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RELATIVISTIC SHOCKS: THE TAUB ADIABAT*

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ABSTRACT

This paper presents a detailed analysis of the adiabat for a relativistic shock wave in a fluid. This analysis extends and supplements previous analyses by Lichnerowicz.

Subject headings: hydrodynamics — relativity — shock waves

I. INTRODUCTION AND OVERVIEW

The relativistic theory of hydrodynamical shocks was developed by Taub (1948); see also pp. 504–505 of Landau and Lifshitz (1959). Since then a number of investigators have applied Taub's equations to a variety of problems in relativistic astrophysics (see, e.g., Colgate and Johnson 1960; Colgate and White 1966; May and White 1966; Johnson and McKee 1971; Eltgroth 1971; Wilson 1972). But astrophysicists have generally ignored, or been unaware of, detailed theorems about the relativistic shock adiabat ("Taub adiabat")—theorems due to the mathematician A. Lichnerowicz (1967, 1970, 1971).[1] The purpose of this paper is twofold: (i) to strengthen and extend Lichnerowicz's theorems, and (ii) to make Lichnerowicz's analysis more accessible to astrophysicists.

This paper strengthens and extends Lichnerowicz's results in two ways. (i) It re-proves Lichnerowicz's results without making any use of his compressibility assumption

$$[\partial(\mu V)/\partial s]_p > 0 \qquad (\tau_s' > 0 \text{ in his notation}) .$$

(ii) It analyzes weak shocks in particularly great detail; Lichnerowicz's analysis largely ignored the weak-shock limit. In other respects, Lichnerowicz's analysis is more detailed and complete than this paper; for example, it includes magnetohydrodynamic shocks, whereas this paper ignores magnetic fields.

Since most astrophysicists learn the nonrelativistic theory of shock waves from the book *Fluid Mechanics* (Landau and Lifshitz 1959, especially §§ 83 and 84), this paper is patterned after the Landau-Lifshitz analysis.

II. THERMODYNAMIC NOTATION

We shall restrict attention to relativistic perfect fluids with no heat transport, no shear stresses, and no viscosity (except in the shock front, which will be idealized as a discontinuity). Our notation will be that of Misner, Thorne, and Wheeler (1973, hereafter MTW), chapter 22. In particular, the following symbols will denote thermodynamic quantities measured in the local rest frame of the fluid:

$$n \ldots\ldots\ldots\ldots \text{number density of baryons} , \tag{1a}$$

$$V \equiv 1/n \ldots\ldots\ldots\ldots \text{specific volume} , \tag{1b}$$

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[1] *Note added in proof.*—It has been called to my attention that many of the theorems presented by Lichnerowicz actually originated (for the case of zero magnetic field) with Israel (1960).

ρ . . . . total density of mass-energy, (1c)

p . . . . pressure, (1d)

T . . . . temperature, (1e)

s . . . . entropy per baryon, (1f)

$\mu = (\partial\rho/\partial n)_s = (\rho + p)/n$ . . . . chemical potential, (1g)

$\Gamma = (\partial \ln p/\partial \ln n)_s$ . . . . adiabatic index, (1h)

$v_S = \left[\left(\frac{\partial p}{\partial \rho}\right)_s\right]^{1/2} = \left(\frac{\Gamma p}{\rho + p}\right)^{1/2}$ . . . . . . velocity of sound relative to fluid (assumed throughout to be less than speed of light), (1i)

$u_S \equiv v_S\gamma_S \equiv v_S/(1 - v_S{}^2)^{1/2}$ . . . . "4-velocity" of sound relative to fluid; (1j)

we use units in which the speed of light is one; and we write the first law of thermodynamics in the form (MTW eq. [22.6])

$$d\rho = \frac{\rho + p}{n}\, dn + nTds \tag{2a}$$

or, equivalently,

$$d\mu = Vdp + Tds\,. \tag{2b}$$

Diagrams in which p is plotted against μV will play a key role in our analysis of the Taub adiabat. Notice that in such a diagram a Poisson adiabat (curve of constant entropy) has the slope

$$\left[\frac{\partial p}{\partial(\mu V)}\right]_s = -\left(\frac{u_S}{V}\right)^2 = -\left(\frac{\text{4-velocity of sound}}{\text{specific volume}}\right)^2 < 0\,. \tag{3}$$

(To derive this relation combine equations [1b, g, i, j] and [2a, b].)

It is sometimes instructive to take Newtonian limits of relativistic equations. Our Newtonian variables will follow the conventions of Landau and Lifshitz (1959, denoted henceforth as LL):

ρ_N . . . . (rest) mass density, (4a)

$V_N = 1/\rho_N$ . . . . specific volume, (4b)

p_N . . . . pressure, (4c)

ϵ_N . . . . energy per unit rest mass, (4d)

s_N . . . . entropy per unit rest mass, (4e)

T_N . . . . temperature, (4f)

$w_N = \left[\frac{\partial(\epsilon_N\rho_N)}{\partial\rho_N}\right]_{s_N} = \epsilon_N + \frac{p_N}{\rho_N}$ . . . . enthalpy, (4g)

$\Gamma_N = (\partial \ln p_N/\partial \ln \rho_N)_{s_N}$ . . . . adiabatic index, (4h)

$v_{SN} = [(\partial p_N/\partial\rho_N)_{s_N}]^{1/2} = (\Gamma_N p_N/\rho_N)^{1/2}$ . . . . . . speed of sound. (4i)

In taking Newtonian limits, one must convert various quantities from a "per baryon" basis to a "per (rest) mass" basis—a conversion that requires the use of

$$m_B \equiv \text{(average) rest mass per baryon}\,. \tag{4j}$$

The following Newtonian limits should be obvious:

$$n \to \rho_N/m_B\,, \tag{5a}$$

$$V \to m_B V_N\,, \tag{5b}$$

$$\rho \to \rho_N(1+\epsilon_N) \approx \rho_N\,, \tag{5c}$$

$$p \to p_N\,, \tag{5d}$$

$$s \to m_B s_N\,, \tag{5e}$$

$$\mu \to m_B(1+w_N)\,, \tag{5f}$$

$$\Gamma \to \Gamma_N\,, \tag{5g}$$

$$v_S \to v_{SN}\,, \tag{5h}$$

$$u_S \to v_{SN}/(1-v_{SN}^2)^{1/2} \approx v_{SN}\,. \tag{5i}$$

III. FUNDAMENTALS OF SHOCK-WAVE THEORY

The standard analysis of relativistic shocks is valid in curved spacetime as well as flat; in the Dicke-Brans-Jordan theory of gravity as well as general relativity; indeed, in any "metric theory of gravity."[2] In that analysis one picks a particular event $\mathscr{P}$ on the shock front. In the neighborhood of $\mathscr{P}$ one introduces a local Lorentz frame ("rest frame of the shock") in which (i) the shock is momentarily at rest; (ii) the shock is the surface $y = z = 0$; and (iii) on both sides of the shock the fluid is moving in the x direction, i.e., perpendicular to the shock front ("normal shock"; see fig. 1). That such a local Lorentz frame exists in general one can prove quite easily (see, e.g., Taub 1948). In accordance with the conventions of LL denote the "*front*" side of the shock (side *from* which the fluid moves) by a "1," and denote the "back" side (side *toward* which the fluid moves) by a "2"; see figure 1. Denote the velocity of the fluid, as measured in the rest frame of the shock, by

$$v_1 = (dx/dt)_1 = \text{ordinary velocity on front side}\,, \tag{6a}$$

$$v_2 = (dx/dt)_2 = \text{ordinary velocity on back side}\,, \tag{6b}$$

$$u_1 = v_1\gamma_1 = v_1/(1-v_1^2)^{1/2} = \text{"4-velocity" on front side}\,, \tag{6c}$$

$$u_2 = v_2\gamma_2 = v_2/(1-v_2^2)^{1/2} = \text{"4-velocity" on back side}\,. \tag{6d}$$

(Note that these "4-velocities" are scalars, not vectors.)

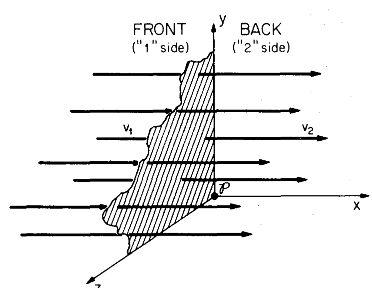

FIG. 1.—A relativistic shock wave viewed in the local-Lorentz rest frame of the shock front at an event $\mathscr{P}$.

[2] By "metric theory" we mean a theory with a metric in whose local Lorentz frames the non-gravitational laws of physics assume their standard special-relativistic forms (equivalence principle). Ni (1972) has given a partial catalog of metric theories.

In the rest frame of the shock the law of baryon conservation is equivalent to continuity of the *baryon flux*:

$$j \equiv n_1 u_1 = n_2 u_2 . \tag{7a}$$

Similarly, energy and momentum conservation are equivalent to continuity of energy flux and momentum flux:

$$\{(\rho + p)\gamma u\}_1 = \{(\rho + p)\gamma u\}_2 ; \tag{7b}$$

$$\{(\rho + p)u^2 + p\}_1 = \{(\rho + p)u^2 + p\}_2 . \tag{7c}$$

These junction conditions are more easily understood by writing them in a form analogous to the Rankine-Hugoniot equations of Newtonian theory: First take the law of baryon conservation (7a) and turn it into equations for the fluid 4-velocities

$$u_1 = jV_1 , \qquad u_2 = jV_2 . \tag{8a}$$

Then take the law of momentum conservation (7c); rewrite it in terms of μ, V, j, and p using equations (1b, g) and (7a); and solve for the baryon flux to obtain

$$j^2 = -\frac{p_2 - p_1}{\mu_2 V_2 - \mu_1 V_1} . \tag{8b}$$

Finally, rewrite the energy equation (7b) in the form

$$\mu_1 \gamma_1 = \mu_2 \gamma_2$$

by using equations (1g) and (7a); divide equation (8b) by $(\mu_1 V_1 + \mu_2 V_2)$, and combine with $j = u_1/V_1 = u_2/V_2$ (eqs. [7a] and [1b]) to obtain

$$(\mu_2 u_2)^2 - (\mu_1 u_1)^2 = (p_1 - p_2)(\mu_1 V_1 + \mu_2 V_2) ;$$

then subtract this from the square of the energy equation, $(\mu_2\gamma_2)^2 - (\mu_1\gamma_1)^2 = 0$, to obtain

$$\mu_2{}^2 - \mu_1{}^2 = (p_2 - p_1)(\mu_1 V_1 + \mu_2 V_2) . \tag{8c}$$

Equations (8a, b, c) *are Taub's* (*1948*) *junction conditions for shock waves.* Their Newtonian limits (obtained using equations [5]) are the standard Rankine-Hugoniot equations:

$$v_{\mathrm{N}1} = j_{\mathrm{N}} V_{\mathrm{N}1} , \qquad v_{\mathrm{N}2} = j_{\mathrm{N}} V_{\mathrm{N}2} \tag{8a,N}$$

(j_{N} = "mass flux" of Newtonian theory),

$$j_{\mathrm{N}}{}^2 = -\frac{p_{\mathrm{N}2} - p_{\mathrm{N}1}}{V_{\mathrm{N}2} - V_{\mathrm{N}1}} , \tag{8b,N}$$

$$w_{\mathrm{N}2} - w_{\mathrm{N}1} = \tfrac{1}{2}(p_{\mathrm{N}2} - p_{\mathrm{N}1})(V_{\mathrm{N}1} + V_{\mathrm{N}2}) . \tag{8c,N}$$

Just as the form of the Newtonian junction conditions (8b,N) and (8c,N) motivates one to use p_{N} and V_{N} as one's independent thermodynamic variables when analyzing Newtonian shocks, so the form of the relativistic junction conditions (8b) and (8c) motivates one to use p and μV as one's independent variables for relativistic shocks.

Everything up to this point is the well-known theory of relativistic shocks as developed by Taub and used extensively by others. We now turn to the raison d'être of this paper: a detailed analysis of the "Taub adiabat" (8c) for relativistic shocks.

Our analysis will be patterned after the Newtonian analysis of LL, §§ 83 and 84. The reader may wish to take Newtonian limits of each equation in our analysis and see it reduce to the corresponding LL equation.

IV. WEAK RELATIVISTIC SHOCKS

Consider a family of shocks, each with the same thermodynamic state on the front face (same $\mu_1 V_1$, p_1, etc.), but with different states on the back face (different $\mu_2 V_2$, p_2, etc.). As in Newtonian theory, this family of shocks is a one-parameter family. Thus, if one plots all back-face states $(\mu_2 V_2, p_2)$, in the $(\mu V, p)$-plane, they lie on a single curve—the Taub adiabat—passing through the point $(\mu_1 V_1, p_1)$.

One can also plot, in the $(\mu V, p)$-plane, the Poisson adiabat (curve of constant entropy) passing through $(\mu_1 V_1, p_1)$. We shall show below that these two adiabats typically have the relative locations and shapes shown in figure 2.

We begin by considering weak shocks—i.e., points on the Taub adiabat near $(\mu_1 V_1, p_1)$. Following the Newtonian approach (LL, § 83) we expand equation (8c) for the Taub adiabat as a power series in pressure and entropy, keeping terms up to third order in $p_2 - p_1$, but only up to first order in $s_2 - s_1$:

$$\mu_2{}^2 - \mu_1{}^2 = (p_2 - p_1)\left\{2\mu_1 V_1 + \left[\frac{\partial(\mu V)}{\partial p}\right]_{s,1}(p_2 - p_1) + \frac{1}{2}\left[\frac{\partial^2(\mu V)}{\partial p^2}\right]_{s,1}(p_2 - p_1)^2\right\} + O[(p_2 - p_1)(s_2 - s_1)] + \cdots .$$

The first law of thermodynamics (2b) dictates that, for this shock-wave change in the state of the fluid, as for any change,

$$\mu_2{}^2 - \mu_1{}^2 = 2\{\mu_1 T_1(s_2 - s_1) + \mu_1 V_1(p_2 - p_1) + \tfrac{1}{2}[\partial(\mu V)/\partial p]_{s,1}(p_2 - p_1)^2 + \tfrac{1}{6}[\partial^2(\mu V)/\partial p^2]_{s,1}(p_2 - p_1)^3\} + O[(p_2 - p_1)(s_2 - s_1)] + \cdots .$$

By equating these two expressions for $\mu_2{}^2 - \mu_1{}^2$, we obtain

$$s_2 - s_1 = \left\{\frac{1}{12\mu T}\left[\frac{\partial^2(\mu V)}{\partial p^2}\right]_s\right\}_1 (p_2 - p_1)^3 + O[(p_2 - p_1)^4] . \tag{9}$$

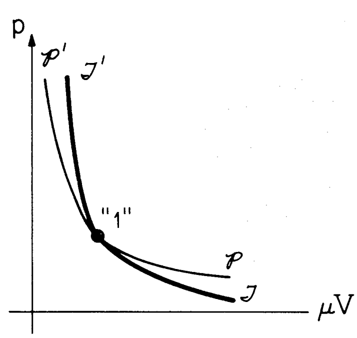

FIG. 2.—The Poisson (i.e. isentropic) adiabat $\mathscr{P} - \mathscr{P}'$ and the Taub (i.e., shock) adiabat $\mathscr{T} - \mathscr{T}'$ plotted in the $(\mu V, p)$-plane. Shown here are only those adiabats which pass through the point "1"—i.e., $(\mu_1 V_1, p_1)$ corresponding to the state of the fluid on the front face of the shock. In the text it is shown that (i) the two adiabats are tangent and have the same second derivative at "1"; and (ii) their common slope at "1" is negative. If, in addition, $[\partial^2 p/\partial(\mu V)^2]_s > 0$ at "1" ("usual case"; cf. text) and $[\partial(\mu V)/\partial s]_p > 0$ at "1" (Lichnerowicz's compressibility assumption; not assumed anywhere in this paper), then above and below the point "1" the adiabats have the shapes and relative locations shown here.

This equation for the jump in entropy as a function of the jump in pressure has as its Newtonian limit

$$s_{N2} - s_{N1} = \left[\frac{1}{12T_N}\left(\frac{\partial^2 V_N}{\partial p_N^2}\right)_{s_N}\right]_1 (p_{N2} - p_{N1})^3 + O[(p_{N2} - p_{N1})^4] \tag{9,N}$$

(LL eq. [83.1]). [In Newtonian analyses one used (V_N, p_N) diagrams, rather than $(\mu V, p)$ diagrams.]

The relativistic equation (9), like its Newtonian counterpart (9,N), guarantees that there are no first-order or second-order changes in entropy as one moves an infinitesimal "distance" $[\Delta(\mu V), \Delta p]$ along the shock adiabat away from the point $(\mu_1 V_1, p_1)$. Hence, the Taub adiabat and the Poisson adiabat are tangent and also have the same second derivatives at $(\mu_1 V_1, p_1)$; cf. figure 2. The common first derivative (tangent) is always negative, according to equation (3):

$$\frac{dp}{d(\mu V)} = -\left(\frac{u_S}{V}\right)^2 < 0 \begin{cases}\text{for Poisson adiabat anywhere;}\\ \text{for Taub adiabat at } (\mu_1 V_1, p_1)\,.\end{cases} \tag{10a}$$

The common second derivative can be written in the following forms (obtained by manipulating the relativistic laws of thermodynamics at fixed entropy, eqs. [1], [2], and [3]):

$$\frac{d^2 p}{d(\mu V)^2} = \left(\frac{u_S}{V}\right)^6 \frac{d^2(\mu V)}{dp^2} = \frac{v_S{}^2 \gamma_S{}^6}{\mu V^3}\left[\frac{2}{\gamma_S{}^2} + \left(\frac{\partial \ln v_S{}^2}{\partial \ln n}\right)_s\right]$$

$$= \frac{v_S{}^2 \gamma_S{}^6}{\mu V^3}\left[\frac{2}{\gamma_S{}^2} + \frac{(\Gamma - 1)\rho - p}{(\rho + p)} + \left(\frac{\partial \ln \Gamma}{\partial \ln n}\right)_s\right] \tag{10b}$$

$$\begin{cases}\text{for Poisson adiabat anywhere;}\\ \text{for Taub adiabat at } (\mu_1 V_1, p_1)\,.\end{cases}$$

To make this second derivative negative anywhere would require a highly unusual equation of state: one for which the speed of sound and adiabatic index plummet exceedingly rapidly with increasing density. One does not meet such equations of state in practice—either in Newtonian theory or in relativity (cf. LL, p. 327). But such equations of state would not violate any of the laws of thermodynamics. Henceforth, except in parenthetical remarks, we shall assume that the second derivative is positive throughout the $(\mu V, p)$-plane[3]:

$$\left[\frac{\partial^2 p}{\partial(\mu V)^2}\right]_s > 0 \text{ everywhere}\,. \tag{10c}$$

Then the Poisson adiabat is everywhere concave upward as drawn in figure 2; and the Taub adiabat is concave upward at the point $(\mu_1 V_1, p_1)$ as drawn in figures 2 and 3*a*.

The above results are a foundation for deriving the fundamental properties of weak shocks: Equation (9) for the Taub adiabat requires that the entropy change (albeit slightly) across a shock; and the second law of thermodynamics requires that change to be an increase

$$s_2 > s_1\,. \tag{11a}$$

[3] *Note added in proof:*—Israel (1960) shows that this condition is necessary to ensure that a compressive shock, once formed, will be maintained.

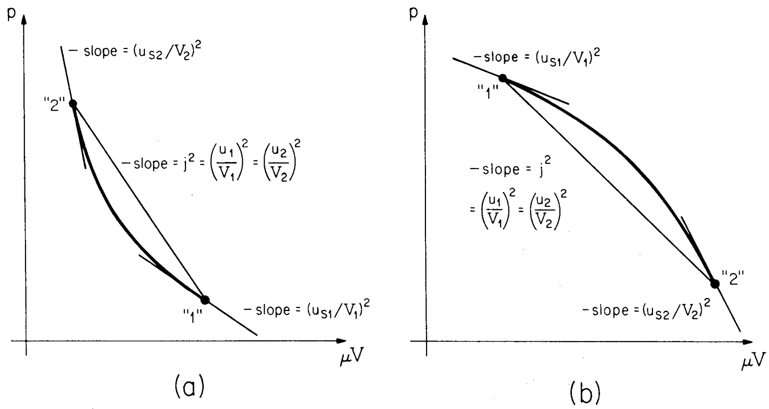

FIG. 3.—The Taub adiabat for weak shocks (same as Poisson adiabat except for third-order corrections, which are ignored here). Figure 3*a* is drawn for the realistic case $[\partial^2 p/\partial(\mu V)^2]_s > 0$—i.e., adiabat concave upward. Figure 3*b* is drawn for the hypothetical case $[\partial^2 p/\partial(\mu V)^2]_s < 0$—i.e., adiabat concave downward. The slopes of the chord and the tangents follow from equations (8a, b) and (3).

Consequently, for weak shocks the pressure also increases (cf. eq. [9]):

$$p_2 > p_1 . \tag{11b}$$

This means that the only physically relevant part of the Taub adiabat is that lying above the point $(\mu_1 V_1, p_1)$ in figure 2—i.e., the "upper branch." Equation (8c), combined with (11b), implies that the chemical potential increases across the shock:

$$\mu_2 > \mu_1 . \tag{11c}$$

Since μV must decrease (cf. fig. 2 or eqs. [10a] and [11b]), V must decrease even more than μ increases (gas gets compressed in shock):

$$V_2 < V_1 . \tag{11d}$$

This compression, together with simple graphical relations (fig. 3*a*), guarantees

$$u_2 < u_1 \tag{11e}$$

(fluid behind shock has a slower speed relative to shock than fluid in front),

$$u_1 > u_{S1} \tag{11f}$$

(flow in front of shock is supersonic),

$$u_2 < u_{S2} \tag{11g}$$

(flow behind shock is subsonic).

[In the hypothetical case of a fluid with $[\partial^2 p/\partial(\mu V)^2]_s < 0$ at the shock front, the entropy must still increase across the shock

$$s_2 > s_1 ; \tag{12a}$$

but now, for weak shocks, equation (9) demands a decrease in pressure

$$p_2 < p_1 . \tag{12b}$$

Thus, the lower branch of the Taub adiabat (region below point "1") is the physically relevant branch. Equation (8c), combined with (12b), implies a decrease in chemical potential

$$\mu_2 < \mu_1 . \tag{12c}$$

Since μV must increase (cf. fig. 3*b* or eqs. [10a] and [12b]), V must increase even more than μ decreases (gas expands in shock):

$$V_2 > V_1 . \tag{12d}$$

This expansion, together with simple graphical relations (fig. 3*b*), guarantees

$$u_2 > u_1 \tag{12e}$$

(fluid behind shock has a faster speed relative to shock than fluid in front),

$$u_1 > u_{S1} \tag{12f}$$

(flow in front of shock is supersonic), and

$$u_2 < u_{S2} \tag{12g}$$

(flow behind shock is subsonic). Note that all inequalities are reversed when one goes from the standard case of $[\partial^2 p/\partial(\mu V)^2]_s > 0$ to this hypothetical case of $[\partial^2 p/\partial(\mu V)^2]_s < 0$, with three exceptions: the entropy still increases across the shock, eq. (12a); the flow remains supersonic in front of the shock, eq. (12f); and the flow remains subsonic behind the shock, eq. (12g).]

One can derive an explicit expression for the Mach number $M = v/v_s$ in front of and behind a weak shock in the following way: expand equation (8b) in the forms

$$\begin{aligned} -\frac{1}{j^2} &= \left[\frac{\partial(\mu V)}{\partial p}\right]_{s,1} + \frac{1}{2}\left[\frac{\partial^2(\mu V)}{\partial p^2}\right]_{s,1}(p_2 - p_1) + \cdots \\ &= \left[\frac{\partial(\mu V)}{\partial p}\right]_{s,2} - \frac{1}{2}\left[\frac{\partial^2(\mu V)}{\partial p^2}\right]_{s,2}(p_2 - p_1) + \cdots ; \end{aligned}$$

insert expressions (8a) for $1/j^2$, (3) for $[\partial(\mu V)/\partial p]_s$, and (9) for $[\partial^2(\mu V)/\partial p^2]_{s,1}$; and manipulate using equation (1j). The results, valid independent of the sign of $[\partial^2(\mu V)/\partial p^2]_s$, are

$$M_1 = 1 + \left(\frac{3\mu T v_S^2}{V^2}\right)_1 \frac{s_2 - s_1}{(p_2 - p_1)^2} > 1 , \tag{13a}$$

$$M_2 = 1 - \left(\frac{3\mu T v_S^2}{V^2}\right)_1 \frac{s_2 - s_1}{(p_2 - p_1)^2} < 1 . \tag{13b}$$

V. STRONG RELATIVISTIC SHOCKS

Abandon, henceforth, the restriction to weak shocks, and consider shocks of arbitrary strength. An argument patterned after LL § 84 shows that *nowhere on the Taub adiabat can the "chord" (straight line linking points "1" and "2") become tangent to the adiabat (no point like $\mathcal{Q}$ in fig. 4).*

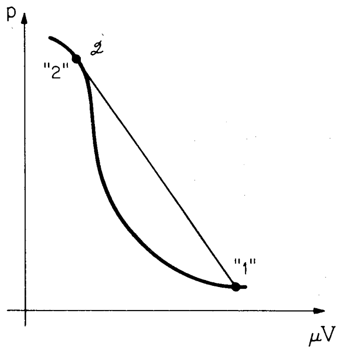

FIG. 4.—A hypothetical Taub adiabat for shocks of arbitrary strength, with a point $\mathscr{Q}$ at which the chord becomes tangent to the adiabat. Arguments given in the text show that such a point can never occur—if $[\partial^2 p/\partial(\mu V)^2]_s$ is positive everywhere.

The proof begins by constructing a differential equation for the squared baryon flux j^2 as a function of entropy s_2: take the equation for the Taub adiabat (8c) and differentiate it (holding fixed the thermodynamic variables on the front face of the shock) to obtain

$$0 = -2\mu_2 d\mu_2 + (\mu_1 V_1 + \mu_2 V_2)dp_2 + (p_2 - p_1)d(\mu_2 V_2) ;$$

to this equation add $2\mu_2 \times$ [first law of thermodynamics (2b)], obtaining

$$2\mu_2 T_2 ds_2 = (\mu_1 V_1 - \mu_2 V_2)dp_2 + (p_2 - p_1)d(\mu_2 V_2) ;$$

then combine this equation with the derivative of the Taub flux equation (8b) to obtain

$$dj^2 = \frac{2\mu_2 T_2}{(\mu_2 V_2 - \mu_1 V_1)^2}\, ds_2 \text{ along the Taub adiabat .} \tag{14}$$

Now, *assume* there exists a point at which the chord is tangent to the adiabat (point $\mathscr{Q}$ of fig. 4). Then, because the slope of the chord is $-j^2$, we must have

$$dj^2/dp_2 = 0 \text{ along the Taub adiabat at } \mathscr{Q} , \tag{15a}$$

which is equivalent, by equation (14), to

$$ds_2/dp_2 = 0 \text{ along the Taub adiabat at } \mathscr{Q} . \tag{15b}$$

Thus, the chord, the Taub adiabat, and the Poisson adiabat are all tangent at $\mathscr{Q}$. Their common slope is

$$(\text{slope of chord}) = -j^2 = -(u_2/V_2)^2$$
$$= (\text{slope of tangent}) = \left[\frac{\partial p}{\partial(\mu V)}\right]_s = -\left(\frac{u_{S2}}{V_2}\right)^2 ;$$

which means that the gas on the back of the shock at $\mathscr{Q}$ is moving sonically

$$u_2 = u_{S2} \text{ at } \mathscr{Q} . \tag{15c}$$

It is easy to verify that conditions (15a, b, c) are equivalent: at any point where one holds, the other two must hold. Choose for $\mathscr{Q}$ the *first* point, above $(\mu_2 V_2, p_2)$ along the Taub adiabat, at which these three conditions hold. Then, as one moves along the

Taub adiabat from point "1" to point $\mathscr{Q}$, the ratio $(u_{S2}/u_2)^2$ must begin at unity (limit of exceedingly weak shock; eq. [13b]); it must at first increase above unity (the back face of a weak shock is subsonic); and it must then decrease, falling to unity at $\mathscr{Q}$. Clearly, then,

$$\left[\frac{d(u_{S2}/u_2)^2}{dp_2}\right]_{\text{along Taub adiabat at }\mathscr{Q}} \leq 0\,. \tag{16a}$$

But equations (3) and (8a) imply that

$$\left(\frac{u_{S2}}{u_2}\right)^2 = \left[\frac{-1}{j^2}\left(\frac{\partial p}{\partial \mu V}\right)_s\right]_2;$$

and this, plus the tangency of Taub and Poisson adiabats at $\mathscr{Q}$, plus equation (15a), implies

$$\left[\frac{d(u_{S2}/u_2)^2}{dp_2}\right]_{\text{along Taub adiabat at }\mathscr{Q}} = -\left\{\frac{1}{j^2}\left[\frac{\partial(\mu V)}{\partial p}\right]_s\left[\frac{\partial^2 p}{\partial(\mu V)^2}\right]_s\right\}_{\text{at }\mathscr{Q}}$$

$$= \left\{\left(\frac{V}{ju_S}\right)^2\left[\frac{\partial^2 p}{\partial(\mu V)^2}\right]_s\right\}_{\text{at }\mathscr{Q}} > 0 \tag{16b}$$

(except in the hypothetical case of negative $[\partial^2 p/\partial(\mu V)^2]_s$). Equations (16a, b) contradict each other. Thus, the assumption must be wrong. No point of the form $\mathscr{Q}$ can exist. None of conditions (15a, b, c) can occur along the upper branch of the Taub adiabat.

This theorem enables one to conclude that the weak-shock inequalities (12) are valid also for strong shocks, and that certain quantities increase monotonically as one moves along the Taub adiabat away from point "1": In particular, consider all hypothetical adiabats in the $(\mu V, p)$-plane. They are all subject to the constraints that (i) the slope of the chord is always between 0 and $-\infty$ (eq. [8b]); (ii) the chord is nowhere tangent to the adiabat (nonexistence of points like $\mathscr{Q}$). By drawing one hypothetical adiabat after another, one quickly sees that (i)

$$-(\text{slope of chord}) = j^2 \qquad \text{increases monotonically along the adiabat} \tag{17a}$$

(the "stronger" the shock, the greater the baryon flux); (ii) p_2 might not always increase along the adiabat, but at least it always remains greater than p_1,

$$p_2 > p_1\,; \tag{17b}$$

and (iii) $\mu_2 V_2$ might not always decrease along the adiabat, but at least it always remains smaller than $\mu_1 V_1$,

$$\mu_2 V_2 < \mu_1 V_1\,. \tag{17c}$$

Equations (14) and (17a, c) imply that

$$s_2 \text{ increases monotonically along the adiabat} \tag{17d}$$

(the "stronger" the shock, the greater the jump in entropy). Equations (8c) and (17b) guarantee that the chemical potential always increases across the shock:

$$\mu_2 > \mu_1\,. \tag{17e}$$

This, combined with condition (17c), guarantees that the gas is always compressed in the shock:

$$V_2 < V_1 . \tag{17f}$$

The proof that points like $\mathscr{Q}$ cannot exist showed, as a corollary, that the flow behind the shock is always subsonic:

$$u_2 < u_{S2} . \tag{17g}$$

Because the chord steepens monotonically along the adiabat,

$$\frac{(\text{slope of chord})^2}{(\text{slope of tangent at "1"})^2} = \frac{-j^2}{-(u_{S1}/V_1)^2} = \left(\frac{u_1}{u_{S1}}\right)^2$$

increases monotonically; i.e., the flow in front of the shock becomes more and more supersonic:

$$u_1 > u_{S1} , \tag{17h}$$

$$u_1/u_{S1} \text{ increases monotonically along the adiabat} . \tag{17i}$$

VI. CONCLUDING REMARKS

It is impressive that the Newtonian analysis of shock adiabats can be generalized to relativity theory with such ease, and that the relativistic equations are virtually as simple as the Newtonian equations. Here, as elsewhere in physics, one has evidence of the power and elegance of a relativistic viewpoint.

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