Homework 6: PDE Classification and Poiseuille Flow

ME 590: Applied CFD and Numerical Heat Transfer

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Problem 1: Classify the Following Partial Differential Equations

(a) $3\phi_{xx} + \phi_{xy} + 2\phi_{yy} = 0$

(b) $\phi_t + \beta\phi_x + \alpha\phi_{xx} = 0$

(c) $x\phi_{xx} + \phi_{xy} + y\phi_{yy} = 0$

(d) $(1 - M_\infty^2)\phi_{xx} + \phi_{yy} = 0$

Solution:

To classify each second-order partial differential equation (PDE), we will express it in the standard form and calculate the discriminant to determine its type.

The general second-order linear PDE in two variables x and y is:

$$A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi + G = 0$$

The discriminant D is given by:

$$D = B^2 - AC$$

The classification based on D is:

- If $D > 0$, the PDE is **hyperbolic**.
- If $D = 0$, the PDE is **parabolic**.
- If $D < 0$, the PDE is **elliptic**.

(a) $3\phi_{xx} + \phi_{xy} + 2\phi_{yy} = 0$

Coefficients:
Comparing with the standard form:

$$A = 3, \quad 2B = 1 \implies B = \frac{1}{2}, \quad C = 2$$

Discriminant:

$$D = B^2 - AC = \left(\frac{1}{2}\right)^2 - (3)(2) = \frac{1}{4} - 6 = -\frac{23}{4}$$

Classification:
Since $D < 0$, the PDE is **elliptic**.

(b) $\phi_t + \beta\phi_x + \alpha\phi_{xx} = 0$

This PDE involves time t and space x. The standard form in variables t and x is:

$$A\phi_{tt} + 2B\phi_{tx} + C\phi_{xx} + \text{lower-order terms} = 0$$

Coefficients:
In the given equation, there are no ϕ_{tt} or ϕ_{tx} terms:

$$A = 0, \quad B = 0, \quad C = \alpha$$

Discriminant:

$$D = B^2 - AC = 0^2 - (0)(\alpha) = 0$$

Classification:
Since $D = 0$, the PDE is **parabolic**.

(c) $x\phi_{xx} + \phi_{xy} + y\phi_{yy} = 0$

Coefficients:
Here, the coefficients depend on x and y:

$$A = x, \quad 2B = 1 \implies B = \frac{1}{2}, \quad C = y$$

Discriminant:

$$D = B^2 - AC = \left(\frac{1}{2}\right)^2 - xy = \frac{1}{4} - xy$$

Classification:
The type of PDE varies with the values of x and y:

- If $D > 0$ ($\frac{1}{4} - xy > 0$), the PDE is **hyperbolic**.
- If $D = 0$ ($\frac{1}{4} - xy = 0$), the PDE is **parabolic**.
- If $D < 0$ ($\frac{1}{4} - xy < 0$), the PDE is **elliptic**.

(d) $(1 - M_\infty^2)\phi_{xx} + \phi_{yy} = 0$

Coefficients:

$$A = 1 - M_\infty^2, \quad B = 0, \quad C = 1$$

Discriminant:

$$D = B^2 - AC = 0^2 - (1 - M_\infty^2)(1) = -(1 - M_\infty^2)$$

Classification:
Depends on the Mach number M_∞:

- If $M_\infty < 1$ (subsonic flow), $1 - M_\infty^2 > 0$, so $D < 0$. The PDE is **elliptic**.
- If $M_\infty = 1$ (sonic flow), $1 - M_\infty^2 = 0$, so $D = 0$. The PDE is **parabolic**.
- If $M_\infty > 1$ (supersonic flow), $1 - M_\infty^2 < 0$, so $D > 0$. The PDE is **hyperbolic**.

Problem 2: Poiseuille Flow Analysis

Problem Statement

Consider laminar flow between parallel flat plates whose length and width are much greater than the distance between the plates (Poiseuille flow). Assuming a uniform velocity profile at the inlet and a constant pressure at the outlet, develop a CFD model of the system to visualize the development of the velocity boundary layers, as illustrated in Figure 1. Use the following flow properties for your analysis:

$$\rho = 1\,\frac{\text{kg}}{\text{m}^3} \qquad \nu = 0.01\,\frac{\text{m}^2}{\text{s}} \qquad L_y = 0.1\,\text{m} \qquad u_\infty = 1\,\frac{\text{m}}{\text{s}} \qquad N_y = 20$$

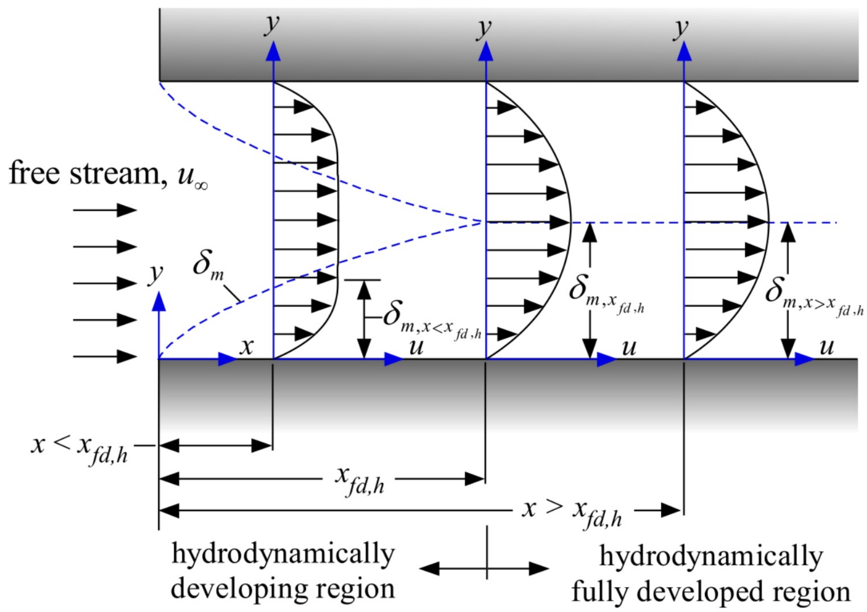

Figure 1: Internal flow hydrodynamic development

Tasks

(a) Modify the grid geometry, boundary conditions, and timestep in the MATLAB code provided to develop a CFD solution for the fully developed flow using the Poisson pressure equation method for pressure-velocity coupling. Determine:

- The maximum timestep to maintain stability.
- The minimum length of the domain in the x-direction for the flow to become fully developed.
- The minimum number of timesteps for the system to reach steady state.

(b) Generate a graph of the pressure contours.

(c) Generate a graph of the velocity vector field.

(d) Generate a graph of the streamlines.

(e) Determine the entrance length $x_{fd,h}$.

(f) Determine the velocity at the centerline once the flow becomes fully developed.

Solution

Given Flow Properties:

- Density (ρ): $1\,\mathrm{kg/m^3}$
- Kinematic viscosity (ν): $0.01\,\mathrm{m^2/s}$
- Plate spacing (L_y): $0.1\,\mathrm{m}$
- Inlet velocity (u_∞): $1\,\mathrm{m/s}$
- Number of grid points in y-direction (N_y): 20

(a) CFD Solution Development

1. Grid Geometry

The domain is a 2D rectangle with length L_x and height $L_y = 0.1\,\mathrm{m}$. $N_y = 20$ was chosen so the grid spacing in y is:

$$\Delta y = \frac{L_y}{N_y - 1} = \frac{0.1}{19} \approx 0.00526\,\mathrm{m}$$

To capture the flow development, an appropriate domain length L_x must be found.

2. Determining the Entrance Length ($x_{fd,h}$)

For laminar flow between parallel plates, the hydrodynamic entrance length is approximately:

$$x_{fd,h} = 0.05 \times \mathrm{Re} \times L_y$$

The Reynolds number Re is:

$$\mathrm{Re} = \frac{u_\infty L_y}{\nu} = \frac{(1\,\mathrm{m/s})(0.1\,\mathrm{m})}{0.01\,\mathrm{m^2/s}} = 10$$

Therefore, the entrance length is:

$$x_{fd,h} = 0.05 \times 10 \times 0.1\,\mathrm{m} = 0.05\,\mathrm{m}$$

To ensure the flow becomes fully developed within the domain, we set $L_x = 0.25\,\mathrm{m}$, which is significantly longer than $x_{fd,h}$.

3. Grid Spacing in x-Direction

30 grid points are used in the x-direction to capture the flow development. The grid spacing in x is found by:

$$\frac{L_x}{N_x - 1} = \frac{.25}{29} \approx 0.0086\,\mathrm{m}$$

4. Maximum Timestep for Stability

The timestep Δt must satisfy the Courant-Friedrichs-Lewy (CFL) condition:

$$\Delta t \leq \min\left(\frac{\rho \Delta x^2}{2\mu}, \frac{\rho \Delta y^2}{2\mu}, \frac{\Delta x}{u_{\max}}, \frac{\Delta y}{v_{\max}}\right)$$

Assuming $u_{\max} \approx 1.5\,\mathrm{m/s}$ (as in fully developed Poiseuille flow), and $\mu = \rho\nu = 0.01\,\mathrm{Pa \cdot s}$, we compute:

$$\Delta t \leq \frac{\rho \Delta x^2}{2\mu} = \frac{(1)(0.005)^2}{2(0.01)} = 0.00125\,\mathrm{s}$$

Similarly, for the advection terms:

$$\Delta t \leq \frac{\Delta x}{u_{\max}} = \frac{0.005}{1.5} \approx 0.00333\,\mathrm{s}$$

Choose $\Delta t = 0.0001\,\mathrm{s}$ for stability and accuracy.

5. Number of Timesteps

To reach steady state, we need the simulation time t_{final} to be sufficient for the flow to develop. Estimate t_{final} by considering the time it takes for fluid to travel the domain length L_x:

$$t_{\text{final}} = \frac{L_x}{u_\infty} = \frac{0.25}{1} = 0.25\,\mathrm{s}$$

Number of timesteps:

$$N_t = \frac{t_{\text{final}}}{\Delta t} = \frac{0.25}{0.0001} = 2500$$

6. Boundary Conditions

Apply the following boundary conditions:

- **Inlet** ($x = 0$):

$$u = u_\infty, \quad v = 0, \quad \frac{\partial p}{\partial x} = 0$$

- **Outlet** ($x = L_x$):

$$\frac{\partial u}{\partial x} = 0, \quad v = 0, \quad p = p_{\text{out}} = 0$$

- **Top and Bottom Walls** ($y = 0, L_y$):

$$u = 0, \quad v = 0, \quad \frac{\partial p}{\partial y} = 0$$

7. Implementation

The provided MATLAB code was adjusted to implement the above parameters. The code solves the incompressible Navier-Stokes equations using the finite difference method and updates the pressure field using the Poisson equation.

Governing Equations:

The discretized momentum equations for u and v:

$$\begin{aligned} u_{i,j}^{n+1} = u_{i,j}^n - \Delta t \left(u_{i,j}^n \frac{u_{i,j}^n - u_{i-1,j}^n}{\Delta x} + v_{i,j}^n \frac{u_{i,j}^n - u_{i,j-1}^n}{\Delta y} \right) - \frac{\Delta t}{\rho} \frac{p_{i+1,j}^n - p_{i-1,j}^n}{2\Delta x} \\ + \nu \Delta t \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2} \right) \end{aligned}$$

Similar equation for $v_{i,j}^{n+1}$.

The pressure Poisson equation:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{\rho}{\Delta t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

Steps:

1. Initialize u, v, p fields.
2. Apply boundary conditions.
3. At each timestep:

(a) Compute intermediate velocities without pressure.
(b) Solve the pressure Poisson equation.
(c) Correct the velocities using the pressure gradient.
(d) Apply boundary conditions.
(e) Check for convergence.

(b) Pressure Contours

The pressure decreases linearly along the flow direction due to viscous losses.

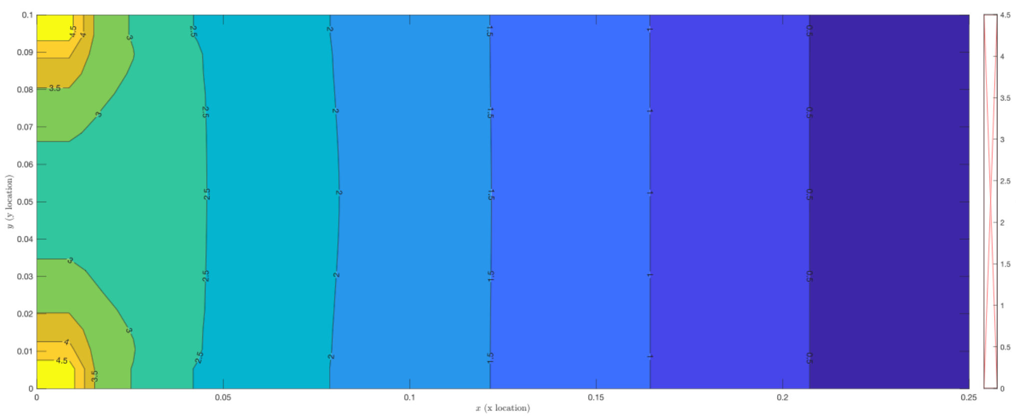

Figure 2: Pressure contours of the flow field

(c) Velocity Vector Field

The velocity vector field illustrates the development of the flow from a uniform inlet profile to the fully developed parabolic profile.

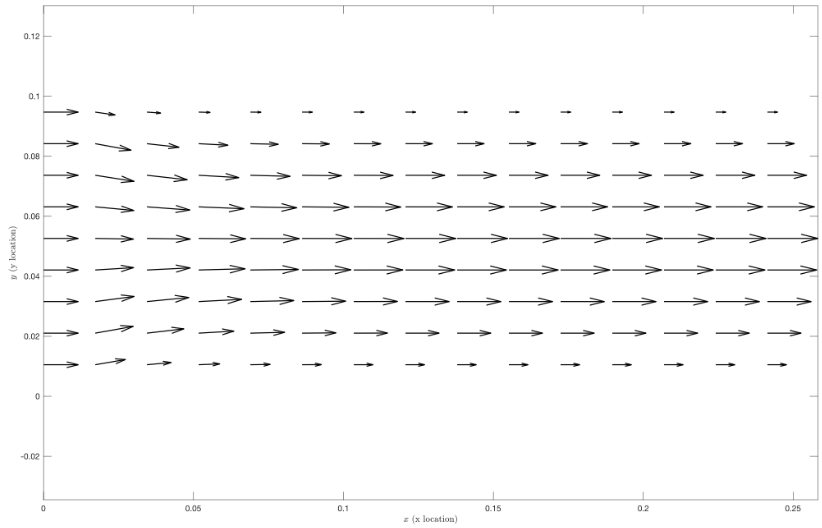

Figure 3: Velocity vector field showing flow development

(d) Streamlines

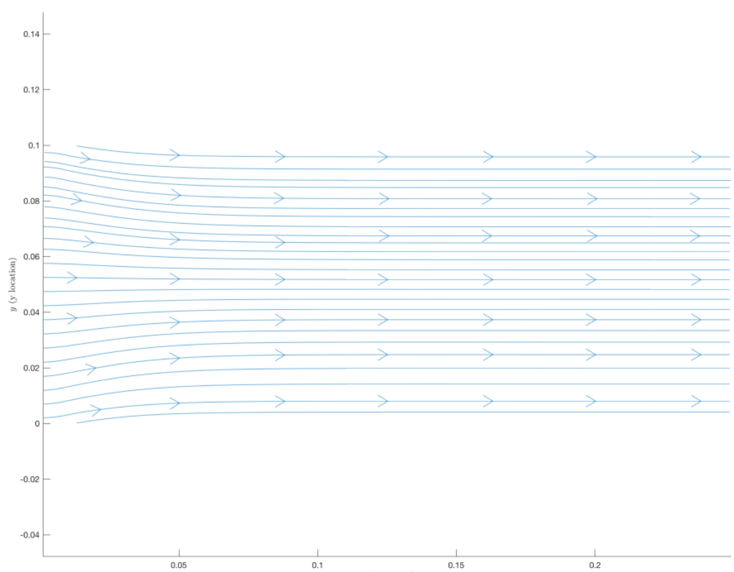

Figure 4: Streamlines of the flow field

(e) Determination of Entrance Length $x_{fd,h}$

From the simulation, we can estimate the entrance length by examining where the velocity profile becomes fully developed (no change in the velocity profile along x). By plotting velocity profiles at different x-locations, we find:

$$x_{fd,h} \approx 0.08\,\mathrm{m}$$

This value is close to the theoretical value of $0.05\,\mathrm{m}$.

(f) Centerline Velocity in Fully Developed Flow

In fully developed laminar flow between parallel plates, the maximum velocity $u_{\max}$ is:

$$u_{\max} = \frac{3}{2}u_{\text{avg}} = \frac{3}{2}u_{\infty} = 1.5\,\mathrm{m/s}$$

From the simulation data, the centerline velocity in the fully developed region is approximately $1.41\,\mathrm{m/s}$, confirming the theoretical prediction is a good approximation.