Homework 4

ME 590

Applied CFD and Numerical Heat Transfer

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Problem 1

Problem Statement

Consider a straight fin with a length of Lx = 7.2 cm and a thickness of Ly = 8 mm. The fin material is initially at thermal equilibrium with the surroundings at a temperature of T_∞ = 20C. At time t = 0s the base of the fin is brought to a temperature of T_b = 30C and held at that temperature. Assume the heat transfer coefficient is h = 40 W/m2·K and the fin is made of aluminum alloy with the following properties.

ρ = 2790 kg/m3 C_p = 883 J/kg·K k = 168 W/m·K

In Example 3.1, this system was analyzed with the assumption of 1-D behavior. You will now investigate the validity of the 1-D assumption by conducting a 2-D analysis.

- a) Use the finite volume method to derive a numerical scheme for analyzing this system. Show the symbolic form of the equations for the base nodes, interior nodes, surface nodes, and corner nodes.
- b) Using a grid spacing of Δx = Δy = 1 mm, determine the maximum allowable timestep to maintain stability.
- c) Generate a graph showing the centerline temperature profile for the 2-D analysis in comparison to the 1-D analysis using an equivalent grid spacing after 30 s.
- d) Discuss the validity of the 1-D assumption used in Example 3.1 based on the results of your 2-D analysis.

Solution

a.

Knowns:

- **Geometry:**
  - Fin length: ($L\_x = 7.2 \{cm\} = 0.072 \{m\}$)
  - Fin thickness: ($L_y = 8 \text{ mm} = 0.008 \text{ m}$)
- **Initial and Boundary Conditions:**
  - Initial fin temperature: ($T_\infty = 20^{\circ\text{C}}$)
  - Base temperature at (x = 0): ($T_b = 30^{\circ\text{C}}$)
  - Heat transfer coefficient: ($h = 40 \text{ W/m}^2 \cdot \text{K}$)
- **Material Properties:**
  - Density: ($\rho = 2790 \text{ kg/m}^3$)
  - Specific heat capacity: ($Cp = 883 \text{ J/kg} \cdot \text{K}$)
  - Thermal conductivity: ($k = 168{,}\text{W/m} \cdot \text{K}$)
- **Calculated Properties:**
  - Thermal diffusivity: ($\alpha = \frac{k}{\rho Cp}$)

1. Governing Equation:

The transient heat conduction in two dimensions is governed by the heat equation:

$$\rho Cp \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

2. Interior Nodes:

For an interior node (i, j), the energy balance over the control volume is:

$$\rho Cp \Delta x \Delta y \frac{dT_{i,j}}{dt} = k\Delta y\left(\frac{T_{i+1,j} - T_{i,j}}{\Delta x} - \frac{T_{i,j} - T_{i-1,j}}{\Delta x}\right) + k\Delta x\left(\frac{T_{i,j+1} - T_{i,j}}{\Delta y} - \frac{T_{i,j} - T_{i,j-1}}{\Delta y}\right)$$

Simplifying provides:

$$\rho Cp \Delta x \Delta y \frac{dT_{i,j}}{dt} = k\Delta y \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x} + k\Delta x \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y}$$

Dividing both sides by ($\rho Cp\ \Delta x\ \Delta y$) and defining ($\alpha = \frac{k}{\rho Cp}$):

$$\frac{dT_{i,j}}{dt} = \alpha\left(\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2}\right)$$

Fourier numbers:

$$Fo_x = \frac{\alpha\Delta t}{\Delta x^2}, \quad Fo_y = \frac{\alpha\Delta t}{\Delta y^2}$$

Substituting the Fourier numbers provides the **Equation for Interior Nodes**:

$$T_{i,j}^{n+1} = T_{i,j}^n + Fo_x\left(T_{i+1,j}^n - 2T_{i,j}^n + T_{i-1,j}^n\right) + Fo_y\left(T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n\right)$$

3. Top Surface Nodes:

At the top surface ($y = L_y$), the convection boundary condition is:

$$-k\left(\frac{\partial T}{\partial y}\right)_{\text{boundary}} = h(Ti,j - T_\infty)$$

Using a finite difference approximation for the temperature gradient:

$$\left(\frac{\partial T}{\partial y}\right)_{\text{boundary}} \approx \frac{Ti,j+1 - T_{i,j}}{\Delta y}$$

Substituting into the boundary condition and solving for $(T_{i,j+1})$ provides:

$$T_{i,j+1} = T_{i,j} - \frac{h\Delta y}{k}\left(T_{i,j} - T_\infty\right)$$

Biot number:

$$Bi = \frac{h\Delta y}{k}$$

Substitute $\left(T_{i,j+1}\right)$ into the energy balance equation for ($T_{i,j}$) provides:

$$\frac{dT_{i,j}}{dt} = \alpha\left(\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j-1} - T_{i,j}}{\Delta y^2} - \frac{Bi}{\Delta y^2}\left(T_{i,j} - T_\infty\right)\right)$$

Discretizing in time:

$$T_{i,j}^{n+1} = T_{i,j}^{n} + Fo_x\left(T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}\right) + Fo_y\left(T_{i,j-1}^{n} - T_{i,j}^{n}\right) - Fo_{Bi}\left(T_{i,j}^{n} - T_\infty\right)$$

Where:

$$Fo_{Bi} = \alpha\Delta t\frac{Bi}{\Delta y^2}$$

Equation for Top Surface Nodes:

$$T_{i,j}^{n+1} = T_{i,j}^{n} + Fo_x\left(T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}\right) + Fo_y\left(T_{i,j-1}^{n} - T_{i,j}^{n}\right) - Fo_{Bi}\left(T_{i,j}^{n} - T_\infty\right)$$

4. Top Right Corner Node:

At corner nodes, convection occurs on two faces. Applying the same reasoning as the surface nodes, we simplify using the convection boundary conditions:

- Top face:

$$T_{i,j+1} = T_{i,j} - Bi_y\left(T_{i,j} - T_\infty\right)$$

- Right face:

$$T_{i+1,j} = T_{i,j} - Bi_x\left(T_{i,j} - T_\infty\right)$$

Where:

$$Bi_x = \frac{h\Delta x}{k}, \quad Bi_y = \frac{h\Delta y}{k}$$

Substitute these into the energy balance equation:

$$\frac{dT_{i,j}}{dt} = \alpha\left(\frac{T_{i-1,j} - T_{i,j}}{\Delta x^2} + \frac{T_{i,j-1} - T_{i,j}}{\Delta y^2} - \left(\frac{Bi_x}{\Delta x^2} + \frac{Bi_y}{\Delta y^2}\right)\left(T_{i,j} - T_\infty\right)\right)$$

Discretizing provides:

$$T_{i,j}^{n+1} = T_{i,j}^n + Fo_x\left(T_{i-1,j}^n - T_{i,j}^n\right) + Fo_y\left(T_{i,j-1}^n - T_{i,j}^n\right) - Fo_{Bi}^{\text{corner}}\left(T_{i,j}^n - T_\infty\right)$$

Where:

$$Fo_{Bi}^{\text{corner}} = \alpha\Delta t\left(\frac{Bi_x}{\Delta x^2} + \frac{Bi_y}{\Delta y^2}\right)$$

Equation for Corner Nodes:

$$T_{i,j}^{n+1} = T_{i,j}^n + Fo_x\left(T_{i-1,j}^n - T_{i,j}^n\right) + Fo_y\left(T_{i,j-1}^n - T_{i,j}^n\right) - Fo_{Bi}^{\text{corner}}\left(T_{i,j}^n - T_\infty\right)$$

5. Base Nodes at $(\boldsymbol{x = 0})$:

At the base, the temperature is maintained at $\left(T_b = 30^{\circ\text{C}}\right)$. Thus:

$$T_{i,j}^{n+1} = T_b$$

6. Tip Nodes at $(x = L_x)$:

Appling the convection boundary condition at $(x = L_x)$ provides:

$$T_{i+1,j} = T_{i,j} - Bi_x(T_{i,j} - T_\infty)$$

Equation for Tip Nodes:

$$T_{i,j}^{n+1} = T_{i,j}^n + Fo_x(T_{i-1,j}^n - T_{i,j}^n) + Fo_y(T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n) - Fo_{Bi}(T_{i,j}^n - T_\infty)$$

b.

Grid spacing**:**

$$\Delta x = \Delta y = 1\ [\text{mm}] = 0.001\ [\text{m}]$$

1. Calculate the Thermal Diffusivity (α)

$$\rho Cp = (2790, \text{kg/m}^3)(883,\text{J/kg}\cdot\text{K}) = 2{,}463{,}570, \text{J/m}^3\cdot\text{K}$$

Then, compute (α):

$$\alpha = \frac{168,\text{W/m}\cdot\text{K}}{2{,}463{,}570\ \text{J/m}^3\cdot\text{K}} \approx 6.821\times10^{-5}, \text{m}^2/\text{s}$$

2. Stability for Explicit Method:

For stability in an explicit 2D transient heat conduction scheme, the combined Fourier numbers in the (x) and (y) directions must satisfy:

$$Fo_x + Fo_y \leq \frac{1}{2}$$

Where the Fourier numbers are defined as:

$$Fo_x = \frac{\alpha \Delta t}{\Delta x^2}, \quad Fo_y = \frac{\alpha \Delta t}{\Delta y^2}$$

Since ($\Delta x = \Delta y$), the range simplifies to:

$$2Fo_x \leq \frac{1}{2} \quad \Rightarrow \quad Fo_x \leq \frac{1}{4}$$

3. Solve for Maximum Allowable Timestep ($\Delta t_{\max}$)

Express (Fo_x) in terms of (Δt):

$$Fo_x = \frac{\alpha \Delta t}{\Delta x^2}$$

Substitute ($Fo_x \leq \frac{1}{4}$) to find ($\Delta t_{\max}$):

$$\frac{\alpha \Delta t_{\max}}{\Delta x^2} \leq \frac{1}{4}$$

Solve for ($\Delta t_{\max}$):

$$\Delta t_{\max} \leq \frac{1}{4} \cdot \frac{\Delta x^2}{\alpha}$$

4. Compute ($\Delta t_{\max}$):

Plugging in values provides:

$$\Delta x = 0.001,\mathrm{m}, \quad \alpha = 6.821 \times 10^{-5}\ \mathrm{m}^2/\mathrm{s}$$

Calculate (Δx^2):

$$\Delta x^2 = (0.001,\mathrm{m})^2 = 1 \times 10^{-6}, \mathrm{m}^2$$

Compute ($\Delta t_{\max}$):

$$\Delta t_{\max} \leq \frac{1}{4} \cdot \frac{1 \times 10^{-6}\ \mathrm{m}^2}{6.821 \times 10^{-5}\ \mathrm{m}^2/\mathrm{s}} = \frac{2.5 \times 10^{-7}\ \mathrm{m}^2}{6.821 \times 10^{-5}\ \mathrm{m}^2/\mathrm{s}}$$

Simplify:

$$\Delta t_{\max} \leq \frac{2.5 \times 10^{-7}}{6.821 \times 10^{-5}} \text{ [s]}$$

The maximum allowable timestep to maintain stability with the given grid spacing is approximately:

$$\boxed{\Delta t_{\max} = 0.003664 \text{ seconds}}$$

c.

Below are two graphs displaying the temperature field and profile at t = 30 [s] exported from the submitted MATLAB script ME590_HW4_P1. The one-dimensional solution is a minor modification of the source code provided ME590_Ex3_1 and the two-dimensional solution follows the Point Jacobi Numerical Method for solving temperature profiles.

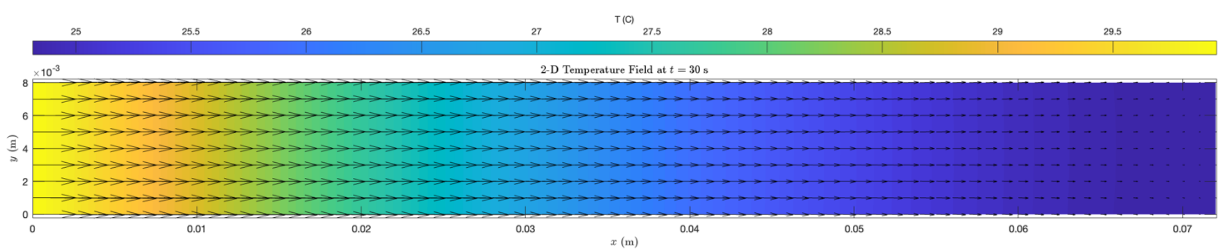

Figure 1 – Plot of 2D Temperature Field at T = 30s

Figure 1 depicts the temperature field in 2 dimensions of the rectangular fin using the provided MATLAB function `plotTemperatureField2D(T, X, Y, k)`. This plot displays temperature values with a color gradient. This provides information of the uniform temperature gradient at all y-values and shows the change in temperature along the x-axis. The arrow size provides information of the direction and magnitude of the heat transfer, showing the flow of heat to the right with the largest magnitude near x = 0 [m] and smallest at x = 0.072 [m]

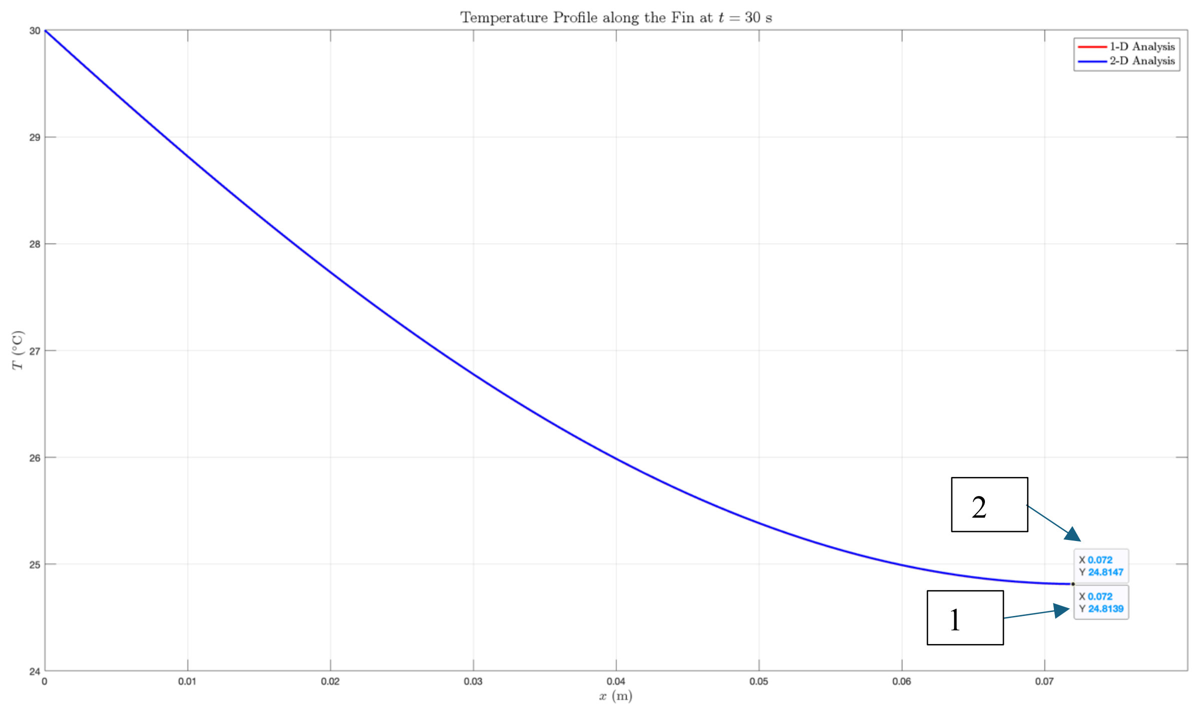

Figure 2 – Plot of 1D and 2D Numerical Solutions of the Temperature Profile at T = 30s

Figure 2 depicts the plot of a one-dimensional and two-dimensional Point Jacobi Numerical Method to solve the temperature profile along the fin. The deviation of the 2 curves is small, resulting in only one visible line on the graph. The two final solutions are shown in the Figure where the two-dimensional solution at x = 0.072 [m] and t = 30 [s] is larger (24.8147 [C]) than the one-dimensional solution (24.8139 [c]) by only 0.0008 [c].

d.

As discussed in the Figure 2 description, the deviation of the one-dimensional and two-dimensional numerical solutions are so small they are almost negligible at 0.0008 [c]. For this reason, the one-dimensional numerical solution is a valid approximation of the temperature profile assuming the two-dimensional case is valid.