Homework 5: Governing Equations

ME 590: Applied CFD and Numerical Heat Transfer

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1 Problem Statement

The full energy equation in Cartesian coordinates is:

$$\begin{aligned}\rho\left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}+w\frac{\partial h}{\partial z}\right)=&\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)+S_h+\Phi\\&+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)-\vec{\nabla}\cdot\vec{q}_r\end{aligned} \tag{1}$$

where h is enthalpy, p is pressure, S_h is a heat generation source term, $\vec{q}_r$ is the radiation heat flux vector, T is temperature, k is thermal conductivity, ρ is density, and Φ is the dissipation function defined as:

$$\begin{aligned}\Phi=\mu\Bigg\{&2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]\\&+\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]\Bigg\}\\&+\lambda\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]^2\end{aligned} \tag{2}$$

Simplify the full energy equation for the case of:

- No radiation
- No internal energy generation
- No viscous dissipation
- Incompressible flow (negligible pressure work)
- 2 dimensional (x and y)
- Let $h = c_pT$ where c_p is the constant pressure specific heat and can be treated as a constant.

Write the end equation in terms of temperature.

2 Problem Statement

Starting with the vector form of the continuity equation:

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{V})=0 \tag{3}$$

Use the vector identities discussed in the videos and listed in the appendix of this assignment to expand this expression to produce the continuity equation in:

(a) Cartesian coordinates (x, y, z)

(b) Cylindrical coordinates (R, θ, z)

(c) Spherical coordinates (r, θ, ϕ)

3 Problem Statement

Starting with the vector form of the momentum equations:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} p + \mu \nabla^2 \vec{V} + \lambda \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) + \rho \vec{f} \tag{4}$$

Use the vector identities listed in the appendix of this assignment to expand this expression for either cylindrical or spherical coordinates.

Problem 1: Simplification of the Energy Equation

Given the full energy equation:

$$\rho\left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}+w\frac{\partial h}{\partial z}\right)=\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)+S_h+\Phi$$
$$+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)-\nabla\cdot\mathbf{q}_r \tag{5}$$

Simplifications

Applying the given assumptions:

- **No radiation:** $\nabla\cdot\mathbf{q}_r=0$
- **No internal energy generation:** $S_h=0$
- **No viscous dissipation:** $\Phi=0$
- **Incompressible flow:** $\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}=0$
- **2-dimensional flow in x and y:** $\frac{\partial}{\partial z}=0$, $w=0$
- **Enthalpy:** $h=c_pT$, with c_p constant

Simplified Energy Equation

Using $h=c_pT$ and the fact that c_p is constant, the left-hand side (LHS) becomes:

$$\rho\left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}\right)=\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right) \tag{6}$$

Since the pressure work term is zero (due to incompressibility) and S_h, Φ, and $\nabla\cdot\mathbf{q}_r$ are zero, the right-hand side (RHS) simplifies to:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \tag{7}$$

Final Equation

Combining the simplified LHS and RHS, the energy equation reduces to:

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \tag{8}$$

This is the simplified energy equation in terms of temperature under the given assumptions.

Problem 2: Expansion of the Continuity Equation

Starting with:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{9}$$

We expand this equation in different coordinate systems.

(a) Cartesian Coordinates

In Cartesian coordinates, $\mathbf{V} = u, \hat{\imath} + v, \hat{\jmath} + w, \hat{k}$, and the divergence operator is:

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{10}$$

Therefore, the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{11}$$

(b) Cylindrical Coordinates

In cylindrical coordinates, $\mathbf{V} = V_R, \hat{e}R + V\theta, \hat{e}_\theta + V_z, \hat{e}_z$, and the divergence operator is:

$$\nabla \cdot \mathbf{V} = \frac{1}{R}\frac{\partial}{\partial R}(RV_R) + \frac{1}{R}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} \tag{12}$$

Thus, the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho V_R) + \frac{1}{R}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0 \tag{13}$$

(c) Spherical Coordinates

In spherical coordinates, $\mathbf{V} = V_r, \hat{e}r + V\theta, \hat{e}\theta + V\phi, \hat{e}_\phi$, and the divergence operator is:

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi} \tag{14}$$

Therefore, the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho V_\phi) = 0 \tag{15}$$

Problem 3: Expansion of the Momentum Equations in Cylindrical Coordinates

Starting with the vector momentum equation:

$$\rho \frac{D\overrightarrow{V}}{Dt} = -\overrightarrow{\nabla} p + \mu \nabla^2 \overrightarrow{V} + \lambda \overrightarrow{\nabla}(\overrightarrow{\nabla} \cdot \overrightarrow{V}) + \rho \overrightarrow{f} \tag{16}$$

This expression will be expanded cylindrical coordinates, using the vector identities provided.

Solution

Step 1: Expand the Material Derivative of Velocity

In cylindrical coordinates, the velocity vector is:

$$\overrightarrow{V} = V_R \hat{e}_R + V_\theta \hat{e}_\theta + V_z \hat{e}_z \tag{17}$$

The material derivative of the velocity vector is:

$$\frac{D\overrightarrow{V}}{Dt} = \frac{\partial \overrightarrow{V}}{\partial t} + (\overrightarrow{V} \cdot \overrightarrow{\nabla})\overrightarrow{V} \tag{18}$$

From the appendix, the material derivative components in cylindrical coordinates are:

$$\text{Radial component: } \frac{DV_R}{Dt} = \frac{\partial V_R}{\partial t} + V_R \frac{\partial V_R}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_R}{\partial \theta} + V_z \frac{\partial V_R}{\partial z} - \frac{V_\theta^2}{R} \tag{19}$$

$$\text{Azimuthal component: } \frac{DV_\theta}{Dt} = \frac{\partial V_\theta}{\partial t} + V_R \frac{\partial V_\theta}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_R V_\theta}{R} \tag{20}$$

$$\text{Axial component: } \frac{DV_z}{Dt} = \frac{\partial V_z}{\partial t} + V_R \frac{\partial V_z}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} \tag{21}$$

Therefore, the material derivative term on the left-hand side becomes:

$$\text{Radial component: } \rho\frac{DV_R}{Dt} = \rho\left(\frac{\partial V_R}{\partial t} + V_R \frac{\partial V_R}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_R}{\partial \theta} + V_z \frac{\partial V_R}{\partial z} - \frac{V_\theta^2}{R}\right) \tag{22}$$

$$\text{Azimuthal component: } \rho\frac{DV_\theta}{Dt} = \rho\left(\frac{\partial V_\theta}{\partial t} + V_R \frac{\partial V_\theta}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_R V_\theta}{R}\right) \tag{23}$$

$$\text{Axial component: } \rho\frac{DV_z}{Dt} = \rho\left(\frac{\partial V_z}{\partial t} + V_R \frac{\partial V_z}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z}\right) \tag{24}$$

Step 2: Expand the Pressure Gradient Term

The gradient of a scalar function in cylindrical coordinates is:

$$\overrightarrow{\nabla} p = \frac{\partial p}{\partial R}\hat{e}_R + \frac{1}{R}\frac{\partial p}{\partial \theta}\hat{e}_\theta + \frac{\partial p}{\partial z}\hat{e}_z \tag{25}$$

Therefore, the pressure gradient term is:

$$-\overrightarrow{\nabla} p = -\frac{\partial p}{\partial R}\hat{e}_R - \frac{1}{R}\frac{\partial p}{\partial \theta}\hat{e}_\theta - \frac{\partial p}{\partial z}\hat{e}_z \tag{26}$$

Step 3: Expand the Viscous Terms

The Laplacian of a vector in cylindrical coordinates is given by:

$$\nabla^2 \overrightarrow{V} = \left(\nabla^2 V_R - \frac{V_R}{R^2} - \frac{2}{R^2}\frac{\partial V_\theta}{\partial \theta}\right)\hat{e}_R + \left(\nabla^2 V_\theta - \frac{V_\theta}{R^2} + \frac{2}{R^2}\frac{\partial V_R}{\partial \theta}\right)\hat{e}_\theta + \left(\nabla^2 V_z\right)\hat{e}_z \tag{27}$$

Where the scalar Laplacian operator ∇^2 applied to a scalar function f is:

$$\nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{28}$$

Therefore, the viscous term $\mu\nabla^2\overrightarrow{V}$ has components:

$$\text{Radial component: } \mu\left(\nabla^2 V_R - \frac{V_R}{R^2} - \frac{2}{R^2}\frac{\partial V_\theta}{\partial \theta}\right) \tag{29}$$

$$\text{Azimuthal component: } \mu\left(\nabla^2 V_\theta - \frac{V_\theta}{R^2} + \frac{2}{R^2}\frac{\partial V_R}{\partial \theta}\right) \tag{30}$$

$$\text{Axial component: } \mu\nabla^2 V_z \tag{31}$$

Step 4: Expand the Gradient of the Divergence Term

First, compute the divergence of $\overrightarrow{V}$:

$$\overrightarrow{\nabla}\cdot\overrightarrow{V} = \frac{1}{R}\frac{\partial}{\partial R}(RV_R) + \frac{1}{R}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} \tag{32}$$

Then, compute the gradient of the divergence:

$$\overrightarrow{\nabla}(\overrightarrow{\nabla}\cdot\overrightarrow{V}) = \left(\frac{\partial}{\partial R}(\overrightarrow{\nabla}\cdot\overrightarrow{V})\right)\hat{e}_R + \left(\frac{1}{R}\frac{\partial}{\partial \theta}(\overrightarrow{\nabla}\cdot\overrightarrow{V})\right)\hat{e}_\theta + \left(\frac{\partial}{\partial z}(\overrightarrow{\nabla}\cdot\overrightarrow{V})\right)\hat{e}_z \tag{33}$$

Therefore, the components of the term $\lambda\overrightarrow{\nabla}(\overrightarrow{\nabla}\cdot\overrightarrow{V})$ are:

$$\text{Radial component: } \lambda\frac{\partial}{\partial R}(\overrightarrow{\nabla}\cdot\overrightarrow{V}) \tag{34}$$

$$\text{Azimuthal component: } \lambda\frac{1}{R}\frac{\partial}{\partial \theta}(\overrightarrow{\nabla}\cdot\overrightarrow{V}) \tag{35}$$

$$\text{Axial component: } \lambda\frac{\partial}{\partial z}(\overrightarrow{\nabla}\cdot\overrightarrow{V}) \tag{36}$$

Step 5: Combine All Terms for Each Component

Now, we can write the momentum equation components by combining all the terms.

Radial Component

$$\begin{aligned}\rho\left(\frac{\partial V_R}{\partial t} + V_R\frac{\partial V_R}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_R}{\partial \theta} + V_z\frac{\partial V_R}{\partial z} - \frac{V_\theta^2}{R}\right) = &-\frac{\partial p}{\partial R} \\ &+\mu\left(\nabla^2 V_R - \frac{V_R}{R^2} - \frac{2}{R^2}\frac{\partial V_\theta}{\partial \theta}\right) \\ &+\lambda\frac{\partial}{\partial R}(\overrightarrow{\nabla}\cdot\overrightarrow{V}) \\ &+\rho f_R\end{aligned} \tag{37}$$

Azimuthal Component

$$\begin{aligned}\rho\left(\frac{\partial V_\theta}{\partial t}+V_R\frac{\partial V_\theta}{\partial R}+\frac{V_\theta}{R}\frac{\partial V_\theta}{\partial \theta}+V_z\frac{\partial V_\theta}{\partial z}+\frac{V_RV_\theta}{R}\right)=&-\frac{1}{R}\frac{\partial p}{\partial \theta}\\&+\mu\left(\nabla^2V_\theta-\frac{V_\theta}{R^2}+\frac{2}{R^2}\frac{\partial V_R}{\partial \theta}\right)\\&+\lambda\frac{1}{R}\frac{\partial}{\partial \theta}(\overrightarrow{\nabla}\cdot\overrightarrow{V})\\&+\rho f_\theta\end{aligned}\tag{38}$$

Axial Component

$$\begin{aligned}\rho\left(\frac{\partial V_z}{\partial t}+V_R\frac{\partial V_z}{\partial R}+\frac{V_\theta}{R}\frac{\partial V_z}{\partial \theta}+V_z\frac{\partial V_z}{\partial z}\right)=&-\frac{\partial p}{\partial z}\\&+\mu\nabla^2V_z\\&+\lambda\frac{\partial}{\partial z}(\overrightarrow{\nabla}\cdot\overrightarrow{V})\\&+\rho f_z\end{aligned}\tag{39}$$

A Coordinate Systems

A.1 Cartesian

$$
\begin{aligned}
\overrightarrow{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\
\overrightarrow{V} &= u\hat{\imath} + v\hat{\jmath} + w\hat{k} \\
d\overrightarrow{l} &= dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}
\end{aligned}
$$

A.2 Cylindrical

$R \geq 0$, $0 \leq \theta \leq 2\pi$, $-\infty \leq z \leq \infty$

$$
\begin{aligned}
x = R\cos\theta, \quad y &= R\sin\theta, \quad z = z \\
\overrightarrow{r} &= R\hat{e}_R + \theta\hat{e}_\theta + z\hat{e}_z \\
\overrightarrow{V} &= v_R\hat{e}_R + v_\theta\hat{e}_\theta + v_z\hat{e}_z \\
d\overrightarrow{l} &= dR\hat{e}_R + Rd\theta\hat{e}_\theta + dz\hat{e}_z
\end{aligned}
$$

A.3 Spherical

$r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$

$$
\begin{aligned}
x = r\sin\theta\cos\phi, \quad y &= r\sin\theta\sin\phi, \quad z = r\cos\theta \\
\overrightarrow{r} &= r\hat{e}_r + \theta\hat{e}_\theta + \phi\hat{e}_\phi \\
\overrightarrow{V} &= v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_\phi\hat{e}_\phi \\
d\overrightarrow{l} &= dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi
\end{aligned}
$$

B Vector Operations

In the following, ϕ is any scalar and $\overrightarrow{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$ is any vector. The appropriate expansions will be given in Cartesian, then cylindrical, then spherical coordinates.

$$
\begin{aligned}
\text{Operation} &= \text{Cartesian Expansion} \\
&= \text{Cylindrical Expansion} \\
&= \text{Spherical Expansion}
\end{aligned}
$$

The operations in this section will be based on $\overrightarrow{\nabla}$, the gradient operator:

$$
\begin{aligned}
\overrightarrow{\nabla} &= \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \\
&= \hat{e}_R\frac{\partial}{\partial R} + \hat{e}_\theta\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z} \\
&= \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}
\end{aligned}
$$

B.1 Gradient

$$
\begin{aligned}
\overrightarrow{\nabla}\phi &= \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k} \\
&= \frac{\partial\phi}{\partial R}\hat{e}_R + \frac{1}{R}\frac{\partial\phi}{\partial \theta}\hat{e}_\theta + \frac{\partial\phi}{\partial z}\hat{e}_z \\
&= \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial \phi}\hat{e}_\phi
\end{aligned}
$$

B.2 Divergence

$$\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{a} &= \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \\
&= \frac{1}{R}\frac{\partial}{\partial R}(R a_1) + \frac{1}{R}\frac{\partial a_2}{\partial \theta} + \frac{\partial a_3}{\partial z} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(a_2 \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial a_3}{\partial \phi}
\end{aligned}$$

B.3 Curl

$$\begin{aligned}
\nabla \times \overrightarrow{a} &= \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)\hat{k} \\
&= \left(\frac{1}{R}\frac{\partial a_3}{\partial \theta} - \frac{\partial a_2}{\partial z}\right)\hat{e}_R + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial R}\right)\hat{e}_\theta + \left(\frac{1}{R}\frac{\partial}{\partial R}(R a_2) - \frac{1}{R}\frac{\partial a_1}{\partial \theta}\right)\hat{e}_z \\
&= \frac{1}{\sin\theta}\left[\frac{\partial}{\partial \theta}(a_3 \sin\theta) - \frac{\partial a_2}{\partial \phi}\right]\hat{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_1}{\partial \phi} - \frac{\partial}{\partial r}(r a_3)\right]\hat{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r a_2) - \frac{\partial a_1}{\partial \theta}\right]\hat{e}_\phi
\end{aligned}$$

B.4 Laplacian

$$\begin{aligned}
\nabla^2 \phi &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \\
&= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \phi}{\partial \phi^2}
\end{aligned}$$

$$\begin{aligned}
\nabla^2 \overrightarrow{a} &= (\nabla^2 a_1)\hat{\imath} + (\nabla^2 a_2)\hat{\jmath} + (\nabla^2 a_3)\hat{k} \\
&= \left(\nabla^2 a_1 - \frac{a_1}{R^2} - \frac{2}{R^2}\frac{\partial a_2}{\partial \theta}\right)\hat{e}_R + \left(\nabla^2 a_2 - \frac{a_2}{R^2} + \frac{2}{R^2}\frac{\partial a_1}{\partial \theta}\right)\hat{e}_\theta + (\nabla^2 a_3)\hat{e}_z \\
&= \left[\nabla^2 a_1 - \frac{2}{r^2}\left(a_1 + \frac{\partial a_2}{\partial \theta} + a_2 \cot\theta + \frac{1}{\sin\theta}\frac{\partial a_3}{\partial \phi}\right)\right]\hat{e}_r \\
&+ \left[\nabla^2 a_2 + \frac{2}{r^2}\left(\frac{\partial a_1}{\partial \phi} - \frac{a_2}{2\sin^2\theta} - \frac{\cot\theta}{\sin\theta}\frac{\partial a_3}{\partial \phi}\right)\right]\hat{e}_\theta \\
&+ \left[\nabla^2 a_3 + \frac{2}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial a_1}{\partial \phi} + \frac{\cos\theta}{\sin^2\theta}\frac{\partial a_2}{\partial \phi} - \frac{a_3}{2\sin^2\theta}\right)\right]\hat{e}_\phi
\end{aligned}$$

B.5 Advective Operator

$$\begin{aligned}
\overrightarrow{a} \cdot \overrightarrow{\nabla} &= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z} \\
&= a_1\frac{\partial}{\partial R} + \frac{a_2}{R}\frac{\partial}{\partial \theta} + a_3\frac{\partial}{\partial z} \\
&= a_1\frac{\partial}{\partial r} + \frac{a_2}{r}\frac{\partial}{\partial \theta} + \frac{a_3}{r\sin\theta}\frac{\partial}{\partial \phi}
\end{aligned}$$

B.6 Material Derivative

Also called: convective derivative, advective derivative, substantive derivative, substantial derivative, Lagrangian derivative, Stokes derivative, particle derivative, hydrodynamic derivative, derivative following the motion, or total derivative.

$$\begin{aligned}
\frac{D\phi}{Dt} &= \frac{\partial \phi}{\partial t} + \vec{V} \cdot \vec{\nabla}\phi \\
\frac{D\phi}{Dt} &= \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z} \\
&= \frac{\partial \phi}{\partial t} + v_R\frac{\partial \phi}{\partial R} + \frac{v_\theta}{R}\frac{\partial \phi}{\partial \theta} + v_z\frac{\partial \phi}{\partial z} \\
&= \frac{\partial \phi}{\partial t} + v_r\frac{\partial \phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial \phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial \phi}{\partial \phi}
\end{aligned}$$

There also exists the material derivative of a vector, which yields another vector.

$$\begin{aligned}
\frac{D\vec{a}}{Dt} &= \frac{\partial \vec{a}}{\partial t} + \vec{V} \cdot \vec{\nabla}\vec{a} \\
\frac{D\vec{a}}{Dt} &= \left(\frac{Da_1}{Dt}\right)\hat{\imath} + \left(\frac{Da_2}{Dt}\right)\hat{\jmath} + \left(\frac{Da_3}{Dt}\right)\hat{k} \\
&= \left(\frac{Da_1}{Dt} - \frac{v_\theta a_2}{R}\right)\hat{e}_R + \left(\frac{Da_2}{Dt} + \frac{v_R a_2}{R}\right)\hat{e}_\theta + \left(\frac{Da_3}{Dt}\right)\hat{e}_z \\
&= \left(\frac{Da_1}{Dt} - \frac{v_\theta a_2 + v_\phi a_3}{r}\right)\hat{e}_r + \left(\frac{Da_2}{Dt} + \frac{v_\theta a_1}{r} - \frac{v_\phi a_3}{r}\cot\theta\right)\hat{e}_\theta + \left(\frac{Da_3}{Dt} + \frac{v_\phi a_1}{r} + \frac{v_\phi a_2}{r}\cot\theta\right)\hat{e}_\phi
\end{aligned}$$