MIT 18.06 Exam 3, Fall 2018 - SOLUTIONS

Problem 1 (33 points):

The following matrix is *kind of like* a Markov matrix:

$$A = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$

except that each *row* sums to 1 (unlike a Markov matrix where each *column* sums to 1).

(a) Give one eigenvalue and a corresponding eigenvector of A.

(b) If $x \in \mathbb{R}^3$ is some vector $\neq 0$, give *brief* answers to the following questions about what *might* be true of $A^n x$ as $n \to \infty$:

(i) Can $A^n x$ approach a nonzero constant vector? If so, give the direction of the constant vector.

(ii) Can $A^n x$ approach the zero vector? *If yes*, describe (without calculating) what must be true of x for this to happen.

(iii) Can $A^n x$ diverge? What property of the eigenvalues of A explains your answer?

(iv) Can $A^n x$ oscillate forever (without growing or decaying)? What property of the eigenvalues of A explains your answer?

Solution:

If the rows of A sum to 1, then the columns of A^T will sum to 1. So A^T is a standard Markov matrix. In particular, since all of the entries of A are positive, A^T is a positive Markov matrix. This means that A^T will have one eigenvalue $\lambda = 1$, while the other two eigenvalues have $|\lambda| < 1$. Since A and A^T have the same eigenvalues, A will also have one eigenvalue $\lambda = 1$, and two other eigenvalues have $|\lambda| < 1$.

(a) The easiest eigenvalue and eigenvector pair to find is for $\lambda = 1$. Since the rows of A all sum to 1, we can see that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and so $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to the eigenvalue $\lambda = 1$.

(b) Solutions:

(i) Since we have one eigenvector with $\lambda = 1$ and two other eigenvalues with $|\lambda| < 1$, $A^n x$ will converge to a constant vector in the direction of $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, provided that x has a component in that direction; otherwise it converges to the zero vector.

(ii) It is possible for $A^n x$ to approach the zero vector, provided that x has no component in the direction of v_1.

(iii) $A^n x$ cannot diverge for any x because all of the eigenvalues of A have $|\lambda| \leq 1$.

(iv) $A^n x$ cannot oscillate forever. This is because all of the entries of A are positive, so that A^T is a positive Markov matrix. The only eigenvalue of A with $|\lambda| = 1$ is then the eigenvalue $\lambda = 1$, and all other eigenvalues have $|\lambda| < 1$. (Positive Markov matrices can still have negative or complex eigenvalues, which oscillate as they decay. This particular matrix has eigenvalues of 1, ≈ -0.34495, and ≈ 0.14495, so there is an oscillating but decaying term.)

Problem 2 (33 points):

The 3×3 *real* matrix A has eigenvalues $\lambda_1 = -1$, $\lambda_2 = -3+4i$, and $\lambda_3 = -3-4i$, with corresponding eigenvectors x_1, x_2, and x_3.

(a) What are the trace and determinant of $2A$?

(b) If $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$, what is x_3?

(c) Which of the following might *possibly give a diverging solution*, i.e. a solution vector whose magnitude blows up as $t \to \infty$ or $n \to \infty$ for some vector y? Circle all that apply:

(i) $A^n y$ as $n \to \infty$

(ii) $A^{-n} y$ as $n \to \infty$

(iii) The solution of $\frac{dx}{dt} = Ax$ as $t \to \infty$ for $x(0) = y$.

(iv) The solution of $\frac{dx}{dt} = -Ax$ as $t \to \infty$ for $x(0) = y$.

(v) The solution of $\frac{dx}{dt} = A^T Ax$ as $t \to \infty$ for $x(0) = y$.

(vi) The solution of $\frac{dx}{dt} = -A^T Ax$ as $t \to \infty$ for $x(0) = y$.

(d) Write down the exact solution $x(t)$ to $\frac{dx}{dt} = Ax$ for the initial condition $x(0) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

Solution:

(a) $2A$ will have eigenvalues $-2, -6+8i$ and $-6-8i$. Recall that the trace of a matrix is the sum of the eigenvalues, while the determinant is the product of the eigenvalues. So:

$$\begin{aligned} \text{trace(2A)} &= -2 + (-6+8i) + (-6-8i) = -14 \\ \det(2\text{A}) &= -2 \times (-6+8i) \times (-6-8i) = -200 \end{aligned}$$

Equivalently, the trace of A is -7 and the determinant of A is -50, and $2A$ has double the trace (because the diagonal entries are doubled) and $2^3 = 8$ times the determinant.

(b) There are two ways to do this. The easiest way is to recall that the eigenvalues and eigenvectors of any real matrix must come in complex-conjugate pairs, so we must have

$$x_3 = \overline{x_2} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}.$$

(or any multiple of this vector).

Alternatively, since A is a real-symmetric matrix, it must have orthogonal eigenvectors, and we can look for an x_3 that is orthogonal to x_1 and x_2 — but be careful, it has to be orthogonal in the sense that $x_2^H x_3 = 0$. So, for example, if you want to use the cross product, you have to conjugate. For example, you could use $x_1 \times \overline{x_3}$ (but then you have to multiply by $-i$ to get the convenient choice above).

(c) Note that the eigenvalues all have negative real parts, $\mathrm{Re}(\lambda_i) < 0$, and are greater than or equal to 1 in absolute value, $|\lambda_i| \geq 1$:

(i) $A^n y$ will generally diverge, since $|\lambda_i| \geq 1$. The only way they will not diverge is if y is parallel to x_1.

(ii) $A^{-n} y$ cannot diverge for any y since the eigenvalues of A^{-1} are λ_i^{-1}, and $|\lambda_i^{-1}| \leq 1$.

(iii) The solution of $\frac{dx}{dt} = Ax$ as $t \to \infty$ cannot diverge for any y since $\mathrm{Re}(\lambda_i) < 0$.

(iv) The solution of $\frac{dx}{dt} = -Ax$ as $t \to \infty$ will diverge for all $y \neq 0$ since the eigenvalues of $-A$ will all have positive real part.

(v) The solution of $\frac{dx}{dt} = A^T Ax$ as $t \to \infty$ will diverge for all $y \neq 0$ since $A^T A$ is a positive-definite matrix (A is full rank), and so has real and strictly positive eigenvalues.

(vi) The solution of $\frac{dx}{dt} = -A^T Ax$ as $t \to \infty$ cannot diverge for any y since $A^T A$ is a negative-definite matrix, and so has real and strictly negative eigenvalues.

(d) The general solution to $\frac{dx}{dt} = Ax$ is

$$x(t) = c_1 e^{-t} x_1 + c_2 e^{(-3+4i)t} x_2 + c_3 e^{(-3-4i)t} x_3$$

for some constants c_1, c_2 and c_3. For real initial conditions we expect $c_3 = \overline{c_2}$. In order to satisfy the initial condition, we require that

$$x(0) = c_1 x_1 + c_2 x_2 + c_3 x_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

By inspection, this is satisfied for $c_1 = c_2 = c_3 = 1$ and so the exact solution is

$$x(t) = e^{-t} x_1 + e^{(-3+4i)t} x_2 + e^{(-3-4i)t} x_3 = e^{-t} x_1 + 2\,\mathrm{Re}\left[e^{(-3+4i)t} x_2\right].$$

Problem 3 (34 points):

A is a real 3×3 matrix. The matrix $B = A + A^T$ has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 0$, and $\lambda_3 = 1$, with corresponding eigenvectors $x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, and $x_3 = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$.

(a) What is the matrix e^B? You can leave your answer as a product of several matrices, as long as you write down each matrix explicitly.

(b) Let $C = (I - B)(I + B)^{-1}$.

(i) What are the eigenvalues of C? (Not much calculation is needed!)

(ii) Suppose that we compute

$$y = C^{100} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Give a good approximation for the vector y in terms of a single eigenvector.

Solution:

(a) Since B has three independent eigenvectors, it is diagonalizble with $B = X\Lambda X^{-1}$. The matrix exponential is then given by $e^B = Xe^\Lambda X^{-1}$, where

$$\boxed{e^\Lambda = \begin{pmatrix} e^2 & & \\ & 1 & \\ & & e \end{pmatrix}}.$$

X is a matrix whose columns are the corresponding eigenvectors. However, since B is a real symmetric matrix, it has orthogonal eigenvectors. We can then normalize each of the eigenvectors to obtain an orthonormal set:

$$q_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \; q_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \; q_3 = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}.$$

Then we have $\boxed{e^B = Qe^\Lambda Q^{-1}}$, where $Q^{-1} = Q^T$ and

$$\boxed{Q = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{pmatrix}}.$$

Alternatively, since $X^T X = D$ is a diagonal matrix by orthogonality, we have $X^{-1} = D^{-1}X^T$, so then

$$e^B = Xe^\Lambda D^{-1}X^T$$

where $D^{-1} = \begin{pmatrix} 1/6 & & \\ & 1/5 & \\ & & 1/30 \end{pmatrix}$ is just the inverses of the squared lengths. Alternatively, you could compute X^{-1} by the Gauss–Jordan method, but that is a lot more work and is easy to get wrong!

(b) If $C = (I - B)(I + B)^{-1}$ then:

(i) The eigenvalues of C are just $\frac{1-\lambda_i}{1+\lambda_i}$, i.e. $\frac{1-2}{1+2} = -\frac{1}{3}$, 1 and $\frac{1-1}{1+1} = 0$, with the same corresponding eigenvectors x_i

(ii) The vector y, where

$$y = C^{100} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

will be almost parallel to the eigenvector corresponding to the largest-magnitude eigenvalue. The largest-magnitude eigenvalue of C is 1, with normalized eigenvector q_2, and so $y = C^{100} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \approx \frac{\alpha}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, where

$$\alpha = q_2^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{-1}{\sqrt{5}}$$

so that

$$\boxed{y \approx \frac{1}{5} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = q_2 q_2^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{x_2 x_2^T}{x_2^T x_2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}.$$