}^T + W_{k+1}^{-1}]^{-1} H_{k+1} \hat{\mathbf{x}}_k \tag{3.140}$$

Therefore, since $H_{k+1}^T [H_{k+1} P_k H_{k+1}^T + W_{k+1}^{-1}]^{-1} H_{k+1}$ is positive definite, then we have $\Delta V(\hat{\mathbf{x}}) < 0$, and the sequential estimator is asymptotically stable. Further details on Lyapunov stability can be found in the references cited in this section.

3.7 Attitude Kinematics and Rigid Body Dynamics

This section reviews the equations and concepts of rotational attitude kinematics and dynamics. These equations form the basis for spacecraft, aircraft, and robotic dynamical systems. Only a brief review of the concepts are presented in this chapter.

3.7.1 Attitude Kinematics

The attitude of a vehicle is defined as its orientation with respect to some reference frame. If the reference frame is non-moving, then it is commonly referred to as an *inertial* frame. To describe the attitude two coordinate systems are usually defined: one on the vehicle body and one on the reference frame. For most dynamical applications these coordinate systems have orthogonal unit vectors that follow the right-hand rule. The *attitude matrix* (A), often referred to as the direction cosine matrix or rotation matrix, maps one frame to another (for spacecraft and aircraft kinematics this mapping is usually from the reference frame to the vehicle body frame). A graphical representation of this concept is shown in Figure 3.4.

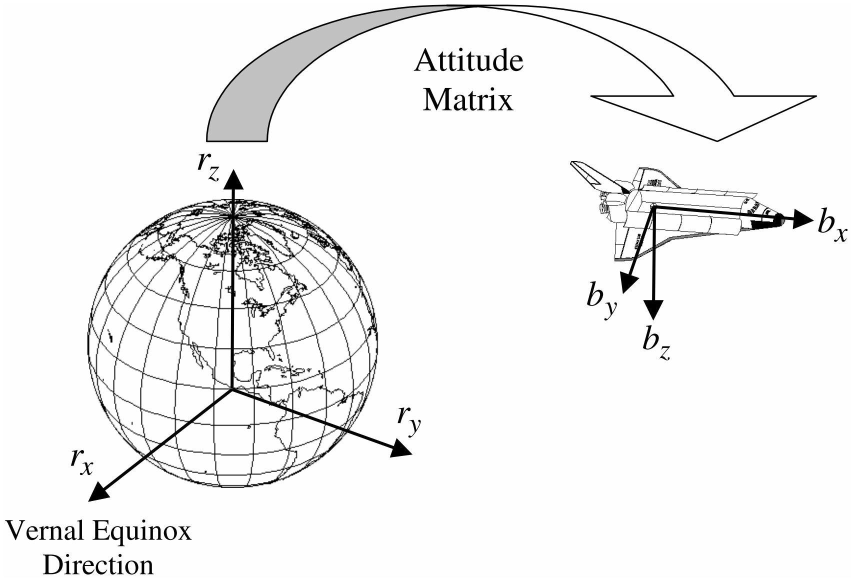

Figure 3.4: Relationship between Reference and Body Frames

Mathematically, the mapping from the reference frame to the body frame is given by

$$\boxed{\mathbf{b} = A\mathbf{r}} \tag{3.141}$$

where $\mathbf{b} = \begin{bmatrix} b_x & b_y & b_z \end{bmatrix}^T$ is the body-frame vector and $\mathbf{r} = \begin{bmatrix} r_x & r_y & r_z \end{bmatrix}^T$ is the reference-frame vector. These vectors are sometimes given by a sum of unit vectors, with orthonormal bases:

$$\mathbf{b} = b_x \hat{\mathbf{b}}_1 + b_y \hat{\mathbf{b}}_2 + b_z \hat{\mathbf{b}}_3 \tag{3.142a}$$

$$\mathbf{r} = r_x \hat{\mathbf{r}}_1 + r_y \hat{\mathbf{r}}_2 + r_z \hat{\mathbf{r}}_3 \tag{3.142b}$$

As an aside, we note the projections of the $\hat{\mathbf{b}}_i$ unit vectors onto the $\hat{\mathbf{r}}_i$ unit vectors are accomplished by the same matrix as

$$\begin{Bmatrix} \hat{\mathbf{b}}_1 \\ \hat{\mathbf{b}}_2 \\ \hat{\mathbf{b}}_3 \end{Bmatrix} = A \begin{Bmatrix} \hat{\mathbf{r}}_1 \\ \hat{\mathbf{r}}_2 \\ \hat{\mathbf{r}}_3 \end{Bmatrix} \tag{3.143}$$

where *vectrix* notation is used in eqn. (3.143) (see Ref. [23] for details). The matrix A is in fact an *orthogonal* matrix since its inverse is given by its transpose. Also, for right-handed systems the determinant of A is given by $+1$.[24] In other words, the attitude matrix is a *proper real orthogonal* matrix. Many parameterizations exist for the attitude matrix, including: the Euler angles, Euler axis/angle, the quaternion, Cayley-Klein parameters, Gibb's vector, modified Rodrigues parameters, etc.[25]

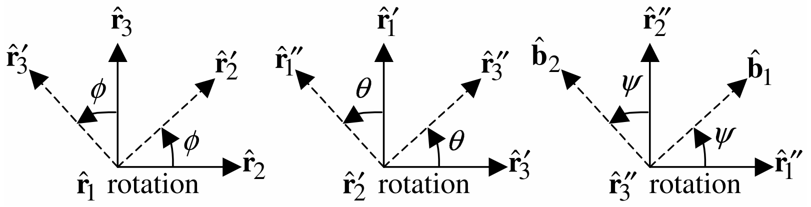

Figure 3.5: Euler Angles for a 1-2-3 Rotation Sequence

Euler angles are commonly used to parameterize the attitude matrix since they give a physical representation. The classical Euler angles are denoted by the roll (ϕ), pitch (θ), and yaw (ψ) angles. Consider a 1-2-3 Euler angle sequence, as shown by Figure 3.5. This sequence performs a rotation from the reference vector ($\mathbf{r}$) to the body vector ($\mathbf{b}$) through a rotation about the $\hat{\mathbf{r}}_1$ vector (the 1-axis rotation) first, with

$$\mathbf{r}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \mathbf{r} \tag{3.144}$$

Then a rotation about the $\hat{\mathbf{r}}_2'$ vector is performed (the 2-axis rotation), with

$$\mathbf{r}'' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \mathbf{r}' \tag{3.145}$$

Finally a rotation about the $\hat{\mathbf{r}}_3'$ vector is performed (the 3-axis rotation), with

$$\mathbf{b} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{r}'' \tag{3.146}$$

Substituting eqn. (3.144) into eqn. (3.145), and substituting the resulting equation into eqn. (3.146) leads to the following form for the attitude matrix:

$$A = \begin{bmatrix} \mathrm{c}\psi\,\mathrm{c}\theta & \mathrm{s}\psi\,\mathrm{c}\phi + \mathrm{c}\psi\,\mathrm{s}\theta\,\mathrm{s}\phi & \mathrm{s}\psi\,\mathrm{s}\phi - \mathrm{c}\psi\,\mathrm{s}\theta\,\mathrm{c}\phi \\ -\mathrm{s}\psi\,\mathrm{c}\theta & \mathrm{c}\psi\,\mathrm{c}\phi - \mathrm{s}\psi\,\mathrm{s}\theta\,\mathrm{s}\phi & \mathrm{c}\psi\,\mathrm{s}\phi + \mathrm{s}\psi\,\mathrm{s}\theta\,\mathrm{c}\phi \\ \mathrm{s}\theta & -\mathrm{c}\theta\,\mathrm{s}\phi & \mathrm{c}\theta\,\mathrm{c}\phi \end{bmatrix} \tag{3.147}$$

where $\mathrm{c}\psi \equiv \cos\psi$, $\mathrm{s}\phi \equiv \sin\phi$, etc. There are in fact twelve possible rotation sequences: six asymmetric (1-2-3, 1-3-2, 2-1-3, 2-3-1, 3-1-2, 3-2-1) and six symmetric (1-2-1, 1-3-1, 2-1-2, 2-3-2, 3-1-3, 3-2-3). An interesting case for the attitude matrix occurs when the Euler angles are small so that the cosine of the angle is approximately one and the sine of the angle is approximately the angle. In this case the attitude matrix is adequately approximated by

$$A \approx \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix} = I_{3\times 3} - [\boldsymbol{\alpha}\times] \tag{3.148}$$

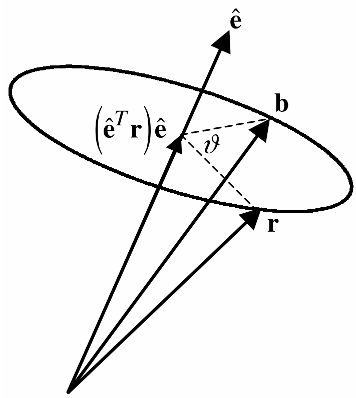

Figure 3.6: Euler Axis and Angle

where $\boldsymbol{\alpha} \equiv \begin{bmatrix}\phi & \theta & \psi\end{bmatrix}^T$, $I_{3\times 3}$ is a 3×3 identity matrix, and $[\boldsymbol{\alpha}\times]$ is referred to as a cross product matrix because $\boldsymbol{\alpha}\times\boldsymbol{\beta} = [\boldsymbol{\alpha}\times]\boldsymbol{\beta}$, with

$$[\boldsymbol{\alpha}\times] \equiv \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix} \tag{3.149}$$

Another attitude parameterization is given by the Euler axis $\hat{\mathbf{e}}$ and angle ϑ. Euler's theorem states that the most general motion of a rigid body with one point fixed is a rotation about some axis. This is represented by Figure 3.6, and can mathematically be written as

$$\mathbf{b} = \left(\hat{\mathbf{e}}^T\mathbf{r}\right)\hat{\mathbf{e}} + \cos\vartheta\left[\mathbf{r} - \left(\hat{\mathbf{e}}^T\mathbf{r}\right)\hat{\mathbf{e}}\right] - \sin\vartheta\left(\hat{\mathbf{e}}\times\mathbf{r}\right) \tag{3.150}$$

Comparing eqn. (3.150) with eqn. (3.141) gives the following attitude matrix:

$$\boxed{A = (\cos\vartheta)I_{3\times 3} + (1-\cos\vartheta)\hat{\mathbf{e}}\hat{\mathbf{e}}^T - \sin\vartheta[\hat{\mathbf{e}}\times]} \tag{3.151}$$

We also note that the Euler axis $\hat{\mathbf{e}}$ is unchanged by the attitude matrix, so that $A\hat{\mathbf{e}} = \hat{\mathbf{e}}$. This is true since any proper orthogonal 3×3 matrix has at least one eigenvector with unity eigenvalue.[24]

One of the most useful attitude parameterization is given by the *quaternion*.[26] Like the Euler axis/angle parameterization, the quaternion is also a four-dimensional vector, defined as

$$\mathbf{q} \equiv \begin{bmatrix} \boldsymbol{\varrho} \\ q_4 \end{bmatrix} \tag{3.152}$$

with

$$\boldsymbol{\varrho} \equiv \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T = \hat{\mathbf{e}}\sin(\vartheta/2) \tag{3.153a}$$

$$q_4 = \cos(\vartheta/2) \tag{3.153b}$$

Since a four-dimensional vector is used to describe three dimensions, the quaternion components cannot be independent of each other. The quaternion satisfies a single constraint given by $\mathbf{q}^T\mathbf{q} = 1$, which is analogous to requiring that $\hat{\mathbf{e}}$ be a unit vector in the Euler axis/angle parameterization. The attitude matrix is related to the quaternion by

$$\boxed{A(\mathbf{q}) = \Xi^T(\mathbf{q})\Psi(\mathbf{q})} \tag{3.154}$$

with

$$\Xi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times3} + [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{3.155a}$$

$$\Psi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3\times3} - [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{3.155b}$$

An advantage to using quaternions, which will be exploited in Chapter 4, is that the attitude matrix is quadratic in the parameters and also does not involve transcendental functions. For small angles the vector part of the quaternion is approximately equal to half angles so that $\boldsymbol{\varrho} \approx \boldsymbol{\alpha}/2$ and $q_4 \approx 1$.

The attitude kinematics equation can be derived by considering a state transition matrix $\Phi(t+\Delta t, t)$ that maps the attitude from one time to the next:

$$A(t+\Delta t) = \Phi(t+\Delta t, t)A(t) \tag{3.156}$$

Obviously $\Phi(t+\Delta t, t)$ must also be an attitude matrix, which can be given by eqn. (3.148) plus higher-order terms. Then, from the definition of the derivative we have

$$\lim_{\Delta t\to 0}\left\{\frac{A(t+\Delta t) - A(t)}{\Delta t}\right\} = -\lim_{\Delta t\to 0}\left\{\frac{1}{\Delta t}[\boldsymbol{\alpha}(t)\times]\right\}A(t) \tag{3.157}$$

where the higher-order terms vanish in the limit. Hence, the following kinematics equation can be derived:

$$\boxed{\dot{A} = -[\boldsymbol{\omega}\times]A} \tag{3.158}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the body frame relative to the reference frame. The Euler angle kinematics equation is given by substituting eqn. (3.147) into eqn. (3.158), leading to

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{\cos\theta}\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \cos\theta\,\sin\psi & \cos\theta\,\cos\psi & 0 \\ -\sin\theta\,\cos\psi & \sin\theta\,\sin\psi & \cos\theta \end{bmatrix}\boldsymbol{\omega} \tag{3.159}$$

We clearly see that the Euler angle kinematics become singular when θ is either 90 or 270 degrees. In fact all three-dimensional (minimal) parameterizations have a singularity, which can cause difficulties in a particular application. The inverse kinematics are given by

$$\boldsymbol{\omega} = \begin{bmatrix} \cos\theta\,\cos\psi & \sin\psi & 0 \\ -\cos\theta\,\sin\psi & \cos\psi & 0 \\ \sin\theta & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{3.160}$$

The quaternion kinematics equation are given by

$$\boxed{\dot{\mathbf{q}} = \frac{1}{2}\Xi(\mathbf{q})\boldsymbol{\omega} = \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q}} \tag{3.161}$$

where

$$\Omega(\boldsymbol{\omega}) \equiv \begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \tag{3.162}$$

The matrix $\Xi(\mathbf{q})$ obeys the following helpful relations:

$$\Xi^T(\mathbf{q})\Xi(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})I_{3\times 3} \tag{3.163a}$$

$$\Xi(\mathbf{q})\Xi^T(\mathbf{q}) = (\mathbf{q}^T\mathbf{q})I_{4\times 4} - \mathbf{q}\mathbf{q}^T \tag{3.163b}$$

$$\Xi^T(\mathbf{q})\mathbf{q} = \mathbf{0}_{3\times 1} \tag{3.163c}$$

$$\Xi^T(\mathbf{q})\boldsymbol{\lambda} = -\Xi^T(\boldsymbol{\lambda})\mathbf{q} \quad \text{for any } \boldsymbol{\lambda}_{4\times 1} \tag{3.163d}$$

Also, another useful identity is given by

$$\Psi(\mathbf{q})\boldsymbol{\omega} = \Gamma(\boldsymbol{\omega})\mathbf{q} \tag{3.164}$$

where

$$\Gamma(\boldsymbol{\omega}) \equiv \begin{bmatrix} [\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \tag{3.165}$$

The inverse kinematics are given by multiplying eqn. (3.161) by $\Xi^T(\mathbf{q})$, and using the identity in eqn. (3.163a), leading to

$$\boldsymbol{\omega} = 2\,\Xi^T(\mathbf{q})\dot{\mathbf{q}} \tag{3.166}$$

A major advantage of using quaternions is that the kinematics equation is linear in the quaternion and is also free of singularities. Another advantage of quaternions is that successive rotations can be accomplished using quaternion multiplication. Here we adopt the convention of Lefferts, Markley, and Shuster[27] who multiply the quaternions in the same order as the attitude matrix multiplication (in contrast to the usual convention established by Hamilton[26]). Suppose we wish to perform a successive rotation. This can be written using

$$A(\mathbf{q}')A(\mathbf{q}) = A(\mathbf{q}'\otimes\mathbf{q}) \tag{3.167}$$

The composition of the quaternions is bilinear, with

$$\mathbf{q}'\otimes\mathbf{q} = \left[\Psi(\mathbf{q}')\ \mathbf{q}'\right]\mathbf{q} = \left[\Xi(\mathbf{q})\ \mathbf{q}\right]\mathbf{q}' \tag{3.168}$$

Also, the inverse quaternion is defined by

$$\mathbf{q}^{-1} \equiv \begin{bmatrix} -\boldsymbol{\varrho} \\ q_4 \end{bmatrix} \tag{3.169}$$

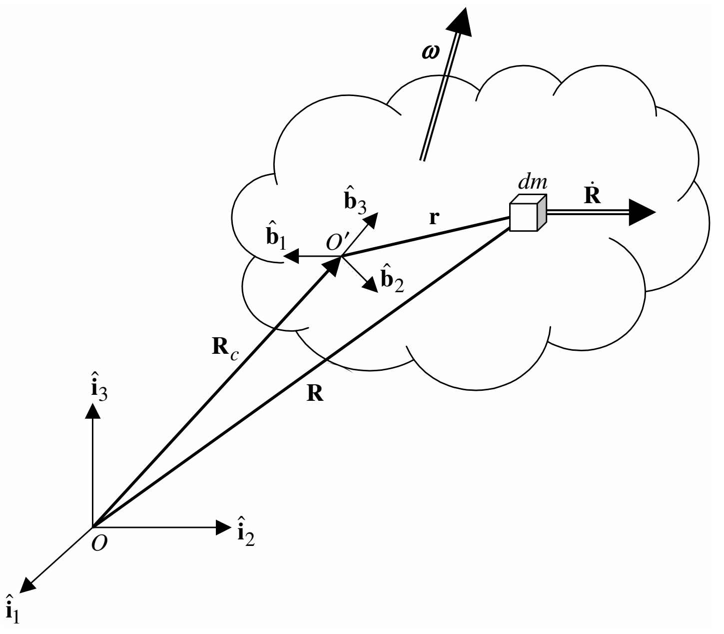

Figure 3.7: General Rigid Body Motion

Note that $\mathbf{q} \otimes \mathbf{q}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$, which is the identity quaternion. A computationally efficient algorithm to extract the quaternion from the attitude matrix is given in Ref. [28]. A more thorough review of the attitude representations shown in this section, as well as others, can be found in the excellent survey paper by Shuster[25] and in the book by Kuipers.[29]

3.7.2 Rigid Body Dynamics

The rigid body equations of motion of a vehicle in both translation and rotation with respect to some inertial frame are obtained from Newton's second law. We first consider the angular momentum $\mathbf{H}_{\text{tot}}$ of a body defined as an integral over a continuous mass density (see Figure 3.7):

$$\mathbf{H}_{\text{tot}} = \int_B \mathbf{R} \times \dot{\mathbf{R}}\, dm \tag{3.170}$$

From Figure 3.7 the following vector relation is given:

$$\mathbf{R} = \mathbf{R}_c + \mathbf{r} \tag{3.171}$$

In order to determine the derivative of eqn. (3.171), since the velocity vector of $\mathbf{r}$ is defined to be an inertial derivative, we must employ the *transport theorem*:[23]

$$\dot{\mathbf{r}} \equiv \frac{{}^{\mathcal{N}}d}{dt}(\mathbf{r}) = \frac{{}^{\mathcal{B}}d}{dt}(\mathbf{r}) + \boldsymbol{\omega} \times \mathbf{r} \tag{3.172}$$

where ${}^{\mathcal{N}}d/dt$ denotes the derivative with respect to the inertial frame, ${}^{\mathcal{B}}d/dt$ denotes the derivative with respect to the body frame, and $\boldsymbol{\omega}$ is the angular velocity of the body relative to the inertial frame. Since we have assumed that the body is rigid, then ${}^{\mathcal{B}}d/dt(\mathbf{r})$ is zero. Therefore, the derivative of eqn. (3.171) with respect to the inertial frame is given by

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_c + \boldsymbol{\omega} \times \mathbf{r} \tag{3.173}$$

Substituting eqns. (3.171) and (3.173) into eqn. (3.170), and assuming that the point O' is the center of mass (so that $\int_B \mathbf{r}\, dm = 0$) leads to

$$\mathbf{H}_{\text{tot}} = \mathbf{H} + m\mathbf{R}_c \times \dot{\mathbf{R}}_c \tag{3.174}$$

where the contribution of the mass relative to the center of mass is defined by

$$\mathbf{H} \equiv \int_B \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm = \left(\int_B -[\mathbf{r}\times][\mathbf{r}\times]\, dm \right) \boldsymbol{\omega} \tag{3.175}$$

This is most often written in compact form:

$$\mathbf{H} = J\,\boldsymbol{\omega} \tag{3.176}$$

with

$$J \equiv \int_B -[\mathbf{r}\times][\mathbf{r}\times]\, dm \tag{3.177}$$

The matrix J is called the *moment of inertia* or simply *inertia matrix*, which is a positive definite, symmetric matrix (with three orthogonal eigenvectors). The off-diagonal terms are sometimes referred to as *products of inertia*. The moment of inertia about some given axis is related simply to the moment about a parallel axis through the center of mass, which can be computed using the *parallel axis theorem*.[24, 30]

The rate of change of the angular momentum with respect to the inertial frame is equal to the applied torque $\mathbf{L}$:

$$\dot{\mathbf{H}} = \mathbf{L} \tag{3.178}$$

Using the transport theorem on eqn. (3.178) gives

$$\dot{\mathbf{H}} = \frac{{}^{\mathcal{B}}d}{dt}(\mathbf{H}) + \boldsymbol{\omega} \times \mathbf{H} = \mathbf{L} \tag{3.179}$$

Substituting eqn. (3.176) into eqn. (3.179) gives *Euler's equations of motion*:

$$\boxed{J\dot{\boldsymbol{\omega}} = -[\boldsymbol{\omega}\times]J\boldsymbol{\omega} + \mathbf{L}} \tag{3.180}$$

Equation (3.180) represents a set of three coupled, first-order, nonlinear differential equations. Closed-form solutions exist for special cases only.[31]

The linear force of a body relative to a body's center of mass is given by Newton's law:

$$\mathbf{F} = m\dot{\mathbf{v}} \tag{3.181}$$

where $\mathbf{F}$ is the total external force acting on the rigid body and $\mathbf{v}$ is the absolute velocity of the center of mass. In order to determine the acceleration in the body frame, the transport theorem must be again used:[32]

$$\boxed{\dot{\mathbf{v}} = \frac{^{\mathcal{B}}d}{dt}(\mathbf{v}) + \boldsymbol{\omega} \times \mathbf{v}} \tag{3.182}$$

Substituting eqn. (3.182) into eqn. (3.181) leads to the following scalar equations for the force:

$$f_1 = m(\dot{v}_1 + v_3\omega_2 - v_2\omega_3) \tag{3.183a}$$

$$f_2 = m(\dot{v}_2 + v_1\omega_3 - v_3\omega_1) \tag{3.183b}$$

$$f_3 = m(\dot{v}_3 + v_2\omega_1 - v_1\omega_2) \tag{3.183c}$$

The components of $\boldsymbol{\omega}$ can be obtained from the solution of eqn. (3.180). The force equations have been derived for a frame fixed to the body. In order to determine the position of the body a transformation of the velocity components v_1, v_2, and v_3 to the reference frame must be made using the attitude matrix, which are then integrated to obtain the absolute position.

3.8 Spacecraft Dynamics and Orbital Mechanics

This section reviews the basic equations for spacecraft dynamics and orbital mechanics. The equations are fairly straightforward, but carry deep meaning and revolutionary concepts, as attested to by the numerous publications in these areas since their conception. We only present the equations necessary to demonstrate the basics of attitude estimation and orbit determination of vehicles.

3.8.1 Spacecraft Dynamics

To fully describe the rotational motion of a rigid spacecraft, a kinematic and a dynamic equation of motion are required. For most modern spacecraft applications the quaternion kinematics equation is preferred. Therefore, the following equations are used:

$$\dot{\mathbf{q}} = \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q} \tag{3.184a}$$

$$J\dot{\boldsymbol{\omega}} = -[\boldsymbol{\omega}\times]J\boldsymbol{\omega} + \mathbf{L} \tag{3.184b}$$

If a spacecraft is equipped with reaction wheels (which are common on most spacecraft) the angular momentum can be modified as[31]

$$\mathbf{H} = J\boldsymbol{\omega} + \mathbf{h} \tag{3.185}$$

where $\mathbf{h}$ is the angular momentum due to the rotation of the wheels relative to the spacecraft, and the inertia J now contains the mass of the wheels. Using eqn. (3.185) in eqn. (3.179) gives

$$\dot{\mathbf{H}} = -[J^{-1}(\mathbf{H}-\mathbf{h})\times]\mathbf{H}+\mathbf{L} \tag{3.186}$$

Equation (3.186) can also be rewritten in Euler's form as

$$J\dot{\boldsymbol{\omega}} = -[\boldsymbol{\omega}\times](J\boldsymbol{\omega}+\mathbf{h})+\mathbf{L}-\dot{\mathbf{h}} \tag{3.187}$$

Equation (3.186) is often preferred since it does not involve the derivative of the wheel momentum.

An interesting and useful case of Euler's rotational equations of motion is given by defining the body coordinate system to coincide with the principal axes (i.e., along the eigenvectors of J). In this case the inertia matrix J is diagonal with elements denoted by J_1, J_2, and J_3 (i.e., the eigenvalues of J). Euler's equations then become:

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + L_1 \tag{3.188a}$$
$$J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + L_2 \tag{3.188b}$$
$$J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + L_3 \tag{3.188c}$$

The stability of rotation about the principal axes can be shown by assuming a constant rotation about one of the axes, e.g., axis 3, and allowing a small perturbation. This indicates that the motion is stable if J_3 is the largest or smallest principal moment of inertia.[33]

We now consider the torque-free case (i.e., $\mathbf{L}=\mathbf{0}$) with two of the principal moments of inertia equal (say $J_1 = J_2 \equiv J_T$), which is the *axially symmetric* case. Euler's equations become

$$J_T\dot{\omega}_1 = -(J_3 - J_T)\omega_2\omega_3 \tag{3.189a}$$
$$J_T\dot{\omega}_2 = (J_3 - J_T)\omega_3\omega_1 \tag{3.189b}$$
$$J_3\dot{\omega}_3 = 0 \tag{3.189c}$$

Equation (3.189c) clearly indicates that ω_3 is constant, with $\omega_3(t) = \omega_3(t_0)$. Next we impose that $\omega_3 > 0$, which can be accomplished by choosing the proper sense of the third principal axis. This leads to the following equations for ω_1 and ω_2:

$$\dot{\omega}_1 - \omega_n\omega_2 = 0 \tag{3.190a}$$
$$\dot{\omega}_2 + \omega_n\omega_1 = 0 \tag{3.190b}$$

where $\omega_n = (1 - J_3/J_T)\omega_3(t_0)$ is a constant. The solutions for ω_1 and ω_2 are given by

$$\omega_1(t) = \omega_1(t_0)\cos\omega_n t + \omega_2(t_0)\sin\omega_n t \tag{3.191a}$$
$$\omega_2(t) = \omega_2(t_0)\cos\omega_n t - \omega_1(t_0)\sin\omega_n t \tag{3.191b}$$

This indicates that the system is *marginally stable*.[2] The constant ω_n is known as the *body nutation rate*. Also, the magnitude of the angular momentum can be shown to be given by

$$||\mathbf{H}|| = \left\{ J_T^2[\omega_1^2(t_0) + \omega_2^2(t_0)] + J_3^2\omega_3^2(t_0) \right\}^{1/2} \tag{3.192}$$

which is constant and inertially fixed along the third principal axis. This also indicates that energy is conserved. The angular momentum in body coordinates can be computed using the attitude matrix:

$$^{\mathcal{B}}\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = A \, ^{\mathcal{I}}\begin{bmatrix} 0 \\ 0 \\ ||\mathbf{H}|| \end{bmatrix} \tag{3.193}$$

Since the body is spinning about its axis of symmetry (the third axis) a convenient parameterization of the attitude matrix is the 3-1-3 sequence. This leads to

$$H_1 = J_T\omega_1 = ||\mathbf{H}|| \sin\theta \, \sin\psi \tag{3.194a}$$

$$H_2 = J_T\omega_2 = ||\mathbf{H}|| \sin\theta \, \cos\psi \tag{3.194b}$$

$$H_3 = J_3\omega_3 = ||\mathbf{H}|| \cos\theta \tag{3.194c}$$

Since H_3 and $||\mathbf{H}||$ are constants then $\theta = \cos^{-1}(H_3/||\mathbf{H}||)$ is constant as well. This angle is known as the *nutation angle*. The solution for the yaw angle ψ is given by $\psi = \tan^{-1}(H_1/H_2)$. The solution for the roll angle ϕ is given from the 3-1-3 kinematics equation and can be shown to be given by $\dot{\phi} = ||\mathbf{H}||/J_T$. The asymmetric case with $J_1 \neq J_2$ can be solved in closed-form using *Jacobian elliptic functions*.[31]

As mentioned previously, a thorough treatise of spacecraft dynamics would entail significant effort. Other topics, such as dual-spin spacecraft, kinetic-energy and angular momentum ellipsoids, variable mass, passive and active control techniques, attitude torque disturbances, etc., can be found in the references in this section. Other reference includes works by Kane, Likens, and Levinson,[34] Hughes,[35] Kaplan,[36] Wiesel,[37] and Junkins and Turner.[38]

3.8.2 Orbital Mechanics

The study of bodies in orbit has attracted the world's greatest mathematicians in the past, and still is a flourishing subject area in the present. In fact many useful mathematical concepts, such as Bessel functions and nonlinear least squares, can be directly traced back to the study of orbital motion. As with spacecraft dynamics, a thorough treatise of orbital mechanics is not possible in the present text. We again only treat the basic equations and concepts that are required to demonstrate orbit determination.

An unperturbed orbiting body follows Kepler's three laws, originally given by

1. The orbit of each planet is an ellipse, with the Sun at a focus.

2. The line joining the planet to the sun sweeps out equal areas in equal times.

3. The square of the period of a planet is proportional to the cube of its mean distance from the sun.

These powerful statements define the shape of planetary orbits, the velocity at which planets travel around the sun, and the time required from a planet to complete an orbit. These laws can be proven mathematically from Newton's universal law of gravitation, which states: any two bodies with mass M and m attract each other by a force that is proportional to the product of their masses and inversely proportional to the square of the distance r between them. Mathematically, this statement is given by

$$F_g = \frac{GMm}{r^2} \tag{3.195}$$

where G is the *universal gravitation constant*.[39, 40] Consider the two bodies in Figure 3.8. The axes $\hat{\mathbf{i}}_1$, $\hat{\mathbf{i}}_2$, and $\hat{\mathbf{i}}_3$ are an inertial frame, and the axes $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ are a non-rotating frame with origin coincident with the center of mass. Applying Newton's law in the inertial frame for each body we obtain

$$M\ddot{\mathbf{r}}_M = \frac{GMm}{||\mathbf{r}||^3}\mathbf{r} \tag{3.196a}$$

$$m\ddot{\mathbf{r}}_m = -\frac{GMm}{||\mathbf{r}||^3}\mathbf{r} \tag{3.196b}$$

The negative sign in eqn. (3.196a) is due to the opposite direction of the force. Since, as shown in Figure 3.8, $\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$ then from eqn. (3.196) we obtain

$$\ddot{\mathbf{r}} = -\frac{G(M+m)}{||\mathbf{r}||^3}\mathbf{r} \tag{3.197}$$

If the mass m is much smaller than M (which is a very accurate assumption for orbiting spacecraft) then we can effectively ignore m so that

$$\boxed{\ddot{\mathbf{r}} = -\frac{\mu}{||\mathbf{r}||^3}\mathbf{r}} \tag{3.198}$$

where $\mu \equiv GM$ is called the *gravitational parameter*. The gravitational parameter is more commonly used in orbital mechanics of spacecraft since it can be measured to high precision, unlike the mass M.

Equation (3.198) is the most fundamental equation used in orbital mechanics, and can be used to prove Kepler's laws. In particular one can show that mechanical energy and angular momentum are conserved. The conservation of mechanical energy gives rise to the *vis-viva integral*.[39] Since angular momentum is related to $\mathbf{r} \times \dot{\mathbf{r}}$, which is constant, then the spacecraft's motion must be confined to a plane inertially fixed in space. The two-body relative equations represent a coupled nonlinear set of differential equations. Fortunately, analytical solutions to this set of equations exist. Herrick[41] establishes the solution of eqn. (3.198), given initial conditions $\mathbf{r}(t_0)$ and

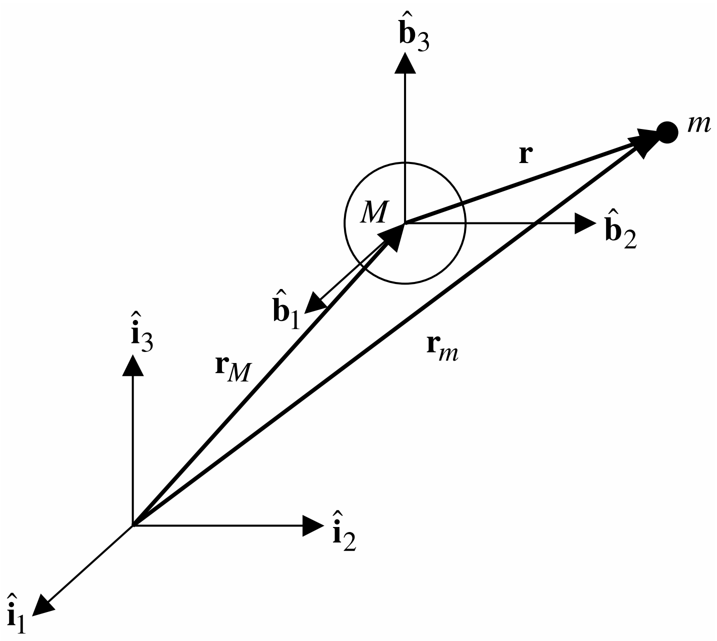

Figure 3.8: Relative Motion of Two Bodies

$\dot{\mathbf{r}}(t_0)$. First we compute the semimajor axis (a) using the vis-viva integral:

$$a = \left(\frac{2}{||\mathbf{r}(t_0)||} - \frac{||\dot{\mathbf{r}}(t_0)||^2}{\mu} \right)^{-1} \tag{3.199}$$

Then, given the current time of interest (t), we solve the following equation for ϕ (using Newton's method):*

$$t - t_0 = \frac{a^{3/2}}{\mu^{1/2}} \left[\phi - \left(1 - \frac{||\mathbf{r}(t_0)||}{a} \right) \sin\phi + \frac{\mathbf{r}^T(t_0)\dot{\mathbf{r}}(t_0)}{(\mu a)^{1/2}} (1 - \cos\phi) \right] \tag{3.200}$$

Next, compute the following variables:

$$f = 1 - a(1 - \cos\phi)/||\mathbf{r}(t_0)|| \tag{3.201a}$$

$$g = (t - t_0) - a^{3/2}(\phi - \sin\phi)/\mu^{1/2} \tag{3.201b}$$

$$r = a[1 - (1 - ||\mathbf{r}(t_0)||/a)\cos\phi] + \mathbf{r}^T(t_0)\dot{\mathbf{r}}(t_0)(a/\mu)^{1/2}\sin\phi \tag{3.201c}$$

$$\dot{f} = -(r||\mathbf{r}(t_0)||)^{-1}(\mu a)^{1/2}\sin\phi \tag{3.201d}$$

$$\dot{g} = 1 - a(1 - \cos\phi)/r \tag{3.201e}$$

*We note ϕ has the geometric interpretation as the change in eccentric anomaly.

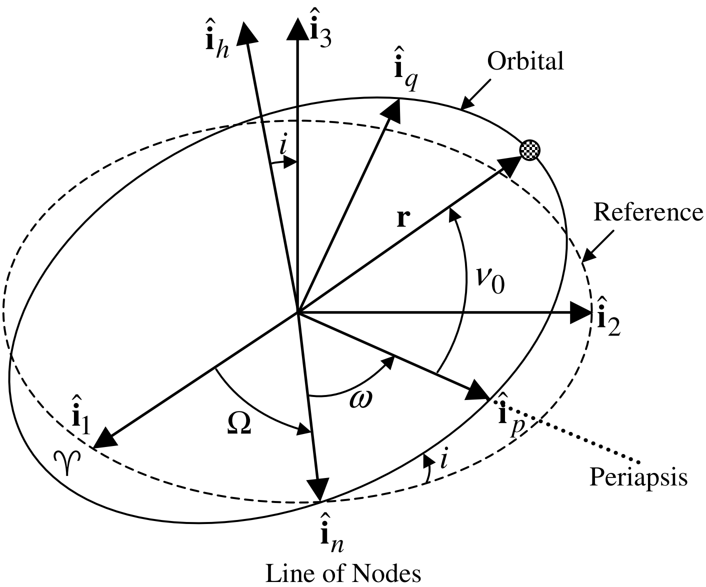

Figure 3.9: Coordinate System Geometry and Orbital Elements

Then, the solution to eqn. (3.198) is given by

$$\mathbf{r}(t) = f\,\mathbf{r}(t_0) + g\,\dot{\mathbf{r}}(t_0) \tag{3.202a}$$

$$\dot{\mathbf{r}}(t) = \dot{f}\,\mathbf{r}(t_0) + \dot{g}\,\dot{\mathbf{r}}(t_0) \tag{3.202b}$$

Unfortunately knowing $\mathbf{r}(t_0)$ and $\dot{\mathbf{r}}(t_0)$ does not provide a physical meaning of the orbit. To characterize an orbit six classical Keplerian orbital elements are given in the place of $\mathbf{r}(t_0)$ and $\dot{\mathbf{r}}(t_0)$, which do provide a physical meaning. Figure 3.9 shows the orbit system geometry and orbital elements. The dimensional elements are given by

- a = semimajor axis (size of the orbit)
- e = eccentricity (shape of the orbit)
- τ = time reference of periapsis or perigee

The orientation elements are given by

- i = inclination (angle between orbit plane and reference plane)
- Ω = right ascension of the ascending node (angle between vernal equinox direction and the line of nodes)
- ω = argument of periapsis or perigee (angle between the ascending node direction and periapsis or perigee direction)

The line of nodes vector is given by the intersection of the reference plane (e.g., the Earth's equatorial plane) and the orbital plane.

From these classical elements it is possible to determine $\mathbf{r}(t_0)$ and $\dot{\mathbf{r}}(t_0)$. Before we state the solution of this problem, we first define some other well known orbital quantities. The *mean motion* is defined by

$$n = \sqrt{\frac{\mu}{a^3}} \tag{3.203}$$

The *mean anomaly* is given by

$$M = n(t - \tau) \tag{3.204}$$

where M is not to be confused with the mass M, as defined previously. Note that M often replaces τ for one of the classical elements (e.g., see §7.1.1). To determine the position vector $\mathbf{r}(t_0)$ the initial *true anomaly*, ν_0, must be first determined. From Figure 3.9 the initial true anomaly is defined as the angle between the periapsis direction and the position vector. Unfortunately, this quantity cannot be determined in a straightforward manner. To facilitate this task Kepler used an intermediate step. First, given M and e, Kepler's equation is solved for the *eccentric anomaly* E:

$$\boxed{M = E - e \sin E} \tag{3.205}$$

The eccentric anomaly can be determined using Newton's method (see exercise 1.15). A series expansion of eqn. (3.205) gives the following approximation for E, which is accurate up to third-order in the eccentricity:[39]

$$E = M + \frac{e \sin M}{1 - e \cos M} - \frac{1}{2}\left(\frac{e \sin M}{1 - e \cos M}\right)^3 + \cdots \tag{3.206}$$

Equation (3.206) can be used as the starting guess in Newton's method. The true anomaly is then given by

$$\nu_0 = \text{atan2}\left[\frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}, \frac{\cos E - e}{1 - e \cos E}\right] \tag{3.207}$$

where atan2 is a four quadrant inverse tangent function. Next, the *semilatus rectum* is computed by

$$p = a(1 - e^2) \tag{3.208}$$

Also, the magnitude of the momentum vector is given by

$$||\mathbf{H}|| = \sqrt{\mu\, p} \tag{3.209}$$

Then, using the equation of an ellipse in polar coordinates, the magnitude of the position vector is given by

$$||\mathbf{r}(t_0)|| = \frac{p}{1 + e \cos \nu_0} \tag{3.210}$$

Finally, the initial position and velocity vectors are determined using a coordinate transformation,[39] given by

$$\mathbf{r}(t_0) = ||\mathbf{r}(t_0)|| \begin{bmatrix} \cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i \\ \sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i \\ \sin\theta\sin i \end{bmatrix} \tag{3.211}$$

and

$$\dot{\mathbf{r}}(t_0) = -\frac{\mu}{||\mathbf{H}||} \begin{bmatrix} (\sin\theta + e\sin\omega)\cos\Omega + (\cos\theta + e\cos\omega)\sin\Omega\cos i \\ (\sin\theta + e\sin\omega)\sin\Omega - (\cos\theta + e\cos\omega)\cos\Omega\cos i \\ (\cos\theta + e\cos\omega)\sin i \end{bmatrix} \tag{3.212}$$

where $\theta = \omega + \nu_0$.

The orbital equations of motion described herein are sufficient to demonstrate the basic concepts of orbit determination and estimation. The two-body problem can also be extended to the n-body problem. The analysis of even the two- and three-body problem provides a wealth of information, which will not be addressed in the present text. Also, perturbation methods discussed in §3.2 can be used for both the problem of determining precision orbits and the problem of ensuring that a spacecraft in orbit will meet certain boundary conditions.[39] The interested reader is encouraged to pursue the vast knowledge base and developments on orbital mechanics in the open literature and texts such as Battin.[39]

3.9 Aircraft Flight Dynamics

This section presents a summary of the equations of motion of aircraft. Once again, we only introduce the fundamentals required within the scope of the present text. Aircraft flight dynamics is only one of three disciplines which encompass flight mechanics; the other two being performance and aeroelasticity.[42] Performance deals with determining various quantities (such as climb rate, range, etc.) that give an indication of the basic characteristics of a particular aircraft. Aeroelasticity involves the structural flexibility of modern aircraft. We will cover the basics of flexibility in §3.10.

We begin our discussion of flight dynamics by defining a number of various aircraft angles (see Figure 3.10): angle of attack (α), sideslip angle (β), flight path angle (γ), and pitch angle (θ). Referring to Figure 3.10, the angle of attack is the angle between the $\hat{\mathbf{b}}_1$ body axis and the projected free-stream velocity vector ($\mathbf{v}_{p_1}$) onto the $\hat{\mathbf{b}}_1$-$\hat{\mathbf{b}}_3$ (body axis) plane. The sideslip angle is the angle between the $\hat{\mathbf{b}}_1$ body axis and the projected free-stream velocity vector ($\mathbf{v}_{p_2}$) onto the $\hat{\mathbf{b}}_1$-$\hat{\mathbf{b}}_2$ (body axis) plane. The flight path angle is the angle between the horizon (which is assumed to be inertial) and the $\mathbf{v}_{p_1}$ axis. The pitch angle is the angle between the horizon and

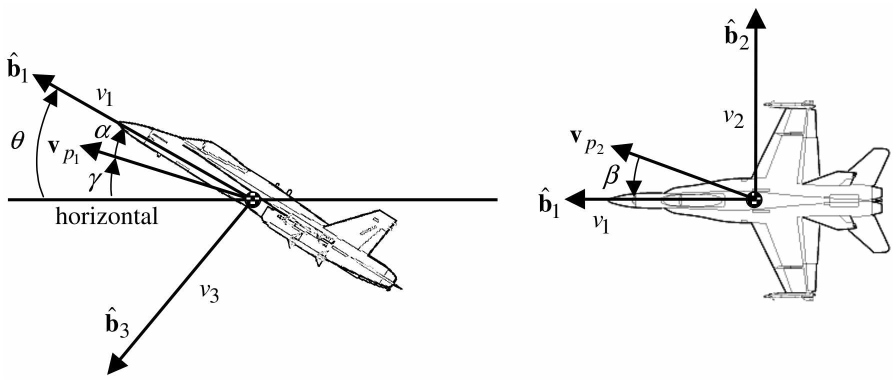

Figure 3.10: Definition of Various Aircraft Angles (Positive Senses Shown)

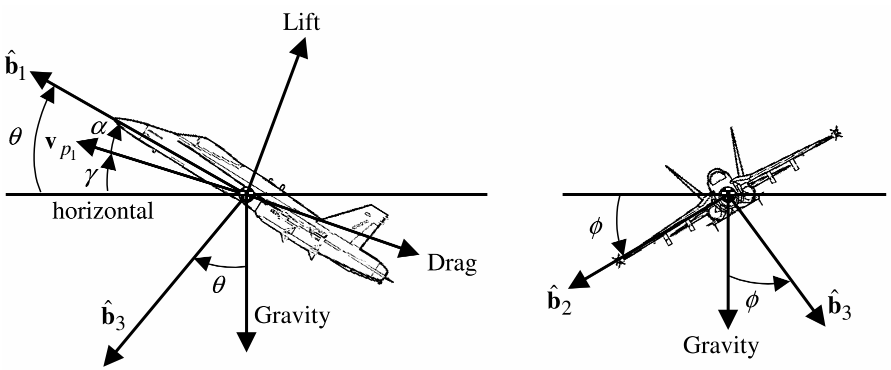

Figure 3.11: Aircraft Forces

the $\hat{\mathbf{b}}_1$ body axis, which is also given by

$$\theta = \alpha + \gamma \tag{3.213}$$

The equations for α and β are given by

$$\alpha = \tan^{-1}\frac{v_3}{v_1} \tag{3.214a}$$

$$\beta = \sin^{-1}\frac{v_2}{||\mathbf{v}||} \tag{3.214b}$$

where v_1, v_2, and v_3 are the free-stream velocity components along the $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$, and $\hat{\mathbf{b}}_3$ axes, respectively, and $||\mathbf{v}||$ is the free-stream velocity magnitude, given by

$$||\mathbf{v}|| = (v_1^2 + v_2^2 + v_3^2)^{1/2} \tag{3.215}$$

The rigid body equations of motion of an aircraft can be derived from Newton's second law, as described in §3.7.2. Figure 3.11 shows the forces acting on an aircraft.

The roll angle ϕ is defined as the angle between the horizon and the $\hat{\mathbf{b}}_2$ body axis. It is important to realize that drag is opposite the velocity vector, not the body axis vector (also, lift is perpendicular to the velocity vector). The force equations are derived from eqn. (3.183), with the addition of gravity, aerodynamic forces, and thrust forces. These equations are given by

$$T_1 - D\cos\alpha + L\sin\alpha - mg\sin\theta = m(\dot{v}_1 + v_3\omega_2 - v_2\omega_3) \tag{3.216a}$$

$$Y + mg\cos\theta\sin\phi = m(\dot{v}_2 + v_1\omega_3 - v_3\omega_1) \tag{3.216b}$$

$$T_3 - D\sin\alpha - L\cos\alpha + mg\cos\theta\cos\phi = m(\dot{v}_3 + v_2\omega_1 - v_1\omega_2) \tag{3.216c}$$

where D is the drag force, Y is the side force due to rudder, L is the lift force, and T_1 and T_3 are the thrust components along $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_3$, respectively. The total drag equation , side-force equation, and lift equation are given by

$$D = C_D\,\bar{q}\,S \tag{3.217a}$$

$$Y = C_Y\,\bar{q}\,S \tag{3.217b}$$

$$L = C_L\,\bar{q}\,S \tag{3.217c}$$

where C_D, C_Y, and C_L are the total drag, side-force, and lift coefficients, respectively, S is the known reference area, and $\bar{q}$ is the dynamic pressure which is a function of the known air density (ρ) and velocity magnitude:

$$\bar{q} = \frac{1}{2}\rho||\mathbf{v}||^2 \tag{3.218}$$

The aerodynamic coefficients are given by

$$C_D = C_{D_0} + C_{D_\alpha}\alpha + C_{D_{\delta_E}}\delta_E \tag{3.219a}$$

$$C_Y = C_{Y_0} + C_{Y_\beta}\beta + C_{Y_{\delta_R}}\delta_R + C_{Y_{\delta_A}}\delta_A \tag{3.219b}$$

$$C_L = C_{L_0} + C_{L_\alpha}\alpha + C_{L_{\delta_E}}\delta_E \tag{3.219c}$$

where δ_E, δ_R, and δ_A are the elevator (or stabilizer), rudder, and aileron angle deflections. The other terms in eqn. (3.219) are the known aerodynamic coefficients (defined by the particular aircraft of interest). These reflect the contributions of the individual quantities (e.g., C_{D_α} is the drag coefficient contribution due to angle of attack, C_{D_0} is the drag coefficient for $\alpha = \delta_E = 0$, etc.). Note, the aerodynamic coefficients are first-order Taylor series with an infinite number of terms (we have chosen to show these with only a few of the most basic terms). Also, instead of eqn. (3.219a), the *drag polar*[43] is often used to approximate the drag coefficient.

The aircraft rotational equations of motion are given by eqn. (3.180). For conventional aircraft configurations the $\mathbf{b}_1$-$\mathbf{b}_3$ plane is usually a plane of symmetry so that $J_{23} = J_{12} = 0$. Therefore, Euler's equations in component form are given by

$$J_{11}\dot{\omega}_1 - J_{13}\dot{\omega}_3 - J_{13}\omega_1\omega_2 + (J_{33} - J_{22})\omega_2\omega_3 = L_{A_1} + L_{T_1} \tag{3.220a}$$

$$J_{22}\dot{\omega}_2 + (J_{11} - J_{33})\omega_1\omega_3 + J_{13}(\omega_1^2 - \omega_3^2) = L_{A_2} + L_{T_2} \tag{3.220b}$$

$$J_{33}\dot{\omega}_3 - J_{13}\dot{\omega}_1 + J_{13}\omega_2\omega_3 + (J_{22} - J_{11})\omega_1\omega_2 = L_{A_3} + L_{T_3} \tag{3.220c}$$

where L_{A_1}, L_{A_2}, and L_{A_3} are the aerodynamic torques, and L_{T_1}, L_{T_2}, and L_{T_3} are the known thrust torques. The aerodynamic torque equations are given by

$$L_{A_1} = C_l\,\bar{q}\,S\,b \tag{3.221a}$$
$$L_{A_2} = C_m\,\bar{q}\,S\,\bar{c} \tag{3.221b}$$
$$L_{A_3} = C_n\,\bar{q}\,S\,b \tag{3.221c}$$

where C_l, C_m, and C_n are the rolling, pitching, and yawing torque coefficients, respectively, b is the known wing span, and $\bar{c}$ is the known mean geometric chord.[43] The torque coefficients are given by

$$C_l = C_{l_0} + C_{l_\beta}\beta + C_{l_{\delta_R}}\delta_R + C_{l_{\delta_A}}\delta_A + C_{l_p}\frac{\omega_1 b}{2||\mathbf{v}||^2} + C_{l_r}\frac{\omega_3 b}{2||\mathbf{v}||^2} \tag{3.222a}$$
$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta_E}}\delta_E + C_{m_q}\frac{\omega_2 \bar{c}}{2||\mathbf{v}||^2} \tag{3.222b}$$
$$C_n = C_{n_0} + C_{n_\beta}\beta + C_{n_{\delta_R}}\delta_R + C_{n_{\delta_A}}\delta_A + C_{n_p}\frac{\omega_1 b}{2||\mathbf{v}||^2} + C_{n_r}\frac{\omega_3 b}{2||\mathbf{v}||^2} \tag{3.222c}$$

By integrating eqns. (3.216) and (3.220) the body linear velocities and angular velocities can be determined. To determine the linear velocities with respect to the reference frame we utilize the inverse attitude matrix, which is usually defined by the 3-2-1 sequence, so that

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \mathrm{c}\theta\,\mathrm{c}\psi & \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi - \mathrm{c}\phi\,\mathrm{s}\psi & \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi + \mathrm{s}\phi\,\mathrm{s}\psi \\ \mathrm{c}\theta\,\mathrm{s}\psi & \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi + \mathrm{c}\phi\,\mathrm{c}\psi & \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi - \mathrm{s}\phi\,\mathrm{c}\psi \\ -\mathrm{s}\theta & \mathrm{s}\phi\,\mathrm{c}\theta & \mathrm{c}\phi\,\mathrm{c}\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{3.223}$$

where $\dot{x}$, $\dot{y}$, and $\dot{z}$ are the velocity components with respect to the reference frame. The aircraft's position relative to the reference frame can be determined by integrating eqn. (3.223). In a similar fashion the Euler rates can be expressed using the 3-2-1 kinematics equations:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\,\tan\theta & \cos\phi\,\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\,\sec\theta & \cos\phi\,\sec\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{3.224}$$

The roll (ϕ), pitch (θ), and yaw (ψ) angles can be determined by integrating the set in eqn. (3.224).

The equations presented in this section allow one to simulate the basic motion of an aircraft. As is the case with spacecraft dynamics, a thorough treatise of aircraft flight dynamics would entail significant effort which is beyond the scope of this text. Other important topics such as small-disturbance theory, atmospheric inputs, flying qualities, etc., can be found in Nelson[42] and Roskam.[43]

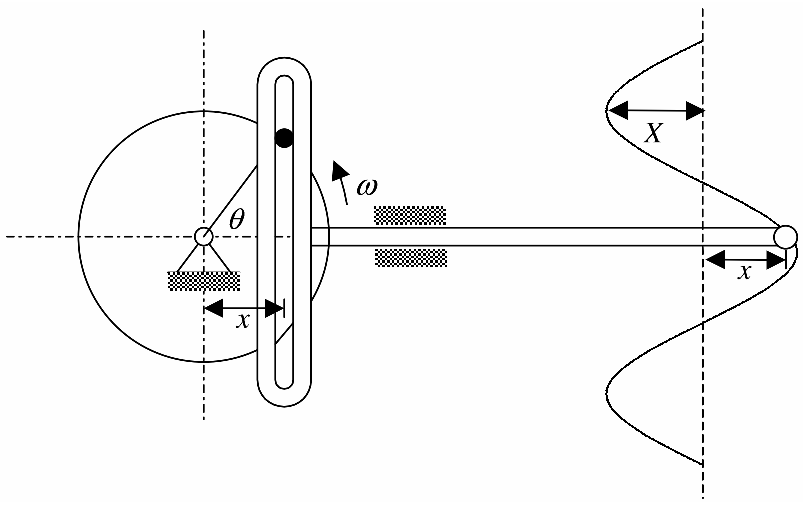

Figure 3.12: Harmonic Motion in a Yoke

3.10 Vibration

Vibration is a kind of motion where an object oscillates with respect to some reference frame. Any body that possesses mass and elasticity, such as flexible structures, aircraft wings, bridges, buildings, strings, etc., can vibrate. Vibration thus covers a wide range of disciplines, which still has a thriving research thrust to this day (especially in the control of vibratory systems). Many devastating failures have resulted when the effects of vibration on structures have not been adequately investigated (e.g., the infamous Tacoma Narrows bridge collapse due to wind-induced vibration[44]).

In order to introduce the concepts involved with vibration, we begin our discussion with the simplest form of periodic motion, known as *harmonic motion*. To illustrate this motion, we consider a simple mechanism called a yoke,[45] shown in Figure 3.12. A pin is attached to a wheel, which can slide freely in a slot attached to a stem. The stem then moves in a periodic manner, which can be expressed by the equation

$$x = X\cos\theta = X\cos\omega t \tag{3.225}$$

where X is the radius of the wheel, and ω is the angular velocity. Taking two time derivatives of eqn. (3.225) and back substituting yields

$$\ddot{x} + \omega^2 x = 0 \tag{3.226}$$

Therefore, in harmonic motion the acceleration is proportional to the displacement.

Harmonic motion can be related to Newton's second law of motion, which states that acceleration is proportional to force. Consider the spring-mass-damper system in Figure 3.13. From Newton's law we have:

$$m\ddot{x} + c\dot{x} + kx = F \tag{3.227}$$

We now consider the free response case only with $F = 0$, and assume an exponential solution for x, given by $x = Ae^{st}$. Taking time derivatives of x and substituting the resultants into eqn. (3.227) leads to

$$(ms^2 + cs + k)Ae^{st} = 0 \tag{3.228}$$

Since Ae^{st} is never zero, eqn. (3.228) holds true if and only if

$$ms^2 + cs + k = 0 \tag{3.229}$$

Equation (3.229) is called the *characteristic equation* of the system. The same equation can also be derived by taking the Laplace transform of eqn. (3.227), with $F = 0$ again. The roots of this equation are clearly given by

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \tag{3.230}$$

Three possibilities for $s_{1,2}$ exist: 1) the roots are real and unequal for $c^2 - 4mk > 0$; 2) the roots are real and repeated for $c^2 - 4mk = 0$; and 3) the roots are complex conjugates for $c^2 - 4mk < 0$. The solution for each of these cases is given by

$$\text{real and unequal} \quad x = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{3.231a}$$

$$\text{real and repeated} \quad x = A_1 e^{s_1 t} + t\, A_2 e^{s_1 t} \tag{3.231b}$$

$$\text{complex conjugates} \quad x = Be^{-at} \sin(bt + \phi) \tag{3.231c}$$

where $a = c/2m$ and $b = \sqrt{4mk - c^2}/2m$. The constants A_1, A_2, ϕ, and B are determined from initial conditions $x(t_0)$ and $\dot{x}(t_0)$:[2]

$$\text{real and unequal} \quad A_1 = \frac{\dot{x}(t_0) - s_2 x(t_0)}{s_1 - s_2}, \quad A_2 = x(t_0) - A_1 \tag{3.232a}$$

$$\text{real and repeated} \quad A_1 = x(t_0), \quad A_2 = \dot{x}(t_0) - s_1 x(t_0) \tag{3.232b}$$

$$\text{complex conjugates} \quad \phi = \text{atan2}[bx(t_0),\, \dot{x}(t_0) + ax(t_0)], \quad B = \frac{x(t_0)}{\sin\phi} \tag{3.232c}$$

Another way to represent the characteristic equation is given by

$$\boxed{s^2 + 2\zeta\omega_n s + w_n^2 = 0} \tag{3.233}$$

where the *damping ratio* ζ and *natural frequency* ω_n are defined as

$$\zeta = \frac{c}{2\sqrt{mk}} \tag{3.234a}$$

$$\omega_n = \sqrt{\frac{k}{m}} \tag{3.234b}$$

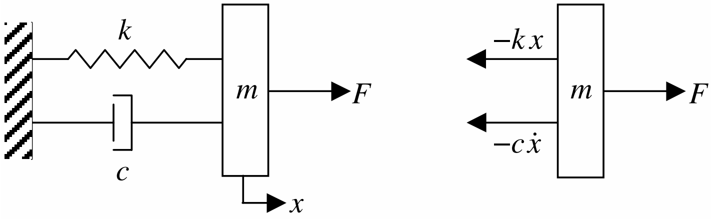

Figure 3.13: Simple Spring-Mass-Damper System

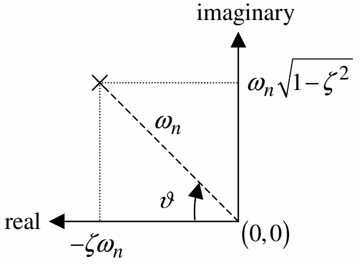

Figure 3.14: Root Location in the Complex Plane

The roots of the characteristic equation are now given by

$$s_{1,2} = -\zeta\,\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{3.235}$$

The three cases shown in eqn. (3.231) depend on three variables (m, c, and k). The convenient notation in eqn. (3.234a) allows us to represent these three cases from the characteristic value of ζ only: 1) the roots are real and unequal for $\zeta > 1$; 2) the roots are real and repeated for $\zeta = 1$; and 3) the roots are complex conjugates for $0 \le \zeta < 1$. A graphical representation of case 3 is shown in Figure 3.14. Since the natural frequency is the magnitude from the origin to the root, all roots with the same natural frequency must lie on a circle centered at the origin. The damping ratio is given by $\zeta = \cos\vartheta$, where ϑ is the angle between the natural frequency line and the negative real axis. If $\zeta = 0$ then the system reduces to the simple harmonic oscillator in eqn. (3.226) with $\omega_n = \omega$. Also, the *damped natural frequency* is defined by $\omega_d \equiv \omega_n\sqrt{1-\zeta^2}$, which is equivalent to b (the frequency of oscillation) in eqn. (3.231c).

Newton's law can easily be extended for a system of particles. In this text we consider a *lumped parameter system*,[46] where each mass corresponds to one degree of freedom. Many systems, such as bridges, trusses, aircraft structures, etc., can be sufficiently modelled using the lumped parameter concept. In order to demonstrate a lump parameter system with multiple springs, masses, and dampers we first consider the system shown in Figure 3.15. This system has two degrees of freedom (with

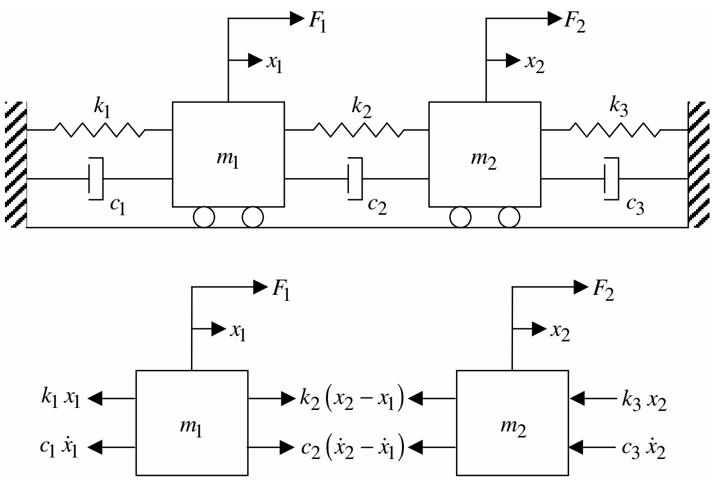

Figure 3.15: Multiple Spring-Mass-Damper System

mass positions given by x_1 and x_2). Applying Newton's law to this system yields

$$\begin{aligned}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} (c_1+c_2) & -c_2 \\ -c_2 & (c_2+c_3) \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\end{aligned} \tag{3.236}$$

Equation (3.236) can be put into compact form using matrix notation:

$$\boxed{M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\,\mathbf{x} = \mathbf{F}} \tag{3.237}$$

with obvious definitions of M (the mass matrix), C (the damping matrix), K (the stiffness matrix), $\mathbf{x}$, and $\mathbf{F}$. The matrices M, C, and K are symmetric and must be positive definite to ensure stability.

In order to investigate the properties of a lump parameter system we first consider an undamped system (i.e., $C = 0$) with no forced input:

$$M\ddot{\mathbf{x}} + K\,\mathbf{x} = \mathbf{0} \tag{3.238}$$

subject to the given initial conditions $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$. An exponential solution to eqn. (3.238) is assumed with[47]

$$\mathbf{x}(t) = e^{st}\mathbf{u} \tag{3.239}$$

where s and $\mathbf{u}$ are constants. Taking two time derivatives of eqn. (3.239) and substituting the resultant into eqn. (3.238) leads to

$$(K - \lambda M)\mathbf{u} = \mathbf{0} \tag{3.240}$$

where $\lambda = -s^2$. Equation (3.240) corresponds to eigenvalue/eigenvector problem with $s = \pm\lambda j$. We seek to find a physical solution that does not entail complex numbers. It is common to perform a linear state transformation using $\mathbf{x} = M^{-1/2}\mathbf{z}$, which leads to the following differential equation:

$$\ddot{\mathbf{z}} + M^{-1/2} K M^{-1/2} \mathbf{z} = \mathbf{0} \tag{3.241}$$

This transformation is performed since the matrix $M^{-1/2}KM^{-1/2}$ is a symmetric matrix, whereas $M^{-1}K$ is generally not symmetric. As shown in §3.1.4 the eigenvalues of the system are invariant to this transformation. The eigenvectors of $M^{-1/2}KM^{-1/2}$ are denoted $\mathbf{v}_i$ for $i = 1, 2, \ldots, p$, where p is the number of degrees of freedom. The solution for $\mathbf{z}$ is given by[48]

$$\mathbf{z}(t) = \sum_{i=1}^{p} a_i \sin(\omega_i t + \phi_i)\mathbf{v}_i \tag{3.242}$$

where the natural frequencies are given by $\omega_i = \sqrt{\lambda_i}$, and the constants ϕ_i and a_i are given by

$$\phi_i = \tan^{-1}\left[\frac{\omega_i \mathbf{v}_i^T \mathbf{z}(t_0)}{\mathbf{v}_i^T \dot{\mathbf{z}}(t_0)}\right] \tag{3.243}$$

$$a_i = \frac{\mathbf{v}_i^T \mathbf{z}(t_0)}{\sin\phi_i} \tag{3.244}$$

The vectors $\mathbf{v}_i$ are called the *mode shapes* since they give an indication of the "shape" of the vibration for each mass, and the constants a_i are the *modal participation factors* since their value indicates how each mode influences the overall response. Once $\mathbf{z}(t)$ has been determined then $\mathbf{x}(t)$ can be found by simply using $\mathbf{x}(t) = M^{-1/2}\mathbf{z}(t)$.

Analytical solutions for the full system in eqn. (3.237) with $\mathbf{F} = \mathbf{0}$ cannot be found in general. However, special cases do exist where the equations of motion decouple. These cases exist if any of the following conditions exist:[48, 49]

1. $C = \alpha M + \beta K$, where α and β are any real scalars.
2. $C = \sum_{i=1}^{p} \gamma_{i-1} K^{i-1}$, where γ_i are real scalars.
3. $CM^{-1}K = KM^{-1}C$.

If any of these conditions holds true then the eigenvectors of eqn. (3.237) are the same as the eigenvectors with $D = 0$. Such systems are known as *normal mode systems*. These systems can be decoupled by the eigenvector matrix of K. Let V be the matrix of eigenvectors of $M^{-1/2}KM^{-1/2}$:

$$V = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p\end{bmatrix} \tag{3.245}$$

Define a normalized matrix of eigenvectors, given by $S = M^{-1/2}V$. The decoupled system is then given by

$$S^T M S = I \tag{3.246a}$$

$$S^T K S = \text{diag}[\omega_i^2] \tag{3.246b}$$

$$S^T C S = \text{diag}[2\zeta_i \omega_i] \tag{3.246c}$$

where *modal frequencies* ω_i^2 are the eigenvalues of the matrix K and ζ_i are the *modal damping ratios*. The decoupled equations are given by

$$\ddot{y}_i + 2\zeta_i \omega_i \dot{y}_i + \omega_i^2 y_i = 0, \quad i = 1, 2, \ldots, p \tag{3.247}$$

The solution of eqn. (3.237) with $\mathbf{F} = \mathbf{0}$ can be found from $\mathbf{x}(t) = S\mathbf{y}(t)$.

This section presented the basic equations and concepts of vibration. The treatise shown here is not complete by any means. Other subjects such as distributed parameter systems, Hamilton's principle, Lagrange's equations, finite element methods, etc., can be found in the references provided in this section.

3.11 Summary

The essence-oriented discussion of differential equations and dynamical systems, while adequate background for following the discussion of Chapters 4 and 7, will likely prove incomplete in many applications. In particular, conspicuous by its lack of coverage here is perturbation theory; Refs. [13] and [14] document perturbation methods which are exceptionally valuable tools for solving weakly nonlinear differential equations. The results of the present chapter do provide an adequate basis for solving differential equations encountered in a substantial fraction of practical applications, and provide a foundation for further study.

A particularly useful tool for the practicing engineer is the state space approach to represent a system of ODEs. This tool will prove invaluable in representing high order systems, commonly found in many applications (e.g., vibration models of tall buildings). Equally valuable is the concept of observability introduced in §3.4. In many applications some states will be able to be "monitored" better than others. By examining the properties of the observability matrix in eqn. (3.105) one can deduce the relative degree of observability of each state. This provides a powerful and useful tool for making tradeoffs between sensor placement requirements and monitoring of states through state estimation techniques.

The terse review of dynamical systems covering spacecraft dynamics, orbital mechanics, aircraft flight dynamics, and vibration is adequate to provide the basic concepts required to demonstrate practical applications of estimation theory. This review serves as a springboard for the various branches in all areas of dynamics. The many fascinating recent discoveries, such as chaotic behavior, since the classical developments by Newton, Lagrange, and Hamilton (to name a few) provide an ongoing research venue in the foreseeable future. Indeed, it is our hope that the interested reader will be motivated to pursue these developments in the open literature.

A summary of the key formulas presented in this chapter is given below.

- State Space Approach

$$\begin{aligned}\dot{\mathbf{x}} &= F\,\mathbf{x} + B\,\mathbf{u}\\ \mathbf{y} &= H\,\mathbf{x} + D\,\mathbf{u}\end{aligned}$$

- Homogeneous Linear Systems

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t), \quad \mathbf{x}(t_0) \text{ known}$$

$$\mathbf{x}(t) = \Phi(t,t_0)\,\mathbf{x}(t_0)$$

$$\Phi(t,t_0) = I + \int_{t_0}^{t} F(\tau_1)\,\Phi(\tau_1,t_0)\,d\tau_1$$

$$\Phi(t,t_0) = e^{F(t-t_0)}, \quad \text{for } F = \text{constant}$$

- Forced Linear Systems

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t)$$

$$\mathbf{x}(t) = \Phi(t,t_0)\,\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,\mathbf{u}(\tau)\,d\tau$$

- Nonlinear Systems

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(t,\mathbf{x},\mathbf{u})\\ \mathbf{y} &= \mathbf{h}(t,\mathbf{x},\mathbf{u})\end{aligned}$$

$$\begin{aligned}\boldsymbol{\delta}\dot{\mathbf{x}}(t) &= F(t)\,\boldsymbol{\delta}\mathbf{x}(t) + B(t)\,\boldsymbol{\delta}\mathbf{u}(t)\\ \boldsymbol{\delta}\mathbf{y}(t) &= H(t)\,\boldsymbol{\delta}\mathbf{x}(t) + D(t)\,\boldsymbol{\delta}\mathbf{u}(t)\end{aligned}$$

$$F(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_N,\mathbf{u}_N}, \quad B(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{x}_N,\mathbf{u}_N}$$

$$H(t) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}_N,\mathbf{u}_N}, \quad D(t) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_{\mathbf{x}_N,\mathbf{u}_N}$$

- Observability

$$\begin{aligned}\dot{\mathbf{x}} &= F\,\mathbf{x} + B\,\mathbf{u}\\ \mathbf{y} &= H\,\mathbf{x} + D\,\mathbf{u}\end{aligned}$$

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

- Discrete-Time Systems

$$\begin{aligned} \mathbf{x}_{k+1} &= \Phi\,\mathbf{x}_k + \Gamma\,\mathbf{u}_k \\ \mathbf{y}_k &= H\,\mathbf{x}_k + D\,\mathbf{u}_k \end{aligned}$$

$$\begin{aligned} \Phi &= I + F\Delta t + \frac{1}{2!}F^2\Delta t^2 + \frac{1}{3!}F^3\Delta t^3 + \cdots \\ \Gamma &= \left[I\,\Delta t + \frac{1}{2!}F\Delta t^2 + \frac{1}{3!}F^2\Delta t^3 + \cdots \right] B \end{aligned}$$

$$\mathcal{O}_d = \begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \\ \vdots \\ H\Phi^{n-1} \end{bmatrix}$$

- Lyapunov Stability

$$\begin{aligned} F^T P + PF &= -Q \\ \Phi^T P\Phi - P &= -Q \end{aligned}$$

- Spacecraft Dynamics

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{1}{2}\Omega(\boldsymbol{\omega})\mathbf{q} \\ J\dot{\boldsymbol{\omega}} &= -[\boldsymbol{\omega}\times]J\boldsymbol{\omega} + \mathbf{L} \end{aligned}$$

- Orbital Mechanics

$$\begin{aligned} \ddot{\mathbf{r}} &= -\frac{\mu}{||\mathbf{r}||^3}\mathbf{r} \\ M &= E - e\sin E \end{aligned}$$

- Aircraft Flight Dynamics

$$\theta = \alpha + \gamma$$

$$\begin{aligned} \alpha &= \tan^{-1}\frac{v_3}{v_1} \\ \beta &= \sin^{-1}\frac{v_2}{||\mathbf{v}||} \end{aligned}$$

$$||\mathbf{v}|| = (v_1^2 + v_2^2 + v_3^2)^{1/2}$$

$$\begin{aligned} T_1 - D\cos\alpha + L\sin\alpha - mg\sin\theta &= m(\dot{v}_1 + v_3\omega_2 - v_2\omega_3) \\ Y + mg\cos\theta\sin\phi &= m(\dot{v}_2 + v_1\omega_3 - v_3\omega_1) \\ T_3 - D\sin\alpha - L\cos\alpha + mg\cos\theta\cos\phi &= m(\dot{v}_3 + v_2\omega_1 - v_1\omega_2) \end{aligned}$$

$$\begin{aligned} D &= C_D\,\bar{q}\,S \\ Y &= C_Y\,\bar{q}\,S \\ L &= C_L\,\bar{q}\,S \end{aligned}$$

$$\bar{q} = \frac{1}{2}\rho||\mathbf{v}||^2$$

$$\begin{aligned} C_D &= C_{D_0} + C_{D_\alpha}\alpha + C_{D_{\delta_E}}\delta_E \\ C_Y &= C_{Y_0} + C_{Y_\beta}\beta + C_{Y_{\delta_R}}\delta_R + C_{Y_{\delta_A}}\delta_A \\ C_L &= C_{L_0} + C_{L_\alpha}\alpha + C_{L_{\delta_E}}\delta_E \end{aligned}$$

$$\begin{aligned} J_{11}\dot{\omega}_1 - J_{13}\dot{\omega}_3 - J_{13}\omega_1\omega_2 + (J_{33} - J_{22})\omega_2\omega_3 &= L_{A_1} + L_{T_1} \\ J_{22}\dot{\omega}_2 + (J_{11} - J_{33})\omega_1\omega_3 + J_{13}(\omega_1^2 - \omega_3^2) &= L_{A_2} + L_{T_2} \\ J_{33}\dot{\omega}_3 - J_{13}\dot{\omega}_1 + J_{13}\omega_2\omega_3 + (J_{22} - J_{11})\omega_1\omega_2 &= L_{A_3} + L_{T_3} \end{aligned}$$

$$\begin{aligned} L_{A_1} &= C_l\,\bar{q}\,S\,b \\ L_{A_2} &= C_m\,\bar{q}\,S\,\bar{c} \\ L_{A_3} &= C_n\,\bar{q}\,S\,b \end{aligned}$$

$$\begin{aligned} C_l &= C_{l_0} + C_{l_\beta}\beta + C_{l_{\delta_R}}\delta_R + C_{l_{\delta_A}}\delta_A + C_{l_p}\frac{\omega_1 b}{2||\mathbf{v}||^2} + C_{l_r}\frac{\omega_3 b}{2||\mathbf{v}||^2} \\ C_m &= C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta_E}}\delta_E + C_{m_q}\frac{\omega_2\bar{c}}{2||\mathbf{v}||^2} \\ C_n &= C_{n_0} + C_{n_\beta}\beta + C_{n_{\delta_R}}\delta_R + C_{n_{\delta_A}}\delta_A + C_{n_p}\frac{\omega_1 b}{2||\mathbf{v}||^2} + C_{n_r}\frac{\omega_3 b}{2||\mathbf{v}||^2} \end{aligned}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \mathrm{c}\theta\,\mathrm{c}\psi & \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi - \mathrm{c}\phi\,\mathrm{s}\psi & \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{c}\psi + \mathrm{s}\phi\,\mathrm{s}\psi \\ \mathrm{c}\theta\,\mathrm{s}\psi & \mathrm{s}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi + \mathrm{c}\phi\,\mathrm{c}\psi & \mathrm{c}\phi\,\mathrm{s}\theta\,\mathrm{s}\psi - \mathrm{s}\phi\,\mathrm{c}\psi \\ -\mathrm{s}\theta & \mathrm{s}\phi\,\mathrm{c}\theta & \mathrm{c}\phi\,\mathrm{c}\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\,\tan\theta & \cos\phi\,\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\,\sec\theta & \cos\phi\,\sec\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

- Vibration

$$s^2 + 2\zeta\omega_n s + w_n^2 = 0$$
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$\begin{aligned} \text{real and unequal} \quad & x = A_1 e^{s_1 t} + A_2 e^{s_2 t} \\ \text{real and repeated} \quad & x = A_1 e^{s_1 t} + t\, A_2 e^{s_1 t} \\ \text{complex conjugates} \quad & x = B e^{-at} \sin(bt + \phi) \end{aligned}$$

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{F}$$

Exercises

3.1 Consider the following linear time-varying system: $\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}$. Denote the state transition matrix of $F(t)$ by $\Phi(t, t_0)$. The differential equation for $\Phi(t, t_0)$ obeys eqn. (3.19). Show that the differential equation for $\Phi(t_0, t)$ obeys

$$\dot{\Phi}(t_0, t) = -\Phi(t_0, t)\, F(t)$$

with $\Phi(t_0, t_0) = I$.

3.2 Consider the following system of equations:

$$\ddot{z} + 3\dot{z} - 2z = 0$$
$$\dot{y} - 3z - 3y = 0$$

Determine the state space matrices (F, B, H, D) with $\mathbf{x} = \begin{bmatrix} z & \dot{z} & y \end{bmatrix}$ for an output y. Is this system observable? Is the system observable for an output z?

3.3 Consider the following system: $\dot{\mathbf{x}} = F\mathbf{x}$, with

$$F = \begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$$

and the transformation $\mathbf{x} = T\mathbf{z}$, with

$$T = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

Find a nonzero a and b such that the transformed equation $\dot{\mathbf{z}} = \Upsilon\mathbf{z}$ has the form given by

$$\Upsilon = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

3.4 Consider the following state equations for a simple circuit:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/(R_1 C) & 0 \\ 0 & -R_2/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/(R_1 C) \\ 1/L \end{bmatrix} u$$
$$y = \begin{bmatrix} -1/R_1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (1/R_1) u$$

For what value of L in terms of R_1, R_2, and C is the system unobservable?

3.5 Consider the following system matrices, which represent the linearized equations of motion for a spacecraft:

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_n^2 & 0 & 0 & 2\omega_n \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega_n & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where ω_n is the angular frequency of the reference circular orbit. Also, the states x_1 and x_3 are radial and angular deviations for the reference circular orbit. Prove that this system is observable using both observations (i.e., using the full H matrix). Also, is the system observable using only one observation (try each one separately)?

3.6 Given the coupled nonlinear second-order system

$$\ddot{x} = -x + axy$$
$$\ddot{y} = -y + bxy$$

where a and b are constants. Rearrange these equations to the form of eqn. (3.73a). Also, determine the associated linear differential equations whose solutions yield the derivative matrices:

$$\Phi(t,t_0) = \begin{bmatrix} \frac{\partial x(t)}{\partial x(t_0)} & \frac{\partial x(t)}{\partial y(t_0)} & \frac{\partial x(t)}{\partial \dot{x}(t_0)} & \frac{\partial x(t)}{\partial \dot{y}(t_0)} \\ \frac{\partial y(t)}{\partial x(t_0)} & \frac{\partial y(t)}{\partial y(t_0)} & \frac{\partial y(t)}{\partial \dot{x}(t_0)} & \frac{\partial y(t)}{\partial \dot{y}(t_0)} \\ \frac{\partial \dot{x}(t)}{\partial x(t_0)} & \frac{\partial \dot{x}(t)}{\partial y(t_0)} & \frac{\partial \dot{x}(t)}{\partial \dot{x}(t_0)} & \frac{\partial \dot{x}(t)}{\partial \dot{y}(t_0)} \\ \frac{\partial \dot{y}(t)}{\partial x(t_0)} & \frac{\partial \dot{y}(t)}{\partial y(t_0)} & \frac{\partial \dot{y}(t)}{\partial \dot{x}(t_0)} & \frac{\partial \dot{y}(t)}{\partial \dot{y}(t_0)} \end{bmatrix}$$

and

$$\Psi(t,t_0) = \begin{bmatrix} \frac{\partial x(t)}{\partial a} & \frac{\partial x(t)}{\partial b} \\ \frac{\partial y(t)}{\partial a} & \frac{\partial y(t)}{\partial b} \\ \frac{\partial \dot{x}(t)}{\partial a} & \frac{\partial \dot{x}(t)}{\partial b} \\ \frac{\partial \dot{y}(t)}{\partial a} & \frac{\partial \dot{y}(t)}{\partial b} \end{bmatrix}$$

3.7 Consider the following continuous-time system:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Is the continuous system observable? Next, convert this system into the discrete-time representation shown in eqn. (3.111) for a sampling interval Δt. Check the discrete-time observability for various sampling intervals. Is the system observable for $\Delta t = 2\pi$ seconds? Explain your results by checking the discrete-time eigenvalues of the matrix Φ in eqn. (3.111a).

3.8 Prove that the observability of a system is invariant under a similarity transformation for both continuous-time and discrete-time systems.

3.9 Find the equilibrium points for the following systems and determine their stability by Lyapunov's linearization method:

(A) $\ddot{x} + \dot{x} = 0$

(B) $\dot{x} + 4x - x^3 = 0$

(C) $\ddot{x} + \dot{x} + \sin x = 0$

Can you show global stability for any of these systems using Lyapunov's direct method?

3.10 ♣ For the discrete matrix Lyapunov equation in eqn. (3.135) prove that if P is positive definite, then Q is positive definite if and only if all the eigenvalues of Φ are within the unit circle.

3.11 Show that the cross product matrix $[\mathbf{a}\times]$ is always singular. Also, show that the nonzero eigenvalues are given by $\pm||\mathbf{a}||\,j$.

3.12 Show that the matrix $(I \pm [\mathbf{a}\times])$ is always non-singular.

3.13 Prove the following identities:

(A) $[\mathbf{a}\times]\mathbf{a} = \mathbf{0}$

(B) $[\mathbf{a}\times][\mathbf{b}\times] = \mathbf{b}\,\mathbf{a}^T - (\mathbf{b}^T\mathbf{a})I$

(C) $[\mathbf{a}\times][\mathbf{b}\times] - [\mathbf{b}\times][\mathbf{a}\times] = \mathbf{b}\,\mathbf{a}^T - \mathbf{a}\,\mathbf{b}^T$

3.14 ♣ Prove that the following matrix: $-[\mathbf{a}\times]^2$, with $\mathbf{a}^T\mathbf{a}=1$ is a projection matrix (see §1.6.4).

3.15 Prove the identities in eqn. (3.163).

3.16 Show that the determinant of an orthogonal matrix is given by ± 1.

3.17 Show that the magnitude of any row or column of an orthogonal matrix is 1.

3.18 Derive the attitude matrix for a 3-1-3 rotation sequence. If the small angle approximation is used, what is the linear approximation for this attitude matrix? How does this matrix differ from eqn. (3.148)?

3.19 ♣ Show that $(I-[\mathbf{a}\times])(I+[\mathbf{a}\times])^{-1}=\dfrac{1}{1+\mathbf{a}^T\mathbf{a}}\left\{(1-\mathbf{a}^T\mathbf{a})I+2\mathbf{a}\mathbf{a}^T-2[\mathbf{a}\times]\right\}$, and show that this is an orthogonal matrix.

3.20 Show that the kinematics equation $\dot{A}=-[\omega\times]A$ holds true for any orthogonal matrix A.

3.21 From the definitions of $\Xi(\mathbf{q})$, $\Psi(\mathbf{q})$, $\Omega(\omega)$, $\Gamma(\omega)$, and $A(\mathbf{q})$ in §3.7.1, prove the following identities:

$$\Omega(\omega)\Xi(\mathbf{q})=-\Xi(\mathbf{q})[\omega\times]-\mathbf{q}\omega^T$$

$$\Gamma(\omega)\Psi(\mathbf{q})=\Psi(\mathbf{q})[\omega\times]-\mathbf{q}\omega^T$$

$$\Omega(\omega)\Psi(\mathbf{q})=-\left\{\Xi(\mathbf{q})[\omega\times]+\mathbf{q}\omega^T\right\}A(\mathbf{q})$$

$$\Omega(\omega)\Psi(\mathbf{q})=\left[-q_4 I_{4\times 4}+\Omega(\varrho)\right]\begin{bmatrix}[\omega\times]\\ \omega^T\end{bmatrix}-\begin{bmatrix}2(\varrho^T\omega)I_{3\times 3}\\ \mathbf{0}^T_{3\times 1}\end{bmatrix}$$

$$\Xi^T(\mathbf{q})\Omega(\omega)\Xi(\mathbf{q})=-[\omega\times]$$

$$\Xi^T(\mathbf{q})\Gamma(\omega)\Xi(\mathbf{q})=[A(\mathbf{q})\omega\times]$$

$$\Gamma(\omega)\Xi(\mathbf{q})=\Xi(\varpi)$$

$$\Omega(\omega)\Psi(\mathbf{q})=\Psi(\chi)$$

where $\varpi\equiv\Psi(\mathbf{q})\omega$ and $\chi\equiv\Xi(\mathbf{q})\omega$. Note that $\mathbf{q}^T\mathbf{q}=1$. Also, show that the matrices $\Omega(\omega)$ and $\Gamma(\boldsymbol{\lambda})$ commute, i.e., $\Omega(\omega)\Gamma(\boldsymbol{\lambda})=\Gamma(\boldsymbol{\lambda})\Omega(\omega)$ for any ω and $\boldsymbol{\lambda}$.

3.22 A *symplectic matrix* A is a $2n\times 2n$ matrix with the defining property

$$A^T J A = J$$

where J is the matrix analogy of the scalar complex number $j^2=-1$; J is defined as the $2n\times 2n$ matrix

$$J=\begin{bmatrix}0 & I_{n\times n}\\ -I_{n\times n} & 0\end{bmatrix},\quad J\,J=-I_{2n\times 2n}$$

An important consequence of the symplectic property is that the inverse can be obtained by the simple rearrangement of A's elements as

$$A^{-1} = -J\,A^T J = \begin{bmatrix} A_{22}^T & -A_{12}^T \\ -A_{21} & A_{11}^T \end{bmatrix}$$

where A is partitioned into $n \times n$ sub-matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

This non-numerical inversion is a most important computational advantage that symplectic matrices have in common with orthogonal matrices. The 6×6 state transition matrix $\Phi(t,t_0)$ for the orbit model in eqn. (3.198) satisfies

$$\dot{\Phi}(t,t_0) = \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix} \Phi(t,t_0)$$

Show that G is given by

$$G = \frac{3\mu}{||\mathbf{r}||^5} \begin{bmatrix} (r_1^2 - ||\mathbf{r}||^2/3) & r_1 r_2 & r_1 r_3 \\ r_1 r_2 & (r_2^2 - ||\mathbf{r}||^2/3) & r_2 r_3 \\ r_1 r_3 & r_2 r_3 & (r_3^2 - ||\mathbf{r}||^2/3) \end{bmatrix}$$

Next show that $\Phi(t,t))$ is symplectic.

3.23 In the torque-free response of spacecraft motion the "energy ellipsoid" is given by

$$1 = \frac{J_1^2\omega_1^2}{2J_1 T} + \frac{J_2^2\omega_2^2}{2J_2 T} + \frac{J_3^2\omega_3^2}{2J_3 T}$$

where the kinetic energy T is given by

$$T = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$$

The "momentum ellipsoid" is given by

$$||\mathbf{H}||^2 = J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2$$

In order for the angular velocity $\boldsymbol{\omega}$ to be feasible, the solution must satisfy both the energy and momentum ellipsoid equations. Show that eqn. (3.191) is a feasible solution.

3.24 Write a computer program to simulate the attitude dynamics of a spacecraft modelled by eqn. (3.184). Consider the following diagonal inertia matrix:

$$J = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \text{N m s}$$

Integrate eqn. (3.184) for an 8-hour simulation. Use the identity quaternion for the initial attitude condition and set $\mathbf{L} = \mathbf{0}$. Use the following initial condition for the angular velocity: $\boldsymbol{\omega}(t_0) = \begin{bmatrix} 1\times10^{-3} & 1\times10^{-3} & 1\times10^{-3} \end{bmatrix}^T$ rad/sec.

Check your results with eqn. (3.191). Next, consider the following inertia matrix:

$$J = \begin{bmatrix} 150 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix} \text{N m s}$$

Use the same initial attitude from before, but now try the following initial conditions for the initial angular velocity vector:

(A) $\boldsymbol{\omega}(t_0) = \left[0 \;\; 1\times 10^{-3} \;\; 0\right]^T$.

(B) $\boldsymbol{\omega}(t_0) = \left[1\times 10^{-5} \;\; 1\times 10^{-3} \;\; 1\times 10^{-5}\right]^T$.

The first case is an intermediate axis spin with no perturbations in the other axes. The second case has slight perturbations in the other axes. Can you explain the vastly different results between these cases?

3.25 Program the analytical solution for the elliptic two-body given by eqns. (3.199) to (3.202). Compute the state histories at an interval of 10 seconds for 5000 seconds. The initial conditions are given by

$$\mathbf{r}(t_0) = \left[7000 \;\; 10 \;\; 20\right]^T \text{ km}$$
$$\dot{\mathbf{r}}(t_0) = \left[4 \;\; 7 \;\; 2\right]^T \text{ km/sec}$$

Compare the analytical solution with a numerical solution by integrating the nonlinear orbit model in eqn. (3.198).

3.26 Prove for an orbiting body that the angular momentum vector $\mathbf{h} = \mathbf{r}\times\dot{\mathbf{r}}$ is constant. This proves that a spacecraft's motion must be confined to a plane which is fixed in space since $\mathbf{r}$ and $\dot{\mathbf{r}}$ always remain in the same plane.

3.27 ♣ Prove Kepler's first law using eqn. (3.198).

3.28 ♣ Derive the coordinate transformations shown in eqns. (3.211) and (3.212).

3.29 In an aircraft, a trimmed condition exists if the forces and moments acting on the aircraft are in equilibrium. This is given when the pitching moment in eqn. (3.222b) is zero and when the lift force in eqn. (3.217c) is equal to mg. For this case determine expressions for the trimmed angle of attack α and elevator δ_E angles in terms of the dynamic pressure ($\bar{q}$), known reference area (S), mass (m), gravity (g), and aerodynamic coefficients.

3.30 Write a program to simulate the motion of a 747 aircraft using the equations of motion in §3.9. The aerodynamic coefficients, assuming a low cruise, for the 747 are given by

$$C_{D_0} = 0.0164 \quad C_{D_\alpha} = 0.20 \quad C_{D_{\delta_E}} = 0$$
$$C_{Y_0} = 0 \quad C_{Y_\beta} = -0.90 \quad C_{Y_{\delta_R}} = 0.120 \quad C_{Y_{\delta_A}} = 0$$
$$C_{L_0} = 0.21 \quad C_{L_\alpha} = 4.4 \quad C_{L_{\delta_E}} = 0.32$$
$$C_{l_0} = 0 \quad C_{l_\beta} = -0.160 \quad C_{l_{\delta_R}} = 0.008 \quad C_{l_{\delta_A}} = 0.013$$

$$C_{l_p} = -0.340 \quad C_{l_r} = 0.130$$
$$C_{m_0} = 0 \quad C_{m_\alpha} = -1.00 \quad C_{m_{\delta_E}} = -1.30 \quad C_{m_q} = -20.5$$
$$C_{n_0} = 0 \quad C_{n_\beta} = 0.160 \quad C_{n_{\delta_R}} = -0.100 \quad C_{n_{\delta_A}} = 0.0018$$
$$C_{n_p} = -0.026 \quad C_{n_r} = -0.280$$

The reference geometry quantities and density are given by

$$S = 510.97\,\text{m}^2 \quad \bar{c} = 8.321\,\text{m} \quad b = 59.74\,\text{m} \quad \rho = 0.6536033\,\text{kg/m}^3$$

The mass data and inertia quantities are given by

$$m = 288,674.58\,\text{kg} \quad J_{13} = 1,315,143\,\text{kg m}^2$$
$$J_{11} = 24,675,882\,\text{kg m}^2 \quad J_{22} = 44,877,565\,\text{kg m}^2 \quad J_{33} = 67,384,138\,\text{kg m}^2$$

The flight conditions for low cruise at an altitude of $6,096$ m are given by

$$||\mathbf{v}|| = 205.13\,\text{m/s} \quad \bar{q} = 13,751.2\,\text{N/m}^2$$

Using these flight conditions compute the trim values for the angle of attack and elevator (see exercise 3.29). Using these trim values compute the drag using eqn. (3.217a). Let the thrust equal the computed drag (assume that the thrust torque quantities in eqn. (3.220) are zero), and set the aileron and rudder angles to 0 degrees in your simulation. Integrate the equations of motion for a 200-second simulation for some initial linear velocities (let $\boldsymbol{\omega}_0 = \mathbf{0}$, $x_0 = 0$, $y_0 = 0$, $z_0 = 6,096$, and $\phi_0 = 0$, $\theta_0 = 0$, and $\psi_0 = 0$). Next, perform a simple maneuver starting at 10 seconds in the simulation by setting the elevator angle equal to its trim value minus 1 degree, and set the aileron angle equal to 1 degree, holding each control surface for a 10-second interval (returning the elevator back to its trimmed condition and setting the aileron angle equal to 0 degrees after the interval). Show plots of aircraft position, velocity, orientation, etc. Perform other maneuvers by changing the thrust, elevator, etc.

3.31 Pick the correct form using eqn. (3.231) for the solution of the following second-order differential equations:

(A) $\ddot{x} + 2\dot{x} + x = 0$

(B) $\ddot{x} + 2\dot{x} + 2x = 0$

(C) $\ddot{x} + 3\dot{x} + 2x = 0$

(D) $\ddot{x} + 4x = 0$

3.32 Consider the following mass, damping, and stiffness matrices:

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$$

Prove that this system is a normal mode system. Convert this system into state space form and numerically determine state trajectories for some given initial conditions. Compare the solutions with the decoupled solutions using eqn. (3.247).

3.33 ♣ Consider the following mass, damping, and stiffness matrices:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Can you find values for m_1, m_2, c_1, c_2, k_1, and k_2 such that the system does not oscillate?

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