MIT 18.06 Final Exam, Fall 2018
Johnson

Your name: ____________________

Recitation: ____________________

problem	**score**
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/10
total	/100

Problem 1 (5+10 points):

The matrix A has the diagonalization $A = X\Lambda X^{-1}$ with

$$X = \begin{pmatrix} 1 & 1 & -1 & 0 \\ & 1 & 2 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}, \ \Lambda = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & -2 & \\ & & & -1 \end{pmatrix}.$$

(a) Give a basis for the nullspace $N(M)$ of the matrix $M = A^4 - 2A^2 - 8I$. (Hint: not much calculation required!)

(b) Write down the solution $x(t)$ to the ODE $\frac{dx}{dt} = Ax$ for $x(0) = \begin{pmatrix} 2 \\ 3 \\ 1 \\ -1 \end{pmatrix}$.

Your final answer should contain no matrix exponentials or matrix inverses, just a sum of vectors (whose components you give explicitly as numbers) multiplied by given scalar coefficients (that may depend on t).

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Problem 2 (3+3+3+3+3 points):

The real $m \times n$ matrix A has a QR factorization $A = QR$ of the form

$$Q = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{pmatrix}, \; R = \begin{pmatrix} 1 & -2 & 2 & 0 & 0 & 0 \\ & 2 & -3 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 3 & 1 & -1 \\ & & & & 1 & 2 \\ & & & & & 1 \end{pmatrix}.$$

where $q_1, \ldots, q_6$ are six orthonormal vectors in $\mathbb{R}^m$.

(a) Give as much true information as possible about m, n, and the rank of A.

(b) If a_5 is the 5th column of A, write it in the basis $q_1, \ldots, q_6$, i.e. write it as $a_5 = c_1 q_1 + c_2 q_2 + \cdots + c_6 q_6$, by giving the numerical values of the coefficients $c_1, \cdots, c_6$.

(c) What is $\|a_5\|$?

(d) This pattern of zero entries in R means that columns .... of A must be .... to columns .... of A.

(e) If A is a square matrix, what is $|\det A|$ (the absolute value of the determinant)?

(blank page for your work if you need it)

Problem 3 (8+3+4 points):

You are given the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

(a) Give bases for the four fundamental subspaces of A.

(b) Which of $Ax = b$ or $A^T y = c$ will *not* have *unique* solutions for x or y (assuming a solution exists)?

(c) Which of $Ax = b$ or $A^T y = c$ may *not have a solution*? For that equation, **give a right-hand side** (b or c) for which a solution exists, and that has only *two nonzero* entries in the right-hand side.

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Problem 4 (5+10 points):

A vector $\hat{x}$ minimizes $\|Ax - b\|$ over all possible vectors x.

(a) If $\hat{x} = 0$, then b must be in which fundamental subspace of A?

(b) If $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, what is the minimum possible $\|Ax - b\|$? Describe (with as much detail as you can) *all possible* $\hat{x}$ that give this minimum.

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Problem 5 (5+10 points):

Suppose that $A = A^T$ is a real-symmetric 10×10 matrix whose eigenvalues are $11, 10, 9, 8, 7, 6, 5, 4, 3, 2$. Corresponding eigenvectors of A, each normalized to have length $\|x\| = 1$, are (to three significant digits) the columns of the matrix:

$$X = \begin{pmatrix} -0.179 & 0.415 & -0.173 & 0.115 & -0.125 & 0.079 & -0.676 & -0.066 & 0.3 & -0.423 \\ -0.239 & -0.168 & -0.53 & -0.08 & 0.232 & -0.281 & -0.386 & -0.223 & -0.353 & 0.413 \\ -0.35 & 0.116 & -0.201 & -0.41 & 0.547 & 0.197 & 0.182 & 0.373 & 0.38 & 0.029 \\ -0.399 & -0.238 & -0.079 & -0.46 & -0.279 & -0.139 & 0.259 & -0.427 & -0.028 & -0.468 \\ -0.378 & -0.132 & 0.013 & 0.28 & -0.374 & -0.332 & 0.096 & 0.056 & 0.582 & 0.4 \\ -0.227 & 0.352 & 0.195 & 0.336 & 0.388 & 0.137 & 0.242 & -0.657 & 0.042 & 0.109 \\ -0.404 & 0.327 & 0.566 & -0.143 & 0.024 & -0.399 & -0.123 & 0.277 & -0.371 & 0.03 \\ -0.001 & 0.628 & -0.369 & -0.162 & -0.46 & 0.172 & 0.3 & 0.05 & -0.202 & 0.261 \\ -0.303 & -0.07 & -0.331 & 0.6 & 0.07 & -0.052 & 0.288 & 0.329 & -0.29 & -0.389 \\ -0.425 & -0.285 & 0.195 & 0.047 & -0.213 & 0.732 & -0.194 & 0.051 & -0.194 & 0.195 \end{pmatrix}.$$

(That is, the first column of X is an eigenvector for $\lambda = 11$, and so on.)

Consider the recurrence relation $x_{n+1} - x_n = -\alpha A x_n$ on vectors $x_n \in \mathbb{R}^{10}$, where $\alpha > 0$ is some positive real number.

(a) For what values of α (if any) can the recurrence have solutions that *diverge* in magnitude as $n \to \infty$?

(b) For $\alpha = 1$, give a good approximation for the vector x_{100} given $x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. You can leave your answer in the form of some vector times some coefficient(s) without carrying out the multiplications, but give all the numbers in your coefficients and vectors to 3 significant digits.

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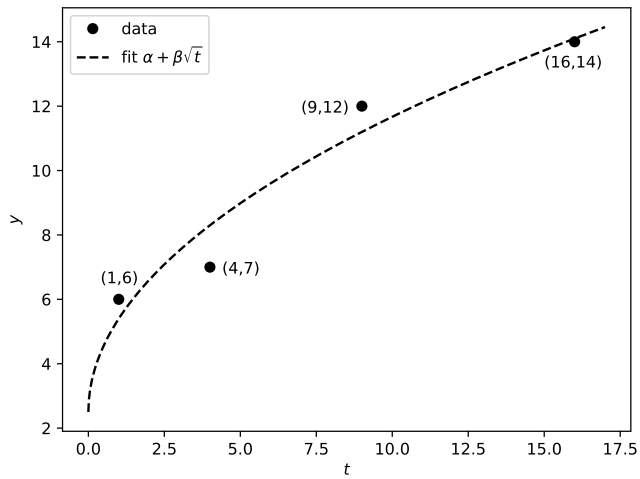

Figure 1: Fitting four data points (t_k, y_k) to $y(t) = \alpha + \beta\sqrt{t}$

Problem 6 (5+10 points):

As shown in Fig. 1, you are trying to fit of a sequence of four data points $(t_k, y_k) = (1, 6), (4, 7), (9, 12), (16, 14)$ to a function of the form $y(t) = \alpha + \beta\sqrt{t}$ for unknown coefficients α and β .

(a) Write down a minimization problem of the form "minimize .... over all possible α, β" that we could solve for a "best fit" curve (resembling the dashed line in Fig. 1) using 18.06 techniques. (Don't solve it.)

(b) Write down a 2×2 system of equations of the form

$$\begin{pmatrix} \text{some} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \text{some} \\ \text{vector} \end{pmatrix}$$

whose solution gives the "best" fit coefficients α and β for your minimization problem from part (a). You don't need to solve it! You can leave the matrix and vector as a product of other matrices/vectors/transposes, but give the numerical values of each of the terms (no matrix inverses allowed).

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Problem 7 (10 points):

Suppose that you want to multiply each of the three matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 4 & 5 \end{pmatrix},$$

by each of the two vectors $x, y \in \mathbb{R}^2$. That is, you want to compute the *six* vectors $x_A = Ax$, $x_B = Bx$, $x_C = Cx$, $y_A = Ay$, $y_B = By$, and $y_C = Cy$. Write *all* of these products in the form of a *single* matrix–matrix multiplication:

$$\begin{pmatrix} \text{some matrix in terms of} \\ x_A, x_B, x_C, y_A, y_B, y_C \end{pmatrix} = \begin{pmatrix} \text{some matrix in terms of} \\ A, B, C \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix}.$$

Note that the third matrix here is the 2×2 matrix whose columns are x and y. That is, **give the sizes of the other two matrices and how their contents are arranged.**

[For example, a possible but wrong(!) answer for the second matrix would be the 2×6 matrix $\begin{pmatrix} A & B^T & C^T \end{pmatrix}$.]

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