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LETTER TO THE EDITOR

A note on the relativistic Euler equations

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Abstract
We present an alternative way to write the Euler equations for an ideal isentropic fluid as a symmetric hyperbolic system of evolution equations for a timelike vector field. An equation of state has to be provided which relates the length of this vector to the sound velocity. The relation to the conventional formulation is established and some of the consequences are discussed.

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In this letter we present an alternative way to write the relativistic Euler equations as a system of symmetric hyperbolic equations without introducing a 3 + 1 decomposition. It is well known (see e.g. [2]) that the relativistic Euler equations are obtained from the energy–momentum tensor

$$T^{ab} = (\rho + p)u^a u^b - pg^{ab} \tag{1}$$

by requiring that it be divergence free,

$$\nabla_b T^{ab} = 0.$$

The timelike unit vector u^a is the 4-velocity of the fluid, ρ and p are its energy density and pressure, respectively. To close the system, an equation of state $\rho = \rho(p)$ needs to be specified.

To obtain a covariant symmetric hyperbolic system, we start out with a symmetric tensor T_{ab} which has the same algebraic properties as the energy–momentum tensor (1)

$$T^{ab} = f(x)w^a w^b - g(x)g^{ab}. \tag{2}$$

The vector w^a is assumed to be timelike and future-pointing so that $w^2 = w_a w^a > 0$ with $w > 0$. The functions f and g are real-valued functions of the parameter $x = 1/w$. They will be determined shortly. To obtain the Euler equations, we require that this tensor be divergence free and get the equation[1]

$$0 = \nabla_b T^{ab} = -x^3 f' w_c w^b (\nabla_b w^c) w^a + f w^a (\nabla_b w^b) + f w^b (\nabla_b w^a) + x^3 g' w_c (\nabla^a w^c). \qquad (3)$$

We write this equation in the form $A^r_{ab} \nabla_r w^b = 0$ and obtain

$$\left(-x^3 f' w_b w_a w^r + f \delta^r_b w_a + f w^r g_{ab} + x^3 g' w_b \delta^r_a\right) \nabla_r w^b = 0. \qquad (4)$$

Symmetry of this system requires that $A^r_{[ab]} = 0$ which implies the relation

$$f - x^3 g' = 0. \qquad (5)$$

Comparing the two energy–momentum tensors (1) and (2), we find agreement if

$$w^a = w u^a, \qquad f(x)/x^2 = \rho + p, \qquad g = p. \qquad (6)$$

From these and (5), we find that the function g is implicitly determined for any given equation of state by the equation $x = x_0 \Phi(g(x))$, where

$$\Phi(p) = \exp\left(\int_{p_0}^{p} \frac{\mathrm{d}\tilde{p}}{\rho(\tilde{p}) + \tilde{p}}\right)$$

is the Lichnerowicz index of the fluid (see e.g. [1]). Differentiating the second equation in (6) with respect to x and using the third equation and (5) yields

$$x f' - 2f = \left(\frac{\mathrm{d}\rho}{\mathrm{d}p} + 1\right) f.$$

Introducing the velocity of sound in the fluid by $s^2 = \mathrm{d}p/\mathrm{d}\rho$, we obtain the equation

$$\frac{1}{s^2} = \left(\frac{x f'}{f} - 3\right)$$

which allows us to write (4) in the form

$$\left(\left(3 + \frac{1}{s^2}\right) \frac{w_b w_a}{w^2} w^r - 2\delta^r_{(b} w_{a)} - w^r g_{ab}\right) \nabla_r w^b = 0. \qquad (7)$$

In this form, the basic variable is the vector w^a whose inverse length is the Lichnerowicz index of the fluid, and the equation of state relates the velocity of sound to the index of the fluid. Note that the closely related vector field $T^a = 1/w^2 w^a$ is the so-called Taub current[2] [3] in terms of which the Euler equations take on a particularly simple form.

Now we discuss the question of hyperbolicity of this system. A system $A^r_{ab} \nabla_r w^a = 0$ will be symmetric hyperbolic if there is a timelike unit vector t^a such that the bilinear form $t_r A^r_{ab}$ is positive definite. Given t^a, we can find a spacelike unit vector z^a with $t_a z^a = 0$ so that $w^a = \alpha\,(t^a + v z^a)$ for some α and v. In fact, we have $\alpha^2 = w^2/(1 - v^2)$ with $v^2 < 1$. Now we complete t^a, z^a with two spacelike vectors x^a, y^a to an orthonormal basis. Taking components of

$$t_r A^r_{ab} = \left(3 + \frac{1}{s^2}\right) \frac{w_b w_a}{w^2} \alpha - 2 t_{(b} w_{a)} - \alpha g_{ab}$$

in that basis we obtain a matrix of the form

$$\alpha \begin{pmatrix} a & b & & \\ b & c & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

[1] The prime denotes the derivative of a function with respect to its argument.

[2] This was pointed out to me by R Beig.

with

$$a = \frac{3 + 1/s^2}{1 - v^2} - 3, \qquad b = v - \frac{3 + 1/s^2}{1 - v^2} v, \qquad c = 1 + \frac{3 + 1/s^2}{1 - v^2} v^2.$$

The matrix will be positive definite if its trace and determinant are positive, i.e, $a + c > 0$ and $ac - b^2 > 0$. The condition on the trace yields

$$\frac{1}{s^2} \frac{1 + v^2}{1 - v^2} + 3 \left(\frac{1 + v^2}{1 - v^2} - 1 \right) > 0$$

which is always satisfied because $v^2 < 1$. The condition on the determinant gives

$$\frac{1}{s^2} - v^2 > 0.$$

Since, for physically reasonable situations the sound velocity is less than the speed of light, we have $s^2 < 1$ so that the inequality is satisfied for all $v^2 < 1$. Therefore, the system is hyperbolic for all timelike directions. However, even if the sound velocity becomes greater than unity, there will be timelike directions for which the system is hyperbolic.

We can also determine the characteristics of the system. We ask for those covectors p_a which annihilate the characteristic polynomial

$$\det \left(p_r A^r_{ab} \right) = (w^a p_a)^2 \left((w^a p_a)^2 \left(\frac{1}{s^2} - 1 \right) + (w_a w^a)(p_a p^a) \right).$$

Then the characteristics are found to be the lines along w^a and the cone given by the equation $w^2 q_a q^a = (1 - s^2)(w_a q^a)^2$. This is exactly the sound cone i.e., the cone defined by the propagation of sound signals.

Finally, we discuss the question of how to reconstruct ρ and p from $x = 1/w$ and a function $s^2(x)$. We define the function

$$N(x) = \exp \left(\int_{x_0}^{x} \frac{1}{s^2(\xi)} \frac{\mathrm{d}\xi}{\xi} \right),$$

where x_0 is an arbitrary but fixed value. Using the field equation (7), it is easy to verify that the vector field $j^a = xN(x)w^a = N(x)u^a$ is divergence free so that N is the baryon number density of the fluid. Then the function $f(x)$ is given by

$$f(x) = \frac{\rho_0}{x_0} x^3 N(x),$$

for some constant ρ_0. The pressure $p(x)$ with $p(x_0) = 0$ is obtained from (5) and (6) as

$$p(x) = \frac{\rho_0}{x_0} \int_{x_0}^{x} N(\xi)\, \mathrm{d}\xi$$

and the density $\rho(x)$ with $\rho(x_0) = \rho_0$ can be found from (6) to be

$$\rho(x) = \frac{\rho_0}{x_0} \left[xN(x) - \int_{x_0}^{x} N(\xi)\, \mathrm{d}\xi \right].$$

These expressions look as if it would not be possible to describe situations in which pressure and density vanish simultaneously. However, for the equations of state $p = \rho^\alpha$ with $\alpha > 1$ for which pressure and density vanish simultaneously, the sound velocity also vanishes at $\rho = 0$. Then the integral in the definition of $N(x)$ diverges at the lower end so that $N(x_0) = 0$, hence $\rho(x_0) = 0$ so that we can recover the simultaneous vanishing of pressure and density.

As a final note, we remark that the system (7) contains only the square of sound velocity which is always positive and the timelike vector w^a. In a situation where matter regions and vacuum regions are matched together, the sound velocity will have a finite value (depending on the equation of state and possibly zero) on the interface between the two regions. Using this value, we can formally continue the system (7) beyond the matter region and find that it formally makes sense even in the vacuum regions.

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