Numerical Hydrodynamics

- Needed to follow the formation and evolution of *baryonic structures* inside the potential wells of Dark Matter in non-linear regime
- Two large classes of numerical methods: **lagrangian** and **eulerian**
- **Eulerian** methods: follow the *fluxes* of gas end energies in space. Derivatives evaluated at fixed points in space
- **Lagrangian** methods: follow the evolution of *fluid elements.* Derivatives in a coordinate system following the element.

- **Here I follow mainly:**
  Monaghan, Rep. Prog. Phys.,2005, 68, 1793
  Rosswog, New.A., 2009, 53, 78

Euler equations

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho}$$

Euler (dynamics)
Change in velocity

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{P}{\rho}\nabla \cdot \mathbf{v}$$

First law of thermodynamics
Change in energy

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\nabla\mathbf{v} = 0$$

Continuity
Change in mass

$$P = (\gamma - 1)\rho u$$

Equation of state
Ideal monoatomic (5/3) gas

v is the fluid velocity; **P** is the pressure; **u** is the internal energy; **ρ** is the density

$$\frac{d}{dt} = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{\partial}{\partial t} = \vec{v}\cdot\nabla + \frac{\partial}{\partial t}$$ Lagrangian derivative

From *mass conservation*, $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$ using $d\rho/dt = \vec{v}\cdot\nabla\rho + \partial\rho/\partial t$

we obtain the Lagrangian continuity equation, $\frac{d\rho}{dt} = -\rho\nabla\cdot\vec{v}.$

From first law of the thermodynamics, $dU = dQ - PdV$, using

$dV \to d(1/\rho) = -d\rho/\rho^2$ and the continuity equation (dQ is zero in adiabatic case)

$$du = \frac{P}{\rho^2}d\rho \longrightarrow \frac{du}{dt} = \frac{P}{\rho^2}\frac{d\rho}{dt} = -\frac{P}{\rho}\nabla\cdot\vec{v}.$$

Smoothed **P**article **H**ydrodynamics

- Basic principle: the fluid is sampled with points (*particles*)
- Hydrodynamic quantities are carried by each particle, but their value is *smoothed* on a given number of neighbouring particles
- Particle evolves under the Euler equations making use of the smoothed quantities
- After a *time-step*, quantities are re-evaluated….

Smoothing: the Kernel.

- A function f(r) is approximated by:

$$\tilde{f}_h(\vec{r}) = \int f(\vec{r'}) W(\vec{r} - \vec{r'}, h)\ d^3r'$$

  where **W** is the **kernel** (or window function)

- **h** is the *smoothing lenght* (physical scale over which the smoothing is done)
- To recover the original function in the limit of small h:

$$\lim_{h \to 0} \tilde{f}_h(\vec{r}) = f(\vec{r}) \qquad \int W(\vec{r} - \vec{r'}, h)\ d^3r' = 1$$

Discretization

$$\tilde{f}_h(\vec{r}) = \int \frac{f(\vec{r'})}{\rho(\vec{r'})} W(\vec{r} - \vec{r'}, h)\, \rho(\vec{r'}) d^3 r'$$

$$f(\vec{r}) = \sum_b \frac{m_b}{\rho_b} f_b W(\vec{r} - \vec{r}_b, h)$$

from which we obtain the SPH estimate of the density, when f=ρ

$$\boxed{\rho(\vec{r}) = \sum_b m_b W(\vec{r} - \vec{r}_b, h)}$$

This is the «central» formula of SPH formulation…
and is the continuity equation (when particle mass is kept fixed)

Derivatives

- By differentiating the discretization formula:

$$\nabla f(\vec{r}) = \sum_b \frac{m_b}{\rho_b} f_b \nabla W(\vec{r} - \vec{r}_b, h)$$

- …this does not vanishes if f=cost. Way to enforce it:

$$\frac{\partial A}{\partial x} = \frac{1}{\Phi}\left(\frac{\partial(\Phi A)}{\partial x} - A\frac{\partial \Phi}{\partial x}\right)$$

In SPH this is:

$$\left(\frac{\partial A}{\partial x}\right)_a = \frac{1}{\Phi_a}\sum_b m_b \frac{\Phi_b}{\rho_b}(A_b - A_a)\frac{\partial W_{ab}}{\partial x_a}$$

Zero if A=cost

- Taking $\Phi = 1$ or $\Phi = \rho$

$$\frac{\partial A_a}{\partial x_a} = \sum_b \frac{m_b}{\rho_b}(A_b - A_a)\frac{\partial W_{ab}}{\partial x_a} \qquad \frac{\partial A_a}{\partial x_a} = \frac{1}{\rho_a}\sum_b m_b(A_b - A_a)\frac{\partial W_{ab}}{\partial x_a}$$

where W_{ab} is $W(\boldsymbol{r}_a - \boldsymbol{r}_b, h)$

Generalizing to divergence for e.g. the continuity equation:

$$\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = \sum_b m_b \boldsymbol{v}_{ab} \cdot \nabla_a W_{ab} \qquad \frac{\mathrm{d}\rho_a}{\mathrm{d}t} = \rho_a \sum_b \frac{m_b}{\rho_b} \boldsymbol{v}_{ab} \cdot \nabla_a W_{ab}$$

Second order derivatives

- Differentiating again: too noisy
- A good approximation is:

$$(\nabla^2 f)_a = 2\sum_b \frac{m_b}{\rho_b}(f_a - f_b)\frac{w_{ab}}{r_{ab}}$$

- …however, higher order derivatives are a pain for SPH…
- Not straightforward to solve diffusion equation!

Kernel function

- Additional properties: radial (conservation of angular momentum); compact support (avoid n^2 interactions per particle) $W(\vec{r}-\vec{r}', h) = W(|\vec{r}-\vec{r}'|, h)$
- Accuracy: control error with

$$f(\vec{r'}) = \sum_{k=0}^{\infty} \frac{(-1)^k h^k f^{(k)}(\vec{r})}{k!} \left(\frac{\vec{r}-\vec{r}'}{h}\right)^k$$

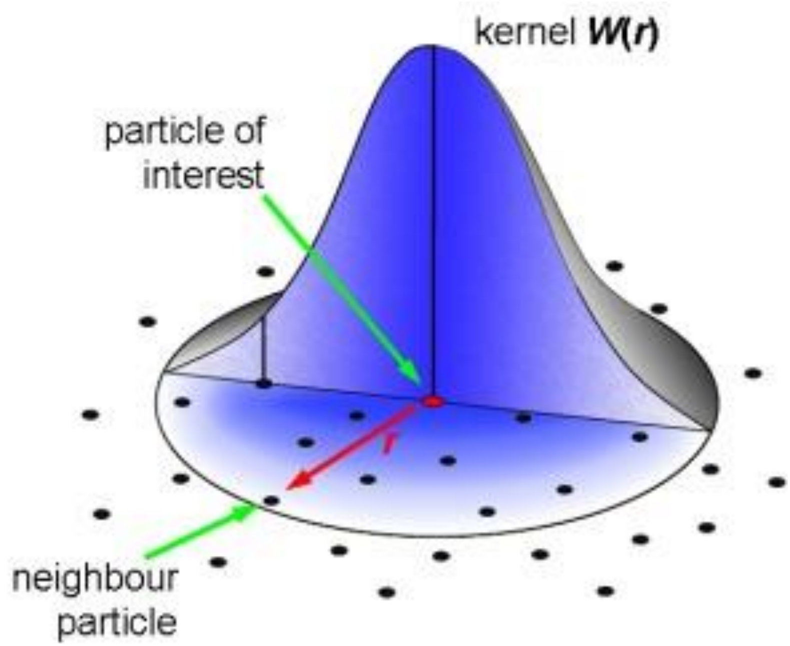
  write down integrals and construct desired order.
- In practice, **cubic spline kernel**:

$$W(q) = \frac{1}{\pi h^3} \begin{cases} 1 - \frac{3}{2}q^2 + \frac{3}{4}q^3 & \text{for } 0 \le q \le 1 \\ \frac{1}{4}(2-q)^3 & \text{for } 1 < q \le 2 \\ 0 & \text{for } q > 2 \end{cases}$$

Errors: $$\tilde{f}_h(\vec{r}) = f(\vec{r}) + C\ h^2 + O(h^4)$$

C contains 2° order derivatives of f(r). Constants and linear funcions are exact.

- Gaussian kernel used mainly for theoretical calculations (non compact)
- Higher order now studied (more accurate)
- ....the meaning...:



- Other sources of errors (numerical noise, derivatives...)

(Properties of radial kernels)

Throughout the text, we use the notation $\vec{r}_{bk} = \vec{r}_b - \vec{r}_k$, $r_{bk} = |\vec{r}_{bk}|$ and $\vec{v}_{bk} = \vec{v}_b - \vec{v}_k$. Derivatives resulting from the smoothing lengths are ignored for the moment, they will receive particular attention in Sect. 3. By straight-forward component-wise differentiation one finds

$$\frac{\partial}{\partial \vec{r}_a} |\vec{r}_b - \vec{r}_k| = \frac{(\vec{r}_b - \vec{r}_k)(\delta_{ba} - \delta_{ka})}{|\vec{r}_b - \vec{r}_k|} = \hat{e}_{bk}(\delta_{ba} - \delta_{ka}), \tag{21}$$

where $\hat{e}_{bk}$ is the unit vector from particle k to particle b,

$$\frac{\partial}{\partial \vec{r}_a} \frac{1}{|\vec{r}_b - \vec{r}_k|} = -\frac{\hat{e}_{bk}(\delta_{ba} - \delta_{ka})}{|\vec{r}_b - \vec{r}_k|^2} \tag{22}$$

and δ_{ij} is the usual Kronecker symbol. Another frequently needed expression is

$$\begin{aligned} \frac{dr_{ab}}{dt} &= \frac{\partial r_{ab}}{\partial x_a}\frac{dx_a}{dt} + \frac{\partial r_{ab}}{\partial y_a}\frac{dy_a}{dt} + \frac{\partial r_{ab}}{\partial z_a}\frac{dz_a}{dt} + \frac{\partial r_{ab}}{\partial x_b}\frac{dx_b}{dt} + \frac{\partial r_{ab}}{\partial y_b}\frac{dy_b}{dt} + \frac{\partial r_{ab}}{\partial z_b}\frac{dz_b}{dt} \\ &= \nabla_a r_{ab} \cdot \vec{v}_a + \nabla_b r_{ab} \cdot \vec{v}_b = \nabla_a r_{ab} \cdot \vec{v}_a - \nabla_a r_{ab} \cdot \vec{v}_b = \nabla_a r_{ab} \cdot \vec{v}_{ab} = \hat{e}_{ab} \cdot \vec{v}_{ab}, \end{aligned} \tag{23}$$

where $\partial r_{ab}/\partial x_b = -\partial r_{ab}/\partial x_a$ etc. was used. For kernels that only depend on the magnitude of the separation, $W(\vec{r}_b - \vec{r}_k) = W(|\vec{r}_b - \vec{r}_k|) \equiv W_{bk}$, the derivative with respect to the coordinate of an arbitrary particle a is

$$\nabla_a W_{bk} = \frac{\partial}{\partial \vec{r}_a} W_{bk} = \frac{\partial W_{bk}}{\partial r_{bk}} \frac{\partial r_{bk}}{\partial \vec{r}_a} = \frac{\partial W_{bk}}{\partial r_{bk}} \hat{e}_{bk}(\delta_{ba} - \delta_{ka}) = \nabla_b W_{kb}(\delta_{ba} - \delta_{ka}), \tag{24}$$

where Eq. (21) was used. This yields in particular the important property

$$\boxed{\nabla_a W_{ab} = \frac{\partial}{\partial \vec{r}_a} W_{ab} = \frac{\partial W_{ab}}{\partial r_{ab}} \frac{\partial r_{ab}}{\partial \vec{r}_a} = \frac{\partial W_{ab}}{\partial r_{ab}} \hat{e}_{ab} = -\frac{\partial W_{ab}}{\partial r_{ab}} \frac{\partial r_{ab}}{\partial \vec{r}_b} = -\frac{\partial}{\partial \vec{r}_b} W_{ab} = -\nabla_b W_{ab}.} \tag{25}$$

The time derivative of the kernel is given by

$$\boxed{\frac{dW_{ab}}{dt} = \frac{\partial W_{ab}}{\partial r_{ab}} \frac{dr_{ab}}{dt} = \frac{\partial W_{ab}}{\partial r_{ab}} \frac{(\vec{r}_a - \vec{r}_b) \cdot (\vec{v}_a - \vec{v}_b)}{r_{ab}} = \frac{\partial W_{ab}}{\partial r_{ab}} \hat{e}_{ab} \cdot \vec{v}_{ab} = \vec{v}_{ab} \cdot \nabla_a W_{ab}.} \tag{26}$$

Euler equation

- Brute force differentiation:

$$\frac{d\vec{v}_a}{dt} = -\frac{1}{\rho_a}\sum_b \frac{m_b}{\rho_b} P_b \nabla_a W_{ab}$$

- But:

$$\vec{F}_{ba} = \left(m_a \frac{d\vec{v}_a}{dt} \right)_b = -\frac{m_a}{\rho_a}\frac{m_b}{\rho_b} P_b \nabla_a W_{ab}$$

$$\vec{F}_{ab} = \left(m_b \frac{d\vec{v}_b}{dt} \right)_a = -\frac{m_b}{\rho_b}\frac{m_a}{\rho_a} P_a \nabla_b W_{ba} = \frac{m_a}{\rho_a}\frac{m_b}{\rho_b} P_a \nabla_a W_{ab}$$

and if Pa ≠ Pb, momentum not conserved

- Thus use $\nabla\left(\frac{P}{\rho}\right)=\frac{\nabla P}{\rho}-P\frac{\nabla\rho}{\rho^2}$ to obtain $\nabla P/\rho$

- Writing gradients in SPH formulations we get

$$\begin{aligned}\frac{d\vec{v}_a}{dt} &= -\frac{P_a}{\rho_a^2}\sum_b m_b \nabla_a W_{ab} - \sum_b \frac{m_b}{\rho_b}\frac{P_b}{\rho_b}\nabla_a W_{ab} \\ &= -\sum_b m_b\left(\frac{P_a}{\rho_a^2}+\frac{P_b}{\rho_b^2}\right)\nabla_a W_{ab}.\end{aligned}$$

- Since eq. is symmetric in a,b and $\nabla_a W_{ab} = -\nabla_b W_{ba}$ now momentum is conserved.
- If the smoothing lenght varies, to have conservation:

$$\nabla_a W_{ab} \longrightarrow [\nabla_a W(r_{ab},h_a)+\nabla_a W(r_{ab},h_b)]/2$$

Energy equation

$$\frac{du_a}{dt} = \frac{P_a}{\rho_a^2}\frac{d\rho_a}{dt} = \frac{P_a}{\rho_a^2}\frac{d}{dt}\left(\sum_b m_b W_{ab}\right) = \frac{P_a}{\rho_a^2}\sum_b m_b \vec{v}_{ab}\cdot\nabla_a W_{ab}$$

where we used:

$$\frac{dW_{ab}}{dt} = \frac{\partial W_{ab}}{\partial r_{ab}}\frac{dr_{ab}}{dt} = \frac{\partial W_{ab}}{\partial r_{ab}}\frac{(\vec{r}_a - \vec{r}_b)\cdot(\vec{v}_a - \vec{v}_b)}{r_{ab}} = \frac{\partial W_{ab}}{\partial r_{ab}}\hat{e}_{ab}\cdot\vec{v}_{ab} = \vec{v}_{ab}\cdot\nabla_a W_{ab}.$$

Equation of state

$$P_a = (\gamma - 1)\rho_a u_a$$

…these make a full set of SPH equations.

Conservation properties

- SPH equations do conserve momentum, energy and angular momentum
- Exercise: show that the above quantities are conservet, that is, show that the quantities

$$\sum_a m_a \frac{d\vec{v}_a}{dt} \qquad \frac{d\vec{L}}{dt} = \sum_a \vec{M}_a \qquad \frac{dE}{dt} = \frac{d}{dt}\sum_a \left(m_a u_a + \frac{1}{2} m_a v_a^2 \right)$$

(where $\vec{M}_a = \vec{r}_a \times \vec{F}_a$ is the torque)

are all zero.

Hint: use simmetry properties. Write SPH equations only for energy.

Solution: Rosswog pages 15-16

Adaptive resolution

- Meaning of the smoothing lenght: minimum lenght scale above which results have a physical meaning
- Adaptive resolution needed when the problems involves a wide dynamical range (…example: formation of cosmic structures!)
- Many prescriptions: the most common is to keep the number of gas particles fixed, or to keep fixed the gas mass enclosed in a sphere of radius h if particle mass can vary
- *Formally SPH equations hold if h is fixed. We need to rederive them to allow a varying h.*

Lagrangian derivation of SPH equations

$$L = \int \rho \left(\frac{v^2}{2} - u(\rho, s) \right) dV \longrightarrow L_{\mathrm{SPH,h}} = \sum_b m_b \left(\frac{v_b^2}{2} - u(\rho_b, s_b) \right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_a} \right) - \frac{\partial L}{\partial \vec{r}_a} = 0$$

Euler-Lagrange equations

$$\frac{\partial L}{\partial \vec{v}_a} = \frac{\partial}{\partial \vec{v}_a} \left[\sum_b m_b \left(\frac{v_b^2}{2} - u(\rho_b, s_b) \right) \right] = m_a \vec{v}_a \longrightarrow$$

First term is $m_a \frac{d\vec{v}_a}{dt}$

$$\frac{\partial L}{\partial \vec{r}_a} = \frac{\partial}{\partial \vec{r}_a} \left[\sum_b m_b \left(\frac{v_b^2}{2} - u(\rho_b, s_b) \right) \right] = -\sum_b m_b \left. \frac{\partial u_b}{\partial \rho_b} \right|_s \cdot \frac{\partial \rho_b}{\partial \vec{r}_a}$$

Using: $\left(\frac{\partial u}{\partial \rho} \right)_s = \frac{P}{\rho^2}$

We get: $$m_a \frac{d\vec{v}_a}{dt} = -\sum_b m_b \frac{P_b}{\rho_b^2} \frac{\partial \rho_b}{\partial \vec{r}_a}$$

- To take into account the change oh the smoothing lenght:

$$\frac{d\rho_a}{dt} = \frac{d}{dt}\left(\sum_b m_b W_{ab}(h_a)\right) = \sum_b m_b \left\{\frac{\partial W_{ab}(h_a)}{\partial r_{ab}}\frac{dr_{ab}}{dt} + \frac{\partial W_{ab}(h_a)}{\partial h_a}\frac{dh_a}{dt}\right\}$$

$$= \sum_b m_b \frac{\partial W_{ab}(h_a)}{\partial r_{ab}}\hat{e}_{ab}\cdot\vec{v}_{ab} + \sum_b m_b \frac{\partial W_{ab}(h_a)}{\partial h_a}\cdot\frac{\partial h_a}{\partial \rho_a}\frac{d\rho_a}{dt}$$

$$= \sum_b m_b \vec{v}_{ab}\cdot\nabla_a W_{ab}(h_a) + \frac{\partial h_a}{\partial \rho_a}\frac{d\rho_a}{dt}\sum_b m_b \frac{\partial W_{ab}(h_a)}{\partial h_a},$$

Using $\boxed{\Omega_a \equiv \left(1 - \frac{\partial h_a}{\partial \rho_a}\sum_b m_b \frac{\partial W_{ab}(h_a)}{\partial h_a}\right)}$ We obtain:

$$\boxed{\frac{d\rho_a}{dt} = \frac{1}{\Omega_a}\sum_b m_b \vec{v}_{ab}\cdot\nabla_a W_{ab}(h_a)}$$

And similarly for the spatial derivative:

$$\frac{\partial \rho_b}{\partial \vec{r}_a} = \sum_k m_k \left\{\nabla_a W_{bk}(h_b) + \frac{\partial W_{bk}(h_b)}{\partial h_b}\frac{\partial h_b}{\partial \rho_b}\frac{\partial \rho_b}{\partial \vec{r}_a}\right\}$$

$$= \frac{1}{\Omega_b}\sum_k m_k \nabla_a W_{bk}(h_b).$$

- ...recovering $\frac{du_a}{dt} = \frac{P_a}{\rho_a^2}\frac{d\rho_a}{dt}$ and inserting the new $\frac{d\rho_a}{dt}$:

$$\frac{du_a}{dt} = \frac{1}{\Omega_a}\frac{P_a}{\rho_a^2}\sum_b m_b \vec{v}_{ab} \cdot \nabla_a W_{ab}(h_a)$$

- For the momentum equation (derived fom E-L):

$$m_a \frac{d\vec{v}_a}{dt} = -\sum_b m_b \frac{P_b}{\rho_b^2}\nabla_a \rho_b = -\sum_b m_b \frac{P_b}{\rho_b^2}\left(\frac{1}{\Omega_b}\sum_k m_k \nabla_a W_{bk}(h_b)\right)$$

Usıng $\nabla_a W_{bk} = \nabla_b W_{kb}(\delta_{ba} - \delta_{ka})$:

$$\begin{aligned}
m_a \frac{d\vec{v}_a}{dt} &= -\sum_b m_b \frac{P_b}{\rho_b^2}\frac{1}{\Omega_b}\sum_k m_k \nabla_b W_{kb}(h_b)\ (\delta_{ba} - \delta_{ka}) \\
&= -m_a \frac{P_a}{\rho_a^2}\frac{1}{\Omega_a}\sum_k m_k \nabla_a W_{ka}(h_a) + \sum_b m_b \frac{P_b}{\rho_b^2}\frac{1}{\Omega_b} m_a \nabla_b W_{ab}(h_b) \\
&= -m_a \frac{P_a}{\rho_a^2}\frac{1}{\Omega_a}\sum_b m_b \nabla_a W_{ba}(h_a) - m_a \sum_b m_b \frac{P_b}{\rho_b^2}\frac{1}{\Omega_b}\nabla_a W_{ab}(h_b) \\
&= -m_a \sum_b m_b \left(\frac{P_a}{\Omega_a \rho_a^2}\nabla_a W_{ab}(h_a) + \frac{P_b}{\Omega_b \rho_b^2}\nabla_a W_{ab}(h_b)\right).
\end{aligned}$$

- …thus, finally we have

$$\frac{d\vec{v}_a}{dt} = -\sum_b m_b \left(\frac{P_a}{\Omega_a \rho_a^2} \nabla_a W_{ab}(h_a) + \frac{P_b}{\Omega_b \rho_b^2} \nabla_a W_{ab}(h_b) \right)$$

- Here, formally equations are correct even if h vary. Note the «automatic» simmetrization.

Entropy formulation and Lagrangian constraints

- Standard formulation of SPH does not conserve explicitly entropy. To get conservation:
- Define $P = A(s)\rho^{\gamma}$ where P is the pressure, A(s) an *entropic function*
- From eq. of state, we get $u = \frac{A(s)}{\gamma - 1}\rho^{\gamma-1}$
- …integrating $\frac{dA}{dt} = 0$ is equivalent to integrating the energy equation
- *Lagrangian constraints*: we require that a sphere of radius *h* contains a fixed mass

$$\phi_i(\boldsymbol{q}) \equiv \frac{4\pi}{3} h_i^3 \rho_i - M_{\rm sph} = 0$$

- The Lagrangian is now:

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i^2 - \frac{1}{\gamma - 1}\sum_{i=1}^{N} m_i A_i \rho_i^{\gamma-1}$$

where independent variables are $\boldsymbol{q} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N, h_1, \ldots, h_N)$

- Euler-Lagrange equations become:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{N} \lambda_j \frac{\partial \phi_j}{\partial q_i}$$

- Second N equations give:

$$\lambda_i = \frac{3}{4\pi}\frac{m_i}{h_i^3}\frac{P_i}{\rho_i^2}\left[1 + \frac{3\rho_i}{h_i}\left(\frac{\partial \rho_i}{\partial h_i}\right)^{-1}\right]^{-1}$$

- Using which, the first N:

$$m_i \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\sum_{j=1}^{N} m_j \frac{P_j}{\rho_j^2}\left[1 + \frac{h_j}{3\rho_j}\frac{\partial \rho_j}{\partial h_j}\right]^{-1} \nabla_i \rho_j$$

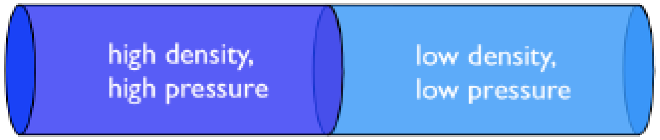
- Now using $$\nabla_i \rho_j = m_i \nabla_i W_{ij}(h_j) + \delta_{ij} \sum_{k=1}^{N} m_k \nabla_i W_{ki}(h_i)$$

- We have $$\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\sum_{j=1}^{N} m_j \left[f_i \frac{P_i}{\rho_i^2} \nabla_i W_{ij}(h_i) + f_j \frac{P_j}{\rho_j^2} \nabla_i W_{ij}(h_j) \right]$$

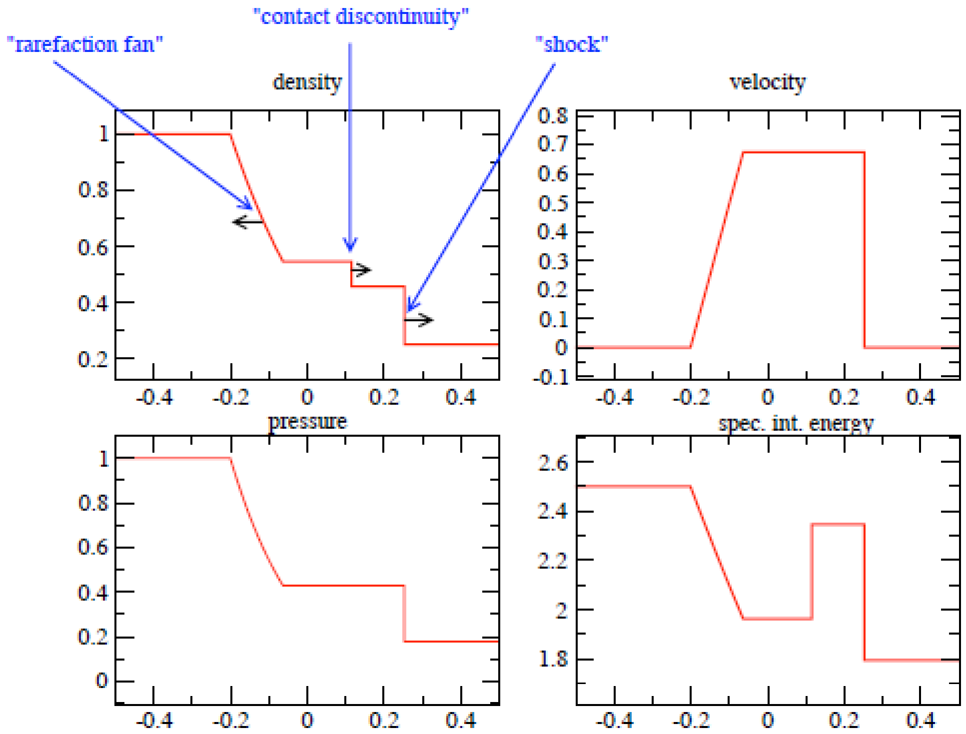
$$f_i = \left(1 + \frac{h_i}{3\rho_i} \frac{\partial \rho_i}{\partial h_i}\right)^{-1}$$

- …similar to the previous derivation, but holds formally for varying h with the constraint that the spere they define around each SPH particle contains a fixed gas mass M_{sph}.
- This derivation from Springel & Hernquist, 2005, MNRAS, 333,649

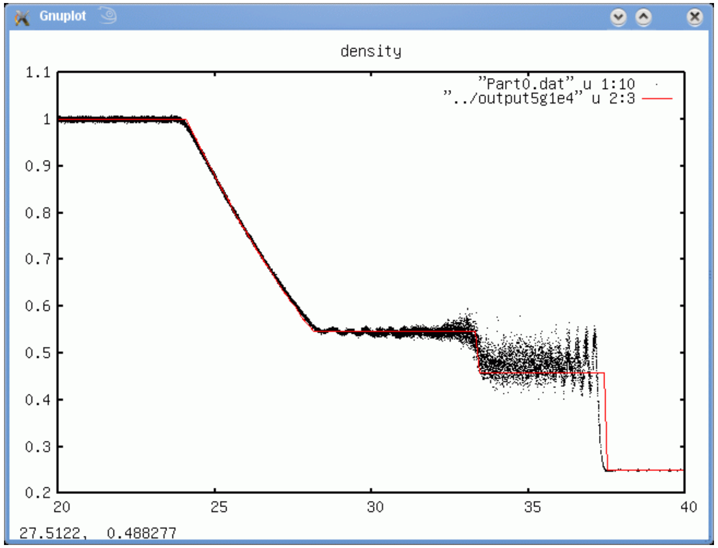
…now what?

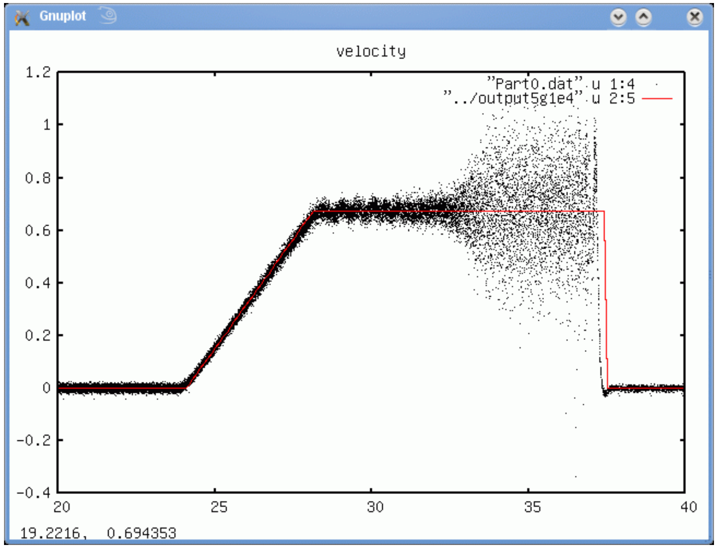


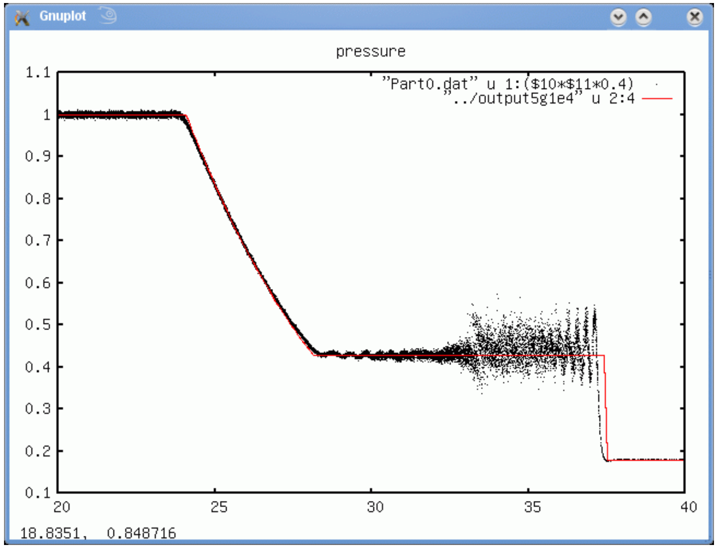
- Let's use these SPH equation to integrate a simple test case, the shock tube.
- 1D; exact solution (..later we'll see..)



- Let’s use a SPH code and perform the experiment….:







!!!!!

Artificial viscosity

- To avoid that, we have to add a VISCOSITY TERM to the equations!

$$\left(\frac{P_a}{\rho_a^2}+\frac{P_b}{\rho_b^2}\right)\rightarrow\left(\frac{P_a}{\rho_a^2}+\frac{P_b}{\rho_b^2}+\Pi_{ab}\right)$$

- Bulk viscosity: $q_{\mathrm{b}}=-c_1\rho c_{\mathrm{s}}l(\nabla\cdot\vec{v})$ von Neumann-Richmyer: $q_{\mathrm{NR}}=c_2\rho l^2(\nabla\cdot\vec{v})^2$
- Taylor expand: $\left(\frac{\partial v}{\partial x}\right)_a=\frac{v_b-v_a}{x_b-x_a}+O\left((x_b-x_a)^2\right)$
- From which, the form is: ($x_{ab}=x_a-x_b$ $v_{ab}=v_a-v_b$ $\frac{1}{x_{ab}}\rightarrow\frac{x_{ab}}{x_{ab}^2+\epsilon\bar{h}_{ab}^2}$)

$$\Pi_{ab,\mathrm{bulk}}=\begin{cases}-c_1\frac{\bar{c}_{s,ab}}{\bar{\rho}_{ab}}\mu_{ab} & \text{for } x_{ab}\,v_{ab}<0\\ 0 & \text{otherwise}\end{cases},\text{where}\quad \mu_{ab}=\frac{\bar{h}_{ab}\,x_{ab}\,v_{ab}}{x_{ab}^2+\epsilon\bar{h}_{ab}^2}$$

- Summing up bulk and vNR, and with usual notation for this ARTIFICIAL viscosity is

$$\Pi_{ab} = \Pi_{ab,\mathrm{bulk}} + \Pi_{ab,\mathrm{NR}} = \begin{cases} \frac{-\alpha \bar{c}_{ab}\mu_{ab} + \beta\mu_{ab}^2}{\bar{\rho}_{ab}} & \text{for } \vec{r}_{ab}\cdot\vec{v}_{ab} < 0 \\ 0 & \text{otherwise} \end{cases}$$

- SPH equations with artificial viscosity:

$$\frac{d\vec{v}_a}{dt} = -\sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} + \Pi_{ab} \right) \nabla_a W_{ab}.$$

$$\frac{du_a}{dt} = \sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{1}{2}\Pi_{ab} \right) \vec{v}_{ab}\cdot\nabla_a W_{ab}$$

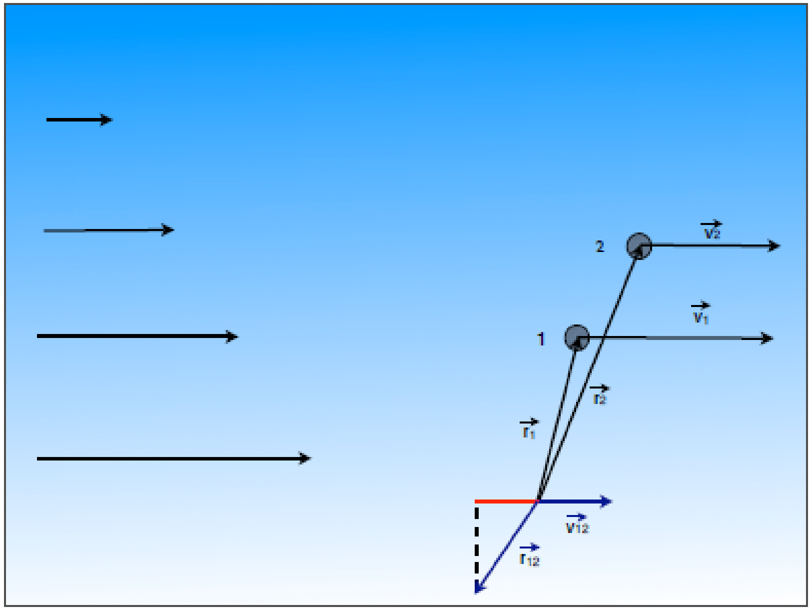
$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = \frac{1}{2}\frac{\gamma-1}{\rho_i^{\gamma-1}} \sum_{j=1}^{N} m_j \Pi_{ij} \boldsymbol{v}_{ij} \cdot \nabla_i \bar{W}_{ij}$$

Reduced viscosity

- «Balsara switch»

$$f_a = \frac{|\langle \nabla \cdot \vec{v} \rangle_a|}{|\langle \nabla \cdot \vec{v} \rangle_a| + |\langle \nabla \times \vec{v} \rangle_a| + 0.0001 c_{s,a}/h_a}$$

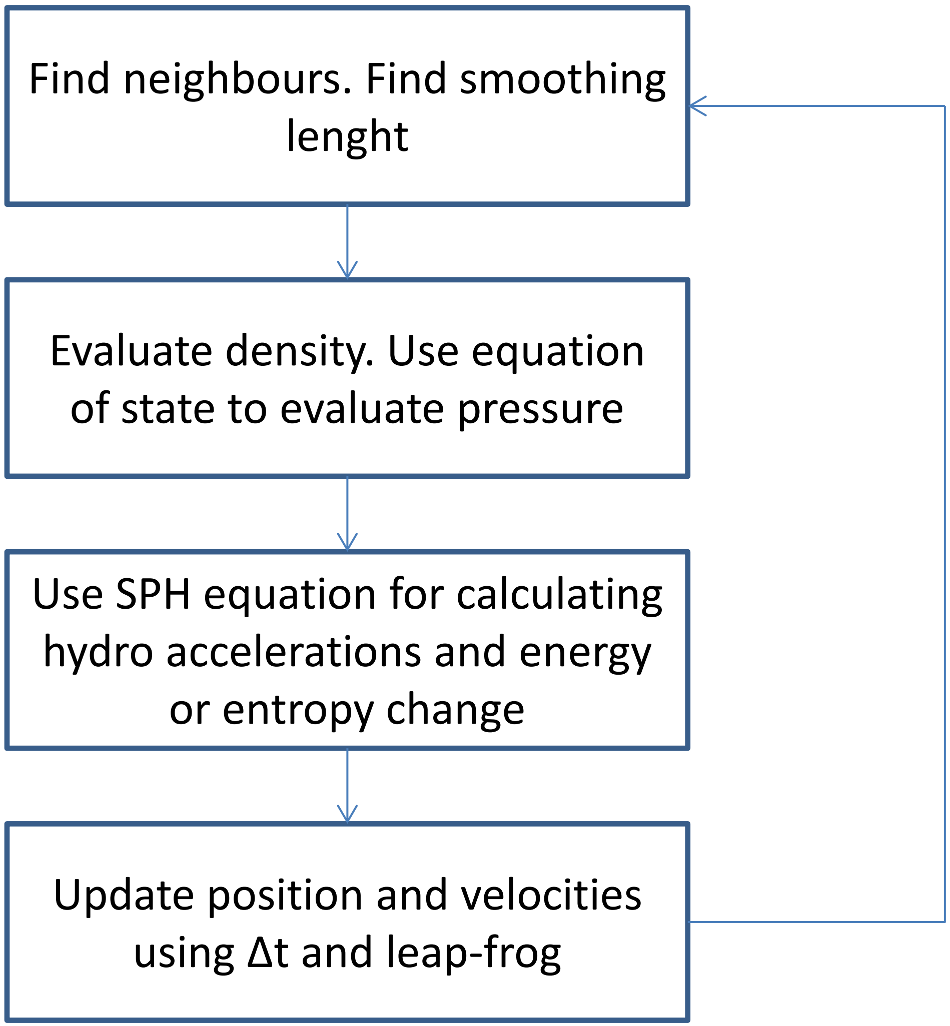
$$\Pi'_{ab} = \Pi_{ab} \; \bar{f}_{ab}$$



Other attempt to introduced form of reduced viscosity have been introduced in literature

Integration and timestepping

- Simple(r) scheme for an SPH:



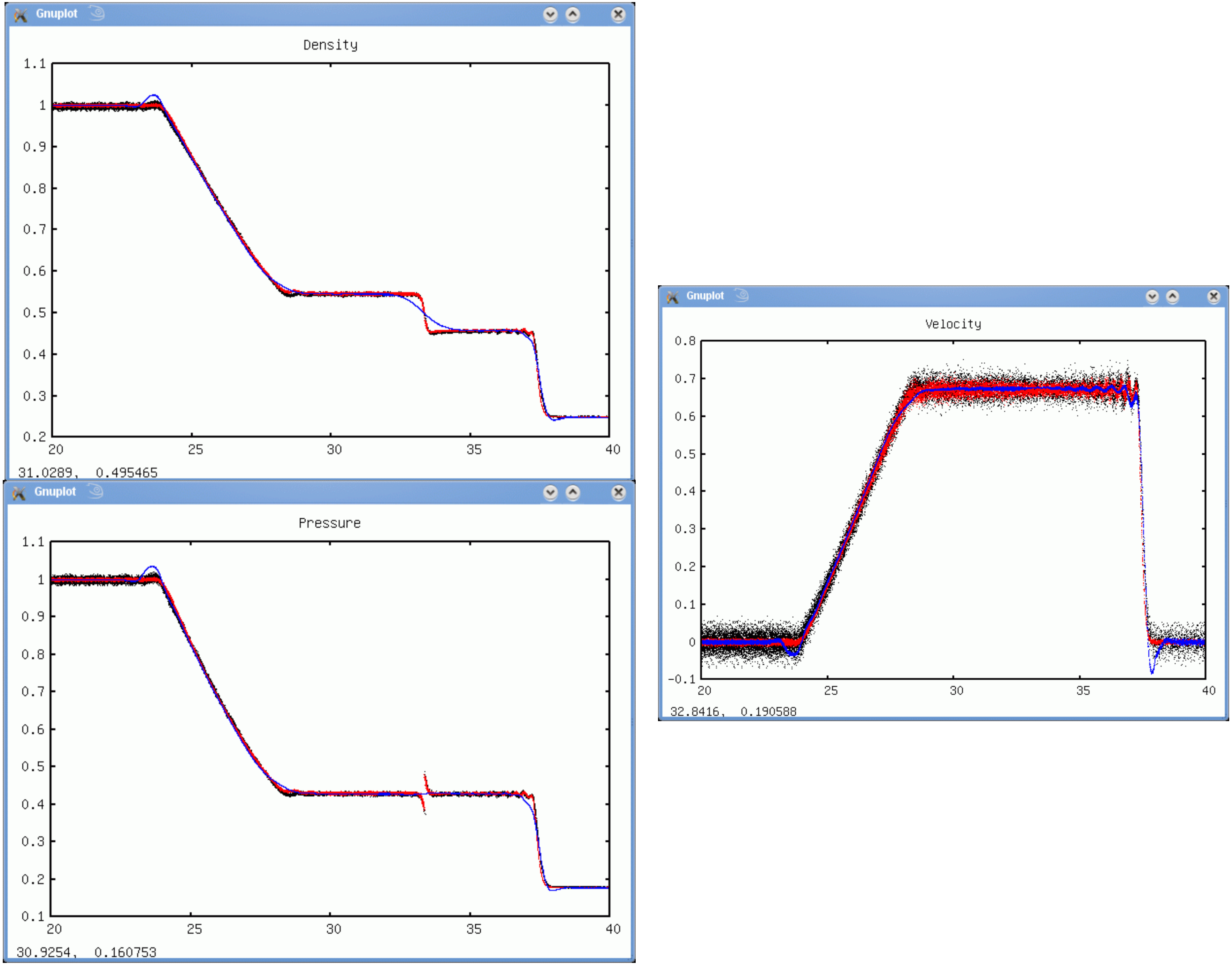
Courant criterion

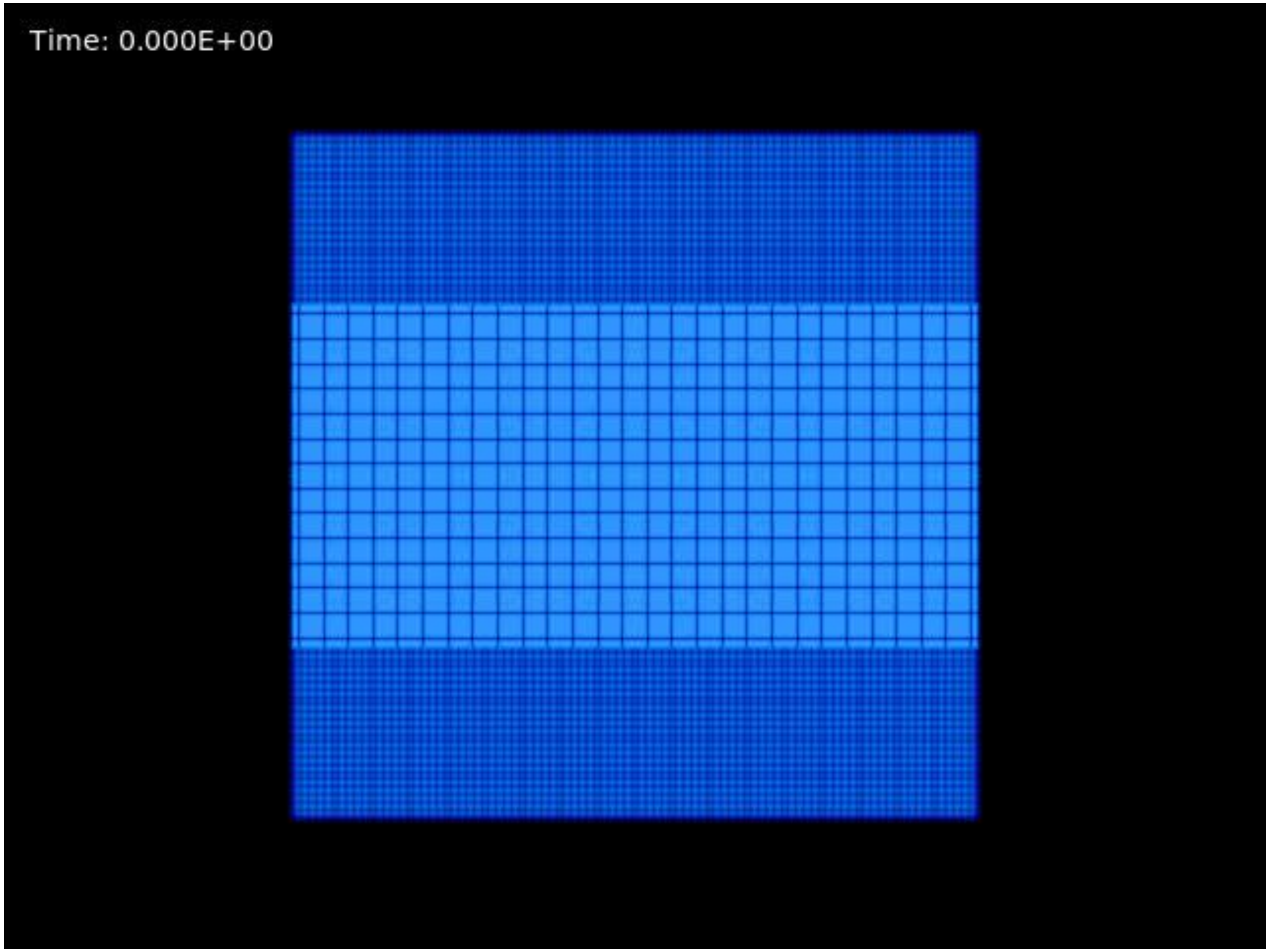
- Additional condition to appropriately capture changes in velocity and energy:

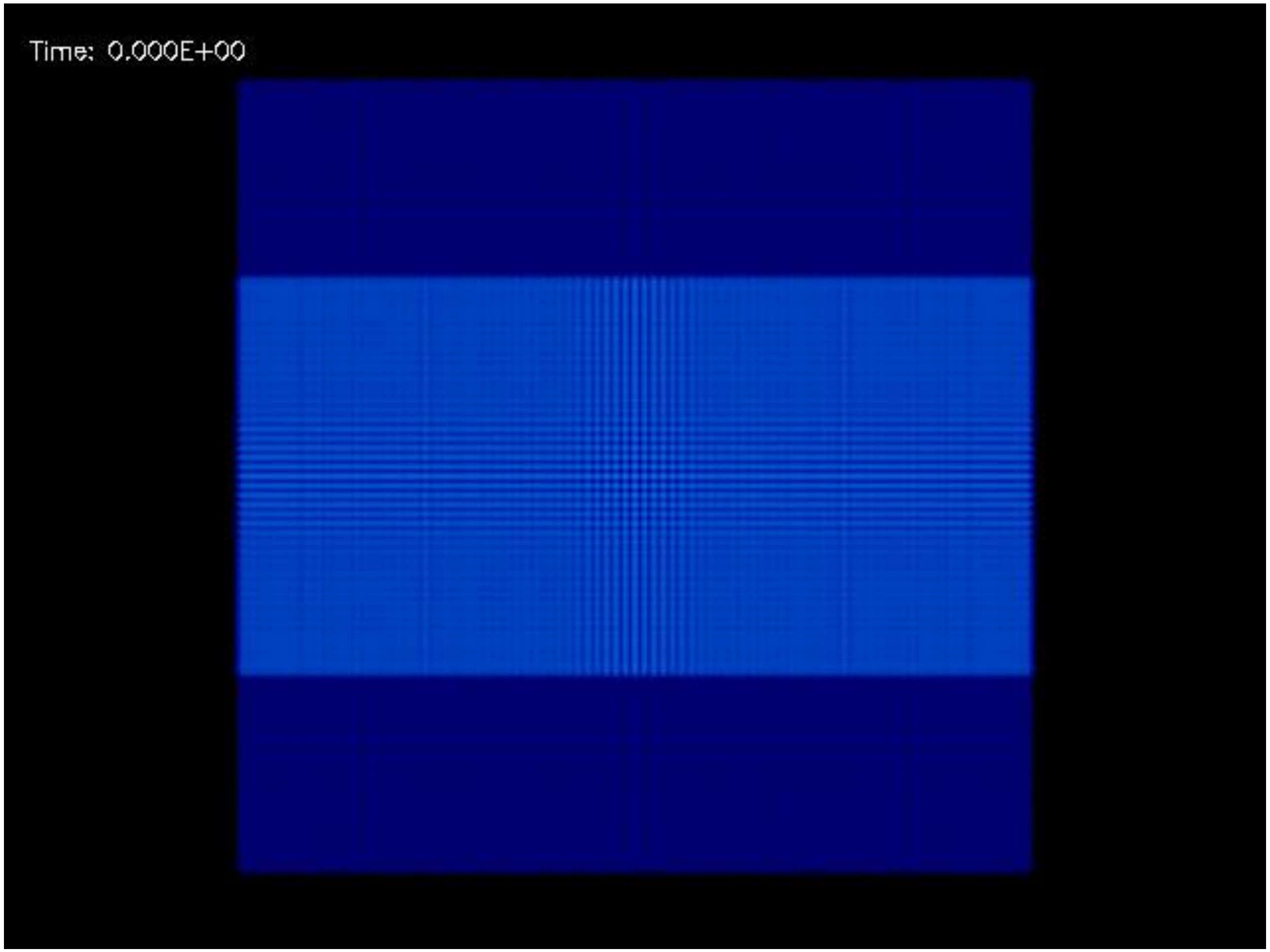
$$\Delta t < \Delta x / c_s$$

- Applying to SPH, with viscosity:

$$\Delta t_{CV,a} \propto \frac{h_a}{v_{\mathrm{s},a} + 0.6(c_{\mathrm{s},a} + 2\max_b \mu_{ab})}$$


Gnuplot
Density
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
20
25
30
35
40
31.0289, 0.495465
Gnuplot
Pressure
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
20
25
30
35
40
30.9254, 0.160753
Gnuplot
Velocity
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
20
25
30
35
40
32.8416, 0.190588


Time: 0.000E+00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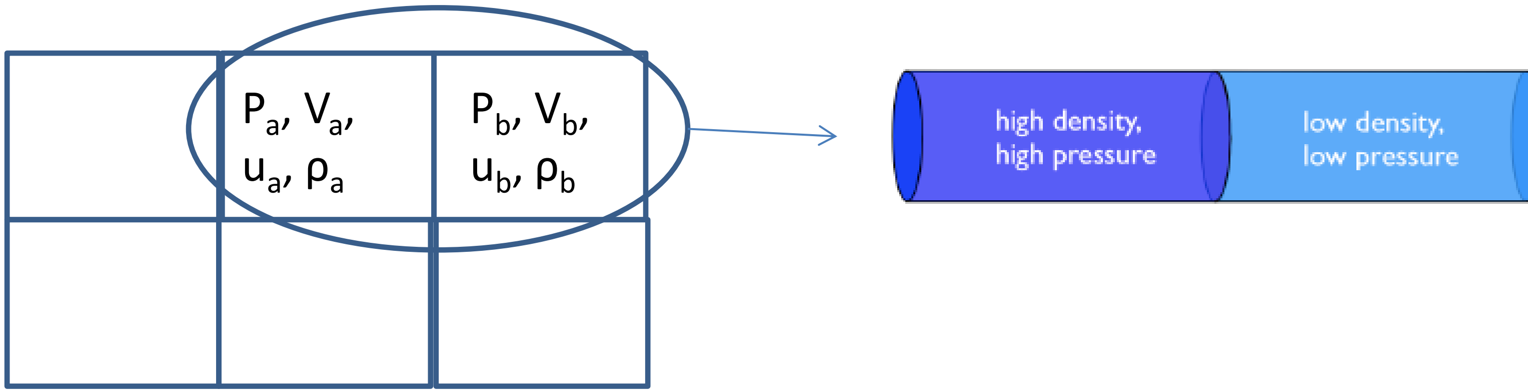
They happen in real life!!

Hints on Eulerian techniques

- Space is partitioned onto a MESH (e.g., cartesian)
- Hydrodynamical quantities assigned to mesh points



- …but the Riemann problem can be solved *exactly*!
- Solve Riemann problem -> find fluxes -> update quantities
- (things are by far more complex than this)

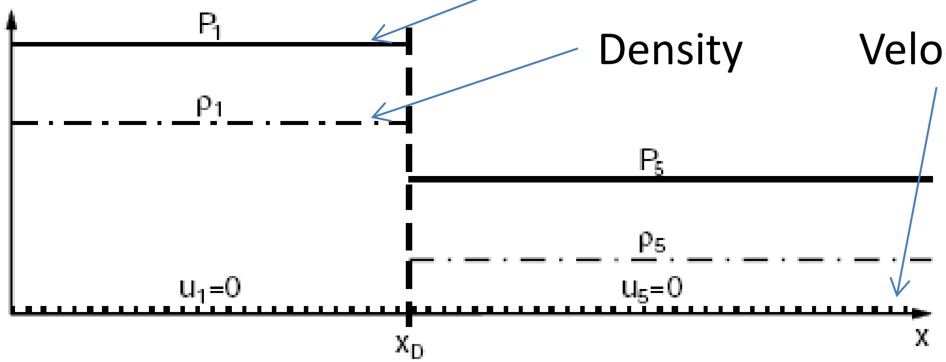
- Here I follow Borgani et al, Space Science Review, 2008, 134, 229

The solution of the Riemann problem (shock tube)

Pressure

Density

Velocity



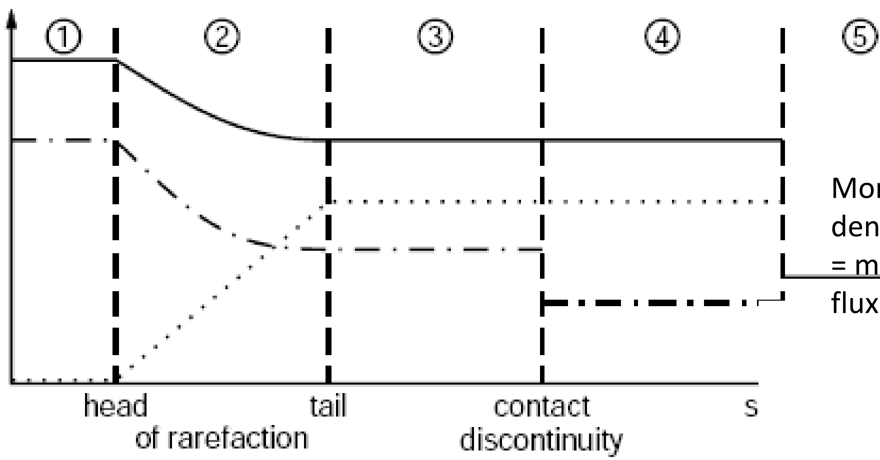
We fix:

$v_1 = v_5 = 0$

We need:

$\rho_3,\ P_3,\ v_3$

$\rho_4,\ P_4,\ v_4$



Rankine-Hugoniot (JUMP) conditions (conservation of mass, momentum, energy):

Momentum density * velocity = momentum flux

density*velocity=flux

Pressure IS a momentum flux

$$\rho_l v_l = \rho_r v_r$$

$$\rho_l v_l^2 + P_l = \rho_r v_r^2 + P_r$$

$$v_l \left(\rho_l (v_l^2/2 + u_l) + P_l \right) = v_r \left(\rho_r (v_r^2/2 + u_r) + P_r \right)$$

Energy density

A momentum flux IS an energy density…

- No mass flux and pressure costant across contact discontinuity:

$$v_3 = v_4 = v_c \qquad P_3 = P_4 = P_c$$

- From first condition (SR moving with the shock)

$W_5=V_5 - V_s$
$W_4=V_4 - V_s$

$$m = \rho_5 v_s = \rho_4 (v_s - v_c) \longrightarrow v_s = \frac{\rho_4 v_c}{\rho_4 - \rho_5}$$

- From second condition:

$$m v_c = \rho_4 (v_s - v_c) v_c = P_c - P_5$$

- ..combined with first:

$$\rho_4 \left(\frac{\rho_4 v_c}{\rho_4 - \rho_5} - v_c \right) v_c = P_c - P_5 \longrightarrow \text{(A)} \quad (P_5 - P_c) \left(\frac{1}{\rho_5} - \frac{1}{\rho_4} \right) = -v_c^2$$

- From third condition:

$$m(\epsilon_4 + \frac{v_c^2}{2} - \epsilon_5) = P_c v_c \xrightarrow{\text{Eliminating m}} \epsilon_4 - \epsilon_5 = \frac{P_c + P_5}{P_c - P_5} \frac{v_c^2}{2}$$

- Which becomes, using ideal gas equation of state and (A):

$$\frac{1}{\gamma-1}\left(\frac{P_c}{\rho_4}-\frac{P_5}{\rho_5}\right)=\frac{P_c+P_5}{2\rho_4\rho_5} \longrightarrow \quad \text{(B)} \quad \frac{P_c-P_5}{P_c+P_5}=\gamma\frac{\rho_4-\rho_5}{\rho_4+\rho_5}$$

- Entropy is constant through rarefaction wave:

$$S \propto \ln P/\rho^{\gamma} \longrightarrow \quad \text{(C)} \quad \frac{P_1}{P_c}=\left(\frac{\rho_1}{\rho_3}\right)^{\gamma}$$

- The Riemann invariant $v+\int\frac{c}{\rho}\mathrm{d}\rho$ is constant, thus

$$v_c+\int\frac{c_3}{\rho_3}\mathrm{d}\rho=v_1+\int\frac{c_1}{\rho_1}\mathrm{d}\rho,$$ but $c=\sqrt{\gamma P/\rho}$ and solving:

$$\int\frac{c}{\rho}\mathrm{d}\rho=\frac{2}{\gamma-1}\sqrt{\frac{\gamma P}{\rho}} \longrightarrow \quad \text{(D)} \quad v_c+\frac{2}{\gamma-1}\sqrt{\frac{\gamma P_c}{\rho_3}}=\frac{2}{\gamma-1}\sqrt{\frac{\gamma P_1}{\rho_1}}.$$

- Putting togheter (A), (B), (C), (D):

$$\frac{\rho_1}{\rho_5}\frac{1}{\lambda}\frac{(1-P)^2}{\gamma(1+P)-1+P} = \frac{2\gamma}{(\gamma-1)^2}\left[1-\left(\frac{P}{\lambda}\right)^{(\gamma-1)/(2\gamma)}\right]^2$$

- Where $\lambda = \rho_1/\rho_5$ and $P = P_c/P_5$;

…..from which we get $\mathbf{P_c}$

- All of the other quantities can be obtained from (A…D)
- Solution only depends on states 1,5 (left and right)
- From the solution, fluxes of quantities between interface can be obtained.

- More general solutions for the evolution of a discontinuity separating two states can be obtained.

(Riemann invariants)

- Constant along the characteristic curves of quasi-linear hyperbolic first order partial differential equations
- Characteristic curves: curves (tercnically, hypersurfaces) along which the PDE can be transformed in an ODE and solved
- The solution is transformed in a solution of the original PDE
- For the Euler equation, it can be shown that twoof these are

$$d\rho - \frac{dP}{c^2} \quad , dv \pm \frac{dP}{\rho c} \quad \rightarrow dv - \left(\frac{\mathrm{c}}{\rho}\right) d\rho = cost$$

which, using the equation of state, can be integrated to

$$(\frac{1}{\gamma})d(\rho^{\gamma}/P)$$

and if the entropy is $S=P/\rho^{\gamma}$, it's constant along the path.

Equations in conservative form

- Conserved quantities: $q = \{ \rho, \rho v, \rho(u + v^2/2) \}$ – they correspond to flux densities
- Integral form of equations:

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{F} = 0$$

where **F** are the corresponding flux densities operators

$$\{ \rho v, \rho v^2 + P, \rho(u + v^2/2) + P) \}$$

- Continuity equation: in primitive variables is

$$\frac{\partial \rho}{\partial t} + v \nabla v + \rho \nabla v = 0$$

- …deriving the first flux density operator, we get it

- Euler, in Eulerian coordinates:

$$\frac{\partial v}{\partial t} - v\nabla v = -\frac{\nabla P}{\rho}$$

Deriving the second flux density we get:

$$\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} = -\nabla(\rho v^2 + P) = -2\rho v\nabla v - v^2\nabla\rho - \nabla P$$

But from the continuity equation, multiplied by v:

$$v^2\nabla\rho = -v\frac{\partial \rho}{\partial t} - \rho v\nabla v$$

Substituting we get

$$\rho\frac{\partial v}{\partial t} - \rho v\nabla v = -\nabla P$$

…which is the Euler equation multiplied by ρ.

- Try the Energy equation!

- This formulation allows us to use the theory of hyperbolic partial differential equations

$$\partial_t \mathbf{Q} + \mathbf{A}\partial_x \mathbf{Q} = \mathbf{0}$$

- …which gives flexible solutions for the quantities once we have the fluxes from the Riemann problem (…no more details!)
- Applied to a cell **l**, the equations read

$$\frac{\partial q(\mathbf{l},t)}{\partial t} + \sum_{k=1}^{3} \frac{[F_k(\ell_k + 1/2, t) - F_k(\ell_k - 1/2, t)]}{\Delta x} = 0$$

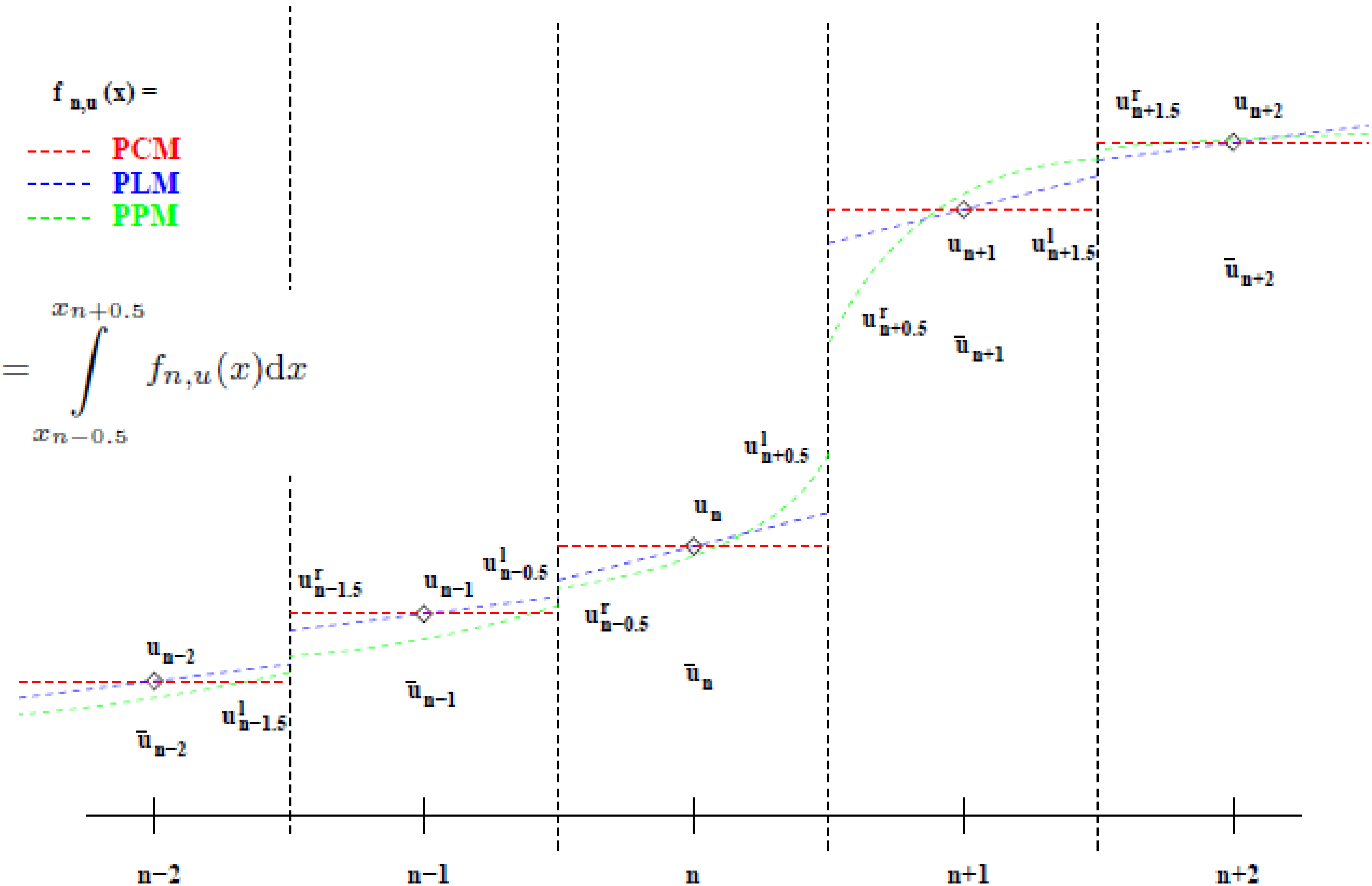
from which the temporal evolution is simply:

$$q(\mathbf{l}, t + \Delta t) = q(\mathbf{l}, t) - \frac{\Delta t}{\Delta x} \sum_{k=1}^{3} [F_k(\ell_k + 1/2, t) - F_k(\ell_k - 1/2, t)]$$

Problem: fluxes at the interfaces, values in grid cells (or averages into the cell). What values should we use to solve the Riemann problem?

Reconstruction and limiters

- The simplest scheme is very **dissipative**: N shocks per cell!

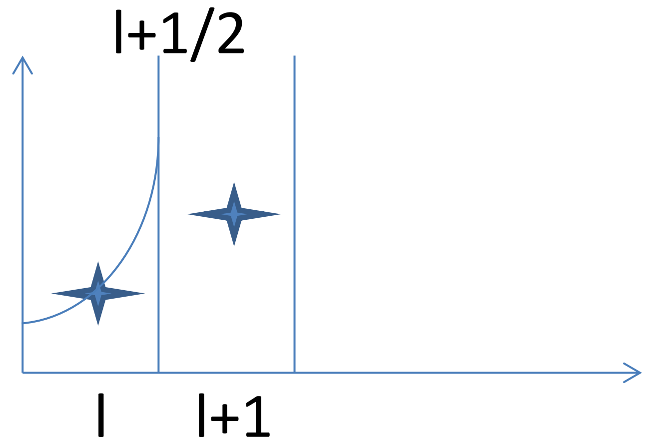


$$\hat{u}_n = \int_{x_{n-0.5}}^{x_{n+0.5}} f_{n,u}(x)\mathrm{d}x$$

- For the ***reconstruction***, informations from neighbouring cells (*stencils*) are used (e.g., gradients). Many different types of reconstructions exist.
- ***Limiters*** are used, as additional constraint on the reconstruction, to avoid unphysical oscillations in the solution
- Older schemes (PLM; PPM): **P**iecewise **L**inear **M**ethod, **P**iecewise **P**arabolic **M**ethod.
- Newer schemes (ENO, WENO, MP): (Weighted) Essentially Non Oscillatory, Monotonicity Preserving. Use more grid points. They provide high order reconstriction in smooth part of the fluid, and sharp discontinuities at the shocks.
- **From 1D to 3D: operator splitting**. Recently, *unsplit* methods have been introduced.

Limiter

- …basically if values vary strongly, interpolated quantities at the interface could be outside the interval defined by the cell-averaged values

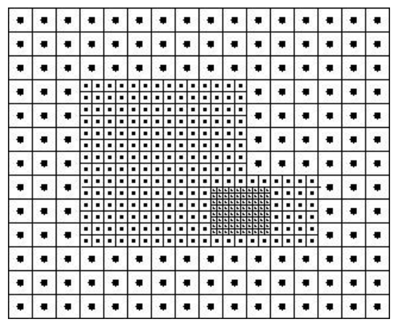


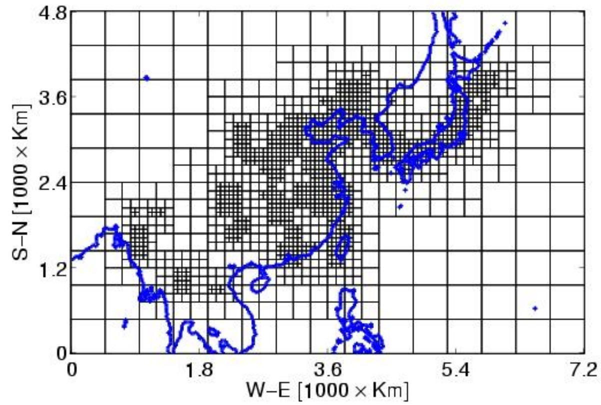
- This can determine **unphysical oscillations** in the solution

$$\delta Q_\ell \longrightarrow \delta_m Q_\ell = \begin{cases} \min(|\delta Q_\ell|, 2|Q^n_\ell - Q^n_{\ell-1}|, 2|Q^n_\ell - Q^n_{\ell+1}|) \times \mathrm{sgn}(\delta Q_\ell) \\ \qquad\qquad \text{if} \quad (Q^n_{\ell+1} - Q^n_\ell)(Q^n_\ell - Q^n_{\ell-1}) > 0; \\ 0 \quad \text{otherwise.} \end{cases}$$

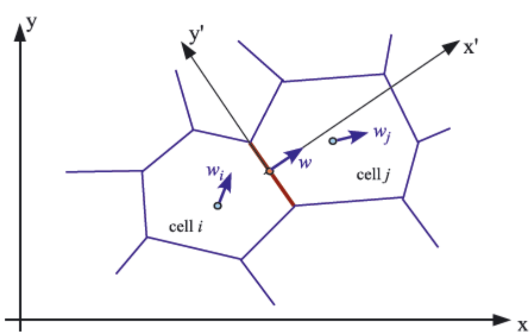
Eulerian Codes

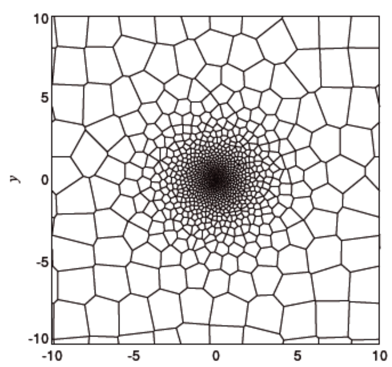
- Uniform grid: fixed resolution.
- AMR, Adaptive Mesh Refinement: a hierarchy of nested grid are introduced to increase resolution where it is needed

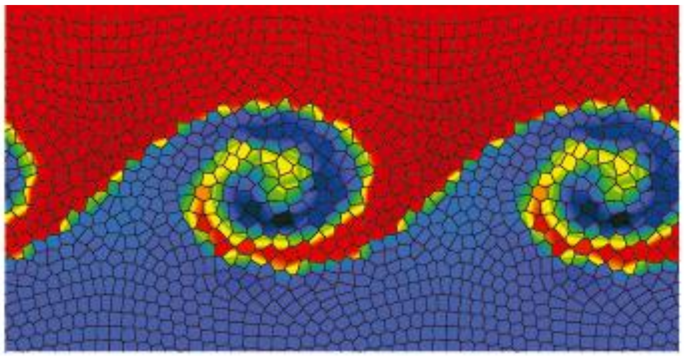




- Moving mesh (e.g., AREPO)







- …meshless.

Eulerian *vs* Lagrangian: pros and cons

- SPH automatically increases resolution where needed
- Easily coupled with N-Body codes
- Discretization makes it a *model* of Euler equations: difficulties in capturing (potentially important) hydro instabilities

- Eulerian methods are an exact solution of fluid dynamics
- Not manifestely Galileian invariant
- Difficult to increase resolution

- …what method to use depends on the physical problem!

- In both fields, efforts to circumvent problems.

Cooling

- Baryon physics is **radiative**. Gas cools.
- Include additional term to the energy equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{P}{\rho}\nabla \cdot \mathbf{v} - \frac{\Lambda(u,\rho)}{\rho}$$

- Assuming gas optically thin, in ionization equilibrium, ignoring three-body cooling:

$$\Lambda(\rho, u) \rightarrow \frac{\Lambda(u)}{n^2}$$

cooling rate *per squared density*, that is, the cooling rate goes with the density squared.

- Dependance on temperature usually tabulated
- Metal cooling is important!

From Katz et al, 1996:

$$\Gamma_{\mathrm{eH_0}} n_{\mathrm{e}} n_{\mathrm{H_0}} + \Gamma_{\gamma \mathrm{H_0}} n_{\mathrm{H_0}} = \alpha_{\mathrm{H_+}} n_{\mathrm{H_+}} n_{\mathrm{e}}, \quad (25)$$

$$\Gamma_{\mathrm{eHe_0}} n_{\mathrm{He_0}} n_{\mathrm{e}} + \Gamma_{\gamma \mathrm{He_0}} n_{\mathrm{He_0}} = (\alpha_{\mathrm{He_+}} + \alpha_{\mathrm{d}}) n_{\mathrm{He_+}} n_{\mathrm{e}}, \quad (26)$$

$$\Gamma_{\mathrm{eHe_+}} n_{\mathrm{He_+}} n_{\mathrm{e}} + \Gamma_{\gamma \mathrm{He_+}} n_{\mathrm{He_+}} + (\alpha_{\mathrm{He_+}} + \alpha_{\mathrm{d}}) n_{\mathrm{He_+}} n_{\mathrm{e}} = \alpha_{\mathrm{He_{++}}} n_{\mathrm{He_{++}}} n_{\mathrm{e}} + \Gamma_{\mathrm{eHe_0}} n_{\mathrm{He_0}} n_{\mathrm{e}} + \Gamma_{\gamma \mathrm{He_0}} n_{\mathrm{He_0}}, \quad (27)$$

$$\alpha_{\mathrm{He_{++}}} n_{\mathrm{He_{++}}} n_{\mathrm{e}} = \Gamma_{\mathrm{eHe_+}} n_{\mathrm{He_+}} n_{\mathrm{e}} + \Gamma_{\gamma \mathrm{He_+}} n_{\mathrm{He_+}}. \quad (28)$$

Process	Species	Rate[a]
Collisional excitation	H^0	$7.50 \times 10^{-19}\, e^{-118348.0/T}(1+T_5^{1/2})^{-1} n_{\mathrm{e}} n_{\mathrm{H_0}}$
	He^+	$5.54 \times 10^{-17}\, T^{-0.397} e^{-473638.0/T}(1+T_5^{1/2})^{-1} n_{\mathrm{e}} n_{\mathrm{He_+}}$
Collisional ionization	H^0	$1.27 \times 10^{-21}\, T^{1/2} e^{-157809.1/T}(1+T_5^{1/2})^{-1} n_{\mathrm{e}} n_{\mathrm{H_0}}$
	He^0	$9.38 \times 10^{-22}\, T^{1/2} e^{-285335.4/T}(1+T_5^{1/2})^{-1} n_{\mathrm{e}} n_{\mathrm{He_0}}$
	He^+	$4.95 \times 10^{-22}\, T^{1/2} e^{-631515.0/T}(1+T_5^{1/2})^{-1} n_{\mathrm{e}} n_{\mathrm{He_+}}$
Recombination	H^+	$8.70 \times 10^{-27}\, T^{1/2} T_3^{-0.2}(1+T_6^{0.7})^{-1} n_{\mathrm{e}} n_{\mathrm{H_+}}$
	He^+	$1.55 \times 10^{-26}\, T^{0.3647} n_{\mathrm{e}} n_{\mathrm{He_+}}$
	He^{++}	$3.48 \times 10^{-26}\, T^{1/2} T_3^{-0.2}(1+T_6^{0.7})^{-1} n_{\mathrm{e}} n_{\mathrm{He_{++}}}$
Dielectronic recombination	He^+	$1.24 \times 10^{-13}\, T^{-1.5} e^{-470000.0/T}\left(1+0.3e^{-94000.0/T}\right) n_{\mathrm{e}} n_{\mathrm{He_+}}$
free-free	all ions	$1.42 \times 10^{-27}\, g_{\mathrm{ff}} T^{1/2}(n_{\mathrm{H_+}} + n_{\mathrm{He_+}} + 4n_{\mathrm{He_{++}}}) n_{\mathrm{e}}$

Table 1: Cooling Rates ($\mathrm{erg\,s^{-1}\,cm^{-3}}$)

$$\begin{aligned}
\alpha_{\mathrm{H_+}} &= 8.4 \times 10^{-11}\, T^{-1/2} T_3^{-0.2}(1+T_6^{0.7})^{-1} \\
\alpha_{\mathrm{He_+}} &= 1.5 \times 10^{-10}\, T^{-0.6353} \\
\alpha_{\mathrm{d}} &= 1.9 \times 10^{-3}\, T^{-1.5} e^{-470000.0/T}\left(1+0.3e^{-94000.0/T}\right) \\
\alpha_{\mathrm{He_{++}}} &= 3.36 \times 10^{-10}\, T^{-1/2} T_3^{-0.2}(1+T_6^{0.7})^{-1} \\
\Gamma_{\mathrm{eH_0}} &= 5.85 \times 10^{-11}\, T^{1/2} e^{-157809.1/T}(1+T_5^{1/2})^{-1} \\
\Gamma_{\mathrm{eHe_0}} &= 2.38 \times 10^{-11}\, T^{1/2} e^{-285335.4/T}(1+T_5^{1/2})^{-1} \\
\Gamma_{\mathrm{eHe_+}} &= 5.68 \times 10^{-12}\, T^{1/2} e^{-631515.0/T}(1+T_5^{1/2})^{-1}
\end{aligned}$$

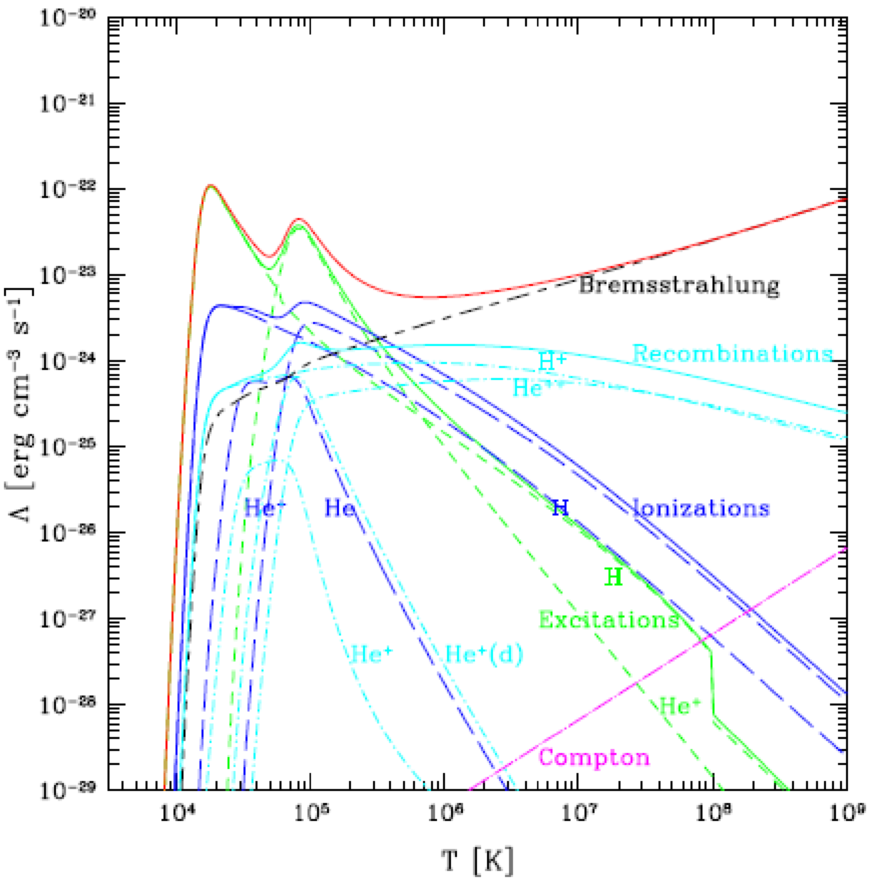
Table 2: Recombination and collisional ionization rates ($\mathrm{cm^3\,s^{-1}}$)

$$g_{\mathrm{ff}} = 1.1 + 0.34 \exp\left[-(5.5 - \log_{10} T)^2/3.0\right]$$

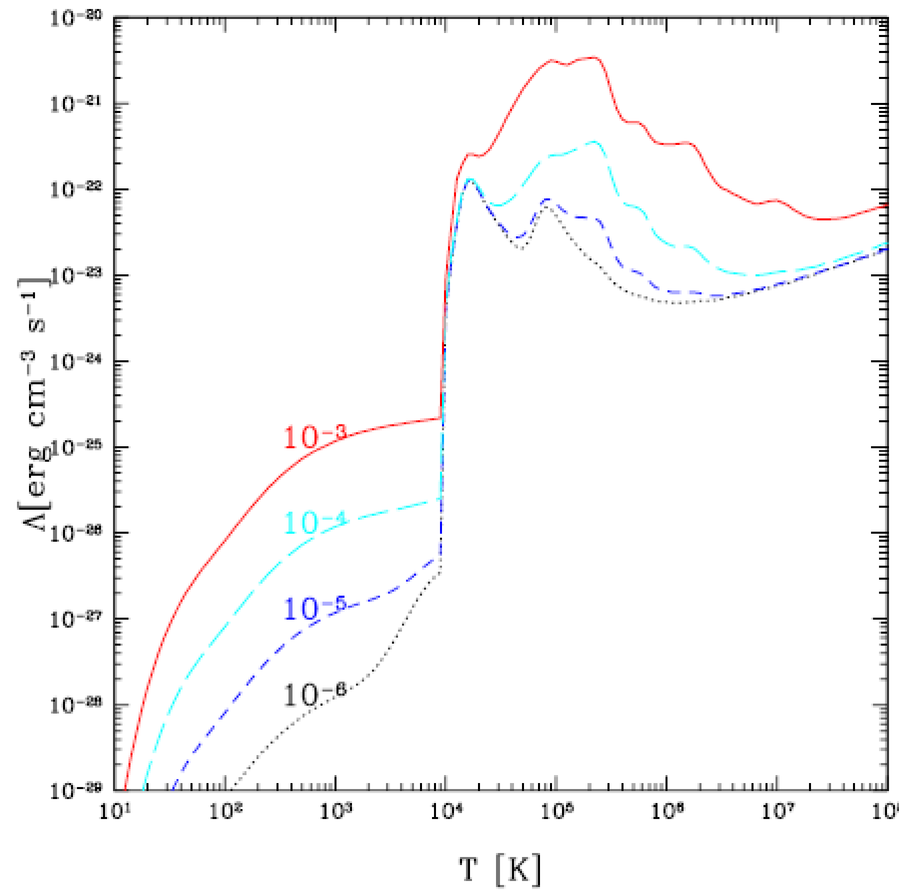
$$
\begin{aligned}
n_{\mathrm{H}_+} &= n_{\mathrm{H}} - n_{\mathrm{H}_0}, \\
n_{\mathrm{e}} &= n_{\mathrm{H}_+} + n_{\mathrm{He}_+} + 2n_{\mathrm{He}_{++}} \\
(n_{\mathrm{He}_0} + n_{\mathrm{He}_+} + n_{\mathrm{He}_{++}})/n_{\mathrm{H}} &= y \equiv Y/(4-4Y),
\end{aligned}
$$

From this, given the temperature and the density of the gas, it is possible to calculate the total cooling rate.

Note that the cooling rate depends on the **square** of the density and on the square root of the temperature.

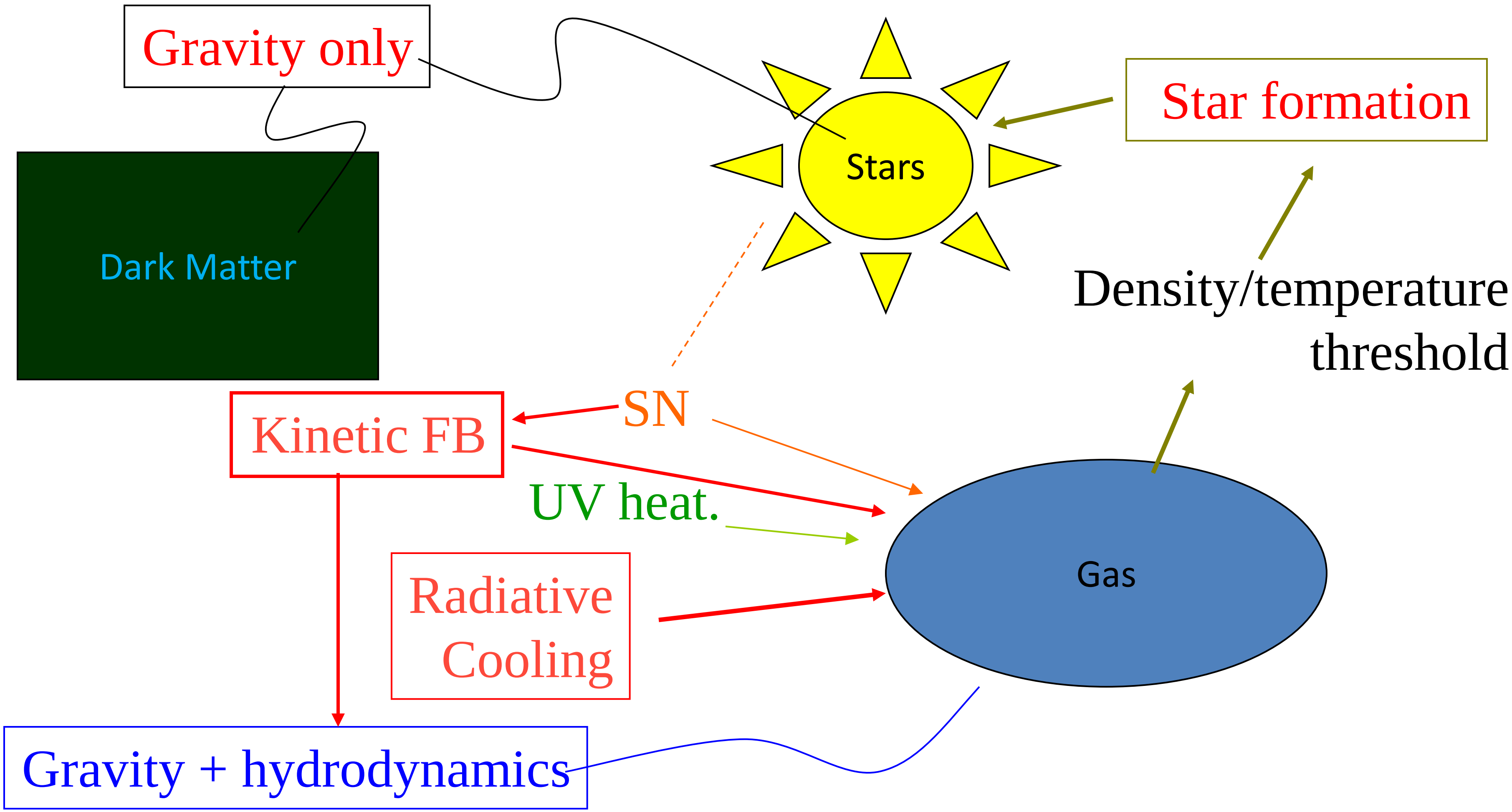


Λ [erg cm⁻³ s⁻¹]
T [K]
Bremsstrahlung
Recombinations
H⁺
He⁺⁺
He⁺
He
H
Ionizations
H
Excitations
He⁺
He⁺(d)
He⁺
Compton



Λ[erg cm⁻³ s⁻¹]
T [K]
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶

Star formation, feedback… simple schemes



- Gas is Jeans unstable: $\frac{h_i}{c_i} > \frac{1}{\sqrt{4\pi G\rho_i}}$

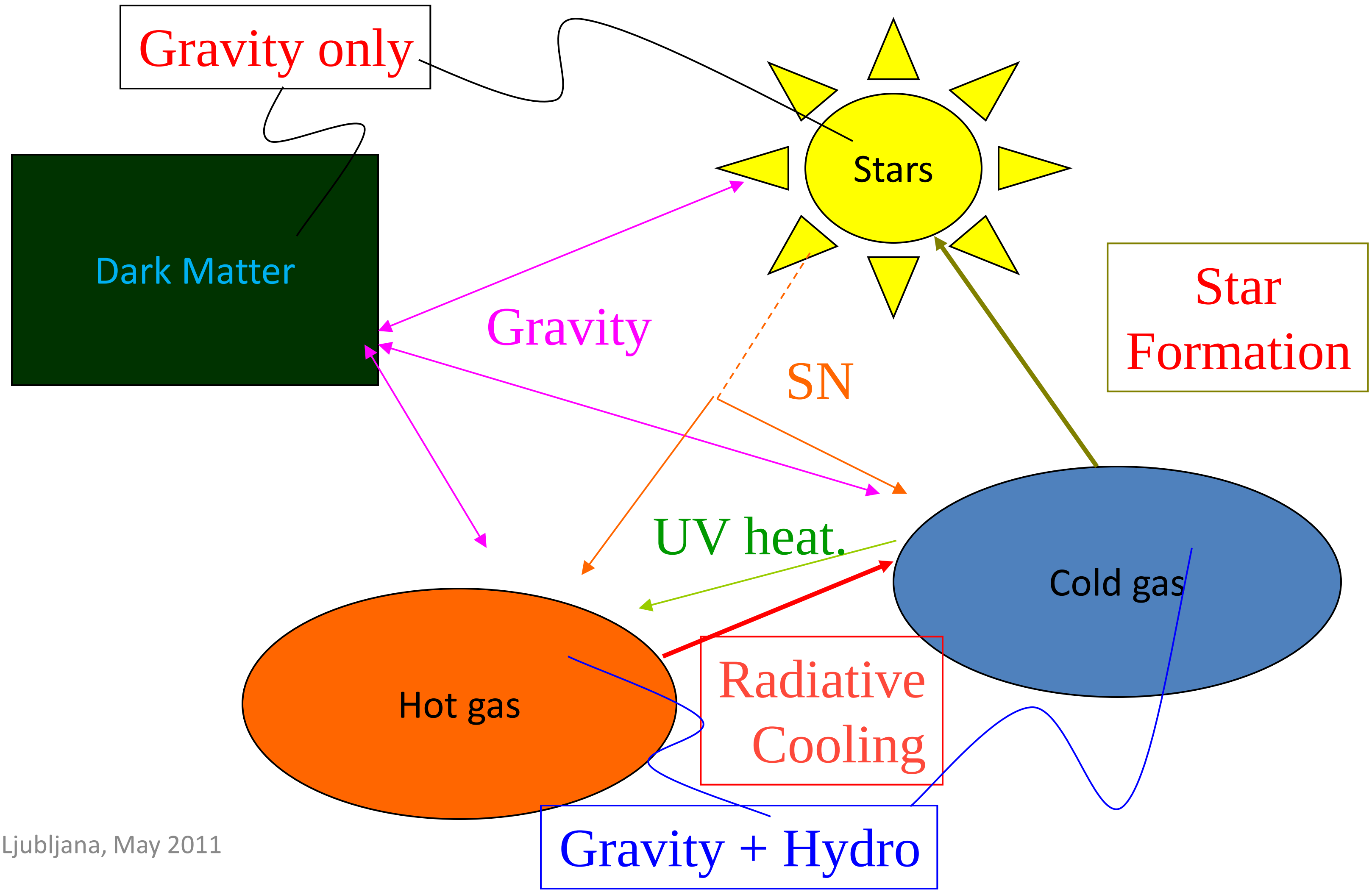
- Star formation rate:

$$\frac{d\rho_\star}{dt} = -\frac{d\rho_g}{dt} = \frac{c_\star \rho_g}{t_g}$$

c_* is the star formation efficiency, t_g the typical gas consumption time

Katz et al 96: SMe give **thermal** energy to the surrounding gas
IMF gives the number of Sne, energy per SN is fixed to 10^{51} erg

Star formation, feedback…
MultiPhase schemes



$$\frac{\mathrm{d}\rho_\star}{\mathrm{d}t} = \frac{\rho_c}{t_\star} - \beta\frac{\rho_c}{t_\star} = (1-\beta)\frac{\rho_c}{t_\star}$$

Stars form from COLD gas

β is star mass fraction in supernovae

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}(\rho_h u_h)\right|_{\mathrm{SN}} = \epsilon_{\mathrm{SN}}\frac{\mathrm{d}\rho_\star}{\mathrm{d}t} = \beta u_{\mathrm{SN}}\frac{\rho_c}{t_\star}$$

SNe energy heats up HOT gas

$$\left.\frac{\mathrm{d}\rho_c}{\mathrm{d}t}\right|_{\mathrm{EV}} = A\beta\frac{\rho_c}{t_\star}$$

SNe evaporates a fraction of cold gas

$$\left.\frac{\mathrm{d}\rho_c}{\mathrm{d}t}\right|_{\mathrm{TI}} = -\left.\frac{\mathrm{d}\rho_h}{\mathrm{d}t}\right|_{\mathrm{TI}} = \frac{1}{u_h - u_c}\Lambda_{\mathrm{net}}(\rho_h, u_h)$$

HOT gas cools to COLD gas

$$\frac{\mathrm{d}\rho_c}{\mathrm{d}t} = -\frac{\rho_c}{t_\star} - A\beta\frac{\rho_c}{t_\star} + \frac{1-f}{u_h - u_c}\Lambda_{\mathrm{net}}(\rho_h, u_h)$$

$$\frac{\mathrm{d}\rho_h}{\mathrm{d}t} = \beta\frac{\rho_c}{t_\star} + A\beta\frac{\rho_c}{t_\star} - \frac{1-f}{u_h - u_c}\Lambda_{\mathrm{net}}(\rho_h, u_h).$$

…pressure equilibrium between hot and cold phase

Evolution of energy of cold and hot phases:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_c u_c) = -\frac{\rho_c}{t_\star}u_c - A\beta\frac{\rho_c}{t_\star}u_c + \frac{(1-f)u_c}{u_h - u_c}\Lambda_{\rm net},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_h u_h) = \beta\frac{\rho_c}{t_\star}(u_{\rm SN} + u_c) + A\beta\frac{\rho_c}{t_\star}u_c - \frac{u_h - f u_c}{u_h - u_c}\Lambda_{\rm net}.$$

f=0 if $\rho > \rho_c$, 1 otherwise

Parameters are fixed to reproduce a number of observations (e.g. Schmidt-Kennicutt relations)

In SH03 the system is SOLVED and solutions are used to calculate star formation rates given the gas density and the cooling rate.

Kinetic winds

- Gas particles are «kicked» with a given constant velocity and decoupled for a period from the ambient gas

$$\dot{M}_{\rm w} = \eta \dot{M}_{\star}$$

$$\frac{1}{2}\dot{M}_{\rm w} v_{\rm w}^2 = \chi \epsilon_{\rm SN} \dot{M}_{\star}$$

$$v_{\rm w} = \sqrt{\frac{2\beta\chi u_{\rm SN}}{\eta(1-\beta)}}.$$

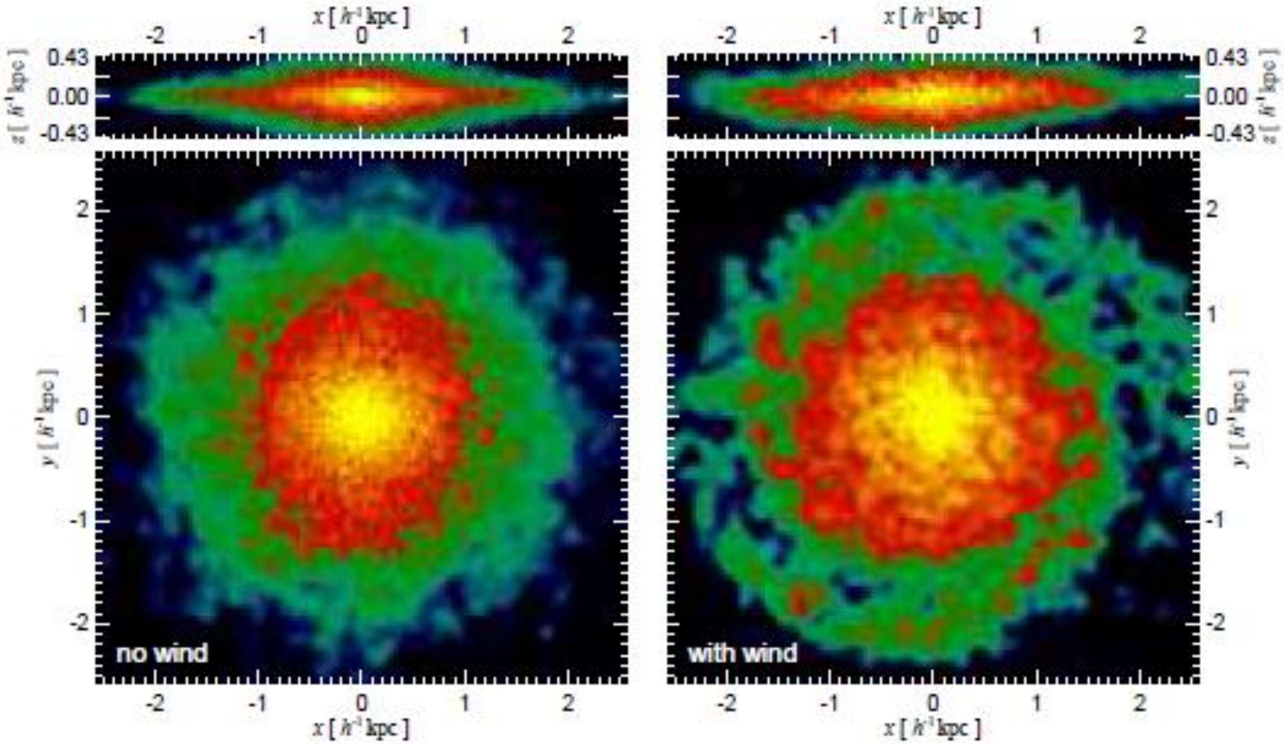


Figure 11. The stellar disk that formed after 3 Gyr in a halo of mass $M_{\rm vir} = 10^{10}\,h^{-1}\,{\rm M}_{\odot}$, seen both face on and edge on. On the left, the simulation does not include winds, on the right it does. The colour-contours in the pictures have been set such that they contain roughly the same fraction of the total light in each case. Note that the the morphology of the two disks appears similar, but the model that includes the wind has only 36% of the luminosity of the run that does not.

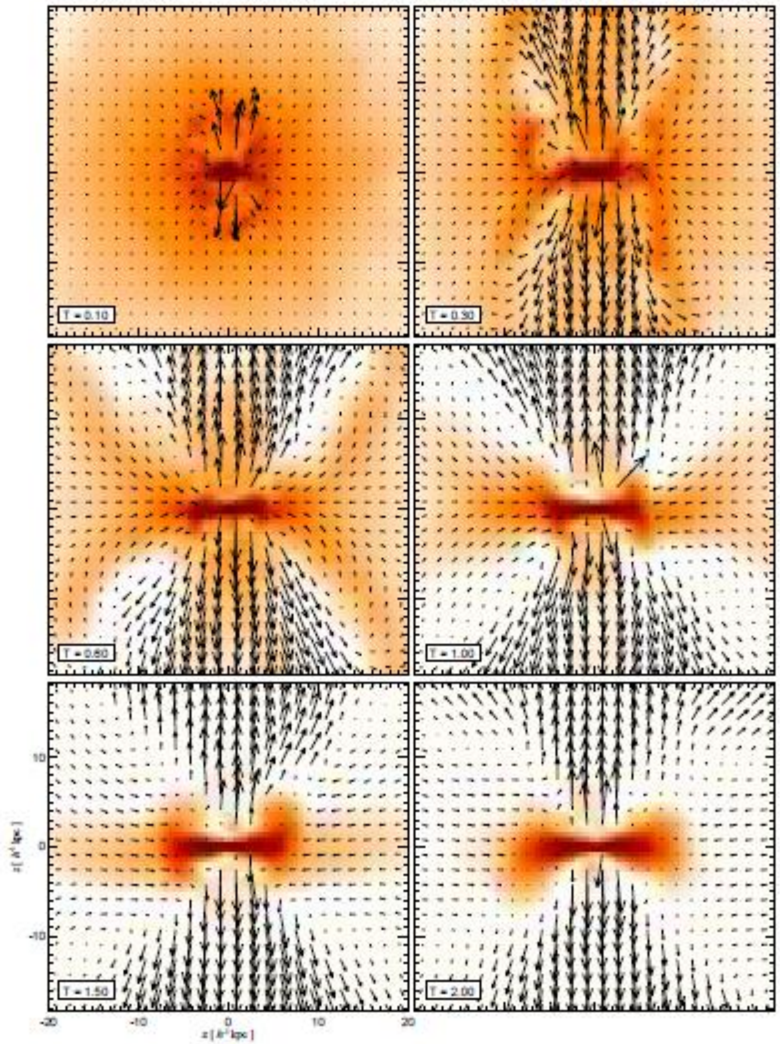


Figure 7. Time evolution of the gas flow in a halo of total mass $M_{\rm vir} = 10^{10}\,h^{-1}{\rm M}_\odot$. The velocity field is represented by arrows and the logarithm of the gas density is indicated as a colour-scale. Labels in each panel give the elapsed time in h^{-1}Gyr since the start of the simulation. A wind of speed $242\,{\rm km\,s^{-1}}$ is included in this model, considerably higher than the escape speed of $v_{\rm esc} \simeq 130\,{\rm km\,s^{-1}}$ from this halo. As a result, a galactic super-wind develops which blows out of the galaxy, entraining a significant fraction of the gas from the halo.