Homework 3

ME 590
Applied CFD and Numerical Heat Transfer

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Problem 1

Problem Statement

Consider a cylindrical pin fin with a length of L = 3 cm and a radius of r = 5 mm. The fin material is initially at thermal equilibrium with the surroundings at a temperature of T∞ = 20◦C. The fin is attached to an electronic device that is powered on and produces a constant heat rate of qf = 1 W supplied to the base of the fin. Assume the heat transfer coefficient is h = 100 W/(m2 · K) and the fin is made of pure copper with the following properties:

Rho =8933 kg/m^3 cp= 385 J/kg*K k = 401 W/m*K

(a) Use the finite volume method to derive a numerical scheme for analyzing this system. Show the symbolic form the equations for the base node, interior nodes, and tip node.
(b) Using a grid with 10 nodes, determine the maximum allowable timestep to maintain stability.
(c) Determine the base temperature and the tip temperature after 30s
(d) Generate a graph showing the temperature profile at 10s intervals for 2 minutes in comparison to the exact steady state solution
(e) Repeat parts 1b and 1c for aluminum allot with the following properties

Rho = 2790 kg/m3 cp = 883 J/kg*K k = 168 W/m*K

Solution

A.

Discretization of the Fin

Divide the fin along its length into N_x equally spaced nodes with spacing

$$\Delta x = L/N_x$$

- **Cross-sectional area:** $A_c = \pi r^2$
- **Perimeter:** $P = 2\pi r$
- **Surface area for convection over a control volume:** $A_s = P\Delta x$

Energy Balance

For interior nodes $(1 < i < N\_x)$**, the energy balance over each control volume is:**

$$\rho c_p A_c \Delta x \frac{dT_i}{dt} = kA_c \left(\frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} \right) - hP\Delta x (T_i - T_\infty)$$

Non-Dimensionalization

Fourier number:

$$Fo = \frac{\alpha \Delta t}{(\Delta x)^2}, \quad \text{where } \alpha = \frac{k}{\rho c_p}$$

Biot number:

$$Bi = \frac{hP\Delta x}{kA_c} = \frac{hA_s}{kA_c}$$

Using an explicit time integration scheme the time derivative is discretized as:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} - \frac{hP}{\rho c_p A_c}(T_i^n - T_\infty)$$

Substitute Fo and Bi into the discretized equation:

$$T_i^{n+1} = T_i^n(1 - 2Fo - Bi \cdot Fo) + Fo(T_{i-1}^n + T_{i+1}^n) + Bi \cdot Fo \cdot T_\infty$$

This is the general equation for interior nodes ($1 < i < N\_x$).

Base Node ($i = 1$)

At the base node, the heat supplied q_f must be included in the energy balance:

$$\rho c_p A_c \frac{\Delta x}{2} \frac{dT_1}{dt} = q_f + kA_c\left(\frac{T_2 - T_1}{\Delta x}\right) - hP\frac{\Delta x}{2}(T_1 - T_\infty)$$

Divide both sides by $\rho c_p A_c \frac{\Delta x}{2}$ and discretize:

$$T_1^{n+1} = T_1^n(1 - 2Fo - Bi \cdot Fo) + 2FoT_2^n + 2Fo\left(\frac{q_f \Delta x}{kA_c}\right) + Bi \cdot FoT_\infty$$

Tip Node ($i = N_x$)

At the tip node, considering convection from both the lateral surface and the tip area, the energy balance is:

$$\rho c_p A_c \frac{\Delta x}{2}\frac{dT_{N_x}}{dt} = kA_c\left(\frac{T_{N_x} - T_{N_x-1}}{\Delta x}\right) - h\left(P\frac{\Delta x}{2} + A_c\right)\left(T_{N_x} - T_\infty\right)$$

Simplifying and discretizing arrives at:

$$T_{N_x}^{n+1} = T_{N_x}^n\left(1 - 2Fo - 2FoBi\left(\frac{A_s}{2A_c} + 1\right)\right) + 2FoT_{N_x-1}^n + 2FoBiT_\infty\left(\frac{A_s}{2A_c} + 1\right)$$

Subbing Variables arrives at:

Base Node ($i = 1$):

$$T_1^{n+1} = Fo\left(2\Delta x\frac{q_f}{kA_c} + 2T_2^n + Bi\frac{A_s}{A_c}T_\infty\right) + T_1^n\left(1 - 2Fo - FoBi\frac{A_s}{A_c}\right)$$

Interior Nodes ($1 < i < N_x$)**:**

$$T_i^{n+1} = Fo\left(T_{i-1}^n + T_{i+1}^n + Bi\frac{A_s}{A_c}T_\infty\right) + T_i^n\left(1 - 2Fo - FoBi\frac{A_s}{A_c}\right)$$

Tip Node ($i = N_x$):

$$T_{N_x}^{n+1} = 2Fo\left(T_{N_x-1}^n + BiT_\infty\left(\frac{A_s}{2A_c} + 1\right)\right) + T_{N_x}^n\left(1 - 2Fo - 2FoBi\left(\frac{A_s}{2A_c} + 1\right)\right)$$

Where:

- T_i^n **: Temperature at node i and time step n .**
- Fo **: Fourier number**

- **Bi : Biot number**
- **A_s/A_c : Ratio of surface area to cross-sectional area**
- **q_f : Heat supplied at the base**

B.

Below are the inequalities and solution equations for each node (Base, Interior, and Tip). The coefficients, for $T_i^{\,n}$, of the previous time step must be greater than or equal to zero to solve for the max-time step.

Base

$$1 - Fo\left(2 + Bi\frac{A_s}{A_c}\right) \geq 0 \qquad \frac{\Delta x^2}{\alpha(2 + Bi\frac{A_s}{A_c})} \geq \Delta t$$

$$\Delta t_{base} \leq .0476\ s$$

Interior

$$1 - Fo\left(2 + Bi\frac{A_s}{A_c}\right) \geq 0. \qquad \frac{\Delta x^2}{\alpha(2 + Bi\frac{A_s}{A_c})} \geq \Delta t$$

$$\Delta t_{interior} \leq .0476\ s$$

Tip

$$(1 - 2Fo - 2FoBi\left(1 + \frac{A_s}{2A_c}\right) \geq 0 \qquad \frac{\Delta x^2}{\alpha(2 + 2Bi(\frac{A_s}{A_c} + 1)} \geq \Delta t$$

$$\Delta t_{tip} \leq .04758$$

Since the minimum max-time step is the tip, with a value of 0.04758 the max-time step is

$$\Delta t_{max} \leq .04758$$

C.

The base and tip temperatures at 30 seconds can be found by using the equations derived in part A over many time steps. Solving these steps arrives at this result:

$$T_1^{30s} = 23.41\ °C$$
$$T_{10}^{30s} = 22.92\ °C$$

D.

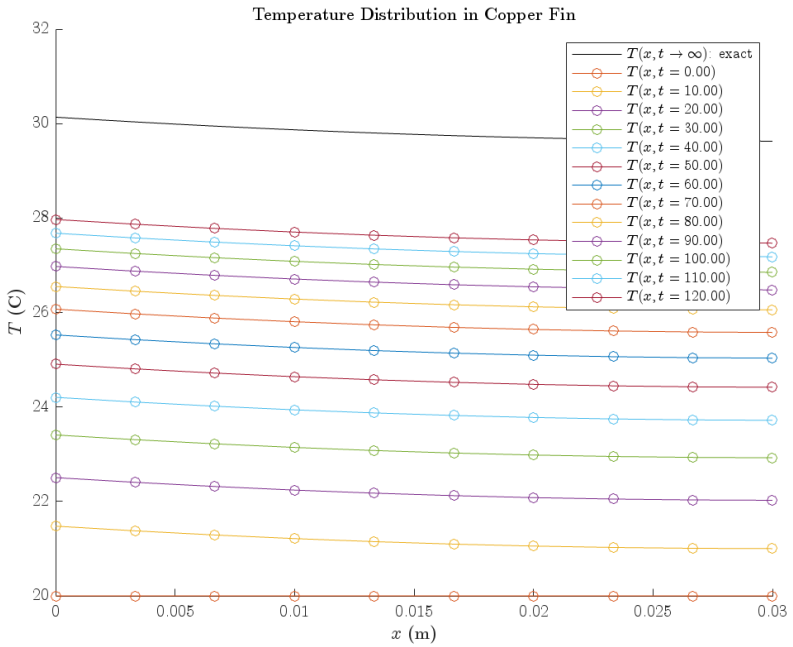

Figure 1: Steady state exact solution against time step approximations for Copper

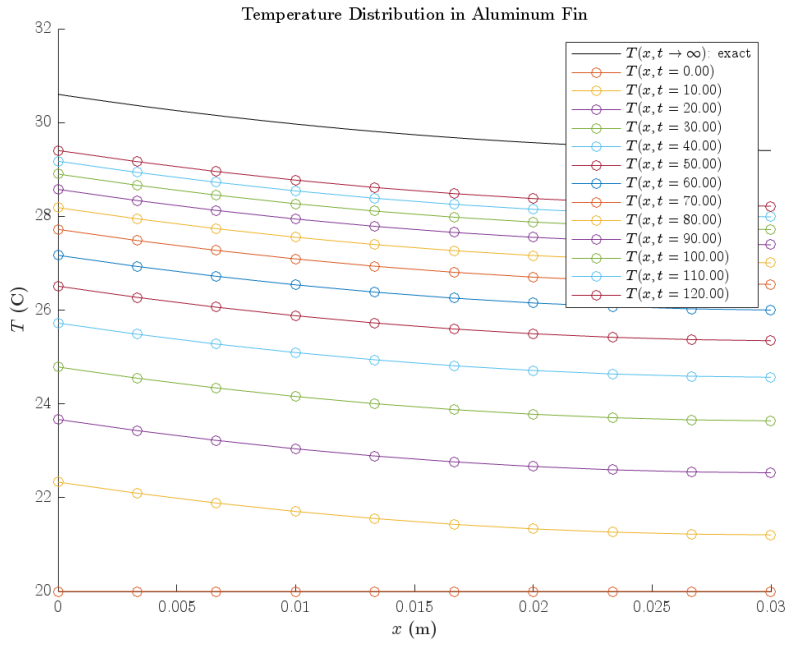

Figure 2: Steady state exact solution against time step approximations for Aluminum

```
>> HW3
Enter the desired timestep (in seconds): .025
Enter material choice (c for copper, a for aluminum, b for both): b
Maximum timestep: 0.047621 s
Copper fin at t=30s:
Base temperature: 23.41°C
Tip temperature: 22.92°C

Maximum timestep: 0.081360 s
Aluminum fin at t=30s:
Base temperature: 24.79°C
Tip temperature: 23.64°C
```

Figure 3: Example Output of HW3 MATLAB Script

E.

Repeating the steps in parts b and c to solve for the material properties of aluminum follow the same steps only altering the physical material properties. The MATLAB script I have written allows for both the aluminum and copper materials to be solve independently or simultaneously. The equation driven results match the MATLAB output below.

The maximum allowable timestep for aluminum is

$$\Delta t_{alum,max} \leq .0814\ (s)$$

And the base and tip temperature of aluminum at 30 seconds are as follows

$$T_1^{30s} = 24.79\ °C$$
$$T_{10}^{30s} = 23.64\ °C$$