MIT 18.06 Exam 3, Fall 2018
Johnson

Your name: ______________________________

Recitation: ______________________________

problem	**score**
1	/33
2	/33
3	/34
total	/100

Problem 1 (33 points):

The following matrix is *kind of like* a Markov matrix:

$$A = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$

except that each *row* sums to 1 (unlike a Markov matrix where each *column* sums to 1).

(a) Give one eigenvalue and a corresponding eigenvector of A.

(b) If $x \in \mathbb{R}^3$ is some vector $\neq 0$, give *brief* answers to the following questions about what *might* be true of $A^n x$ as $n \to \infty$:

(i) Can $A^n x$ approach a nonzero constant vector? If so, give the direction of the constant vector.

(ii) Can $A^n x$ approach the zero vector? *If yes*, describe (without calculating) what must be true of x for this to happen.

(iii) Can $A^n x$ diverge? What property of the eigenvalues of A explains your answer?

(iv) Can $A^n x$ oscillate forever (without growing or decaying)? What property of the eigenvalues of A explains your answer?

(blank page for your work if you need it)

Problem 2 (33 points):

The 3×3 *real* matrix A has eigenvalues $\lambda_1 = -1$, $\lambda_2 = -3+4i$, and $\lambda_3 = -3-4i$, with corresponding eigenvectors x_1, x_2, and x_3.

(a) What are the trace and determinant of $2A$?

(b) If $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$, what is x_3?

(c) Which of the following might *possibly give a diverging solution*, i.e. a solution vector whose magnitude blows up as $t \to \infty$ or $n \to \infty$ for some vector y? Circle all that apply:

(i) $A^n y$ as $n \to \infty$

(ii) $A^{-n} y$ as $n \to \infty$

(iii) The solution of $\frac{dx}{dt} = Ax$ as $t \to \infty$ for $x(0) = y$.

(iv) The solution of $\frac{dx}{dt} = -Ax$ as $t \to \infty$ for $x(0) = y$.

(v) The solution of $\frac{dx}{dt} = A^T Ax$ as $t \to \infty$ for $x(0) = y$.

(vi) The solution of $\frac{dx}{dt} = -A^T Ax$ as $t \to \infty$ for $x(0) = y$.

(d) Write down the exact solution $x(t)$ to $\frac{dx}{dt} = Ax$ for the initial condition $x(0) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

(blank page for your work if you need it)

Problem 3 (34 points):

A is a real 3×3 matrix. The matrix $B = A + A^T$ has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 0$, and $\lambda_3 = 1$, with corresponding eigenvectors $x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, and $x_3 = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$.

(a) What is the matrix e^B? You can leave your answer as a product of several matrices, as long as you write down each matrix explicitly.

(b) Let $C = (I - B)(I + B)^{-1}$.

  (i) What are the eigenvalues of C? (Not much calculation is needed!)

  (ii) Suppose that we compute

$$y = C^{100} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

  Give a good approximation for the vector y in terms of a single eigenvector.

(blank page for your work if you need it)