ON THE CONSTRUCTION OF WINDOW FUNCTIONS WITH CONSTANT-OVERLAP-ADD CONSTRAINT FOR ARBITRARY WINDOW SHIFTS

Christian Borß[*]

Fraunhofer Institute for Integrated Circuits IIS
91058 Erlangen, Germany

Rainer Martin

Institute of Communication Acoustics
Ruhr-Universität Bochum
44801 Bochum, Germany

ABSTRACT

In this paper we present a construction method for window functions with constant-overlap-add (COLA) constraint for spectral analysis-synthesis with a given percentage of window overlap. The window functions are derived from the shortest possible COLA window i.e., the rectangular window. The construction method allows to adjust the side-lobe fall-off and to fulfill the COLA constraint for arbitrary window shifts. The procedure results in a family of window functions which includes some well known members (the Bartlett and the Hann window) for 50% overlap.

***Index Terms*—** constant-overlap-add, window family

1. INTRODUCTION

A frequently used method for processing an audio signal is to split it into overlapping segments and to transform them into the frequency domain using a window function and the short-time Fourier transform [1]. Then, a synthesis window and the overlap-add method can be used to reconstruct the output signal from the processed segments which are transformed back into time domain using the inverse Fourier transform. A desirable constraint for such a signal processing architecture is that unprocessed segments result in an output signal which is - apart from a time shift - a perfect reconstruction of the input signal. Many windows[1] have been proposed for this purpose. For instance, the members of the general Hamming window family, like the well known Hamming and Hann window, as well as the Barlett (triangular) window fulfill this COLA constraint for 50% overlap and certain other percentages. The members of the three-term Blackman window family [2, 3], require 75% or a even greater percentage of overlap. In general, every member of the Blackman-Harris window family i.e., a window of the form $w(t) = \sum_{k=0}^{K} a_k \cos\left(\frac{2\pi k t}{T_T}\right), t \in \left[-\frac{T_T}{2}, \frac{T_T}{2}\right]$, fulfills the COLA constraint for $100\% - \frac{50\%}{N\,\mathrm{lcm}(1,\ldots,K)}, N \in \mathbb{N}$, overlap where lcm$(\cdot)$ denotes the least common multiple. The construction method for COLA windows which we propose in this paper is intended for analysis/synthesis systems where the restriction to certain window shifts is a limitation.

This paper contains two main parts and is organized as follows: in Sec. 2 we will present a method for the construction of COLA windows for a given frame shift and transformation period. Then, in Sec. 3, we introduce a new COLA window family which features a maximally steep side-lobe fall-off for a given window order.

[*]The content of this paper was developed when the author was with the Institute of Communication Acoustics, Ruhr-Universität Bochum.

[1]This refers to the product of the analysis and the synthesis window.

2. WINDOW CONSTRUCTION

In the following we will use the term *COLA window* to denote a window function $w(t)$ which fulfills the COLA constraint,

$$\sum_{k=-\infty}^{\infty} w(t - kT_S) = 1 \quad , \tag{1}$$

where T_S denotes the frame shift of the periodically applied window.

A function which fulfills this constraint is the rectangular window of length T_S,

$$r_S(t) = \mathrm{rect}(t/T_S) \tag{2}$$

$$\mathrm{rect}(t) = \begin{cases} 1, & \text{if } t \in \left]-\frac{1}{2}, \frac{1}{2}\right] \\ 0, & \text{else} \end{cases} \quad . \tag{3}$$

This is the shortest possible COLA window and due to this fundamental property we will use $r_S(t)$ as basis for the construction of COLA windows.

The COLA constraint (1) is linear and time-invariant. Hence, the sum of shifted and weighted COLA windows also fulfills the COLA constraint, if the sum of the weights is equal to 1. This can be further generalized by considering an infinite large number of shifted windows with infinitesimal small weights. The limiting process results in a convolution integral,

$$w(t) = r_S(t) * g(t) = \int_{-\infty}^{\infty} r_S(\tau) g(t-\tau)\mathrm{d}\tau \quad , \tag{4}$$

where $g(t)$ denotes a superposition density. The correctness of this proposition can be shown by inserting (4) into (1) and changing the order of the summation and integration,

$$\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} r_S(\tau - kT_S) g(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} g(t-\tau) \sum_{k=-\infty}^{\infty} r_S(\tau - kT_S)\mathrm{d}\tau = \int_{-\infty}^{\infty} g(t)\mathrm{d}t = 1 \quad . \tag{5}$$

Hence, $g(t)$ can be any integrable function which fulfills the normalization constraint (5).

In practice where $w(t)$ is used to determine the short-time Fourier transform over a transformation period T_T, $w(t)$ needs to be time-constrained. In the design of $g(t)$ this constraint can be enforced by a multiplication of an arbitrary integrable function $f(t)$ with a rectangular function $r_L(t)$ of length $T_L = T_T - T_S$,

$$g(t) = f(t)r_L(t) \tag{6}$$
$$r_L(t) = \mathrm{rect}(t/T_L) \quad , \tag{7}$$

where $f(t)$ fulfills the following normalization constraint:

$$\int_{-\frac{T_L}{2}}^{\frac{T_L}{2}} f(t)\mathrm{d}t = 1 \quad . \tag{8}$$

For a given frame shift T_S and transformation period T_T this allows for constructing a COLA window as follows:

$$w(t) = r_S(t) * (r_L(t)f(t)) \quad . \tag{9}$$

The choice of $f(t)$ is detailed in the following section.

3. GENERALIZED WINDOW FAMILY WITH ADJUSTABLE SIDE-LOBE FALL-OFF

As window functions are used for spectrum analysis, their spectral properties are of essential importance. Hence, for designing a COLA window it is favorable to use a frequency domain formulation of the construction equation by applying the Fourier transform to (9),

$$W(f) = R_S(f)\,(R_L(f) * F(f)) \quad . \tag{10}$$

As a matter of fact, the proper choice of $F(f)$ or the method for its construction depends on the application. For instance, the criteria proposed by Adams [4] or Slepian [5, 6] could be used to obtain an optimal window for a given analysis/synthesis system.

A simpler solution is obtained by using a conventional window like the Kaiser window [7], the Dolph-Chebyshev window [8], or a member of the Blackman-Harris window family [2]. The latter are polynomially attenuated functions where the order of the polynomial defines the main lobe width. In order to provide some structure and to gain a deeper insight into this window family, we extend this family in the following paragraphs and derive a new family of COLA windows with maximally steep side-lobe fall-off.

3.1. Trivial Choice

The first member of our window family results from choosing the Dirac delta,

$$F_0(f) = c_0\delta(f) \quad , \tag{11}$$

with the normalization coefficient $c_0 = \frac{1}{T_L}$. The index, that is also used in the following paragraphs, is utilized to distinguish between the different members of the window family and indicates the order of the window. As the Dirac delta is the identity element with respect to convolution, the spectrum $W_0(f)$ of the 0^{th}-order window is given by the product of $R_S(f)$ and $R_L(f)$. The spectra of these rectangular windows have the form of a scaled sinc function,

$$\mathcal{F}\{\mathrm{rect}\,(t/T)\} = T\mathrm{sinc}\,(\pi T f) \quad , \tag{12}$$

which has zero crossings at integer multiples of $1/T$. Hence, $W(f)$ has zero crossings at integer multiples of $1/T_S$ and $1/T_L$, and the width of the resulting main-lobe is the minimum of the main-lobe width of $R_S(f)$ and $R_L(f)$.

In time domain, the 0^{th}-order window is given by the convolution

$$w_0(t) = c_0 r_S(t) * r_L(t) \tag{13}$$

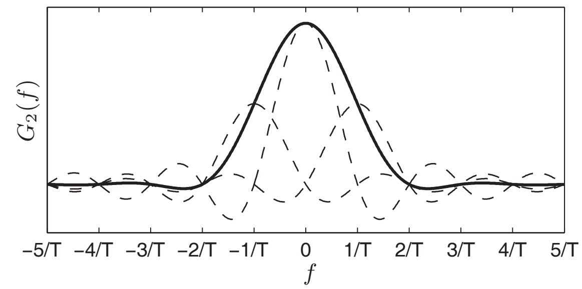

Fig. 1. $G_2(f)$: spectrum of a Hann window (solid) which results from the superposition of 3 shifted sinc functions (dashed)

which results in a tapered window with linear slopes. A special case is produced for $T_S = T_L = T_T/2$ (50% overlap) where the tapered window becomes the well known Bartlett (triangular) window.

3.2. Side-Lobe Reduction

The $w_0(t)$ window exhibits significant side-lobe levels, that can be reduced by applying the same principle which is used for the construction of the Hann window [2]. Such a window results from the sum of a cosine and a constant bias, multiplied by a rectangular window. In frequency domain, this corresponds to the superposition of three shifted sinc functions which compensate the side-lobes as illustrated in Fig. 1. The same principle can be applied by choosing

$$F_1(f) = c_1\delta\left(f + \frac{1}{2T_L}\right) + c_1\delta\left(f - \frac{1}{2T_L}\right) \tag{14}$$

with $c_1 = \frac{\pi}{4T_L}$ to fulfill the normalization constraint. Convolved with $R_L(f)$, this results in the spectrum $G_1(f)$ which is shown in Fig. 2. Applying the inverse Fourier transform to (14) yields the time domain representation,

$$f_1(t) = 2c_1 \cos\left(\frac{\pi t}{T_L}\right) \quad . \tag{15}$$

Accordingly, $g_1(t)$ is the well known cosine window. And as $w(t)$ results from the convolution of $r_S(t)$ and $g(t)$, the 1^{st}-order window $w_1(t)$ is a Tukey window for $T_S > T_L$ and a Hann window for $T_S = T_L$ [9].

3.3. Higher-Order Windows

For the 1^{st}-order window $w_1(t)$, we utilized the property that adding two functions which have the form $\frac{sin(x)}{x}$ and which are shifted by

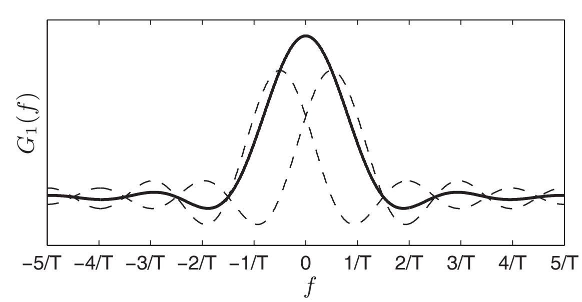

Fig. 2. Reduced side-lobes of the spectrum $G_1(f)$

one period, where $x = \pi T_L f$, results in reduced side-lobes,

$$\begin{aligned} G_0(x) &= \frac{1}{x}\sin(x) \\ G_1(x) &= G_0\left(x+\frac{\pi}{2}\right)+G_0\left(x-\frac{\pi}{2}\right) \\ &= \frac{-4\pi}{4x^2-\pi^2}\cos(x) \quad . \end{aligned} \tag{16}$$

As the cos term is attenuated by a rational function with a strictly monotonic decreasing asymptote, the side-lobes can be further reduced by applying the same method again and again,

$$G_{n+1}(x) = G_n\left(x+\frac{\pi}{2}\right)+G_n\left(x-\frac{\pi}{2}\right) \tag{17}$$

$$G_2(x) = \frac{-2\pi^2}{x^3-\pi^2 x}\sin(x) \tag{18}$$

$$G_3(x) = \frac{96\pi^3}{16x^4-40\pi^2x^2+9\pi^4}\cos(x) \tag{19}$$

$$\dots \quad .$$

Every iteration increments the polynomial order of the denominator and thus improves the side-lobe fall-off by 6 dB / octave.

Consequently, we define the higher-order members of our window family by the construction spectra which are defined as follows,

$$F_n(f) = c_n F_1(f) * F_{n-1}(f) \quad , \tag{20}$$

where c_n results from fulfilling the normalization constraint (8). This iteration results in a series of Dirac deltas where the individual weights are the binomial coefficients. The resulting $G_n(f)$ are the spectra of the time-constrained $\cos^n$ windows. Consequently, $G_2(f)$ is the spectrum of a Hann window (see Fig. 1), and $w_2(t)$ is the result of a Hann window convolved with a rectangular window. The even order spectra $G_n(f)$ are specific members of the Blackman-Harris window family while the odd order spectra are members of an *extended* Blackman-Harris window family which also includes windows of the form $w(t) = \sum_{k=1}^{K} a_k \cos\left(\frac{\pi(2k-1)t}{T_T}\right)$. Figure 3 illustrates the first members of the presented window family. Note, that a time domain approach for the extension of the Blackman-Harris window family was recently (and independently) proposed in [10]. However, our formulation in the frequency domain provides a quantification of the polynomial attenuation which is discussed in more detail below.

3.4. Properties of $G_n(f)$

The definition (20) results in a spectrum $G_n(f)$ which has the form

$$G_{2K}(x) = \sin(x)\sum_{k=0}^{2K}\frac{a_k}{x-x_k} \tag{21}$$

$$G_{2K+1}(x) = \cos(x)\sum_{k=0}^{2K+1}\frac{a_k}{x-x_k} \quad , \tag{22}$$

where the sum of the partial fractions can be combined to a rational function of the form $\frac{P_N(x)}{P_D(x)}$. The poles of this rational function are compensated by the zeros of the sin/cos term. The order of $P_N(x)$ and $P_D(x)$ define the order of the asymptote that attenuates the sin/cos term. For the chosen weights i.e., the binomial coefficients with alternating signs, the numerator polynomial $P_N(x)$ is reduced to 0^{th}-order. Other weights could be used e.g., to reduce the maximum side-lobe level, but this would also reduce the order of the asymptote. Hence, the proposed window family is optimal with regard to the asymptotic decay (the side-lobe fall-off).

As can be observed in Fig. 3, the level of the first side lobe of $W_n(f)$ increases with its order, if the width of the main lobe of $G_n(f)$ largely exceeds the main lobe of $R_S(f)$. Hence, the order is limited to a reasonable range and high orders are of practical use only for a large window overlap.

4. CONCLUSIONS

The presented construction method for window functions produces windows with constant-overlap-add (COLA) constraint for any percentage of overlap. In our formulation, the shortest possible COLA window is used as a template to produce COLA windows with given properties.

We introduced a family of window functions with increasing order n featuring an optimal side-lobe fall-off with a $(n+2)6$ dB / octave slope. We showed that the Bartlett (triangular) and the Hann window are the first members of this family for 50% overlap. Hence, the presented family can be regarded as a generalization for arbitrary frame shifts and adjustable side-lobe fall-offs.

5. REFERENCES

[1] Oppenheim, A. V. and Schafer, R. W., *Digital Signal Processing*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1975.

[2] Harris, F. J., "On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform," *Proceedings of the IEEE*, vol. 66, no. 1, pp. 51–83, 1978.

[3] Nuttal, A. H., "Some Windows with Very Good Sidelobe Behavior," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-29, no. 1, pp. 84–91, 1981.

[4] Adams, J. W., "A New Optimal Window," *IEEE Transactions on Signal Processing*, vol. 39, no. 8, pp. 1753–1769, Aug. 1991.

[5] Slepian, D. and Pollak, H. O., "Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty - I," *Bell System Tech. J.*, vol. 40, no. 1, pp. 43–64, 1961.

[6] Landau, H. J. and Pollak, H. O., "Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty - II," *Bell System Tech. J.*, vol. 40, no. 1, pp. 65–84, 1961.

[7] Kaiser, J. and Schafer, R., "On the Use of the I0-sinh Window for Spectrum Analysis ," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-28, no. 1, pp. 105–107, 1980.

[8] Dolph, C. L., "A Current Distribution for Broadside Arrays Which Optimizes the Relationship between Beam Width and Side-Lobe Level," *Proceedings of the IRE*, vol. 35, no. 5, pp. 489–492, May 1947.

[9] Tukey, J. W., "An Introduction to the Calculations of Numerical Spectrum Analysis," in *Spectral Analysis of Time Series*, Haris, B., Ed., pp. 25–46. Wiley, New York, 1967.

[10] Helmrich, C. R., "On the Use of Sines in the Design of Signal Windows," in *13th Int. Conference on Digital Audio Effects (DAFx-10)*, Graz, Austria, Sep. 2010.

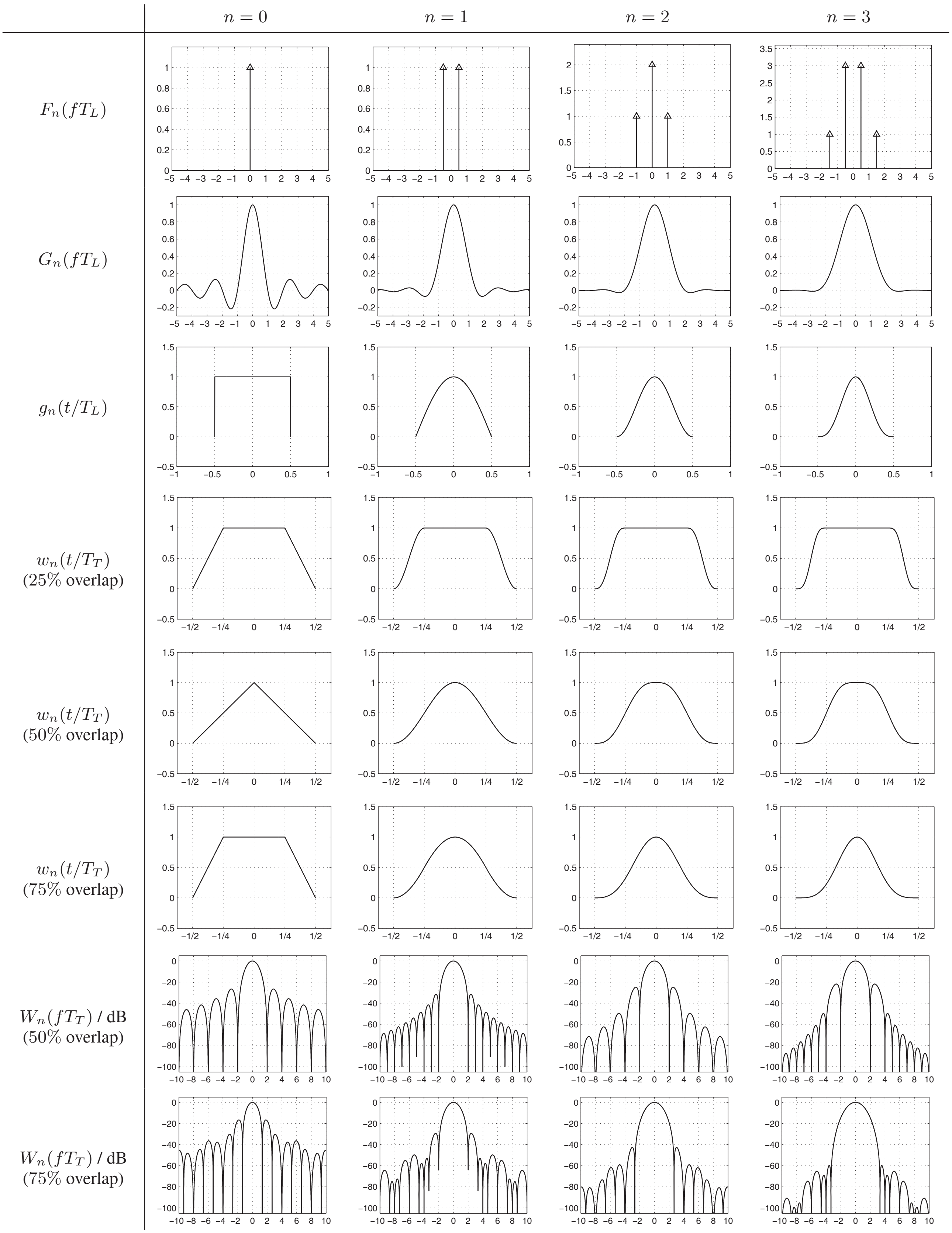

Fig. 3. First members of the window family and their construction spectra (normalized for better comparability)