APPLICATIONS OF CANONICAL TRANSFORMATIONS AND NONTRIVIAL VACUUM SOLUTIONS TO FLAVOR MIXING AND CRITICAL PHENOMENA IN QUANTUM FIELD THEORY

BY

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a thesis submitted to the graduate faculty of

north carolina state university

in partial fulfillment of the

requirements for the degree of

doctor of philosophy

department of physics

raleigh

October 2004

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Abstract

In this dissertation we consider two recent applications of Bogoliubov Transformation to the phenomenology of quantum mixing and the theory of critical phenomena. In recent years quantum mixing enjoyed increasing attention due to its unparalleled sensitivity to SM parameters and indications of neutrino mixing that put the mixing phenomenology in the very center of the searches for New Physics. It was recently suggested that Bogoliubov Transformation may be important in proper definition of the flavor states that otherwise presents a problem in simpler perturbative treatment. As first part of this dissertation we investigate this conjecture and develop a complete formulation of such a mixing field theory involving introduction of general formalism, analysis of space-time conversion and phenomenological implications.

As second part of this dissertation we focus our attention on Oscillator Representation Method relevant to the study of degrees-of-freedom rearrangement during phase transitions in which vacuum condensation and mass change are analyzed using Bogoliubov Transformation. Given parallels with the duality between the quarks and hadron as well as constituent and current quarks, this method presents an attractive and interesting idea. We review this method and consider its applications to nonlinear sigma model and other models. We also discuss possible schemes for its improvement.

Motivated by variational extensions of the above mentioned method, we further introduce and discuss a novel variational approach in QFT which we call Symmetric Decomposition Problem. In this approach we attempt to substitute variational problem in terms of complicated Fock space with a constrained minimization problem in terms of only relevant quantum operators expectation values. Application of this principle to quadratic operators and full derivation of the exact constraints on the expectation values in this case are discussed along with an example of study of the ground state in a variant of nonlinear sigma model.

Biography

I was born on 5th of June, 1978 in Boiarka, which is near the capital city of Ukraine Kiev. When I was about one year old my family moved to town Tynda, which is in Russian Siberia. In Tynda I grew up till age of about 10. My primary school and half of my childhood passed there. In 1988, after almost 10 years, my mother decided to return to Ukraine. That was a big change for me - change of place, change of class, change of life. In Ukraine we returned to a house in Boiarka that my family owned since 1970th. There I also continued on in school until age of 14. Since then first signs of my interest to physics and mathematics became apparent. I still remember myself sitting for hours in front of TV and watching educational astronomical programs. My special interests included space mechanics and introductory physics and chemistry school texts. I also learned differential and integral calculus at that time. While the tale of those times is worth telling in its own, this isn't the proper place for it.

In 1995 I took examination and was admitted to specialized board school of "Kiev Physical-Mathematical Lyceum", which is a division of Kiev Taras Shevchenko University. In the Lyceum my interest in natural sciences took sharper focus and shape of theoretical physics so that when the time came in 1995 to choose the path of life between math, cybernetics or physics, my choice was quick. That is to say, I passed examinations to Physics Department of Kiev Taras Shevchenko University and was enrolled in the Quantum Field Theory division.

In Taras Shevchenko University passed next 5 years of my life. That was fun time of learning and getting to know the world around us. I should say little about

this time as I would have to say too much otherwise. There I began to work with Prof. I. Simenog on the applications of variational approach in Quantum Mechanics and Field Theory. My first research project, completed for Bachelor of Science degree, was concerned with high-precision variational calculations of the ground state of 3-body meso-molecules. The beauty of this problem is in that it is completely defined by Coulomb interaction and still the task of achieving record precision is by far a nontrivial one. We used so called Galyorkin expansion in Gaussian basis with special emphasis on the basis optimization which allowed us to reduce 10-fold the number of Gaussians needed to reach precision comparable with the best groups in the world using merely a home PC. Still, pounding the digits of numerically generated answer for higher and higher precision for the years to come didn't make me very excited. In 2000, for my Master Degree project, I concentrated on applications of variational principle for mass renormalization and nonperturbative relativistic 2-body Shrodinger equation in scalar Yukawa quantum field theory. Using a special form of coherent trial state we were able to obtain a full numerical solution of this problem for any values of coupling and also developed analytical approximations useful in weak and strong couplings. Also, at that time I continued to work on 3-body Coulomb problem.

Nonetheless, by the time I graduated from Kiev Taras Shevchenko University with a M.Sc. worsening economical situation in the country left me little choice if the physical research would stay in my primary focus. In 2000 I applied and was admitted to Physics Department of North Carolina State University in Raleigh where I eventually started in August of 2000. Here I concentrated on the phenomenology of flavor mixing and thereafter some topics in the field theory of phase transitions and critical phenomena that eventually lead to this dissertation.

Preface

This thesis deals with recent applications of Bogoliubov Transformation to the field-theoretic formulation of flavor mixing and to the study of critical behavior of field theories within Oscillator Representation Method. The dissertation contains a brief review of Canonical Transformations with a special emphasis on quantum linear Canonical Transformation (or Bogoliubov Transformation) as they appear in Classical and Quantum Physics and their applications in superfluidity and low-energy QCD. The field theory of flavor oscillations is formulated with Bogoliubov Transformation and the Oscillator Representation Method in QFT are fully introduced and illustrated. Original contributions are presented. An independent variational approach in QFT, method of Symmetric Decomposition Problem, originally motivated by the ORM is also fully introduced and illustrated for the application to the analysis of the ground state in a variant of nonlinear sigma model.

Acknowledgments

I owe a debt of gratitude to my academic adviser Dr. Chueng-Ryong Ji for his guidance and support throughout the term of my PhD study not only via many discussions but also by encouraging me at all times during this dissertation research.

I am greatly thankful to Dr. Dean Lee for many useful comments, advises and his constant enthusiasm to help with any question.

I would like to thank my other committee members Dr. David Brown and Dr. Ronald Fulp for their time, help and advises on topics related to my dissertation as well as many others.

I am also thankful to SURA for two years of partial financial support through SURA/JLAB fellowship which played a significant role in making research, in part incorporated in this dissertation, possible.

I would like to express my appreciation to Graduate Program Director Dr. Michael Paesler for continuing care about the Graduate Program and Graduate Students. Also, without our Graduate Program Secretary Jenny Allen my PhD student life would definitely be much more complicated.

Finally, I want to thank all professors at North Carolina State University as well as at Taras Shevchenko Kiev University whose courses I have taken and who have made, undoubtedly, significant contribution into my growth and maturing as a scientist.

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Introduction

With continuing progress in both understanding the low-energy QCD properties such as duality relations, sum-rules, etc and unraveling experimental investigation of the parton distribution functions, hadrons structure and nuclear matter properties, non-perturbative aspects of Quantum Field Theory (QFT) gain an increasing importance in the particle physics. While perturbative approach not only provided the foundation for our understanding of the elementary particles and quantum processes but also served well in solving the puzzles of electromagnetic and weak interactions, these days physics phenomenology puts forward new demands to theoretical methods in order to understand the low-energy and high-density behavior of QCD, critical phenomena and phase transitions in condensed matter physics and even cosmology. A number of successful approaches to deal with these applications have been developed in the past, most general stemming from Feynman Path Integration formalism which lately evolved into an extensive field of lattice QCD. Numerous accomplishments in lattice QCD have been already scored. Examples are the calculation of quark-quark potential in quenched lattice QCD, the observation of string formation and breakdown in unquenched formulation, calculation of strong coupling constant and the quark-gluon deconfinement at high temperatures/densities known as Quark Gluon Plasma (QGP). Due to the *ab initio* nature of lattice calculations, this approach is deemed to be most promising for the fundamental inquiries in nonperturbative QCD.

Despite a number of successes, lattice QCD is essentially an experimental approach due to its nature and lacks important physical insight. Even though the results in

lattice QCD are not obtained by direct experimentation, still it requires to an extensive numerical simulations of certain experimental configurations. Lattice also requires unparalleled amount of computational resources in order to achieve acceptable reliability in the results. Nonetheless, with a proper use of lattice in field theory a substantial theoretical advance can be made by obtaining and combining pivotal numerical results, e.g. the QCD equation of state, the quark-quark potential and the parton distribution functions, with more general phenomenological approaches. Indeed, this can be seen as one of the most promising directions in understanding the features of strongly interacting QCD.

Yet, due to the lack of clear physical intuition as well as serious numerical demands and severe limitations in choosing possible problems, which are inherent to current state of lattice calculations, the simpler analytical approaches capable of dealing with nonperturbative issues in QFT are of great interest and importance. One of the major hints for this direction is offered by the Nature in the phenomenon of duality. In general sense, duality reveals itself in QCD as the presence of distinct regimes where properties of QCD can be viewed as originated from different, yet related, fundamental particles or fundamental degrees of freedom. Particle scattering, hadron resonance spectra, etc can be remarkably well understood in terms of massive constituent quarks at low energies and in terms of practically massless current quarks at high energies. One can see from this duality a quantum manifestation of the feature which has long been known in Classical Mechanics - namely the possibility to describe a physical system in different generalized coordinates and the existence of great simplifications with the appropriate choice of the latter. Dramatic transformation from the fundamental properties in physical theories to the observable features, characteristic to quantum physics, leads to our perception of such "optimal" coordinates as the "real" particles.

The importance of the proper choice of the degrees of freedom can be hardly underestimated in physics. Taking the form of Canonical Transformations and generalized coordinates formalism in Classical Mechanics, the transformations of degrees of freedom had been taken to a qualitatively different level in Quantum Physics and

are responsible for such dramatic phenomena as phase-transitions, superconductivity and quark-hadron duality. Representations of physical theory in terms of different generalized coordinates in Classical Mechanics was emphasized and elaborated in the theory of Canonical Transformations (CT) using Hamiltonian dynamics and Jacobi Equation. Canonical Transformations play a central role in Classical Mechanics both by providing a powerful tool for solving dynamics and by setting up a conceptual framework to establish bridges between various physical models.

Unfortunately, the general theory of CT fails in Quantum Mechanics due to the operator nature of the generalized coordinates that leads to the ordering ambiguities in generating functions. Although some conjectures existed that identified CT in Quantum Mechanics with unitary transformations and even some classes of non-unitary transformations, the true relevance of CT in Quantum Physics is yet to be established. Out of a large body of CT known in Classical Mechanics, the only transformation that survived the quantum transition is what is known today as Bogoliubov Transformation (BT). This is regretful because the phenomenon of duality is likely the most unambiguous hint for the importance of the degrees of freedom change and transformation in Quantum Physics.

BT is a linear CT usually formulated in terms of creation and annihilation operators. It introduces quasi-particle objects, described as coherent superposition of the original particles and holes (absence of particle). It was originally suggested by Russian physicist N. Bogoliubov as a base for microscopic theory of superfluidity. In superfluidity, BT helps to obtain a concise answer to otherwise intractable problem and yields its full solution, e.g. energy spectrum and vacuum state structure. In the recent time BT found growing applications in many areas of field theory. Among these, perhaps one of the most prominent areas may be the QCD. Appearing under the name of BCS or RPA, modifications of BT had long been used to obtain answers about the structure of QCD vacuum, in particular, its superconducting and condensation properties. In the recent years, applications of BT had been extended to new areas including QFT of the critical phenomena and the vacuum structure in quantum

mixing of flavors.

As was shown in the literature, serious difficulties exist with definition of the flavor states in field theory of flavor mixing. This provided a serious blow to the usual wave-packet treatment in Field Theory and Quantum Mechanics. Recently it was suggested by some authors that a version of BT can be used to diagonalize the mixing Hamiltonian and thus to define a Fock space for the flavor states. An extensive research program into this approach had been carried out over the past 10 years and it was shown that such treatment introduces new nontrivial effects into the flavor oscillation formulas due to the interactions with the nontrivial flavor condensation in the vacuum.

Another approach, dealing with quite different topic in QFT, is the Oscillator Representation Method (ORM) which is concerned with the degrees-of-freedom change and rearrangement in critical phenomena and the phase transitions in field-theoretic models. The ORM uses BT as a central ingredient to define a set of alternative degrees-of-freedom in terms of which the phase transition can be considered and described in detail. Nontrivial vacuum condensation and dynamic mass generation had been successfully described using this approach. Also, the phenomenon of duality had been observed. In these settings, a strong-interacting field-theory with light elementary particles can be related by duality to a weak-interacting theory with heavy particles with dynamically generated mass. These features of the ORM are promising with respect to the dynamic generation of mass for constituent quarks and the duality between current and constituent quarks in QCD.

In this dissertation, we concentrate on the above two recent applications of BT and nontrivial vacuum in QFT. We consider the role of BT in the field theory of mixing and the definition of the flavor quantum states and make an attempt toward a complete formulation of such a field theory of mixing. This involves an introduction of general formalism, a study of space-to-time conversion and an analysis of phenomenological implications. We also discuss the general application of BT in study

of critical phenomena and the role of nontrivial vacuum in the ORM. We pay special attention to the specific application of the ORM in the nonlinear sigma model. We test the extent, reliability and conclusions of the ORM from the example of the phase transition in the 1+1 dimensional scalar ϕ^4 model for which properties of exact solution is better known. Finally, we concentrate on the analysis of the ORM as a general field-theoretic technique and seek the possibilities for its improvement. As a part of this study, we focus on the nonperturbative aspects of the ORM and discuss the prospects for its systematic improvement via variational and Quantum Effective Action (QEA) extensions.

As a final part of this dissertation, we will present and discuss a novel variational technique, initially inspired by the ORM, which we call Symmetric Decomposition Problem. In this approach, we attempt to substitute the variational problem in terms of complicated Fock space by a constrained minimization problem in terms of only relevant quantum expectation values. The application of this principle to quadratic operators and the full derivation of the exact constraints on the expectation values in this case are considered along with an example of study of the ground state in a variant of nonlinear sigma model.

Chapter 1

Canonical Transformations in Quantum Physics

The Canonical Transformation (CT) is a powerful tool of Classical Mechanics. Yet, the strength of CT is yet to be fully realized in Quantum Mechanics [1,2]. The ground for CT in classical mechanics is laid down by the Hamilton formalism in which the Lagrangian function and the dynamics in terms of generalized coordinates

$$\frac{\partial}{\partial \dot{q}_i}\mathcal{L}(q,\dot{q},t) - \frac{\partial}{\partial q_i}\mathcal{L}(q,\dot{q},t) = 0 \tag{1.1}$$

is translated into dynamics in terms of twice as many coordinates and momenta and the Hamiltonian function

$$\begin{aligned} \dot{q}_i &= \tfrac{\partial}{\partial p_i}\mathcal{H}(q,p,t), \\ \dot{p}_i &= -\tfrac{\partial}{\partial q_i}\mathcal{H}(q,p,t). \end{aligned} \tag{1.2}$$

Hamiltonian formalism is advantageous over Lagrangian formalism in that it reduces the system's dynamics to solution of a set of first order differential equations. On the other hand, one has to deal with twice as many variables. The main advantage of the Hamiltonian formalism is, however, in the conceptual framework. Note that in Lagrangian formalism the only "fundamental" degrees of freedom are the generalized coordinates and the momenta are merely derivatives $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. In Hamiltonian formalism both coordinates and momenta are treated on equal footing and are independent except for the dynamical link established by Hamilton's equations (1.2).

Such independence provides most important piece of the foundation for the theory of Canonical Transformations.

Thus, in classical mechanics CT is such a change of the phase space variables $(q, p) \to (Q, P)$ that preserves the Poisson bracket

$$[q, p] = 1 = [Q, P]. \tag{1.3}$$

The Poisson bracket is defined as usual,

$$[f, g] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{1.4}$$

Main property of CT is the way the *action* is transformed

$$\int dt(p\dot{q} - \mathcal{H}(p, q, t)) = \int dF(P, Q, t) + \int dt(P\dot{Q} - \mathcal{H}'(P, Q, t)), \tag{1.5}$$

where the full differential is given by

$$dF = p \cdot dq - P \cdot dQ. \tag{1.6}$$

As long as Eq.(1.5) holds, the dynamics of the system in new coordinates is described with Hamilton formalism and, in this sense, is similar to that of the original description. $F(P, Q, t)$ is often used to characterize classical CT and is typically called a generating function [3]. CTs are extremely helpful tools that allow to change the system's Hamiltonian to a simpler form thus leading to great simplifications in the equations of motion. A textbook example in this respect is the oscillator dynamics which can be transformed to a trivial problem with decoupled variables with [3]

$$\begin{aligned} P &= \tfrac{1}{\sqrt{2}}(ip + x) \\ Q &= \tfrac{1}{\sqrt{2}}(-ip + x), \end{aligned} \tag{1.7}$$

conventionally denoted as a and $a^\dagger$. CT can be also used to generate families of exactly solvable Hamiltonians out of a single Hamiltonian where the dynamics is known, thus, providing a set of "toy" models whose properties can be studied exactly. Finally, CT plays a central role in the Jacobi theory where a special transformation is sought that

reduces the dynamics to a trivial one $\mathcal{H} \to \mathcal{H}' = 0$ and thus provides immediately the full solution to the classical equations of motion [3].

In Quantum Mechanics, however, the use of CTs is practically completely lost. Three major problems exist in the translation of the formalism of CTs to the Quantum Mechanics [4,5]: the ordering of operators must be specified, the inverse and fractional powers of operators that may appear in the transformation must be handled and the possibility of non-unitary CT must be addressed. While in Classical Mechanics three major roles of canonical transformations (evolution, physical equivalence and solving theory) are blurred together, in Quantum Mechanics they are distinct. In Quantum Mechanics the evolution is produced by unitary transformations, while the physical equivalence is proved with isometric transformations (norm preserving isomorphisms between different Hilbert spaces) and the solution of a theory is achieved by general transformations which may involve non-unitary transformations [6].

A number of approaches had been pursued in a general attempt to resolve these issues [1,2,5,7–10]. The conjecture that CTs in Quantum Mechanics are one and the same with the unitary transformations is one of the oldest such attempts [7]. Non-hermitian linear transformations [9] and general form integral transformations [8] had been also considered in this respect. Such approaches typically experience the problem that the procedure necessary to build and apply a CT is not at all simpler than finding the solution of the original Schrodinger equation itself, thus undermining the very first idea of the use of CT for simplification of original problem [7,8].

Rather different approach embraces Path Integral as the base for further development of CT in the quantum theory [10]. Feynman Path Integral (PI) written in the phase space of p and q provides one of the most startling paradoxes in the breakdown between classical and quantum CTs. Conventionally, one writes the PI in the Euclidean space as

$$\mathcal{U}(q,q';t) = \int \mathcal{D}q(t)\mathcal{D}p(t)e^{\int dt(ip\dot{q}-H(p,q,t))}. \tag{1.8}$$

Similarity between Eq.(1.8) and Eq.(1.5) may tempt one to use this relation as the

basis for the program of quantization of CT [10]. Nonetheless it had been noted in the literature that the main obstacle along this line consists in the discretized nature of the expression (1.8) [2]. Specifically, one need to remember that Eq.(1.8) is only a formal representation in which the time derivatives, e.g., should be properly defined. In this sense Eq.(1.5) can not hold if $p \cdot dq - P \cdot dQ \neq dF(P,Q)$ in a finite-difference form. Also, an explicit application of canonical transformation to formal expression (1.8) to derive, e.g., analog of Jacobi equation for the propagator $\mathcal{U}(q,q')$ immediately yields an inconsistent answer.

Breakdown of Eq.(1.6) in quantum case can be seen as the primary source of the lack of correspondence between quantum and classical CTs. It may be of interest to examine this point more closely. Consider, e.g. the point transformation $q \to Q(q)$ in the usual Lagrangian form of the Path Integral

$$\begin{array}{l} \mathcal{U}(q,q';t) = \int \mathcal{D}q(t) e^{-\int dt(\dot{q}^2/2 + V(q))} \to \\ \mathcal{U}(Q,Q';t) = \int \mathcal{D}Q(t) J(Q) e^{-\int dt[(\frac{dQ}{dq})^2 \dot{Q}^2/2 + V(Q)]}. \end{array} \tag{1.9}$$

This transformation is widely applied in the field theory and generally known to work well even though Eq.(1.9) is only a formal representation of a properly discretized expression. One reason for this is that the contributions to the PI in Lagrangian form come only from the trajectories which are continuous, thus, justifying the use of transformation (1.9). Really, one considers a trajectory which has a discontinuity $q(\tau) \to q(\tau + 0) = q(\tau) + \Delta$ at some time τ. If sampled with time step dt, such trajectory would contribute to the integral a quantity

$$e^{-\Delta^2/2dt} e^{-\int\limits_{t \neq \tau} dt(\dot{q}^2/2 + V(q))}. \tag{1.10}$$

As $dt \to 0$, this contribution becomes exponentially suppressed relative to the contributions coming from the comparable continuous trajectories. Then, it can be said that the support of the integral in Eq.(1.9) consists only from the continuous trajectories $q(t)$ and this is why the formal operation with the integrand in Eq.(1.9) works.

In the case of the phase space form of PI (1.8), one may immediately observe that integration over discontinuous trajectories $(q(t), p(t))$ is not suppressed. Really, the contribution of such a trajectory would come with merely a factor of $e^{p\Delta q}$ where Δq is the discontinuity in $q(t)$. Moreover, to derive the Lagrangian PI from Eq.(1.9), one needs to integrate each $p(x)$ from $-\infty$ to $+\infty$ regardless of the values of $p(x)$ at the neighboring points. Thus, in Eq.(1.8) the discretized nature of the integral is important and Eq.(1.6), infinitesimally correct, cannot be generally used. It is useful to note, however, that Eq.(1.6) will retain its general validity in finite-differenced form if the canonical transformation is *linear*. Indeed, it had been known for quite some time that linear canonical transformation can be applied successfully in Quantum Mechanics (most typical example is, again, quantum oscillator) [1].

Despite such severe limitations of the apparatus of CTs in Quantum Mechanics, the use of linear CT had proved to be of a great advantage in the study of nonperturbative features of quantum systems. The quantum linear CT was first put forward in 1947 by Russian physicist N. Bogoliubov in order to build a microscopic theory of superconductivity and often bears his name [11]. Bogoliubov Transformation (BT) is usually formulated in terms of creation/annihilation operators of quantum many-body problem. Two forms of BT (one for fermions and another for bosons) are known. For bosons BT reads

$$\begin{aligned} A_k &= u(k)a_k - v(k)b^\dagger_{-k} \\ B^\dagger_{-k} &= v(k)a_k - u(k)b^\dagger_{-k}, \end{aligned} \tag{1.11}$$

where $u(k)$ and $v(k)$ are transformation amplitudes such that $u^2(k) - v^2(k) = 1$. For fermions

$$\begin{aligned} A_k &= u(k)a_k + v(k)b^\dagger_{-k} \\ B^\dagger_{-k} &= -v(k)a_k + u(k)b^\dagger_{-k} \end{aligned} \tag{1.12}$$

and $u^2(k) + v^2(k) = 1$.

Bogoliubov Transformation can be viewed mathematically as a rotation of basis in the linear space of quantum fields built on $(a_k, b_k, h.c.)$. BT is a unitary transformation

and can be represented in the form

$$U = e^{i\sum\limits_{k} \rho(k)(a_k^\dagger b_{-k}^\dagger - a_k b_{-k})}. \tag{1.13}$$

Physically, due to the manifestation of fundamental physical properties as observable effects inherent to Quantum Physics, this transformation describes an alternative set of degrees of freedom of field-theoretic model that appear as quasi-particles with different properties than those of the original particles.

Over the years, BT found wide range of applications in various areas of quantum physics from condensed matter theory [12] to strongly interacting QCD [13, 14]. It provided a powerful nonperturbative tool that helps to understand many central features of macroscopic behavior of field-theoretic models. To illustrate the power of BT, let us consider its first success in the theory of superconductivity in greater details.

Superfluidity is the phenomenon of loss of viscous friction in a flowing fluid at superlow temperatures. Superfluidity was discovered by Petr Kapitza and Arno Allen Penzias and Robert Woodrow Wilson in 1938 [15]. P. Kapitza later received Nobel prize for this discovery. The theoretical explanation of superfluidity was obtained in the works of L. Landau and N. Bogoliubov [11, 16, 17]. According to Landau, viscous friction is due to the transfer of momentum between neighboring fluid elements toward the walls of the fluid container via the gradient in the fluid velocity. At low temperature, the viscous drag is transfered to the fluid from the stationary walls by means of elementary excitations. If such elementary excitation is created at the wall with momentum p, it will transfer to the fluid a momentum $P_0 = p$ and an energy $E_0 = \epsilon(p)$ so that the momentum and energy of the fluid become

$$\begin{aligned} P &= Mv + P_0 \\ E &= \tfrac{1}{2}Mv^2 + E_0 + P_0 v. \end{aligned} \tag{1.14}$$

The central point of this argument is the notion that the viscous drag excitations can be created at the walls only if there is free energy to do so, i.e. if the energy of the

fluid will decrease when such excitation is created. That means that if

$$v < \epsilon(p)/p \tag{1.15}$$

then none such elementary excitation can be created. For any fluid in which elementary excitations at low temperature are phonons with speed

$$u = \lim_{T \to 0} \frac{\epsilon(p)}{p}, \tag{1.16}$$

such condition occurs when $v < u$.

In our argument, we intentionally dismissed the role of momentum transfer by the population of thermal elementary phonons that always exist in the liquid and that is capable of carrying momentum from one point to another and thus provide a source of viscous friction. The momentum, that thermal elementary excitations may support, is finite and decrease with the temperature. It may be shown that such momentum is proportional to fluid velocity and is

$$P = M_{ex} v. \tag{1.17}$$

Whenever $M_{ex} < M$ the momentum capacity of thermal phonons becomes insufficient to decelerate the fluid, i.e. superfluidity is observed. In these conditions, one component of mass M_{ex} acts like an ordinary fluid being subject to viscous forces and the remainder experiences zero viscous effect and forms the superfluid component. The amount of viscous component at given temperature can be calculated in the microscopic theory of superfluidity, which we will describe below, and can be shown to fall as T^4 as the temperature decreases.

The microscopic theory of superfluidity had been suggested originally by N. Bogoliubov [11, 12] and made use of three fundamental points which are thought to be valid for low-temperature real fluid. The three fundamental assumptions of Bogoliubov's treatment are that at low temperature a macroscopic number N_0 of particles in the fluid occupies one single-particle state, e.g. $k = 0$, the interaction between particles is essentially short range and its main effect consists in scattering particles

in/out of the Bose condensate, thus making pairs of particles with momenta k and $-k$ as the second largest population in the system. With these assumptions the original many-body Hamiltonian

$$\mathcal{H} = \sum_k \epsilon_k^0 a_k^\dagger a_k + \frac{1}{2V} \sum_{k_1,k_2,k_3,k_4} V_{k_1-k_3} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \delta_{k_1+k_2,k_3+k_4} \tag{1.18}$$

can be transformed into

$$\mathcal{H} = \frac{gN^2}{2V} + \frac{1}{2} \sum_{k\neq 0} \left[(\epsilon_k^0 + ng)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + ng(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) \right], \tag{1.19}$$

where we also neglected "small" commutator $[a_0, a_0^\dagger]$ relative to N_0. Here, g is the strength of effective short range interaction between the particles in the superfluid fluid and $n = N/V$ is the particle density.

It was further observed by N. Bogoliubov that Hamiltonian (1.19) can be exactly diagonalized with the linear canonical transformation of the form (1.11) where

$$v_k^2 = u_k^2 - 1 = \frac{1}{2}\left(\frac{\epsilon_k^0 + ng}{E_k} - 1 \right) \tag{1.20}$$

and

$$E_k = \sqrt{(\epsilon_k^0 + ng)^2 - (ng)^2} = \sqrt{(\epsilon_k^0)^2 + 2ng\epsilon_k^0}. \tag{1.21}$$

After BT, the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2} gn^2 V - \frac{1}{2} \sum_{k\neq 0} (\epsilon_k^0 + ng - E_k) + \frac{1}{2} \sum_{k\neq 0} E_k (A_k^\dagger A_k + A_{-k}^\dagger A_{-k}). \tag{1.22}$$

Eq.(1.22) solves the original problem in its entirety yielding the energy spectrum of the elementary excitations, ground state energy and vacuum structure. In particular, all elementary excitations correspond to coherent superpositions of particles $a_k^\dagger$ and holes a_{-k} in the Bose condensate with the wave vector k and energy E_k. In long-wavelength limit ($k \to 0$) the elementary excitations represent sound waves with the propagation speed

$$s = \sqrt{\frac{ng}{M}}. \tag{1.23}$$

For short wavelength, the spectrum is that of a free particle shifted upward by a constant gn arising from the interaction with the Bose condensate.

From Eq.(1.21), the thermal spectrum of the excitations can be easily derived and shown to be that of a Bose gas and all other thermodynamic properties of superfluid can be found [12]. In particular, one can easily show that the momentum, that can be carried by thermal excitations, at low temperatures is proportional to T^4

$$P = -\int \frac{d^3p}{(2\pi\hbar)^3} p \cdot n(\epsilon - pv) \sim vT^4, \tag{1.24}$$

where $n(\epsilon)$ is the Bose-Einstein distribution. In this brief example, the power of the application of CT can be vividly seen because an exact solution to a highly nontrivial problem is obtained with ease and full information about the system becomes available.

Even when BT does not lead to a full diagonalization of the model Hamiltonian, the reduction gained by its use may be beneficial. In QCD, BT has been used continuously to describe nontrivial structure of QCD vacuum, its superconducting properties and values of quark and glue condensations [13, 14, 18]. In a number of variational and field-theoretic works, it was suggested that the quantum configurations with quark pairs may have a lower free energy than perturbative QCD vacuum and thus be energetically preferred leading to existence of nontrivial color condensation in QCD vacuum. Such models had been also extended to describe properties of dressed constituent quarks and to derive from fundamental QCD hadron structure and mass spectrum [18]. In BT treatment, the QCD vacuum $|\Omega\rangle$ is modeled as the BT vacuum annihilated by quasi-particle operators obtained from the original current-quark ladder operators with a rotation

$$\begin{aligned} A_\lambda(k) &= \cos\theta_k a_\lambda(k) - \lambda \sin\theta_k b^\dagger_\lambda(k), \\ B_\lambda(k) &= \cos\theta_k b_\lambda(k) + \lambda \sin\theta_k a^\dagger_\lambda(k). \end{aligned} \tag{1.25}$$

Here, λ is the helicity of the quark. Such vacuum can be explicitly related to the

perturbative vacuum by

$$|\Omega\rangle = \exp\left[-\sum_{\lambda}\int \frac{d^3k}{(2\pi)^3}\lambda \tan\theta_k a^{\dagger}_{\lambda}(k) b^{\dagger}_{\lambda}(k)\right]|0\rangle, \tag{1.26}$$

where the gap angle θ_k is free parameter used to minimize the energy of the trial QCD vacuum. The condition of the energy minimization typically results in a nonlinear integral gap equation which needs to be solved in order for the explicit structure of vacuum to become transparent. Such approach is able to describe the vacuum condensations in QCD as well as superconducting BCS features of QCD vacuum. Operators $A_{\lambda}(k)$ can be seen to describe the dressed quarks and may be further used to model dynamical mass generation of the constituent quarks in Constituent Quark Model (CQM) and to produce a CQM-like description of the hadrons starting from the current quarks and fundamental QCD Hamiltonian [18].

Along this line, one employs the above QCD vacuum to construct meson states as produced on top of $|\Omega\rangle$ by means of meson creation operator

$$Q^{\dagger}_{nJP} = \sum_{\gamma\delta}\int \frac{d^3k}{(2\pi)^3}\Psi^{nJP}_{\gamma\delta}(k) A^{\dagger}_{\lambda}(k) B^{\dagger}_{\delta}(-k). \tag{1.27}$$

Application of variational principle, or Tamm-Dancoff truncation, to

$$|nJP\rangle = Q^{\dagger}_{nJP}|\Omega\rangle \tag{1.28}$$

leads to Schrodinger type equation which can be solved for the spectrum and wave-functions of the mesons [18]. Further improvements of this approach, which relies on QCD vacuum improved by an introduction of two and four quasi-particle correlations, is known in QCD as Random Phase Approximation (RPA) [18, 19]. RPA, based on taking into account particle-particle and particle-hole correlations in mean field, also applied in many other areas of Quantum Physics.

In all of these applications, BT serves as a powerful tool to help gain nonperturbative knowledge about quantum-mechanical and field-theoretical properties of physical systems. BT, stemming from the classical apparatus of CTs, is more an

exception rather than a rule given that the systematic translation of classical CTs into quantum framework experiences detrimental difficulties. Nonetheless, since its introduction in 1947, BT found a wide area of applications ranging from problems of condensed matter physics to strong interacting QCD as well as meson physics. It is repeatedly employed to describe nontrivial correlations in physical systems responsible for some dramatic physical behavior such as superfluidity, superconductivity, phase transitions, nontrivial vacuum condensations. In this dissertation we will focus our attention on two rather recent areas of application of BT that appeared in the field theory of flavor oscillations and in the field theory of critical phenomena. In the upcoming sections, we will examine these developments in greater details.

Chapter 2

Quantum Field Theory of Flavor Mixing

2.1 Introduction

Quantum mixing of particles is among the most interesting and important topics in Particle Physics [20–29]. First discovered in 1963, mixing of K^0 and $\bar{K}^0$ provided an evidence of CP-violation in weak interactions [30] and till today meson mixing is used immensely to experimentally determine the precise profile of CKM unitarity triangle [21, 29, 31, 32]. The Standard Model of particle physics involves quantum mixing in the form of Kobayashi-Maskawa (CKM) mixing matrix [31], a generalization of the original Cabibbo mixing between d and s quarks [32]. In the boson sector, the mixing of K^0 with $\bar{K}^0$ via weak currents provided the first evidence of CP violation [30] and the $B^0\bar{B}^0$ mixing plays important role in determining the parameters of CKM quark mixing in Wolfenstein parameter space [21, 29, 33]. The η-η' mixing in the $SU(3)$ flavor group provides a unique opportunity for testing QCD and the Constituent Quark Model (CQM). It is believed that the condensate structure of QCD vacuum is tightly related to the oscillations in η-η' system [21]. The non-trivial nature of the vacuum is expected to hold the answer to many of the most salient questions regarding confinement and the symmetry breaking mechanism in QCD. Recently, in lepton sector convincing evidences have been provided by Super-Kamiokande and SNO experiments [24–28] that neutrino mix, thus suggesting neutrino oscillations as

the most likely resolution for the solar neutrino puzzle [20] and evidence of neutrino masses [34].

Since the middle of the century, when the quantum mixing was first observed in meson systems, this phenomenon has played significant role in the phenomenology of particle physics and till today mixing phenomenology is thought to be important for possible searches of new physics beyond the Standard Model. Upgraded high-precision B-meson experiments would be vital to search for deviations from the unitarity in CKM matrix which can put important constraints on the physics beyond the Standard Model [29]. At the same time, in the fermion sector, the discovery of neutrino mixing and neutrino masses challenged our fundamental understanding of P-violation and, thus, of the Standard Model itself. A number of experiments are underway to improve our understanding of both long-base and short-base neutrino mixing [24–29].

Regarding the vanishing magnitudes of the expected new physics effects (such as the unitarity violation in CKM matrix and/or neutrino masses), it is imperative that the theoretical aspects of the quantum mixing are precisely understood. In this respect, it was noticed recently that the conventional treatment of flavor mixing, where the flavor states are defined in the Fock space of the energy-eigenstates, suffers from the problem with the probability conservation [35]. This suggested that the mixed states should be treated rather independently from the energy-eigenstates. It was found, indeed, that the flavor mixing in QFT introduces very non-trivial relationships between the flavor and the energy quantum states which lead to *unitary nonequivalence* between the Fock space of the interacting fields and that of the free fields [36–39]. This is quite different from the conventional perturbation theory where one expects the vacuum of the interacting theory to be essentially the same with one of the free theory (up to a phase factor e^{iS_0} [40, 41]).

The importance of mixing transformations has prompted their fundamental examination from a field-theoretical perspective. The investigation of two-field unitary mixing in the fermion sector by Blasone and Vitiello [35, 39, 42, 43] demonstrated a rich structure of the interacting-field vacuum as SU(2) coherent state and altered the

oscillation formula to include the antiparticle degrees of freedom. Subsequent analysis of the boson case revealed a similar but much richer structure of the vacuum of the interacting fields [44, 45]. Especially, the pole structure in the inner product between the vacuum of the free theory and the vacuum of the interacting theory was found and related to the convergence radius of the perturbation series [45]. Simpler quantum mechanical result is only reproduced in the relativistic limit of the field theory so that one may wonder about the magnitude of the field-theoretical effects and their significance for the new physics in mixing phenomena. Mathematically rigorous study of 2-flavor quantum field mixing in the framework of Axiomatic Field Theory has been carried out by Hannabuss and Latimer [46]. Attempts to look at the mixing of more than two flavors have also been carried out [47, 48]. Mixing of three and more flavors can be considered in full analogy with the calculations of $SU(2)$ mixing. Due to growing complexity of $SU(N)$ general parameterization for larger N, it makes better sense to consider such cases in numerical form, applying formalism of Ref.citeJM011 to explicitly given mixing matrix. Nonetheless, some results for mixing in $SU(3)$ are known in the literature in Standard [31] and Wolfenstein parameterization [33]. Due to the size of the relevant formulas, we do not find it possible to explicitly list these results here. Interested reader should look at the relevant publications [47, 48]. When treating such problem in general, it may be useful to keep in mind few comments following from the discussion we will later present. In particular, mixing of N-flavors have similar structure for any spin with spin-information being encoded implicitly in $\alpha_{\mu i}$, $\beta_{\mu i}$ mixing matrices [47]. Once the result is written, immediate generalization to other spins can be obtained. Furthermore, there exist close parallels between mixing of bosons and fermions. If one goes from fermion to boson case, the sign of β^2 terms shall be changed and α, β parameters should be reinterpreted according to the spin change (see also Section 2.5). For these reasons, in fact, it is sufficient to know only one case of mixing, say fermions, in given $SU(N)$ parameterization to extract the general result. In particular, [48] presents $SU(3)$ calculations in Standard parameterization which can be appropriately generalized to describe boson case. Ref. [47]

considers both boson and fermion case in Wolfenstein parameterization of $SU(3)$, although one shall keep in mind that this is only approximate parameterization.

A significant research effort had been further undertaken in the quantum field theory of mixing [35, 39, 44, 45, 47–50]. Still, the general theoretical results obtained therein cannot be immediately applied to the phenomenologically interesting cases. Specifically, previously the mixing was considered for fields that are charged, for which the particles and the antiparticles can be distinguished and are different. This may seem feasible for dirac neutrinos, however, in the meson sector the mixing particles and antiparticles are not totally independent (e.g. $K_0 - \bar{K}_0$, $B_0 - \bar{B}_0$), thus necessitating specific adjustments to the results obtained previously. Moreover, except neutrinos, all known mixed systems are subject to decay and thus the effect of particle's final life-time should be also taken into account.

In this chapter we are going to concentrate on these and other issues regarding flavor oscillations in QFT.

2.2 Flavor Oscillations in Quantum Mechanics

Quantum mixing is a fascinating phenomenon first observed in weak interactions where the interacting states of a particle turned out to be dramatically different from the free-propagation states. During mixing the particle, say, produced in a weak decay evolves over time into a drastically different weak-interaction state with very different weak decay signature. One usually thinks of this phenomenon in terms of weak-interaction (or flavor $A, B, \dots$) and free-propagation (or energy $a, b, \dots$) eigenstates. The flavor state produced in weak interaction shall be treated as a superposition of energy-eigenstates which then propagate independently from each other. Should weak decay happen once more, the evolved superposition of energy-eigenstates should be thought again in terms of flavor-eigenstates to find the appropriate weak-decay signatures [Fig.(2.1)].

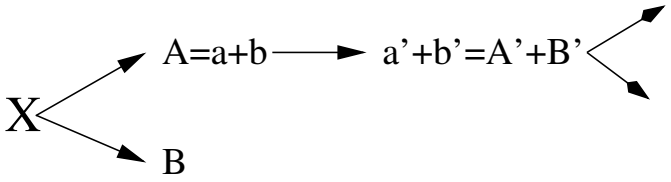

Figure 2.1: Mixing illustration: flavor eigenstates, produced in weak interaction, propagate as superposition of energy eigenstates which then decay as different flavor eigenstates

In particle phenomenology quantum oscillations are possible because of the flavor-changing term of the Standard Model Lagrangian

$$\mathcal{L} = \frac{g}{\sqrt{2}}(\bar{u}, \bar{c}, \bar{t})_L V_{CKM} \gamma_\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L W^\mu + h.c. \tag{2.1}$$

This interaction results in box diagram involving internal loop with two W bosons producing transition matrix element between $B^0 - \bar{B}^0$ and $B_s^0 - \bar{B}_s^0$ and, thus, introducing their mixing. Here V_{CKM} is the CKM mixing matrix [31]. Along with the parameterization originally suggested by Kobayashi and Maskawa [31], which now is often regarded as the Standard, a popular parameterization of this matrix is that of Wolfenstein [33] which expands each term in powers if the Cabibbo angle $\lambda \approx 0.22$ and is valid up to $O(\lambda^4)$

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \approx \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda(1 + iA^2\lambda^4\eta) & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}. \tag{2.2}$$

In Quantum Mechanics weak B-states can be described as quantum states $|B\rangle$ and $|\bar{B}\rangle$. The time evolution of, say, a pure $|B\rangle$ state is easily calculated using standard perturbation theory and in general the contribution develops both for $|\bar{B}\rangle$ and continuum (decay) components. We can think of this process as if the time

evolution of states is governed by Hamiltonian

$$\mathcal{H} = \begin{pmatrix} m_0 + \delta E & W_{12} + \delta E_{12} \\ W_{12}^* + \delta E_{12}^* & m_0 + \delta E \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma \end{pmatrix}. \tag{2.3}$$

Diagonalization of this Hamiltonian gives eigenvectors and eigenvalues [51]

$$\begin{aligned} |B_L\rangle &= p|B\rangle + q|\bar{B}\rangle \\ |B_H\rangle &= p|B\rangle - q|\bar{B}\rangle \end{aligned} \tag{2.4}$$

$$\begin{aligned} \lambda &= M \pm \Delta m/2 - \tfrac{i}{2}(\Gamma \pm \Delta\Gamma/2) \\ \Delta m/2 &= Re\sqrt{|M_{12}|^2 - \tfrac{|\Gamma_{12}|^2}{4} - iRe(M_{12}\Gamma_{12}^*)} \\ \Delta\Gamma/2 &= 2Im\sqrt{|M_{12}|^2 - \tfrac{|\Gamma_{12}|^2}{4} - iRe(M_{12}\Gamma_{12}^*)} \end{aligned} \tag{2.5}$$

Eigenvalues of Eq.(2.3) correspond to energy eigenstates, or free-propagating states. They are conventionally denoted as H(eavy) and L(ight) in B-mixing or S(hort) and L(ong) in K-mixing. Eigenvalues $\lambda_{H,L}$ correspond to the energies of Heavy and Light states and

$$\frac{q}{p} = \sqrt{\frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}}} \approx e^{2i\phi}. \tag{2.6}$$

The time evolution of the flavor states $|B\rangle$, $|\bar{B}\rangle$ can be found by re-expressing them in terms of the energy eigenstates and results in [51]

$$\begin{aligned} |B(t)\rangle &= C(|B_L(t)\rangle + |B_H(t)\rangle) \\ &= \tfrac{1}{2}e^{-iMt}e^{-\frac{\Gamma}{2}t}\big((e^{\frac{\Delta\Gamma}{4}t + i\frac{\Delta m}{2}t} + e^{-\frac{\Delta\Gamma}{4}t - i\frac{\Delta m}{2}t})|B\rangle \\ &\quad + \tfrac{q}{p}(e^{\frac{\Delta\Gamma}{4}t + i\frac{\Delta m}{2}t} - e^{-\frac{\Delta\Gamma}{4}t - i\frac{\Delta m}{2}t})|\bar{B}>\big), \\ |\bar{B}(t)\rangle &= C'(|B_L(t)\rangle - |B_H(t)\rangle) \\ &= \tfrac{1}{2}e^{-iMt}e^{-\frac{\Gamma}{2}t}\big(\tfrac{q}{p}(e^{\frac{\Delta\Gamma}{4}t + i\frac{\Delta m}{2}t} - e^{-\frac{\Delta\Gamma}{4}t - i\frac{\Delta m}{2}t})|B\rangle \\ &\quad + (e^{\frac{\Delta\Gamma}{4}t + i\frac{\Delta m}{2}t} + e^{-\frac{\Delta\Gamma}{4}t - i\frac{\Delta m}{2}t})|\bar{B}>\big). \end{aligned} \tag{2.7}$$

This yields the following probabilities for original B-meson to oscillate into meson $\bar{B}$ and decay,

$$P_2(t) \sim |\langle\bar{B}|B(t)\rangle|^2 \sim e^{-\Gamma t}(\cosh\frac{\Delta\Gamma}{2}t - \cos\Delta m t) \tag{2.8}$$

Similarly the probability for the original meson B to retain its flavor after time t is given by

$$P_1(t) \sim e^{-\Gamma t}(\cosh\frac{\Delta\Gamma}{2}t + \cos\Delta m t). \tag{2.9}$$

These are the formulas for particle oscillations in the presence of decay and are known as Gell-Mann-Pais formulas.

This treatment straightforwardly generalizes to mixing of more than two flavors. If we denote flavor state μ by $|\mu\rangle = a^\dagger_\mu|0\rangle$[1], where $a^\dagger_\mu$ is creation operator for flavor particle μ, the mixing of flavors can be described by the interaction Hamiltonian

$$\mathcal{H}_I = \frac{1}{2}\sum_{\mathbf{k}}\sum_{\mu,\nu=A,B,\ldots} h_{\mu\nu}(a^\dagger_{\mu,\mathbf{k}}a_{\nu,\mathbf{k}} + h.c.), \tag{2.10}$$

where $a^\dagger_{A,B,\ldots}(a_{A,B,\ldots})$ are creation (annihilation) operators for the flavor states. The full Hamiltonian

$$\mathcal{H} = \sum_{\mathbf{k}}\sum_{\mu=A,B,\ldots}\epsilon_{\mu,\mathbf{k}}a^\dagger_{\mu,\mathbf{k}}a_{\mu,\mathbf{k}} + H_I \tag{2.11}$$

can be straightforwardly diagonalized by introducing energy states $|i,\mathbf{k}\rangle$

$$|\mu,\mathbf{k}\rangle = \sum_i U_{\mu i}|i,\mathbf{k}\rangle \tag{2.12}$$

so that

$$\begin{aligned}\mathcal{H} &= \sum_{\mathbf{k}}\sum_{i=a,b,\ldots}\epsilon_{i,\mathbf{k}}a^\dagger_{i,\mathbf{k}}a_{i,\mathbf{k}}+\text{const}\\ a_{i,\mathbf{k}} &= \sum_{\mu=A,B,\ldots}U^\dagger_{i\mu}a_{\mu,\mathbf{k}}.\end{aligned} \tag{2.13}$$

Here $U^\dagger_{i\mu}$ is appropriately chosen mixing matrix. This simple transformation allows one to immediately solve the time dynamics of the flavor states and arrive at the oscillation formulas. In general, the transition probability between initial state of flavor μ at time $t = 0$ and position $x = 0$ and a final state of flavor ν at time t and position x is given by

$$|\mathcal{A}_{\mu\to\nu}(t,x)|^2 = |\langle\nu|\exp(-i\mathcal{H}t + iPx)|\mu\rangle|^2 = |\sum_i U_{\mu i}U^\dagger_{i\nu}e^{-i\phi_i}|^2. \tag{2.14}$$

[1]We use the Latin indexes $i,j,k,\ldots$ and small Latin letters $a,b,\ldots$ to label the mass-eigenstates and the Greek indexes $\mu,\nu,\rho,\ldots$ and capital Latin letters $A,B,\ldots$ to label the flavor-eigenstates.

where $\phi_i = \epsilon_{i,\mathbf{k}_i} t - \mathbf{k}_i \mathbf{x}$. The factors $\mathbf{k}_i \mathbf{x}$ are often dropped in derivation either because the reference frame is assumed to be that of the flavor particle rest frame or because momenta of the energy-eigenstates are assumed to be the same. Although neither of these assumptions can be justified, they serve as a good approximation for phenomenologically important cases. In the case of two flavors with no decay one recovers from (2.14) so called Pontecorvo oscillation formula

$$\begin{aligned} |\langle A|A;t\rangle|^2 &= 1 - \sin^2(2\theta)\sin^2(\tfrac{E_a - E_b}{2}t), \\ |\langle B|A;t\rangle|^2 &= \sin^2(2\theta)\sin^2(\tfrac{E_a - E_b}{2}t). \end{aligned} \tag{2.15}$$

Here θ is a $SU(2)$ mixing angle.

The flavor mixing treatment outlined above is known as the plain wave approach [52]. Although simple and illustrative, it has a number of conceptual difficulties when considered with respect to the measurable quantities. Specifically, certain controversy exist in the way these results should be converted into space oscillations, rather than time oscillations. In fact, such conversion cannot be satisfactorily carried out because a perfectly known energy-momentum and an infinite uncertainty on the space-time localization of the oscillating particles was implied in which case quantum oscillations are, in fact, destroyed [52, 53]. A correct oscillation formula should include observability conditions in such a way that oscillation terms vanish whenever the energy-momentum uncertainty is less than the mass difference between energy eigenstates or the oscillation length is smaller than the uncertainty in the position of the source or of the detector.

These problems are resolved by quantum mechanical wave-packets approach. Without going through detailed calculations and referring an interested reader to extensive literature on the subject [53–59], we note that wave-packet method considers the flavor particle as superposition of energy-eigenstates existed originally as an extended wavefunction, e.g. a narrow gaussian, with finite uncertainty in momentum and position. The evolution of the flavor particle is described via interference between these wave-packets and the final oscillation formulas can be obtained as the convolution of the plain-wave oscillation amplitude with the wavefunctions of the energy-eigenstate

wave-packets

$$P(t,x) = |\langle D(x)|\mathcal{U}(t)|A(\Delta p, \Delta x)\rangle|^2. \tag{2.16}$$

Here $|D(x)\rangle$ is the wavefunction representing the detector at position x, $\mathcal{U}(t)$ is propagation operator for the flavor state like $\mathcal{A}_{\mu\to\nu}(t,x)$ in Eq.(2.14) and $|A\rangle$ is the original flavor state - a coherent superposition of energy-eigenstate wavefunctions with finite uncertainty in momentum and position. This approach successfully describes such features of quantum oscillations as vanishing due to decoherence caused by too extended a wave-packet (uncertainty in position is larger than the oscillation length) or due to decoherence caused by separation of the mass eigenstates moving with different group velocities (uncertainty in energy-momentum is less than difference in mass).

Although description of flavor oscillations in quantum mechanical plain-wave or wave-packet approach provides a concise and phenomenologically successful description of quantum oscillations, few authors would deny that the most rigorous treatment of oscillations should be done in the QFT framework. However, although the quantum field computations in the literature all reproduce in some limit the quantum-mechanical formulas, there exist not yet an agreement in which respect they differ from these formulas.

The field-theoretical approach to particle oscillations is quite old. Already in 1963 Sachs [60, 61] applied S-matrix method to neutral kaon interference. This approach is also known as Jacob-Sachs model or external wave packet method. The particle here is represented by its propagator and propagates between a source and a detector where it interacts with the wave packets representing "detectable" particles. Similar simplified model using propagator description of the flavor particles with account for external wave packets had been suggested by Beuthe, Lopez Castro and Pestieau [62]. In regard to neutrinos, Kobzarev, Martemyanov, Okun and Shchepkin [63] analyzed neutrino oscillations with a field-theoretical model in which the source and the detector are both represented by heavy nuclei. Giunti, Kim, Lee and Lee [64] studied neutrino oscillations within a Gaussian external wave packet model. They derive a localization condition and a coherence length beyond which the oscillations

must vanish. These results agree with those obtained in Quantum Mechanics [53–55]. Later, the same authors presented essentially identical result to Ref. [64] but with more generality [65]. Attempts along the same line with different modifications and improvements had been undertaken also by many other authors [67–79].

Another approach (source-propagator model) had been suggested by Srivastave, Widom and Sassaroli [80, 81] where the neutrino propagator is coupled to a source, but not a detector. Since external wave-packets are absent, time-space conversion problem cannot be avoided here and lead to nonstandard correlation lengths or recoil oscillations. Shtanov [82] used the source-propagator model to claim strong dependence of the oscillation formula on the neutrino masses if the source and the detector are very well localized in space-time.

In 1991 important article by Giunti, Kim and Lee [66] stroke a blow to the quantum-mechanical wave-packet approach and its generalizations to QFT by showing that it is impossible to build a Fock space for flavor states because the mixing of the ladder operators for mass eigenstates does not yield canonical flavor ladder operators for flavor eigenstates (i.e. commutation relations cannot be satisfied). Along this line in a completely different line of thoughts, Blasone and Vitiello (BV) have attempted to define a Fock space of flavor eigenstates to derive nonperturbative oscillation formula using Bogoliubov transformation [35, 39, 83]. In oscillation formulas of BV theory the oscillation frequency depends surprisingly not only on the difference but also on the sum of the energies of the different mass eigenstates. Although this study is very interesting from a fundamental point of view, it is not obvious whether new features put forward in BV theory may produce effect observable in practice. Also, BV theory was shown to suffer from parametrization ambiguity, which has not yet been completely resolved.

These observations do not detract from the theoretical worth of this investigation. In the following sections we will concentrate specifically on BV field theory of flavor oscillations and on our contribution to generalization and phenomenological applications of this theory.

2.3 Quantum Field Theory of Mixing of Blasone and Vitiello

Difficulty encountered in QFT of flavor oscillations with proper definition of flavor states demonstrated much more complicated structure of the field theory of mixing than previously thought. Intrinsic possibility of antiparticle admixture in field-theoretical mixing is capable of introducing dramatic new effects in flavors dynamics: the flavor vacuum state may acquire rich coherent structure and oscillation formulas may change significantly with additional high-frequency terms [84, 85]. Whenever description of classical (or quantum-mechanical) mixing involves one unitary matrix $U_{\mu j}$, in QFT in general one needs two matrices $\alpha_{\mu j}$ and $\beta_{\mu j}$, representing particle-particle and particle-antiparticle mixing sectors, to describe the same mixing transformation [84]. In field theory these matrices should be consistently related to their classical analog $U_{\mu j}$. A way to establish such relationship was proposed by quantum field theory of mixing of Blasone and Vitiello [39].

Blasone and Vitiello based their theory on the observation that an explicit quantum transformation $\Lambda(U,t)$ in the linear space of quantum fields can be constructed out of fields φ and their canonical momenta π that provides representation of the mixing relations (2.12),

$$\phi_A(t) = \sum_i U_{Ai}\varphi_i(t) = \Lambda(U,t)^\dagger \varphi_a(t)\Lambda(U,t), \text{ etc.} \tag{2.17}$$

In the case of two-scalar mixing, e.g., such transformation can be explicitly found as

$$\Lambda(\theta,t) = e^{i\theta\int d\mathbf{x}(\pi_a^\dagger(\mathbf{x},t)\varphi_b(\mathbf{x},t) - \pi_b^\dagger(\mathbf{x},t)\varphi_a(\mathbf{x},t) + h.c.)}.$$

This generates $SU(2)$ mixing relations, as can be found by straightforward calculation,

$$\begin{aligned}\phi_A &= \cos(\theta)\varphi_a + \sin(\theta)\varphi_b \\ \phi_B &= -\sin(\theta)\varphi_a + \cos(\theta)\varphi_b\end{aligned}$$

and diagonalizes the field-theoretical Hamiltonian. In the associate Fock-space $\Lambda(U,t)$

acts similarly,

$$|A\rangle = \Lambda(U,0)^\dagger |a\rangle, \text{ etc}, \tag{2.18}$$

so that the ladder operators are transformed as in

$$a_A(t) = \Lambda(U,t)^\dagger a_a(t) \Lambda(U,t), \text{ etc.} \tag{2.19}$$

Obviously, these definitions are consistent in the sense that

$$\begin{aligned} &|A+1\rangle = a_A^\dagger(0)|A\rangle = \Lambda(U,0)^\dagger a_a^\dagger(0)\Lambda(U,0)\Lambda(U,0)^\dagger|a\rangle = \\ &\Lambda(U,0)^\dagger a_a^\dagger(0)|a\rangle = \Lambda(U,0)^\dagger|a+1\rangle \end{aligned} \tag{2.20}$$

and the flavor vacuum state satisfies

$$a_A(0)|\Omega\rangle = \Lambda(U,0)^\dagger a_a(0)\Lambda(U,0)\Lambda(U,0)^\dagger|0\rangle = 0. \tag{2.21}$$

In their approach BV concentrated explicitly on finding the structure of the unitary transformation $\Lambda(U,t)$ and on applying $\Lambda(U,t)$ to the Fock space of the free-propagation states to define flavor states and obtain nonperturbative oscillation formulas. $SU(2)$ mixing and $SU(3)$ mixing (where explicit building of $\Lambda(U,t)$ is a much more difficult task) for both spin-1/2 fermions and scalar bosons had been analyzed along this line in the literature. The explicit mixing relations and oscillation formulas for flavor charge, $Q(t)$, and number, $N(t)$, operators had been presented in [39, 42–45, 47, 48] and elsewhere. We note that particle number oscillations $N(t)$ as well as charge oscillations $Q(t)$ in general depend on the choice of so called mass parameters used to define the free-field Fock space, which was used as the base for the flavor Fock space. One may explicitly see this on the example of the charge operator Eq.(2.129) which does not depend on the choice of mass parameters m_μ only for real mixing matrix $U_{\mu j}$. Otherwise, as in the case of CP-violating CKM matrix, there will be a nontrivial mass dependence from the imaginary part of $U_{\mu j}$. Blasone and Vitiello maintained that the set of the free-field masses $m_{i=a,b,\ldots}$ shall be chosen as such mass parameter m_μ.

Alternative view consider BV procedure Eq.(2.19) as result of expanding the flavor fields $\phi_\mu(x)$ in the free-fields basis parametrized with masses m_μ where BV choice

corresponds to setting m_μ to free-field masses observed in experiment for energy-eigenstates,

$$\phi_\alpha = \int \frac{d\vec{k}}{(2\pi)^{3/2}} \left(u_{\alpha k} a_{\alpha k}(t) + v_{\alpha,-k} b^\dagger_{\alpha,-k}(t) \right) e^{i\vec{k}\vec{x}}. \tag{2.22}$$

On the other hand, one may as well expand the flavor fields in a basis with the flavor mass parameters m_μ left free,

$$\phi_\alpha = \int \frac{d\vec{k}}{(2\pi)^{3/2}} \left(\tilde{u}_{\alpha k} \tilde{a}_{\alpha k}(t) + \tilde{v}_{\alpha,-k} \tilde{b}^\dagger_{\alpha,-k}(t) \right) e^{i\vec{k}\vec{x}}, \tag{2.23}$$

corresponding to choosing $\tilde{u}^\mu_{\vec{k}\sigma}$, $\tilde{v}^\mu_{-\vec{k}\sigma}$ as different free-field amplitudes with the flavor mass m_μ in Eqs.(2.136)-(2.139) [37, 38]. This ambiguity can be understood in terms of the original BV idea as follows. For any $\Lambda(U,t)$ that is a valid representation of the classical mixing transformation, $\Lambda'(U,t) = I(t)\Lambda(U,t)$, that can be obtained by means of a similarity transformation mixing $a_{\alpha\mathbf{k}}(t)$ and $b^\dagger_{\alpha-\mathbf{k}}(t)$ but leaving their covariant combination $\phi(\vec{k})$ unchanged (i.e. $\phi_\alpha(\mathbf{k},t) = I(t)^\dagger \phi_\alpha(\mathbf{k},t) I(t)$), is a valid representation of the mixing transformation (2.12). The ladder operators, defined in Eq.(2.19), therefore, depend additionally on the choice of $I(t)$ or, equivalently, the choice of the mass parameters m_μ associated with the base free-fields Fock space.

Although there are different opinions about whether or not the measurable quantities in the theory shall depend on the mass parameters [38,43–45,83], we note that the mass parametrization problem, indeed, is not at all specific to quantum mixing but exists in almost any instance of QFT. Consider a free field theory with Hamiltonian

$$:\mathcal{H}_0 := \sum_{\vec{k}\sigma} \left(\epsilon_{\vec{k}} a^\dagger_{\vec{k}\sigma} a_{\vec{k}\sigma} + \epsilon_{\vec{k}} b^\dagger_{\vec{k}\sigma} b_{\vec{k}\sigma} \right), \tag{2.24}$$

one may change the mass parametrization $m \to m_\mu$ similarly to Ref. [37];

$$\begin{pmatrix} \tilde{a}(t) \\ \tilde{b}^\dagger(t) \end{pmatrix} = I^{-1}(t) \begin{pmatrix} a \\ b^\dagger \end{pmatrix} I(t) = \begin{pmatrix} e^{i(\tilde{\epsilon}_{\vec{k}} - \epsilon_{\vec{k}})t} \rho^*_{\vec{k}} & e^{i(\tilde{\epsilon}_{\vec{k}} + \epsilon_{\vec{k}})t} \lambda_{\vec{k}} \\ e^{-i(\tilde{\epsilon}_{\vec{k}} + \epsilon_{\vec{k}})t} \lambda^*_{\vec{k}} & e^{-i(\tilde{\epsilon}_{\vec{k}} - \epsilon_{\vec{k}})t} \rho_{\vec{k}} \end{pmatrix} \begin{pmatrix} a(0) \\ b^\dagger(0) \end{pmatrix}, \tag{2.25}$$

where $\tilde{\epsilon}_k = \sqrt{k^2 + {m_\mu}^2}$ and $\epsilon_k = \sqrt{k^2 + m^2}$. When we compare directly Eq.(2.22) and Eq.(2.23), we observe that

$$\tilde{a}_i = (\tilde{u}^\dagger_{k,i} u_{k,i}) a_{k,i} e^{-i\epsilon_{k,i}t} + (\tilde{u}^\dagger_{k,i} v_{-k,i}) b^\dagger_{-k,i} e^{i\epsilon_{k,i}t} = \rho_k a_{k,i} e^{-i\epsilon_{k,i}t} + \lambda_k b^\dagger_{-k,i} e^{i\epsilon_{k,i}t} \tag{2.26}$$

where we follow the notations of Refs. [37]. The number operator (for the free fields) in such transformation is not conserved, e.g. for fermions

$$\langle \tilde{N} \rangle = |\{\tilde{a}, \tilde{a}^\dagger(t)\}|^2 = \left| |\rho_k|^2 e^{-i\epsilon_k t} + |\lambda_k|^2 e^{i\epsilon_k t} \right|^2 , \tag{2.27}$$

that may lead to obviously wrong assertion that the number of particles in the free field case *is not* an observable quantity [45].

Mathematically this is understood once we note that the above transformation is equivalent to the splitting of the initial Hamiltonian into

$$\mathcal{H}_0 = \mathcal{H}_0' + \mathcal{H}_I' = \int d^4p \left(\left\{ (\hat{p}\psi)^\dagger (\hat{p}\psi) - {m_\mu}^2 \psi^\dagger \psi \right\} + ({m_\mu}^2 - m^2) \psi^\dagger \psi \right) . \tag{2.28}$$

Thus, additional self-interaction term is responsible for driving oscillation of $\langle \tilde{N}(t) \rangle$.

Physically, the transformation given by Eq.(2.25) shall be viewed as a redefinition of one-particle physical states. The tilde quantities thus correspond to a new quasi-particle objects so that the tilde number operator describes a different type of particles and does not have to be invariant under such transformation, although the particle-antiparticle difference is conserved in the transformation (2.25). The situation here may be analogous to dependence of physical observables upon the choice of coordinate system. Although the Casimir operator (e.g. $\vec{S}^2$ for spin observables) is independent from the coordinate system, other physical quantities (e.g. S_x, S_y and S_z) depend on the coordinate system. To compare the value of S_z, say, between theory and experiment, one should first fix the coordinate system. Similarly, we think specific mass parameters should be selected from the physical reasons to compare theoretical results (*e.g.* the occupation number expectation values) with the experiment.

From the above example it is clear that the mass parametrization problem is present in any regular perturbation theory as well once one attempts to redefine one-particle states as in Eq.(2.25). In the free theory and the perturbation theory this issue is resolved by the presence of the mass scale of well defined asymptotic physical states, which therefore fix the mass parameters. In this sense we agree with Blasone and Vitiello in that the mass scale of the energy-eigenstates is the most feasible option for setting m_μ in the field theory of flavor mixing.

While one may find the above analogy with coordinate systems crude and still deem mass parameters in Blasone and Vitiello theory "unphysical" or "arbitrary" and thus conclude that the measured quantities should not depend on such an arbitrary parameter, we must emphasize once again that as we demonstrate above such "unphysical" mass parameters are present in any and every instance of quantum field theory and most of the observables, that we are used to in field theory, do depend on the choice of such parameters. While in a free field case we know a physical ground to uniquely fix such parameters, in flavor mixing we are not yet aware of such unambiguous way. However, one cannot simply dismiss the very same observables and declare them unphysical in flavor mixing. Unless a lucky coincidence, the measured quantities $\langle a|\hat{O}|a\rangle$ will depend on arbitrary parameters relevant to the choice of quantum state $|a\rangle$, i.e. 1-flavor-particle state in mixing theory. The resolution to this problem lies neither on the way of restricting observables only to those that do not depend on the choice of $|a\rangle$ [38] nor in artificially modifying observables to fit such requirement [43]. Rather, the resolution comes in specifying properly what state $|a\rangle$ should actually be to describe relevant experimental setup.

2.4 Blasone and Vitiello Theory in Mixing of Two Bosons

This section shall help us exemplify and illustrate the applications of BV theory on a simpler example of $SU(2)$ scalar mixing where we will follow closely the original BV paper [39]. At the same time, scalar mixing introduces new interesting features into BV theory due to bosonic nature of mixed fields, such as large particle numbers contribution in mixing and divergence of perturbation theory which we will be able to investigate in details. We start our analysis by considering the Pontecorvo mixing

relationship for two fields [86]

$$\begin{aligned} \phi_A &= \cos\theta\varphi_1 + \sin\theta\varphi_2 \\ \phi_B &= -\sin\theta\varphi_1 + \cos\theta\varphi_2, \end{aligned} \tag{2.29}$$

where $\varphi_{1,2}$ are the free fields with definite masses $m_{1,2}$ and $\phi_{A,B}$ are the interacting fields with definite flavors A, B respectively. The above mentioned relations naturally arise by considering the mixing problem for two quantum fields with the Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_{0,\alpha} + \mathcal{L}_{0,\beta} - g(\phi_A^\dagger\phi_B + \phi_B^\dagger\phi_A), \tag{2.30}$$

where $\mathcal{L}_{0,A(B)}$ are the free flavor-field Lagrangian (i.e. $\mathcal{L}_{0,A(B)} = \frac{1}{2}(\partial\phi^\dagger_{A(B)}\partial\phi_{A(B)} - m^2_{A(B)}\phi^\dagger_{A(B)}\phi_{A(B)})$) and g is the coupling constant responsible for mixing. It is straightforward to show that the above Lagrangian can be immediately diagonalized by the transformation (2.29) with appropriate choice of mixing angle θ. The parameters of diagonalized Lagrangian can be expressed in terms of the flavor-field masses (m_A, m_B) and the interaction constant (g) as

$$\tan(2\theta) = \frac{4g}{m_A^2 - m_B^2} \tag{2.31}$$

and

$$m^2_{1,2} = \frac{(m_A^2 + m_B^2) \pm \sqrt{(m_A^2 - m_B^2)^2 + 16g^2}}{2}. \tag{2.32}$$

We write the free fields $\varphi_{1,2}$ as usual in terms of Fourier transform

$$\varphi_i = \sum_{\vec{k}} \frac{1}{\sqrt{2\epsilon_i(k)}}(u_{\vec{k}i}a_{\vec{k}i}e^{-ikx} + v_{\vec{k}i}b^\dagger_{\vec{k}i}e^{ikx}), \tag{2.33}$$

where $a_{\vec{k}i}$ and $b_{\vec{k}i}$ are respectively the particle and the anti-particle ladder operators that satisfy standard equal-time commutation relations

$$\begin{aligned} [a_{\vec{k}i}, a^\dagger_{\vec{k}'i'}] &= \delta_{\vec{k},\vec{k}'}\delta_{i,i'}, \\ [b_{\vec{k}i}, b^\dagger_{\vec{k}'i'}] &= \delta_{\vec{k},\vec{k}'}\delta_{i,i'}. \end{aligned} \tag{2.34}$$

Here, $kx = k_0x_0 - \vec{k}\cdot\vec{x}$ and $\epsilon_i(k) = k_0(k) = \sqrt{\vec{k}^2 + m_i^2}$. For spin-0 the free-field amplitudes $u_{\vec{k}i}$ and $v_{\vec{k}i}$ are just numbers, i.e. $u_{\vec{k}i} = v_{\vec{k}i} = 1$. The interacting fields

introduced by Eq.(2.29) are the solutions of the Euler-Lagrange equation for Eq.(2.30) and are completely determined in terms of two free-fields (2.33) and the mixing angle θ.

Such rather simple correspondence, however, gives rise to highly nontrivial relationship between the Fock-space of free-fields and that of interacting fields. To build the Fock space for the flavor-eigenstates we consider the representation of the transformation (2.29) in the linear space of quantum fields. Using the Baker-Hausdorff lemma [87], we can check that the generator of this transformation has following form

$$\begin{aligned} \hat{S} = & \int d^3x(\dot{\phi}^\dagger_\alpha(x)\phi_\beta(x) + \phi^\dagger_\beta(x)\dot{\phi}_\alpha(x) - \dot{\phi}^\dagger_\beta(x)\phi_\alpha(x) - \phi^\dagger_\alpha(x)\dot{\phi}_\beta(x)), \quad (2.35) \\ = & \int d^3x(\dot{\varphi}^\dagger_1(x)\varphi_2(x) + \varphi^\dagger_2(x)\dot{\varphi}_1(x) - \dot{\varphi}^\dagger_2(x)\varphi_1(x) - \varphi^\dagger_1(x)\dot{\varphi}_2(x)), \end{aligned}$$

so that

$$\begin{aligned} \phi_A &= e^{-i\hat{S}\theta}\varphi_1 e^{i\hat{S}\theta}, \\ \phi_B &= e^{-i\hat{S}\theta}\varphi_2 e^{i\hat{S}\theta} \end{aligned} \quad (2.36)$$

and

$$G(\theta) = e^{i\hat{S}\theta}. \quad (2.37)$$

The similarity transformation (2.36) relates free-field operators $\varphi_{1,2}$ to the interacting fields $\phi_{\alpha,\beta}$. It also acts in the associated Fock space as

$$|A\rangle = e^{-i\hat{S}\theta}|a\rangle \text{ etc} \quad (2.38)$$

and relates the Hilbert spaces of the mass-eigenstates $\mathcal{H}_{1,2}$ with the flavor-eigenstates $\mathcal{H}_{\alpha,\beta} = G^{-1}(\theta)\mathcal{H}_{1,2}$ and also can be viewed as the "rotation" of the basis in the Hilbert space of quantum states diagonalizing the bilinear Lagrangian given by Eq.(2.30), i.e. a generalization of Bogoliubov Transformation.

The operator $\hat{S}$ also can be written in terms of ladder operators $a_{\vec{k}i}$ and $b_{\vec{k}i}$ as follows

$$\hat{S} = \sum_{\vec{k}} \quad \tfrac{i}{2} \quad \{\gamma_+(a_{\vec{k}1}a^\dagger_{\vec{k}2} + b_{-\vec{k}1}b^\dagger_{-\vec{k}2} - a^\dagger_{\vec{k}1}a_{\vec{k}2} - b^\dagger_{-\vec{k}1}b_{-\vec{k}2}) +$$

$$+ \quad \gamma_-(a_{\vec{k}1}b_{-\vec{k}2} + a_{\vec{k}2}b_{-\vec{k}1} - a^\dagger_{\vec{k}1}b^\dagger_{-\vec{k}2} - b^\dagger_{-\vec{k}1}a^\dagger_{\vec{k}2})\big\}, \tag{2.39}$$

where we denote $\gamma_+ = \sqrt{\frac{\epsilon_1(k)}{\epsilon_2(k)}} + \sqrt{\frac{\epsilon_2(k)}{\epsilon_1(k)}}$ and $\gamma_- = \sqrt{\frac{\epsilon_1(k)}{\epsilon_2(k)}} - \sqrt{\frac{\epsilon_2(k)}{\epsilon_1(k)}}$ and $\gamma_+^2 - \gamma_-^2 = 4$. We shall note that Eq.(2.36) makes each cluster $\Omega_{\vec{k}}$, defined by linear superposition of operators $(a_{\vec{k}1}, a_{\vec{k}2}, b^\dagger_{-\vec{k}1}, b^\dagger_{-\vec{k}2})$, and $\Omega^\dagger_{\mathbf{k}}$ transform into itself, i.e. $e^{-i\hat{S}\theta}\Omega_{\vec{k}}e^{i\hat{S}\theta} = \Omega_{\vec{k}}$. This means that we can consider the transformation given by Eq.(2.36) restricted to such cluster with a specific momentum $\vec{k}$ and concentrate on

$$\begin{aligned}\hat{S}_{\vec{k}} = \quad & \tfrac{i}{2} \quad \big\{\gamma_+(a_{\vec{k}1}a^\dagger_{\vec{k}2} + b_{-\vec{k}1}b^\dagger_{-\vec{k}2} - a^\dagger_{\vec{k}1}a_{\vec{k}2} - b^\dagger_{-\vec{k}1}b_{-\vec{k}2}) + \\ & + \quad \gamma_-(a_{\vec{k}1}b_{-\vec{k}2} + a_{\vec{k}2}b_{-\vec{k}1} - a^\dagger_{\vec{k}1}b^\dagger_{-\vec{k}2} - b^\dagger_{-\vec{k}1}a^\dagger_{\vec{k}2})\big\}.\end{aligned} \tag{2.40}$$

The total transformation is $e^{i\hat{S}\theta} = \prod_{\vec{k}} e^{i\hat{S}_{\vec{k}}\theta}$.

It is convenient to express $\hat{S}_{\vec{k}}$ as $\hat{S}_{\vec{k}} = \sqrt{2}(\hat{T}^\dagger_{\vec{k}} + \hat{T}_{\vec{k}})$ with the operator $\hat{T}_{\vec{k}}$ defined by

$$\hat{T}_{\vec{k}} = -\frac{i}{2\sqrt{2}}(\gamma_+(a^\dagger_{\vec{k}1}a_{\vec{k}2} - b_{-\vec{k}1}b^\dagger_{-\vec{k}2}) + \gamma_-(a^\dagger_{\vec{k}1}b^\dagger_{-\vec{k}2} - a_{\vec{k}2}b_{-\vec{k}1})), \tag{2.41}$$

where the commutation relation $[\hat{T}_{\vec{k}}, \hat{T}^\dagger_{\vec{k}}] = 1$ is satisfied between $\hat{T}_k$ and $\hat{T}^\dagger_k$ just the same way as the particle creation and annihilation operators satisfy the commutation relations (2.34). With the operators $\hat{T}_k$ and $\hat{T}^\dagger_k$, we can directly apply Eq.(2.37) to the mass-eigenstate vacuum to obtain

$$|\Omega(\theta)\rangle_{\vec{k}} = \sum_{n=0}^{\infty} \frac{(-i\theta\sqrt{2})^n}{n!} \sum_{l=0}^{n} \binom{n}{l}(\theta)(\hat{T}^\dagger_{\vec{k}})^l \hat{T}^{n-l}_{\vec{k}}|0\rangle, \tag{2.42}$$

where $\binom{n}{l}(\theta)$ are the generalized binomial coefficients that can be found after appropriate orderings of $\hat{T}$ and $\hat{T}^\dagger$ are carried out. In the expression given by Eq.(2.42), one can treat the operators $\hat{T}_{\vec{k}}$ $(\hat{T}^\dagger_{\vec{k}})$ as the annihilation (creation) operator of the vacuum fluctuations. Let us directly estimate the norm of the flavor vacuum state using the perturbative expansion in powers of θ (2.42). Truncating the series for (2.42) to first N terms, we have the term with the largest number of particles coming from $\frac{\gamma_-}{2}\left(a^\dagger_1 b^\dagger_{-2} + a^\dagger_1 b^\dagger_{-2}\right)$ in the $\left(-\theta\hat{S}\right)^N$. Thus, the truncated series of $G_N^{-1}(\theta)|0\rangle$ can be

written as[2]

$$\begin{aligned} G_N^{-1}(\theta)|0\rangle &= X + \tfrac{1}{N!}\left(-\tfrac{i\gamma_-\theta}{2}\right)^N \left(a_1^\dagger b_{-2}^\dagger + a_2^\dagger b_{-1}^\dagger\right)^N |0\rangle \\ &= X + \tfrac{1}{N!}\left(-\tfrac{i\gamma_-\theta}{2}\right)^N \sum_{n=0}^{N} \binom{N}{n} \left(a_1^\dagger\right)^n \left(b_{-2}^\dagger\right)^n \left(a_2^\dagger\right)^{N-n} \left(b_{-2}^\dagger\right)^{N-n} |0\rangle, \end{aligned} \tag{2.43}$$

where X denotes all terms with the total number of particles and antiparticles less than $2N$. For the norm of above expression we get

$$\begin{aligned} \left\|G_N^{-1}(\theta)|0\rangle\right\|^2 &= \|X\|^2 + \left(\tfrac{\gamma_-\theta}{2}\right)^{2N} \tfrac{1}{N!^2} \sum_{n=0}^{N} n!n!\,(N-n)!\,(N-n)!\tfrac{N!^2}{n!^2(N-n)!^2} \\ &= \|X\|^2 + (N+1)\left(\tfrac{\gamma_-\theta}{2}\right)^{2N}. \end{aligned} \tag{2.44}$$

Thus, when $\gamma_-\theta > 2$ the norm of the $|\Omega(\theta)\rangle_N = [G_N^{-1}(\theta)|0\rangle]_N$ is growing as the number of terms kept in the expansion of the $G(\theta)$ grows and therefore the transformation operator $G_N^{-1}(\theta)$ is not well defined operator in the mass-eigenstates Fock space.

One may also try to check directly the identity $G(\theta)G^{-1}(\theta) = 1$. Here we define

$$\begin{aligned} G(\theta) &= \lim_{N\to\infty} G_N(\theta) \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \tfrac{\left(i\theta\cdot\hat{S}\right)^n}{n!}. \end{aligned} \tag{2.45}$$

Then, we shall show that $\lim\limits_{N\to\infty} \left\|G_N(\theta)G_N^{-1}(\theta) - \hat{1}\right\| = 0$, *i.e.*

$$\lim_{N\to\infty} \left\|(G_N(\theta)G_N^{-1}(\theta) - \hat{1})|x\rangle\right\| = 0$$

for any mass-eigenstate $|x\rangle$. When multiplying $G_N(\theta)$ and $G_N^{-1}(\theta)$ one typically gets all coefficients vanish till the power of N and then have a "tail" up to the order $\hat{S}^{2N}$. If $G(\theta)$ is well defined, this tail is expected to vanish when N is taken to infinity. However, this does not happen in our perturbative expansion. Consider the last term of the "tail" given exactly by $\frac{\hat{S}^{2N}}{N!N!}$. Recalling that $\hat{S}$ generator contains $\frac{\gamma_-}{2}\left(a_1^\dagger b_{-2}^\dagger + a_2^\dagger b_{-1}^\dagger\right)$ combination, we can write the state $\left(G_N G_N^{-1} - 1\right)|0\rangle$ as

$$\left(G_N G_N^{-1} - 1\right)|0\rangle = Y + \frac{(\theta\gamma_-)^{2N}}{2^{2N}N!^2} \sum_{t=0}^{2N} \binom{2N}{t} \left(a_1^\dagger\right)^t \left(b_{-2}^\dagger\right)^t \left(a_2^\dagger\right)^{2N-t} \left(b_{-1}^\dagger\right)^{2N-t} |0\rangle, \tag{2.46}$$

[2]here we suppress momentum indexes for brevity, i.e. $a_1 = a_{\vec{k},1}$ and $b_{-2} = b_{-\vec{k},2}$

where Y denotes all states with less then $4N$ number of particles and antiparticles. The norm of this state is

$$\begin{aligned}&\left\|\left(G_N G_N^{-1}-1\right)|0\rangle\right\|^2= \\ &\|Y\|^2+\tfrac{1}{N!^4}\left(\tfrac{\gamma_-\theta}{2}\right)^{2N}\sum_{t=0}^{2N}\left(\tbinom{2N}{t}\right)^2 t!t!\,(2N-t)!\,(2N-t)! \\ &>(2N+1)\left(\tfrac{\gamma_-\theta}{2}\right)^{2N}.\end{aligned} \tag{2.47}$$

Again, when $\gamma_-\theta>2$ the above expression is not convergent and the $G(\theta)\,G^{-1}(\theta)$ is not well defined. For small values of θ, however, the perturbative expansions (2.42) does make sense. The radius of convergence can be related to the pole of $\mathcal{Z}=\langle 0|\Omega(\theta)\rangle$ on the complex plane of θ and is given by $\gamma_-\sinh(\theta_{critical})=2$ (see below). It can be seen from here that a direct computation of the vacuum structure in perturbation series fails, unlike in the case of fermions [39], due to infinite number of states present in each bosonic mode.

To approach this problem from somewhat different side, we try to consider the flavor vacuum state in the most general form

$$|\Omega(\theta)\rangle=\sum_{n,l,m,k}\frac{C'_{nlmk}(\theta)}{n!l!}(a_1^\dagger)^n(a_2^\dagger)^l(b_{-1}^\dagger)^m(b_{-2}^\dagger)^k|0\rangle. \tag{2.48}$$

Applying the flavor annihilation operators to Eq.(2.48), one can obtain an infinite system of coupled linear equations for the coefficients $C'_{nlmk}(\theta)$

$$\begin{aligned}a_{A,B}\quad &|\Omega(\theta)\rangle=0; \\ b_{-A,-B}\quad &|\Omega(\theta)\rangle=0,\end{aligned} \tag{2.49}$$

and using explicit form of the ladder operators (2.69),

$$\begin{gathered}CC'_{n+1,lmk}+S_+C'_{n,l+1,mk}+S_-C'_{nlm,k-1}=0, \\ CC'_{n,l+1,mk}-S_+C'_{n+1,lmk}+S_-C'_{nl,m-1,k}=0, \\ n,l,m,k=0,1,2,3,\ldots\end{gathered} \tag{2.50}$$

To solve this infinite set of equations, we express $C'_{n+1,lmk}, C'_{n,l+1,km}$ in terms of $C'_{nl,m-1,k}, C'_{nlm,k-1}$ so that step by step we can reduce the $L=n+l$ total number of

particles in C'_{nlmk} down to zero. Denoting

$$\begin{pmatrix} Z_{12} & Z_{11} \\ Z_{22} & Z_{21} \end{pmatrix} = -S_- \begin{pmatrix} C & S_+ \\ -S_+ & C \end{pmatrix}^{-1} \tag{2.51}$$

we write Eq.(2.50) as

$$\begin{aligned} {C'}_{n+1,lmk} &= Z_{12}{C'}_{nlm,k-1} + Z_{11}{C'}_{nl,m-1,k} \\ {C'}_{n,l+1,mk} &= Z_{22}{C'}_{nlm,k-1} + Z_{21}{C'}_{nl,m-1,k}. \end{aligned} \tag{2.52}$$

One also can write this in a symbolic manner introducing shifting operators $\hat{k}$ and $\hat{m}$

$$\begin{aligned} \hat{k}{C'}_{nlmk} &= {C'}_{nlm,k-1}, \\ \hat{m}{C'}_{nlmk} &= {C'}_{nl,m-1,k} \end{aligned} \tag{2.53}$$

as

$$\begin{aligned} {C'}_{n+1,lmk} &= (Z_{12}\hat{k} + Z_{11}\hat{m}){C'}_{nlmk} = (Z_{12}\hat{k} + Z_{11}\hat{m})^2{C'}_{n-1,lmk} = \dots \\ {C'}_{n,l+1,mk} &= (Z_{22}\hat{k} + Z_{21}\hat{m}){C'}_{nlmk} = (Z_{22}\hat{k} + Z_{21}\hat{m})^2{C'}_{n,l-1,mk} = \dots \end{aligned} \tag{2.54}$$

so that finally we get

$$\begin{aligned} &{C'}_{nlmk} = (Z_{12}\hat{k} + Z_{11}\hat{m})^n(Z_{22}\hat{k} + Z_{21}\hat{m})^l{C'}_{00mk} = \\ &\left(\sum_{m'=0}^{n}\sum_{t'=0}^{l}\binom{n}{m'}\binom{l}{t'}Z_{11}^{m'}Z_{12}^{n-m'}Z_{21}^{t'}Z_{22}^{l-t'}\hat{k}^{n+l-(m'+t')}\hat{m}^{m'+t'}\right){C'}_{00mk}. \end{aligned} \tag{2.55}$$

One should note that, since the total momentum of vacuum state should be zero, ${C'}_{00mk} = 0$ unless $m = k = 0$. Therefore, in Eq.(2.55) only terms with $(m' + t' = m, n + l - (m' + t') = k)$ must survive. Also from Eq.(2.51) we can get $Z_{11} = -Z_{22}, Z_{12} = Z_{21}$ to write

$$|\Omega(\theta)\rangle = \mathcal{Z}\sum_{n,l=0}^{\infty}\sum_{m=0}^{n+l}\frac{B_{nlm}}{n!l!}(\hat{a}_1^\dagger)^n(\hat{a}_2^\dagger)^l(\hat{b}_{-1}^\dagger)^m(\hat{b}_{-2}^\dagger)^{n+l-m}|0\rangle, \tag{2.56}$$

where

$$B_{nlm} = \sum_{\substack{m'+t'=m \\ 0\le m'\le n \\ 0\le t'\le l}} \binom{n}{m'}\binom{l}{t'}Z_{11}^{l+m'-t'}Z_{12}^{n-m'+t'}(-1)^{l-t'}. \tag{2.57}$$

By a direct expansion one can verify that this cumbersome expression is equivalent to

$$|\Omega(\theta)\rangle = \mathcal{Z}\sum_{n,l}\frac{1}{n!l!}(Z_{11}a_1^\dagger b_{-1}^\dagger + Z_{12}a_1^\dagger b_{-2}^\dagger)^n(-Z_{11}a_2^\dagger b_{-2}^\dagger + Z_{12}a_2^\dagger b_{-1}^\dagger)^l|0\rangle. \tag{2.58}$$

Here $\mathcal{Z} = \langle 0|\Omega(\theta)\rangle$ is the normalization factor to be fixed by $\langle\Omega(\theta)|\Omega(\theta)\rangle = 1$ and the coefficients Z_{11} and Z_{12} are given by

$$\begin{aligned} Z_{11} &= \frac{\gamma_+\gamma_-\sin^2\theta}{4(\cos^2\theta + \frac{\gamma_+^2}{4}\sin^2\theta)} = \frac{\gamma_+\gamma_-\sin^2\theta}{4(1 + \frac{\gamma_-^2}{4}\sin^2\theta)} \\ Z_{12} &= \frac{-\gamma_-\sin 2\theta}{4(\cos^2\theta + \frac{\gamma_+^2}{4}\sin^2\theta)} = \frac{-\gamma_-\sin 2\theta}{4(1 + \frac{\gamma_-^2}{4}\sin^2\theta)}. \end{aligned} \tag{2.59}$$

We must note that coefficients Z_{11} and Z_{12} can be also written in this form

$$Z_{11} = \chi\cdot x, Z_{12} = \chi\cdot y, \tag{2.60}$$

where

$$\begin{aligned} \chi &= \frac{\gamma_-\sin\theta}{2\sqrt{1 + \frac{\gamma_-^2\sin^2\theta}{4}}} \\ x &= \frac{\gamma_+\sin\theta}{2\sqrt{\cos^2\theta + \frac{\gamma_+^2\sin^2\theta}{4}}} \\ y &= \frac{-\cos\theta}{\sqrt{\cos^2\theta + \frac{\gamma_+^2\sin^2\theta}{4}}}. \end{aligned} \tag{2.61}$$

Thus, the flavor vacuum state (2.58) shall be rewritten in a more concise form as

$$|\Omega(\theta)\rangle = \mathcal{Z}\sum_{n,l}\frac{\chi^{n+l}}{n!l!}\left(xa_1^\dagger b_{-1}^\dagger + ya_1^\dagger b_{-2}^\dagger\right)^n\left(-xa_2^\dagger b_{-2}^\dagger + ya_2^\dagger b_{-1}^\dagger\right)^l|0\rangle. \tag{2.62}$$

This result can be further reduced to

$$|\Omega(\theta)\rangle = \mathcal{Z}\sum_{n,l}\frac{\chi^{n+l}}{n!l!}(a_1^\dagger c_1^\dagger)^n(a_2^\dagger c_2^\dagger)^l|0\rangle \tag{2.63}$$

by defining

$$\begin{aligned} c_1 &= xb_{-1} + yb_{-2}, \\ c_2 &= -xb_{-2} + yb_{-1}. \end{aligned} \tag{2.64}$$

Note that $[c_1, c_1^\dagger] = [c_2, c_2^\dagger] = x^2 + y^2 = 1$, $[c_1, c_2^\dagger] = 0$. Thus, it becomes possible to compute directly the value of $\mathcal{Z}$ because

$$\langle \Omega(\theta)|\Omega(\theta)\rangle = \mathcal{Z}^2 \sum_{n,l} \frac{\chi^{2(n+l)}}{n!^2 l!^2} n!^2 l!^2 = \mathcal{Z}^2 \left(\sum_n \chi^{2n} \right)^2 = \frac{\mathcal{Z}^2}{(1-\chi^2)^2}. \tag{2.65}$$

We find

$$\mathcal{Z} = 1 - \chi^2 = \frac{1}{1 + \gamma_-^2 \sin^2 \theta/4}. \tag{2.66}$$

We see that, despite divergence of the perturbation series, the flavor vacuum state exists and the normalization factor $\mathcal{Z}(\theta)$ is finite. The same expression was obtained in Ref. [44] by solving a differential equation of $\langle 0|\Omega(\theta)\rangle = \langle 0|G(\theta)|0\rangle$. While such method of derivation using a differential equation has been known for some time [39,44,88], our algebraic method is a new development which also yields the complete structure of the flavor vacuum and not just $\mathcal{Z}(\theta)$ as in [39, 44, 88]. We observe that expression (2.66) has singularity on the complex plane at $\gamma_- \sin\theta = 2i$. Thus, the flavor vacuum $|\Omega(\theta)\rangle$ in terms of series in θ shall have a finite convergence radius and this would result in the divergence of the Taylor expansion for $\langle 0|\Omega(\theta)\rangle$ in powers of θ, as we have previously observed. This is in a remarkable difference from the fermion case where the corresponding result $\mathcal{Z}_{fermion} = 1 - \gamma_-^2 \sin^2 \theta/4$ doesn't have any singularity on the complex plane.

The above result proves also the unitary nonequivalence between the Fock spaces of the energy and the flavor eigenstates in the infinite volume limit, following the procedure discussed in Ref. [39],

$$\lim_{V\to\infty} \langle 0|\Omega(\theta)\rangle = \lim_{V\to\infty} \exp\left(\frac{V}{2\pi^3} \int d^3k \ln \mathcal{Z} \right) = 0. \tag{2.67}$$

Now, that we have built the representation of the mixing transformation (2.29) in the space of $\varphi_{1,2}$, let us further investigate these representations to obtain oscillation formulas for physically measurable quantities.

Using Eqs.(2.33) and (2.36), one can immediately obtain ladder operators for the mixed (flavor) fields that are consistent with the Pontecorvo mixing relations;

$$\begin{aligned} a_{\alpha,\beta} &= G^{-1}(\theta) a_{1,2} G(\theta), \\ b_{\alpha,\beta} &= G^{-1}(\theta) b_{1,2} G(\theta). \end{aligned} \tag{2.68}$$

Straightforward application of Baker-Hausdorff lemma to Eq.(2.68) yields

$$\begin{aligned} a_{\alpha} &= a_1 \cos\theta + \frac{\sin\theta}{2}(\gamma_+ a_2 + \gamma_- b^{\dagger}_{-2}), \\ a_{\beta} &= a_2 \cos\theta + \frac{\sin\theta}{2}(-\gamma_+ a_1 + \gamma_- b^{\dagger}_{-1}), \\ b_{-\alpha} &= b_{-1} \cos\theta + \frac{\sin\theta}{2}(\gamma_+ b_{-2} + \gamma_- a^{\dagger}_{2}), \\ b_{-\beta} &= b_{-2} \cos\theta + \frac{\sin\theta}{2}(-\gamma_+ b_{-1} + \gamma_- a^{\dagger}_{1}). \end{aligned} \tag{2.69}$$

It is also not difficult to reverse Eq.(2.69) in order to obtain the mass-eigenstate ladder operators in terms of the flavor ones. Using the above relations we can find the time dependence of the flavor-eigenstate ladder operators in the Heisenberg picture since the time evolution of mass-eigenstate ladder operators is known,

$$\begin{aligned} a_{1,2}(t) &= e^{i\mathcal{H}_0 t} a_{1,2} e^{-i\mathcal{H}_0 t} = e^{-i\epsilon_{1,2} t} a_{1,2}, \\ b_{1,2}(t) &= e^{i\mathcal{H}_0 t} b_{1,2} e^{-i\mathcal{H}_0 t} = e^{-i\epsilon_{1,2} t} b_{1,2}. \end{aligned} \tag{2.70}$$

In particular, after introducing a more compact notation,

$$C = \cos\theta; S_+ = \frac{\sin\theta\gamma_+}{2}; S_- = \frac{\sin\theta\gamma_-}{2}, \tag{2.71}$$

we find

$$\begin{aligned} a_{At} = \quad & \big(C^2 e^{-i\epsilon_1 t} + S_+^2 e^{-i\epsilon_2 t} - S_-^2 e^{i\epsilon_2 t}\big) a_A + C S_+ \big(e^{-i\epsilon_2 t} - e^{-i\epsilon_1 t}\big) a_B + \\ & S_+ S_- \big(e^{i\epsilon_2 t} - e^{-i\epsilon_2 t}\big) b^{\dagger}_{-A} + C S_- \big(e^{i\epsilon_2 t} - e^{-i\epsilon_1 t}\big) b^{\dagger}_{-B}, \end{aligned}$$

$$
\begin{aligned}
a_{Bt} = \quad & \big(C^2 e^{-i\epsilon_2 t} + S_+^2 e^{-i\epsilon_1 t} - S_-^2 e^{i\epsilon_1 t}\big) a_B + CS_+\big(e^{-i\epsilon_2 t} - e^{-i\epsilon_1 t}\big) a_A + \\
& S_+ S_-\big(e^{-i\epsilon_1 t} - e^{i\epsilon_1 t}\big) b_{-B}^\dagger + CS_-\big(e^{i\epsilon_1 t} - e^{-i\epsilon_2 t}\big) b_{-A}^\dagger, \\
b_{-At} = \quad & \big(C^2 e^{-i\epsilon_1 t} + S_+^2 e^{-i\epsilon_2 t} - S_-^2 e^{i\epsilon_2 t}\big) b_{-A} + CS_+\big(e^{-i\epsilon_2 t} - e^{-i\epsilon_1 t}\big) b_{-B} + \\
& S_+ S_-\big(e^{i\epsilon_2 t} - e^{-i\epsilon_2 t}\big) a_A^\dagger + CS_-\big(e^{i\epsilon_2 t} - e^{-i\epsilon_1 t}\big) a_B^\dagger, \\
b_{-Bt} = \quad & \big(C^2 e^{-i\epsilon_2 t} + S_+^2 e^{-i\epsilon_1 t} - S_-^2 e^{i\epsilon_1 t}\big) b_{-B} + CS_+\big(e^{-i\epsilon_2 t} - e^{-i\epsilon_1 t}\big) b_{-A} + \\
& S_+ S_-\big(e^{-i\epsilon_1 t} - e^{i\epsilon_1 t}\big) a_B^\dagger + CS_-\big(e^{i\epsilon_1 t} - e^{-i\epsilon_2 t}\big) a_A^\dagger, \qquad (2.72)
\end{aligned}
$$

from which we can obtain nonequal-time commutation relations:

$$
\begin{aligned}
[a_A, a_{At}^\dagger] &= [b_{-A}, b_{-At}^\dagger] = C^2 e^{i\epsilon_1 t} + S_+^2 e^{i\epsilon_2 t} - S_-^2 e^{-i\epsilon_2 t} = A_{AA}, \\
[a_B, a_{Bt}^\dagger] &= [b_{-B}, b_{-Bt}^\dagger] = C^2 e^{i\epsilon_2 t} + S_+^2 e^{i\epsilon_1 t} - S_-^2 e^{-i\epsilon_1 t} = A_{BB}, \\
[a_B, a_{At}^\dagger] &= [a_A, a_{Bt}^\dagger] = [b_{-B}, b_{-At}^\dagger] \\
&= [b_{-A}, b_{-Bt}^\dagger] = CS_+\big(e^{i\epsilon_2 t} - e^{i\epsilon_1 t}\big) = A_{BA}, \\
[b_{-B}, a_{At}] &= [a_B, b_{-At}] = -[b_{-A}, a_{Bt}]^* \\
&= -[a_A, b_{-Bt}]^* = CS_-\big(e^{i\epsilon_2 t} - e^{-i\epsilon_1 t}\big) = A_{\bar{B}A}, \\
[b_{-A}, a_{At}] &= [a_A, b_{-At}] = S_+ S_-\big(e^{i\epsilon_2 t} - e^{-i\epsilon_2 t}\big) = A_{\bar{A}A}, \\
[b_{-B}, a_{Bt}] &= [a_B, b_{-Bt}] = S_+ S_-\big(e^{-i\epsilon_1 t} - e^{i\epsilon_1 t}\big) = A_{\bar{B}B}. \qquad (2.73)
\end{aligned}
$$

All other commutators are either zero or can be expressed in terms of Eq.(2.73). Eqs.(2.69), (2.72), (2.73) define all dynamics of mixing for two scalar quantum fields. To show how these relations can be used to calculate the dynamical quantities, we can consider the time evolution of cluster $\Omega_{\vec{k}}$, as defined above. To calculate the number of particles with a definite mass condensed in the flavor vacuum state $|\Omega\rangle$ let us consider the condensation of the particle with a definite mass, e.g. $Z_1 = \langle\Omega|N_1|\Omega\rangle$. Using the inverse relation of Eq.(2.69):

$$
a_1 = a_A \cos\theta - \frac{\sin\theta}{2}(\gamma_+ a_B + \gamma_- b_{-B}^\dagger), \qquad (2.74)
$$

we get

$$
Z_1 = \langle\Omega|a_1^\dagger a_1|\Omega\rangle = \frac{\sin^2\theta\gamma_-^2}{4}\langle\Omega|b_{-B} b_{-B}^\dagger|\Omega\rangle. \qquad (2.75)
$$

One can show that the same result is true for $Z_2 = \langle\Omega|N_2|\Omega\rangle$. Thus, the condensate

density of particles with a definite mass in the flavor vacuum is

$$Z_1 = Z_2 = S_-^2 = \frac{\sin^2\theta\gamma_-^2}{4}. \tag{2.76}$$

Apparently, the condensate densities for particles with definite flavor in the mass vacuum, *i.e.* $\langle 0|N_{\alpha(\beta)}|0\rangle$, are also given by S_-^2.

Let us now consider the number of particles with a definite flavor in the flavor vacuum, for example $Z_A(t) = \langle\Omega|N_A(t)|\Omega\rangle$. Using Eq.(2.72), one can easily show that

$$\begin{aligned} Z_A(t) = \langle\Omega|\big(S_+S_-\big(e^{i\epsilon_2 t} - e^{-i\epsilon_2 t}\big)b^\dagger_{-A} + CS_-\big(e^{i\epsilon_2 t} - e^{-i\epsilon_1 t}\big)b^\dagger_{-B}\big)^\dagger\cdot \\ \big(S_+S_-\big(e^{i\epsilon_2 t} - e^{-i\epsilon_2 t}\big)b^\dagger_{-A} + CS_-\big(e^{i\epsilon_2 t} - e^{-i\epsilon_1 t}\big)b^\dagger_{-B}\big)|\Omega\rangle \end{aligned} \tag{2.77}$$

and thus

$$Z_A(t) = 4S_-^2S_+^2\sin^2(\epsilon_2 t) + 4S_-^2C^2\sin^2(\frac{\epsilon_1+\epsilon_2}{2}t). \tag{2.78}$$

Similarly, we get for B-particles

$$Z_B(t) = 4S_-^2S_+^2\sin^2(\epsilon_1 t) + 4S_-^2C^2\sin^2(\frac{\epsilon_1+\epsilon_2}{2}t). \tag{2.79}$$

We see that the number of particles with a definite flavor in the flavor vacuum is not zero. This is due to the fact that the flavor vacuum is not an energy eigenstate of the Hamiltonian $\mathcal{H}(\theta)$ and changes with time producing and destroying coherently virtual flavor particle/antiparticle pairs. It shows a significant difference from the ordinary quantum mechanical treatment without account for the flavor vacuum effect which yields apparently $Z_{A(B)} = 0$ at any time. We emphasize that here the flavor vacuum is not perturbative but exact. This is different from the approach of Sachs and others [60, 61, 89] where the energy-eigenstates vacuum $|0\rangle$ was used in place of the flavor vacuum to define flavor eigenstates, *e.g.* $|A\rangle = a_A^\dagger|0\rangle$. If the flavor vacuum $|\Omega\rangle$ was replaced by the mass vacuum $|0\rangle$, then we would have obtained $Z_1 = Z_2 = 0$ instead of Eq.(2.76). As we will discuss in the next section, such definition of the flavor states results in a number of anomalies and normalization problems for flavor states and oscillation formulas (see also [84]). In fact the energy-eigenstates vacuum $|0\rangle$ is not annihilated by $a_{A,B}$ operators up to the terms proportional to $O(\gamma_-)$ so

that the error of the order of $O(\gamma_-^2)$ should be expected from the results of such perturbative vacuum treatment.

We now consider the flavor oscillations of a single particle with flavor A and momentum $\vec{k}$. In Heisenberg picture, the average number of particles with flavor A or B in the beam can be related to

$$\langle N_A(t)\rangle = \langle A|N_A(t)|A\rangle = \langle\Omega|a_A a_A^\dagger(t)a_A(t)a_A^\dagger|\Omega\rangle. \tag{2.80}$$

In Eq.(2.80) we note that we use the flavor vacuum to obtain nonperturbative flavor oscillation formula.

Using Eqs.(2.69), (2.72), (2.73) we directly apply the standard QFT method - since the flavor vacuum is annihilated by $a_{A,B}$, we move a_A in Eq.(2.80) to the most right position and $a_A^\dagger$ to the most left position to annihilate the flavor vacuum. What is left is uniquely determined by nonequal time commutation relations (2.73) and we find

$$\begin{aligned}
\langle A|N_{At}|A\rangle &= \langle\Omega|a_{At}^\dagger a_{At}|\Omega\rangle + |[a_A, a_{At}^\dagger]|^2 = Z_A + |A_{AA}|^2;\\
\langle A|N_{-\bar{A}t}|A\rangle &= \langle\Omega|b_{-At}^\dagger b_{-At}|\Omega\rangle + |[a_A, b_{-At}]|^2 = Z_A + |A_{\bar{A}A}|^2;\\
\langle A|N_{Bt}|A\rangle &= \langle\Omega|a_{Bt}^\dagger a_{Bt}|\Omega\rangle + |[a_A, a_{Bt}^\dagger]|^2 = Z_B + |A_{BA}|^2;\\
\langle A|N_{-\bar{B}t}|A\rangle &= \langle\Omega|b_{-Bt}^\dagger b_{-Bt}|\Omega\rangle + |[a_A, b_{-Bt}]|^2 = Z_B + |A_{\bar{B}A}|^2.
\end{aligned} \tag{2.81}$$

Using the notation of C, $S_\pm$, our results are summarized as:

$$\begin{aligned}
\langle A|N_{At}|A\rangle &= 1 + 8C^2S_-^2\sin^2(\frac{\epsilon_1+\epsilon_2}{2}t) + 8S_-^2S_+^2\sin^2(\epsilon_2 t) - 4C^2S_+^2\sin^2(\frac{\epsilon_1-\epsilon_2}{2}t),\\
\langle A|N_{Bt}|A\rangle &= 4C^2S_-^2\sin^2(\frac{\epsilon_1+\epsilon_2}{2}t) + 4S_-^2S_+^2\sin^2(\epsilon_1 t) + 4C^2S_+^2\sin^2(\frac{\epsilon_1-\epsilon_2}{2}t),\\
\langle A|N_{-\bar{A}t}|A\rangle &= 4C^2S_-^2\sin^2(\frac{\epsilon_1+\epsilon_2}{2}t) + 8S_-^2S_+^2\sin^2(\epsilon_2 t),\\
\langle A|N_{-\bar{B}t}|A\rangle &= 8C^2S_-^2\sin^2(\frac{\epsilon_1+\epsilon_2}{2}t) + 4S_-^2S_+^2\sin^2(\epsilon_1 t).
\end{aligned} \tag{2.82}$$

As shown in Eq.(2.82),the time dependence of the average number of particles with a definite flavor is rather complicate. It contains oscillating contributions both from the $A \to B$ conversion and from the virtual pair creation in a dynamically "rotating"

flavor vacuum. As discussed in Ref. [36], the $A \to B$ conversion process generates the term proportional to $\sin^2(\frac{\epsilon_1-\epsilon_2}{2}t)$. The terms involving $\epsilon_1+\epsilon_2, \epsilon_1, \epsilon_2$-frequencies in Eq.(2.82) are, however, related to the creation of virtual pairs. For example, virtual pair creation violates energy conservation within the uncertainty time, i.e. $\Delta E \Delta t \approx 1$ (in our units $\hbar = 1$) and thus both creation and annihilation of, say, $(1+\bar{2})$ virtual pair must occur within $\tau \approx \frac{1}{\epsilon_1+\epsilon_2}$ time interval. Thus, the terms in Eq.(2.82) involving $\epsilon_1+\epsilon_2, \epsilon_1, \epsilon_2$-frequencies can be related to the creation of different types of virtual pairs, while the terms involving $\epsilon_1 - \epsilon_2$ is related to the actual $A \to B$ conversion.

Using Eq.(2.82), we can also calculate the expectation value of the flavor charge operator defined by $Q_{A(B)} = N_{A(B)} - N_{-\bar{A}(-\bar{B})}$,

$$\begin{aligned}
\langle Q_A \rangle &= 1 - 4C^2S_+^2 \sin^2(\frac{\epsilon_1-\epsilon_2}{2}t) + 4CS_-^2 \sin^2(\frac{\epsilon_1+\epsilon_2}{2}t), \\
\langle Q_B \rangle &= 4C^2S_+^2 \sin^2(\frac{\epsilon_1-\epsilon_2}{2}t) - 4C^2S_-^2 \sin^2(\frac{\epsilon_1+\epsilon_2}{2}t),
\end{aligned} \tag{2.83}$$

or with $\gamma_\pm$ parameters,

$$\begin{aligned}
\langle Q_A \rangle &= 1 - \gamma_+^2 \sin^2(2\theta)\sin^2(\frac{\epsilon_1-\epsilon_2}{2}t) + \gamma_-^2 \sin^2(2\theta)\sin^2(\frac{\epsilon_1+\epsilon_2}{2}t), \\
\langle Q_B \rangle &= \gamma_+^2 \sin^2(2\theta)\sin^2(\frac{\epsilon_1-\epsilon_2}{2}t) - \gamma_-^2 \sin^2(2\theta)\sin^2(\frac{\epsilon_1+\epsilon_2}{2}t).
\end{aligned} \tag{2.84}$$

From this result, one can see that there is term proportional to $\sin^2(\frac{\epsilon_1+\epsilon_2}{2}t)$ in the flavor charge oscillations in addition to usual Pontecorvo formula. As discussed above, the origin of this term is understood as contribution from virtual pair creation in "rotating" flavor vacuum.

We also calculate the time evolution of a coherent state [90] in field-theoretic $SU(2)$ flavor mixing,

$$|CA\rangle = e^{Ca_A^\dagger}|\Omega\rangle. \tag{2.85}$$

Extending the above calculation, it is not difficult to verify that for the state containing n particles with flavor A

$$\langle n|N_{At}|n\rangle = \frac{1}{n!}\langle\Omega|a_A^n N_{At}(a_A^\dagger)^n|\Omega\rangle = \langle N_{At}\rangle + n|A_{AA}|^2. \tag{2.86}$$

Besides $n|A_{AA}|^2$, which is simply n times the probability of $A \to A$ transition, we see in Eq.(2.86) that the condensate contribution is present. Applying this result directly to the coherent state, we obtain

$$\begin{aligned} \langle CA|N_{At}|CA\rangle &= Z_A + |C|^2|A_{AA}|^2, \\ \langle CA|N_{Bt}|CA\rangle &= Z_B + |C|^2|A_{BA}|^2. \end{aligned} \tag{2.87}$$

Thus, the expectation values of the flavor charge operator $Q_{(A,B)} = N_{(A,B)} - N_{(-\bar{A},-\bar{B})}$ turns out to be

$$\begin{aligned} \langle CA|Q_A|CA\rangle &= |C|^2\langle Q_A\rangle \\ \langle CA|Q_B|CA\rangle &= |C|^2\langle Q_B\rangle \end{aligned} \tag{2.88}$$

In Eq.(2.88), the vacuum contributions $Z_{(A,B)}$ are removed from the flavor charge and the result for the coherent state is simply $|C|^2$ times the expectation value of the flavor charge for a single particle initial state.

We now apply these results to oscillations in $\eta\eta'$. The masses are known to be 549 MeV and 958 MeV, respectively [91]. The phenomenologically allowed mixing angle ($\theta_{SU(3)}$) range of the $\eta\eta'$ system is found to be between -10° and -23° [91], where the mixing angle $\theta_{SU(3)}$ is defined by Eq.(36) of Ref. [92]. This angle represents the breaking of the SU(3) symmetry, the eigenstates of which are already rotated -35.26° from $u\bar{u} + d\bar{d}$ and $s\bar{s}$ to $\alpha = u\bar{u} + d\bar{d} - 2s\bar{s}$ and $\beta = u\bar{u} + d\bar{d} + s\bar{s}$. Thus, our mixing angle should be defined by $\theta = \theta_{SU(3)} - 35.26^\circ$. Recent analysis of the $\eta\eta'$ mixing angle using a CQM based on the Fock states quantized on the light-front can be found in Ref. [92] and the references therein. The value found for $\theta_{SU(3)}$ was $\approx -19^\circ$ and thus $\theta \approx -54^\circ$. We use these values in Eqs.(2.82) and (2.83) (or equivalently (2.84)) to determine the evolution of definite flavor particle number and flavor charge.

In Fig.(2.2), we present both $\langle A|N_{At}|A\rangle$ (thick solid line) and $\langle Q_A\rangle$ (dotted line) as a function of time when the particle momentum is low ($k = 0.1$GeV). For comparison, we also show the previous approximate result (thin solid line) based on the perturbative vacuum [36] ($\langle A|N_{At}|A\rangle$ and $\langle Q_A\rangle$ coincide in this approximation). As we show in Fig.(2.2), the population density $\langle A|N_{At}|A\rangle$ (thick solid line) is distorted

(deviation up to 40%) due to interaction with flavor vacuum while only renormalized Pontecorvo result (thin solid line) is obtained for the approximate perturbative vacuum treatment of Ref. [36]. However, one cannot see the same amount of deviation in $\langle Q_A \rangle$ and the previous result [36] based on the perturbative vacuum is in a good agreement with the flavor charge oscillations modulo the accuracy of order $O(\gamma_-^2)$. More details of our results on the time evolution of the particle number with the momentum $k = 0.1$ GeV are shown in Fig.(2.3), where thick solid and dashed lines are $\langle A|N_{At}|A\rangle$ and $\langle A|N_{Bt}|A\rangle$, respectively, and thin solid and dotted lines are respectively the antiparticle contributions $\langle A|N_{\bar{A}t}|A\rangle$ and $\langle A|N_{\bar{B}t}|A\rangle$.

The $\eta\eta'$ is one of the most severely mixed systems due to the great difference in masses of mixed particles and large mixing angle. As we have stated earlier, the simple harmonic structure of particle number usually obtained in Quantum Mechanics or in an approximate QFT treatment is completely altered as the result of nontrivial interaction with the flavor vacuum. What we see is the superposition of two different modes as described in Eq.(2.82). From the initial moment of time the population of both A-particles (thick solid line) and B-particles (thick dashed line) increases. Although the increase in number of B-particles is well understood due to $A \to B$ conversion, the initial increase of A-population is quite unexpected and caused by virtual pairs production from the vacuum. The contribution from this process is rather fast so that the general tendency of exchanging between A and B particles can be seen quite well. In Fig.(2.3), we see the oscillations of the number of antiparticles. This effect is given in the order of γ_-^2 and is absent in perturbative vacuum models or Quantum Mechanics. Besides the beams of A and B particles moving in $\vec{k}$ direction, we necessarily have antiparticle beam traveling in the opposite direction. The population density in this beam is correlated with particle-beam so that the total flavor is preserved. The existence of beam is caused by disturbance of flavor vacuum at the time when A-particle was created. Thus, the mixed particle of definite flavor not only produces the ordinary flavor oscillations but also is accompanied with the beam of flavor antiparticles traveling in the direction opposite to the beam of particles.

In Fig.(2.4), we plot oscillations of flavor charge for the same momentum $k = 0.1$GeV. The thick solid and dashed lines are $\langle Q_A \rangle$ and $\langle Q_B \rangle$, respectively. One can see that these are the simple oscillations similar to Quantum Mechanical result [22,51]. Distortions due to interaction with vacuum are of the order of γ_-^2, *i.e.* about 10% in this example. Interesting feature is presence of regions where the flavor charge is overwhelmed by antiparticle presence and changes sign. This feature is understood as result of virtual pair production at the times when the number of, e.g., A-particles is small due to $A \to B$ conversion.

It is also interesting that the efficiency of the conversion processes and the flavor-vacuum contribution depends on the energy of the original particle. The dependence is proportional to the difference in the relativistic masses of the energy-eigenstates and is decreasing with the energy growth. The effect is given by the amplitudes $\gamma_+(k)$ and $\gamma_-(k)$, which determine the contribution from a_2 and $b^\dagger_{-2}$ terms in a_A (See Eq.(2.69)). In Fig.(2.5), we plot dependence of γ_+ and γ_- on the momentum. As we see in Fig.(2.5), γ_+ amplitude falls as k increases and goes to 2 for $k \to \infty$. In this limit, γ_+ defines mixing due to a simple rotation between a_1 and a_2 states, it gives quantum-mechanical Pontecorvo formula. On the other hand, γ_- appears with an antiparticle creation operator and describe Bogoliubov rotation between a_1 and $b^\dagger_{-1}$. This term is responsible for $\frac{\epsilon_1+\epsilon_2}{2}$ high frequency contributions and antiparticle beams. As we see in Fig.(2.5), it decreases as $k \to \infty$ and the mass difference becomes washed out by the relativistic gain of mass. This also means that at ultra-relativistic limit the QFT-mixing effects vanish so that the simple Pontecorvo formula is restored for flavor-oscillation.

To demonstrate the energy dependence of mixing in BV theory, we show in Fig.(2.6) the plot of population densities evolving with time for the larger momentum $k = 0.5$GeV. The line assignments are same as in Fig.(2.3). As easily seen in Fig.(2.6), the intensity of antiparticle beam decreases dramatically to about 10% [in contrast to 20-40% in Fig.(2.3)] of initial intensity. The initial increase in the population density fluctuation in particle beams also reduces even though the quantum mechanical

oscillations with $\frac{\epsilon_1-\epsilon_2}{2}$ frequency are still visibly distorted. Two beams demonstrate strong correlation of the same kind as EPR-correlation in Quantum Mechanics so that the total flavor charge is conserved as it should be.

2.5 General Quantum Field Theory of Flavor Mixing

We now concentrate our attention on generalization of the above treatment to understand general physical background behind BV theory and to obtain a formulation of BV theory applicable in general to $SU(N)$ mixing with $N > 2$.

In QFT mixing is described with interaction Hamiltonian density

$$\mathcal{H}_I(\phi(x)) = \frac{1}{2}\sum_{\mu,\nu=A,B,...} m_{\mu\nu}\phi^\dagger_\mu(x)\phi_\nu(x) + h.c. \tag{2.89}$$

so that full classical Hamiltonian can be diagonalized similarly with appropriately chosen linear transformation from the flavor-fields ϕ_μ to the mass-fields φ_i

$$\phi_\mu \to \varphi_i = \sum_{\mu=A,B,...} U^\dagger_{i\mu}\phi_\mu, \tag{2.90}$$

$$\mathcal{H}_0(\phi(x)) + \mathcal{H}_I(\phi(x)) \to \mathcal{H}'_0(\varphi(x)). \tag{2.91}$$

Here $\mathcal{H}_0$ is a free-theory Hamiltonian and the free-fields φ_i are written in terms of their Fourier transform as usual

$$\varphi_i = \sum_\sigma \int \frac{d\mathbf{k}}{\sqrt{2\epsilon_{i\mathbf{k}}}}\left(u^i_{\mathbf{k}\sigma}a_{i\mathbf{k}\sigma}\left(t\right)e^{i\mathbf{kx}} + v^i_{\mathbf{k}\sigma}b^\dagger_{i\mathbf{k}\sigma}\left(t\right)e^{-i\mathbf{kx}}\right). \tag{2.92}$$

$a_{i\mathbf{k}\sigma}(t) = e^{-i\epsilon_{i\mathbf{k}}t}a_{i\mathbf{k}\sigma}$ and $b_{i\mathbf{k}\sigma}(t) = e^{-i\epsilon_{i\mathbf{k}}t}b_{i\mathbf{k}\sigma}$ with the standard equal time commutation/anticommutation relations. In Eq.(2.92), $u^i_{\mathbf{k}\sigma}$ and $v^i_{\mathbf{k}\sigma}$ are free particle and antiparticle amplitudes, respectively, and σ is the helicity quantum number

$$(\mathbf{n}\cdot\mathbf{s})\,u^i_{\mathbf{k}\sigma} = \sigma u^i_{\mathbf{k}\sigma}, (\mathbf{n}\cdot\mathbf{s})\,v^i_{\mathbf{k}\sigma} = \sigma v^i_{\mathbf{k}\sigma}. \tag{2.93}$$

Here $\mathbf{s}$ is the spin operator and $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$.

Main difference from Quantum Mechanics is that in QFT the transformation (2.90) *does not* immediately imply a specific form for the mixing relations between the flavor and the energy eigenstates. In fact, the intrinsic presence of antiparticle degrees of freedom introduces a dramatic difference. As is known, in field theory any operators with the same conserved quantum numbers can mix. This means that in general in Eq.(2.12) not only the flavor particle annihilation operators with momentum $\mathbf{k}$ and helicity σ will mix, but also the flavor antiparticle creation operators with momentum $-\mathbf{k}$ and helicity $-\sigma$ may enter. Thus, the most general linear mixing relation in QFT becomes

$$\begin{aligned} a_{\mu\mathbf{k}\sigma} &= \sum_{i=a,b,\ldots} \left(\alpha_{\mu i}(\mathbf{k}) a_{i\mathbf{k}\sigma} + \beta_{\mu i}(\mathbf{k}) b^{\dagger}_{i-\mathbf{k}-\sigma} \right), \\ b^{\dagger}_{\mu-\mathbf{k}-\sigma} &= \sum_{i=a,b,\ldots} \left(\alpha_{\mu i}(\mathbf{k}) b^{\dagger}_{i-\mathbf{k}-\sigma} + \eta \beta_{\mu i}(\mathbf{k}) a_{i\mathbf{k}\sigma} \right), \end{aligned} \tag{2.94}$$

where $a_\mu(a_i)$ stands for flavor-eigenstate (mass-eigenstate) particle annihilation operator and $b_\mu(b_i)$ stands for flavor-eigenstate (mass-eigenstate) antiparticle annihilation operator. In these relations we explicitly imply that antiparticles and "particle-holes" are treated on equal footing, as they enter covariant field $\varphi(x)$. Factor $\eta = (-1)^{2S}$ with S being the spin of the mixed fields (η is +1 for bosons and -1 for fermions) has the quantum-statistics related origin and is required to guarantee $[a_{\mu\mathbf{k}\sigma}, b_{\mu-\mathbf{k}-\sigma}]_{\pm} = 0$. Consequently, we have introduced two mixing matrices, $\alpha_{\mu i}$ and $\beta_{\mu i}$, describing particle-particle mixing and particle-antiparticle cross-mixing. For brevity, we will suppress momentum notation implying that all quantities are taken at given momentum $\mathbf{k}$ and helicity σ or $-\mathbf{k}$, $-\sigma$ as indicated by the sign in front of the flavor/mass-eigenstate index (i.e. a_i stands for $a_{i\mathbf{k}\sigma}$ and b_{-i} for $b_{i-\mathbf{k}-\sigma}$).

Few simple properties of quantum mixing transformation are in place. For Eq.(2.94) to preserve commutation/anticommutation relations it should hold

$$\begin{cases} |\alpha_{\mu i}|^2 + |\beta_{\mu i}|^2 = |U_{\mu i}|^2 \ \text{ fermions}, \\ |\alpha_{\mu i}|^2 - |\beta_{\mu i}|^2 = |U_{\mu i}|^2 \ \text{ bosons}, \end{cases} \tag{2.95}$$

so that one can relate $\alpha_{\mu i}$ and $\beta_{\mu i}$ to cos and sin for fermions or cosh and sinh for

bosons, respectively;

$$\begin{aligned} \alpha_{\mu i} &= U_{\mu i} \begin{cases} \cos(\theta_{\mu i}) \text{ fermions} \\ \cosh(\theta_{\mu i}) \text{ bosons} \end{cases} , \\ \beta_{\mu i} &= U_{\mu i} \begin{cases} \sin(\theta_{\mu i}) \text{ fermions} \\ \sinh(\theta_{\mu i}) \text{ bosons} \end{cases} . \end{aligned} \tag{2.96}$$

Eqs.(2.94) must also furnish a representation of the classical mixing transformation (2.90) in the linear space of quantum fields. Then, it follows that following relation should hold

$$\theta_{\mu i} - \theta_{\mu i'} = \theta_{i' i} \tag{2.97}$$

independent of μ.

The possibility of antiparticle admixture in (2.94) has important consequences. In particular, it is clear that the vacuum $|0\rangle$, associated with the free fields φ_i, is not annihilated by a_μ. In fact, use of $|0\rangle$ to define flavor quantum states introduces various normalization problems. If one proceeds to define the flavor states as in Ref. [36], e.g.

$$|A\rangle = a_A^\dagger |0\rangle,$$

then as follows from Eq.(2.95)

$$\langle A|A\rangle = \sum_i |\alpha_{Ai}|^2 \neq 1.$$

This forces one to introduce much artificial normalization factors as in, e.g., Ref. [35, 36]. In a more general form a similar normalization problem had been demonstrated in regular perturbation theory in the mixing of two fermions of spin 1/2 in Ref. [35].

Following Ref. [35], QFT of 2-flavor mixing is a theory simple enough for the perturbation series to be summed exactly. For example, one may obtain for a time-ordered two-point function $S_{AA} = \langle 0|T[\psi_A \bar{\psi}_A]|0\rangle$

$$\begin{aligned} S_{AA} &= S_A(1 + m_{AB}^2 S_B S_A + m_{AB}^4 S_B S_A S_B S_A + \dots) \\ &= S_A(1 - m_{AB}^2 S_B S_A)^{-1} = \cos^2(\theta) \tfrac{\not{k} + m_a}{k^2 - m_a^2 + i\delta} + \sin^2(\theta) \tfrac{\not{k} + m_b}{k^2 - m_b^2 + i\delta} \end{aligned} \tag{2.98}$$

where the "bare" propagators are $S_{A,B} = (\not{k} - m_{A,B} + i\delta)^{-1}$. The transition amplitude for a fermion A created at time $t = 0$ to go into the same particle at time t then is given by

$$\begin{aligned} P^r_{AA}(\mathbf{k}, t) &= i u^{r\dagger}_{\mathbf{k},a} e^{i\epsilon_{a,\mathbf{k}} t} S^{>}_{AA}(\mathbf{k}, t) \gamma^0 u^r_{\mathbf{k},a} \\ &= \cos^2(\theta) + \sin^2(\theta) |U_{\mathbf{k}}|^2 e^{i(\epsilon_{\mathbf{k},a} - \epsilon_{\mathbf{k},b})t}, \end{aligned} \tag{2.99}$$

where $S^{>}_{AA}(t)$ is forward ($t > 0$) propagation function, $u_{a,b}$ are bi-spinors used to expand mass-eigenstate fields and

$$|U_{\mathbf{k}}|^2 = \frac{1}{2} \sum_{r,s} |u^{r\dagger}_{\mathbf{k},b} u^s_{\mathbf{k},a}|^2.$$

Upon computing $|U_{\mathbf{k}}|^2$ one can explicitly observe that $|U_{\mathbf{k}}|^2 < 1$ and thus $P_{AA}(t \to +0) \neq 1$ [35]. These results indicate that a special care needs to be taken in QFT to properly define flavor states. In particular, the flavor vacuum state $|\Omega\rangle$ shall be properly defined as the state annihilated by flavor particle/antiparticle annihilation operators and flavor quantum states shall be built on top of $|\Omega\rangle$.

The explicit structure of flavor vacuum state can be obtained in general by solving the set of equations

$$a_\mu |\Omega\rangle = 0, b_\mu |\Omega\rangle = 0. \tag{2.100}$$

Let us express the flavor vacuum state as a linear combination of the mass eigenstates, i.e. in the most general form,

$$|\Omega\rangle = \sum_{(n),(l)} \frac{1}{n_1! n_2! \dots n_k!} B_{(n)(l)} \left(a^\dagger_1\right)^{n_1} \dots \left(a^\dagger_k\right)^{n_k} \left(b^\dagger_{-1}\right)^{l_1} \dots \left(b^\dagger_{-k}\right)^{l_k} |0\rangle, \tag{2.101}$$

with $(n) = (n_1 n_2 n_3 \dots)$. To explicitly solve flavor vacuum structure we first consider boson case. After applying Eq.(2.100) to Eq.(2.101) we get an infinite set of coupled equations. The part of Eq.(2.100) involving antiparticle annihilation operators results in a dependent set of equations and thus can be omitted. Expanding Eq.(2.100), we find,

$$\sum_j \left(\alpha_{ij} B_{(n_j+1)(l)} + \beta_{ij} B_{(n)(l_j-1)}\right) = 0, \text{all } (n), (l) \tag{2.102}$$

where $(n_j \pm 1)$ stands for $(n_1, n_2, \ldots, n_j \pm 1, \ldots n_k)$ and k is the number of flavor fields. To solve Eq.(2.102) we introduce symbolic operators which decrease the subscript index of $B_{(n)(l)}$, i.e. $d_{-j} B_{(n)(l)} = B_{(n)(l_j - 1)}$. Then solving each set of equations in (2.102) with respect to $B_{(n_j+1)(l)}$ we find

$$\begin{aligned} B_{(n_i+1)(l)} &= (\sum_j Z_{ij} d_{-j}) B_{(n)(l)} \text{ and consequently} \\ B_{(n)(l)} &= \prod_i (\sum_j Z_{ij} d_{-j})^{n_i} B_{(0)(l)} \end{aligned} \tag{2.103}$$

with matrix $\hat{Z} = -\hat{\alpha}^{-1} \cdot \hat{\beta}$. Considering the momentum conservation and the original Eq.(2.102), it can be shown that only $B_{(0)(l=0)}$ must be non-zero among all (l). Thus, applying symbolic operators d_{-j} and leaving only term $B_{(0)(0)}$ in the expansion, we get

$$B_{(n)(l)} = \sum_{\left\{ \begin{array}{c} (j_p^i) \\ \sum_p j_p^i = n_i \\ \sum_i j_p^i = l_p \end{array} \right\}} \prod_i \frac{n_i!}{j_1^i! \ldots j_k^i!} Z_{i1}^{j_1^i} \ldots Z_{ik}^{j_k^i} B_{(0)(0)}. \tag{2.104}$$

It is possible to re-express this complicated equation in a more compact form;

$$|\Omega\rangle = \frac{1}{\mathcal{Z}} \sum_{(k)} \prod_i \frac{1}{k_i!} (\sum_j Z_{ij} a_i^\dagger b_{-j}^\dagger)^{k_i} |0\rangle, \tag{2.105}$$

that can be shown by noticing that to obtain $B_{(n)(l)}$ from Eq.(2.105) one needs to leave only those terms in the expansion that give the correct power of particle and antiparticle creation operators, i.e. the total powers of all $a_i^\dagger$'s are n_i's and $b_i^\dagger$'s are l_i. This is the same procedure to extract $B_{(n)(l)}$ from Eq.(2.103). Or Eq.(2.105) can be shown by directly expanding the above expressions. The constant $\mathcal{Z}$ is introduced instead of $B_{(0)(0)}$ and serves as a normalization factor determined by $\langle \Omega | \Omega \rangle = 1$.

Eq.(2.105) can be further simplified to

$$\begin{aligned} |\Omega\rangle &= \tfrac{1}{\mathcal{Z}} \sum_{(k)} \prod_i \tfrac{1}{k_i!} (\sum_j Z_{ij} a_i^\dagger b_{-j}^\dagger)^{k_i} |0\rangle = \\ &= \tfrac{1}{\mathcal{Z}} \prod_i \sum_{k_i=0}^{\infty} \tfrac{1}{k_i!} (\sum_j Z_{ij} a_i^\dagger b_{-j}^\dagger)^{k_i} |0\rangle = \\ &= \tfrac{1}{\mathcal{Z}} \exp(\sum_{i,j=1}^{N} Z_{ij} a_i^\dagger b_{-j}^\dagger) |0\rangle. \end{aligned} \tag{2.106}$$

Let us now proceed to the fermion case. We employ the same idea with the symbolic shifting operators. One needs to be careful about the sign convention here. If $\hat{C}_{(n)(l)}$ stands for creation operator for fermion state $|(n),(l)\rangle$, we want then

$$\begin{aligned} a_i B_{(n_i+1)(l)} \hat{C}_{(n_i+1)(l)} |0\rangle &= \pm B_{(n_i+1)(l)} \hat{C}_{(n)(l)} |0\rangle = d_{+i} B_{(n)(l)} \hat{C}_{(n)(l)} |0\rangle \\ b_i^\dagger B_{(n)(l_i-1)} \hat{C}_{(n)(l_i-1)} |0\rangle &= \pm B_{(n)(l_i-1)} \hat{C}_{(n)(l)} |0\rangle = d_{-i} B_{(n)(l)} \hat{C}_{(n)(l)} |0\rangle \end{aligned} \tag{2.107}$$

with the correct sign. Eq.(2.102) then can be written in the form

$$\sum_j (\alpha_{ij} d_{+j} + \beta_{ij} d_{-j}) B_{(n)(l)} = 0 \tag{2.108}$$

which binds together the shifting operators that increase and decrease the index. This set can be solved as

$$d_{+i}[B_{(n)(l)}] = \sum_j Z_{ij} d_{-j}[B_{(n)(l)}] \tag{2.109}$$

with the same matrix $\hat{Z}$ as in the boson case. From the definition of shifting operators it can be inferred that they obey anticommutation property (i.e. $d_{\pm i} d_{\pm j} = -d_{\pm j} d_{\pm i}$) and thus it can be shown further that for $i_1 > i_2 > \ldots > i_n$

$$\begin{aligned} d_{+i_n} d_{+i_{n-1}} \ldots d_{+i_1} B_{(0)(l)} &= B_{(i)(l)} \\ d_{-i_1} d_{-i_2} \ldots d_{-i_l} B_{(n)(l)} &= B_{(n)(l-i)} \end{aligned} \tag{2.110}$$

so that the solution can be written again as

$$B_{(n)(l)} = \prod_i (\sum_j Z_{ij} d_{-j})^{n_i} B_{(0)(l)}, \tag{2.111}$$

where only $B_{(0)(0)}$ survives. Here, n_i can be only 0 or 1 and the anticommutation rules for the ordering are applied. It is remarkable that Eq.(2.105) can still be used for

the fermion vacuum. This can be verified by a direct expansion keeping in mind the anticommutation nature of the ladder operators. Thus, for either boson or fermion case the flavor vacuum state is

$$|\Omega\rangle = \frac{1}{\mathcal{Z}} \exp(\sum_{i,j=1}^{N} Z_{ij} a_i^\dagger b_{-j}^\dagger)|0\rangle. \tag{2.112}$$

We now proceed to find the normalization constant $\mathcal{Z}$. For this, we consider

$$||\Omega\rangle|^2 = |\exp(\sum_{i,j=1}^{N} Z_{ij} a_i^\dagger b_{-j}^\dagger)|0\rangle|^2 = \sum_L \frac{1}{L!^2} |(\sum_{i,j=1}^{N} Z_{ij} a_i^\dagger b_{-j}^\dagger)^L|0\rangle|^2, \tag{2.113}$$

where we use the fact that the states of $(\sum_{i,j=1}^{N} Z_{ij} a_i^\dagger b_{-j}^\dagger)^L|0\rangle$ are orthogonal for different L's. We then employ the fact that any matrix $\hat{Z}$ can be transformed to a diagonal form with two unitary transformations, i.e.

$$Z' = \begin{pmatrix} x_1 & 0 & \dots \\ 0 & \ddots & 0 \\ \dots & 0 & x_N \end{pmatrix} = UZV^\dagger. \tag{2.114}$$

We can now introduce additional unitary transformations $a'^\dagger = U^\dagger a^\dagger$, $b'^\dagger = V^\dagger b^\dagger$ to make $\sum_{i,j=1}^{N} Z_{ij} a_i^\dagger b_{-j}^\dagger = \sum_{i=1}^{N} Z'_{ii} a_i'^\dagger b_{-i}'^\dagger$, where a'_i, b'_{-j} satisfy the standard commutation/anticommutation relations. Then, using binomial formula to expand $(\sum_{i=1}^{N} Z'_{ii} a_i'^\dagger b_{-i}'^\dagger)^L$, we find

$$\begin{aligned} &\sum_L \tfrac{1}{L!^2} |(\sum_{i=1}^{N} Z'_{ii} a_i'^\dagger b_{-i}'^\dagger)^L|0\rangle|^2 = \\ &= \sum_L \tfrac{1}{L!^2} \sum_{n_1+\dots+n_N=L} L!^2 \prod_{j=1}^{N} \tfrac{1}{n_j!^2} |(Z'_{jj} a_j'^\dagger b_{-j}'^\dagger)^{n_j}|0\rangle|^2 = \\ &= \sum_L \sum_{n_1+\dots+n_N=L} \prod_{j=1}^{N} \tfrac{n_j!^2}{n_j!^2} |Z'^{n_j}_{jj}|^2 = \sum_{n_1,\dots,n_N} \lambda_1^{n_1} \dots \lambda_N^{n_N}, \end{aligned} \tag{2.115}$$

where λ_i's are eigenvalues of $ZZ^\dagger$. The summation limits in Eq.(2.115) are different for fermions and bosons. For bosons n_i run from 0 to ∞, while for fermions they only

can be 0 or 1. In either case the sum can be evaluated to give

$$|\exp(\sum_{i,j=1}^{N} Z_{ij} a_i^\dagger b_{-j}^\dagger)|0\rangle|^2 = \begin{cases} \prod_i (1+\lambda_i) & \text{fermions} \\ \prod_i \frac{1}{1-\lambda_i} & \text{bosons} \end{cases} = \begin{cases} \det(1+ZZ^\dagger) \\ \det^{-1}(1-ZZ^\dagger). \end{cases} \tag{2.116}$$

Thus, for the flavor vacuum state we find explicitly

$$|\Omega\rangle = \frac{1}{\mathcal{Z}} \exp(\sum_{i,j=1}^{N} Z_{ij} a_i^\dagger b_{-j}^\dagger)|0\rangle, \tag{2.117}$$

where Z_{ij} is an (i,j) element of the matrix $Z = -\alpha^{-1}\cdot\beta$. and $\mathcal{Z}$ is fixed by $\langle\Omega|\Omega\rangle = 1$: $\mathcal{Z} = \det^{1/2}\left(1+\hat{Z}\hat{Z}^\dagger\right)$ for fermions and $\mathcal{Z} = \det^{-1/2}\left(1-\hat{Z}\hat{Z}^\dagger\right)$ for bosons. We see that the flavor vacuum state has a rich coherent structure. This situation is different from the perturbative QFT, where the adiabatic enabling of interaction ensures $|0\rangle_{interacting} \sim |0\rangle_{free}$. Rich nontrivial vacuum renders additional effects in the flavor dynamics. In particular, the normalization constant $\mathcal{Z}$ is always greater than 1 so that in the infinite volume limit, when the density of states is going to infinity, we have

$$\mathcal{Z}_{tot} = \exp\left(\frac{V}{(2\pi)^3}\int d\vec{k}\ln(\mathcal{Z}_{\vec{k}})\right) \to \infty. \tag{2.118}$$

Thus, any possible state for the flavor vacuum shall have an infinite norm in the free-field Fock space and therefore the flavor vacuum state *cannot* be found in the original Fock space. The unitary nonequivalence of the flavor Fock space and the original Fock space is, therefore, established in general case, i.e. $\langle\Omega|0\rangle = \frac{1}{\mathcal{Z}_{tot}} \to 0$, $V \to \infty$.

The time dynamics of quantum theory can be entirely described in terms of its non-equal time commutation/anticommutation relations. We note that in quantum field theory of mixing only $a_{i\mathbf{k}\sigma}$ and $b^\dagger_{i-\mathbf{k}-\sigma}$ operators and their conjugates mix together. We denote the set of quantum fields formed by all linear combinations of these operators and their products (algebra on $a_{i\vec{k}\sigma}$, $b_{i-\vec{k}-\sigma}$ and h.c.) as a cluster $\Omega(\mathbf{k},\sigma)$ with a particular momentum $\mathbf{k}$ and helicity σ. It follows then that $\Omega(\mathbf{k},\sigma)$'s are invariant under mixing transformation (2.94) and we can analyze each cluster independently from the others.

The non-equal time commutators/anticommutators for flavor fields with given $\mathbf{k}$ and σ can be derived from (2.94) using the standard commutation/anticommutation relations for the mass-eigenstate ladder operators;

$$\begin{aligned} F_{\mu\nu}(t) = [a_\mu(t), a^\dagger_\nu]_\pm &= \sum_{k,k'} \left(\alpha_{\mu k}\alpha^*_{\nu k'} \left[a_k e^{-i\epsilon_k t}, a^\dagger_{k'} \right]_\pm + \beta_{\mu k}\beta^*_{\nu k'} \left[b^\dagger_{-k} e^{i\epsilon_k t}, b_{-k'} \right]_\pm \right) \\ &= \sum_k (\alpha_{\mu k}\alpha^*_{\nu k} e^{-i\epsilon_k t} - \eta\beta_{\mu k}\beta^*_{\nu k} e^{i\epsilon_k t}); \\ [b_{-\mu}(t), b^\dagger_{-\nu}]_\pm &= F_{\nu\mu}(t); \\ G_{\mu\nu}(t) = [b_{-\mu}(t), a_\nu]_\pm &= \sum_{k,k'} \left(\alpha^*_{\mu k}\beta_{\nu k'} \left[b_{-k} e^{-i\epsilon_k t}, b^\dagger_{-k'} \right]_\pm + \eta\beta^*_{\mu k}\alpha_{\nu k'} \left[a^\dagger_k e^{i\epsilon_k t}, a_{k'} \right]_\pm \right) \\ &= \sum_k (\alpha^*_{\mu k}\beta_{\nu k} e^{-i\epsilon_k t} - \beta^*_{\mu k}\alpha_{\nu k} e^{i\epsilon_k t}), \end{aligned} \tag{2.119}$$

where $\pm$ in $[\cdot]_\pm$ corresponds to commutation/anticommutation. The two matrices F and G represent the only nontrivial commutators/anticommutators in the sense that all others are either zero or can be written in terms of the elements of these matrices. It is useful to note that, for $t \to +0$, Eq.(2.119) shall be reduced to $F_{\mu\nu}(0) = \delta_{\mu\nu}$ and $G_{\mu\nu}(0) = 0$. We also note that

$$\begin{aligned} F_{\mu\nu}(t)^* &= F_{\nu\mu}(-t), \\ G_{\mu\nu}(t)^* &= -G_{\nu\mu}(t). \end{aligned} \tag{2.120}$$

Eq.(2.119) allows one to compute many mixing quantities directly. For example, the time dynamics of ladder operators for flavor fields can be solved by writing them as $a_\mu(t) = \sum_\nu (f_{\mu\nu} a_\nu(0) + g_{\mu\nu} b^\dagger_{-\nu}(0) + \ldots)$. Then, one can get, straightforwardly, $f^*_{\mu\nu} = [a_\nu(0), a^\dagger_\mu(t)]_\pm = F_{\nu\mu}(-t)$ and $g_{\mu\nu} = [b_{-\nu}(0), a_\mu(t)]_\pm = G_{\nu\mu}(-t)$ while all other coefficients are zeros

$$\begin{aligned} a_\mu(t) &= \sum_\nu \left(F_{\mu\nu}(t) a_\nu + G_{\nu\mu}(-t) b^\dagger_{-\nu} \right); \\ b_{-\mu}(t) &= \sum_\nu \left(F_{\nu\mu}(t) b_{-\nu} + \eta G_{\mu\nu}(t) a^\dagger_\nu \right). \end{aligned} \tag{2.121}$$

We now consider various dynamical quantities in field theory of mixing such as the condensate densities of the mass-eigenstate particles in the flavor vacuum

($Z'_i = \langle\Omega|a_i^\dagger(t)\,a_i(t)\,|\Omega\rangle$), the number of flavor-eigenstate particles in the flavor vacuum ($Z_\nu = \langle\Omega|a_\nu^\dagger(t)\,a_\nu(t)\,|\Omega\rangle$) and the particle number expectation for a single flavor-particle initial state, which is related in the Heisenberg picture to $N_{\rho\nu\rho} = \langle\rho|a_\nu^\dagger(t)\,a_\nu(t)\,|\rho\rangle$, $\bar{N}_{\rho\nu\rho} = \langle\rho|b_{-\nu}^\dagger(t)\,b_{-\nu}(t)\,|\rho\rangle$.

The free-field particle condensates in the flavor vacuum state are computed directly from Eq.(2.94);

$$Z'_i = \sum_j |\beta_{ij}|^2 . \tag{2.122}$$

In the following, the particle-antiparticle symmetry should be taken into account, so that a corresponding antiparticle quantity can be found from the particle expression after a necessary substitution (particles→antiparticles and vice versa). It means that the antiparticle condensate is also given by Eq.(2.122). Similarly, the flavor particle condensates in the free-field vacuum are given by Eq.(2.122) as well. Using Eq.(2.121), we can get the flavor-particle condensates in the flavor vacuum;

$$Z_\nu(t) = \sum_\mu |G_{\nu\mu}(-t)|^2 . \tag{2.123}$$

It is remarkable that this number is not zero but oscillates with time.

The evolution of the particle ($N_{\rho\nu\rho}$) and antiparticle ($\bar{N}_{\rho\nu\rho}$) expectation number with flavor ν can be found using the standard technique of normal ordering, i.e. moving annihilation operators to the right side and creation operators to the left side of the expression. With this technique, in general, we obtain

$$\begin{aligned} N_{\rho\nu\sigma}(t) &= [a_\rho, a_\nu^\dagger(t)]_\pm [a_\nu(t)\,a_\sigma^\dagger]_\pm + \delta_{\rho\sigma}\langle 0|a_\nu^\dagger(t)a_\nu(t)|0\rangle = \\ &= F_{\nu\rho}^*(t)F_{\nu\sigma}(t) + \delta_{\rho\sigma}Z_\nu(t), \\ \bar{N}_{\rho\nu\sigma}(t) &= \eta[a_\rho, b_{-\nu}(t)]_\pm [b_{-\nu}^\dagger(t)\,, a_\sigma^\dagger]_\pm + \delta_{\rho\sigma}\langle 0|b_\nu^\dagger(t)b_\nu(t)|0\rangle = \\ &= \eta G_{\nu\rho}(t)G_{\nu\sigma}(t)^* + \delta_{\rho\sigma}Z_\nu(-t). \end{aligned} \tag{2.124}$$

For a specific case of the number evolution in the beam with a fixed 3-momentum, we find

$$\begin{aligned} N_{\rho\nu\rho} &= \langle 0|a_\rho a_\nu^\dagger(t)\,a_\nu(t)\,a_\rho^\dagger|0\rangle = |F_{\nu\rho}(t)|^2 + Z_\nu(t), \\ \bar{N}_{\rho\nu\rho} &= \langle 0|a_\rho b_{-\nu}^\dagger(t)\,b_{-\nu}(t)\,a_\rho^\dagger|0\rangle = \eta\,|G_{\nu\rho}(t)|^2 + Z_\nu(-t). \end{aligned} \tag{2.125}$$

We emphasize that no general reason can be found in the above theory for $Z_\nu(t)$ to be equal to $Z_\nu(-t)$. In fact, explicit computation shows that if the mixing matrix $U_{\mu i}$ is not real, e.g. CP violating CKM matrix, $Z_\nu(t)-Z_\nu(-t)$ will have nonzero contribution proportional to the imaginary part of $U_{\mu j}$. We must understand this quite unexpected result as manifestation of T-violation in the presence of CP-violating mixing, which *is required* by the CPT-theorem. Interestingly, Eq.(2.120) demonstrates that T-violation can occur only in "particle-antiparticle" cross-mixing, described by $G_{\mu\nu}$, and not in "particle-particle" mixing described by $F_{\mu\nu}$.

The flavor charge $Q_{\rho\nu\rho}$ can be defined by [42, 43, 93]

$$Q_{\rho\nu\rho} = |F_{\nu\rho}(t)|^2 - \eta\,|G_{\nu\rho}(t)|^2\,. \tag{2.126}$$

Explicitly we get

$$\begin{aligned} Q_{\mu\nu\mu} &= \sum_{k,k'}(\alpha_{\mu k}\alpha^*_{\nu k}e^{i\epsilon_k t} - \eta\beta_{\mu k}\beta^*_{\nu k}e^{-i\epsilon_k t})(\alpha^*_{\mu k'}\alpha_{\nu k'}e^{-i\epsilon_{k'} t} - \eta\beta^*_{\mu k'}\beta_{\nu k'}e^{i\epsilon_{k'} t})- \\ &-\eta\sum_{k,k'}(\alpha^*_{\nu k}\beta_{\mu k}e^{-i\epsilon_k t} - \beta^*_{\nu k}\alpha_{\mu k}e^{i\epsilon_k t})(\alpha_{\nu k'}\beta^*_{\mu k'}e^{i\epsilon_{k'} t} - \beta_{\nu k'}\alpha^*_{\mu k'}e^{-i\epsilon_{k'} t}) \\ &= \sum_{k,k'}(\alpha_{\nu k'}\alpha^*_{\nu k} - \eta\beta_{\nu k'}\beta^*_{\nu k})(e^{-i(\epsilon_{k'}-\epsilon_k)t}\alpha^*_{\mu k'}\alpha_{\mu k} - \eta e^{i(\epsilon_{k'}-\epsilon_k)t}\beta_{\mu k}\beta^*_{\mu k'})- \\ &\qquad \eta(\beta^*_{\nu k}\alpha_{\nu k'} - \alpha^*_{\nu k}\beta_{\nu k'})(e^{-i(\epsilon_{k'}+\epsilon_k)t}\beta_{\mu k}\alpha^*_{\mu k'} - e^{i(\epsilon_{k'}+\epsilon_k)t}\alpha_{\mu k}\beta^*_{\mu k'}). \end{aligned} \tag{2.127}$$

Taking into account Eq.(2.97), we can write, e.g. for fermions (S=1/2),

$$\begin{aligned} \alpha_{\nu k'}\alpha^*_{\nu k} + \beta_{\nu k'}\beta^*_{\nu k} &= U_{\nu k'}U^*_{\nu k}(\cos(\theta_{\nu k'})\cos(\theta_{\nu k}) + \sin(\theta_{\nu k'})\sin(\theta_{\nu k})) \\ &= U_{\nu k'}U^*_{\nu k}\cos(\theta_{\nu k'} - \theta_{\nu k}) = U_{\nu k'}U^*_{\nu k}\cos(\theta_{k k'}), \\ \beta^*_{\nu k}\alpha_{\nu k'} - \alpha^*_{\nu k}\beta_{\nu k'} &= U_{\nu k'}U^*_{\nu k}(\cos(\theta_{\nu k'})\sin(\theta_{\nu k}) - \cos(\theta_{\nu k'})\sin(\theta_{\nu k})) \\ &= U_{\nu k'}U^*_{\nu k}\sin(\theta_{\nu k} - \theta_{\nu k'}) = U_{\nu k'}U^*_{\nu k}\sin(\theta_{k' k}). \end{aligned}$$

Thus, we find

$$\begin{aligned} Q_{\mu\nu\mu} = \sum_{k,k'} U_{\nu k'}U^*_{\nu k}U_{\mu k}U^*_{\mu k'}(\cos^2(\theta_{k k'})\cos(\omega_{k' k}t) + i\cos(\theta_{k' k})\cos(\theta_{\mu k} + \theta_{\mu k'})\sin(\omega_{k k'}t)+ \\ \sin^2(\theta_{k' k})\cos(\Omega_{k' k}t) - i\sin(\theta_{k' k})\sin(\theta_{\mu k} + \theta_{\mu k'})\sin(\Omega_{k k'}t)); \end{aligned} \tag{2.128}$$

where $\Omega_{ij} = \epsilon_i + \epsilon_j$ and $\omega_{ij} = \epsilon_i - \epsilon_j$, and finally

$$\begin{aligned} Q_{\mu\nu\mu} &= \sum_{k,k'} Re(U_{\nu k'} U^*_{\nu k} U_{\mu k} U^*_{\mu k'})(\cos^2(\theta_{kk'}) \cos(\omega_{k'k} t) - \eta \sin^2(\theta_{k'k}) \cos(\Omega_{k'k} t)) + \\ &+ \sum_{k,k'} Im(U_{\nu k'} U^*_{\nu k} U_{\mu k} U^*_{\mu k'})(\cos(\theta_{kk'}) \cos(\theta_{\mu k} + \theta_{\mu k'}) \sin(\omega_{k'k} t) - \\ &- \eta \sin(\theta_{k'k}) \sin(\theta_{\mu k} + \theta_{\mu k'}) \sin(\Omega_{k'k} t)). \end{aligned} \tag{2.129}$$

These formulas are valid also for bosons with the substitution cos $\rightarrow$ cosh, sin $\rightarrow$ sinh, $\eta \rightarrow +1$.

Finally, we note that Eq.(2.125) may be viewed as a superposition of two terms: $\rho \rightarrow \nu$ propagation and background contribution Z_ν. Thus, one may introduce the particle-particle and particle-antiparticle oscillations amplitudes, respectively,

$$\begin{aligned} \mathcal{A}_{\rho \rightarrow \nu}(k,t) &= [a_\nu(t), a^\dagger_\rho(0)]_\pm = F_{\nu\rho}(t) \\ \mathcal{A}_{\rho \rightarrow -\bar{\nu}}(k,t) &= [b_{-\nu}(t), a_\rho(0)]_\pm = G_{\nu\rho}(t). \end{aligned} \tag{2.130}$$

Such propagation amplitudes would also appear from the flavor-field Green functions defined on the flavor vacuum $|\Omega\rangle$ as $\langle\Omega|T[\phi_\nu(k,t)\phi^\dagger_\rho(k,0)]|\Omega\rangle$ [35].

In the general theory presented above, as we discussed earlier, a specific model shall be used to connect field-theoretical mixing matrices $\alpha_{\mu j}$ and $\beta_{\mu j}$ with classical mixing matrix $U_{\mu j}$. Here we will use BV theory to establish such a connection. To develop our general formulation we shall pursue a bit different strategy from that employed originally by BV. We shall avoid the step of explicitly building the quantum representation $\Lambda(U,t)$ by directly employing Eq.(2.90) and an observation, pioneered in [37], that the ladder operators can be directly extracted from the covariant fields with a linear operation. For example, for spin-1/2 fermions

$$\begin{aligned} a_{i\mathbf{k}\sigma}(t) &= \tfrac{\sqrt{2\epsilon_{i\mathbf{k}}}}{H^{ii}_{\mathbf{k}\sigma}} u^{i\dagger}_{\mathbf{k}\sigma} \varphi_{i\mathbf{k}}(t), \\ b_{i-\mathbf{k}-\sigma}(t) &= \left[\tfrac{\sqrt{2\epsilon_{i\mathbf{k}}}}{H^{ii}_{\mathbf{k}\sigma}} v^{i\dagger}_{-\mathbf{k}-\sigma} \varphi_{i\mathbf{k}}(t) \right]^\dagger, \end{aligned} \tag{2.131}$$

where H and h parameters are defined as follows

$$\begin{aligned} H^{\mu j}_{\mathbf{k}\sigma} \delta_{\sigma,\sigma'} = u^{\mu\dagger}_{\mathbf{k}\sigma} u^j_{\mathbf{k}\sigma'} = v^{\mu\dagger}_{-\mathbf{k}-\sigma} v^j_{-\mathbf{k}-\sigma'}, \\ h^{\mu j}_{\mathbf{k}\sigma} \delta_{\sigma,\sigma'} = u^{\mu\dagger}_{\mathbf{k}\sigma} v^j_{-\mathbf{k}-\sigma'}. \end{aligned} \tag{2.132}$$

Since the Fourier component $\varphi_{i\mathbf{k}}(t) = \sum_\sigma \frac{1}{\sqrt{\epsilon_{i\mathbf{k}}}} \left(u^i_{\mathbf{k}\sigma} a_{i\mathbf{k}\sigma}(t) + v^i_{-\mathbf{k}\sigma} b^\dagger_{i-\mathbf{k}\sigma}(t) \right)$ is obviously a linear combination of $\varphi_i(\mathbf{x}, t)$, one expressed the ladder operators as a linear combinations of the original covariant fields. Using linearity of the mixing transformation (2.17), we can find the structure of $a_{\mu\mathbf{k}\sigma}(t)$ without explicitly building $\Lambda(U,t)$;

$$\begin{aligned} a_{\mu\mathbf{k}\sigma}(t) &= \tfrac{\sqrt{2\epsilon_{\mu\mathbf{k}}}}{H^{\mu\mu}_{\mathbf{k}\sigma}} u^{\mu\dagger}_{\mathbf{k}\sigma} (\Lambda(U,t)^\dagger \bar{\varphi}_{\mathbf{k}}(t) \Lambda(U,t))_\mu = \\ &= \sum_j \tfrac{\sqrt{2\epsilon_{\mu\mathbf{k}}}}{H^{\mu\mu}_{\mathbf{k}\sigma}} u^{\mu\dagger}_{\mathbf{k}\sigma} U_{\mu j} \varphi_{j\mathbf{k}}(t), \\ b_{\mu-\mathbf{k}-\sigma}(t) &= \sum_j \tfrac{\sqrt{2\epsilon_{\mu\mathbf{k}}}}{H^{\mu\mu}_{\mathbf{k}\sigma}} U^*_{\mu j} \varphi^\dagger_{j\mathbf{k}}(t) v^\mu_{-\mathbf{k}-\sigma}. \end{aligned} \tag{2.133}$$

For the bosons the ladder operators do not separate as in the fermion case, e.g.

$$u^{i\dagger}_{\mathbf{k}\sigma} \varphi_{i\mathbf{k}}(t) = \frac{1}{\sqrt{2\epsilon_{i\mathbf{k}}}} (a_{i\mathbf{k}\sigma}(t) + h^{ii}_{\mathbf{k}\sigma} b^\dagger_{i-\mathbf{k}-\sigma}(t)) \tag{2.134}$$

and in general $h^{ii}_{\mathbf{k}\sigma} \neq 0$. Eq.(2.134) implies that particles and antiparticles in boson case can not be distinguished unless time dynamics is considered. To deal with this problem we define ladder operators for bosons by

$$\begin{aligned} a_{i\mathbf{k}\sigma} &= u^{i\dagger}_{\mathbf{k}\sigma} \left(\sqrt{\tfrac{\epsilon_{i\mathbf{k}}}{2}} \varphi_{i\mathbf{k}}(t) + \tfrac{1}{\sqrt{2\epsilon_{i\mathbf{k}}}} \dot{\varphi}_{i\mathbf{k}}(t) \right), \\ b^\dagger_{i-\mathbf{k}-\sigma} &= v^{i\dagger}_{-\mathbf{k}-\sigma} \left(\sqrt{\tfrac{\epsilon_{i\mathbf{k}}}{2}} \varphi_{i\mathbf{k}}(t) - \tfrac{1}{\sqrt{2\epsilon_{i\mathbf{k}}}} \dot{\varphi}_{i\mathbf{k}}(t) \right). \end{aligned} \tag{2.135}$$

With Eqs.(2.131) and (2.135), we then derive for fermions

$$\begin{aligned} a_\mu &= \tfrac{\sqrt{2\epsilon_\mu}}{H^{\mu\mu}} \sum_{j,\sigma'} (u^{\mu\dagger}_{\mathbf{k}\sigma} u^j_{\mathbf{k}\sigma'} a_j + u^{\mu\dagger}_{\mathbf{k}\sigma} v^j_{-\mathbf{k}-\sigma'} b^\dagger_{-j}) \tfrac{U_{\mu j}}{\sqrt{2\epsilon_j}} = \\ &= \sum_j \left(\sqrt{\tfrac{\epsilon_\mu}{\epsilon_j}} \tfrac{H^{\mu j}}{H^{\mu\mu}} U_{\mu j} a_j + \sqrt{\tfrac{\epsilon_\mu}{\epsilon_j}} \tfrac{h^{\mu j}}{H^{\mu\mu}} U_{\mu j} b^\dagger_{-j} \right); \end{aligned} \tag{2.136}$$

$$\begin{aligned} b_{-\mu} &= \tfrac{\sqrt{2\epsilon_\mu}}{H^{\mu\mu}} \sum_{j,\sigma'} \left(\left(v^{\mu\dagger}_{-\mathbf{k}-\sigma} u^j_{\mathbf{k}\sigma'} \right)^* a^\dagger_j + \left(v^{\mu\dagger}_{-\mathbf{k}-\sigma} v^j_{-\mathbf{k}-\sigma'} \right)^* b_{-j} \right) \tfrac{U^*_{\mu j}}{\sqrt{2\epsilon_j}} = \\ &= \sum_j \left(\sqrt{\tfrac{\epsilon_\mu}{\epsilon_j}} \tfrac{\left(H^{\mu j}\right)^*}{H^{\mu\mu}} U^*_{\mu j} b_{-j} - \sqrt{\tfrac{\epsilon_\mu}{\epsilon_j}} \tfrac{\left(h^{\mu j}\right)^*}{H^{\mu\mu}} U^*_{\mu j} a^\dagger_j \right) \end{aligned} \tag{2.137}$$

and for bosons:

$$\begin{aligned} a_\mu &= \tfrac{\sqrt{2\epsilon_\mu}}{2} \sum_{j,\sigma'} \left(u^{\mu\dagger}_{\mathbf{k}\sigma} u^j_{\mathbf{k}\sigma'} \tfrac{\epsilon_\mu + \epsilon_j}{\epsilon_\mu} a_j + u^{\mu\dagger}_{\mathbf{k}\sigma} v^j_{-\mathbf{k}-\sigma'} \tfrac{\epsilon_\mu - \epsilon_j}{\epsilon_\mu} b^\dagger_{-j} \right) \tfrac{U_{\mu j}}{\sqrt{2\epsilon_j}} = \\ &= \sum_j \left(\tfrac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}} + \sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2} H^{\mu j} U_{\mu j} a_j + \tfrac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}} - \sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2} h^{\mu j} U_{\mu j} b^\dagger_{-j} \right); \end{aligned} \tag{2.138}$$

$$b_{-\mu} = \frac{\sqrt{2\epsilon_\mu}}{2}\sum_{j,\sigma'}\left(v^{\mu\dagger}_{-\mathbf{k}-\sigma}u^{j}_{\mathbf{k}\sigma'}\frac{\epsilon_\mu-\epsilon_j}{\epsilon_\mu}a^\dagger_j + v^{\mu\dagger}_{-\mathbf{k}-\sigma}v^{j}_{-\mathbf{k}-\sigma'}\frac{\epsilon_\mu+\epsilon_j}{\epsilon_\mu}b_{-j}\right)^*\frac{U^*_{\mu j}}{\sqrt{2\epsilon_j}} = \\ = \sum_j\left(\frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}+\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2}(H^{\mu j})^*U^*_{\mu j}b_{-j} + \frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}-\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2}(h^{\mu j})^*U^*_{\mu j}a^\dagger_j\right). \tag{2.139}$$

By defining

$$\alpha_{\mu j} = \gamma^+_{\mu j}U_{\mu j},\ \beta_{\mu j} = \gamma^-_{\mu j}U_{\mu j}, \tag{2.140}$$

where

$$\gamma^+_{\mu j} = \begin{cases} \sqrt{\frac{\epsilon_\mu}{\epsilon_j}}\frac{H^{\mu j}}{H^{\mu\mu}} \text{ fermions,} \\ H^{\mu j}\frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}+\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2} \text{ bosons.} \end{cases} \\ \gamma^-_{\mu j} = \begin{cases} \sqrt{\frac{\epsilon_\mu}{\epsilon_j}}\frac{h^{\mu j}}{H^{\mu\mu}} \text{ fermions,} \\ h^{\mu j}\frac{\sqrt{\frac{\epsilon_\mu}{\epsilon_j}}-\sqrt{\frac{\epsilon_j}{\epsilon_\mu}}}{2} \text{ bosons,} \end{cases} \tag{2.141}$$

this can be put in the form (2.94).

Using the formulas presented in Appendix A, one can explicitly verify for $S = 0, 1/2, 1$ that the conditions (2.95) and (2.97) are satisfied and Eqs.(2.136-2.139) provide a valid representation for the classical mixing transformation. As an example, consider $\frac{\partial\gamma^-_{\mu j}}{\partial m_\mu}$ for spin-1/2 fermions. One notices that $\frac{\partial\gamma^-_{\mu j}}{\partial m_\mu}$ can be reduced to $\frac{\partial\gamma^-_{\mu j}}{\partial m_\mu} = \gamma^+_{\mu j}f(m_\mu)$, e.g. for fermions

$$\frac{\partial\theta_{\mu j'}}{\partial m_\mu} - \frac{\partial\theta_{\mu j}}{\partial m_\mu} = \frac{\frac{\partial\sin(\theta_{\mu j'})}{\partial m_\mu}}{\cos(\theta_{\mu j'})} - \frac{\frac{\partial\sin(\theta_{\mu j})}{\partial m_\mu}}{\cos(\theta_{\mu j})} = f(m_\mu) - f(m_\mu) = 0$$

so that $\theta_{\mu j} = \theta_\mu - \theta_j$, where $\cos(\theta_\mu) = \frac{1}{2\sqrt{\epsilon_\mu}}(\sqrt{\epsilon_\mu + m_\mu} + \sqrt{\epsilon_\mu - m_\mu})$ and $\sin(\theta_\mu) = \frac{1}{2\sqrt{\epsilon_\mu}}(\sqrt{\epsilon_\mu + m_\mu} - \sqrt{\epsilon_\mu - m_\mu})$. Thus Eq.(2.97) is trivially satisfied. One may use then our general results to analyze the flavor dynamics in BV formulation.

Up to this point our discussion was concerned with time dynamics of flavor fields, let us now concentrate on flavor vacuum effect on flavor oscillations in space dimensions. We begin our study with a simple example. Let us consider a particle created initially in a quantum state $|i\rangle$ that propagates in space and time. The number of particles of sort ρ expected at space-time position $x = (t, \mathbf{x})$ may be introduced via

$$N_\rho(x) = \left\langle i|a^\dagger_\rho(x)a_\rho(x)|i\right\rangle, \tag{2.142}$$

where $a^\dagger_\rho(x)(a_\rho(x))$ is creation (annihilation) operator for particles of sort ρ at space-time position x. This can be defined via creation (annihilation) operators for given momentum $\mathbf{k}$

$$a^\dagger_\rho(t,\mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\rho\mathbf{k}}}} e^{-i\mathbf{kx}} a^\dagger_{\rho,\mathbf{k}}(t). \tag{2.143}$$

Substituting the definition (2.143) into Eq.(2.142), we obtain

$$N_\rho(x) = \sum_{\mathbf{k},\mathbf{k}'} \frac{e^{i(\mathbf{k}'-\mathbf{k})\mathbf{x}}}{2\sqrt{\omega_{\rho\mathbf{k}}\omega_{\rho\mathbf{k}'}}} \left\langle i|a^\dagger_{\rho,\mathbf{k}}(t)a_{\rho,\mathbf{k}'}(t)|i\right\rangle. \tag{2.144}$$

We thus find that the number of particles expected at the space-time position x can be found using Eq.(2.144) once $\left\langle i|a^\dagger_{\rho,\mathbf{k}}(t)a_{\rho,\mathbf{k}'}(t)|i\right\rangle$ is known for all $\mathbf{k}$ and $\mathbf{k}'$. In the above example, it is instructive to recognize a more general problem. First of all, note that Eq.(2.144) is analogous to the conventional probability density $|\Psi(x)|^2$ - in case of free fields it yields the square of the Feynman propagation amplitude

$$|\Delta_F(z)|^2 = \left|\sum_{\mathbf{k}} \frac{e^{i(\mathbf{kz}-\omega_{\mathbf{k}}z^0)}}{2\omega_{\mathbf{k}}}\right|^2 = |\int \frac{dk}{(2\pi)^d} \frac{ie^{-ikz}}{k^2-m^2+i\epsilon}|^2. \tag{2.145}$$

Unlike the Feynman propagation amplitude, however, Eq.(2.144) generalizes naturally to the case when the total number of particles of a given sort is not conserved, as is, essentially, in any flavor oscillation theory.

For a system with N flavors of particles we define the initial flavor state $|i\rangle$ by

$$|i\rangle = \sum_{\rho=1;\mathbf{k}}^{N} \left(g_{\rho,\mathbf{k}}a^\dagger_{\rho,\mathbf{k}} + h_{\rho,\mathbf{k}}b^\dagger_{\rho,-\mathbf{k}}\right)|\Omega\rangle. \tag{2.146}$$

Eq.(2.146) represents a single flavor particle initially created in a state such that the probability to observe it as sort ρ-particle (antiparticle) is simply $|g_{\rho,\mathbf{k}}|^2$ ($|h_{\rho,-\mathbf{k}}|^2$). $2N$ functions $g_{\rho,\mathbf{k}}, h_{\rho,\mathbf{k}}$ shall be understood as the form-factors of the initial state $|i\rangle$[3].

[3]In Eq.(2.146) we assumed that particles and antiparticles are distinguishable. Although this is feasible in the case of, e.g., neutrinos, for many cases of meson mixing the field operators are self-adjoint and thus particles may not be distinguished from antiparticles. However, Eq.(2.146) can still be used by redefining $b^\dagger \equiv a^\dagger$ and $g_{\rho,\mathbf{k}} \equiv h_{\rho,-\mathbf{k}}$. Given this remark, we will continue with general formulation keeping in mind that the meson-mixing can be obtained with straightforward adjustments from our final results.

For notation convenience we shall adopt following convention. We will let α be both positive and negative ($\alpha = -N, \ldots, N$, excluding $\alpha = 0$) with negative α enumerating antiparticles and positive α enumerating particles, respectively. In this notation

$$|i\rangle = \sum_{\alpha=-N;\mathbf{k}}^{N} f_{\alpha,\mathbf{k}} a^{\dagger}_{\alpha,\mathbf{k}} |\Omega\rangle , \tag{2.147}$$

where, for convenience, we have introduced $a^{\dagger}_{\alpha,\mathbf{k}} := a^{\dagger}_{\alpha,\mathbf{k}}$ *for* $\alpha > 0$ and $a^{\dagger}_{\alpha,\mathbf{k}} := b^{\dagger}_{-\alpha,-\mathbf{k}}$ *for* $\alpha < 0$. Analogously, $f_{\alpha,\mathbf{k}} := g_{\alpha,\mathbf{k}}$ *for* $\alpha > 0$ and $f_{\alpha,\mathbf{k}} := h_{-\alpha,\mathbf{k}}$ *for* $\alpha < 0$. Also, from now on, in summations over α we imply $\alpha \neq 0$.

From Eq.(2.147), we may further introduce the creation operator for form-factor $F = \{f_{\alpha,\mathbf{k}}, \alpha = -N \cdots -1, 1 \ldots N\}$

$$a^{\dagger}(F) = \sum_{\alpha=-N;\mathbf{k}}^{N} f_{\alpha,\mathbf{k}} a^{\dagger}_{\alpha,\mathbf{k}}, \tag{2.148}$$

so that, concisely, $|i\rangle = a^{\dagger}(F)|\Omega\rangle$. It is straightforward to obtain non-equal time commutation/anticommutation relations for $a(F)$ and $a^{\dagger}(G)$

$$\begin{aligned}
[a(F), a^{\dagger}(G)]_{\pm,\text{equal time}} &= \sum_{\alpha=-N;\mathbf{k}}^{N} f^{*}_{\alpha,\mathbf{k}} g_{\alpha,\mathbf{k}} = (f,g)_{I}, \\
[a_t(F), a^{\dagger}(G)]_{\pm} &= \sum_{\alpha,\beta=-N;\mathbf{k}}^{N} f^{*}_{\alpha,\mathbf{k}} \mathcal{F}_{\alpha\beta;\mathbf{k}}(t) g_{\beta,\mathbf{k}} = (f,g)_{\mathcal{F}}, \\
[a_t(F), a(G)]_{\pm} &= \sum_{\alpha,\beta=-N;\mathbf{k}}^{N} f^{*}_{\alpha,\mathbf{k}} \mathcal{G}_{\alpha\beta;\mathbf{k}}(t) g^{*}_{\beta,\mathbf{k}} = (f,g^{*})_{\mathcal{G}},
\end{aligned} \tag{2.149}$$

where the inner product is defined by

$$(f,g)_{\mathcal{A}} = \sum_{\alpha,\beta=-N;\mathbf{k}}^{N} f^{*}_{\alpha,\mathbf{k}} \mathcal{A}_{\alpha\beta;\mathbf{k}}(t) g_{\beta,\mathbf{k}}.$$

$\mathcal{F}_{\alpha\beta;\mathbf{k}}(t)$, $\mathcal{G}_{\alpha\beta;\mathbf{k}}(t)$ are the non-equal time commutators/anticommutators for momentum $\mathbf{k}$ analogous to Eq.(2.119),

$$\begin{aligned}
[a_{\alpha,\mathbf{k}}(t), a^{\dagger}_{\beta,\mathbf{k}'}]_{\pm} &= \mathcal{F}_{\alpha\beta;\mathbf{k}}(t)\delta(\mathbf{k}-\mathbf{k}') \\
[a_{\alpha,\mathbf{k}}(t), a_{\beta,\mathbf{k}'}]_{\pm} &= \mathcal{G}_{\alpha\beta;\mathbf{k}}(t)\delta(\mathbf{k}-\mathbf{k}').
\end{aligned} \tag{2.150}$$

Explicitly,

$$\begin{aligned}\mathcal{G}_{\alpha\beta}(t) &= \begin{cases} G_{-\alpha,\beta}(t), \alpha<0,\beta>0; \\ \eta G_{-\beta,\alpha}(-t), \alpha>0,\beta<0; \\ 0, \text{ otherwise.}\end{cases} = \begin{pmatrix} 0 & \eta G^T(-t) \\ G(t) & 0 \end{pmatrix}, \\ \mathcal{F}_{\alpha\beta}(t) &= \begin{cases} F_{\alpha,\beta}(t), \alpha>0,\beta>0; \\ F_{-\beta,-\alpha}(t), \alpha<0,\beta<0; \\ 0, \text{ otherwise.}\end{cases} = \begin{pmatrix} F(t) & 0 \\ 0 & F^T(t) \end{pmatrix},\end{aligned} \tag{2.151}$$

Similarly to external wave-packet models, our primary interest in this discussion is the process in which a "flavor" particle is created in some initial state $|i\rangle$ with form-factor given by $G = \{g_{\alpha,\mathbf{k}}\}$ and is detected at time t as state $|f\rangle$ with form-factor $F = \{f_{\alpha,\mathbf{k}}\}$. In that case the object of interest is the expectation value of the number operator $N_F(t) = a_t^\dagger(F)a_t(F)$

$$\langle i|N_F(t)|i\rangle = \left\langle \Omega|a(G)a_t^\dagger(F)a_t(F)a^\dagger(G)|\Omega\right\rangle. \tag{2.152}$$

Using the machinery we just have introduced and after simple algebra, we find

$$\begin{aligned}\langle i|N_F(t)|i\rangle &= |[a_t(F),a^\dagger(G)]_\pm|^2 + \eta|[a(G),a_t(F)]_\pm|^2 + \left\langle a_t^\dagger(F)a_t(F)\right\rangle, \\ &= |(f,g)_\mathcal{F}|^2 + \eta|(f,g^*)_\mathcal{G}|^2 + (f,f)_\mathcal{Z},\end{aligned} \tag{2.153}$$

where the last term is related to the flavor condensation in the vacuum,

$$\begin{aligned}(f,f)_\mathcal{Z} = \left\langle a_t^\dagger(F)a_t(F)\right\rangle &= \sum_{\alpha=-N;\mathbf{k}}^{N} f_{\alpha,\mathbf{k}}\mathcal{Z}_{\alpha,\mathbf{k}}(t)f_{\alpha,\mathbf{k}}, \\ \mathcal{Z}_{\alpha,\mathbf{k}}(t) &= \left\langle \Omega|a_{\alpha,\mathbf{k}}^\dagger(t)a_{\alpha,\mathbf{k}}(t)|\Omega\right\rangle.\end{aligned} \tag{2.154}$$

Generally this term is not zero for $t \neq 0$, in fact, for many choices of F it is infinite. On the other hand, one may notice that this contribution is independent from the initial state $|i\rangle$. Furthermore, in case of a point like detector this contribution also is x-independent and thus may be interpreted as the background due to the vacuum condensation picked up by the detector itself. In this case we can remove it by defining

$$\begin{aligned}\langle i|N_F(t)|i\rangle_r &= \langle i|N_F(t)|i\rangle - \langle\Omega|N_F(t)|\Omega\rangle = \\ &= |\sum_{\alpha,\beta=-N;\mathbf{k}}^{N} f_{\alpha,\mathbf{k}}^*\mathcal{F}_{\alpha\beta;\mathbf{k}}(t)g_{\beta,\mathbf{k}}|^2 + \eta|\sum_{\alpha,\beta=-N;\mathbf{k}}^{N} f_{\alpha,\mathbf{k}}^*\mathcal{G}_{\alpha\beta;\mathbf{k}}(t)g_{\beta,\mathbf{k}}^*|^2.\end{aligned} \tag{2.155}$$

Oscillations of flavor charge can be obtained from Eq.(2.155). For the case of detection of a single particle with flavor β and form-factor $f(\mathbf{k})$

$$\langle Q_\beta(t)\rangle_r = \langle N_{[\beta]}(t)\rangle_r - \langle N_{[-\beta]}(t)\rangle_r, \tag{2.156}$$

where form-factors $[\beta] = \{f_{\alpha,\mathbf{k}} = f(\mathbf{k})\delta_{\alpha,\beta}\}$ and $[-\beta] = \{f_{\alpha,\mathbf{k}} = f(-\mathbf{k})\delta_{-\alpha,\beta}\}$.

It is straightforward to derive from Eqs.(2.155) and (2.156) the oscillation formulas in space, in which case the detector shall be characterized by the form-factor $[\beta] = \{f_{\alpha,\mathbf{k}} = \frac{e^{-i\mathbf{kx}}}{\sqrt{2\omega_{\beta\mathbf{k}}}}\delta_{\alpha,\beta}\}$ for a single particle of sort β or $[-\beta] = \{f_{\alpha,\mathbf{k}} = \frac{e^{i\mathbf{kx}}}{\sqrt{2\omega_{\beta\mathbf{k}}}}\delta_{-\alpha,\beta}\}$ for a single anti-particle of the same sort. Up to a constant background contribution we have

$$\begin{aligned}
\langle i|N_\beta(t,\mathbf{x})|i\rangle_r &= |\sum_{\alpha;\mathbf{k}} \tfrac{e^{i\mathbf{kx}}}{(2\omega_{\beta\mathbf{k}})^{1/2}}\mathcal{F}_{\beta\alpha;\mathbf{k}}(t)g_{\alpha,\mathbf{k}}|^2 + \eta|\sum_{\alpha;\mathbf{k}} \tfrac{e^{i\mathbf{kx}}}{(2\omega_{\beta\mathbf{k}})^{1/2}}\mathcal{G}_{\beta\alpha;\mathbf{k}}(t)g^*_{\alpha,\mathbf{k}}|^2,\\
\langle i|N_{-\beta}(t,\mathbf{x})|i\rangle_r &= |\sum_{\alpha;\mathbf{k}} \tfrac{e^{-i\mathbf{kx}}}{(2\omega_{\beta\mathbf{k}})^{1/2}}\mathcal{F}_{-\beta\alpha;\mathbf{k}}(t)g_{\alpha,\mathbf{k}}|^2 + \eta|\sum_{\alpha;\mathbf{k}} \tfrac{e^{-i\mathbf{kx}}}{(2\omega_{\beta\mathbf{k}})^{1/2}}\mathcal{G}_{-\beta\alpha;\mathbf{k}}(t)g^*_{\alpha,\mathbf{k}}|^2,\\
\langle i|Q_\beta(t,\mathbf{x})|i\rangle_r &= \langle i|N_\beta(t,\mathbf{x})|i\rangle_r - \langle i|N_{-\beta}(t,\mathbf{x})|i\rangle_r .
\end{aligned} \tag{2.157}$$

Our final result depends on the form of the initial state assumed for the flavor particle, e.g., one may consider the initial state as a state with definite momentum $\mathbf{k}$ and definite flavor β,

$$\langle\beta\mathbf{k}|N_\alpha(t,\mathbf{x})|\beta\mathbf{k}\rangle_r = (2\omega_{\alpha\mathbf{k}}\omega_{\beta\mathbf{k}})^{-1}|\mathcal{F}_{\alpha\beta;\mathbf{k}}(t)|^2 + \eta(2\omega_{\alpha\mathbf{k}}\omega_{\beta\mathbf{k}})^{-1}|\mathcal{G}_{\alpha\beta;\mathbf{k}}(t)|^2. \tag{2.158}$$

This, however, has no dependence on $\mathbf{x}$ and thus one can not see any space oscillations. One also might consider a particle of sort β created at position $\mathbf{x}'$ and observed at position $\mathbf{x}$ at time t as a particle of sort α. If $\beta > 0$, one finds

$$\begin{aligned}
\langle\beta\mathbf{x}'|N_\alpha(t,\mathbf{x})|\beta\mathbf{x}'\rangle_r &= |\sum_{\mathbf{k}} \tfrac{e^{i(\mathbf{x}-\mathbf{x}')\mathbf{k}}}{2\sqrt{\omega_{\alpha\mathbf{k}}\omega_{\beta\mathbf{k}}}}\mathcal{F}_{\alpha\beta;\mathbf{k}}(t)|^2 = |{}^*\mathcal{F}_{\alpha\beta}(\mathbf{x}'-\mathbf{x},t)|^2,\\
\langle\beta\mathbf{x}'|N_{-\alpha}(t,\mathbf{x})|\beta\mathbf{x}'\rangle_r &= \eta|\sum_{\mathbf{k}} \tfrac{e^{i(\mathbf{x}-\mathbf{x}')\mathbf{k}}}{2\sqrt{\omega_{\alpha\mathbf{k}}\omega_{\beta\mathbf{k}}}}\mathcal{G}_{\alpha\beta;\mathbf{k}}(t)|^2 = \eta|{}^*\mathcal{G}_{\alpha\beta}(\mathbf{x}'-\mathbf{x},t)|^2,
\end{aligned} \tag{2.159}$$

where ${}^*\mathcal{F}(\mathbf{z})$ is the Fourier transform of $(2\sqrt{\omega_{\alpha\mathbf{k}}\omega_{\beta\mathbf{k}}})^{-1}\mathcal{F}(\mathbf{k})$ and ${}^*\mathcal{G}$ is defined similarly.

In practice we are interested in the case when a flavor particle was produced originally in a small (but finite) region of space with (nonzero) momentum $\mathbf{k}$. This

can be represented by a well-peaked initial state $|i\rangle$ with form-factor $g(\mathbf{k})$ such that a single particle of sort β appears as a narrow wave-packet of momentum $\mathbf{k}_0$. Taking now the explicit form of $\mathcal{F}$ and $\mathcal{G}$ from Eq.(2.119) and leaving the detector point-like, we obtain

$$\begin{gathered}\langle \beta g|N_\alpha(t,\mathbf{x})|\beta g\rangle_r = \left|\sum_\gamma (a_{\alpha\beta;\gamma}e^{-iw_\gamma t}e^{-\sigma^2(\mathbf{v}_\gamma t-\mathbf{x})^2/2} + b_{\alpha\beta;\gamma}e^{iw_\gamma t}e^{-\sigma^2(-\mathbf{v}_\gamma t-\mathbf{x})^2/2})\right|^2 + \\ \eta\left|\sum_\gamma (c_{\alpha\beta;\gamma}e^{-iw_\gamma t}e^{-\sigma^2(\mathbf{v}_\gamma t-\mathbf{x})^2/2} + d_{\alpha\beta;\gamma}e^{iw_\gamma t}e^{-\sigma^2(-\mathbf{v}_\gamma t-\mathbf{x})^2/2})\right|^2; \\ \mathbf{v}_\gamma = \tfrac{dw_\gamma(\mathbf{k})}{d\mathbf{k}}|_{\mathbf{k}_0}.\end{gathered} \tag{2.160}$$

In the above derivation we used the following identity, which can be proved using the stationary phase approximation for the function $g(\mathbf{k})$ sharply peaked around $\mathbf{k}=\mathbf{k}_0$ and the slow-varying functions $f(\mathbf{k})$ and $S(\mathbf{k})$,

$$\begin{gathered}\int d\mathbf{k} g(\mathbf{k})f(\mathbf{k})e^{iS(\mathbf{k})}e^{i\mathbf{k}\mathbf{x}} \approx (2\pi\sigma^2)^{3/2}g(\mathbf{k}_0)f(\mathbf{k}_0)e^{i(S(\mathbf{k}_0)+\mathbf{k}_0\mathbf{x})}\exp(-\sigma^2(\mathbf{x}+\tilde{\nabla}S(\mathbf{k}_0))^2/2), \\ \sigma^2 = -\tfrac{g(\mathbf{k}_0)}{g''(\mathbf{k}_0)}.\end{gathered} \tag{2.161}$$

The explicit form of $\mathcal{F}$ and $\mathcal{G}$ is parametrized as

$$\begin{aligned}(\sqrt{2\omega_{\alpha\mathbf{k}}})^{-1}\mathcal{F}_{\alpha\beta,\mathbf{k}}(t) &= \sum_{\gamma=1}^{N}(a_{\alpha\beta;\gamma}(\mathbf{k})e^{-iw_{\gamma\mathbf{k}}t} + b_{\alpha\beta;\gamma}(\mathbf{k})e^{iw_{\gamma\mathbf{k}}t}), \\ (\sqrt{2\omega_{\alpha\mathbf{k}}})^{-1}\mathcal{G}_{\alpha\beta,\mathbf{k}}(t) &= \sum_{\gamma=1}^{N}(c_{\alpha\beta;\gamma}(\mathbf{k})e^{-iw_{\gamma\mathbf{k}}t} + d_{\alpha\beta;\gamma}(\mathbf{k})e^{iw_{\gamma\mathbf{k}}t}),\end{aligned} \tag{2.162}$$

and all amplitudes in Eq.(2.160) are taken at momentum $\mathbf{k}=\mathbf{k}_0$. Eq.(2.160) physically represents the expectation value for the number of α-sort particles observed at position $\mathbf{x}$ at time t when a single particle of sort β and form-factor $g(\mathbf{k})$ had been emitted. What we observe, hence, is a single wave-packet propagating through the space: one can see that $\langle\beta g|N_t(\alpha\mathbf{x})|\beta g\rangle$ reaches maximum only when the "center" of the wave-packet passes over the observation point $\mathbf{x}$, i.e.

$$\langle\mathbf{v}\rangle t \approx \mathbf{x} \text{ or } \langle -\mathbf{v}\rangle t \approx \mathbf{x}. \tag{2.163}$$

To explicitly observe space oscillations one should take an average over time in

Eq.(2.157) which would correspond to an observation continuous in time [58, 59, 94],

$$\begin{aligned} W_g(\alpha\mathbf{x}) &\sim \lim_{T\to\infty}\int_T dt\,\langle g|N_\alpha(t,\mathbf{x})|g\rangle_r \\ &\sim \lim_{T\to\infty}\int_T dt\left(|\sum_{\beta;\mathbf{k}}\tfrac{e^{i\mathbf{kx}}}{\sqrt{2\omega_{\alpha\mathbf{k}}}}\mathcal{F}_{\alpha\beta;\mathbf{k}}(t)g_{\beta,\mathbf{k}}|^2+\eta|\sum_{\beta;\mathbf{k}}\tfrac{e^{i\mathbf{kx}}}{\sqrt{2\omega_{\alpha\mathbf{k}}}}\mathcal{G}_{\alpha\beta;\mathbf{k}}(t)g^*_{\beta,\mathbf{k}}|^2\right). \end{aligned} \tag{2.164}$$

Using Eq.(2.162), we may rewrite this as

$$\begin{aligned} \sum_{\gamma,\gamma'}\int\int d\mathbf{k}d\mathbf{k}'\delta(w_{\gamma\mathbf{k}}-w_{\gamma'\mathbf{k}'})[(a_{\alpha\beta;\gamma}(\mathbf{k})a^*_{\alpha\beta;\gamma'}(\mathbf{k}')+b_{\alpha\beta;\gamma}(\mathbf{k})b^*_{\alpha\beta;\gamma'}(\mathbf{k}'))+ \\ \eta(c_{\alpha\beta;\gamma}(\mathbf{k})c^*_{\alpha\beta;\gamma'}(\mathbf{k}')+d_{\alpha\beta;\gamma}(\mathbf{k})d^*_{\alpha\beta;\gamma'}(\mathbf{k}'))]g_{\beta,\mathbf{k}}g^*_{\beta,\mathbf{k}'}e^{i(\mathbf{k}-\mathbf{k}')\mathbf{x}}. \end{aligned}$$

When the mass difference $m^2_\gamma - m^2_{\gamma'}$ is small, the functional

$$\begin{aligned} &\delta(w_{\gamma\mathbf{k}}-w_{\gamma'\mathbf{k}'})[(a_{\alpha\beta;\gamma}(\mathbf{k})a^*_{\alpha\beta;\gamma'}(\mathbf{k}')+b_{\alpha\beta;\gamma}(\mathbf{k})b^*_{\alpha\beta;\gamma'}(\mathbf{k}'))\pm \\ &(c_{\alpha\beta;\gamma}(\mathbf{k})c^*_{\alpha\beta;\gamma'}(\mathbf{k}')+d_{\alpha\beta;\gamma}(\mathbf{k})d^*_{\alpha\beta;\gamma'}(\mathbf{k}'))]g_{\beta,\mathbf{k}}g^*_{\beta,\mathbf{k}'}e^{i(\mathbf{k}-\mathbf{k}')\mathbf{x}} \end{aligned} \tag{2.165}$$

has its maximum at $\mathbf{k}\approx\mathbf{k}'\approx\mathbf{k}_0$. Then for the initial single flavor particle β with sharply-peaked $g_{\beta,\mathbf{k}}$ we can use the stationary phase approximation again to find

$$\begin{aligned} W(\mathbf{x}) \sim \sum_{\gamma,\gamma'} &\big[(a_{\alpha\beta;\gamma}(\mathbf{k})a^*_{\alpha\beta;\gamma'}(\mathbf{k})+b_{\alpha\beta;\gamma}(\mathbf{k})b^*_{\alpha\beta;\gamma'}(\mathbf{k})) \\ &+\eta(c_{\alpha\beta;\gamma}(\mathbf{k})c^*_{\alpha\beta;\gamma'}(\mathbf{k})+d_{\alpha\beta;\gamma}(\mathbf{k})d^*_{\alpha\beta;\gamma'}(\mathbf{k}))\big]\,e^{-(\frac{\Delta k_{\gamma\gamma'}}{2k_0})^2\sigma^2\mathbf{x}^2+i\Delta\mathbf{k}_{\gamma\gamma'}\mathbf{x}}, \end{aligned} \tag{2.166}$$

where $\mathbf{k}\approx\mathbf{k}_0$, $\Delta\mathbf{k}_{\gamma\gamma'}=\frac{\Delta m^2_{\gamma\gamma'}}{2|\mathbf{k_0}|}\hat{\mathbf{k}}_0=\frac{m^2_\gamma-m^2_{\gamma'}}{2|\mathbf{k}_0|}\hat{\mathbf{k}}_0$. For the mixing of two flavors we recover the oscillation length as

$$L\approx\frac{2k}{\Delta m^2_{12}}. \tag{2.167}$$

We should point out here that the field-theoretical corrections are generally found only to change the amplitude of the oscillations while no major distortion in the structure of the oscillations is found. This is quite different from the case of the flavor oscillations in time where the additional high-frequency term was prominent. Indeed, we found that only one mode of space oscillations, with the wave-length related to $\Delta m^2_{\gamma\gamma'}$ by Eq.(2.167), survives average over time. While this result may be in part

attributed to the approximation we used, one may also notice that in Eq.(2.164)

$$\int dt e^{i(\omega_{\gamma\mathbf{k}} \pm \omega_{\gamma'\mathbf{k}'})t} \neq 0$$

only when $\omega_{\gamma\mathbf{k}} \pm \omega_{\gamma'\mathbf{k}'} = 0$ and for that the frequencies must come in with the opposite signs. Thus, no high-frequency terms may survive in the integration over time. However, the field-theoretical effect is still noticeable in the shape of the oscillations as one may see in Section 2.6.3. Finally, we note that the decoherence factor $e^{-\sigma^4 \mathbf{x}^2/2}$ in Eq.(2.166) leads eventually to the separation of the mass-eigenstates so that they propagate independently.

2.6 Applications of General Quantum Field Theory of Flavor Mixing

2.6.1 Vector meson mixing (S=1) for two flavors

We now consider the unitary mixing of 2 fields with spin 1 (vector mesons). U(2) parametrization consists of 4 parameters: 3 phases that can be absorbed in the phase redefinition of fields and there is one essential real angle that is left, so that

$$U = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{2.168}$$

Using Appendix A, we define $\gamma^{\pm}_{\mu i} = \frac{1}{2}\left(\sqrt{\frac{\epsilon_\mu}{\epsilon_i}} \pm \sqrt{\frac{\epsilon_i}{\epsilon_\mu}}\right)$ for $\sigma = \pm 1$ and

$$\begin{aligned} \gamma^{+}_{\mu i} &= \tfrac{1}{2}\tfrac{\epsilon_\mu \epsilon_i - k^2}{m_\mu m_i}\left(\sqrt{\tfrac{\epsilon_\mu}{\epsilon_i}} + \sqrt{\tfrac{\epsilon_i}{\epsilon_\mu}}\right), \\ \gamma^{-}_{\mu i} &= \tfrac{1}{2}\tfrac{k^2 + \epsilon_\mu \epsilon_i}{m_\mu m_i}\left(\sqrt{\tfrac{\epsilon_\mu}{\epsilon_i}} - \sqrt{\tfrac{\epsilon_i}{\epsilon_\mu}}\right) \end{aligned} \tag{2.169}$$

for $\sigma = 0$. For the free-field mass parameters m_μ $\gamma^{+}_{12} = \gamma^{+}_{21} = \gamma_+$, $\gamma^{-}_{12} = -\gamma^{-}_{21} = \gamma_-$ (note $\frac{1}{2}$ convention difference from Section 2.4).

The mixing matrices α and β are then

$$\begin{aligned}\alpha &= \begin{pmatrix} \cos(\theta) & \gamma_+ \sin(\theta) \\ -\gamma_+ \sin(\theta) & \cos(\theta) \end{pmatrix}, \\ \beta &= \begin{pmatrix} 0 & \gamma_- \sin(\theta) \\ -\gamma_- \sin(\theta) & 0 \end{pmatrix}.\end{aligned} \tag{2.170}$$

For the flavor charge oscillation we then obtain the result that is independent from the mass parametrization,

$$\begin{aligned}Q_{111} &= 1 + \sin^2(2\theta)\left(\gamma_-^2 \sin^2\left(\tfrac{\Omega_{12}t}{2}\right) - \gamma_+^2 \sin^2\left(\tfrac{\omega_{12}t}{2}\right)\right), \\ Q_{121} &= \sin^2(2\theta)\left(\gamma_+^2 \sin^2\left(\tfrac{\omega_{12}t}{2}\right) - \gamma_-^2 \sin^2\left(\tfrac{\Omega_{12}t}{2}\right)\right).\end{aligned} \tag{2.171}$$

We see that this result, with an exception of greater complexity of $\gamma_\pm$, is identical to the case of spin 0 [44, 45]. According to the general theory, in fact, this should be expected for the two-flavor mixing with any integer spin. For $S = 1$ we see that an essential difference from the scalar/pseudoscalar meson mixing, such as the complication of momentum dependence of $\gamma_\pm$, occurs only for the mixing of longitudinally polarized particles. The mixing of transverse components is essentially the same as in the case of spin-zero particles.

The details of non-equal time commutators are given by

$$F = \begin{pmatrix} e^{-i\epsilon_1 t}\cos^2\theta + \sin^2\theta\left(e^{-i\epsilon_2 t}\gamma_+^2 - e^{i\epsilon_2 t}\gamma_-^2\right), & \gamma_+ \sin\theta\cos\theta\left(e^{-i\epsilon_2 t} - e^{-i\epsilon_1 t}\right) \\ \gamma_+ \sin\theta\cos\theta\left(e^{-i\epsilon_2 t} - e^{-i\epsilon_1 t}\right), & e^{-i\epsilon_2 t}\cos^2\theta + \sin^2\theta\left(e^{-i\epsilon_1 t}\gamma_+^2 - e^{i\epsilon_1 t}\gamma_-^2\right) \end{pmatrix} \tag{2.172}$$

$$G = \begin{pmatrix} \gamma_+\gamma_- \sin^2\theta\left(e^{-i\epsilon_2 t} - e^{i\epsilon_2 t}\right), & \gamma_- \sin\theta\cos\theta\left(e^{-i\epsilon_1 t} - e^{i\epsilon_2 t}\right) \\ \gamma_- \sin\theta\cos\theta\left(e^{-i\epsilon_2 t} - e^{i\epsilon_1 t}\right), & \gamma_+\gamma_- \sin^2\theta\left(e^{i\epsilon_1 t} - e^{-i\epsilon_1 t}\right) \end{pmatrix}. \tag{2.173}$$

The condensates of free-field particles are

$$Z_1' = Z_2' = \gamma_-^2 \sin^2(\theta) \tag{2.174}$$

and the condensates of the flavor particles in the vacuum are

$$\begin{aligned}Z_1 &= 4\gamma_-^2 \sin^2(\theta)\left(\cos^2(\theta)\sin^2\left(\tfrac{\Omega_{12}t}{2}\right) + \gamma_+^2 \sin^2(\theta)\sin^2\left(\tfrac{\Omega_{22}t}{2}\right)\right), \\ Z_2 &= 4\gamma_-^2 \sin^2(\theta)\left(\cos^2(\theta)\sin^2\left(\tfrac{\Omega_{12}t}{2}\right) + \gamma_+^2 \sin^2(\theta)\sin^2\left(\tfrac{\Omega_{11}t}{2}\right)\right).\end{aligned} \tag{2.175}$$

The flavor vacuum structure is defined by the matrix $\hat{Z}$ (note different index convention from Section 2.4):

$$\hat{Z} = \frac{-1}{\left(\cos^2(\theta) + \gamma_+^2 \sin^2(\theta)\right)} \begin{pmatrix} -\gamma_+\gamma_- \sin^2(\theta) & \gamma_- \cos(\theta)\sin(\theta) \\ \gamma_- \cos(\theta)\sin(\theta) & \gamma_+\gamma_- \sin^2(\theta) \end{pmatrix} \tag{2.176}$$

with the normalization constant $\mathcal{Z} = (1 - \frac{\gamma_-^2 \sin^2(\theta)}{\cos^2(\theta)+\gamma_+^2 \sin^2(\theta)})^{-1} = 1 + \gamma_-^2 \sin^2(\theta)$. One can see that $\frac{1}{\mathcal{Z}}$ as function of $\gamma_-^2 \sin^2(\theta)$ has a pole at imaginary $\gamma_- \sin(\theta) = i$, so that the perturbative expansion in $\gamma_-^2 \sin^2(\theta)$ has a finite convergence radius.

The time evolution of the flavor particle number (if #1 was emitted) is given by:

$$\begin{aligned} N_{111} &= 1 + \sin^2(\theta)\{8\gamma_-^2 \cos^2(\theta)\sin^2\left(\tfrac{\Omega_{12}t}{2}\right) - 4\gamma_+^2 \cos^2(\theta)\sin^2\left(\tfrac{\omega_{12}t}{2}\right) + \\ &\quad + 8\gamma_+^2\gamma_-^2 \sin^2(\theta)\sin^2\left(\tfrac{\Omega_{22}t}{2}\right)\}, \\ \bar{N}_{111} &= 4\gamma_-^2 \sin^2(\theta)\left(2\gamma_+^2 \sin^2(\theta)\sin^2\left(\tfrac{\Omega_{22}t}{2}\right) + \cos^2(\theta)\sin^2\left(\tfrac{\Omega_{12}t}{2}\right)\right), \end{aligned} \tag{2.177}$$

$$\begin{aligned} N_{121} &= \sin^2(\theta)\{4\gamma_+^2 \cos^2(\theta)\sin^2\left(\tfrac{\omega_{12}t}{2}\right) + 4\gamma_-^2 \cos^2(\theta)\sin^2\left(\tfrac{\Omega_{12}t}{2}\right) + \\ &\quad + 4\gamma_+^2\gamma_-^2 \sin^2(\theta)\sin^2\left(\tfrac{\Omega_{11}t}{2}\right), \\ \bar{N}_{121} &= 4\gamma_-^2 \sin^2(\theta)\left(2\cos^2(\theta)\cos^2\left(\tfrac{\Omega_{12}t}{2}\right) + \gamma_+^2 \sin^2(\theta)\sin^2\left(\tfrac{\Omega_{11}t}{2}\right)\right). \end{aligned} \tag{2.178}$$

Also we note that the scalar and pseudoscalar case follows immediately from the above presentation when $\gamma_{\mu i}^{\pm} = \frac{1}{2}\left(\sqrt{\frac{\epsilon_\mu}{\epsilon_i}} \pm \sqrt{\frac{\epsilon_i}{\epsilon_\mu}}\right)$. In this respect, the spin-zero mixing is equivalent to the mixing of transverse components of vector fields, described by Eqs.(2.171), (2.172), (2.176) and (2.177). These results are in accord with [44,45].

2.6.2 Fermion mixing (S=1/2) for two flavors

We also present here the calculations for $S = 1/2$ case. For the consistent notation with the previous works [39,83][4], we define

$$\begin{aligned} U &= \frac{\sqrt{(\epsilon_1+m_1)(\epsilon_2+m_2)}+\sqrt{(\epsilon_1-m_1)(\epsilon_2-m_2)}}{2\sqrt{\epsilon_1\epsilon_2}}, \\ V &= \sigma\frac{\sqrt{(\epsilon_1-m_1)(\epsilon_2+m_2)}-\sqrt{(\epsilon_1+m_1)(\epsilon_2-m_2)}}{2\sqrt{\epsilon_1\epsilon_2}}. \end{aligned} \tag{2.179}$$

[4] In our notation $U = \gamma_+$, $V = \gamma_-$

The charge fluctuations are given by

$$\begin{aligned} Q_{111} &= 1-\sin^2(2\theta)\left(U^2\sin^2\left(\tfrac{\omega_{12}t}{2}\right)+V^2\sin^2\left(\tfrac{\Omega_{12}t}{2}\right)\right), \\ Q_{121} &= \sin^2(2\theta)\left(U^2\sin^2\left(\tfrac{\omega_{12}t}{2}\right)+V^2\sin^2\left(\tfrac{\Omega_{12}t}{2}\right)\right) \end{aligned} \tag{2.180}$$

and the mixing matrices are

$$\begin{aligned} \alpha &= \begin{pmatrix} \cos(\theta) & U\sin(\theta) \\ -U\sin(\theta) & \cos(\theta) \end{pmatrix}, \\ \beta &= \begin{pmatrix} 0 & V\sin(\theta) \\ V\sin(\theta) & 0 \end{pmatrix}, \end{aligned} \tag{2.181}$$

which are the same with [39, 83].

We can give more details on the fermion mixing dynamics. The non-equal time anticommutators are given by

$$F = \begin{pmatrix} e^{-i\epsilon_1 t}\cos^2\theta+\sin^2\theta\left(e^{-i\epsilon_2 t}U^2+e^{i\epsilon_2 t}V^2\right), & U\sin\theta\cos\theta\left(e^{-i\epsilon_2 t}-e^{-i\epsilon_1 t}\right) \\ U\sin\theta\cos\theta\left(e^{-i\epsilon_2 t}-e^{-i\epsilon_1 t}\right), & e^{-i\epsilon_2 t}\cos^2\theta+\sin^2\theta\left(e^{-i\epsilon_1 t}U^2+e^{i\epsilon_1 t}V^2\right) \end{pmatrix} \tag{2.182}$$

$$G = \begin{pmatrix} UV\sin^2\theta\left(e^{-i\epsilon_2 t}-e^{i\epsilon_2 t}\right), & V\sin\theta\cos\theta\left(e^{-i\epsilon_1 t}-e^{i\epsilon_2 t}\right) \\ V\sin\theta\cos\theta\left(e^{-i\epsilon_2 t}-e^{i\epsilon_1 t}\right), & UV\sin^2\theta\left(e^{i\epsilon_1 t}-e^{-i\epsilon_1 t}\right) \end{pmatrix}. \tag{2.183}$$

The condensates of the free-field particles are

$$Z_1' = Z_2' = V^2\sin^2(\theta) \tag{2.184}$$

and the condensates of the flavor particles are

$$\begin{aligned} Z_1 &= 4V^2\sin^2(\theta)\left(\cos^2(\theta)\sin^2\left(\tfrac{\Omega_{12}t}{2}\right)+U^2\sin^2(\theta)\sin^2\left(\tfrac{\Omega_{22}t}{2}\right)\right), \\ Z_2 &= 4V^2\sin^2(\theta)\left(\cos^2(\theta)\sin^2\left(\tfrac{\Omega_{12}t}{2}\right)+U^2\sin^2(\theta)\sin^2\left(\tfrac{\Omega_{11}t}{2}\right)\right). \end{aligned} \tag{2.185}$$

The vacuum structure is defined by the matrix $\hat{Z}$:

$$\hat{Z} = \frac{-1}{\cos^2(\theta)+U^2\sin^2(\theta)}\begin{pmatrix} -UV\sin^2(\theta) & V\cos(\theta)\sin(\theta) \\ V\cos(\theta)\sin(\theta) & UV\sin^2(\theta) \end{pmatrix}$$

with the normalization constant being $\mathcal{Z} = \frac{1}{\cos^2(\theta)+U^2\sin^2(\theta)} = \frac{1}{1-V^2\sin^2(\theta)}$.

The time evolution of the flavor particle number (if #1 was emitted) is given by:

$$\begin{aligned} N_{111} &= 1 - 4U^2 \sin^2(\theta) \cos^2(\theta) \sin^2\left(\tfrac{\omega_{12} t}{2}\right), \\ \bar{N}_{111} &= 4V^2 \sin^2(\theta) \cos^2(\theta) \sin^2\left(\tfrac{\Omega_{12} t}{2}\right), \end{aligned} \tag{2.186}$$

$$\begin{aligned} N_{121} = 4\sin^2(\theta) \{ & U^2 \cos^2(\theta) \sin^2\left(\tfrac{\omega_{12} t}{2}\right) + V^2 \cos^2(\theta) \sin^2\left(\tfrac{\Omega_{12} t}{2}\right) + \\ & + U^2 V^2 \sin^2(\theta) \sin^2\left(\tfrac{\Omega_{11} t}{2}\right)\}, \\ \bar{N}_{121} = 4U^2 V^2 & \sin^4(\theta) \sin^2\left(\tfrac{\Omega_{11} t}{2}\right). \end{aligned} \tag{2.187}$$

2.6.3 Space-oscillations in two flavors mixing

In this section we shall investigate in greater details the effect of the field-theoretical corrections on the shape of the oscillations of flavor charge in space. For that, we will apply the formalism developed in Section 2.5 to the mixing of two scalar particles created originally in a state with given flavor and a gaussian form-factor. We will consider the time evolution of a single particle born in flavor $\alpha = 1$ with a sharp gaussian form-factor centered at the average momentum $\mathbf{k}_0$. Making use of Eq.(2.160), we notice that for given α and β either $\mathcal{F}_{\alpha\beta}$ (e.g. $\beta > 0 \to \alpha > 0$) or $\mathcal{G}_{\alpha\beta}$ (e.g. $\beta > 0 \to \alpha < 0$) is not equal to zero but never both of them are zero together. Then, using Eq.(2.162) and Eq.(2.172) and Eq.(2.173), we shall set (up to not essential normalization factor $1/\sqrt{2\omega_{\alpha\mathbf{k}_0}}$)

$$\begin{aligned} & a_{11;1} = \cos^2(\theta); a_{11;2} = \gamma_+^2 \sin^2(\theta); b_{11;1} = 0; b_{11;2} = -\gamma_-^2 \sin^2(\theta), \\ & a_{12;1} = a_{12;2} = -\gamma_+ \sin(\theta)\cos(\theta); b_{12;1} = b_{12;2} = 0, \\ & c_{11;1} = 0; c_{11;2} = \gamma_+\gamma_- \sin^2(\theta); d_{11;1} = 0; d_{11;2} = -\gamma_+\gamma_- \sin^2(\theta), \\ & c_{12;1} = \gamma_- \sin(\theta)\cos(\theta); c_{12;2} = 0; d_{12;1} = 0; d_{12;2} = -\gamma_- \sin(\theta)\cos(\theta). \end{aligned} \tag{2.188}$$

Here we imply that $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are nonzero only for the particle-particle sector while $c_{\alpha\beta}$ and $d_{\alpha\beta}$ are nonzero only for the particle-antiparticle sector (i.e. c_{11} actually is $c_{1\to -1}$).

In our study we've chosen specific values of m_1 and m_2 close to the parameters of η-η' system, i.e. $m_1 = 540MeV$, $m_2 = 930MeV$ and $\theta = \pi/4$. We simulated

the time evolution of the initial gaussian wave-packet according to Eq.(2.160) for a range of incident particle energies. As can be seen in Fig.(2.9), a typical wave-packet propagates in the right direction oscillating at the same time into the other flavor. After a certain time, $\alpha = 1$ particle is almost completely converted into $\alpha = 2$ flavor after which the reverse process takes place. With time evolution, the original gaussian wave-packet deforms so that two separate gaussians eventually emerge. This corresponds to two mass-eigenstates completely separated in space: no flavor oscillations occur after that point in agreement with the concept of coherence length. In particular, if σ is large (or almost point source) and $k_0 \neq 0$, the two mass-eigenstates separate almost immediately and propagate independently producing no flavor oscillations at all [see Fig.(2.10)]. The additional effect due to the nontrivial flavor vacuum is rather small and is most noticeable only at the moments when one of the flavors has almost completely disappeared due to the flavor conversion, as shown in Fig.(2.9). More remarkable is the presence of the traces of negative charge propagating in the opposite direction to that of the main wave-packet. This coherent beam of "recoil" anti-particles is due to the terms of the form $\exp[(-\mathbf{v}t - \mathbf{x})^2]$ in Eq.(2.160); it is correlated with the positive wing at all times via the mechanism similar to the EPR-effect, [see Fig.(2.10)]. The contribution from the high-frequency term, prominent in time-evolution, translates in space as an interference between these parts of the wave-packet, propagating to the right and to the left respectively, and dies out almost immediately.

So far, we have studied the propagation of a single flavor particle through the space. The space oscillations of flavor in conventional sense can be seen through the change in the amplitude of the wave-packet as the particle flies through the space. To observe space oscillations explicitly, we numerically traced the position of the maximum of the wave-packet (2.160) with time. We found that the maximum propagates at approximately constant speed consistent with

$$v = \frac{2k_0}{\sqrt{k_0^2 + m_1^2} + \sqrt{k_0^2 + m_2^2}}. \tag{2.189}$$

With that in mind, we reproduced the plot of the maximum amplitude of the wave-packet vs. distance. In Fig.(2.11), indeed, we observe the space oscillations with the wave-length $L \approx \frac{2k_0}{\Delta m^2}$. We also observed that, when the momentum of the particle is sufficiently low, the form of the space-oscillations is noticeably distorted from the quantum-mechanical prescription; for example, at certain points the flavor charge becomes negative. However, the field-theoretical corrections decrease as the energy increases and also die out with the distance.

2.7 Phenomenological Aspects of the Field-Theoretical Oscillation Formula

In this section we are going to examine consequences of the BV theory of flavor oscillations for phenomenologically important systems. While general theory of BV oscillations had been introduced and in details considered above, for systems like $K_0 - \bar{K}_0$ some more care needs to be taken in applying this formalism. Specifically, in $K^0 - \bar{K}^0$ mixing K^0 is not truly neutral since $K^0 \neq \bar{K}^0$. This is neither the case of mixing of two different charged particles, rather here particle is mixed with its antiparticle. To establish connection with our general theory, it is important to identify properly the mixed degrees of freedom. Note that in $K^0 - \bar{K}^0$ mixing there are three distinct modes, namely the strong-interaction eigenstates $K^0 - \bar{K}^0$, the mass eigenstates $K_L - K_S$ and the CP eigenstates $K_1 - K_2$. Each pair can be written as a linear combination of the other ones, e.g.

$$\begin{aligned} K_1 = \tfrac{1}{\sqrt{2}}(K^0 + \bar{K}^0), \quad K_2 = \tfrac{1}{\sqrt{2}}(K^0 - \bar{K}^0); \\ K^0 = \tfrac{e^{i\delta}}{\sqrt{2}}(K_L + K_S), \quad \bar{K}^0 = \tfrac{e^{-i\delta}}{\sqrt{2}}(K_L - K_S); \\ K_1 = \tfrac{1}{\sqrt{1+|\epsilon|^2}}(K_S + \epsilon K_L), \quad K_2 = \tfrac{1}{\sqrt{1+|\epsilon|^2}}(K_L + \epsilon K_S); \end{aligned} \tag{2.190}$$

with $e^{i\delta}$ being a complex phase and $\epsilon = i\delta$ being the imaginary CP-violation parameter. In a sense, $K^0 - \bar{K}^0$ are produced as strong-interaction eigenstates, propagate as mass eigenstates K_L, K_S and decay as CP-eigenstates K_1, K_2.

The mass eigenstates K_L and K_S are defined as the +1 and -1 CPT eigenstates, respectively, so that they can be represented in terms of self-adjoint scalar fields ϕ_1, ϕ_2 as

$$K_L = \phi_1, \quad K_S = i\phi_2. \tag{2.191}$$

Therefore the mixing in this system is similar to the case of neutral fields with complex mixing matrix. Since the complex mixing matrix in SU(2) can be always transformed into the real one by suitable redefinition of the field phases, which would not affect the expectation values, the mixing in this case is still equivalent to the mixing of two neutral fields.

The oscillating observables may be that of the strange charge (in the system K^0 and $\bar{K}^0$ taken as flavor A and B, respectively) with the trivial mixing angle $\theta = \pi/4$, as follows from Eq.(2.190). Phenomenologically relevant is the oscillation of CP-eigenvalue which determines the ratio of experimentally measured $\pi\pi$ to $\pi\pi\pi$ decay rates. CP-oscillations are given in terms of K_1 and K_2 flavors with small mixing angle $\cos(\theta) = 1/\sqrt{1+|\epsilon|^2}$. With these reservations in Figs.(2.7) we compare the Quantum Mechanical result and the BV result for the oscillations of strange charge.

Important to phenomenology, we shall also include the effect of the particle decay into our oscillation formulas. We follow rather common approach often used in quantum mechanics to introduce decays: we account for particle decay by appropriately inserting the factor $e^{-\gamma t}$ in our final formulas. This can be formally accomplished by making the following substitution in the annihilation operators (which also can be justified via introduction of imaginary decay width into position of the pole $m^2 + i\Gamma$) $a_{\mathbf{k},i} \to a_{\mathbf{k},i} e^{-\frac{\gamma_i}{2}t}$. Then, the oscillation formulas for decaying particles in $SU(2)$ mixing

can be written as

$$\begin{aligned}
Q_{\mathbf{k},A}(t) &= \left|\left[a_{\mathbf{k},A}(t), a^{\dagger}_{\mathbf{k},A}(0)\right]\right|^2 - \left|\left[a^{\dagger}_{-\mathbf{k},A}(t), a^{\dagger}_{\mathbf{k},A}(0)\right]\right|^2 \\
&= \left(\cos^2\theta e^{-\frac{\gamma_1}{2}t} + \sin^2\theta e^{-\frac{\gamma_2}{2}t}\right)^2 \\
&- \sin^2(2\theta) e^{-\frac{\gamma_1+\gamma_2}{2}t}\left[|U_{\mathbf{k}}|^2 \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) - |V_{\mathbf{k}}|^2 \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right] \\
Q_{\mathbf{k},B}(t) &= \left|\left[a_{\mathbf{k},B}(t), a^{\dagger}_{\mathbf{k},A}(0)\right]\right|^2 - \left|\left[a^{\dagger}_{-\mathbf{k},B}(t), a^{\dagger}_{\mathbf{k},A}(0)\right]\right|^2 \\
&= \sin^2(2\theta)\left(\left[\frac{e^{-\frac{\gamma_1}{2}t} - e^{-\frac{\gamma_2}{2}t}}{2}\right]^2\right. \\
&+ \left. e^{-\frac{\gamma_1+\gamma_2}{2}t}\left[|U_{\mathbf{k}}|^2 \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}t\right) - |V_{\mathbf{k}}|^2 \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}t\right)\right]\right).
\end{aligned} \tag{2.192}$$

We note the difference between these oscillation formulas and the quantum mechanical Gell-Mann–Pais formulas. The field-theoretic corrections appear as the additional high-frequency oscillation terms.

As we have opportunity to observe in Eq.(2.192), the field-theoretical effect is proportional to $|V_{\mathbf{k}}|^2$. In estimating the maximal magnitude of this term it is useful to write $|V_{\mathbf{k}}|^2$ in terms of the dimensionless momentum $p \equiv \sqrt{\frac{2|\mathbf{k}|^2}{m_1^2+m_2^2}}$ and the dimensionless parameter $a \equiv \frac{m_2^2-m_1^2}{m_1^2+m_2^2}$

$$|V(p,a)|^2 = \frac{p^2+1}{2\sqrt{(p^2+1)^2-a^2}} - \frac{1}{2}. \tag{2.193}$$

$|V_{\mathbf{k}}|^2$ is maximal at $p=0$ ($|V_{max}|^2 = \frac{(m_1-m_2)^2}{4m_1m_2}$) and goes to zero for large momenta (i.e. for $|\mathbf{k}|^2 \gg \frac{m_1^2+m_2^2}{2}$). The optimal observation scale for field-theoretical effect in meson mixing, therefore, is $k=0$ and the maximal correction is of the order of $|V|^2 \sim \frac{\Delta m^2}{m^2}$. It is straightforward to find that field-theoretical effect in K^0-$\bar{K}^0$, D^0-$\bar{D}^0$, B^0-$\bar{B}^0$ and B_s^0-$\bar{B}_s^0$ is very small and generally does not exceed 10^{-26}. For η-η' and ω-ϕ, however, field-theoretical corrections may be as large as $5-20\%$ and thus one needs to be careful about taking them into account should these systems be used in mixing experiments of some sort.

Similarly, in the fermion sector we can employ similar method. Neutrinos are known to be stable so that no additional adjustments are necessary to our previous

results. We can write the field-theoretical correction amplitude $|V_{\mathbf{k}}|^2$ as a function of the dimensionless momentum $p = \frac{|\mathbf{k}|}{\sqrt{m_1 m_2}}$ and dimensionless parameter $a = \frac{m_2^2 - m_1^2}{m_1 m_2}$, as follows,

$$|V(p,a)|^2 \;=\; \frac{1}{2}(1 - \frac{p^2+1}{\sqrt{(p^2+1)^2 + ap^2}}) \; . \tag{2.194}$$

From Fig.(2.8) we see that the effect is maximal when $p = 1$ ($|V_{max}|^2 \approx \frac{(m_1-m_2)^2}{16 m_1 m_2}$) and $|V|^2$ goes to zero for large momenta (i.e. for $|\mathbf{k}|^2 \gg \frac{m_1^2+m_2^2}{2}$) as $|V|^2 \approx \frac{\Delta m^2}{4k^2}$.

Since we do not know yet the values of neutrino masses, we cannot properly specify the optimal scale for observation of field-theoretical effect in this sector. However, certainly this scale cannot be much larger than a fraction of eV. So far the experimentally observed neutrinos are always extremely relativistic and, therefore, the value of $|V|^2$ may be estimated as $|V|^2 \sim \frac{\Delta m}{k^2} \sim 10^{-18}$. Only for extremely low energies (like those in neutrino cosmological background) the field-theoretical corrections may be large and account for few percents.

2.8 Summary

The quantum field mixing effects may be understood by considering interplay between the two Fock-spaces of the free-fields and the flavor fields. As we demonstrate, this interplay is highly non-trivial and gives rise to deviations from the simple quantum mechanical approach due to antiparticle content in the mixing. Starting from original work of BV, we have now extended nontrivial flavor vacuum approach to obtain full solution of the field-theoretical mixing problem with arbitrary number of fields with boson or fermion statistics. Our results fall into the same scheme and can be easily unified. We investigated the time dynamics by calculating non-equal time commutators for such theory. We found an explicit solution for the Fock space of the interacting fields and the corresponding vacuum structure that turned out to be a generalized coherent state. We then showed the unitary nonequivalence between the mixed-field Fock space and the free-field Fock space in the infinite volume limit. After

we built a formal framework, we applied it to solve mixing dynamics of 2 vector mesons ($S = 1$) and fermions ($S = 1/2$). We found that the scalar/pseudoscalar ($S = 0$) boson mixing is the same as the mixing of transverse components of the vector fields, while for the longitudinal component of the vector field we found richer momentum dependence than in the spin-zero case. We presented a new algebraic method which is distinct from the conventional method of using a differential equation for $\mathcal{Z}$. While the unitary nonequivalence occurs only in the infinite volume limit for the boson case, we find an intrinsic difference between the fermion and boson cases. As shown in this work, the normalization factor $\mathcal{Z}$ for the boson given by Eq.(2.66) has a singularity on the complex plane while the corresponding result for the fermion doesn't have any singularity. This singularity corresponds to the divergence of the Taylor expansion in powers of mixing angle θ. We also addressed the problem of time-to-space conversion in quantum field theory of mixing and explicitly considered the effect from nontrivial flavor vacuum on oscillations of flavor in space.

The general field theory of mixing introduces following differences, relative to quantum mechanics, into oscillating quantities. Oscillation formulas typically involve all possible low-frequency and high-frequency energy combinations. The amplitudes of the oscillation terms are essentially momentum dependent. We have also observed the coherent antiparticle beam generated from the starting definite-flavor particle state and presented its dynamics.

Our general approach does not use any specific continuous parametrization of the mixing group but directly takes the values of the matrix elements. This allows an analysis to be carried out in a unified closed form. In our formalism it may be preferable to solve the mixing problems without going through the intermediate parametrization step for the mixing matrix. Even if one wants to use a specific parametrization scheme, it is rather straightforward to formulate our general framework into a symbolic calculation system, like maple or mathematica, and carry out calculations involving mixing parameters in relatively short period of time.

We also considered phenomenological aspects of the quantum field theoretical

mixing formalism. We have estimated the magnitude of the field-theoretical effect in known mixed systems. We found that for most of known mixed systems both in meson and neutrino sectors this effect is very negligible. Only in strongly mixed systems, such as $\eta - \eta'$ or $\omega - \phi$, or for very low-energy neutrino effects the corrections may be as large as 10% and thus may need additional attention should these systems be used in experiment of some sort. The nontrivial vacuum effect is the most prominent when the particles are produced at low momentum.

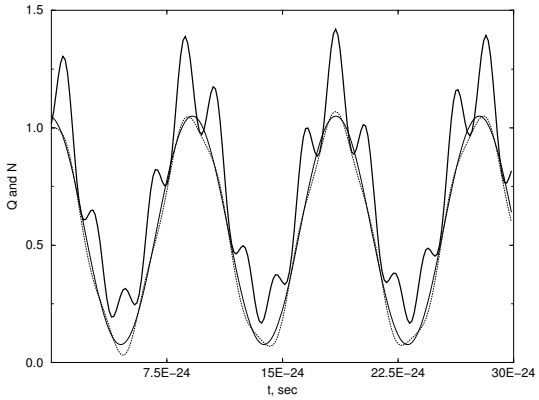

Figure 2.2: Population density in $SU(2)$ scalar mixing at $k = 0.1$GeV in QFT and QM

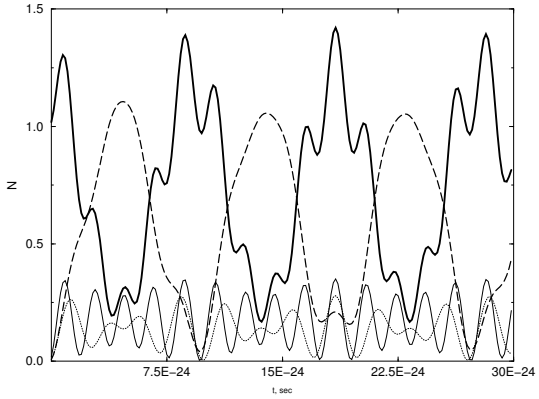

Figure 2.3: Population densities in $SU(2)$ scalar mixing at $k = 0.1$GeV in QFT

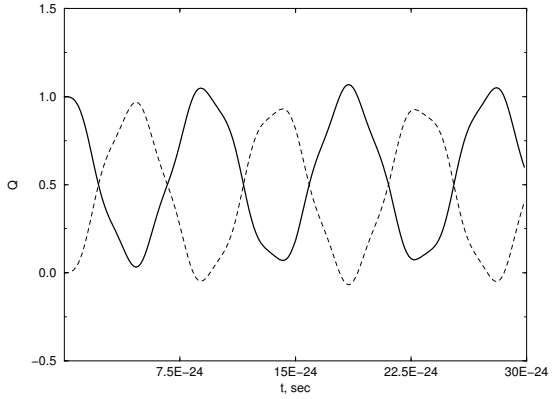

Figure 2.4: Flavor charge oscillations in $SU(2)$ scalar mixing in QFT

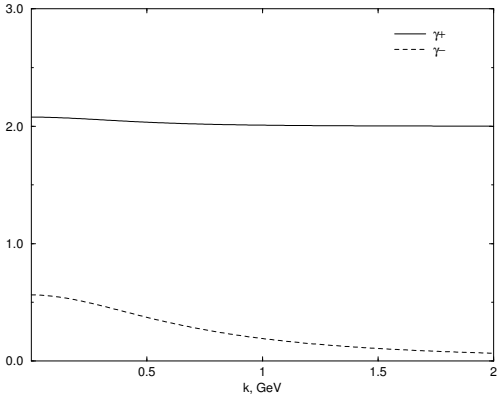

Figure 2.5: Mixing amplitudes for $SU(2)$ scalar mixing

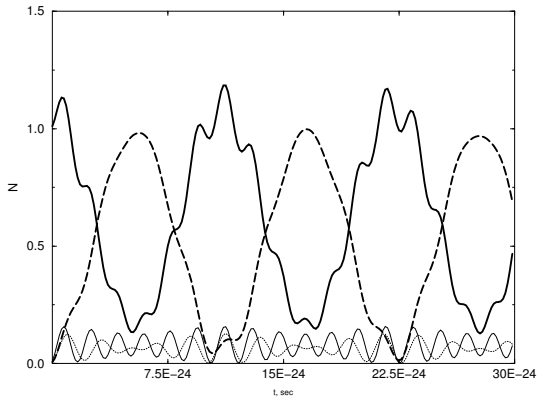

Figure 2.6: Population densities in $SU(2)$ scalar mixing at $k = 0.5$GeV in QFT

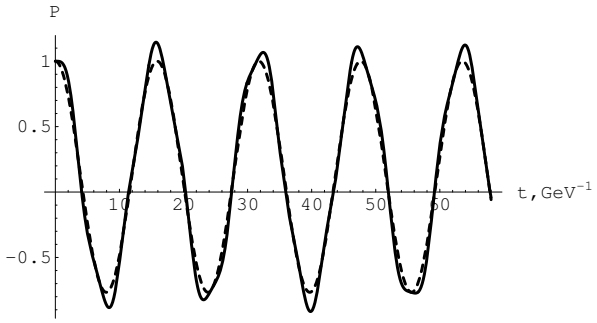

Figure 2.7: Strange charge $Q_{\mathbf{k},A}(t)$ oscillations in time for "$\eta - \eta'$" like mixing. Solid line - BV prescription, dashed line - QM prescription.

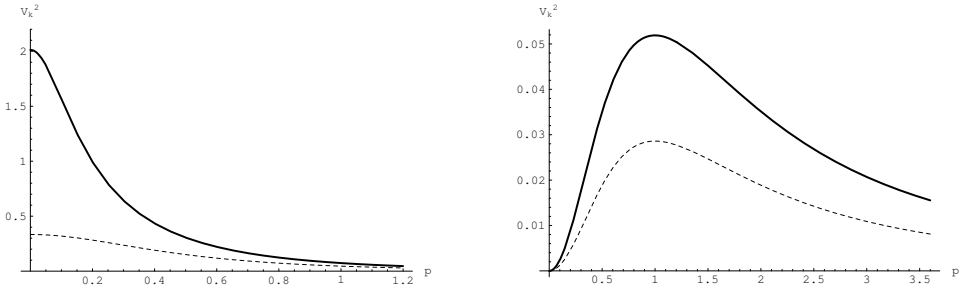

Figure 2.8: The bosonic (left) and fermionic (right) condensation density $|V(p,a)|^2$ as a function of momentum p for $a = 0.98$ (solid line) and $a = 0.8$ ($a = 0.5$ for fermions) (dashed line).

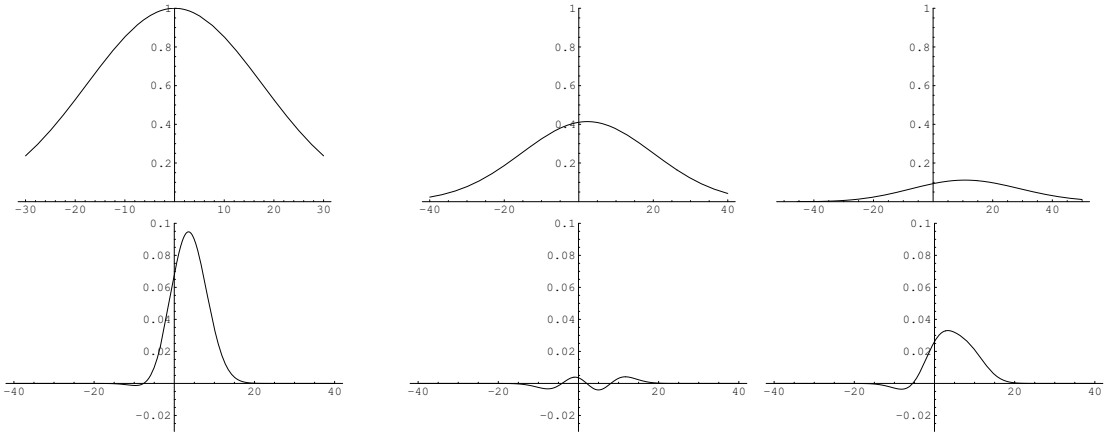

Figure 2.9: Example of propagation of gaussian wave-packet for particle of flavor $\alpha = 1$ at $k_0 \approx 0.35$GeV, (upper row, time flow left-to-right) and example of QFT fluctuation at the time when flavor $\alpha = 1$ has almost disappeared. In here and in what follows the distance scale is GeV^{-1}.

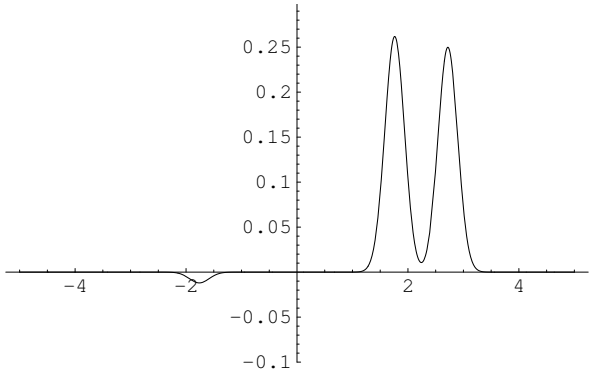

Figure 2.10: A snapshot of the flavor charge space distribution for a wave-packet originally well localized in space: an example of the coherence loss by a point-like flavor source and of EPR-correlated antiparticle wave-packet traveling in the opposite direction.

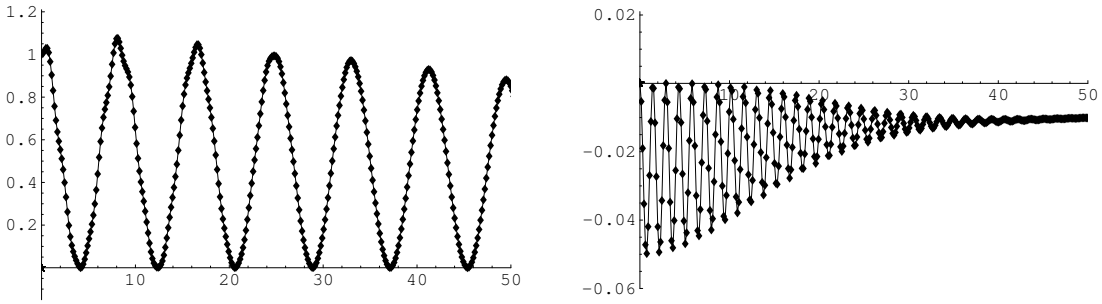

Figure 2.11: Space oscillations for particle (left) and antiparticle (right) populations (for flavor $\alpha = 1$). ($k \approx 0.35$GeV)

Chapter 3

Critical Phenomena and Oscillator Representation Method

3.1 Introduction

With the current advances of Relativistic Heavy Ion Collision (RHIC) physics, it becomes increasingly interesting to discuss the nonperturbative methods in relativistic quantum field theories. In particular, study of phase transitions in QFT is highly nontrivial problem. The conventional approach is based on the Renormalization Group Equations (RGEs) which are calculated using the perturbation theory. Significant success had been achieved on this way using RGE formulated for the massless theory at the phase transition point. This is also known as ϵ-expansion. However, one essential drawback of RGE is that the ϵ-expansion is formulated at the critical point and provides no detailed information on dynamics of the system away from the critical region. RGE approaches using calculations in massive phase, or off the critical point, demonstrated that the phase transition itself is highly nonperturbative phenomenon so that the perturbation theory quickly looses its reliability in the near-critical region and it becomes difficult to extend such calculations close toward the critical point. Many other nonperturbative techniques have been developed to resolve this drawback including the large N expansions, Hartree-Fock approximation, Gaussian

Effective Potential Method, lattice simulations, etc.

The Oscillator Representation Method (ORM), that we are investigating in this chapter, is one of the nonperturbative approaches to study the critical properties of the quantum systems. The ORM is based on the comparison of alternative canonical representations of the commutation relations. It was explicitly formulated by G. Efimov [95] and is at length described in a monograph [96]. In this monograph [96], the ORM had been successfully applied to a range of field-theoretic models including d-dimensional ϕ^4 theory, O(N) invariant ϕ^4 theory and others. Although the results of the ORM are essentially consistent with the results of the other nonperturbative approaches, it was noted more recently that the mass of the quasi-particle in the ambient nontrivial vacuum is in detail different from the one obtained in the Hartree-Fock approximation [97, 98] or the Gaussian Effective Potential (GEP) Method [99–101]. Also, the bifurcated vacuum solution obtained by the ORM had been discussed in depth in that analysis [102].

The basic idea of the ORM is to introduce a shift in the quantization point of the interacting fields and simultaneously redefine their masses in a self-consistent manner so that the system is effectively described in the new degrees of freedom with nontrivial vacuum condensate and different dynamical mass. This effect is realized in the nature by spontaneous symmetry breaking mechanism and thus one can expect that it may be a general feature of quantum field systems with interaction. Formally, the ORM is settled by requiring that the Hamiltonian should be written in terms of creation and annihilation operators of a free-field basis with an appropriate mass and in the correct form defined by the following requirements:

1. the total Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ is written in normal ordered form,

2. the Hamiltonian $\mathcal{H}_0$ is quadratic in the field operators,

3. the interaction Hamiltonian $\mathcal{H}_I$ contains field operators in powers higher than two.

Description of a quantum system requires choosing the Hilbert space of states in which the canonical variables are defined as operators. In Quantum Mechanics all such possible choices are equivalent (Stone-Von Neumann Theorem [103]). Contrary, in QFT unitary inequivalent representations may exist due to infinite number of degrees of freedom. Each such representation corresponds to a specific physical picture that is characterized, in particular, by symmetry properties. From a physical viewpoint, unitary inequivalent representations correspond to a set of distinct vacua of the system among which the internal dynamics of the system chooses one with the minimal free energy as a true physical ground state. One conjectures, therefore, that the possible phases in a field-theoretic model are relevant to its unitary inequivalent representations [88, 95, 102, 104].

Conventionally in QFT the Hilbert space of the free particles of certain mass, i.e. $\mathcal{L}_0 = (\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2)/2$, is used to define the canonical variables in the theory. If in the full Hamiltonian

$$H = \int dx \frac{1}{2}[\pi(x)^2 + (\vec{\partial}\varphi(x))^2 + m_0^2\varphi(x)^2 + g\mathcal{H}_I(\varphi(x), \partial_\mu\varphi(x), x)] \qquad (3.1)$$

the interaction is weak, this choice can be physically justified by the notion that the interacting particles can be with a good accuracy described as free after the self-interaction is properly taken into account (i.e. renormalization procedure). As discovered in the early years of Quantum Electro-Dynamics (QED), calculations of the scattering amplitudes involving self-interactions led to diverging expressions. To extract meaningful results from QED one needs to regularize it by introducing a cutoff α for the diverging integrals and render them finite, i.e. $\int \mapsto \int_\alpha$. In general this procedure would result in scattering amplitudes that depend on the value of the cutoff α^{-1}. One then postulates the notion of the bare parameters, which are, in a sense, true parameters of the theory but which cannot be observed themselves due to self-interaction effects. One can adjust the bare parameter for each given value of cutoff to fit the observables of the theory, i.e. observed electron charge and mass in QED. This procedure is known as renormalization and it expresses the scattering

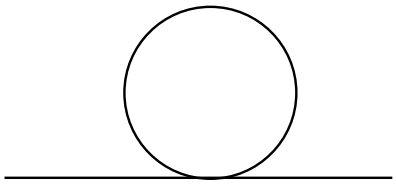

Figure 3.1: Divergent loop diagram in 1+1 ϕ^4 theory.

amplitudes as limit $\alpha^{-1} \to \infty$ with the bare parameters $e_b(\alpha) \to \infty$, $m_b^2(\alpha) \to \infty$ in such a manner that the observables, or renormalized parameters, are held as constants $e_r = const$, $m_r^2 = const$.

Consider, for simplicity, 1+1 dimensional scalar φ^4 theory [105, 106]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2) - \frac{\lambda}{4}\varphi^4. \tag{3.2}$$

Calculations of the particle mass including 1-loop is divergent here, e.g. Fig.(3.1). One can regularize it by introducing momentum cutoff $\int \mapsto \int\limits_{|k|<1/\alpha}$. For the observed mass, one then obtains

$$m_r^2 = m_b^2 - \frac{3g}{4\pi} \ln(\alpha^2 m_b^2), \tag{3.3}$$

where m_r^2 is renormalized mass and m_b^2 is the bare mass entering the Lagrangian (3.2). One can see that, when the cutoff is implemented, one must start with the Lagrangian with the bare mass

$$m_b^2 = m_r^2 + \frac{3g}{4\pi} \ln(\alpha^2 m_r^2), \tag{3.4}$$

in order to get the correct observable mass. This can be also represented by adding to the original Lagrangian an additional local term depending on the cutoff which would guarantee the correct renormalized mass

$$\mathcal{L}_0 = \mathcal{L}_r[\varphi, m_r, \lambda_r] + \frac{\delta m^2(\alpha \cdot m_r)}{2}\varphi^2. \tag{3.5}$$

As one can see, the classical expression for the Lagrangian $\mathcal{L}(m_r, g_r)$ alone is meaningless in QFT context. Indeed, it shall be supplemented with regularization procedure, characterized by cutoff parameter α, and α-dependent counter-terms

$\delta\mathcal{L}(m_r, g_r; \alpha)$. A quantum field theory is called renormalizable if two requirements are met:

- for any α and up to any order in perturbation theory only a finite number of different kinds of counter-terms is required to properly reproduce the observables (usually particle mass and interaction constants as defined by scattering processes at fixed momentum scale $p^2 \to \mu^2$);
- the observables of the theory are well-defined in the limit $\alpha^{-1} \to \infty$ with m_r and g_r kept constant, i.e. all physical observables are approaching certain values as this limit is implemented.

Sometimes a field theory is called renormalizable if renormalization of the masses, coupling constants and fields is sufficient to render theory finite, i.e. no counter-terms of the structure different from $\mathcal{L}(\phi)$ are needed. In these settings the bare Lagrangian can be written by

$$\mathcal{L}_b(\phi_r, m_r, g_r) = \mathcal{L}(\varphi_b(m_r, g_r; \alpha), m_b(m_r, g_r; \alpha), g_b(m_r, g_r; \alpha)). \tag{3.6}$$

Once infinite self-interaction contributions are taken into account and if interaction is weak, the particles in the theory can be well described as almost free with interactions appearing as small corrections to otherwise free motion. If coupling constants begin to grow, however, the legitimacy of such reasoning becomes questionable. Indeed, a phase transition may occur rearranging the dynamics of the system in such a way that other degrees of freedom become more appropriate. QCD is, likely, most vivid phenomenological example of such scenario.

In Oscillator Representation Method, one tries to investigate the effect of such alternative degrees of freedom generated by a unitary transformation $U(b, t)$ of field shift $\{\varphi \to b + \Phi,\ \pi \to \Pi\}$ followed by unitary transformation of mass change $m \to t \cdot m$. Consider, again, 1+1 dimensional ϕ^4 scalar theory. For the quantum fields, we shall postulate that field ϕ and its conjugate momentum π are operators that satisfy canonical equal time commutation relations

$$[\phi(x), \pi(y)]_{x_0=y_0} = i\delta(\mathbf{x} - \mathbf{y}), \tag{3.7}$$

and then should attempt to solve Hamilton's equations

$$\begin{aligned} \dot{\phi} &= \tfrac{\partial}{\partial \pi}\mathcal{H}, \\ \dot{\pi} &= -\tfrac{\partial}{\partial \phi}\mathcal{H}, \end{aligned} \tag{3.8}$$

with the boundary conditions given by Eq.(3.7). Since it is typically impossible to solve explicitly Eq.(3.8) whenever interaction is nonlinear, the conventional approach in interacting fields is to start with the free field which dynamics is known. Then ϕ and π operators are represented as Fourier transform,

$$\begin{aligned} \phi(x) &= \sum_k \tfrac{1}{\sqrt{2V\omega_m}}(a_{\mathbf{k}}e^{ikx} + a^{\dagger}_{-\mathbf{k}}e^{-ikx}), \\ \pi(x) &= \sum_k \tfrac{ik_0}{\sqrt{2V\omega_m}}(a_{\mathbf{k}}e^{ikx} - a^{\dagger}_{-\mathbf{k}}e^{-ikx}), \end{aligned} \tag{3.9}$$

where $k = (\omega_m, \mathbf{k})$, $\omega_m = \sqrt{\mathbf{k}^2 + m^2}$ and the operators of creation and annihilation of the particle of mass m and momentum $\mathbf{k}$ satisfy commutation relations

$$\begin{aligned} [a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] &= \delta(\mathbf{k} - \mathbf{k}'), \\ [a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a^{\dagger}_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] &= 0. \end{aligned} \tag{3.10}$$

With these definitions state $a^{\dagger}_{\mathbf{k}}|0\rangle$ is an eigenstate of both free Hamiltonian and momentum operators with corresponding eigenvalues ω_m and $\mathbf{k}$.

In the case of interacting fields the states $a^{\dagger}_{\mathbf{k}}|0\rangle$ are no longer eigenstates of the full Hamiltonian. Nevertheless, one still uses the above representation of the interacting fields as a good starting point for perturbation theory. The Hamiltonian density operator is then written in terms of the free-field basis as

$$\mathcal{H} \to \mathcal{H}_m = \frac{1}{2}\pi_m^2 + \frac{1}{2}(\nabla\phi_m)^2 + \frac{1}{2}m^2\phi_m^2 + \frac{g}{4}\phi_m^4. \tag{3.11}$$

Here we emphasize that it is based on the free-field operators (3.9).

Since quantum interactions commonly introduce dynamical mass different from the bare mass m_b^2, the free field representation Eq.(3.11) may not be unique. This is related to renormalization group transformation when a special simultaneous variation of mass, coupling constant and field normalization is performed in such a way

that all physical quantities remains the same. The RG transformation changes the renormalization point of a quantum theory, although the physical contents of the theory is preserved. Yet, there exists another possibility to change parameters of the theory.

Let us consider the essential point of these parameterizations, i.e the change of mass and field quantization point shift. For the free field operators the change of mass $m \to M$ is described by Bogoliubov-Valatin transformation:

$$\begin{aligned} \tilde{a}_{\mathbf{k}} &= \cosh\zeta_{\mathbf{k}} a_{\mathbf{k}} - \sinh\zeta_{\mathbf{k}} a^{\dagger}_{-\mathbf{k}}, \\ \tilde{a}^{\dagger}_{\mathbf{k}} &= \cosh\zeta_{\mathbf{k}} a^{\dagger}_{\mathbf{k}} - \sinh\zeta_{\mathbf{k}} a_{-\mathbf{k}}, \\ \zeta_{\mathbf{k}} &= \ln\sqrt{\tfrac{\omega_m(\mathbf{k})}{\omega_M(\mathbf{k})}}, \end{aligned} \tag{3.12}$$

i.e. the free field operators with mass M can be expressed in terms of the free field operators with mass m.

A consequence of the ambiguity in the mass definition due to interactions comes from the ambiguity in the choice of the initial representation of the interacting fields (3.9). This was emphasized in Coleman's paper [107]. He also proposed a useful technique on how to redefine a normal ordered product of any number of field operators with respect to a new value of mass in (1+1) scalar field theory [107, 108],

$$N_m(\frac{1}{2}\pi_m^2 + \frac{1}{2}(\nabla\phi_m)^2) = N_M(\frac{1}{2}\pi_M^2 + \frac{1}{2}(\nabla\phi_M)^2) + \frac{1}{8\pi}(M^2 - m^2), \tag{3.13}$$

$$N_m(e^{\imath\beta\phi_m}) = \left(\frac{M^2}{m^2}\right)^{\beta^2/8\pi} N_M(e^{\imath\beta\phi_M}), \tag{3.14}$$

where β is some arbitrary c-number, $N_m(N_M)$ stands for the normal ordering with respect to the vacuum of the free-fields with initial mass m (new mass M), ϕ_m is the free quantum field defined by Eq.(3.9) and ϕ_M is the quantum field of independent quasi-particles for which representation (3.9) is valid after the mass change. The expression (3.14) can be manipulated to produce set of equalities as in

$$N_m(\phi_m^n) = n!\sum_{j=0}^{[\frac{n}{2}]}\left(\frac{1}{8\pi}\ln(\frac{M^2}{m^2})\right)^j \frac{(-1)^j}{j!(n-2j)!} N_M(\phi_M^{n-2j}), \tag{3.15}$$

where $\left[\frac{n}{2}\right]$ is the integer part of $\frac{n}{2}$.

According to the ORM [96], normal ordering corresponds to incorporating most important quantum contributions into the vacuum state. Also, 1+1 ϕ^4 scalar theory is fully regularized with normal ordering. Using Eq.(3.13-3.15), after the particle mass is changed and the vacuum condensation is introduced, $\phi_M(x) \rightarrow \phi_M(x) + b$, the Hamiltonian density can be written as,

$$N_M(\mathcal{H}) = N_M(\mathcal{H}_M^{correct}) + \mathcal{H}_1 + \varepsilon_M, \tag{3.16}$$

where $\mathcal{H}_M^{correct}$ is the Hamiltonian in the "correct" form

$$\mathcal{H}_M^{correct} = \frac{1}{2}\pi_M^2 + \frac{1}{2}(\nabla\phi_M)^2 + \frac{M^2}{2}\phi_M^2 + \frac{g}{4}\phi_M^4 + gb\phi_M^3, \tag{3.17}$$

ε_M is energy density of the new vacuum,

$$\varepsilon_M = \frac{m^2b^2}{2} + \frac{gb^4}{4} - \frac{m^2\ln(t)}{8\pi} - \frac{3gb^2\ln(t)}{8\pi} + \frac{3g\ln(t)^2}{64\pi^2} + \frac{M^2-m^2}{8\pi} \tag{3.18}$$

and

$$\mathcal{H}_1 = \{\frac{1}{2}(m^2 - M^2) + \frac{3}{2}gb^2t - \frac{3g}{8\pi}\ln(t)\}\phi_M^2 + \{m^2b + gb^3 - \frac{3gb\ln(t)}{4\pi}\}\phi_M. \tag{3.19}$$

The ORM maintains that proper dynamical description of the system corresponds to the quadratic part $\mathcal{H}_0$ having "correct" form, i.e. $\mathcal{H}_1 = 0$,

$$\begin{cases} b(m^2 + gb^2 - \frac{3g}{4\pi}\ln(t)) = 0, \\ \frac{m^2}{2}(1-t) + \frac{3}{2}gb^2 - \frac{3g}{8\pi}\ln(t) = 0. \end{cases} \tag{3.20}$$

Eq.(3.20) is the ORM equation of state for ϕ^4 theory. It has one symmetric (S) $b = 0$ (s_0) and two broken-symmetry (BS) $b \neq 0$ (s_+, s_-) solutions,

$$\begin{gathered} b = 0,\ t = 1, \text{ any } g; \\ b^2 = \frac{3}{4\pi}\frac{t\ln(t)}{2+t},\ g = \frac{2\pi m^2}{3}\frac{2+t}{\ln(t)}. \end{gathered} \tag{3.21}$$

BS-solution has lower energy than S-solution for large g and also has small effective coupling $G \sim \frac{1}{\ln g}$, thus, introducing duality relation between S-solution with large g

and BS-solution with small coupling [Fig.(3.10) and Fig.(3.11)], i.e. large $g\ \phi^4$ scalar theory can be treated fully perturbatively using Hamiltonian (3.17) in BS-phase. In the following sections, we will concentrate on more detailed discussion of this approach and its applications.

3.2 Formulating Oscillator Representation Method

Two key points of the ORM are presented in Ref. [104]:

- different phases of a quantum field system are revealed as nonequivalent representations of the canonical (anti)commutation relations;
- normal ordered structure of a quantum field theory contains the information about its phases.

The problem of unitary inequivalent representations was first encountered in QFT in the Van Hove model

$$H = \int dk w_k a_k^\dagger a_k + g \int dk J(k)[a_k + a_k^\dagger] \tag{3.22}$$

with $J(k)$ being c-function external source. $a_k^\dagger$, a_k are defined as creation and annihilation operators for the Fock space of the free field with mass m

$$\begin{gathered} \varphi(x) = \sum_k \tfrac{1}{\sqrt{2w_k}}[a_k e^{ik\cdot x} + a_k^\dagger e^{-ik\cdot x}], \\ w_k = \sqrt{k^2+m^2}, [a_k, a_{k'}^\dagger] = \delta_{k,k'}. \end{gathered} \tag{3.23}$$

One can see that each mode k has structure of the harmonic oscillator so that for each mode we can associate its oscillator Hilbert space

$$\mathcal{H}_k = span\{\frac{(a_k^\dagger)^n}{\sqrt{n!}} \left|0,k\right\rangle, n = 0,1,2,\ldots\}. \tag{3.24}$$

The Hilbert space is the direct product of $\mathcal{H}_k$. Let us denote a state vector of the Hilbert space as $\left|\{n_k\}\right\rangle$

$$N_k \left|\{n_k\}\right\rangle = a_k^\dagger a_k \left|\{n_k\}\right\rangle = n_k \left|\{n_k\}\right\rangle. \tag{3.25}$$

Since each mode $\mathcal{H}_k$ contains a countable number of states w and for volume V there exist a countable number of modes w, the dimension of the Hilbert space is $C = w^w$, i.e. continuum, and the set $\{|\{n_k\}\rangle\}$ is uncountable and cannot be used as a basis for separable Hilbert space. This problem is avoided by postulating that we are only interested in the states with finite number of particles so that

$$N\,|\{n_k\}\rangle = \sum_k N_k\,|\{n_k\}\rangle = \sum_k n_k < \infty. \tag{3.26}$$

This set is countable and forms a basis for a separable Hilbert space also known as 0-set, or the Fock space. Remarkably, however, one may choose also other countable subset of $\{|\{n_k\}\rangle\}$ to build a different Fock space. In fact, there is an infinite number of such possibilities. If two subsets provide basis for representations of canonical commutation relations, then these representations are unitary inequivalent in the sense that a vector from one of them is not a finite-norm superposition of the vectors from the other. When the number of particles is conserved, i.e. for free fields, one Fock space suffice to develop a field theory. For interacting fields the situation is more complicated, which is reflected in the well-known Haag's theorem - under conditions of *Poincare covariance of the theory, positive definiteness of the norm, locality, positive definiteness of the spectrum, equal time canonical relations, completeness of the set of canonical variables, existence of unique invariant and normalizable vacuum state* there exist no quantum field theory different from the free field theory [109]. Remarkable consequence of this is the statement that the interaction representation is mathematically unfounded in QFT.

A simplified argument for the Haag's theorem helps clarify this statement. Note that for a translation invariant state $|\Psi\rangle$

$$|H\Psi|^2 = \int dxdy\,\langle\Psi, H(x)H(y)\Psi\rangle = \begin{cases} 0, H\Psi = 0, \\ \infty, H\Psi \neq 0. \end{cases} \tag{3.27}$$

Thus in Euclidean quantum field theory, based on the Fock representation, the Hamiltonian is only a well-defined operator if it annihilates translation invariant states. In

the Fock space only $|0\rangle$ is translation invariant so that it must be

$$H_0 |0\rangle = 0 \ \& \ H |0\rangle = (H_0 + gH_I) |0\rangle = 0. \tag{3.28}$$

These relations obviously cannot be satisfied simultaneously for any $g \neq 0$, thus one can see that the operator

$$V(t) = e^{iHt} e^{-iH_0 t} \tag{3.29}$$

is ill-defined on the Fock space. As was first observed in Van Hove model, the practical consequence of this is that the true ground state of the theory $H|\Omega\rangle = 0$ does not belong to the Fock space (i.e. have infinite norm in the Fock space).

The crucial point of the Haag's theorem is uniqueness of the vacuum. If one weaken this condition and accepts possibility of existence of a set of normalized and invariant vectors, which are necessarily unitary inequivalent, then the usual identification of the vacuum and ground state will be no longer correct. The true ground state should be defined by the internal dynamics of the system as a state with lowest free energy.

To consider this point, one should start with the classical Hamiltonian $\mathcal{H}(\pi, \varphi) = \mathcal{H}_{0,m}(\pi, \varphi) + \mathcal{H}_I(\pi, \varphi)$. To render theory finite one should introduce normal ordering of the Hamiltonian with respect to the Fock space vacuum state $|0\rangle$ and also introduce all necessary counter-terms $\delta H_\alpha(\pi, \varphi)$. For given cutoff the bare Hamiltonian will be defined by

$$\mathcal{H}(\pi, \varphi) = N_{|0\rangle} \left[\mathcal{H}_{0,m}(\pi, \varphi) + \mathcal{H}_I(\pi, \varphi) + \delta \mathcal{H}_\alpha(\pi, \varphi) \right]. \tag{3.30}$$

To analyze the system in different unitary inequivalent representation, the ORM suggests to rewrite Eq.(3.30) in new degrees of freedom given by $\Phi = \varphi - b$, $\Pi = \pi$ and normal order the resulting Hamiltonian with respect to the new Fock space vacuum state with mass $t \cdot m$ $|0; t, b\rangle$ using [107]

$$N_{t \cdot m}(\varphi^n) = \sum_{j=0}^{[n/2]} \frac{n!}{(n-2j)! j!} \left(\frac{\Delta(m, tm)}{2} \right)^j N_m(\varphi^{n-2j}), \tag{3.31}$$

where

$$\Delta(m, tm) = \int_\alpha \frac{d^d k}{(2\pi)^d} [\frac{1}{\sqrt{k^2 + t^2 m^2}} - \frac{1}{\sqrt{k^2 + m^2}}] = \int_{\alpha, m} \frac{d^d k}{(2\pi)^d} \frac{1}{\sqrt{k^2 + t^2 m^2}} \tag{3.32}$$

and $\int_{\alpha,m}$ means that the integral has been regularized by condition $\Delta(\mu,m)|_{\mu=m}=0$. One may also think about $\mathcal{H}(\Pi,\Phi)$ after such transformation as

$$\begin{gathered}\mathcal{H}(\Pi,\Phi)=\mathcal{H}(\pi=\Pi,\varphi=\Phi+b)= \\ =N_{|0;t,b\rangle}\left(\mathcal{H}_{0,tm}(\Pi,\Phi)+\mathcal{H}_I^{'}(\Pi,\Phi)+\delta\mathcal{H}_\alpha^{'}(\Pi,\Phi)\right)+\mathcal{H}_{ext}(\Pi,\Phi),\end{gathered} \tag{3.33}$$

where $\mathcal{H}_I^{'}$ is transformed classical interaction Hamiltonian, $\delta\mathcal{H}_\alpha^{'}$ is the proper counterterm generated by $\mathcal{H}_I^{'}$ and $\mathcal{H}_{ext}$ is polynomial of no higher than 2nd degree in φ given by

$$\begin{aligned}\mathcal{H}_{ext}(\Pi,\Phi)= \quad & N_{|0\rangle}[\mathcal{H}(\pi=\Pi,\varphi=\Phi+b)]- \\ & N_{|0;t,b\rangle}[\mathcal{H}_{0,tm}(\Pi,\Phi)+\mathcal{H}_I^{'}(\Pi,\Phi)+\delta\mathcal{H}_\alpha^{'}(\Pi,\Phi)].\end{aligned} \tag{3.34}$$

According to [104] for (Π,Φ) to provide an appropriate dynamical variables for the model $\mathcal{H}(\pi=\Pi,\varphi=\Phi+b)$ should have the "correct" form, i.e. exactly equal

$$N_{|0;t,b\rangle}\left[\mathcal{H}_{0,tm}(\Pi,\Phi)+\mathcal{H}_I^{'}(\Pi,\Phi)+\delta\mathcal{H}_\alpha^{'}(\Pi,\Phi)\right],$$

so that one arrives at the requirement of the "correct" form for the Hamiltonian

$$\mathcal{H}_{ext}(\Pi,\Phi)=0. \tag{3.35}$$

Solution of Eq.(3.35) defines possible stable phases of the system among which the vacuum energy density $\langle 0;t,b|H_{in}|0,t,b\rangle$ and the effective coupling constant $G=g/t^2m^2$ define the ground state.

As a practical illustration of this approach consider 2+1 scalar φ^4 theory following Ref. [104]. Here the classical Hamiltonian is given by

$$\mathcal{H}=\int dx[\frac{1}{2}(\partial_\mu\varphi)^2+\frac{1}{2}m_r^2\varphi^2+\frac{\lambda_r}{4}\varphi^4], \tag{3.36}$$

with $\mathcal{H}_0=\int dx[\frac{1}{2}(\partial_\mu\varphi)^2+\frac{1}{2}m_r^2\varphi^2]$ and $\mathcal{H}_I=\int dx\frac{\lambda_r}{4}\varphi^4$. When normal ordering is implemented, the only divergent primitive diagram is Fig.(3.2) so that the counterterm is given by $\delta\mathcal{H}_\alpha=\int dx[3\lambda_r^2\Sigma(m_r)\varphi^2(x)+E(m_r,\lambda_r)]$ and

$$\begin{gathered}\Sigma_\alpha(m_r)=\textstyle\int_\alpha dxD^3(x,m_r), \\ D(x,m_r)=\int\frac{d^3k}{(2\pi)^3}\frac{e^{-ikx}}{k^2+m_r^2}=\frac{1}{4\pi}\frac{e^{-m_rr}}{r}.\end{gathered} \tag{3.37}$$

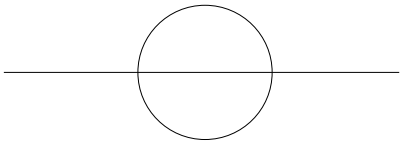

Figure 3.2: Divergent loop diagram in 2+1 ϕ^4 theory.

For the bare Hamiltonian we obtain, therefore,

$$\begin{aligned} H(\pi,\phi) = \quad & N_{|0\rangle}\big[\int dx[\tfrac{1}{2}(\partial_\mu\varphi)^2+\tfrac{1}{2}m_r^2\varphi^2+\tfrac{\lambda_r}{4}\varphi^4]+ \\ & \int dx[3\lambda^2\Sigma_\alpha(m_r)\varphi^2(x)+E_\alpha(m_r,\lambda_r)]\big]. \end{aligned} \tag{3.38}$$

Once change of variables is performed in Eq.(3.38) the Hamiltonian takes the form

$$\begin{aligned} H(\Pi,\Phi) = \quad & N_{|0\rangle}\big[\int dx[\tfrac{1}{2}(\partial_\mu\Phi)^2+ \\ & (m_r^2b+\lambda_rb^3)\Phi+(\tfrac{1}{2}m_r^2+\tfrac{3}{2}\lambda_rb^2)\Phi^2+\lambda_rb\Phi^3+\tfrac{\lambda_r}{4}\Phi^4]+ \\ & \int dx[3\lambda_r^2\Sigma_\alpha(m_r)\Phi^2+6\lambda_r^2\Sigma_\alpha(m_r)\Phi+E'_\alpha(m_r,\lambda_r)]\big]. \end{aligned} \tag{3.39}$$

After normal ordering to the new vacuum state we get

$$\begin{aligned} H_{in}(\Pi,\Phi) = N_{|0;t,m\rangle}[\int dx[\tfrac{1}{2}(\partial_\mu\Phi)^2+(m_r^2b+\lambda_rb^3+3\lambda_r\Delta_r(m,tm))\Phi+ \\ \tfrac{1}{2}(m_r^2+3\lambda_rb^2+3\lambda_r\Delta_r(m,tm))\Phi^2+\lambda_rb\Phi^3+\tfrac{\lambda_r}{4}\Phi^4]+ \\ \int dx[3\lambda_r^2\Sigma_\alpha(m_r)\Phi^2+6\lambda_r^2b\Sigma_\alpha(m_r)\Phi+E'(m_r,\lambda_r)]]. \end{aligned} \tag{3.40}$$

On the other side one may view Eq.(3.40) as

$$\begin{aligned} H_{out}(\Pi,\Phi) = N_{|0;t,m\rangle}(\int dx[\tfrac{1}{2}(\partial_\mu\Phi)^2+\tfrac{1}{2}t^2m_r^2\Phi^2+\lambda_rb\Phi^3+\tfrac{\lambda_r}{4}\Phi^4]+ \\ \int dx[3\lambda_r^2\Sigma_\alpha(tm_r)\Phi^2+6\lambda_r^2b\Sigma_\alpha(tm_r)\Phi+E''_\alpha(tm_r,\lambda_r)]), \end{aligned} \tag{3.41}$$

where the counter-terms are generated by interaction parts ϕ^3 and ϕ^4. Then

$$\begin{aligned} H_{in}(\Pi,\Phi)-H_{out}(\Pi,\Phi) = \int dx\big\{\tfrac{1}{2}(m_r^2-t^2m_r^2+3\lambda_r(b^2+\Delta_r(m,tm))+ \\ 6\lambda_r^2(\Sigma_\alpha(m_r)-\Sigma_\alpha(tm_r)))\Phi^2+(\lambda_rb+6\lambda_r^2b(\Sigma_\alpha(m_r)-\Sigma_\alpha(tm_r)))\Phi\big\}=0. \end{aligned} \tag{3.42}$$

Eq.(3.42) gives conditions on t and b for which the final Hamiltonian would in the "correct" form. Solution of Eq.(3.42) along with the difference in the vacuum energy densities $E(m,\lambda)$ defines the phase structure of the theory according to the ORM.

We must point out that applications of the ORM in no way should be limited to the canonical transformation of field shift. Let us consider 1+1 dimensional Yukawa model,

$$\mathcal{L} = \bar{\psi}(x) i \not{\partial} \psi(x) + \frac{1}{2}(\partial\phi(x))^2 - m^2\phi^2(x) - y\phi(x)\bar{\psi}(x) i\gamma_5\psi(x) - \frac{g}{4}\phi^4(x). \quad (3.43)$$

Here we use : : symbol to denote normal ordering. In canonical variables the Hamiltonian is given by

$$\begin{aligned} &\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I + \mathcal{H}_{ct} \\ &\mathcal{H}_0 = \int dx_1 : \tfrac{1}{2}\left[(\pi^2 + (\partial_1\phi)^2 + m^2\phi^2) + \bar{\psi} i\gamma_1\partial_1\psi\right] : \\ &\mathcal{H}_I = \int dx_1 : y\phi\bar{\psi} i\gamma_5\psi + \tfrac{g}{4}\phi^4 : \\ &\mathcal{H}_{ct} = \int dx_1 : \tfrac{1}{2}\delta m^2\phi^2 : \end{aligned} \quad (3.44)$$

The dirac matrices in 1+1 dimensions are $\gamma_0 = \sigma_3$, $\gamma_1 = i\sigma_2$, $\gamma_5 = \sigma_1$. This theory is super-renormalizable and there is only one diagram in self-energy which needs to be renormalized (not counting divergent vacuum energy bubble). In [104], authors consider canonical transformation of the form

$$\begin{aligned} &\{i\psi^\dagger, \psi\} \to \{i\psi^\dagger e^{-i\frac{\alpha}{2}\gamma_5}, e^{i\frac{\alpha}{2}\gamma_5}\psi\} \\ &\{\pi, \phi\} \to \{\Pi, \Phi + b\} \end{aligned} \quad (3.45)$$

accompanied with fermion and boson fields mass change $M_F^2 = fm^2$ and $M_B^2 = tm^2$.

After the transformation the Hamiltonian is

$$\begin{aligned} \mathcal{H}_0' &= \int dx_1 : \tfrac{1}{2}(\Pi^2 + (\partial_1\Phi)^2 + M_B^2\Phi^2) + \bar{\Psi}(i\gamma_1\partial_1 + M_F)\Psi) : \\ \mathcal{H}_I' &= \int dx_1 : \bar{\Psi}(\sin\alpha - i\gamma_5\cos\alpha)\Psi + \tfrac{g}{4}(\Phi^4 + 4b\Phi^3) : \\ \mathcal{H}_{ct}' &= \int dx_1 \tfrac{1}{2}\delta M_B^2 : \Phi^2 : \\ \mathcal{H}_1' &= \int dx_1 : \tfrac{1}{2}(m^2 - M_B^2 - 3g\Delta(t) + 3gb^2 + \delta m^2 - \delta M_B^2)\Phi^2 + \\ &\quad (m^2 b - 3gb\Delta(t) + gb^3 + \delta m^2 b - y\sin\alpha\tilde{\Delta}(f))\Phi + \\ &\quad (yb\sin\alpha - M_F)\bar{\Psi}\Psi - yb\cos\alpha\bar{\Psi} i\gamma_5\Psi : . \end{aligned} \quad (3.46)$$

The correct form of the Hamiltonian requires that $H_1' = 0$ and, thus,

$$\begin{array}{l} yb\sin\alpha - M_F = 0 \ (\sqrt{2Y}b\sin\alpha - \sqrt{f} = 0), \\ yb\cos\alpha = 0 \ (\sqrt{2Y}b\cos\alpha = 0), \\ m^2 - M_B^2 - 3g\Delta(t) + 3gb^2 + \delta m^2 - \delta M_B^2 = 0, \\ (1 - t - \frac{3G\ln t}{2} + Y\ln f + Y(1-4\frac{f}{t})F(\frac{f}{t}) + 6Gb^2 = 0), \\ m^2 b - 3gb\Delta(t) + gb^3 - y\sin\alpha\tilde{\Delta}(f) + \delta m^2 b = 0, \\ (b\left[1 - \frac{3G\ln t}{2} + Y\ln f + 2Gb^2\right] = 0). \end{array} \tag{3.47}$$

Here $Y = \frac{y^2}{2\pi m^2}$ and $G = \frac{g}{2\pi m^2}$ are dimensionless couplings and

$$F(z) = \int\limits_0^1 \frac{dx}{x(1-x) - z}. \tag{3.48}$$

The vacuum energy density can be obtained in the following form

$$\begin{array}{rl} E = & \frac{m^2}{8\pi}\big\{4b^2 + t - 1 - \ln t + \\ & 2f\ln f + G(4b^4 - 6b^2\ln t + \frac{3}{4}\ln^2 t) - \frac{Y\ln^2 t}{2} + YJ(\frac{t}{f})\big\} \\ & J(z) = 2\int\limits_0^1 dx \frac{(1-x^2)\left(\frac{x}{x-1}\ln x - \ln(1-x)\right)}{x((1-x)^2 + zx)} \end{array} \tag{3.49}$$

The phase structure of the theory can be investigated starting from the above equations of state. Particularly, it was found that as long as $g = 0$ for any values of y no dynamical generation of the fermion mass can be obtained in this way. Generally, however, if $g \neq 0$ symmetry breaking in ϕ-sector may lead to P-violation and fermion mass generation [96]. Also, notably, requirements of the correct form (3.47) are not minimization conditions for Eq.(3.49).

In the form outlined above, the ORM can be applied only in $d \leq 4$ since $\Delta(m, tm)$ defined by

$$\begin{array}{c} \Delta(m, tm) = \int_\alpha \frac{d^dk}{(2\pi)^d}\left[\frac{1}{\sqrt{k^2 + t^2m^2}} - \frac{1}{\sqrt{k^2+m^2}}\right] = \\ m^{d-2}\int_\alpha \frac{d^dk}{(2\pi)^d}\left[\frac{1}{\sqrt{k^2+t^2}} - \frac{1}{\sqrt{k^2+1}}\right] \end{array} \tag{3.50}$$

contains singular contribution proportional to the mass for $d \geq 4$. Also, other densities, e.g. $(\Sigma_\alpha(m_r) - \Sigma_\alpha(tm_r))$, $(E_\alpha(tm_r) - E_\alpha(m_r))$ acquire infinite contributions

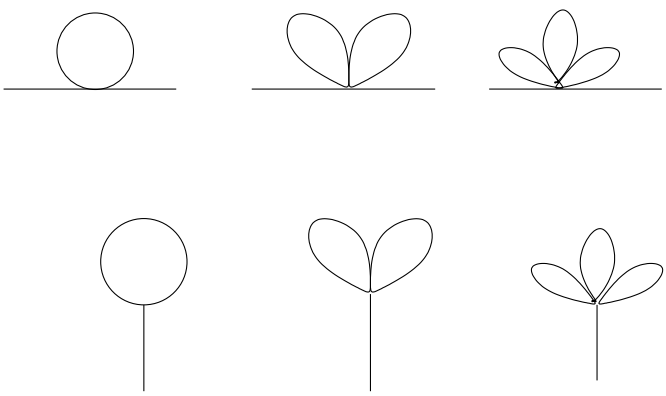

Figure 3.3: Daisy loop diagrams in polynomial field theories

proportional to mass as well, so that the ORM in the form, presented above, looses its applicability.

It may be noted that normal ordering in polynomial field theories is equivalent to regularization of the "daisy" loop diagrams Fig.(3.3). E.g., in the discussed above φ^4 theory the normal ordering is equivalent to regularizing the diagram Fig.(3.1). Thus, instead of requiring from the Hamiltonian to be normal ordered one may require from it to contain proper counter-terms for all divergent diagrams including the "daisy" loop diagrams, e.g.

$$\begin{gathered} H(\pi,\phi) = N_{|0\rangle}\big[\int dx[\tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{2}m_r^2\varphi^2 + \tfrac{\lambda_r}{4}\varphi^4] + \\ \int dx[3\lambda^2\Sigma_\alpha(m_r)\varphi^2(x) + E_\alpha(m_r,\lambda_r)]\big] = \\ \int dx[\tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{2}m_r^2\varphi^2 + \tfrac{\lambda_r}{4}\varphi^4] + \\ \int dx[\tfrac{1}{2}(6\lambda_r^2\Sigma_\alpha(m_r) + 3\lambda_r\Delta_\alpha(m_r))\varphi^2(x) + E'_\alpha(m_r,\lambda_r)]. \end{gathered} \tag{3.51}$$

where counter-term $3/2\lambda_r\Delta_\alpha(m_r)\varphi^2(x)$ appears from the renormalization of the diagram Fig.(3.1), while $3\lambda_r^2\Sigma_\alpha(m_r)\varphi^2(x)$ comes from the loop diagram Fig.(3.2). In this form the ORM can be reformulated with no reference to normal ordering [104]. Advantage of this formulation is that it can be extended beyond $d = 3$ limit. If certain theory is given by its Hamiltonian $H(\pi,\varphi) = H_0(\pi,\varphi) + H_I(\pi,\varphi)$ with counter-terms $\delta H_{\alpha,\mu}(\pi,\varphi)$ in given regularization procedure characterized by cutoff parameter α^{-1}

and renormalization point μ (e.g. physical mass for on-shell renormalization), then the bare Hamiltonian is

$$\begin{gathered} H(\pi,\varphi,m_r^2,\lambda_r) = H_0(\pi,\varphi) + H_I(\pi,\varphi) + \delta H_{\alpha,\mu}(\pi,\varphi) = \\ = H(Z_2^{-1}(\alpha,\mu)\pi, Z_2(\alpha,\mu)\varphi, m_r^2(\mu) + \delta m_r^2(\alpha,\mu), Z_1(\alpha,\mu)\lambda_r(\mu)). \end{gathered} \tag{3.52}$$

We should perform change of variables

$$\varphi \to \frac{z}{e} ta(\Phi + b), \pi \to \zeta^{-1}\Pi \tag{3.53}$$

accompanied by change of the renormalization point $\mu \to t\mu$ corresponding to scaling of the renormalized mass by factor t.

Suppose one works in on-shell renormalization $\mu = m_r$. After the transformation to (Φ, Π) degrees of freedom one should assume different renormalized mass for Φ-field $M^2 = t^2 m^2$. New counter-terms then should be calculated with this new mass so that

$$m_r^2(tm_r) = m_r^2 + \delta m^2(m_r) - \delta m^2(tm_r) = t^2 m_r^2$$

and $\delta m^2(tm_r)$ (or equivalently $m^2(tm_r)$) can be evaluated from the perturbation theory or from renormalization group equations Eq.(3.58).

After the transformation the Hamiltonian takes form

$$\begin{gathered} H'(\Pi,\Phi) = H(\pi = \zeta\Pi, \varphi = \zeta^{-1}(\Phi + b)) = \\ = H(Z_2^{-1}(\alpha,\mu)\zeta^{-1}\Pi, Z_2(\alpha,\mu)\zeta(\Phi + b), m_r^2(\mu) + \delta m_r^2(\alpha,\mu), Z_1(\alpha,\mu)\lambda_r(\mu)). \end{gathered} \tag{3.54}$$

This is also

$$\begin{gathered} H'(\Pi,\Phi) = H'(Z_2^{-1}(\alpha,t\mu)\Pi, Z_2(\alpha,t\mu)\Phi, t^2 m_r^2(t\mu) + \\ \delta m_r^2(\alpha,t\mu), Z_1(\alpha,t\mu)\lambda_r(t\mu); b) \end{gathered} \tag{3.55}$$

so that we obtain

$$\begin{gathered} H(Z_2(\alpha,\mu)^{-1}\zeta^{-1}\Pi, Z_2(\alpha,\mu)\zeta(\Phi + b), m_r^2 + \delta m_r^2(\alpha,\mu), Z_1(\alpha,\mu)\lambda_r) - \\ - H'(Z_2(\alpha,t\mu)^{-1}\Pi, Z_2(\alpha,t\mu)\Phi, t^2 m_r^2(t\mu) + \delta m_r^2(\alpha,t\mu), Z_1(\alpha,t\mu)\lambda_r(t\mu); b) = 0. \end{gathered} \tag{3.56}$$

Choosing $\zeta = Z_2(\alpha, t\mu)^{-1} Z_2(\alpha, \mu)$ one obtains

$$\begin{gathered} H(Z_2(\alpha,t\mu)^{-1}\Pi, Z_2(\alpha,t\mu)(\Phi+b), m_r^2+\delta m_r^2(\alpha,\mu), Z_1(\alpha,\mu)\lambda_r) - \\ -H'(Z_2(\alpha,t\mu)^{-1}\Pi, Z_2(\alpha,t\mu)\Phi, m_r^2(t\mu)+\delta m_r^2(\alpha,t\mu), Z_1(\alpha,t\mu)\lambda_r(t\mu); b) = 0 \end{gathered} \tag{3.57}$$

and $Z_1(\alpha, t\mu)$, $\delta m_r^2(\alpha, t\mu)$ and $\lambda_r(tm)$ are governed by the Renormalization Group Equations (RGE)

$$\begin{gathered} t\frac{d}{dt} g_r(t\mu) = \beta(g_r(t\mu)), \\ -\frac{t}{m^2(t\mu)}\frac{d}{dt} m^2(t\mu) = \rho(g_r(t\mu)). \end{gathered} \tag{3.58}$$

We note that in this form the ORM relies on the RGE and is correct perturbatively up to the order to which the RG functions are known.

To illustrate this point, consider a general case of the ORM transformation with $b = 0$. In this case the problem reduces to

$$\frac{m_r^2(t\mu)}{m_r^2(\mu)} = t^2. \tag{3.59}$$

With RGE

$$\begin{gathered} \frac{1}{t}\frac{dt}{dg_r} = \beta(g_r)^{-1}, \\ \frac{1}{m_r^2}\frac{dm_r^2}{dg_r} = -\rho(g_r)\beta(g_r)^{-1}, \end{gathered} \tag{3.60}$$

this can be integrated regardless of the explicit form of RG functions

$$\begin{gathered} \ln\frac{t(g_1)}{t(g_0)} = \int\limits_{g_o}^{g_1} dg \frac{1}{\beta(g)}, \\ \ln\frac{m^2(g_1)}{m^2(g_0)} = -\int\limits_{g_o}^{g_1} dg \frac{\rho(g)}{\beta(g)}. \end{gathered} \tag{3.61}$$

Eq.(3.59) then reads

$$2\int\limits_{g_o}^{g_1} dg \frac{1}{\beta(g)} = -\int\limits_{g_o}^{g_1} dg \frac{\rho(g)}{\beta(g)} \tag{3.62}$$

or for initial renormalized coupling g_0 possible alternative phases as defined by

$$\int\limits_{g_o}^{g(g_0)} dg \frac{2+\rho(g)}{\beta(g)} = 0. \tag{3.63}$$

Since the energy density differences for different phases in $d \geq 4$ are typically infinite and cannot be compared directly, the phase with the smallest effective coupling is assumed to be the true ground state in this case.

The typical behavior of $F(g) = \int^{g} dg' \frac{2+\rho(g')}{\beta(g')}$ is given in Fig.(3.7). Intersection of $F(g)$ with any horizontal line gives the set of alternative vacua amongst which one with the lowest effective coupling is chosen as the ground state. Easy to see that whenever $\beta(g) = 0$ and $2 + \rho(g) \neq 0$ $F(g)$ typically has a pole so that two distinct phases $g > g^*$ and $g < g^*$ appear at $\beta(g^*) = 0$. This conclusion (zero of $\beta(g)$ is relevant to critical point), however, is no different from RG conclusion, even though some more information on relation between different phases can be obtained within the ORM.

3.3 Nonlinear Sigma Model

3.3.1 Physical instances of nonlinear σ-model

σ-model was originally formulated as an effective theory for low-energy physics of strong interactions [110–116]. Two primary points were taken into account when constructing this model. One is that the fundamental theory of the strong interaction, QCD, is formulated in terms of the "wrong" degrees of freedom. It is therefore of interest for description of the low-energy phenomena to consider effective field theories that are formulated in terms of the hadronic degrees of freedom and possess the same symmetries as QCD.

Another observation is that in the limit of massless quarks an exact symmetry of the QCD Lagrangian is chiral symmetry, i.e. left-handed and right-handed fields $\Psi_{L/R} = \frac{1\pm\gamma_5}{2}\Psi$ are decoupled and propagate independently,

$$\bar{\Psi}\gamma^\mu D_\mu \Psi = \bar{\Psi}_L \gamma^\mu D_\mu \Psi_L + \bar{\Psi}_R \gamma^\mu D_\mu \Psi_R. \tag{3.64}$$

In QCD the masses of u and d current quarks are small relative to the QCD energy scale so that it appears feasible that chiral symmetry is approximate symmetry of, at

least, *u*-*d* sector of QCD. Since quarks are massive, one shall assume that the chiral symmetry in nuclear physics is spontaneously broken, i.e. the vacuum state of the theory is not chiral invariant [110].

σ-model was originally introduced as such an effective chiral field theory for *u*-*d* physics. It is described with effective Lagrangian

$$\mathcal{L} = \bar{\Psi}\gamma^{\mu}(\partial_{\mu} - gf_{\pi}U)\Psi + \frac{1}{16}f_{\pi}^{2}\mathrm{Tr}(\partial_{\mu}U^{\dagger}\partial^{\mu}U) - G(\mathrm{Tr}(U^{\dagger}U)), \tag{3.65}$$

where $\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$ is fermion isospin doublet and $U(x) = \frac{\sigma(x)+i\gamma_5\vec{\tau}\cdot\vec{\pi}(x)}{f_{\pi}}$ so that

$$\mathcal{L} = \bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - g\bar{\Psi}(\sigma + i\gamma_5\vec{\tau}\vec{\pi})\Psi - \frac{m^2}{2}[(\partial_{\mu}\sigma)^2 + (\partial_{\mu}\vec{\pi})^2] - G(\sigma^2 + \vec{\pi}^2). \tag{3.66}$$

Lagrangian (3.65) is manifestly invariant under vector $\mathrm{SU}(2)_V$ transformation $U = e^{-i\vec{\theta}\cdot\vec{\tau}}$ and conforms with SU(2) symmetry of the strong interactions. It is also invariant under axial $\mathrm{SU}(2)_A$ transformation $U = e^{-i\vec{\zeta}\gamma_5\vec{\tau}}$ and as such is chiral symmetric. Note also that the mass term for fermions is forbidden since it would violate the invariance under the chiral transformation $\mathrm{SU}(2)_A$

$$\delta(\bar{\Psi}\Psi) = -i\frac{\vec{\zeta}}{2}(\bar{\Psi}\gamma_5\vec{\tau}\Psi + \bar{\Psi}\vec{\tau}\gamma_5\Psi). \tag{3.67}$$

To generate fermion masses the chiral symmetry should be broken. One can achieve this by introducing appropriate self-interaction for $U(x)$ so that $G(\mathrm{Tr}(U^{\dagger}U))$ has a minimum when $\mathrm{Tr}(U^{\dagger}U) \neq 0$. The minimal such interaction can be constructed by

$$G(\mathrm{Tr}(U^{\dagger}U)) = \frac{\mu^2}{2}\mathrm{Tr}(U^{\dagger}U) - \frac{\lambda}{4}\mathrm{Tr}(U^{\dagger}U)^2 \tag{3.68}$$

in which case chiral-symmetric state $\langle U\rangle = 0$ corresponds to the local maximum of the potential and is unstable. The true ground state is given by

$$\langle\pi\rangle = 0; \langle\sigma\rangle = f = \sqrt{\frac{\mu^2}{\lambda}}. \tag{3.69}$$

Therefore we now have vacuum which is not itself invariant under $\mathrm{SU}(2)_A$ transformation so that the chiral symmetry is said to be broken in the ground state and thus

realized in the Goldstone mode. The massless Goldstone bosons associated with this spontaneous symmetry breakdown are $\vec{\pi}(x)$ fields and can be identified from the particle spectrum of the Lagrangian (3.65) if expanded around the new vacuum (3.69). After appropriate transformation $\sigma' = \sigma - f$ is carried out, one finds that the fermion has acquires mass $m = g/f$, the σ-meson acquires mass $m_\sigma = \sqrt{2\mu^2}$ and the pion appears as the Goldstone boson with mass zero.

In the model considered so far the pions are massless because of an exact SU(2)$\otimes$SU(2) symmetry. The experimentally observed small, but non zero, mass of the pion requires explicit breaking of the chiral symmetry in the original Lagrangian. This can be done, e.g., by introducing chiral symmetry breaking term [113]

$$\mathcal{L}' = \beta\sigma. \tag{3.70}$$

With this term the axial vector current is no longer conserved and the true ground state is shifted to

$$\begin{aligned} \langle\pi\rangle = 0, \langle\sigma\rangle = f', \\ \beta = \lambda^2 f'(f'^2 - f). \end{aligned} \tag{3.71}$$

with particle spectrum given by

$$\begin{aligned} m_\pi^2 = \lambda^2(f'^2 - f^2) = \beta/f', \\ m_\sigma^2 = \lambda^2(3f'^2 - f^2), \\ m_n^2 = gf'. \end{aligned} \tag{3.72}$$

Note that with symmetry breaking term (3.70) the divergence of the axial current $A_\mu = \bar{\Psi}\gamma_\mu\gamma_5\frac{\tau}{2}\Psi + (\sigma\partial_\mu\pi - \pi\partial_\mu\sigma)$ is given by

$$\partial_\mu A^\mu = -\beta\pi = -f' m_\pi^2 \pi \tag{3.73}$$

so that $f' = f_\pi$ can be associated with the pion decay constant.

The σ-field that appears in the Lagrangian Eq.(3.66) describes a scalar meson with the mass of the order of a few 100 MeV. Such scalar particle has not unequivocally been identified. One possibility then was to observe that for large λ in Eq.(3.66) the fields σ and π are more and more forced to live on the "chiral circle"

$f_\pi^2 \mathrm{Tr}(U^\dagger(x)U(x)) = \sigma^2(x) + \vec{\pi}^2(x) = f_\pi^2$ and the mass of the σ-meson is simultaneously increased. In the limit $\lambda \to \infty$, the Lagrangian of the so called "nonlinear σ-model" is recovered

$$\mathcal{L} = \bar{\Psi}\gamma^\mu(\partial_\mu - g f_\pi U)\Psi + \frac{1}{16} f_\pi^2 \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U), \tag{3.74}$$

with the subsidiary condition

$$U^\dagger(x)U(x) = 1. \tag{3.75}$$

Since σ-field no longer appears explicitly in the Lagrangian it is evident that the chiral symmetry is realized in the Goldstone mode in this model; this can be also seen by noting that the pion is massless and that the σ-field $\sigma = \sqrt{f_\pi^2 - \pi^2}$ has a non-vanishing value.

In the recent years the nonlinear σ-model found its growing applications in a variety of fields dealing with critical phenomena ranging from spin-waves description in magnetic and paramagnetic materials [117], numerous effective theory treatments of low-energy QCD phenomenology and even high-density QCD phase transition [110, 118], QGP and hybrid stars [119]. σ-model also is widely used in the super-symmetric theories to introduce super-symmetry breaking present in the real world [120].

In this form the Lagrangian for the super-symmetric model may be given by

$$\begin{gathered} \mathcal{L}_F = \tfrac{1}{2g^2}(-\partial^\mu n^a \partial_\mu n^a + i\bar{\psi}^a \gamma_\mu \partial_\mu \psi^a + F^a F^a), \\ n^a n^a - 1 = 0, \\ \psi^a n^a = 0, \\ n^a F^a - \tfrac{1}{2}\bar{\psi}^a \psi^a = 0. \end{gathered} \tag{3.76}$$

In Eq.(3.76) n^a, F^a, ψ^a are vectors with respect to O(N) transformation while n, F are scalar and ψ is Majorana spinor with respect to Lorentz O(3,1) transformation. Lagrangian Eq.(3.76) is invariant under the super-symmetry transformation

$$\begin{gathered} \delta n^a = \bar{\epsilon}\psi^a, \\ \delta\psi^a = -i\gamma^\mu \epsilon \partial_\mu n^a + F^a \epsilon, \\ \delta F^a = -i\bar{\epsilon}\gamma^\mu \partial_\mu \psi^a, \end{gathered}$$

with ϵ being anticommuting Majorana spinor. After F^a field is algebraically eliminated, one obtains

$$\mathcal{L}_F = \tfrac{1}{2g^2}(-\partial^\mu n^a \partial_\mu n^a + i\bar{\psi}^a \gamma_\mu \partial_\mu \psi^a + \tfrac{1}{4} g^2 (\bar{\psi}^a \psi^a)^2),$$
$$n^a n^a - 1 = 0,$$
$$\psi^a n^a = 0.$$

As shown in [120], this model exhibit spontaneous symmetry breaking in large N limit so that the breakdown of the super-symmetry can be naturally introduced.

3.3.2 Properties of nonlinear σ-model

Non-linear σ-model is defined as a model with O(N) symmetry and the field being a N-vector of fixed length $\vec{\phi}^2(x) = f^2$ [121,122]. The field can be identified with an element of a homogeneous space O(N)/O(N-1) by $\vec{\phi}(x) = D[g(x)]\vec{u}$, where $\vec{u} = (1, 0, \ldots, 0)$ - a vector in the fundamental representation of O(N)/O(N-1) and $g(x) \in$O(N) with O(N-1) the little group (stabilizer) of $\vec{u}$.

The perturbative analysis of non-linear σ-model typically starts with the generating functional

$$Z[\vec{J}] = A \int [d\vec{\phi}(x)] \delta(\vec{\phi}^2 - 1) \exp\left(\frac{i}{2} f^2 \int dx \partial_\mu \vec{\phi} \partial^\mu \vec{\phi} + i \int dx \vec{J}(x) \vec{\phi}(x) \right). \quad (3.77)$$

σ component of the field $\vec{\phi}(x) = (\sigma(x), \vec{\pi}(x))$ can be trivially integrated out using δ-function. For that and for proper definition of the perturbation expansion the O(N) symmetry should be broken and certain direction $\vec{u}$ should be picked up. This can be done, e.g., by introducing explicit symmetry breaking in the Lagrangian, $\mathcal{L}' = h\sigma$, and eliminating σ field locally by

$$\vec{\phi}(x) = (\sigma(x) = \pm\sqrt{1 - \pi^2(x)}, \vec{\pi}(x)). \quad (3.78)$$

This also generates mass scale for π field and removes infrared divergences if present.

One then obtain the following form for the generating functional:

$$\begin{aligned} Z[\vec{J}] = \; & A \int [d\pi(x)(1-\pi^2(x))^{-1/2}] \\ & \exp\left(\tfrac{i}{2} f^2 \int dx \left[\partial_\mu \vec{\pi} \partial^\mu \vec{\pi} + \tfrac{(\vec{\pi}\partial^\mu \vec{\pi})^2}{1-\vec{\pi}^2}\right] + \right. \\ & \left. i \int dx \vec{J}(x)\vec{\pi}(x) - i \int dx h \sqrt{1-\vec{\pi}^2}\right). \end{aligned} \tag{3.79}$$

If f is large, the main contribution in the functional integral Eq.(3.80) comes from field $|\pi(x)| \sim f^{-2}$ in which case the restriction $\vec{\pi}^2(x) < 1$ is irrelevant and one can integrate freely over $\pi(x)$ from $-\infty$ to $+\infty$ [121]. Therefore, after introducing the coupling constant $g = f^{-2}$ and rescaling the field to $\pi \to \sqrt{g}\pi$, the Lagrangian takes the form

$$\begin{aligned} Z[\vec{J}] = \; & A \int [d\pi(x)] \exp(\tfrac{i}{2} \int dx \left[\partial_\mu \vec{\pi} \partial^\mu \vec{\pi} + g \tfrac{(\vec{\pi}\partial^\mu \vec{\pi})^2}{1-g\vec{\pi}^2}\right] - \\ & \tfrac{1}{2}\delta(0) \int dx \ln(1-g\pi^2(x)) + \\ & i\sqrt{g} \int dx \vec{J}(x)\vec{\pi}(x) - i \int dx h \sqrt{1-g\vec{\pi}^2}). \end{aligned} \tag{3.80}$$

An infinite contribution $\frac{1}{2}\delta(0) \int dx \ln(1-g^2\pi^2(x))$ from the measure can be dropped according to the rules of dimensional regularization $\delta(0) \sim 0$. Strictly speaking, it cancels with the counter-terms arising because of the symmetry breaking procedure we have used. The perturbation theory for the σ-model is formulated using Eq.(3.80) or, equivalently, Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \vec{\pi} \partial^\mu \vec{\pi} + g \frac{(\vec{\pi}\partial^\mu \vec{\pi})^2}{1-g\vec{\pi}^2}\right) - h\sqrt{1-g\vec{\pi}^2}, \tag{3.81}$$

and relies on functional gaussian integral expansion order by order in g [121]. We note that, with further transformation of the integration variables, the Lagrangian can be written in alternative form [122]

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \vec{\pi} \partial^\mu \vec{\pi} (1 + \frac{g}{4}\vec{\pi}^2)^{-2}\right) - h\sqrt{1-g\vec{\pi}^2}. \tag{3.82}$$

Power counting shows that the theory defined by Eq.(3.80) is renormalizable in $d = 2$ and non-renormalizable for $d > 2$. In $d = 2$ renormalization can be studied with help of the generalized Ward-Takahashi identities which allow to conclude that

the theory can be renormalized by rescaling of the field, redefinition of the coupling constant and rescaling of the external "magnetic" field h [121, 122]:

$$\begin{gathered}\mathcal{L}_r = \frac{\mu^{d-2}Z}{2g_rZ_g}\left(\partial_\mu\vec{\pi}_r\partial^\mu\vec{\pi}_r + \partial_\mu\sigma_r\partial^\mu\sigma_r\right) - \mu^{d-2}\frac{h_r}{g_r}\sigma_r, \\ g_r = Z_g(\Lambda/\mu, g_r)g, h_r = \Lambda^{d-2}Z^{1/2}(\Lambda/\mu, g_r)Z_g(\Lambda/\mu, g_r)h, \\ \vec{\pi}_r = Z^{-1/2}(\Lambda/\mu, g_r)\vec{\pi}, \sigma_r = \sqrt{Z^{-1}(\Lambda/\mu, g_r) - \vec{\pi}_r^2},\end{gathered} \tag{3.83}$$

where Λ is a cutoff parameter and μ is the renormalization scale.

Dependence of the renormalized parameters on μ is determined by the renormalization group equations

$$\begin{gathered}\mu\frac{\partial}{\partial\mu}g_r = \beta(g_r), \\ \mu\frac{\partial}{\partial\mu}\ln(Z) = -\zeta(g_r), \\ \mu\frac{\partial}{\partial\mu}\ln h_r = \rho(g_r).\end{gathered} \tag{3.84}$$

To the leading order in g RG-functions are known to be [121]

$$\begin{gathered}\rho(g_r) = 2 - d + \frac{\zeta(g_r)}{2} + \frac{\beta(g_r)}{g_r} = \frac{3-N}{4\pi}g, \\ \beta(g) = \varepsilon g - \frac{N-2}{2\pi}g^2, \\ \zeta(g) = \frac{N-1}{2\pi}g,\end{gathered} \tag{3.85}$$

where $\varepsilon = d - 2$.

As one can see, there is non-trivial zero of the β-function which means that there exist fixed point of RG-flow given by $g = \frac{2\pi\varepsilon}{N-2}$, which in turn implies presence of the phase transition at this value of g whenever $d > 2$. It also implies that nonlinear σ model in 2 dimensions exhibit asymptotic freedom when $N > 2$. The RG functions to the order presently available, i.e. four loops, are given by [121]

$$\begin{gathered}N_d\beta(g) = \varepsilon\tilde{g} - (N-2)\tilde{g}^2\left[1 + \tilde{g} + \frac{N+2}{4}\tilde{g}^2 + b\tilde{g}^3\right], \\ \zeta(g) = (N-1)\tilde{g}\left\{1 + (N-2)\tilde{g}^2\left[\frac{3}{4} + \left(\frac{5-N}{3} + \frac{1-(N-1)/2}{2}\zeta(3)\right)\tilde{g}\right]\right\}, \\ b = -\frac{1}{12}\left(N^2 - 22N + 34\right) + \frac{3}{2}\zeta(3)(N-3), \\ \tilde{g} = N_d g, N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\end{gathered} \tag{3.86}$$

and $\zeta(3) = 1.2020569\ldots$ is Riemann ζ-function. Note that even though to this order β-function has the same qualitative behavior, the fixed point of RG flow is significantly shifted toward $g = 0$ [see Fig.(3.4)].

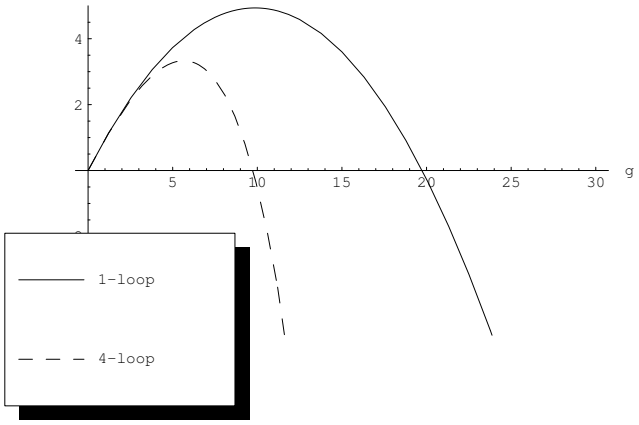

Figure 3.4: β function for nonlinear σ-model in 1- and 4-loop approximations.

Attempts to study non-perturbative properties of nonlinear σ-model are motivated by the conjecture that behavior of the theory in the near-critical regime is essentially non-perturbative. Besides Monte-Carlo simulations [123], this is usually done in the form of large N expansion. In this approach one re-sums the perturbation theory in the orders of N^{-1}. As shown in [122] such re-summation can be done when N is taken to infinity by keeping Ng fixed and produces in the leading order equation of state

$$\left\langle (\partial_\mu \pi)^2 \right\rangle_\varepsilon \left(1 - \frac{1}{2} g Z^{1/2} \left\langle \pi^2 \right\rangle_\varepsilon \right) = 0. \tag{3.87}$$

This equation has two solutions $\left\langle (\partial_\mu \pi)^2 \right\rangle_\varepsilon = \left\langle \pi^2 \right\rangle_\varepsilon = 0$ and $\left\langle \pi^2 \right\rangle_\varepsilon = \frac{2}{gZ^{1/2}}$ which implies mass of the π field as

$$m^2 = \left[\frac{(4\pi)^{d/2}}{gN\Gamma(-\varepsilon/2)} \right]^{2/\varepsilon}. \tag{3.88}$$

Although $\left\langle \pi^2 \right\rangle_\varepsilon < 0$ in dimensional regularization, used in [122], and m^2 is negative, Eq.(3.87) is interpreted as evidence of the phase transition occurring at the critical point of the RG-equation $g_c \approx 2\pi\varepsilon/N$. Considering $g = g_r Z_g \mu^{-\varepsilon}$ where μ is mass scale that renders g_r dimensionless, one can find that $m^2 \approx \mu^2 (1 - g_c/g_r)^{2/\varepsilon}$ when $g_r > g_c$ so that Eq.(3.87) actually describes 2nd order phase transition. Note that

since μ^2 is arbitrary scaling parameter the physical mass $m^2(\mu, g_r)$ should not depend on it and

$$\mu\frac{d}{d\mu}m^2 = \left(\mu\frac{\partial}{\partial\mu}m^2 + \beta(g_r)\frac{\partial}{\partial g_r}m^2\right) = 0, \tag{3.89}$$

which is consistent with $\beta(g_r) \approx g_r\left(\varepsilon - \frac{g_r N}{2\pi}\right)$ as obtained above.

In functional form one can consider the representation of σ-model in the Lagrange multipliers form, i.e. [121]

$$Z = \int [d\phi(x) d\lambda(x)] \exp(\frac{i}{2g}\int dx[(\partial_\mu\phi)^2 + \lambda(\phi^2 - 1)]). \tag{3.90}$$

Since the action is quadratic in ϕ one can explicitly integrate over $N-1$ components of the field to obtain

$$Z = \int [d\sigma(x) d\lambda(x)] \exp(\frac{i}{2g}\int dx[(\partial_\mu\sigma)^2 + \lambda(\sigma^2 - 1)] - \frac{N-1}{2}\mathrm{tr}\ln[-\Delta + \lambda(x)]). \tag{3.91}$$

The saddle point of the effective action is given by

$$\begin{cases} \lambda\sigma = 0, \\ \sigma^2 = 1 - \frac{(N-1)g}{(2\pi)^d}\int\limits_{\Lambda,\mu}\frac{dp}{p^2+\lambda} \end{cases}. \tag{3.92}$$

One again recovers equation of state (3.87) with two phases. In the phase $g > g_c$ $\sigma = 0$ while when $g < g_c$

$$\sigma^2 = 1 - \frac{(N-1)g}{(2\pi)^d}\int_{\Lambda,\mu}\frac{dp}{p^2+\lambda} = 1 - g/g_c. \tag{3.93}$$

At small coupling $\sigma^2 \neq 0$ and thus $\lambda = m^2 = 0$ so that the mass of the π-field vanish. Above $g_c = \frac{(2\pi)^{2d}}{(N-1)\int_{\Lambda,\mu}\frac{dp}{p^2+\lambda}}$ σ vanish thus yielding λ, which is also the square of the mass for σ and π fields,

$$\frac{1}{g_c} - \frac{1}{g} = \lambda^{d/2-1}\frac{(N-1)}{(2\pi)^d}\int_{\Lambda,\mu}\frac{dp}{(p^2+1)p^2} \tag{3.94}$$

resulting in the critical behavior $\lambda \sim (1 - g_c/g)^{2/\varepsilon}$ [121].

3.4 The ORM in Nonlinear Sigma Model

3.4.1 The ORM in nonlinear σ-model with Lagrange multipliers

We now proceed to discussion of the results of our study of the ORM in the nonlinear σ-model. We should start from analyzing σ-model with the constraint enforced via Lagrange multipliers so that the Lagrangian of the model under consideration is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\varphi})^2 - \lambda(\vec{\varphi}^2 - g^{-1}) - \frac{\mu^2}{2}\vec{\varphi}^2 + Z^{-1}\mathcal{L}_{add}, \tag{3.95}$$

where $\mathcal{L}_{add} = [(\partial_\mu \lambda)^2 - \lambda^2]/2$ is added with $Z \to \infty$ to provide correct form for the λ field and $\frac{\mu^2}{2}\vec{\varphi}^2$ is used to introduce mass scale into the theory. In the limit $Z \to \infty$ nonlinear σ-model with Lagrange multiplier would be recovered.

We are specifically interested in the case $d = 2, 3$ where model defined by Eq.(3.95) can be renormalized by normal ordering and thus no counter-terms are explicitly needed in the ORM. The bare Lagrangian is given by

$$H = \int dx N_{\mu_r}[\frac{1}{2}(\partial_\mu \vec{\varphi})^2 + \lambda(\vec{\varphi}^2 - g_r^{-1}) + \frac{\mu_r^2}{2}\vec{\varphi}^2 + Z_r^{-1}[(\partial_\mu \lambda)^2 + \lambda^2]/2]. \tag{3.96}$$

According to the ORM, we want to investigate other degrees of freedom that the theory can be represented with introduced by the transformation

$$\begin{gathered} \vec{\varphi} \equiv (\sigma, \vec{\phi}) \to (\sigma + c, \vec{\phi}), \\ \lambda \to \lambda + b, \\ \mu_r \to (m_\sigma, m_\phi). \end{gathered} \tag{3.97}$$

After the change of variables the Hamiltonian becomes

$$\begin{gathered} H = \int dx N_{\mu_r}[\tfrac{1}{2}(\partial_\mu \vec{\phi})^2 + \tfrac{1}{2}(\partial_\mu \sigma)^2 + \lambda(\sigma^2 + 2\sigma c + c^2 + \vec{\phi}^2 - g_r^{-1}) + \\ + b(\sigma^2 + 2\sigma c + c^2 + \vec{\phi}^2 - g_r^{-1}) + \tfrac{\mu_r^2}{2}\vec{\phi}^2 + \tfrac{\mu_r^2}{2}\sigma^2 + \mu_r^2 \sigma c + \tfrac{\mu_r^2 c^2}{2} + \\ + Z_r^{-1}[(\partial_\mu \lambda)^2 + \lambda^2]/2], \end{gathered} \tag{3.98}$$

which, after normal ordering to the new vacuum, gives

$$\begin{gathered} H = \int dx\{N_M[\tfrac{1}{2}(\partial_\mu\vec{\phi})^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 + \lambda(\sigma^2+\vec{\phi}^2)+ \\ +\sigma^2(b+\tfrac{\mu_r^2}{2}) + \vec{\phi}^2(b+\tfrac{\mu_r^2}{2}) + 2c\lambda\sigma + (\mu_r^2+2b)c\sigma+ \\ +\lambda(\Delta(\mu_r,m_s)+(N-1)\Delta(\mu_r,m_\phi)+c^2-g_r^{-1}+bZ_r^{-1})+ \\ Z_r^{-1}[(\partial_\mu\lambda)^2+\lambda^2]/2] + \delta E\}. \end{gathered} \tag{3.99}$$

Here

$$\begin{gathered} \delta E = \tfrac{1}{2}(\delta(\mu_r,m_\sigma)+(N-1)\delta(\mu_r,m_\phi)+\mu_r^2(\Delta(\mu_r,m_\sigma)+(N-1)\Delta(\mu_r,m_\phi)+c^2)+ \\ +b(\Delta(\mu_r,m_s)+(N-1)\Delta(\mu_r,m_\phi)+c^2-g_r^{-1})+Z_r^{-1}[\delta(1,m_\lambda)+\Delta(1,m_\lambda)+b^2]/2, \end{gathered} \tag{3.100}$$

and

$$\begin{gathered} \delta(m,m') = N_m[(\partial_\mu\varphi)^2] - N_{m'}[(\partial_\mu\varphi)^2] = \int_a \tfrac{d^{d-1}k}{(2\pi)^{d-1}}[\tfrac{w_{m'}(k)^2+k^2}{2w_{m'}(k)} - \tfrac{w_m(k)^2+k^2}{2w_m(k)}], \\ \Delta(m,m') = N_m[\varphi^2] - N_{m'}[\varphi^2] = \int_a \tfrac{d^{d-1}k}{(2\pi)^{d-1}}[\tfrac{1}{2w_{m'}(k)} - \tfrac{1}{2w_m(k)}], \\ w_m(k)^2 = k^2+m^2. \end{gathered} \tag{3.101}$$

Eq.(3.101) can be also written as

$$\begin{gathered} \delta(m,m') = \int_a \tfrac{d^dk}{(2\pi)^d}((k^0)^2+\vec{k}^2)[\tfrac{1}{k^2-m'^2} - \tfrac{1}{k^2-m^2}], \\ \Delta(m,m') = \int_a \tfrac{d^dk}{(2\pi)^d}[\tfrac{1}{k^2-m'^2} - \tfrac{1}{k^2-m^2}]. \end{gathered} \tag{3.102}$$

On the other hand, we treat Eq.(3.99) as Hamiltonian in "correct" form, i.e.

$$\begin{gathered} H = \int dx N_M[\tfrac{1}{2}(\partial_\mu\vec{\phi})^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 + \lambda(\sigma^2+\vec{\phi}^2) + 2c\sigma\lambda \\ +\tfrac{m_\sigma^2}{2}\vec{\phi}^2 + \tfrac{m_\phi^2}{2}\sigma^2 + Z_r^{-1}[(\partial_\mu\lambda)^2+m_\lambda^2\lambda^2]/2 + \delta E], \end{gathered} \tag{3.103}$$

so that identifying corresponding terms we find following ORM equations,

$$\begin{gathered} m_\sigma^2 = m_\phi^2 = m^2 = 2b+\mu_r^2, m_\lambda^2 = 1, \\ (\mu_r^2+2b)c = m^2c = 0, \\ N\Delta(\mu_r,m)+c^2-g_r^{-1} = 0. \end{gathered} \tag{3.104}$$

Eq.(3.104) formally has two solutions

$$\begin{gathered} m^2 = 0, c^2 = g_r^{-1}(1-Ng_r\Delta(\mu_r,0)); \\ c = 0, \Delta(\mu_r,m) = (Ng_r)^{-1}. \end{gathered} \tag{3.105}$$

Eq.(3.105) closely parallels results of the large N limit discussed above in Eq.(3.87) and (3.88). It is interpreted as the phase transition of second kind.

To comply with the notation used in the discussion of the large N limit we recognize in $\Delta(\mu, m)$

$$\Delta_{\mu,\alpha}(m) = D_{\mu,\alpha}(0; m) = \int_{a,\mu} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \tag{3.106}$$

φ-field propagator renormalized at scale μ. We therefore understand μ_r term in Eq.(3.105) as the renormalization scale for field ϕ. Eq.(3.105) describes phase transition at critical coupling

$$g_c = \frac{1}{N\Delta_{\mu,\alpha}(0)}. \tag{3.107}$$

At small coupling ($g_r < g_c$) $m = 0$ and $c^2 = \langle\sigma\rangle^2 = g_r^{-1}(1 - g_r/g_c)$. Above g_c $\langle\sigma\rangle$ vanishes and m^2 is given by

$$N m^{d-2} \int_{a,\mu} \frac{d^d k}{(2\pi)^d} (\frac{1}{k^2 - 1} - \frac{1}{k^2}) = \frac{1}{g_r} - \frac{1}{g_c} \tag{3.108}$$

so that one recovers

$$m^2 \sim g_c^{-2/(d-2)} (1 - g_c/g_r)^{2/(d-2)} = g_c^{-2/\varepsilon} (1 - g_c/g_r)^{2/\varepsilon}. \tag{3.109}$$

In $d = 2$ dimensions the critical temperature vanishes and b has the form

$$b \sim m^2 \sim e^{-\frac{2\pi}{Ng}} \tag{3.110}$$

so that the mass of the field is invisible in all orders of perturbation theory in the coupling constant g.

We see that for $d > 2$ the critical value for the coupling constant depends on the renormalization scale μ, i.e.

$$\begin{gathered} g_c = \frac{1}{N\mu^\varepsilon \int_\alpha \frac{d^d k}{(2\pi)^d} [\frac{1}{k^2} - \frac{1}{k^2 - 1}]} = \bar{g}_c \mu^{-\varepsilon}, \\ m^2 = \mu^2 (1 - \bar{g}_c/(g_r \mu^\varepsilon))^{2/\varepsilon}. \end{gathered} \tag{3.111}$$

$$\bar{g}_c = -2/(N\pi \csc(d\pi/2)) \approx \varepsilon/(\pi N), \varepsilon \to 0. \tag{3.112}$$

This is consistent with dimensional regularization results described in Section 3.3.2 in which dimensionless coupling is introduced by $\bar{g}_r = g_r \mu^{\varepsilon}$, and is relevant to fixing the mass scale in the theory. For small ε, $\bar{g}_c \approx \varepsilon/(\pi N)$ which is qualitatively consistent with 1-loop RG result

$$g_c \approx \frac{2\pi\varepsilon}{N}. \tag{3.113}$$

In special case $d = 2 + 1$ this gives $\bar{g}_c = \frac{2\pi}{N}$.

In regular perturbation theory for σ-model it is common to break O(N) symmetry by external magnetic field, so that the Hamiltonian under question becomes

$$H = \int dx N_{\mu_r}[\frac{1}{2}(\partial_\mu \vec{\varphi})^2 + \lambda(\vec{\varphi}^2 - g_r^{-1}) + \vec{h}\vec{\varphi} + \frac{1}{2} Z_r^{-1}[(\partial_\mu \lambda)^2 + \lambda^2]]. \tag{3.114}$$

In here we assume w.l.o.g. $\vec{\varphi} = (\sigma, \vec{\phi})$ and $\vec{h} = (h, 0)$. After shifting $\sigma \to \sigma + c$ and $\lambda \to \lambda + b$ we obtain

$$\begin{aligned}
&H = \textstyle\int dx N_{\mu_r}[\frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \lambda(\vec{\varphi}^2 + \sigma^2 + 2c\sigma + c^2 - g_r^{-1}) + \\
&+ b(\vec{\varphi}^2 + \sigma^2 + 2c\sigma + c^2 - g_r^{-1}) + hc + h\sigma + Z_r^{-1}[(\partial_\mu \lambda)^2 + \lambda^2]/2] = \\
&= \textstyle\int dx \{N_M[\frac{1}{2}(\partial_\mu \vec{\phi})^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \lambda(\sigma^2 + \vec{\phi}^2) + 2c\lambda\sigma + \\
&+ b\sigma^2 + b\vec{\phi}^2 + (h + 2bc)\sigma + \\
&+ \lambda(\Delta(\mu_r, m_s) + (N-1)\Delta(\mu_r, m_\phi) + c^2 - g_r^{-1} + bZ_r^{-1}) + \\
&\tfrac{1}{2} Z_r^{-1}[(\partial_\mu \lambda)^2 + \lambda^2]] + \delta E\}
\end{aligned} \tag{3.115}$$

$$\begin{aligned}
&\delta E = \tfrac{1}{2}(\delta(\mu_r, m_\sigma) + (N-1)\delta(\mu_r, m_\phi) + \\
&b(\Delta(\mu_r, m_s) + (N-1)\Delta(\mu_r, m_\phi) + c^2 - g_r^{-1}) + \\
&+ hc + \tfrac{1}{2} Z_r^{-1}[\delta(1, m_\lambda) + \Delta(1, m_\lambda) + b^2].
\end{aligned} \tag{3.116}$$

From the "correct" form requirements we obtain ORM equation of state

$$\begin{aligned}
&m^2 = m_\sigma^2 = m_\phi^2 = 2b, m_\lambda^2 = 1, \\
&h + 2bc = 0, \\
&N\Delta_{\alpha,\mu}(m) + c^2 - g_r^{-1} = 0.
\end{aligned} \tag{3.117}$$

Solving Eq.(3.117) we obtain

$$\begin{aligned}
&m^4 = \tfrac{g_r h^2}{1 - N g_r \Delta_{\alpha,\mu}(m)} \\
&c = -\tfrac{h}{m^2}.
\end{aligned} \tag{3.118}$$

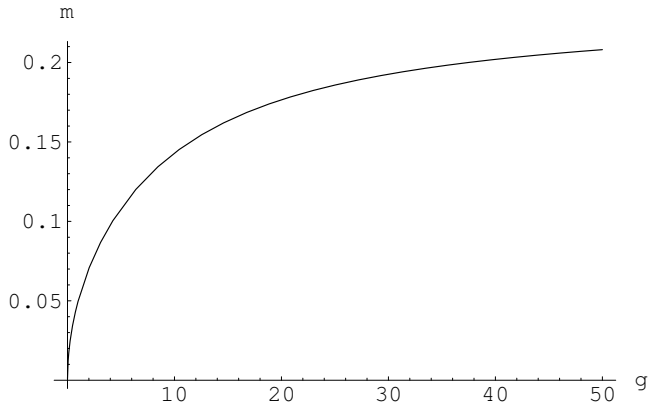

Figure 3.5: Solution for Eq.(3.118) for a choice of $h \neq 0$ and μ.

Indeed, as can be seen from simple analysis, this equation has single continuous solution $m^2(h, g_r; \mu)$ for all g_r so that no phase transition is observed in this case for any values of $h \neq 0$. This is also entirely consistent with our general argument about phase transition disappearance due to introduction of external perturbation in a system undergoing phase transition of the second kind as we will present in Section 3.5.

3.4.2 The ORM in nonlinear σ-model in perturbative form

We now attempt to discuss more controversial, and perhaps more interesting, results or the ORM application in σ-model. In this section we shall investigate nonlinear σ-model in perturbative form, where the constraint is locally resolved in terms of O(N) generators $\vec{\phi}$

$$\sigma = \pm\sqrt{1 - g\vec{\phi}^2}. \tag{3.119}$$

Locally Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}[\partial_\mu\vec{\phi}\partial^\mu\vec{\phi} + g\frac{(\vec{\phi}\partial_\mu\vec{\phi})(\vec{\phi}\partial^\mu\vec{\phi})}{1 - g\vec{\phi}^2}]. \tag{3.120}$$

In canonical variables the Hamiltonian can be obtained in this form

$$\begin{gathered}\vec{\pi} = \partial_0\vec{\phi} + g\vec{\phi}\frac{(\vec{\phi}\partial_0\vec{\phi})}{1-g\vec{\phi}^2}, \\ \mathcal{H} = \vec{\pi}\partial_0\vec{\phi} - \tfrac{1}{2}(\partial_0\vec{\phi})^2 - \tfrac{g}{2}\frac{(\vec{\phi}\partial_0\vec{\phi})^2}{1-g\vec{\phi}^2} + \tfrac{1}{2}[(\vec{\partial}\vec{\phi})^2 + g\frac{(\vec{\phi}\vec{\partial}\vec{\phi})^2}{1-g\vec{\phi}^2}].\end{gathered} \tag{3.121}$$

Relation between $\vec{\pi}$ and $\partial_0\vec{\phi}$ can be explicitly inverted

$$\partial_0\vec{\phi} = \vec{\pi} - g\vec{\phi}(\vec{\phi}\vec{\pi}), \tag{3.122}$$

to produce

$$\mathcal{H} = \frac{1}{2}[\vec{\pi}^2 - g(\vec{\phi}\vec{\pi})^2 + (\vec{\partial}\vec{\phi})^2 + g\frac{(\vec{\phi}\vec{\partial}\vec{\phi})^2}{1-g\vec{\phi}^2}]. \tag{3.123}$$

In the ORM this form is equivalent to

$$\mathcal{H} = \frac{1}{2}[(\partial_\mu\vec{\phi})^2 + g\frac{(\vec{\phi}\vec{\partial}\vec{\phi})^2 - (\vec{\phi}\partial_0\vec{\phi})^2}{1-g\vec{\phi}^2}]. \tag{3.124}$$

According to the general theory, Lagrangian (3.120) can be renormalized by renormalization of the field and coupling constant, so that

$$\mathcal{L} = \frac{1}{2}Z^{-1}_{\alpha,\mu}[\partial_\mu\vec{\phi}_r(\mu)\partial^\mu\vec{\phi}_r(\mu) + g_r(\mu)Z_g^{\alpha,\mu}Z^{-1}_{\alpha,\mu}\frac{(\vec{\phi}_r(\mu)\partial_\nu\vec{\phi}_r(\mu))(\vec{\phi}_r(\mu)\partial^\nu\vec{\phi}_r(\mu))}{1-g_r(\mu)Z_g^{\alpha,\mu}Z^{-1}_{\alpha,\mu}\vec{\phi}_r(\mu)^2}], \tag{3.125}$$

In further we assume that normalization scheme and cutoff scale are fixed so that we have to work with Hamiltonian in the form Eq.(3.124) with $Z_g^{\alpha,\mu}g_r \mapsto g$ and $\vec{\phi} = Z^{-1}_{\alpha,\mu}\vec{\phi}_r$. The Hamiltonian Eq.(3.124) is naturally understood in the sense of its power series expansion, i.e.

$$\mathcal{H} = \frac{1}{2}Z^{-1}[(\partial_\mu\vec{\phi}_r)^2 - \sum_{n=0}^{\infty} g^{n+1}(\vec{\phi}_r\partial_i\vec{\phi}_r)^2(\vec{\phi}_r^2)^n], \tag{3.126}$$

where by $(\vec{\phi}_r\partial_i\vec{\phi}_r)^2$ we shall understand $(\vec{\phi}_r\partial_i\vec{\phi}_r)(\vec{\phi}_r\partial^i\vec{\phi}_r)$, here summation is only over space indexes.

Since in this form $O(N)$ symmetry is explicitly broken, we shall not consider here field shift as one of the degrees of freedom in the ORM. Indeed, if we wished to do so, we would have to keep the measure term in Eq.(3.124). Any shift $\vec{\phi} \to \vec{\phi} + \vec{c}$ then, due to the symmetry, would produce no change in the energy density of the state.

The only degree of freedom that we may employ in the ORM is, thus, the change of mass. This is, of course, consistent with the large N analysis as well that the phase transition in the nonlinear σ-model is related to generators $\vec{\phi}$ acquiring mass.

According to the ORM, we should consider Lagrangian Eq.(3.125) renormalized with respect to a different scale ν and require then that the renormalized Lagrangian has the correct form

$$\begin{array}{c}[(\partial_\mu\vec{\phi}_r)^2+g_r(\nu)\frac{(\vec{\phi}_r\partial_i\vec{\phi}_r)^2}{1-g_r(\nu)\vec{\phi}_r^2}+2h_r(\nu)\sigma_r]= \\ [(\partial_\mu\vec{\phi}_r)^2+g_r(\mu)Z_g(\mu,\nu)\frac{(\vec{\phi}_r\partial_i\vec{\phi}_r)^2}{1-g_r(\mu)Z_g(\mu,\nu)\vec{\phi}_r^2}+2h_r(\mu)Z_h(\mu,\nu)\sigma_r],\end{array} \tag{3.127}$$

where $Z_g(\mu,\nu)$ and $Z_h(\mu,\nu)$ are finite multiplicative renormalizations governed by RG flow. Eq.(3.127) leads to the RGORM problem considered in Section 3.4.3.

Alternatively, we can take first into account the main quantum contributions by performing normal ordering in which case we work with the Hamiltonian

$$\mathcal{H}=\frac{1}{2}Z_\mu^{-1}N_\mu[(\partial_\mu\vec{\phi}_r)^2+\sum_{n=0}^{\infty}g_\mu^{n+1}(\vec{\phi}_r\partial_i\vec{\phi}_r)^2(\vec{\phi}_r^2)^n+\delta E_\mu], \tag{3.128}$$

where Z_μ, $g_\mu=g_rZ_g^\mu Z_\mu^{-1}$, δE_μ are understood as contributions due to the diagrams not regularized by normal ordering (loop diagrams etc). After the change of scale $\mu\to\nu$ the Hamiltonian becomes

$$\mathcal{H}=\frac{1}{2}Z_\nu^{-1}N_\nu[(\partial_\mu\vec{\phi}_r)^2+\sum_{n=0}^{\infty}g_\nu^{n+1}(\vec{\phi}_r\partial_i\vec{\phi}_r)^2(\vec{\phi}_r^2)^n+\delta E_\mu+\Delta E(\mu,\nu)]+\delta\mathcal{H}(\mu,\nu), \tag{3.129}$$

where $\Delta E(\mu,\nu)=\delta E_N(\mu,\nu)+\delta E_\nu-\delta E_\mu$ is the vacuum energy difference consisting from the contributions due to normal ordering δE_N and all other diagrams $\delta E_\nu-\delta E_\mu$. Let us have a closer look at δE_N. After some algebra we find

$$\begin{array}{c}\left\langle\nu|N[(\vec{\phi}_r\partial_i\vec{\phi}_r)^2(\vec{\phi}_r^2)^n]|\nu\right\rangle=\left\langle\nu|N[(\partial_i\phi_r^a\partial^i\phi_r^b)\phi_r^a\phi_r^b(\phi_r^c\phi_r^c)^n]|\nu\right\rangle= \\ =\sum_{a,b}\left\{\begin{array}{c}a\neq b,\left\langle\nu|N'[(\partial_i\phi_r^a\partial^i\phi_r^b)]N[\phi_r^a\phi_r^b(\phi_r^c\phi_r^c)^n]|\nu\right\rangle \\ a=b,\langle\nu|[N'[(\partial_i\phi_r^a\partial^i\phi_r^a)]+d_\mu(\nu)]N[(\phi_r^a)^2(\phi_r^c\phi_r^c)^n]|\nu\right\rangle\end{array}\right.,\end{array} \tag{3.130}$$

where N' represents normal ordering with respect to $|\nu\rangle$ state and N is N_μ normal ordering and we take into account that under normal ordering the order of operators

is not important. For $a \neq b$ we obtain

$$\begin{aligned}\langle\nu|N \quad & [(\partial_i\phi_r^a\partial^i\phi_r^b)\phi_r^a\phi_r^b(\phi_r^c\phi_r^c)^n]|\nu\rangle = \\ & \left\langle \nu \left| N'[(\partial_i\phi_r^a\partial^i\phi_r^b)] \sum_{(c),m\leq n+2} a_{(c)} N'[\phi_r^{c_1}\phi_r^{c_2}\dots\phi_r^{c_{(m)}}] \right| \nu \right\rangle = 0.\end{aligned} \tag{3.131}$$

Since the lowest in fields number term is $\left\langle\nu|N'[(\partial_\mu\phi_r^a\partial^\mu\phi_r^b)]a_{(0)}|\nu\right\rangle = 0$, terms with $a \neq b$ do not contribute in the vacuum energy density.

For $a = b$ we find

$$\begin{aligned}&\langle\nu|N[(\partial_i\phi_r^a\partial^i\phi_r^a)\phi_r^a\phi_r^a(\phi_r^c\phi_r^c)^n]|\nu\rangle = \\ &= \langle\nu|(N'[(\partial_i\phi_r^a\partial^i\phi_r^a)]N[(\phi_r^a)^2(\phi_r^c\phi_r^c)^n]|\nu\rangle + d_\mu(\nu)\,\langle\nu|N[(\phi_r^a)^2(\phi_r^c\phi_r^c)^n]|\nu\rangle\,,\end{aligned} \tag{3.132}$$

and we used notation

$$d_\mu = \left\langle\nu|\partial_i\phi_r\partial^i\phi_r)|\nu\right\rangle = \frac{\delta_\mu - \mu^2\Delta_\mu}{2}.$$

First term in Eq.(3.132) equals zero for the reasons discussed above. Therefore we now need only to compute

$$\left\langle \nu \left| N[(\phi_r^a)^2(\sum_c(\phi_r^c)^2)^n] \right| \nu \right\rangle. \tag{3.133}$$

Note that if $l = N-1$ is the dimension of $\vec{\phi}$ vector then

$$\begin{aligned}\left\langle \nu \left| N[(\phi_r^a)^2(\sum_c(\phi_r^c)^2)^n] \right| \nu \right\rangle &= \left\langle \nu \left| N[(\phi_r^a)^2 \sum_{(c)} \tfrac{n!}{k_1!\dots k_n!}(\phi_r^1)^{2k_1}\dots(\phi_r^l)^{2k_n}] \right| \nu \right\rangle = \\ &= \left\langle \nu \left| N[\sum_{(c)} \tfrac{n!}{k_1!\dots k_n!}(\phi_r^1)^{2k_1}\dots(\phi_r^a)^{2k_a+2}\dots(\phi_r^l)^{2k_n}] \right| \nu \right\rangle.\end{aligned} \tag{3.134}$$

Since ϕ_r^a commute for different a Eq.(3.134) can be rewritten as

$$\begin{aligned}\langle\nu|N \quad & [(\phi_r^a)^2(\sum_c(\phi_r^c)^2)^n]|\nu\rangle = \\ & \left\langle \nu \left| \sum_{(k)} \tfrac{n!}{k_1!\dots k_n!} N[(\phi_r^1)^{2k_1}]\dots N[(\phi_r^a)^{2k_a+2}]\dots N[(\phi_r^l)^{2k_l}] \right| \nu \right\rangle.\end{aligned} \tag{3.135}$$

For each a $N[(\phi_r^a)^{2k}] = \sum\limits_{j=0}^{k} \frac{(2k)!}{(2(k-j))!j!} \left(\frac{\Delta_\mu(\nu)}{2}\right)^j N'[(\phi_r^a)^{2k-2j}]$ and since we are only interested in the term with no $N'[\phi]$ at all, the only contribution is $\frac{(2k)!}{k!}\left(\frac{\Delta_\mu(\nu)}{2}\right)^k$,

$$\left\langle \nu | N[(\phi_r^a)^2 (\sum_c (\phi_r^c)^2)^n] | \nu \right\rangle = \left\langle \nu \left| \sum_{(k)} \frac{n!}{k_1!\ldots k_l!} \frac{(2k_1)!}{k_1!} \left(\frac{\Delta_\mu(\nu_1)}{2}\right)^{k_1} \ldots \frac{(2k_a+2)!}{(k_a+1)!} \left(\frac{\Delta_\mu(\nu_a)}{2}\right)^{k_a+1} \ldots \frac{(2k_n)!}{(k_n)!} \left(\frac{\Delta_\mu(\nu_l)}{2}\right)^{k_l} \right| \nu \right\rangle. \tag{3.136}$$

For the total energy we obtain

$$\begin{aligned} 2Z\delta E_N = \sum_a \big(\delta_\mu(\nu_a) + d_\mu(\nu_a) \sum_{n=0}^{\infty} \tfrac{g^{n+1}}{2^{n+1}} \Delta_\mu(\nu_a) \times \\ \sum_{\substack{k_1+\ldots k_l = n \\ (k)=0..n}} \tfrac{n!}{k_1!\ldots k_l!} \tfrac{(2k_1)!}{k_1!} \Delta_\mu(\nu_1)^{k_1} \ldots \tfrac{(2k_a+2)!}{(k_a+1)!} \Delta_\mu(\nu_a)^{k_a} \ldots \tfrac{(2k_n)!}{(k_n)!} \Delta_\mu(\nu_l)^{k_l} \big), \end{aligned} \tag{3.137}$$

where $Z = Z_\nu/Z_\mu$ is finite field renormalization and $g = g_\nu$ is bare coupling constant at ν-scale.

Mass of the ϕ_a fields comes out from $\frac{\partial \delta E}{\partial \nu_a} = 0$. We also can obtain the mass directly as coefficient at $N'[\phi_a^2]$. Using the same idea as when calculating δE_N, we are only interested in terms in Eq.(3.135) containing one $N'[\phi_c^2]$ and $N'[\phi^0_{b\neq c}]$ fields. In that case we get

$$\begin{cases} a = c, & \sum\limits_{(k)} \frac{n!}{k_1!\ldots k_l!} \frac{(2k_1)!}{k_1!} \left(\frac{\Delta_\mu(\nu_1)}{2}\right)^{k_1} \ldots \frac{(2k_a+2)!}{(k_a)!} \left(\frac{\Delta_\mu(\nu_a)}{2}\right)^{k_a} \ldots \frac{(2k_n)!}{(k_n)!} \left(\frac{\Delta_\mu(\nu_l)}{2}\right)^{k_l} \\ a \neq c, & \sum\limits_{(k)} \frac{n!}{k_1!\ldots k_l!} \frac{(2k_1)!}{k_1!} \left(\frac{\Delta_\mu(\nu_1)}{2}\right)^{k_1} \ldots \frac{(2k_c)!}{(k_c-1)!} \left(\frac{\Delta_\mu(\nu_a)}{2}\right)^{k_c-1} \ldots \\ & \ldots \frac{(2k_a+2)!}{(k_a+1)!} \left(\frac{\Delta_\mu(\nu_a)}{2}\right)^{k_a+1} \ldots \frac{(2k_n)!}{(k_n)!} \left(\frac{\Delta_\mu(\nu_l)}{2}\right)^{k_l} \end{cases} \tag{3.138}$$

so that for $a'_{2,c}$ one obtains

$$\begin{aligned} 2Z a'_{2,c} = \sum_{n=0}^{\infty} \tfrac{g^{n+1}}{2^{n+1}} \sum_{(k)} \tfrac{n!}{k_1!\ldots k_l!} \big[\sum_{a\neq c} d_\mu(\nu_a) \Delta_\mu(\nu_a) \tfrac{(2k_1)!}{k_1!} \Delta_\mu(\nu_1)^{k_1} \ldots \times \\ \times \tfrac{(2k_c)!}{(k_c-1)!} \Delta_\mu(\nu_c)^{k_c-1} \ldots \tfrac{(2k_a+2)!}{(k_a+1)!} \Delta_\mu(\nu_a)^{k_a} \ldots \tfrac{(2k_n)!}{(k_n)!} \Delta_\mu(\nu_l)^{k_l} + \\ d_\mu(\nu_c) \Delta_\mu(\nu_c) \tfrac{(2k_1)!}{k_1!} \Delta_\mu(\nu_1)^{k_1} \ldots \tfrac{(2k_c+2)!}{(k_c)!} \Delta_\mu(\nu_a)^{k_c} \ldots \tfrac{(2k_n)!}{(k_n)!} \Delta_\mu(\nu_l)^{k_l} \big], \end{aligned} \tag{3.139}$$

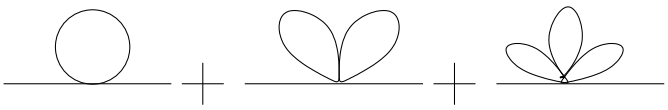

Figure 3.6: Daisy loop-diagrams summation

where summation over index $(k) = (k_1 \dots k_l)$ is in the limits from 0 to n and $k_1 + \dots + k_l = n$ and l is, again, the number of generators in $\vec{\phi}$. This is consistent with δE: $\nu_a^2 = \frac{d}{d\Delta_a}(2Z\delta E_N)$.

Let us now consider specific example when $\vec{\phi}$ field has single-component. In that case the above formulas can be substantially simplified and for the vacuum energy δE_N we obtain

$$2Z\delta E_N = \delta_\mu(\nu) + d_\mu(\nu) \sum_{n=0}^{\infty} \frac{(2n+2)!}{(n+1)!} \left(g \frac{\Delta_\mu(\nu)}{2} \right)^{n+1} . \tag{3.140}$$

In Eq.(3.140) we should emphasize that g, Z, δ_μ, Δ_μ are *finite* bare parameters calculated for fixed renormalization scheme with fixed finite cutoff.

It is easy to see, nonetheless, that summation in Eq.(3.140) $\rho(z) = \sum\limits_{n=1}^{\infty} \frac{(2n)!}{n!} z^n$ diverges as $n \to \infty$ for all $z \neq 0$. We should emphasize that this type of divergence does not come from the UV singularities of the renormalization theory. In terms of perturbation theory normal ordering in Eq.(3.126) equivalent to renormalizing daisy loop diagrams Fig(3.6). Higher n correspond to more loops attached to the vertex in Fig.(3.6), each next loop comes with its own g factor thus making each next loop next order in perturbation theory. From Eq.(3.140) it follows, therefore, that contribution to the vacuum energy of the daisy-loop diagrams in all orders of perturbation theory is infinite for any finite cutoff α and finite bare coupling constant g. This shows that such perturbation series is ill-defined in any finite regularization of the theory. This is rather not surprising result since almost in all field theories the perturbation series are only asymptotically converging, in the best case. Still, this presents substantial difficulty in application of the ORM to nonlinear σ-model in this form since one have

to deal with all orders of perturbation theory at once. One, however, may still assume some validity of the perturbation theory in every finite order, so that we will now try to make sense of our result in perturbation sense and investigate σ-model in the lowest order in g approximation

In this approximation Eq.(3.137) reads

$$2Z\delta E_N = \sum_a \left[\delta_\mu(M_a) - g d_\mu(M_a)\Delta_\mu(M_a)\right]. \tag{3.141}$$

If we plot the vacuum energy density vs $t = M/\mu$,

$$2Z\delta E_N(t) = N\mu^d \delta_1(t) + g\mu^{2d-2} d_1(t)\Delta_1(t), \tag{3.142}$$

for $d = 3$ we obtain

$$2Z_{\alpha,\mu}\delta E_N(t) \sim \mu^3(t^3-1) + \mu^4\tilde{g}\left(\frac{t^3-1}{3} - (1-t)\right)(1-t). \tag{3.143}$$

It can be observed that solution to the ORM equations in this case describe unstable point of the maximum energy; moreover, energy $E(t)$ decreases to $-\infty$ as $t \to \infty$ [see Fig.(3.9)]. It can be see that this situation holds for any dimension d since large t behavior of Eq.(3.142) is

$$2Z\delta E_N \sim \mu^d t^d - \mu^{2d-2} t^{2d-2}.$$

This conclusion is confusing once one realizes that the ORM can be related to variational principle. In fact, if $\delta E_N(t)$ is viewed as $\langle t|H|t\rangle$, Eq.(3.142) implies that for any $-M < 0$ there exist a state $|t\rangle$ such that $\langle t|H|t\rangle < -M$ so that the spectrum of the theory is unbounded from below. One, however, should be careful about UV-divergences, infinite volume of the field theory and other QFT-effects which may render these implications meaningless in strict variational treatment[1].

[1]Note that this question can be also possibly related to instabilities discovered in RG β-function in [124–127]

3.4.3 The RGORM in the nonlinear σ-model

Regarding the kind of infinities we have encountered in the above treatment with the standard formulation of the ORM, we now would like to use RG form of the ORM (RGORM) to investigate the phase structure of nonlinear σ-model. Since the only degree of freedom that is left for the ORM in nonlinear σ-model is mass rescaling, we are confined to the case of RGORM already discussed in the Section 3.2, i.e. solution of the equation

$$\begin{gathered} \frac{m^2(\mu)}{\mu^2} = \frac{m^2(\nu)}{\nu^2}, \\ \mu\frac{dg}{d\mu} = \beta(g), \frac{\mu}{m^2}\frac{dm^2}{d\mu} = \rho(g). \end{gathered} \tag{3.144}$$

As was discussed in Section 3.2, two phases are related to nontrivial solution of

$$F(g) - F(g_0) = \int_{g_0}^{g} dx \frac{2-\rho(x)}{\beta(x)} = 0. \tag{3.145}$$

In 1-loop RG functions for nonlinear σ-model are known to be, see Section 3.3.2,

$$\begin{gathered} \beta(x) = \varepsilon x - (N-2)N_d x^2, \\ \rho(x) = \frac{3-n}{2} N_d x, N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}. \end{gathered} \tag{3.146}$$

so that after integration in Eq.(3.145) we obtain

$$F(g) = \frac{2\ln g}{\varepsilon} + \left(\frac{N-1}{2(N-2)} - \frac{\varepsilon+2}{\varepsilon}\right)\ln((N-2)N_d g - \varepsilon). \tag{3.147}$$

Typical behavior of $F(g)$ is shown in Fig.(3.7). As is expected from the general case, $F(g)$ has pole at $g = g_c$ - fixed point of RG flow which is also the critical point. For $g_0 < g_c$ the only other solution is on the right branch of the graph, so that $g(g_0)$, satisfying Eq.(3.145), is always greater than g_0. Once $g_0 < g_c$, the corresponding phase has smaller coupling and therefore is physically preferred. We conclude therefore that for $g_0 < g_c$ no alternative physical solution exist and only trivial solution $t = 1$ should be selected. Once $g_0 > g_c$ the system exists in the phase $g(g_0) < g_0$ with $t \neq 0$ and thus $\langle\phi^2\rangle \sim \Delta(t) \neq 0$.

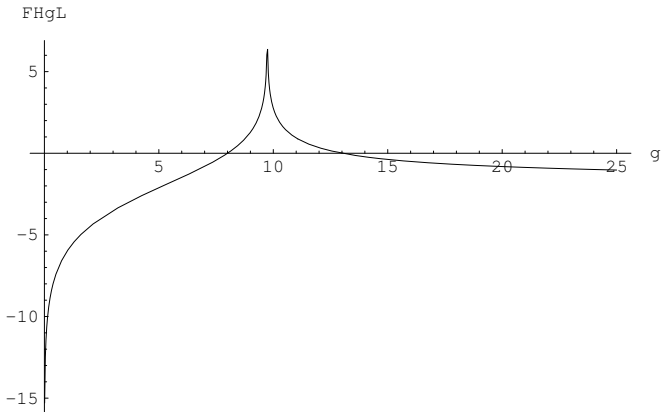

Figure 3.7: F(g) function for RGORM in nonlinear σ-model.

Phase in strong coupling regime is duality related with weakly coupled phase by

$$g \sim (g_0/g_c)^\gamma, \gamma = -\frac{\varepsilon(N-3)}{4(N-2)} < 0, N > 3. \tag{3.148}$$

For $N = 3$ we recover interesting effect, namely $F(g) \to Const$, $g \to \infty$, so that $g = \infty$ is projected to $g^* \approx 5.861$ in $d = 3$ and $g^* \approx 5.056$ in $d = 4$ [see Fig.(3.7)]. For $g < g^*$ no second solution ever exists for Eq.(3.145).

When the best known approximation for RG functions is taken into account, however, we observe that the answer for $N = 3$ changes. In particular $F(g) \to -\infty$ rather slowly, but yet two solution exists for all values of g. Furthermore, γ is also somewhat different from 1-loop calculations. We found

$$\gamma(d=3) = \begin{cases} -0.326, N = 3 \\ -0.445, N = 4 \end{cases} \tag{3.149}$$

and

$$\gamma(d=4) = \begin{cases} -0.652, N = 3 \\ -0.990, N = 4 \end{cases} \tag{3.150}$$

approximately 2 times large than $\gamma(d = 3)$ in consistency with proportionality to $\varepsilon = d - 2$ as in Eq.(3.148). The fixed point of the RG flow is also moved significantly

to its smaller value. Qualitatively, however, the phase relations is the same with the lowest order RG functions.

3.5 The ORM in Scalar ϕ^4 Theory with Magnetic Field

In this section we examine capability of the ORM to properly describe critical phenomena by considering ϕ^4 theory with magnetic field defined by

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{4}\phi^4 + B\phi \tag{3.151}$$

in 1+1 dimension for which many exact results are known [41, 128].

As was discussed in Section 3.1, in the case $B = 0$ the ORM equations of state describe S- and BS-phases. As can be seen from Fig.(3.10) two critical points are present in ORM solution: $g_c/m^2 = \frac{2\pi t_c}{3}, t_c = \exp(1+W(2/e))^2$, such that BS solution does not exist for $g/m^2 < g_c/m^2$, and $g^*/m^2 \approx 10.2108\ldots$, such that the energy of the BS phase $E_{BS} > E_S$ for $g_c/m^2 < g/m^2 < g^*/m^2$. We shall also note that, as can be seen from Fig.(3.11), the effective dimensionless coupling $G = g/M^2$ after the transformation to the new mass M and new quantization point b is small whenever the original coupling was large. Therefore BS-phase obtained with the ORM represents the strong coupling regime of the original theory as a weak coupling regime of the new theory such that the ordinary perturbation theory would still be successful. According to the ORM, we observe that for weak coupling $g/m^2 < g_c/m^2 \approx 9.0459$ only S-phase $\langle\varphi\rangle = 0$ exists. When coupling becomes sufficiently strong ($g/m^2 \geq g_c/m^2$) the BS-phase develops with $\langle\varphi\rangle \neq 0$ but not until $g/m^2 \geq g^*/m^2$ that BS-phase is energetically favorable and is physically realized as the ground state. Thus, at g^*/m^2 the phase transition occurs between S and BS-phases. There exists no continuous connection between the phases and the order parameter $b(g) = \langle\varphi\rangle$ is discontinuous at g^* so that the phase transition is of the first kind according to the ORM.

[2] $W(x)$ is the product log function $W(x)e^{W(x)} = x$

For the purposes of the upcoming discussion let us mention also that one may use somewhat different interpretation of the ORM equation of state. Consider the vacuum energy density (3.18) treating it as a free energy of a system described with two generalized coordinates b and t. Note that (in energy units m^2) Eq.(3.18) can be written in a more familiar form as

$$\varepsilon(b,t) = gU - S \tag{3.152}$$

with "potential" and "entropy" defined by

$$\begin{array}{c} S(b,t) = \frac{\ln(t)+1-t}{8\pi} - \frac{b^2}{2}; \\ U(b,t) = \frac{b^4}{4} - \frac{3b^2\ln(t)}{8\pi} + \frac{3\ln^2(t)}{64\pi^2}. \end{array} \tag{3.153}$$

It is easy to check that $S(b,t)$ is monotonous for $b > 0$ and $t > 1$, symmetric with respect to $b \to -b$ and have maximum at $(b = 0, t = 1)$. Potential $U(b,t)$ is not monotonous in either of its arguments but is symmetric with respect to $b \to -b$. For a quasi-stable state (b^*, t^*) the stationary condition on $\varepsilon(b,t)$ results in

$$\delta\varepsilon = \left(b^* + gb^{*3} - \frac{3gb^*}{4\pi}\ln(t^*)\right)\delta b + \left(-\frac{1}{8\pi t^*} + \frac{1}{8\pi} - \frac{3b^{*2}}{8\pi t^*}g + \frac{3g\ln(t^*)}{32\pi^2 t^*}\right)\delta t = 0, \tag{3.154}$$

which is equivalent to Eq.(3.21). The phase diagram of the system, plotting quasi-stable states vs. one of the parameters, e.g. b, has the form as in Fig.(3.17a).

Note that the energy of the quasi-stable states, plotted along the stability curve, looks like in Fig.(3.17b) so that the states along this curve are energetically not favored as long as $g < g^*$. At $g = g^*$ a sudden jump from the trivial branch $b = 0$ to the nontrivial branch $b \neq 0$ occurs representing 1st order phase transition as we have said previously. The phase transition proceeds through the spontaneous symmetry breaking mechanism.

We now consider how the presence of the magnetic field in Eq.(3.151) affects the equation of state Eq.(3.20). Along the line of the previous discussion, the free energy density (in units m^2) is given by

$$\begin{array}{c} \varepsilon(b,t) = \left(\frac{b^2}{2} - \frac{\ln(t)}{8\pi} + \frac{t-1}{8\pi}\right) + \\ \frac{g}{4}\left(b^4 - \frac{3b^2}{2\pi}\ln(t) + \frac{3}{16\pi^2}\ln^2(t)\right) + Bb. \end{array} \tag{3.155}$$

This leads to the following equation of state derived from the stationary conditions on $\varepsilon(b,t)$:

$$\begin{cases} b + gb^3 - \frac{3bg}{4\pi}\ln(t) + B = 0, \\ \frac{1-t}{2} + \frac{3}{2}gb^2 - \frac{3g}{8\pi}\ln(t) = 0. \end{cases} \tag{3.156}$$

It can be further transformed to 3^d-order equation for vacuum condensation density b:

$$\begin{cases} g = \frac{4\pi}{3}\frac{t-1}{4\pi b^2 - \ln(t)}, \\ \frac{4\pi}{3}(t+2)b^3 - t\ln(t)b + B(4\pi b^2 - \ln(t)) = 0. \end{cases} \tag{3.157}$$

Analytical solution to Eq.(3.157), although possible, is rather cumbersome and is not particularly illuminating. For that reason we studied Eq.(3.157) numerically.

If the external magnetic field is strong, we no longer observe any phase transition. The trivial solution s_0 in this case does not exist and two nontrivial solutions $s_\pm$ are completely decoupled and deformed by the external perturbation (Fig.(3.12)). When the external perturbation decreases, however, an interesting phenomenon occurs. One can clearly observe in that case that the "trivial" solution s_0 for weak perturbation emerges and gets continuously connected with one of the nontrivial solutions (e.g. s_-), while the nontrivial branches s_+ and s_- get decoupled slightly. As the external field B gets weaker, the nontrivial solutions $s_\pm$ approach their non-perturbed analogs closer and closer and eventually a loop is formed in $g-\varepsilon$ plane Fig.(3.13). At point g^*, where the loop is formed, we eyewitness phase transition of the first kind in which both vacuum condensation b and effective heat capacity $c_g = \frac{d\varepsilon}{dg}$ are discontinuous. Note, however, that this phase transition occurs entirely within Broken Symmetry phase without any symmetry rearrangement. This result of the ORM is in contradiction with Grifith-Simon theorem [128] claiming the absence of the phase transition in φ^4-theory with external field $B \neq 0$. We attribute this to approximate nature of the ORM.

In fact, in the phase transition region the coupling constant is quite large both in Symmetric and Broken Symmetry phases so that the perturbative corrections may substantially affect the phase transition region in the ORM. To study the significance

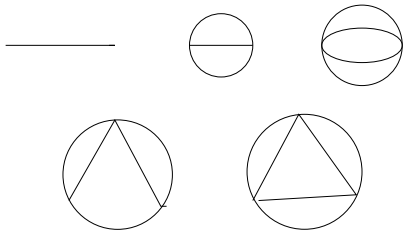

Figure 3.8: The diagrams that contribute to O(g^2) and O(g^3) orders perturbative corrections to vacuum energy in BS phase of 1+1 ϕ^4 theory.

of these effects we calculated perturbative corrections to the vacuum energy in S and BS phases up to g^3 order. We take the Hamiltonian in the most general form as

$$\mathcal{H} = \frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{2}\varphi^2 + a\varphi + b\varphi^3 + \frac{g}{4}\varphi^4. \tag{3.158}$$

Then the diagrams that contribute to $O(g^2)$ and $O(g^3)$ orders are Fig.(3.8). After performing calculation we obtain

$$\delta E = -\frac{a^2}{2m^2} - 0.04452\frac{b^2}{m^2} - 0.00318\frac{g^2}{m^2} + 0.01998\frac{b^2 g}{m^4} + 0.0006507\frac{g^3}{m^4} + \dots \tag{3.159}$$

The perturbative corrections are indeed large for S-phase once $g/m^2 > 5$ and in BS-phase in the near-critical region, so that the ORM conclusion about the nature and the point of the phase transition may be, indeed, approximate [see Fig.(3.15)]. We observe, though, that the corrections in S-phase for small g and in BS phase for large g are small, that is consistent with the transformation of the strong-coupled regime into the weak-coupled BS-phase. Away from the transition point the conclusions of the ORM are increasingly reliable. To improve our perturbative series Eq.(3.159) we further perform a Borel summation,

$$-\alpha g^2 + \beta g^3 \to \int_0^\infty e^{-t} \frac{1 + \frac{\beta}{3\alpha} gt}{1 + \frac{\beta}{3\alpha} gt + \frac{\alpha}{2}(gt)^2} - 1, \tag{3.160}$$

in which case for the vacuum energy density we observe Fig.(3.10). Perturbative corrections to the vacuum energy indeed are significant and shift the transition point

from $g/m^2 \approx 10$ to $g/m^2 \approx 12$. The phase transition is still of the first kind since the energy of the ground state has a cusp at this point. On the other side, as one can see in Fig.(3.10), the magnitude of the jump at g^* is decreased after accounting for first few perturbative corrections. In this sense the perturbative corrections decrease discontinuity and make the phase transition, in a sense, closer to second kind.

The calculations of the vacuum energy density for $B \neq 0$ leads to similar picture. S-phase and BS-phase are yet continuously connected, although through a quite remote section of the graph [not shown in Figs.(3.10) and (3.16)]. The loop, mentioned in the original ORM, still exists and indicates first order phase transition in the region of small perturbations $B/m^2 \leq 0.1$ Fig.(3.16).

As we shall show now, the observed persistence of the 1st order phase transition in the region of small perturbation has its roots in much more general properties of the systems undergoing 1st order phase transition and, thus, is originating from the transition kind itself obtained by the ORM in $B = 0$ case. For that we shall use our analogy with thermodynamics to demonstrate that the sort of phenomenon we observed above is typical for systems undergoing 1st order phase transition.

We consider a 1-dimensional effective system that can be obtained from the initial ORM equation of state Eq.(3.20) by eliminating t-variable from $\frac{\delta\varepsilon}{\delta t} = 0$. We are looking therefore at an effective system described by a single generalized coordinate b with the free energy given by Eq.(3.152). According to thermodynamical viewpoint, possible stable states may exist whenever variation $\delta\varepsilon = 0$. If the external field B is absent, this results in the following

$$\begin{gathered} \delta\varepsilon = 0 = -\delta S + g\delta U \Rightarrow \\ g = \tfrac{\delta S}{\delta U}, \end{gathered} \tag{3.161}$$

where we denoted with $\delta S, \delta U$ derivatives dS/db and dU/db respectively.

Some general properties of equation Eq.(3.161) are appropriate to note. $S(b)$ and $U(b)$ are even functions of b so that $\delta S = \delta U = 0$ if $b = 0$. $b = 0$, therefore, is always a solution of Eq.(3.161) and apparently represents the trivial solution true for all couplings g. For all other (b, g) the diagram of possible stable states qualitatively

looks like in Fig.(3.17a). All points in this diagram are quasi-stable states and their stability is defined by whether $\varepsilon(g, b(g)) < \varepsilon(g, 0)$. Furthermore, the energy of the quasi-stable states, plotted along the curve in Fig.(3.17a), qualitatively behaves like in Fig.(3.17b) so that all states $g < g^*$ are only quasi-stable. Actual phase transition occurs at $g = g^*$ for which $\varepsilon(g^*, b(g^*)) = \varepsilon(g^*, 0)$.

In the presence of the external field B the free energy is modified by perturbation term of the form $B(b)$, notice that the perturbation is odd function in b: $B(b) = -B(-b)$. In this case the condition for stability of state looks like

$$\begin{aligned} 0 &= g\delta U - \delta S + \delta B, \\ g &= \left(\tfrac{\delta S}{\delta U}\right) - \left(\tfrac{\delta B}{\delta U}\right). \end{aligned} \tag{3.162}$$

The diagram at Fig.(3.17a) therefore should be modified by adding $-\frac{\delta B}{\delta U}$ term due to the perturbation. Since $B(b)$ is odd, this contribution has $1/b$-type singular behavior as shown Fig.(3.18a). The result qualitatively looks like in the right panel of Fig.(3.18b). As was already observed, the nontrivial solutions are now separated by the perturbation and the trivial solution got shifted and continuously connected with one of the nontrivial solutions. For example, the left branch in Fig.(3.18b) consists essentially from two regions: one part approaching the original trivial solution $\varepsilon \approx 0$ and the other part approaching the negative branch of the original nontrivial solution $\varepsilon = \varepsilon(g, b(g))$.

To make an inference about how the transition point is affected, we should notice first that, if B is small enough, the energy of the states (g, b) is modified just slightly so that Fig.(3.17b) would be altered very little. In turn that means that for sufficiently small perturbation, when branches of the perturbed solution are sufficiently close to nonperturbed solutions, there yet exists such $g^*(B)$ that $\varepsilon(g^*(B), b_1) = \varepsilon(g^*(B), b_2)$. Therefore, first order phase transition still persists close to g^* for sufficiently small perturbations.

We should emphasize that in the above discussion we didn't use the knowledge of the specific form of $\varepsilon(b)$. Instead, our reasoning was based on fairly general assumptions and properties of $\varepsilon(b)$. Essentially, our conclusions depend only on the following

points:

1. system itself is $b \to -b$ invariant and $U(b)$ and $S(b)$ are even functions of b;

2. nontrivial solution of Eq.(3.161) exists for all b;

3. nonperturbed system is in its trivial phase for $g < g^*$ and in nontrivial phase for $g > g^*$ so that the energy of the nontrivial solutions is greater than the energy of the original phase for $g < g^*$.

Under these quite general assumptions the phase transition of the first kind should persist even when small asymmetric perturbation is added to the system.

For comparison let us consider in a similar fashion a system that undergoes 2nd order phase transition. We still assume that the free energy of the system is written as Eq.(3.152) and the system is invariant under $b \to -b$ transformation. In this case the stability diagram qualitatively looks like in Fig.(3.19a), with nontrivial phase continuously connected with the trivial phase $b = 0$ at some g^* (e.g. the Ising Model and nonlinear σ-model). Also, the energy of the states, plotted along the stability curve, should be always smaller than the corresponding energy of the trivial phase. In the presence of the external perturbation this diagram shall be modified in the same manner as we did for Fig.(3.17a).

Because for $g > g^*$ the energy of the trivial solution was always greater than the energy of the nontrivial solution, differently from Fig.(3.18b) now there may not exist two points on the curve with the energy[3]. Since also now the trivial and the nontrivial solutions are connected smoothly due to the perturbation, we shall conclude that there is no phase transition in this case consistent with our earlier experience in nonlinear σ-model.

From a fairly general viewpoint we have shown therefore that 1st order phase transition still remains when a sufficiently small perturbation is added into the system.

[3]We also imply that the perturbation as function of b is monotonous as is for Bb case. In general, if this is not true, the conclusion may change.

This is different from 2nd order phase transition which is completely destroyed by whatever small external perturbations.

3.6 ORM as a General Field-Theoretic Technique

3.6.1 Nonperturbative interpretation of ORM

As we have seen in the previous sections, applications of ORM lead to interesting and beneficial results including inference about the phase structure of models as well as duality and explicit form of the BS effective Hamiltonian. At the same time in the form ORM is originally formulated by G.Efimov [96] the method leads to upsetting inconsistency even in the simplest case of ϕ^4 scalar field theory in 1+1 dimension and is most obviously is an approximate method. If one was to improve the ORM, the question needs to be answered what underlying nonperturbative principle behind the ORM shall be?

Before attempting discussion of this question, let us emphasize that the ORM as presented in Ref. [96] is formulated in the settings of low-dimensional super-renormalizable field theories and has Hamiltonian formulation and normal ordering as its central ingredients. In that sense the ORM, as suggested in Ref. [96], is limited to the cases considered therein and does not contain an explicit directions as to how these examples can be extended, generalized or improved. It is essentially our task now to gain some insight into possible nonperturbative significance of the ORM and to suggest a feasible improvement or extension of this approach. Such extension is thought to reproduce the ORM as a partial case and to allow introduction of a systematic procedure with which the ORM results can be consistently improved.

In fact, two such ideas had been presented already in Ref. [96]. One such principle was put forward when the authors introduced, along with a number of super-renormalizable theories, the treatment of ϕ^4 in 3+1 dimension and defined RGORM which consists in the following prescription:

- perform canonical transformation on the original renormalized Hamiltonian, e.g. field shift;
- change the renormalization scale $\mu \to \nu$ using RG flow;
- require correct form from this new Hamiltonian, i.e. renormalized coefficient of ϕ^1 is zero and renormalized coefficient of ϕ^2 is $tm^2/2$ accompanied with OR-scale change condition $\nu/\mu = t$.

In this formulation ORM equations for ϕ^4 theory read, e.g.

$$\begin{aligned} m^2(\mu t) + 3g(\mu t)b^2 - \mu^2(\mu)t^2 = 0, \\ b(m^2(\mu t) + g(\mu t)b^2) = 0. \end{aligned} \tag{3.163}$$

This formulation is easily generalizable to nonperturbative and higher dimensional examples by assuming that one uses full RG equations and RG functions can be found with sufficient accuracy. At the same time this formulation has few problems of conceptual and practical character.

One conceptual problem is that the authors of Ref. [96] did not specify what exactly is meant by Renormalization Group in the context of the ORM. If what is usually refereed to as Renormalization Group, i.e. change of mass and coupling with the change of the external momentum scale at which these parameters are observed, is used in the ORM, one encounters upsetting incompatibility. Going back to the examples of 2-dimensional ϕ^4 field theory, considered in Ref. [96], in 1-loop order it is known that the self-energy diagram Fig.(3.1) does not depend on the external momentum and exactly cancels by the counter-term. In fact very often this diagram is not even present in the perturbative treatments for the above reason. Thus, at this level there is no contribution to RG equations and to RGORM. At the same time diagram Fig.(3.1) plays central role in the original treatment carried out in [96].

Furthermore, it was not made clear in Ref. [96] whether RGORM should take into account only infinite renormalizations or also finite renormalizations. In the examples considered by the authors typically only infinite renormalizations had been kept in which case $m^2(\mu)$ corresponds to a finite bare mass parameter of the theory and $m_r^2 \neq m^2$. In this case, however, the significance of ORM "correct" form condition

becomes obscure since there are many ways to define such finite bare mass parameter in practically any QFT.

It is also not clear whether the RG of the original theory or of the transformed theory should be used in the ORM-change of scale $\mu \to \nu$. While the infinite pieces of RG equations are identical, as follows from general requirements of renormalizability, the finite pieces are different and introduce additional ambiguity in the ORM procedure. In their original work, the authors of Ref. [96] have avoided this ambiguity by concentrating only on the infinite renormalizations.

The significance of the ORM-scale change $\mu \to \nu$ also is not clear. The original argumentation of the authors that the same renormalization scheme should be retained can be accepted for order-by-order perturbative calculations, in which case it is equivalent to using the same prescription in computing each next order approximation. However, in the case of calculations in two different theories related only via a variable change this condition - that the same renormalization prescription should be used in either of them - lacks such "obvious" justification.

In the spirit of Section 3.2 we may understand the above ambiguities in the sense of the canonical transformation performed on the bare Hamiltonian with following requirement that the transformed Hamiltonian has correct form, i.e.

$$\mathcal{H}(\phi,\pi)+\delta\mathcal{H}(\phi,\pi) \quad \underset{\phi\to\Phi+b}{\longrightarrow} \quad \mathcal{H}'(\Phi,\Pi)+\delta\mathcal{H}'(\Phi,\Pi) \tag{3.164}$$

with the counter-terms $\delta\mathcal{H}'$ correctly corresponding to the transformed interaction Hamiltonian and new value of the mass, as used in the perturbative propagator. On the example of ϕ^4 theory the condition of invariability of the Hamiltonian and of the

correct form of the Hamiltonian results in

$$\begin{aligned}
&\zeta Z(m,g;\Lambda) = Z'(M,H,G;\Lambda)\\
&\zeta\delta a_1(m,g;\Lambda) + \zeta^2(m^2+\delta m^2(m,g;\Lambda) + \zeta^4(g+\delta g(m,g;\Lambda)))b^3 = \delta a_1'(M,H,G;\Lambda)\\
&\zeta^2(m^2+\delta m^2(m,g;\Lambda)) + 3\zeta^4(g+\delta g(m,g;\Lambda))b^2 = M^2+\delta M^2(M,H,G;\Lambda)\\
&\zeta^4(g+\delta g(m,g;\Lambda))b = H+\delta H(M,H,G;\Lambda)\\
&\zeta^4(g+\delta g(m,g;\Lambda)) = G+\delta G(M,H,G;\Lambda).
\end{aligned} \tag{3.165}$$

where field rescaling ζ, field shift b, final mass M and final couplings H and G are all parameters of the new Hamiltonian to be found. Eq.(3.165) defines one extension of the ORM equations. It also gives meaning to the Renormalization Group as used in the RGORM. Allegedly, change of the mass in the perturbative propagator results in the change of the counter-terms which is described by a RG-like transformation (3.165). We have observed that this transformation coincides with RG-equations for dimensional on-shell regularization where mass scale μ is *identified* with the on-shell particle mass m (and not the renormalization momentum scale). However, the question whether this correspondence will hold in other renormalization prescriptions (e.g. Pauli-Villars renormalization with subtraction at $p=0$ as we commonly used) is of course an open question.

Still, extension of the ORM along this line runs into practical difficulties. In Eq.(3.165) it should be remembered that the infinite and the finite terms should be distinguished. Specifically, requirement that the cut-off scale Λ consistently cancels in (3.165) leads to condition $\zeta^2 g = G$, $H = bG$ and leaves us with three equations and two unknowns

$$\begin{aligned}
&\zeta\delta a_1(m,g;\Lambda) + \zeta^2(m^2+\delta m^2(m,g;\Lambda) + \zeta^4(g+\delta g(m,g;\Lambda)))b^3 = \delta a_1'(M,bG,G;\Lambda)\\
&\zeta^2(m^2+\delta m^2(m,g;\Lambda)) + 3\zeta^4(g+\delta g(m,g;\Lambda))b^2 = M^2+\delta M^2(M,H,G;\Lambda)\\
&\zeta^4(g+\delta g(m,g;\Lambda)) = G+\delta G(M,bG,G;\Lambda).
\end{aligned} \tag{3.166}$$

Note that ζ is uniquely fixed by M and field renormalization condition $\zeta Z(m,g;\Lambda) = Z'(M,bG,G;\Lambda)$. This system is overloaded and is likely to not have a solution different

from the trivial one. The problem arises from different structure of the RG-flow in S- and BS-phases of the theory when the finite pieces are accounted for. If only infinite renormalizations were retained, no such problem would have appeared. Finally, even if Eq.(3.163) are faithfully taken without regard to the above issues, then the ORM relies completely on the knowledge of the RG functions. The only advantage in its use would come from the ability to have the BS Hamiltonian in an explicit form.

Another extension of the ORM had also been discussed in Ref. [96] in the language of Path Integrals (PI). Consider, e.g., PI expression for the generating functional

$$Z[g] = \int \mathcal{D}\phi e^{-\int dx \mathcal{L}(\partial\phi,\phi,x)}. \tag{3.167}$$

This integral is often understood as an integral with Gaussian measure by writing

$$Z[g] = \int d\mu[\phi] e^{-\int dx :g\mathcal{L}_I(\phi):} \tag{3.168}$$

where the gaussian measure is defined as the "kinetic" term of the Lagrangian normalized so as $\int d\mu[\phi] = 1$,

$$d\mu[\phi] = \frac{1}{\sqrt{\det D}} e^{-\frac{1}{2}\phi D^{-1}\phi}, \tag{3.169}$$

and the integrand is written in so called form normal-ordered with respect to gaussian measure $d\mu[\phi]$. Here D is quadratic differential operator that defines measure in the PI and usually is associated with kinetic term in the Lagrangian. It is conjectured in Ref. [96] that by writing PI in normal-ordered form one takes into account main quantum contributions to the vacuum state. Then, for small values of coupling constant the integral (3.168) can be computed perturbatively order by order in g. Using observations of duality, characteristic to the ORM solutions, one suggests that the ORM can be used to redefine the PI for large values of g in a different form with different gaussian measure $d\mu'[\phi]$ which would absorb main contributions and represent large g functional $Z[J]$ in a perturbative form.

For that Eq.(3.168) can be rewritten using transformation of the field shift and different gaussian measure defined by differential operator D'^{-1},

$$Z[g] = e^{-E_0} \int d\mu'[\Phi] e^{-\int dx :g\mathcal{L}'(\Phi):}. \tag{3.170}$$

As Ref. [96] claims, the system under consideration should be near its equilibrium point so that any linear terms for variable ϕ should be absent and the quadratic configuration $\sim \phi^2$, which determines the Gaussian character of the equilibrium point, should be all concentrated in the gaussian measure $d\mu'[\phi]$. By this the requirement of the "correct" form is introduced which imposes constraints on b and D' such that no linear or quadratic terms are left in $\mathcal{L}'(\Phi)$ after the transformation. Then

$$E_0 = \frac{1}{2}\ln\det\frac{D'}{D} - \frac{1}{2}bD^{-1}b + \frac{1}{2}(1-\frac{D'}{D}) + g\mathcal{L}_I(b). \tag{3.171}$$

It was also shown in Ref. [96] that the ORM "correct" form requirement corresponds to minimization of the 0-order vacuum energy density (3.171). This argument, however, has difficulty of including nontrivial renormalizations. Specifically, if infinite self-energy terms are present after normal ordering, then definition of the gaussian measure is ambiguous - should it be the bare mass, mass used in perturbative propagator or renormalized mass - that defines the "correct" $d\mu[\phi]$? The situation is worsen since, formally, one can use any bare mass for the gaussian measure as long as proper counter-term is introduced. The above argumentation to define the "correct" form for PI therefore is generally not applicable except for a few theories completely renormalized by normal ordering.

In this form, nonetheless, the ORM is seen conceptually as the search for canonical transformation related, yet equivalent, representations of the original model. One views ORM-phases as equivalent ways to represent the original theory in terms of quasi-particle degrees of freedom. Duality provides second ingredient to this interpretation - whenever quasi-particle representation is weakly-interacting it gives possibility to describe strong-interacting regime of the original model in terms of a quasi-particle perturbation theory. Even if the Hamiltonian is not completely diagonalized in this transformation, one can account for perturbative corrections to the vacuum energy in the original or the quasi-particle representations of the theory, as we did in Section 3.5, and in principle can obtain exact an answer.

One may extend the requirement of the correct form of the Hamiltonian after

the canonical transformation is applied to the bare Hamiltonian as in Eq.(3.165). As we have already seen, an extension of the ORM along this way may encounter difficulty at higher orders due to inconsistencies between finite peaces of RG in the original and canonically transformed theories leading to impossibility to balance all couplings in the Hamiltonian with only two adjustable parameters. The other way is to abandon the "correct" form requirement altogether, since if two representations of the theory are canonical transformation related, the calculations carried out in any one of them should be completely equivalent, thus, undermining significance of having Hamiltonian in *any special form*. In this sense one should just account for higher perturbative contributions to the vacuum energy on top of any ORM solution and require that the solution with the lowest energy, after higher order corrections are accounted for, wins. In principle, such procedure can be systematically implemented by accounting for higher and higher perturbative corrections to the vacuum energy.

This leads to the following problem. If exactly calculated, the vacuum energy should not change after the canonical transformation. Really, the ORM-transformation is merely an attempt to describe the original theory using different mass and a shifted field variable and, strictly speaking, should produce exactly the same final answers for all quantities in the theory. This point is most vividly seen in PI formalism where the vacuum energy is given by

$$W = i \ln \int \mathcal{D}\phi e^{i \int dx (\mathcal{L}_0(\phi) - \mathcal{L}_I(\phi))}. \tag{3.172}$$

The ORM transformation is relevant to change of variable $\phi \to b + \phi$ and splitting of the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ into different parts

$$W = i \ln \int \mathcal{D}\phi e^{i \int dx (\mathcal{L}'_0(\phi+b) - \mathcal{L}'_I(\phi+b))}. \tag{3.173}$$

Obviously such transformation produces no change in the final answer W. Even if we view the ORM in the sense of a "better canonical representation" of the theory, one need to provide a good definition of what such better representation actually is in order to suggest extensions of the ORM to higher orders. As we have seen above,

original "correct" form requirements loose much of applicability if more complex renormalizations are needed to render field theory finite. One may suggest that the ORM is the choice of the best approximation to the true theory vacuum $|\Omega\rangle$ in terms of the vacua for field with mass M and condensation b. The original ORM is just a zeroth order approximation in that sense: it gives the best variant of the true vacuum in terms of "free-field" vacuum when the vacuum energy is given in zeroth order (3.171). One can as well consider best "free-field" approximation with the vacuum energy computed in the first order of perturbation theory etc.

To express this condition in nonperturbative form, let us turn our attention back to the path integral formulation. We look at

$$E = E_N = \left[i \ln \int \mathcal{D}\phi e^{i \int dx \mathcal{L}} \right]_{N-order} . \tag{3.174}$$

We need to consider change in this energy under infinitesimal transformation of field shift and perturbative propagator mass change,

$$\begin{aligned} &\mathcal{L}(\phi) \to \mathcal{L}(\phi + \delta b) \\ &m^2 \to m^2 + \delta m^2 \end{aligned} \tag{3.175}$$

It can be verified order by order that field shift in $\mathcal{L}_0$ enters perturbatively at orders up to $N-1$ as a counter-term, while field shift in $\mathcal{L}_I$ appears in the interaction part with the opposite sign at orders up to N. Thus,

$$i\delta E_N \sim \left\{ i \left(\frac{\delta Z}{\delta J} \right)_N - i \left(\frac{\delta Z}{\delta J} \right)_{N-1} \right\} \delta b \tag{3.176}$$

The first equation in this extension of the ORM becomes

$$a_1^{(N)} - a_1^{(N-1)} = 0. \tag{3.177}$$

Similar variation of the mass change of the perturbative propagator mass yields

$$\frac{\delta E_N}{\delta m^2} = \frac{1}{2} \left(G_N(0) - G_{N-1}(0) \right) = 0. \tag{3.178}$$

Here $G(0)$ is $\Delta(m^2)$ function used before in our applications of the original ORM. Eqs.(3.177,3.178) define the "free-field" representation in terms of field condensate b

and field mass m that most closely describe the true vacuum $|\Omega\rangle$ within Nth order perturbation theory. If particles in the theory are close to massive particles of some mass M, such solutions would definitely be preferred in this context as giving nearly exact answer to the vacuum energy at the first few orders of perturbation theory.

While such extension of the ORM is transparent and generally consistent, it clearly looses some of the original attractive features of the ORM as a "nonperturbative" method and its relation to phases or critical behavior of the theory. In the next two sections we are going to focus our attention on the interpretation of the ORM as a variational approach or as QEA method which retains one or the other of these features.

3.6.2 ORM as variational method

It could be noted from the previous sections that solutions of the ORM "correct" form equations often minimize the energy density $\delta E(b, c, m)$. Although this is not always the case (e.g. see Section 3.2), similar phenomena we observed in φ^4 and nonlinear σ-model, so that it can be hypothesized that the "correct" form equations may be deemed equivalent to minimization of the δE function. Here we are going to investigate this hypothesis more carefully.

Let us consider a field theoretical model with single scalar field $\varphi(x)$ and Hamiltonian

$$\begin{gathered}\mathcal{H}{=}\mathcal{H}_0(\partial\varphi) + \tfrac{m^2}{2}\varphi^2 + V(\varphi),\\ \mathcal{H}_0(\partial\varphi) = \tfrac{1}{2}(\partial_\mu\varphi)^2.\end{gathered} \tag{3.179}$$

After appropriate renormalization, the normal ordered bare Hamiltonian is given by

$$\mathcal{H}{=}N_\mu[\mathcal{H}_0(\partial\varphi) + \frac{m^2}{2}\varphi^2 + V(\varphi) + V_{ct}^{\alpha,\mu}(\varphi)]. \tag{3.180}$$

Here and in the following discussion we assume that there is no field renormalization. The normal ordering is equivalent to accounting for major quantum corrections, i.e. renormalization of all daisy-loop diagrams Fig.(3.6), and $V_{ct}^{\alpha,\mu}(\varphi)$ includes all other

counter-terms. We further assume that $V(\varphi)$ and $V_{ct}^{\alpha,\mu}(\varphi)$ have form

$$V(\varphi) + V_{ct}^{\alpha,\mu}(\varphi) = \sum_{n>2} a_n \varphi(x)^n. \tag{3.181}$$

so that

$$\mathcal{H} = N_\mu[\mathcal{H}_0(\partial\varphi) + \sum a_n \varphi(x)^n] \tag{3.182}$$

and $a_0 = \delta E$, $a_1 = 0$, $a_2 = m^2/2$.

After shift and normal reordering is done in Eq.(3.182) one obtains

$$\begin{gathered}
\mathcal{H} = N_\mu[\mathcal{H}_0(\partial\varphi) + \sum\sum \tfrac{n!}{m!(n-m)!} a_n c^{n-m} \varphi(x)^m] = \\
= N_{t\mu}[\mathcal{H}_0(\partial\varphi) + \delta_\mu(t\mu) + \sum_n \sum_{j=0}^{[n/2]} \tfrac{n!}{j!(n-2j)!} \left(\tfrac{\Delta_\mu(t\mu)}{2}\right)^j \varphi(x)^{n-2j} \sum_{m\geq n} a_m \tfrac{m!}{(m-n)!n!} c^{m-n}] = \\
= N_{t\mu}[\mathcal{H}_0(\partial\varphi) + \delta_\mu(t\mu) + \sum_k \varphi^k \sum_{j=0}^{\infty} \tfrac{1}{j!k!} \left(\tfrac{\Delta_\mu(t\mu)}{2}\right)^j \sum_{m\geq k+2j} a_m \tfrac{m!}{(m-k-2j)!} c^{m-k-2j}] = \\
= N_{t\mu}[\mathcal{H}_0(\partial\varphi) + \sum a'_n \varphi(x)^n],
\end{gathered} \tag{3.183}$$

where

$$\begin{gathered}
\delta_\mu(m) = \delta(\mu, m) = N_\mu[(\partial_\mu\varphi)^2] - N_m[(\partial_\mu\varphi)^2], \\
\Delta_\mu(m) = \Delta(\mu, m) = N_\mu[\varphi^2] - N_m[\varphi^2].
\end{gathered} \tag{3.184}$$

Note that

$$\begin{gathered}
\delta E = \tfrac{\delta_\mu(t\mu)}{2} + a'_o = \tfrac{\delta_\mu(t\mu)}{2} + \sum_{j=0}^{\infty} \tfrac{1}{j!} \left(\tfrac{\Delta_\mu(t\mu)}{2}\right)^j \sum_{m\geq 2j} a_m \tfrac{m!}{(m-2j)!} c^{m-2j}, \\
a'_1 = \sum_{j=0}^{\infty} \tfrac{1}{j!1!} \left(\tfrac{\Delta_\mu(t\mu)}{2}\right)^j \sum_{m\geq 1+2j} a_m \tfrac{m!}{(m-1-2j)!} c^{m-1-2j} = \tfrac{\partial}{\partial c}\delta E, \\
a'_2 = \tfrac{1}{2} \sum_{j=0}^{\infty} \tfrac{1}{j!} \left(\tfrac{\Delta_\mu(t\mu)}{2}\right)^j \sum_{m\geq 2+2j} a_m \tfrac{m!}{(m-2-2j)!} c^{m-2-2j}.
\end{gathered} \tag{3.185}$$

Easy to check

$$\begin{gathered}
\tfrac{1}{\mu}\tfrac{\partial}{\partial t}\delta E = \tfrac{1}{2}\tfrac{\partial}{\partial t\mu}\delta_\mu(t\mu) + \tfrac{1}{2}\sum_{j=1}^{\infty} \tfrac{1}{(j-1)!} \left(\tfrac{\Delta_\mu(t\mu)}{2}\right)^{j-1} \sum_{m\geq 2j} a_m \tfrac{m!}{(m-2j)!} c^{m-2j} \tfrac{\partial}{\partial t\mu}\Delta_\mu(t\mu) = \\
= \tfrac{1}{2}\tfrac{\partial}{\partial t\mu}\delta_\mu(t\mu) + \tfrac{1}{2}\sum_{j=0}^{\infty} \tfrac{1}{j!} \left(\tfrac{\Delta_\mu(t\mu)}{2}\right)^j \sum_{m\geq 2j+2} a_m \tfrac{m!}{(m-2-2j)!} c^{m-2-2j} \tfrac{\partial}{\partial t\mu}\Delta_\mu(t\mu) = \\
= \tfrac{1}{2}(\tfrac{\partial}{\partial t\mu}\delta_\mu(t\mu) + 2a'_2 \tfrac{\partial}{\partial t\mu}\Delta_\mu(t\mu)).
\end{gathered} \tag{3.186}$$

If we require that after normal reordering $a_1' = 0$ and $a_2' = M^2/2$, then we obtain

$$\begin{gathered} \tfrac{\partial}{\partial c}\delta E = 0, \\ \tfrac{1}{\mu}\tfrac{\partial}{\partial t}\delta E = \tfrac{1}{2}\tfrac{\partial}{\partial t\mu}(\delta_\mu(t\mu) + M^2\Delta_\mu(t\mu)) = 0. \end{gathered} \tag{3.187}$$

Where we've taken into account that

$$\frac{\partial}{\partial m}(\delta_\mu(m) + M^2\Delta_\mu(m)) = 0 \Longleftrightarrow m^2 = M^2 \tag{3.188}$$

that follows from considering a free field Hamiltonian [106]

$$\mathcal{H} = \frac{1}{2}[(\partial_\mu\varphi)^2 + M^2\varphi^2].$$

The same conclusion can be derived from a more general argument. For that we should notice that transformation from the initial representation $|c = 0, t = 1\rangle$ to $|c', t'\rangle$ ($t' = M/m$) can be done as well in two steps $|c = 0, t = 1\rangle \rightarrow |c, t\rangle \rightarrow |c + \delta c, t + \delta t\rangle$. Suppose now that after $|c = 0, t = 1\rangle \rightarrow |c, t\rangle$ transformation the Hamiltonian takes the form

$$\mathcal{H} = N_m[\mathcal{H}_0(\partial\varphi) + \sum a_n\varphi(x)^n] = N_M[\mathcal{H}_0(\partial\varphi) + \sum a_n'\varphi(x)^n] \tag{3.189}$$

where $a_1' = 0$ and $a_2' = M^2/2$. We shall show now that (c, t) is a local minimum of $\delta E = \langle c, t|N_m[\mathcal{H}]|c, t\rangle = a_0'$. For that we consider small perturbation of the final point $(c + \delta c, t + \delta t)$. Due to the note made above we may treat this problem as small perturbations of $|c = 0, t = 1\rangle$ point with Hamiltonian given by $\mathcal{H} = N_M[\mathcal{H}_0(\partial\varphi) + \sum a_n'\varphi(x)^n]$. Then, noting that

$$\begin{gathered} \langle\delta c, \delta t|N[\varphi]|\delta c, \delta t\rangle = \delta c, \\ \langle\delta c, \delta t|N[\varphi^2]|\delta c, \delta t\rangle = \delta c^2 + \Delta_\mu(M + M\delta t), \\ \langle\delta c, \delta t|N[\varphi^n]|\delta c, \delta t\rangle = \sum_{j=0}^{[n/2]} \tfrac{n!}{j!(n-2j)!}\left(\tfrac{\Delta_\mu(M+M\delta t)}{2}\right)^j c^{n-2j} = o(\delta c, \delta t), \end{gathered} \tag{3.190}$$

we find

$$\delta(\delta E)|_{\delta c,\delta t} = a_1'\delta c + \frac{1}{2}\frac{d}{dm}(\delta_\mu(M) + 2a_2'\Delta_\mu(M))M\delta t + o(\delta c, \delta t) \tag{3.191}$$

so that a minimum of $\delta E(c,M)$ is equivalent to $a_1' = 0$ and $a_2' = M^2/2$. We point out a useful formula

$$\delta E(c,M) = \frac{\langle c,M|N_\mu[\mathcal{H}(\varphi)]|c,M\rangle}{\langle c,M|c,M\rangle}. \tag{3.192}$$

Let us give a brief example of use of formula (3.192). As was already considered in the previous section, the Hamiltonian for the nonlinear σ-model in Lagrange multipliers form is given by

$$\mathcal{H} = N_{\mu_r}[\frac{1}{2}(\partial_\mu\vec{\varphi})^2 + \lambda(\vec{\varphi}^2 - g_r^{-1}) + \frac{\mu_r^2}{2}\vec{\varphi}^2 + Z_r^{-1}[(\partial_\mu\lambda)^2 + \lambda^2]/2]. \tag{3.193}$$

Assuming $\langle 0'|\lambda|0'\rangle = b, \langle 0'|\vec{\varphi}|0'\rangle = \vec{c}, \langle 0'|\vec{\varphi}^2|0'\rangle = \vec{c}^2 + N\Delta_\mu(M), \langle 0'|(\partial_\mu\vec{\varphi})^2|0'\rangle = N\delta_\mu(M)$ and assuming fields λ and φ are not correlated, it is easy to find δE, i.e.

$$\begin{gathered}\delta E = \langle 0'|\mathcal{H}|0'\rangle = \tfrac{1}{2}[N\delta_\mu(M) + 2b(\vec{c}^2 + N\Delta_\mu(M) - g_r^{-1}) + \\ \mu_r^2(\vec{c}^2 + N\Delta_\mu(M)) + O(Z_r^{-1})] = \\ = \tfrac{1}{2}[N(\delta_\mu(M) + (2b+\mu_r^2)\Delta_\mu(M)) + (2b+\mu_r^2)\vec{c}^2 - 2bg_r^{-1} + O(Z_r^{-1})],\end{gathered} \tag{3.194}$$

from where we immediately conclude

$$\begin{gathered}\tfrac{\partial}{\partial M}\delta E \doteq \tfrac{N}{2}\tfrac{d}{dM}(\delta_\mu(M) + (2b+\mu_r^2)\Delta_\mu(M)) = 0 \Longrightarrow M^2 = (2b+\mu_r^2); \\ \tfrac{\partial}{\partial c}\delta E = 2(2b+\mu_r^2)|\vec{c}| = 0 \Longrightarrow \vec{c} = 0 \text{ or } M^2 = 0; \\ \tfrac{\partial}{\partial b}\delta E = \vec{c}^2 + N\Delta_\mu(M) - g_r^{-1} = 0.\end{gathered} \tag{3.195}$$

Eq.(3.195) is equivalent to Eq.(3.104) derived by straightforward use of the ORM.

We shall note, however, that the above discussion has the limitation that we assumed that the normal reordering of the Hamiltonian is sufficient after the canonical variables transformation to make it finite. This is not generally true. As was discussed above, in this case the ORM procedure can be formulated as follows,

- Hamiltonian should be written as $\mathcal{H}=N_m[\mathcal{H}_0(\partial\varphi)+\frac{m^2}{2}\varphi^2+V(\varphi)+V_{ct}^{\alpha,\mu}(\varphi)]$,
- canonical variables should be changed to $\varphi \mapsto \varphi + c$, $m \mapsto M$ and new counter-terms should be written so that

$$\begin{gathered}\mathcal{H}=N_M[\mathcal{H}_0(\partial\varphi) + \tfrac{M^2}{2}\varphi^2 + V'(\varphi) + V_{ct}'^{\alpha,\mu}(\varphi) + H_{add}(\varphi)] = \\ =N_M[\mathcal{H}_0(\partial\varphi) + \tfrac{m^2}{2}\varphi^2 + \tilde{V}(\varphi) + \tilde{V}_{ct}^{\alpha,\mu}(\varphi) + m^2c\varphi + \tfrac{m^2c^2}{2} + \tfrac{m^2}{2}\Delta_m(M)],\end{gathered}$$

where we used $\tilde{V}(\varphi)$ to denote the potential obtained in the result of the canonical variables transformation $V(\phi + c)$ and by $V_{ct}^{\prime\alpha,\mu}(\varphi)$ we denoted the counter-terms as required by the transformed potential.

- the correct form requirements then are $V'(\varphi) = \tilde{V}(\varphi)$, with no terms proportional to φ and φ^2, and $H_{add}(\varphi) = 0$.

Therefore the ORM "correct" form requirements read: after the Hamiltonian Eq.(3.182) is transformed as in Eq.(3.183) and mass renormalization is subtracted $a'_{2,r} = M^2/2$, and after the linear renormalization term is subtracted $a'_{1,r} = 0$.

Parallels between the ORM and the variational principle should be read in the following manner: the ORM "correct" form is equivalent to

$$\delta_{\bar{c},\bar{M}} \left\langle \bar{c}, \bar{M} | N_m[\mathcal{H}[\varphi(x)] - \frac{\delta m_\alpha(c,M)^2}{2}\varphi^2(x) - \delta a_{1,\alpha}(c,M)\varphi(x)] | \bar{c}, \bar{M} \right\rangle = 0, \quad (3.196)$$

where the counter-terms are computed at the mass scale which is the solution of Eq.(3.196), i.e. $\bar{c} = c$, $\bar{M} = M$. Note that adding/subtracting higher order φ^n renormalizations to/from the Hamiltonian have no effect on Eq.(3.196) and that normal ordering itself can be seen equivalent to renormalization of specific subset of diagrams. In that case Eq.(3.196) can be equivalently rewritten as minimization of the renormalized energy density

$$\delta_{\bar{c},\bar{M}} \left\langle \bar{c}, \bar{M} | \mathcal{H}_r(c,M)[\varphi(x)] | \bar{c}, \bar{M} \right\rangle = 0, \quad (3.197)$$

where renormalization in $\mathcal{H}_r(c,M)$ is done at the point (c,M), which is the solution of Eq.(3.197)

$$(c,M) = (\bar{c}^*, \bar{M}^*). \quad (3.198)$$

By this we have proved following two statements.

VORM Theorem I The ORM method in scalar QFT with analytic potential $V(\varphi)$ and which is renormalizable with normal ordering is equivalent to minimization of $E(c,M) = \lim_{V\to\infty} \langle c, M | \frac{H[\varphi,V]}{V} | c, M \rangle$.

Corollary In particular, all theories in d=1+1 with analytic potential $V(\varphi)$ are

renormalizable with normal ordering alone so that the ORM in d=1+1 almost always is equivalent to minimization of the above energy density.

VORM Theorem II In general, the ORM in scalar field theories with no field renormalization is equivalent to variational method formulated for renormalized energy density Eq.(3.197) on the set of quantum states $\{|c,M\rangle : \langle c,M|(\partial_\mu\varphi)^2|c,M\rangle = \delta_\mu(M)$ & $\langle c,M|\varphi|c,M\rangle = c$ & $\langle c,M|\varphi^2|c,M\rangle = c^2+\Delta_\mu(M)$ etc.$\}$ and condition that the energy density is renormalized at the mass scale M^* and vacuum condensation c^* at which the variational minimum is reached.

The above discussion of variational meaning of the ORM can be used further to suggest a fully variational analog of the ORM. For that we introduce the ORM inspired trial quantum state in the form

$$\begin{gathered}|\alpha\rangle = \exp\{\sum_k \ln\sqrt{1-\alpha_k^2} - \sum_k \alpha_k a_k^\dagger a_{-k}^\dagger\}|0\rangle, \\ \alpha_k = \alpha_{-k}.\end{gathered} \tag{3.199}$$

First let us obtain some useful averages.

$$\begin{gathered}\langle\alpha|\alpha\rangle = \prod_k(1-\alpha_k^2)\sum_{n,m=0}^{\infty}\alpha_k^{n+m}\frac{\left\langle(a_ka_{-k})^n(a_k^\dagger a_{-k}^\dagger)^m\right\rangle}{n!m!} = \\ \prod_k(1-\alpha_k^2)\sum_{n,m=0}^{\infty}\alpha_k^{n+m}\frac{\delta_{n,m}(n!)^2}{n!m!} = \prod_k(1-\alpha_k^2)\sum_n^{\infty}\alpha_k^{2n} = 1;\end{gathered} \tag{3.200}$$

$$\begin{gathered}\left\langle\alpha|a_k^\dagger a_k|\alpha\right\rangle = (1-\alpha_k^2)\sum_{n,m=0}^{\infty}\alpha_k^{n+m}\frac{\left\langle(a_ka_{-k})^n a_k^\dagger a_k(a_k^\dagger a_{-k}^\dagger)^m\right\rangle}{n!m!} = \\ (1-\alpha_k^2)\left\{\sum_{n,m=0}^{\infty}\alpha_k^{n+m}\frac{\left\langle a_k^{n+1}a_{-k}^n(a_k^\dagger)^{m+1}(a_{-k}^\dagger)^m\right\rangle}{n!m!} - \sum_{n,m=0}^{\infty}\alpha_k^{n+m}\frac{\left\langle(a_ka_{-k})^n(a_k^\dagger a_{-k}^\dagger)^m\right\rangle}{n!m!}\right\} = \\ (1-\alpha_k^2)\left(\sum_n\alpha_k^{2n}(n+1) - \sum_n\alpha_k^{2n}\right) = (1-\alpha_k^2)\tfrac{\alpha_k}{2}\sum_n 2n\alpha_k^{2n-1} = \\ (1-\alpha_k^2)\tfrac{\alpha_k}{2}\tfrac{d}{d\alpha_k}\tfrac{1}{1-\alpha_k^2} = \tfrac{\alpha_k^2}{1-\alpha_k^2};\end{gathered} \tag{3.201}$$

$$\begin{gathered}\langle\alpha|a_ka_{-k}|\alpha\rangle = (1-\alpha_k^2)\sum_{n,m=0}^{\infty}\alpha_k^{n+m}\frac{\left\langle(a_ka_{-k})^n a_ka_{-k}(a_k^\dagger a_{-k}^\dagger)^m\right\rangle}{n!m!} = \\ (1-\alpha_k^2)\sum_{n,m=0}^{\infty}\alpha_k^{n+m}\frac{\left\langle(a_ka_{-k})^{n+1}(a_k^\dagger a_{-k}^\dagger)^m\right\rangle}{n!m!} = (1-\alpha_k^2)\sum_n\alpha_k^{2n-1}n = \\ (1-\alpha_k^2)\tfrac{\alpha_k}{2}\tfrac{d}{d\alpha_k}\tfrac{1}{1-\alpha_k^2} = \tfrac{\alpha_k}{1-\alpha_k^2} = \left\langle\alpha|a_k^\dagger a_{-k}^\dagger|\alpha\right\rangle^*.\end{gathered} \tag{3.202}$$

For quantum state $|\alpha\rangle$ we obtain, therefore,

$$\begin{gathered}\langle\alpha|\alpha\rangle = 1, \left\langle\alpha|a_k^\dagger a_k|\alpha\right\rangle = \frac{\alpha_k^2}{1-\alpha_k^2}, \\ \langle\alpha|a_k a_{-k}|\alpha\rangle = \left\langle\alpha|a_k^\dagger a_{-k}^\dagger|\alpha\right\rangle^* = \frac{\alpha_k}{1-\alpha_k^2}.\end{gathered} \tag{3.203}$$

For field's expectation values we get

$$\begin{gathered}\langle\alpha|N[(\partial_\mu\phi)^2+\mu^2\phi^2]|\alpha\rangle = \left\langle\alpha|2\sum_k \frac{w_k}{V}a_k^\dagger a_k|\alpha\right\rangle = \sum_k \frac{2w_k}{V}\frac{\alpha_k^2}{1-\alpha_k^2}, \\ \langle\alpha|N[\phi^2]|\alpha\rangle = \left\langle\alpha|\sum_k \frac{1}{2w_k V}[a_k^\dagger a_k + a_{-k}^\dagger a_{-k} + a_k^\dagger a_{-k}^\dagger + a_k a_{-k}|\alpha\right\rangle = \\ \sum_k \frac{1}{2w_k V}(2\frac{\alpha_k^2}{1-\alpha_k^2} + 2\frac{\alpha_k}{1-\alpha_k^2}) = \sum_k \frac{1}{w_k V}\frac{\alpha_k}{1-\alpha_k},\end{gathered} \tag{3.204}$$

We also note that the same expectation values could be calculated in a slightly different way. One may note that $|\alpha\rangle$ state is annihilated by operator

$$\tilde{a}_k = \frac{a_k - \alpha_k a_{-k}^\dagger}{\sqrt{1-\alpha_k^2}}. \tag{3.205}$$

Moreover, as is easy to check, $[\tilde{a}_k, \tilde{a}_{k'}^\dagger] = \delta_{k,k'}$ so that $\tilde{a}_k$ and $\tilde{a}_k^\dagger$ define a creation and annihilation operators with vacuum being quantum state $|\alpha\rangle$. We call such state a Quantum Gaussian Packet (QGP) and consider some of its properties also in Chapter 4. Here let us only note that QGPs have important applications in many different problems of Quantum Physics and, in particular, in theory of thermo-fields [88].

It is possible to rewrite, e.g. $\left\langle\alpha|\sum_k w_k a_k^\dagger a_k|\alpha\right\rangle$, in terms of $\tilde{a}_k$ and $\tilde{a}_k^\dagger$ and hence only term $\left\langle\alpha|\tilde{a}_k\tilde{a}_k^\dagger|\alpha\right\rangle$ would contribute into the average. Explicitly

$$\begin{gathered}\left\langle\alpha|\sum_k w_k a_k^\dagger a_k|\alpha\right\rangle = \frac{\left\langle\alpha|\sum_k w_k(\tilde{a}_k^\dagger+\alpha_k a_{-k})(\tilde{a}_k+\alpha_k a_{-k}^\dagger)|\alpha\right\rangle}{1-\alpha_k^2} = \\ = \sum_k w_k \frac{\alpha_k^2}{1-\alpha_k^2}.\end{gathered} \tag{3.206}$$

It is also not difficult to check that this approach works as well for the second formula in Eq.(3.204). We should remember this useful trick when calculating expectation values for the interaction term in the Hamiltonian, where direct computation is complicated.

The ORM procedure corresponds to specific form-factor in state $|\alpha\rangle$,

$$\begin{array}{c}\alpha_k = \tanh(\zeta_k) = \frac{\sqrt{k^0/w_k}-\sqrt{w_k/k^0}}{\sqrt{k^0/w_k}+\sqrt{w_k/k^0}}, \\ w_k = \sqrt{k^2+m^2}, k^0 = \sqrt{k^2+\mu^2}.\end{array} \tag{3.207}$$

In this case one obtains

$$\left\langle N[\phi^2]\right\rangle = \sum_k \frac{1}{Vk^0}\frac{\sqrt{k^0/w_k}-\sqrt{w_k/k^0}}{2\sqrt{w_k/k^0}} = \sum_k \frac{1}{2V}(\frac{1}{\sqrt{k^2+m^2}} - \frac{1}{\sqrt{k^2+\mu^2}}), \tag{3.208}$$

consistent with the ORM formula for $\Delta_\mu(m)$ with $\sum\limits_k \to \frac{V}{(2\pi)^{d-1}}\int$ in large V limit. To calculate $\langle N[(\partial\phi)^2]\rangle$ we should note from Eq.(3.204) that

$$\begin{array}{c}\langle N[(\partial\phi)^2]\rangle = \langle N[(\partial\phi)^2+\mu^2\phi^2]\rangle - \mu^2\langle N[\phi^2]\rangle = \\ \sum\limits_k \frac{2k^0}{V}\frac{\alpha_k^2}{1-\alpha_k^2} - \sum\limits_k \frac{\mu^2}{k^0V}\frac{\alpha_k(1+\alpha_k)}{1-\alpha_k^2} = \\ \sum\limits_k \frac{2(k^0)^2-\mu^2}{Vk^0}\frac{\alpha_k^2}{1-\alpha_k^2} - \sum\limits_k \frac{\mu^2}{k^0V}\frac{\alpha_k}{1-\alpha_k^2}.\end{array} \tag{3.209}$$

Then for $\langle N[(\partial\phi)^2]\rangle$ with α_k given by Eq.(3.207) we obtain

$$\begin{array}{c}\langle N[(\partial\phi)^2]\rangle = \sum\limits_k \frac{2(k^0)^2-\mu^2}{Vk^0}\frac{\tanh(\zeta_k)^2}{1-\tanh(\zeta_k)^2} - \sum\limits_k \frac{\mu^2}{k^0V}\frac{\tanh(\zeta_k)}{1-\tanh(\zeta_k)^2} = \\ = \sum\limits_k \frac{2\vec{k}^2+\mu^2}{Vk^0}\sinh(\zeta_k)^2 - \sum\limits_k \frac{\mu^2}{k^0V}\sinh(\zeta_k)\cosh(\zeta_k) = \\ \sum\limits_k \frac{2\vec{k}^2+\mu^2}{4Vk^0}(\sqrt{k^0/w_k}-\sqrt{w_k/k^0})^2 - \sum\limits_k \frac{\mu^2}{4k^0V}(\sqrt{k^0/w_k}-\sqrt{w_k/k^0})(\sqrt{k^0/w_k}+\sqrt{w_k/k^0}) = \\ \sum\limits_k \frac{2\vec{k}^2+\mu^2}{4Vk^0}(k^0/w_k + w_k/k^0 - 2) - \sum\limits_k \frac{\mu^2}{4k^0V}(k^0/w_k - w_k/k^0) = \\ \sum\limits_k \frac{2\vec{k}^2+\mu^2}{4V(k^0)^2w_k}((k^0)^2 + w_k^2 - 2w_k k^0 - \frac{\mu^2}{2\vec{k}^2+\mu^2}((k^0)^2 - w_k^2)) = \\ \sum\limits_k \frac{1}{4V(k^0)^2w_k}((k^0)^2(2\vec{k}^2+\mu^2-\mu^2) + w_k^2(2\vec{k}^2+\mu^2+\mu^2) - 2(2\vec{k}^2+\mu^2)w_k k^0) = \\ \sum\limits_k \frac{1}{4V}(\frac{2\vec{k}^2}{w_k} + \frac{2w_k^2}{w_k} - 2\frac{2\vec{k}^2+\mu^2}{k^0}) = \sum\limits_k \frac{1}{V}(\frac{2\vec{k}^2+m^2}{2w_k} - \frac{2\vec{k}^2+\mu^2}{2k^0}).\end{array} \tag{3.210}$$

That is, again, in agreement with the ORM formula for $\delta_\mu(m)$. We now can establish

connections between expectation values on $|\alpha\rangle$ and the ORM formulas, i.e.

$$\begin{gathered}\delta_\mu(\alpha)=\langle N[(\partial\phi_0)^2]\rangle=\sum_k\tfrac{2(k^0)^2-\mu^2}{Vk^0}\tfrac{\alpha_k^2}{1-\alpha_k^2}-\sum_k\tfrac{\mu^2}{k^0V}\tfrac{\alpha_k}{1-\alpha_k^2};\\ \Delta_\mu(\alpha)=\langle N[\phi_0^2]\rangle=\sum_k\tfrac{1}{k^0V}\tfrac{\alpha_k}{1-\alpha_k};\\ 2\epsilon_\mu(\alpha)=\delta_\mu(\alpha)+\mu^2\Delta_\mu(\alpha)=\sum_k\tfrac{2k^0}{V}\tfrac{\alpha_k^2}{1-\alpha_k^2};\\ \Delta_\mu^l(\alpha)=\left\langle N[\phi_0^{2l}]\right\rangle=\tfrac{(2l)!}{2^l l!}(\Delta_\mu(\alpha))^l.\end{gathered}\tag{3.211}$$

These relations can be used to define a variational extension of the ORM using general QGP $|\alpha\rangle$. As an example of such application let us consider computation of the energy density on states $|\alpha\rangle$ for nonlinear σ model in the lowest order in g - $\langle N[H_I]\rangle$. The direct calculation of $\langle N[H_I]\rangle$ is complicated, instead it is useful to use the main property of QGP that they are annihilated by a linear combination of the original ladder operators and try to rewrite H_I in terms of the ladder operators $\tilde{a}_k$ associated with the state $|\alpha\rangle$. For that we shall find difference between $N_\mu[H_I]$ and $N_\alpha[H_I]$. Here $N_\alpha[\cdot]$ means expression normal-ordered in terms of $\tilde{a}_k$.

Let us first arrive into general expression for $N[H_I]$ in terms of the field itself. We will use following notations for $\phi(x)=\sum_k\frac{e^{-i\vec{k}\vec{x}}}{\sqrt{2k^0V}}(a_k+a_{-k}^\dagger)=\phi^+(x)+\phi^-(x)$:

$$\phi^-(x)=\sum_k\frac{1}{\sqrt{2Vk^0}}\sqrt{\frac{1+\alpha_k}{1-\alpha_k}}\tilde{a}_ke^{-ikx},\ \phi^+(x)=\sum_k\frac{1}{\sqrt{2Vk^0}}\sqrt{\frac{1+\alpha_k}{1-\alpha_k}}\tilde{a}_k^\dagger e^{-ikx},\tag{3.212}$$

and $\partial_0\phi(x)=\sum_k\frac{ik^0e^{-i\vec{k}\vec{x}}}{\sqrt{2k^0V}}(a_k-a_{-k}^\dagger)=\partial_0\phi^+(x)+\partial_0\phi^-(x)$:

$$\partial_0\phi^-(x)=\sum_k\frac{ik^0}{\sqrt{2Vk^0}}\sqrt{\frac{1-\alpha_k}{1+\alpha_k}}\tilde{a}_ke^{-ikx},\ \partial_0\phi^+(x)=\sum_k\frac{ik^0}{\sqrt{2Vk^0}}\sqrt{\frac{1-\alpha_k}{1+\alpha_k}}\tilde{a}_k^\dagger e^{-ikx},\tag{3.213}$$

$$\begin{gathered}[\phi^-(x),\phi^+(x)]=\sum_k\tfrac{1}{2Vk_0}\tfrac{1+\alpha_k}{1-\alpha_k},\\ [\partial_\mu\phi^-(x),\partial_\mu\phi^+(x)]=\sum_k\left(\tfrac{\vec{k}^2}{2k^0V}\tfrac{1+\alpha_k}{1-\alpha_k}+\tfrac{(k^0)^2}{2k^0V}\tfrac{1-\alpha_k}{1+\alpha_k}\right)=\sum_k\tfrac{k^0}{V}\tfrac{1+\alpha_k^2}{1-\alpha_k^2},\\ [\phi^-(x),\partial_0\phi^+(x)]=\sum_k\tfrac{ik^0}{2Vk_0}\sqrt{\tfrac{1+\alpha_k}{1-\alpha_k}}\sqrt{\tfrac{1-\alpha_k}{1+\alpha_k}}=\sum_k\tfrac{i}{2V}=\tfrac{i(L\Lambda)^3}{2V}=\tfrac{i\Lambda^3}{2}.\end{gathered}\tag{3.214}$$

To compute $N[\phi^2(\partial\phi)^2]$ we are going to start from

$$\phi^2(\partial_i\phi)^2=(N[\phi^2]+\Delta_\mu(\alpha))(N[(\partial_i\phi)^2]+d_\mu(\alpha))\tag{3.215}$$

and note that

$$N[\phi^2]N[(\partial_i\phi)^2] = (\phi^+\phi^+ + 2\phi^+\phi^- + \phi^-\phi^-)(\partial_i\phi^+\partial_i\phi^+ + 2\partial_i\phi^+\partial_i\phi^- + \partial_i\phi^-\partial_i\phi^-) \quad (3.216)$$

can be rewritten as $N[\phi^2]N[(\partial_i\phi)^2] = N[\phi^2(\partial_i\phi)^2] + 4\Sigma_i^- N[\phi\partial_i\phi] + 2\Sigma_i^-\Sigma_i^-$ after all ϕ^- are moved to the right. Here

$$\Sigma_\nu^- = [\phi^-(x), \partial_\nu\phi^+(x)] = \begin{cases} 0, \nu = 1,2,3; \\ \frac{i\Lambda^3}{2}, \nu = 0. \end{cases} . \quad (3.217)$$

Therefore, we obtain

$$\phi^2(\partial_i\phi)^2 = N[\phi^2(\partial_i\phi)^2] + \Delta_\mu N[(\partial_i\phi)^2] + d_\mu N[\phi^2] + \Delta_\mu d_\mu, \quad (3.218)$$

where i is space index, while μ is the renormalization scale, i.e. the mass of the field. Taking into account

$$\begin{aligned} N[(\partial_i\phi)^2] &= (\partial_i\phi)^2 - d_\mu(\alpha), \\ N[\phi^2] &= \phi^2 - \Delta_\mu(\alpha), \\ N[\phi\partial_i\phi] &= \phi\partial_i\phi - \Sigma_i^- \end{aligned} \quad (3.219)$$

we arrive at our final formula

$$\begin{aligned} N[\phi^2 \ \cdot \ (\partial_i\phi)^2] = \\ \phi^2(\partial_i\phi)^2 - \Delta_\mu(\alpha)(\partial_i\phi)^2 - d_\mu(\alpha)\phi^2 + \Delta_\mu(\alpha)d_\mu(\alpha). \end{aligned} \quad (3.220)$$

We can write, therefore,

$$\begin{aligned} \langle\alpha \ |N_\mu[\phi^2(\partial_i\phi)^2]|\alpha\rangle = \\ \langle\alpha|\phi^2(\partial_i\phi)^2 - \Delta_\mu(0)(\partial_i\phi)^2 - d_\mu(0)\phi^2 + \Delta_\mu(0)d_\mu(0)|\alpha\rangle \, . \end{aligned} \quad (3.221)$$

On the other side, taking into account Eq.(3.219,3.220), we rewrite Eq.(3.221) in terms of the $N_\alpha[\cdot]$ and after some manipulations obtain

$$\begin{aligned} \langle\alpha| \ N_\mu[\phi^2(\partial_i\phi)^2]|\alpha\rangle = \\ \Delta_\mu(\alpha)d_\mu(\alpha) - \Delta_\mu(0)d_\mu(\alpha) - d_\mu(0)\Delta_\mu(a) + \Delta_\mu(0)d_\mu(0) \Rightarrow \\ \langle\alpha|N_\mu[\phi^2(\partial_i\phi)^2]|\alpha\rangle = (\Delta_\mu(\alpha) - \Delta_\mu(0))(d_\mu(\alpha) - d_\mu(0)) = \Delta_\mu(\alpha)d_\mu(\alpha). \end{aligned} \quad (3.222)$$

Hence, we obtain for the energy of the trial state $|\alpha\rangle$ in first order in g

$$\begin{array}{c} \langle H_0 + H_I \rangle = \frac{V}{2} \left[Z_\phi \delta_\mu(\alpha) + Z_\mu \mu \Delta_\mu(\alpha) + Z_g g \Delta_\mu(\alpha) d_\mu(\alpha) \right] \Rightarrow \\ E_\alpha = \frac{V}{2} \left[Z_\phi \delta_\mu(\alpha) + Z_\mu \mu \Delta_\mu(\alpha) + Z_g g \Delta_\mu(\alpha) d_\mu(\alpha) \right]. \end{array} \tag{3.223}$$

3.6.3 The ORM as Quantum Effective Potential

Above we considered possible nonperturbative interpretation of the ORM. While these equations can be obtained in closed form in nonperturbative context, e.g. as optimization of the accuracy of a given order perturbative expressions for the vacuum energy, they loose some of the initial conceptual points of the ORM - its connection with critical phenomena and phase transitions. In that respect another interpretation of the ORM may be beneficial if one notes similarity between the ORM and Quantum Effective Action method [101, 129–131].

Quantum Effective Action may be introduced in QFT as the energy of a configuration such that the expectation value of the quantum field $\phi(x)$ is given by a classical field $\psi(x)$ [129, 130, 132, 133]

$$\begin{array}{l} {}_\psi\langle 0|\phi(x)|0\rangle_\psi = \psi(x) \\ \Gamma(\psi) =_\psi \langle 0|\mathcal{H}|0\rangle_\psi. \end{array} \tag{3.224}$$

Consider, for example, field theory described by Lagrangian $\mathcal{L}(\phi)$ in the presence of external source $J(x)$. Then

$$e^{-iW[J]V} = Z[J] = \int \mathcal{D}\phi e^{i\int dx(\mathcal{L}+J\phi)} \tag{3.225}$$

and

$$\langle \phi \rangle [J] = \frac{\delta W[J]}{\delta J}. \tag{3.226}$$

In the presence of external source $J(x)$ the field assumes configuration $\langle \phi(x) \rangle = \langle \phi(x) \rangle [J]$. The energy, associated with this configuration, can be found by

$$\Gamma[J] = W[J] - \int dx J(x) \langle \phi(x) \rangle [J]. \tag{3.227}$$

The effective action is defined by Eq.(3.227) if Eq.(3.226) can be solved for $J(\langle\phi\rangle)$ and, thus, to rewrite Eq.(3.227) as

$$\Gamma[\langle\phi\rangle] = W - \int dx J[\langle\phi\rangle]\langle\phi\rangle. \tag{3.228}$$

$\Gamma[\psi]$ can be viewed as Legendre transform of the vacuum energy $W[J]$ and is analogous to Gibbs potential in thermodynamics. For Quantum Effective Action (QEA) it is true that

$$\frac{\delta\Gamma[\psi]}{\delta\psi} = -J(\psi), \tag{3.229}$$

so that for a configuration of field that can exist in the absence of any external support

$$\frac{\delta\Gamma[\psi]}{\delta\psi} = 0. \tag{3.230}$$

Eq.(3.230) may be regarded as the equation of motion of the field ψ taking into account the quantum corrections. Not only $\Gamma[\psi]$ provide the quantum-corrected field equations, it is also an effective action in the sense that $W[J]$ may be calculated as sum of connected tree graphs where the vertexes are obtained from $\Gamma[\phi]$ instead of $\int dx\mathcal{L}$ [134] - all the loop diagrams are taken into account by $\Gamma[\psi]$. The derivatives of the effective action $\delta\Gamma/\delta\psi_1 \ldots \delta\psi_n$ are the 1PI n-point functions of the original theory, thus $\Gamma[\psi]$ is also regarded as generating functional for 1PI amplitudes. Finally, QEA can be calculated by summing all 1PI diagrams for a field theory obtained from the original Lagrangian by field shift $\mathcal{L}(\phi) \to \mathcal{L}(\phi = \psi + \eta)$.

While quantum effective action can be seen in many respects analogous to the vacuum energy W, its important distinction is possibility of multiple solutions in Eq.(3.230) which stimulated its use in investigation of phase transitions in field theories. Investigation of phase transition in ϕ^4 theory has become such a classical example [129, 132].

In the study of phase transitions instead of QEA we usually deal with so called Quantum Effective Potential (QEP) obtained from $\Gamma[\psi]$ in the special case of a constant classical field $\psi(x) = \psi$. For 1+1 dimensional ϕ^4 theory in 1-loop QEP can be

computed straightforwardly,

$$\begin{aligned} V[\psi] &= V_0 + V_1 \\ V_0 &= m^2\psi^2/2 + \lambda\psi^4/4! \\ V_1 &= \hbar\left\{(A+\tfrac{1}{8\pi})(m^2+\tfrac{\lambda\psi^2}{2}) - \tfrac{1}{8\pi}(m^2+\tfrac{\lambda\psi^2}{2})\ln\tfrac{m^2+\frac{\lambda\psi^2}{2}}{\mu^2}\right\}, \end{aligned} \tag{3.231}$$

where A is a divergent constant and m^2, λ are bare mass and coupling of the theory. μ^2 is regularization scale, i.e. cut-off parameter Λ in Pauli-Villars or sliding scale μ in dimensional regularization. This expression is similar to the ORM vacuum energy density, in particular both ORM equations can be obtained from one loop QEA either using

$$\frac{\partial V[\psi,m^2]}{\partial\psi}=0,\ \frac{\partial V[\psi,m^2]}{\partial m^2}=0 \tag{3.232}$$

or using

$$\frac{\partial V[\psi,m^2]}{\partial\psi}=0,\ \frac{\partial^2 V[\psi,m^2]}{\partial\psi^2}=m^2. \tag{3.233}$$

Eq.(3.232) and (3.233) can be seen to provide another interpretation for systematic extension of the ORM either as a stationary point of QEA with respect to variation of both field shift and mass, or with mass self-consistency condition (3.233).

We shall note that in the original QEA approach the bare mass of the particle m^2 is not subject to variation but is fixed. The physical mass, in fact, is defined by $m_r^2=\partial^2\Gamma[\psi,m^2]/\partial\psi^2$ but it is not necessary to have $m_r^2=m^2$. It is implied, of course, that the mass in the perturbative propagator can be chosen quite freely in accord with the other parameters and counter-terms as long as the renormalized mass stays the same. In a sense, one can always redistribute contributions between the perturbative propagator mass and the counter-terms and end up with the same result. The ORM-motivated self-consistency condition $m_r^2=m^2$ is, in a sense, optimization requirement when the mass used in the perturbative propagator coincides with the exact position of the pole in the physical propagator.

Stationary conditions (3.232) can be also deemed legitimate and viewed as a simplified form of approach in which both 1-point function $\langle\phi\rangle$ and two-point $\langle\phi\phi\rangle$ function $G(x,y)$ are allowed to vary. Indeed, such formulations had been known [135].

Here one assumes a-priory form of the propagation function $G(x,y)$ and field configuration $\psi(x)$ in the computing QEA $\Gamma[\psi,G]$ with two source terms $J(x)$ and $K(x,y)$ for 1- and 2-point functions respectively. One can show then that $J(x)=0$ and $K(x,y)=0$ is equivalent to stationary conditions

$$\begin{array}{l} \frac{\delta\Gamma(\psi,G)}{\delta\psi}=0, \\ \frac{\delta\Gamma(\psi,G)}{\delta G}=0. \end{array} \tag{3.234}$$

While such full functional problem is practically intractable due to complexity of the calculations, one can imagine its simplified version where the propagator $G(x,y)$ is parametrized with a single mass parameter M^2 so that Eq.(3.234b) reduces to stationary condition with respect to M^2.

Eq.(3.232) is also similar to so called Gaussian Effective Potential (GEP) method where the vacuum energy is estimated on a set of "gaussian" quantum states with dispersion parameter Ω as $V(\psi,\Omega)$ [99–101]. GEP is defined as $V(\psi,\Omega^*(\psi))$ for such $\Omega^*(\psi)$ for which $V(\psi,\Omega)$ is minimal for given ψ. Alternative vacuum solutions correspond to the stationary conditions of $V^*(\psi)$ with respect to ψ,

$$\frac{dV(\psi,\Omega^*(\psi))}{d\psi}=0 \tag{3.235}$$

with the physical mass given by

$$m_r^2=\frac{d^2V(\psi,\Omega^*(\psi))}{d\psi^2}. \tag{3.236}$$

For ϕ^4 scalar theory all of these methods produce the same equations of state that coincide with 1-loop QEP. However, they are different in the ways solution is interpreted [102].

Both interpretations of the ORM, in principle, allow for systematic improvement by taking into account higher order perturbative corrections to the Effective Potential. In higher orders QEP can be found to have contributions from 2 and 3 loops as follows,

$$\begin{array}{rl} V_2= & -\hbar^2\frac{3!}{2!}\left(\frac{\lambda\psi}{3!}\right)^2\frac{A_1}{M}+\hbar^2 3\frac{\lambda}{4!}\Delta^2(M), \\ V_3= & -\hbar^3\frac{4!}{2!}\left(\frac{\lambda}{4!}\right)^2\frac{A_2}{M}+\hbar^3\frac{3!^3}{2!}\left(\frac{\lambda\psi}{3!}\right)^2\frac{\lambda}{4!}\frac{A_3}{M^2}-\hbar^3\frac{3!^4 4!}{4!16}\left(\frac{\lambda\psi}{3!}\right)^4\frac{A_4}{M^3}- \\ & \hbar^3\frac{3\cdot 4!}{2}\left(\frac{\lambda}{4!}\right)^2\Delta^2(M)\frac{A_5}{M}+\hbar^3\frac{3!^3}{2}\left(\frac{\lambda\psi}{3!}\right)^2\frac{\lambda}{4!}\Delta(M)\frac{A_6}{M^2}, \end{array} \tag{3.237}$$

where $M = m^2 + \lambda\psi^2/2$, $\Delta(M) = 2A - \frac{\ln\frac{M}{\mu^2}}{4\pi}$ and constants $A_1 \ldots A_6$ are related to corresponding loop integrals and are $A_1 = 0.01484$, $A_2 = 0.00424$, $A_3 = 0.0007404$, $A_4 = 0.000332$, $A_5 = 1/4\pi$, $A_6 = 0.004948 = A_1/3$.

Investigation with QEP up to 3-loops of the critical behavior of ϕ^4 theory can be promptly carried out with Eq.(3.237). In 1-loop one recovers result essentially identical to that of the ORM or the GEP [102]. 1st order phase transition is observed at the coupling constant well consistent with lattice calculations [136]. In 2-loops QEP shows second order phase transition at the value of coupling constant much reduced (by a factor of 2) relative to 1-loop result. This conclusion is identical to 2-loop Generalized GEP known in the literature [137, 138]. One may consider this step as a progress since the order of the phase transition is now correct and in agreement with all universality arguments, lattice simulations and constructive field theory implications [128]. In 3-loop level, however, the picture changes dramatically with no phase transition being observed at all. QEP has a deep minimum at $\psi = 0$ and then behaves quite flat till fast ψ^4 ascend is established [see Fig.(3.20). This result can be viewed both optimistically and pessimistically. Optimistically we may see improvement with regard to convexity problem. Specifically, since second derivative of $V(\psi)$ is associated with the square of the physical mass, QEP is bound to be convex. Most perturbative calculations, however, typically result in QEP which convexity is violated at some points, which is inevitable if one hopes to see nontrivial critical behavior in the theory. It is conjectured normally that the fix for this problem is that, if QEP has two nontrivial minima at $\pm\psi^*$, QEP in between these minima should be given with a flat straight line representing mixing of $\pm\psi$ vacua [129, 132, 139]. Approaching of the flat regime by QEP in 3-loop approximation can be seen as a sign of such improvement in the perturbatively calculated expression. On the other hand, such dramatic change of conclusions about the critical nature of ϕ^4 theory in 1-, 2- and 3-loop approximations in QEP tells us that the perturbative approach is ultimately unreliable in trying to answer this question of the critical behavior.

One can hope to improve the situation here by merging together ideas of the ORM

and QEA and using Eq.(3.237) as input in the ORM extension followed Eq.(3.232). In 1-loop the original ORM equations are recovered, however at higher orders certain inconsistencies develop which do not allow to safely follow this direction of improvement of the ORM. In 3-loop approximation, specifically, the structure of the mass counter-term turns out to be such that the trivial point $M^2 = m^2, \psi = 0$ cease to be the solution of Eq.(3.232). Such setback is quite serious since we want the original vacuum to be one of the phases in the theory. We see the reason for this problem in that, as was mentioned previously, this line of extension can be seen as an approximation to a more complex functional relation (3.234). In lower orders of perturbation theory, specifically in 1-loop order, the effect of perturbative corrections is essentially to renormalize the mass by a constant self-energy contribution. At higher orders, however, momentum dependence of the counter-terms and, therefore, of the renormalized mass becomes important and simple parametrization of $G(p^2)$ by $p^2 - M^2$ fails to accommodate this feature. We believe our problems in defining ORM equations in 3-loops are essentially related to this failure of a simple mass shift to accommodate increasingly complicated $m^2(p^2)$.

Second generalization of the ORM using Eq.(3.233) and Eq.(3.237) had been also attempted recently [140]. In this approach one uses Borel summed version of the QEP (3.237) supplemented with the ORM-motivated mass self-consistent condition $m^2 = d^2V/d\psi^2$, resolved for simplicity not for the Borel-Summed QEP, but for 3rd order perturbative expression for QEP. In [140] perturbation series in coupling constant g (3.159) was used rather than loop expansion (3.237). It was found that Borel Summation of QEP improves situation by yielding second order phase transition with 3rd order of perturbation theory. Critical exponents were also found in this work and it was shown that nontrivial values can be obtained, i.e. $\beta \neq 1/2$ etc. This can be definitely seen as a step forward, although in the light of the above discussion about poor applicability of perturbation series to critical region, significantly more analysis of this result must be performed to insure its reliability. Most importantly, given that 3rd order is the first order in which 2nd kind of the phase transition is

restored, it is desirable to extend the analysis to 4th and 5th orders of perturbation series and confirm that the primary conclusion doesn't change. Also, the final result should not, strictly speaking, depend on the way the mass in the propagator is chosen so that it should be examined by how much conclusions of study in Ref. [140] change if self-consistency relation $m_r^2 = m^2$ is omitted or substituted, e.g., with standard prescription $m^2 \to m^2 + \lambda\psi/2$.

3.7 Summary

The degrees of freedom change and rearrangement in critical phenomena in field theories gain increasing importance given new advances in theoretical and experimental understanding of low-energy and high-density QCD phenomenology. Critical phenomena in QFT reveal themselves as an abrupt change in observables of a model with the continuous change of the parameters and are likely to be responsible for such fundamental issues as the hadronization in QGP, baryon-antibaryon asymmetry in the universe and many others. Unfortunately, critical behavior in field theories proved to be highly nonperturbative phenomenon and the most successful, and up to day most elaborated, tools of QFT such as perturbation theory and RGE cannot be reliably applied here. Significant efforts and many hopes are being put into developing and maintaining lattice calculations in QFT which are seen as the most reliable *ab initio* way for studying strongly interacting field theories. Nonetheless, search for simpler analytical techniques capable of providing the ways for qualitative and quantitative analysis of these issues had never ceased to be a high priority.

In this chapter we have investigated applications of a particular such approach known as the Oscillator Representation Method. Though simple in applications and transparent in the results, this method enjoyed little attention relative to comparable techniques such as Hartree approximation or GEP. The ORM relies on utilization of canonical transformation of field variables to introduce a set of unitary nonequivalent representations of the original theory. Duality, observed in the ORM solutions

in ϕ^4, ϕ^6 and nonlinear σ models, allows to use such representations to introduce perturbative description of strong-interaction regimes of the original model. Different unitary nonequivalent representations of the canonical commutation relations can be also thought to relate to different phases of the theory in which case they may be relevant to describe sudden changes of physical properties of the model occurring as the result of a phase transition.

In our investigation we fully reviewed the ORM and furthermore considered its applications in phase transition in nonlinear σ-model and other field-theoretic models. In nonlinear σ-model we found that the ORM gives results coinciding with large N approximation if applied in the formulation with Lagrange multipliers to enforce $\phi^2 = R^2$ constraint. Second order phase transition is recovered in this case with critical parameters reproducing those of large N treatment. In case a magnetic field is added to the Lagrangian, the second order phase transition is completely destroyed in agreement with our general argument.

In the form in which the constraint is algebraically resolved and the symmetry is broken the ORM ran into terminal difficulties with its estimate of the ground state energy being a divergent series. In this form we could not apply the ORM other than using RGE. Here, the ORM relies completely on the use of RG functions known from the literature and gives only some additional insight into the dynamics of the phase transition. Such study does depend significantly on the quality with which RG functions were given and changes appreciably from order to order of perturbation theory.

We also considered applications of the ORM in ϕ^4 in 1+1 dimensions in the presence of magnetic field and found that, although qualitatively correct in the case without magnetic field [96], the ORM in this case contradicts to strict theorems of Constructive Quantum Field Theory that claim disappearance of the phase transition for whatever small external perturbations [128]. In fact we found that, according to the ORM, the phase transition of the first kind persists even when small perturbation is added. We found that such peculiar behavior is direct consequence of the 1st order

phase transition given by the ORM in 1+1 ϕ^4 theory in the absence of magnetic field. We attribute this result to approximate nature of the ORM. To check this conclusion we further computed first perturbative corrections to the vacuum energy in Symmetric and Broken Symmetry phases. We found that, although these corrections are small in S-phase for small g and in BS-phase for large g, in the region of the phase transition perturbative effects are significant and substantially affect the nature and location of the phase transition point. Remarkably, for large g after the main quantum contributions to the vacuum are accounted for by going to BS representation the BS-phase is weakly coupled and perturbative approach can be successful for increasingly large g. This feature of the ORM may be beneficial for other field theories.

We considered different possibilities for further improvement of the ORM. For that a nonperturbative meaning behind the ORM should be uncovered. While no such principle had been provided by the authors who originally formulated the ORM, we investigated various possibilities including extension of the "correct" form requirement to higher orders or RGE, taking into account finite pieces of RGE, interpreting the "correct" form as the "best" perturbative representation of the original theory, variational analogs and QEA analogs. Unfortunately, no reliable and unquestionably successful such extension was found in our attempts have been hindered by numerous inconsistencies in higher orders of perturbation theory or loss of original conceptual content.

In this respect, variational analog where the trial state "parameterization" is extended from single "mass" parameter, such as in the ORM or GEP, to a form-factor may present a clear advantage both in the sense of nonperturbative character and selection of appropriate degrees of freedom. Additional flexibility of such extension may be important when accounting for higher order renormalizations of the theory where simple "mass" parametrization fails to grasp the nature of the true ground state.

Another interesting extension may be in QEA analogs of the ORM where study had shown improvement when higher order corrections were taken into account using

Borel Summation. While such improvements are certainly welcomed, one should be cautious about them given low reliability of perturbative expansion in the massive phase in the near critical region. Still, investigation along this direction deserve on further attention.

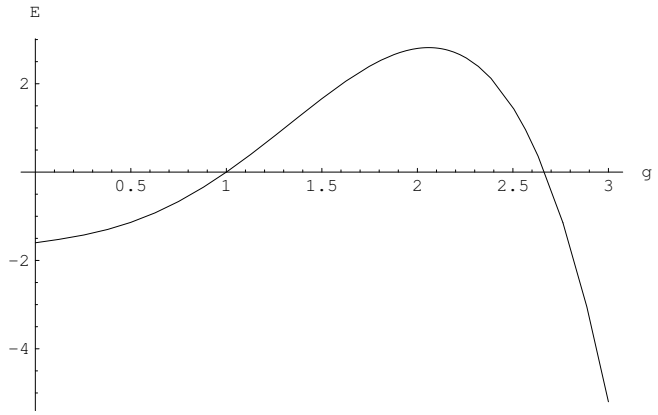

Figure 3.9: The ORM vacuum energy E(t) in nonlinear σ-model at the lowest order in g.

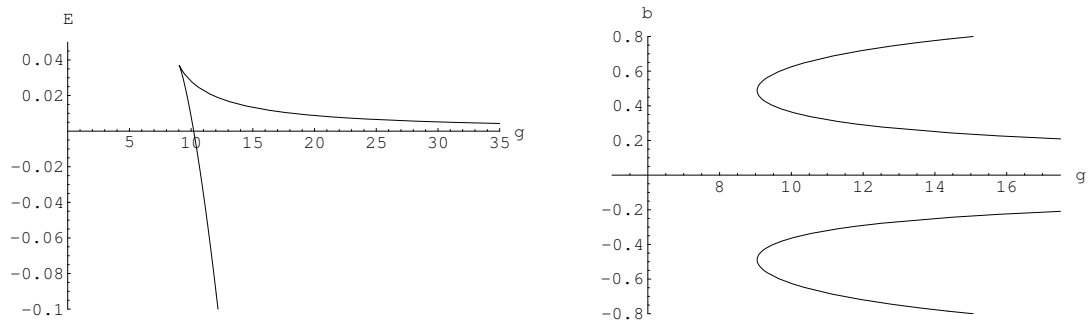

Figure 3.10: Energy density (left panel) and vacuum condensates (right panel) for the solutions of Eq.(3.20). Here s_0 solution coincides with g-axis.

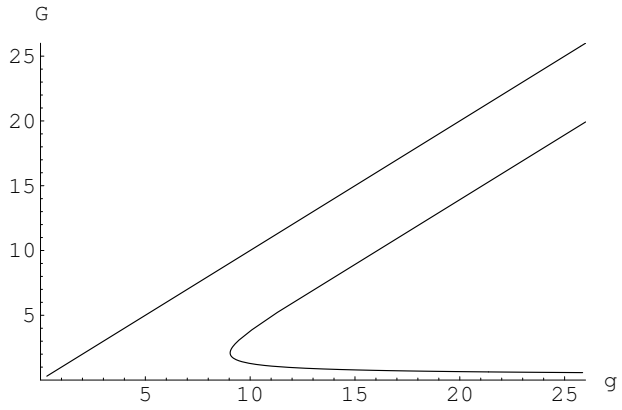

Figure 3.11: Effective dimensionless coupling $G = g/M^2$ for solutions of Eq.(3.20). The straight line corresponds to s_0, $s_\pm$ solutions are equal and correspond to the lower curve.

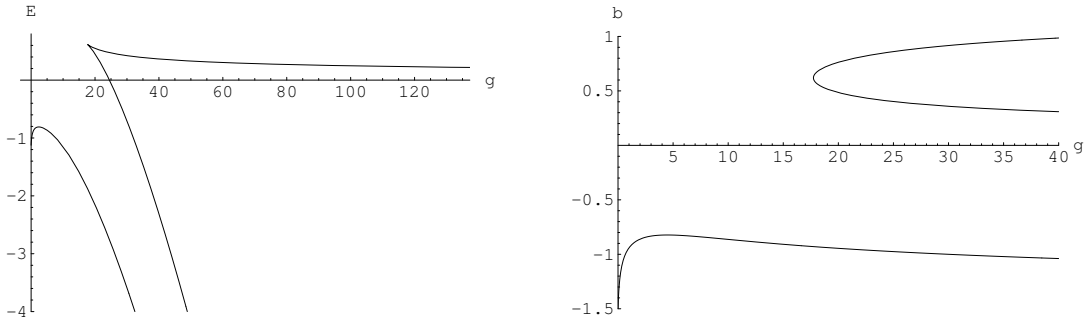

Figure 3.12: The ORM vacuum energy density and vacuum condensates for 1+1 ϕ^4 theory in strong magnetic field ($B/m^2 > 1$).

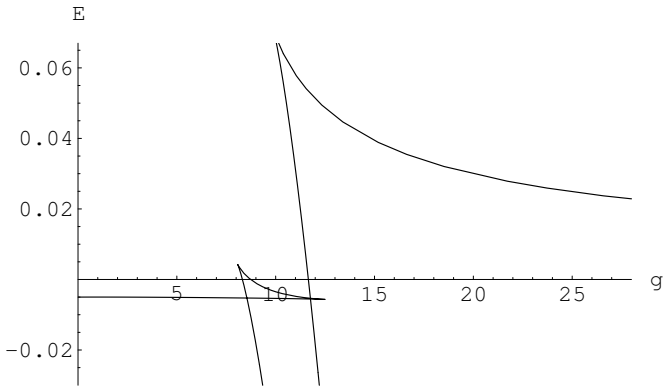

Figure 3.13: The ORM vacuum energy density in 1+1 ϕ^4 theory in weak magnetic field ($B/m^2 \sim 0.01$).

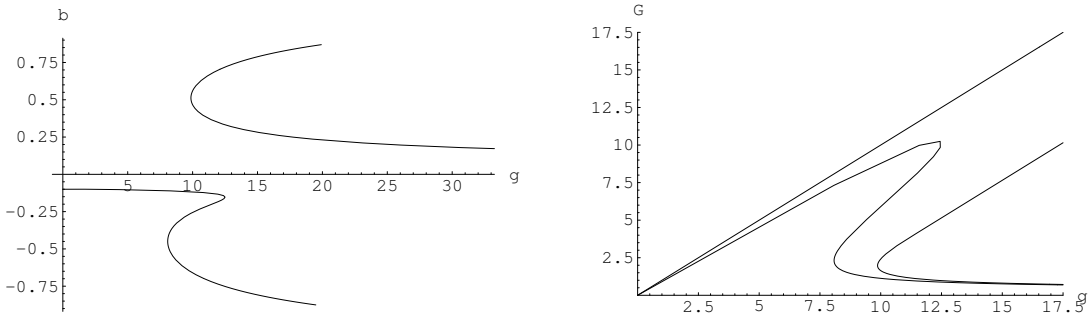

Figure 3.14: Vacuum condensates (left) and the effective dimensionless coupling (right) for 1+1 ϕ^4 theory in weak magnetic field ($B/m^2 \sim 0.01$.

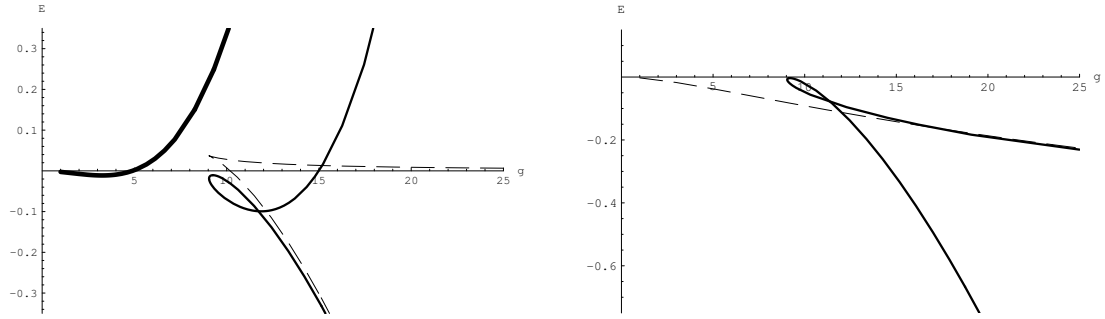

Figure 3.15: The ORM vacuum energy perturbatively corrected ($B = 0$ case, left): S-phase is thick line, dashed and solid lines are 0-order and perturbatively corrected vacuum energies for BS-phase, respectively. Also shown the ORM vacuum energy after Borel summation in S- (dashed line) and BS-phase (solid line) (right).

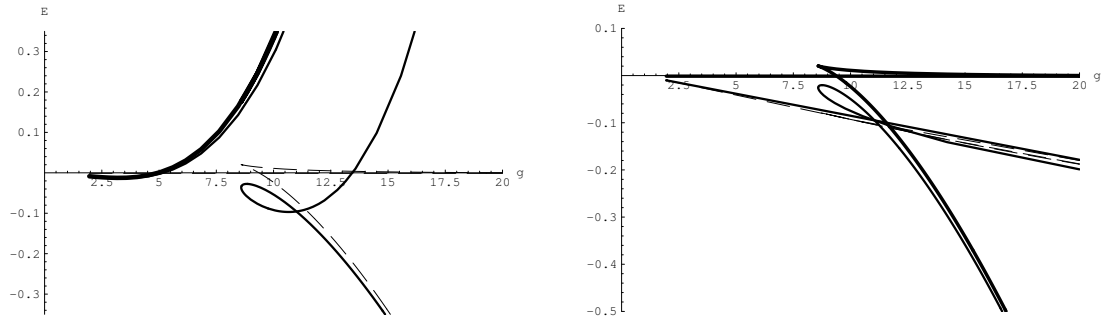

Figure 3.16: The ORM vacuum energy density in S- and BS-phase with $B/m^2 \approx 0.05$. Perturbation series Eq.(3.159) on the left panel and Borel summed perturbation series on the right panel. Dashed line corresponds to $B = 0$ while thick line is $B/m^2 \approx 0.05$.

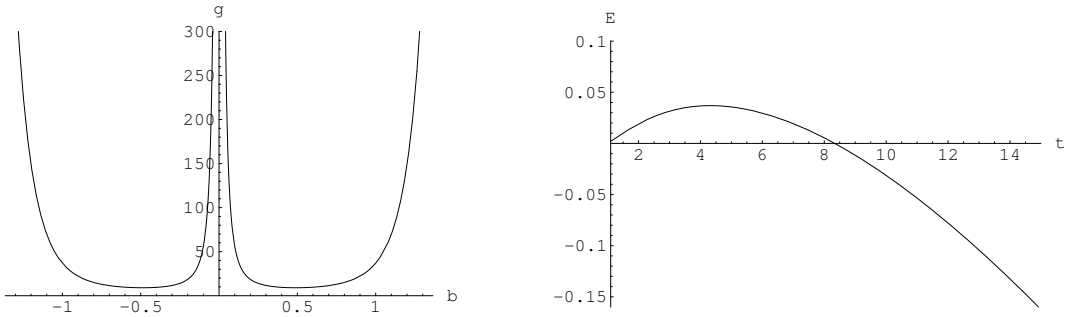

Figure 3.17: Qualitative diagram of possible stable states for a system undergoing 1st order phase transition (left) and free energy of quasi-stable states as traced along the curve. Here t=1 corresponds to $b \to 0$. Increasing t corresponds to moving away from $b = 0$ in either direction (right).

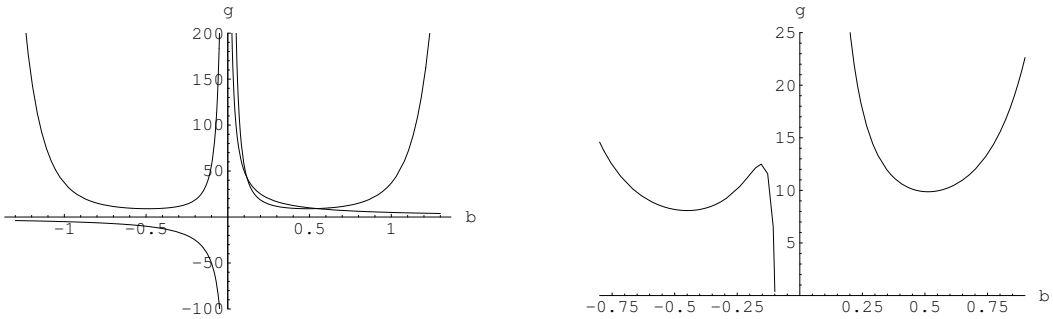

Figure 3.18: $\frac{\delta S}{\delta U}$ (original) and $\frac{\delta B}{\delta U}$ (perturbation) contributions to the equation of state (left panel) and the resulting diagram for quasi-stable states in the presence of a small asymmetric perturbation (right panel).

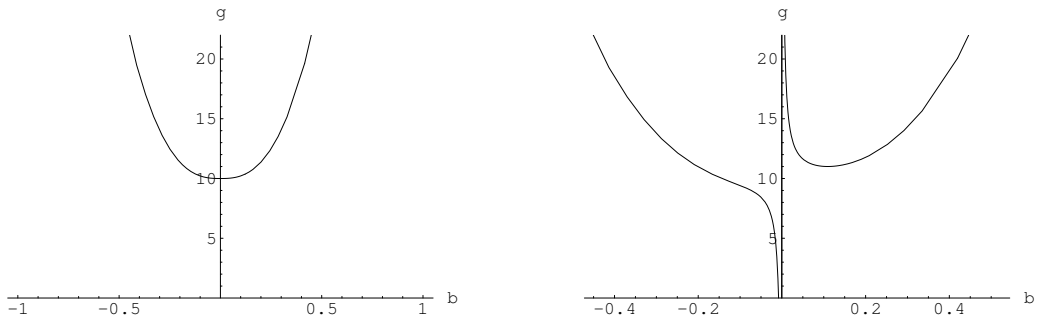

Figure 3.19: Qualitative diagram for quasi-stable states in a system undergoing 2nd order phase transition (left) and qualitative diagram for quasi-stable states in the presence of small perturbation (right).

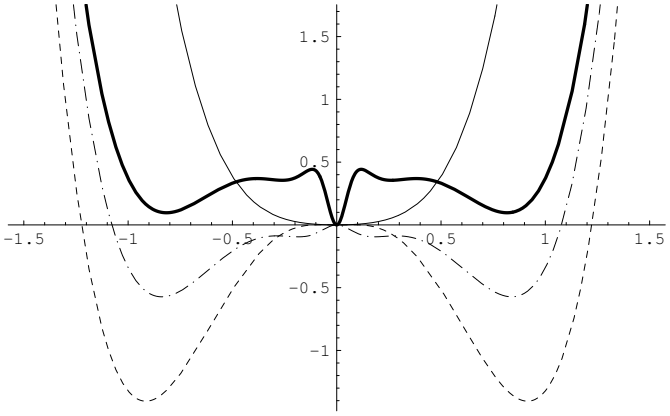

Figure 3.20: Quantum Effective Potential in 1+1 scalar ϕ^4 theory up to 3-loops. Thin line corresponds to 0-loop, dashed and dot-dashed lines are 1- and 2-loop approximations respectively, thick line is 3-loops.

Chapter 4

Quantum Gaussian Packets and Symmetric Decomposition Problem

We would like to present now different interesting approach to the phase transition in nonlinear σ-model. This approach is based on the observation that σ-model may be formulated in the form that involves only operators quadratic in fields and a hypothesis that there exists a unique correspondence between quantum states and the expectations values of the form $\langle\phi\phi\rangle$. For example, as we will show latter, there exists a map between $g(k,k') = \left\langle \eta | a_k^\dagger a_k | \eta \right\rangle$ and the Fock space $|\eta\rangle$. Knowing this correspondence one could reformulate the original problem in terms of the above mentioned expectation values and attempt to solve it at this level.

We shall formulate a variant of nonlinear σ-model with the constraint imposed softly "on average" on the quantum states, similar to the way gauge constraint may be enforced in gauge field theories,

$$\begin{gathered} H = \tfrac{1}{2}\int d\mathbf{x} \sum_i (\vec{\pi}^2 + (\vec{\partial}\vec{\phi})^2 + \mu^2\vec{\phi}^2), \\ \left\langle \vec{\phi}^2 \right\rangle = f^2. \end{gathered} \tag{4.1}$$

Analogously to gauge field theories, we shall call the subspace of the Fock space, introduced via quantization of the Hamiltonian $\mathcal{H}_0 = \frac{1}{2}[(\partial_\mu\vec{\phi})^2 + \mu^2\vec{\phi}^2]$, on which $\left\langle \vec{\phi}^2 \right\rangle = f^2$ the physical subspace. In the light of the above comment we would like to investigate the physical subspace of the model described in terms of the expectation

values $f_\eta = \langle N[(\partial\phi)^2]\rangle$ and $g_\eta = \langle N[\phi^2]\rangle$ and, in particular, we will be interested in its ground state as function of f, i.e.

$$\left\{ \begin{array}{ll} |\eta\rangle : & \min\limits_{\eta}(f_\eta + \mu^2 g_\eta), \\ & g_\eta = f^2. \end{array} \right. \tag{4.2}$$

Our main goal, therefore, is to derive the constraints on $f_\eta = \left\langle \int dx N[(\partial\phi(x))^2]\right\rangle$ following from $g_\eta(x) = \langle N[\phi^2(x)]\rangle = f^2$.

In terms of the quantized degrees of freedom (a_k, b_{-k}) this reads

$$\begin{array}{c} f_\eta = \left\langle \eta| \sum\limits_k \frac{k^0}{V}[a_k^\dagger a_k + b_{-k}^\dagger b_{-k}]|\eta \right\rangle, \\ g_\eta(x) = \left\langle \eta| \sum\limits_{k,k'} \frac{e^{i(\vec{k}-\vec{k}')\vec{x}}}{2V\sqrt{k^0 k'^0}}(a_k^\dagger a_{k'} + b_{-k'}^\dagger b_k + b_{-k}a_{k'} + a_k^\dagger b_{-k'}^\dagger)|\eta \right\rangle. \end{array} \tag{4.3}$$

Therefore, we will be interested in the following expectation values and relations amongst them

$$\begin{array}{c} g_{11}(k,k') = \left\langle \eta|a_k^\dagger a_{k'}|\eta \right\rangle, g_{22}(k,k') = \left\langle \eta|b_{-k'}^\dagger b_k|\eta \right\rangle, \\ g_{12}(k,k') = g_{21}^*(k',k) = \langle \eta|b_{-k}a_{k'}|\eta\rangle, \end{array} \tag{4.4}$$

computed on the quantum states from the symmetrized Fock space

$$|\eta\rangle = \sum_{n,m=0}^{\infty} \int [d\vec{k}]^n [d\vec{k'}]^m c_{n,m}^s(\vec{k}_1, \ldots, \vec{k}'_m) a_{k_1}^\dagger \ldots b_{-k'_m}^\dagger |0\rangle. \tag{4.5}$$

4.1 Symmetric Decomposition Problem

Let us begin with somewhat simpler case, i.e. let us show that the image of the Fock space $\mathcal{F}$ cast by transformation $\Upsilon : |\eta\rangle \to \left\langle \eta|a_k^\dagger a_{k'}|\eta \right\rangle = g_{11}(k,k')$ is the space of all hermitian positive definite linear integral operators with finite trace,

$$\mathcal{A} = \left\{ \begin{array}{c} g_{11}(k,k') = g_{11}^*(k',k) \text{ and} \\ \int\int dk dk' g_{11}(k,k') f^*(k) f(k') \geq 0, \forall f(k) \text{ and} \\ \int dk g_{11}(k,k) < \infty \end{array} \right. . \tag{4.6}$$

Let us first show that $\Upsilon(\mathcal{F}) \subset \mathcal{A}$. Since the Fock space is defined by $\left\langle \eta | \sum_k a_k^\dagger a_k | \eta \right\rangle < \infty$, obviously for any $|\eta\rangle \in \mathcal{F}$ $g_{11}(k,k')$ exists and has finite trace. By trivial properties of the inner product and taking into account that $g_\eta(k,k')$ can be also written as $(a_k\eta, a_{k'}\eta)$,

$$\begin{gathered} g_{11}(k',k) = \langle a_{k'}\eta | a_k\eta \rangle = \langle a_k\eta | a_{k'}\eta \rangle^* = g_{11}^*(k,k') \text{ and} \\ \int\int dkdk' g_{11}(k,k') f^*(k) f(k') = \int\int dkdk' f^*(k) f(k') \langle a_k\eta | a_{k'}\eta \rangle = \langle a(f)\eta | a(f)\eta \rangle \geq 0, \end{gathered} \tag{4.7}$$

where we denoted by $a(f) = \int dk f(k) a_k$. This shows that at least $\Upsilon(\mathcal{F}) \subset \mathcal{A}$. We now shall prove Symmetric Decomposition Theorem I.

Symmetric Decomposition Theorem I: Let $g(x,y)$ be a kernel of a hermitian positive-definite linear integral operator, then for any $n \geq 2$ there exist $f_n(x_1, \ldots, x_n)$, totally symmetric under permutations of $(x_1, \ldots x_n)$, such that

$$g(x,y) = \int dx_1 \ldots dx_{n-1} f_n(x, x_1, \ldots, x_{n-1}) f_n^*(y, x_1, \ldots, x_{n-1}). \tag{4.8}$$

Proof: Under the conditions of the theorem, there exist a set of orthonormal functions $g_m(x)$ such that

$$g(x,y) = \sum_{m=0}^{\infty} \lambda_m g_m(x) g_m^*(y). \tag{4.9}$$

Then choose $f_n(x_1, \ldots, x_n) = \sum_{m=0}^{\infty} \alpha_m^{(n)} g_m(x_1) \ldots g_m(x_n)$. Trivially

$$\begin{gathered} \int dx_1 \ldots dx_{n-1} f_n(x, x_1, \ldots, x_{n-1}) f_n^*(y, x_1, \ldots, x_{n-1}) = \\ \int dx_1 \ldots dx_{n-1} \sum_{m,m'=0}^{\infty} \alpha_m^{(n)} \alpha_{m'}^{(n)*} g_m(x) \ldots g_m(x_{n-1}) g_{m'}^*(y) \ldots g_{m'}^*(x_{n-1}). \end{gathered} \tag{4.10}$$

Due to orthonormality, $\int dx g_m(x) g_{m'}(x) = \delta_{m,m'}$, all $n-1$ integrations in Eq.(4.10) produce $\delta_{m,m'}$ and we are left with

$$\int dx_1 \ldots dx_{n-1} f_n(x, x_1, \ldots, x_{n-1}) f_n^*(y, x_1, \ldots, x_{n-1}) = \sum_{m=0}^{\infty} \left| \alpha_m^{(n)} \right|^2 g_m(x) g_{m'}^*(y). \tag{4.11}$$

Due to positive-definiteness, the eigenvalues of operator $g(x,y)$ λ_m are positive for all m so that it is enough to choose $\left|\alpha_m^{(n)}\right|^2 = \lambda_m$ in Eq.(4.11) to complete the proof.

Then, if $f_n(\{k\})$ is a series of functions from Eq.(4.8), we define

$$|\eta\rangle = \sum_n \frac{\beta_n}{\sqrt{n!}} \int [d\vec{k}]^n f_n^*(\vec{k}_1, \ldots, \vec{k}_n) a_{k_1}^\dagger \ldots a_{k_n}^\dagger |0\rangle \tag{4.12}$$

with β_n chosen in such a way that

$$\begin{gathered}\left\langle \eta | a_k^\dagger a_{k'} | \eta \right\rangle = \sum_n n\beta_n^2 \int [d\vec{k}]^{n-1} f_n(\vec{k}, \ldots, \vec{k}_n) f_n^*(\vec{k}', \ldots, \vec{k}_n) = \sum_n n\beta_n^2 g(k,k') = g(k,k'), \\ \langle \eta | \eta \rangle = \sum_n \beta_n^2 \int [d\vec{k}]^n f_n(\vec{k}_1, \ldots, \vec{k}_n) f_n^*(\vec{k}_1, \ldots, \vec{k}_n) = Tr[g(k,k')] \sum_n \beta_n^2 = 1.\end{gathered} \tag{4.13}$$

Some obvious properties of this representation are in place. For f_n to be integrable

$$\|f_n\|^2 = \int [d\vec{k}]^n f_n(\vec{k}_1, \ldots, \vec{k}_n) f_n^*(\vec{k}_1, \ldots, \vec{k}_n) = \int dk g(k,k) = Tr[g(k,k')] \tag{4.14}$$

we should require that the operator associated with the kernel $g(k,k')$ has finite trace. This is equivalent to $\left\langle \eta | \hat{N} | \eta \right\rangle < \infty$, thus, $\Upsilon(\mathcal{F}) = \mathcal{A}$. We obtain interesting result that for $g_\eta(k,k') = \left\langle \eta | \sum_k a_k^\dagger a_{k'} | \eta \right\rangle$ there are no other restrictions aside from those following from its definition - i.e. inner product properties.

Further implications can be derived from the Symmetric Decomposition Theorem. It is possible to extend our proof to $g(k,k')$ with infinite trace $\mathcal{A}'$. In this case, the states defined by Eq.(4.12) are no longer in the Fock space, but still are in the Hilbert space of the field theory. If definition of $\left\langle \eta | a_k^\dagger a_{k'} | \eta \right\rangle$ is modified by

$$\left\langle \eta | a_k^\dagger a_{k'} | \eta \right\rangle \equiv \lim_{N\to\infty} \frac{{}_N\left\langle \eta | a_k^\dagger a_{k'} | \eta \right\rangle_N}{{}_N\langle \eta | \eta \rangle_N} \tag{4.15}$$

with $|\eta\rangle_N$ being state $|\eta\rangle$ truncated to first N terms, then unitary inequivalent representations can be naturally incorporated as $\Upsilon^{-1}(\mathcal{A}')$. $\left\langle \eta | a_k^\dagger a_{k'} | \eta \right\rangle$ therefore is a convenient way to describe states from extension of the Fock space including all of the conventional Fock space as $g(k,k')$ with finite trace, but also including unitary inequivalent representations in the generalized sense defined by Eq.(4.15). The states

of the total Hilbert space, that are not included in $\Upsilon^{-1}(\mathcal{A}')$, are those having infinite contributions at single modes, e.g. $\frac{\langle\eta|a_k^\dagger a_k|\eta\rangle}{\langle\eta|\eta\rangle} = \infty$ in some region of nonzero volume. These states correspond to infinite concentration of energy in finite volume of coordinate or momentum space and thus violate finite volume properties of QFT. We conjecture that $\Upsilon^{-1}(\mathcal{A}')$ may be relevant extension of the conventional Fock space for the QFT operators.

The Symmetric Decomposition Theorem may be further generalized to include both particle and antiparticle operators.

Symmetric Decomposition Theorem II: If functions $g_{11}(x,y)$, $g_{12}(x,y)$, $g_{21}(x,y)$, $g_{22}(x,y)$ are such that

$$\hat{g}(x,y) = \begin{pmatrix} g_{11}(x,y) & g_{12}(x,y) \\ g_{21}(x,y) & g_{22}(x,y) \end{pmatrix} \tag{4.16}$$

is kernel of a hermitian, positive definite linear integral operator $G : \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rightarrow \begin{pmatrix} f_1' \\ f_2' \end{pmatrix}$, then there exist a quantum state $|\eta\rangle$ in the symmetrized Hilbert space of quantum field theory such that

$$\hat{g}(x,y) = \left\langle \eta \left| \begin{array}{cc} a_x^\dagger a_y & a_x^\dagger b_{-y}^\dagger \\ a_y b_{-x} & b_{-y}^\dagger b_x \end{array} \right| \eta \right\rangle . \tag{4.17}$$

Proof: As before we are going to prove this statement by construction. We consider first an arbitrary quantum state

$$|\eta\rangle = \sum_n \frac{1}{n!} \sum_{(j)} \int [dx]^n f_n(j_1 x_1, \ldots, j_n x_n) a_{x_1}^\dagger(j_1) \ldots a_{x_n}^\dagger(j_n) |0\rangle , \tag{4.18}$$

where $a_x^\dagger(j) = \begin{cases} a_x^\dagger, j=0 \\ b_{-x}^\dagger, j=1 \end{cases}$, $j_i = \{0,1\}$ and $f_n(j_1x_1,\dots,j_nx_n)$ is completely symmetric function with respect to permutation of particles. We note that

$$\begin{gathered}\langle\eta|a_x^\dagger a_y|\eta\rangle = \sum_n \tfrac{1}{(n-1)!}\sum_{(j)}\int[dx]^{n-1} f_n^*(0x,\dots,j_{n-1}x_{n-1})f_n(0y,\dots,j_{n-1}x_{n-1}),\\ \left\langle\eta|b_{-y}^\dagger b_{-x}|\eta\right\rangle = \sum_n \tfrac{1}{(n-1)!}\sum_{(j)}\int[dx]^{n-1} f_n^*(1y,\dots,j_{n-1}x_{n-1})f_n(1x,\dots,j_{n-1}x_{n-1}),\\ \left\langle\eta|a_x^\dagger b_{-y}^\dagger|\eta\right\rangle = \sum_n \tfrac{1}{n!}\sum_{(j)}\int[dx]^{n} f_{n+2}^*(0x,1y,\dots,j_nx_n)f_n(j_1x_1,\dots,j_{n-1}x_{n-1}).\end{gathered} \tag{4.19}$$

Now we can use the same idea as we used in the proof of the first theorem. Under the conditions of the theorem there exist set of orthonormal $w_t(x) = \begin{pmatrix} g_{0,t}^*(x) \\ g_{1,t}(x) \end{pmatrix}$ such that

$$\begin{gathered}\hat{g}(x,y) = \sum_t \lambda_t w_t(x) w_t^\dagger(y),\\ \int dx w_t^\dagger(x) w_{t'}(x) = \delta_{t,t'}.\end{gathered} \tag{4.20}$$

Then

$$\begin{gathered}g_{11}(x,y) = \sum_t \lambda_t g_{0,t}^*(x)g_{0,t}(y), g_{22}(x,y) = \sum_t \lambda_t g_{1,t}(x)g_{1,t}^*(y),\\ g_{12}(x,y) = g_{21}^*(y,x) = \sum_t \lambda_t g_{0,t}^*(x)g_{1,t}^*(y).\end{gathered} \tag{4.21}$$

Set now

$$f_n(j_1x_1,\dots,j_nx_n) = \sum_t \alpha_n(t) g_{j_1,t}(x_1)\dots g_{j_n,t}(x_n), \tag{4.22}$$

then

$$\begin{aligned}\langle\eta|a_x^\dagger \quad & a_y|\eta\rangle = \sum_n \tfrac{1}{(n-1)!}\\ & \sum_{(j)}\int[dx]^{n-1}\sum_{t,t'}\alpha_n^*(t)\alpha_n(t')g_{0,t}^*(x)\dots g_{j_{n-1},t}^*(x_{n-1})g_{0,t}(y)\dots g_{j_{n-1},t}(x_{n-1}).\end{aligned} \tag{4.23}$$

Due to orthogonality all summations and integrations give $\delta_{t,t'}$ so that

$$\langle\eta|a_x^\dagger a_y|\eta\rangle = \sum_n \frac{1}{(n-1)!}\sum_t |\alpha_n(t)|^2 g_{0,t}^*(x)g_{0,t}(y). \tag{4.24}$$

Analogously

$$\begin{aligned}\left\langle \eta | b^{\dagger}_{-y} b_{-x} | \eta \right\rangle &= \sum_n \tfrac{1}{(n-1)!} \sum_t |\alpha_n(t)|^2 g^*_{1,t}(y) g_{1,t}(x), \\ \left\langle \eta | a^{\dagger}_x b^{\dagger}_{-y} | \eta \right\rangle &= \sum_n \tfrac{1}{n!} \sum_t \alpha^*_{n+2}(t) \alpha_n(t) g^*_{0,t}(x) g^*_{1,t}(y).\end{aligned} \tag{4.25}$$

One can always choose $\alpha_n(t)$ such that $\frac{1}{(n-1)!}\sum\limits_n |\alpha_n(t)|^2 = \frac{1}{n!}\sum\limits_n \alpha^*_{n+2}(t)\alpha_n(t) = \lambda_t$ whenever λ_t are positive due to positive definiteness of the operator G. Then, taking into account Eq.(4.21) theorem is proved.

By this we have proved that as long as $\hat{g}(k,k')$ defines a positive definite hermitian operator, there are no other constraints on $g_{11}(k,k'), g_{22}(k,k'), g_{12}(k,k')$. Then the problem given by Eq.(4.2) reads as

$$\begin{aligned}&\min_{\{g_{11},g_{12},g_{22}\}} \sum_k \tfrac{k^0}{2V}[g_{11}(k,k) + g_{22}(k,k)], \\ &\sum_{k,k'} \tfrac{e^{i(k-k')\vec{x}}}{2V\sqrt{k^0 k'^0}} \left(g_{11}(k,k') + g_{22}(k,k') + 2Re[g_{12}(k,k')]\right) = f^2.\end{aligned} \tag{4.26}$$

Unfortunately, Symmetric Decomposition Theorem II is not sufficient to solve this problem. The reason is that, as opposed to the case with a single particle,

$$\left\langle \eta \left| \begin{array}{cc} a^{\dagger}_x a_y & a^{\dagger}_x b^{\dagger}_{-y} \\ a_y b_{-x} & b^{\dagger}_{-y} b_x \end{array} \right| \eta \right\rangle$$

cannot be shown to be positive definite for all states $|\eta\rangle$. As a matter of fact, considering states $|\alpha_k\rangle$, introduced in Section 3.6.2, demonstrates that $\langle \alpha_k | \hat{g} | \alpha_k \rangle$ violates positive definiteness, so that more work is needed to establish better boundaries on $\Upsilon(\mathcal{F})$. Most definitely, the space of all positive-definite hermitian $\hat{g}$ is contained in $\Upsilon(\mathcal{F})$, but more states may be also present thus violating restrictions between g_{11}, g_{22} and g_{12} following only from positive definiteness of $\hat{g}$.

4.2 Quantum Gaussian Packets

To advance our study of the symmetric decomposition problem, we will have to take a side road and look at, important in its own, Quantum Gaussian Packets (QGPs).

We say that a quantum state $|\alpha\rangle = U|0\rangle$ is a QGP iff

$$Ua_tU^\dagger = \sum_{t'}(\alpha_{tt'}a_{t'} + \beta_{tt'}a^\dagger_{t'}), \tag{4.27}$$

where a_t is ladder operator generalized to incorporate both $a_{k\sigma}$ and $b_{k\sigma}$, as needed, via multi-index t and $\sum\limits_{t'}$ is the generalized sum over multi-index, i.e. sum over discrete indexes and integration over continuous ones.

Let us clarify our definition and its relation to word "gaussians". Define

$$\begin{aligned} U_g &= \exp\left(-\tfrac{i}{2}\sum_{tt'} g_{tt'}[a^\dagger_t a^\dagger_{t'} + a_t a_{t'}]\right), \\ V_\kappa &= \exp\left(-\tfrac{i}{2}\sum_{tt'} \kappa_{tt'}[a^\dagger_t a_{t'} + a_{t'} a^\dagger_t]\right). \end{aligned} \tag{4.28}$$

Here $g_{tt'}$ is symmetric real and $\kappa_{tt'}$ is hermitian (e.g. $\kappa_{tt'} = \kappa^*_{t't}$) and the above convention for the ladder operators is adopted. Obviously U_g, V_κ are unitary and

$$\begin{gathered} U_g a_i U_g^\dagger = \cosh(g)_{it}a_t + i\sinh(g)_{it}a^\dagger_t, \\ V_\kappa a_i V_\kappa^\dagger = \exp(i\kappa)_{it}a_t,\ V_\kappa a^\dagger_i V_\kappa^\dagger = \exp(-i\kappa)_{ti}a^\dagger_t \\ \text{summation over } t. \end{gathered} \tag{4.29}$$

For $U^\dagger = V^\dagger_\kappa U^\dagger_g V^\dagger_\lambda$, we, consequently, find

$$\begin{gathered} Ua_iU^\dagger = V_\lambda U_g V_\kappa a_i V^\dagger_\kappa U^\dagger_g V^\dagger_\lambda \\ = [e^{i\kappa}\cdot\cosh(g)\cdot e^{i\lambda}]_{it}a_t + i[e^{i\kappa}\cdot\sinh(g)\cdot e^{-i\lambda^T}]_{it}a^\dagger_t. \end{gathered} \tag{4.30}$$

whereby "$\cdot$" means matrix multiplication. Eq.(4.30) has the same form as our definition Eq.(4.27) with $\alpha = e^{i\kappa}\cdot\cosh(g)\cdot e^{i\lambda}$ and $\beta = e^{i\kappa}\cdot\sinh(g)\cdot e^{-i\lambda^T}$, so that indeed QGP can be generated by an exponential of a quadratic in quantum fields operator, which we shall analogously call Quantum Gaussian Operator.

From the explicit expressions for α and β we conclude that

$$\alpha\alpha^\dagger - \beta\beta^\dagger = 1,\ \alpha\beta^T = \beta\alpha^T. \tag{4.31}$$

It is easy to see, however, that these constraints are nothing special to the gaussian operators but have more general roots in unitary conditions, i.e.

$$\begin{aligned} U[a_i, a_j^\dagger]U^\dagger &= [\sum_{t'}(\alpha_{it'}a_{t'} + \beta_{it'}a_{t'}^\dagger), \sum_{t''}(\alpha_{jt''}^* a_{t''}^\dagger + \beta_{jt''}^* a_{t''})] = (\alpha\alpha^\dagger - \beta\beta^\dagger)_{ij} = \delta_{ij}, \\ U[a_i, a_j]U^\dagger &= [\sum_{t'}(\alpha_{it'}a_{t'} + \beta_{it'}a_{t'}^\dagger), \sum_{t''}(\alpha_{jt''}a_{t''} + \beta_{jt''}a_{t''}^\dagger)] = (\alpha\beta^T - \beta\alpha^T)_{ij} = 0. \end{aligned} \tag{4.32}$$

We shall show now that in general for any (α, β) that satisfy unitary conditions there exists an appropriate operator Eq.(4.30), identified by (g, κ, λ), such that $\alpha = e^{i\kappa}\cosh(g)e^{i\lambda}$, $\beta = e^{i\kappa}\sinh(g)e^{-i\lambda^T}$.

First of all, we shall elaborate on the dependencies that exist between α and β. It is not difficult to show that the most general form satisfying Eq.(4.31) is

$$\begin{aligned} \beta &= e^{ia^T}Be^{-ib^T}, \\ \alpha &= e^{ia^T}\sqrt{1+B^2}e^{i\mathcal{Z}}e^{ib}, \end{aligned} \tag{4.33}$$

where B is real diagonal, e^{ia}, e^{ib} and $e^{i\mathcal{Z}}$ are unitary matrices (consider e.g. $\beta\beta^\dagger = e^{ia^T}B^2e^{-ia^T}$ is a unitary positive definite matrix). From $\alpha\beta^T = \beta\alpha^T$ we conclude that

$$e^{ia^T}\sqrt{1+B^2}e^{i\mathcal{Z}}Be^{ia} = e^{ia^T}Be^{i\mathcal{Z}^T}\sqrt{1+B^2}e^{ia}. \tag{4.34}$$

Introducing real diagonal matrix $X = B/\sqrt{1+B^2}$ this can be cast into

$$\begin{cases} X^{-1}e^{i\mathcal{Z}}X = e^{i\mathcal{Z}^T}, \\ Xe^{-i\mathcal{Z}}X^{-1} = e^{-i\mathcal{Z}^T}. \end{cases} \Rightarrow X^2 = e^{i\mathcal{Z}}X^2e^{-i\mathcal{Z}}, \tag{4.35}$$

from which it follows, except perhaps some pathological cases, that $e^{i\mathcal{Z}}$ and B commute and $e^{i\mathcal{Z}} = e^{i\mathcal{Z}^T}$. We choose κ and λ in such a way that

$$\begin{aligned} e^{-2i\lambda} &= e^{-ib}e^{-i\mathcal{Z}}e^{-ib^T}, \\ e^{i\kappa} &= e^{ia^T}e^{i\mathcal{Z}}e^{ib}e^{-i\lambda}. \end{aligned} \tag{4.36}$$

In that case

$$\begin{aligned} e^{-i\kappa}\beta e^{2i\lambda^T}e^{-i\lambda^T} &= e^{i\lambda}e^{-ib}Be^{ib}e^{-i\lambda}, \\ e^{-i\kappa}\alpha e^{-2i\lambda}e^{i\lambda} &= e^{i\lambda}e^{-ib}\sqrt{1+B^2}e^{ib}e^{-i\lambda}. \end{aligned} \tag{4.37}$$

One can explicitly check that $e^{-i\kappa}\beta e^{i\lambda^T}$ and $e^{-i\kappa}\alpha e^{i\lambda}$ are symmetric and hermitian, therefore they can be simultaneously represented as $\sinh(g)$ and $\cosh(g)$ respectively for some real symmetric g. This completes our proof that for any α, β satisfying unitary conditions, there exists a quantum gaussian operator such that Eq.(4.27) holds, so that the following statement holds.

QGP Theorem Quantum states introduced by definition Eq.(4.27) are equivalent to quantum states generated by operators Eq.(4.30).

This theorem motivates an following trick: whenever an expectation value is to be evaluated over a QGP, one can find it by changing the ladder operators according to Eq.(4.27) and computing the vacuum expectation of the result. Moreover, any QGP can be equivalently specified by respective α and β.

Let us illustrate application of these principles on the following problem. Denote $\left\langle a_t^\dagger a_{t'} \right\rangle = g_{1,tt'}$ and $\langle a_t a_{t'} \rangle = g_{2,tt'}$, where the expectation value is taken on a QGP. What kinds of g_1 and g_2 can be obtained in this way? Using the above theorem we immediately conclude that

$$\begin{aligned} g_1 &= \left\langle a^\dagger a \right\rangle = \beta \cdot \beta^\dagger, \\ g_2 &= \langle aa \rangle = \beta \cdot \alpha^T. \end{aligned} \tag{4.38}$$

We see, hence, that the necessary conditions on g_1 and g_2 are that g_2 is symmetric and g_1 is positive definite hermitian. Surprisingly, these conditions are not specific to QGP but follow rather immediately from the definition of $g_{1,2}$, e.g.

$$(f, g_1 f) = \left\langle a^\dagger(f) a(f) \right\rangle = (a(f)|\eta\rangle, a(f)|\eta\rangle) \geq 0. \tag{4.39}$$

$\alpha\alpha^\dagger - \beta\beta^\dagger = 1$ introduces stronger constraint which can be cast in the following form:

$$g_2^\dagger g_1^{-1} g_2 = 1 + g_1^T. \tag{4.40}$$

QGP-generated $g_{1,2}$ necessarily satisfy the above conditions. Is the converse true, i.e. any $g_{1,2}$ that satisfy (4.40) can be generated with a QGP? Indeed, it is true for which we may choose

$$\alpha = \sqrt{1 + g_1}, \beta = g_2 (1 + g_1^T)^{-1/2}. \tag{4.41}$$

We've found therefore that all $\langle aa \rangle$ and $\langle a^\dagger a \rangle$ that can be represented by QGP are equivalent to g_2 and g_1 such that

$$\begin{cases} g_1 \text{ positive definite hermitian;} \\ g_2 \text{ symmetric;} \\ g_2^\dagger g_1^{-1} g_2 = 1 + g_1^T. \end{cases} \tag{4.42}$$

Finally, in principle one can solve Eq.(4.42) for g_2 for given g_1 which yet has to be symmetric. One may wonder, therefore, are there any additional constraints on g_1 from this condition? After simple substitution, $g_2 = g_1^{1/2} V (1 + g_1^T)^{1/2}$, we find that V must be unitary - $V^\dagger V = 1$. Thus the pair (g_1, g_2) may be uniquely specified with a hermitian positive definite g_1 and a unitary V.

We note that if g_1 is written as $g_1 = UDU^\dagger$ for some positive real diagonal D and unitary U, then

$$\begin{aligned} g_2 &= UD^{1/2} U^\dagger V U^* (1 + D)^{1/2} U^T, \\ g_2^T &= U(1 + D)^{1/2} U^\dagger V^T U^* D^{1/2} U^T. \end{aligned} \tag{4.43}$$

Thus, $U^\dagger V U^* = U^\dagger V^T U^*$ which means that, besides some possible pathological cases, $V^T = V$ is symmetric unitary matrix. Since we can always choose $V = UU^T$, for any g_1 there exists at least one g_2 that satisfies Eq.(4.42). Therefore, there are no additional restrictions on g_1 - it can be chosen arbitrarily with $g_2 = g_1^{1/2} V (1 + g_1^T)^{1/2}$ with some symmetric unitary V so that Eq.(4.43) is satisfied.

QGPs are closely related to many important methods in QFT. Eq.(4.27) defines generalized Bogoliubov transformation, which was used to solve quantum field theory of flavor oscillations. The QGP of the form

$$|\theta\rangle = \frac{1}{\mathcal{Z}} \exp\left(- \sum_{tt'} (\alpha^{-1} \cdot \beta)_{tt'} a_t^\dagger b_{-t'}^\dagger \right) |0\rangle. \tag{4.44}$$

is used in this theory. QGP also is closely related to the ORM method, in particular the ORM trial state $|c, t\rangle$ corresponds to

$$\begin{aligned} a_k &\to c\delta(k) + \cosh(\zeta_k) a_k + \sinh(\zeta_k) b_{-k}^\dagger, \\ b_{-k} &\to d\delta(k) + \cosh(\zeta_k) b_{-k} + \sinh(\zeta_k) a_k^\dagger \end{aligned} \tag{4.45}$$

and (see Eq.(4.49))

$$\begin{aligned} A_{kk'} &= [(c^2+d^2)\delta_{k0} + (\cosh(2\zeta_k)-1)]\,\delta_{kk'}, \\ B_{kk'} &= [2cd\delta_{k0} + \sinh(2\zeta_k)]\,\delta_{kk'}. \end{aligned} \tag{4.46}$$

ζ_k is specific mixing angle defined by

$$\zeta_k = \frac{1}{2}\ln(\frac{\epsilon_k^{new}}{\epsilon_k^{old}}), \tag{4.47}$$

where $\epsilon_k^{old,new}$ energies of the state k for old and new phases respectively. Easy to verify that all ORM relations hold when represented by QGP of this form. Finally, QGP is intrinsically embedded in GEP method [99].

Besides importance of QGP in QFT of flavor mixing, the ORM and GEP, we will also find one other application of QGP in the discussion of symmetric decomposition problem that will follow.

4.3 Symmetric Decomposition Problem: Solution

We will now consider expectation value

$$\begin{pmatrix} \left\langle \eta|a_k^\dagger a_{k'}|\eta\right\rangle & \left\langle \eta|a_k^\dagger b_{-k'}^\dagger|\eta\right\rangle \\ \langle \eta|a_{k'}b_{-k}|\eta\rangle & \left\langle \eta|b_{-k'}^\dagger b_k|\eta\right\rangle \end{pmatrix} = \begin{pmatrix} g_{11}(k,k') & g_{12}(k,k') \\ g_{21}(k,k') & g_{22}(k,k') \end{pmatrix} = \hat{g}(k,k') \tag{4.48}$$

on a quantum state from the symmetrized Fock space, where index k spans both momentum and spin degrees of freedom as needed and $-k$ refers to negation of the momentum. In the Symmetric Decomposition Problem, we are interested in what are the possible values for g_{ij}? From Symmetric Decomposition Theorems I, we know that g_{11} and g_{22} can be set separately and independently to define kernels of any positive definite hermitian linear integral operators. g_{12} however is not independent of g_{11}, g_{22} and the constraint between components of g is of special interest. In the case $\hat{g}$ is a positive definite operator $C(\Re)\times C(\Re) \to C(\Re)\times C(\Re)$ we know that there always exists a suitable quantum state that yields (4.48). A counterexample can be

constructed to show that $\hat{g}$ must not be positive definite, so that this condition on $\hat{g}$ is too weak. In this section we will provide the arguments which make us believe in a specific solution to this problem and formulate and prove Symmetric Decomposition Theorem III.

Let us introduce linear integral operators with kernels

$$\begin{aligned} A_\eta(k,k') &= \left\langle \eta | a_k^\dagger a_{k'} + b_{-k'}^\dagger b_{-k} | \eta \right\rangle, \\ B_\eta(k,k') &= \left\langle \eta | a_k^\dagger b_{-k'}^\dagger + a_{k'} b_{-k} | \eta \right\rangle. \end{aligned} \tag{4.49}$$

We will be interested in what domain of A_η and B_η can be represented with the quantum states from the symmetrized Fock space.

Let us note first that A_η and B_η are hermitian, A_η is positive definite and

$$\begin{aligned} 1 + A_\eta + B_\eta &= \left\langle \eta | (a_k^\dagger + b_{-k})(a_{k'} + b_{-k'}^\dagger) | \eta \right\rangle \succ 0, \\ 1 + A_\eta - B_\eta &= \left\langle \eta | (a_k^\dagger - b_{-k})(a_{k'} - b_{-k'}^\dagger) | \eta \right\rangle \succ 0, \end{aligned} \tag{4.50}$$

positive definite linear integral operators so that

$$1 + A_\eta \succ |B_\eta|. \tag{4.51}$$

This constraint is necessary but not sufficient condition since $1 + A_\eta = B_\eta$ can be never realized on a quantum state with finite norm. We shall now show that QGPs provide a "better" estimate for (A_η, B_η) and that, in fact, QGP provides boundaries for (A_η, B_η) that are exact.

Consider QGP of the form

$$\begin{gathered} |\theta\rangle = \exp\left[\int dk dk' \theta_{kk'} \left(a_k^\dagger b_{-k'}^\dagger - a_k b_{-k'} \right) \right] |0\rangle \\ \theta \text{ real symmetric.} \end{gathered} \tag{4.52}$$

By the QGP Theorem it is easy to find $\alpha = \cosh(\theta)$, $\beta = \sinh(\theta)$ and

$$\begin{aligned} a_k &\to \cosh(\theta)_{kk'} a_{k'} + \sinh(\theta)_{kk'} b_{-k'}^\dagger, \\ b_k &\to \cosh(\theta)_{kk'} b_{-k'} + \sinh(\theta)_{kk'} a_{k'}^\dagger, \end{aligned} \tag{4.53}$$

so that

$$\begin{gathered}\left\langle \theta|a_k^\dagger a_{k'}|\theta\right\rangle = \left\langle \theta|b_{-k'}^\dagger b_{-k}|\theta\right\rangle = [(\sinh(\theta))^2]_{k'k}, \\ \langle\theta|a_k b_{-k'}|\theta\rangle = [\sinh(\theta)\cosh(\theta)]_{k'k}.\end{gathered} \tag{4.54}$$

Then we obtain

$$\begin{aligned}A_\theta &= 2\sinh(\theta)^2 = \cosh(2\theta) - 1, \\ B_\theta &= 2\sinh(\theta)\cosh(\theta) = \sinh(2\theta).\end{aligned} \tag{4.55}$$

So that

$$A_\theta = \sqrt{1 + B_\theta^2} - 1. \tag{4.56}$$

Eq.(4.56) is the relation between A_η and B_η for the states of the form Eq.(4.52).

To take our study one step further we need to make a simple observation. If the trial state is chosen in the form

$$|T\rangle = (f[a^\dagger] + g[b^\dagger] + \kappa[a^\dagger b^\dagger])|0\rangle, \tag{4.57}$$

where by $\kappa[a^\dagger b^\dagger]$ we understand a superposition of terms involving equal number of $a^\dagger$ and $b^\dagger$ (QGP in Eq.(4.52) is such an example), then with simple manipulations we find

$$\begin{aligned}\left\langle T|a_k^\dagger a_{k'} + b_{-k'}^\dagger b_{-k}|T\right\rangle &= \left\langle 0|f[a]a_k^\dagger a_{k'} f[a^\dagger]|0\right\rangle + \left\langle 0|g[b]b_{-k'}^\dagger b_{-k} g[b^\dagger]|0\right\rangle + \\ &\quad \left\langle 0|\kappa[ab](a_k^\dagger a_{k'} + b_{-k'}^\dagger b_{-k})\kappa[a^\dagger b^\dagger]|0\right\rangle; \\ \left\langle T|b_k a_{k'} + a_k^\dagger b_{-k'}^\dagger|T\right\rangle &= \left\langle 0|\kappa[ab](b_k a_{k'} + a_k^\dagger b_{-k'}^\dagger)\kappa[a^\dagger b^\dagger]|0\right\rangle.\end{aligned} \tag{4.58}$$

By Symmetric Decomposition Theorem I, $\left\langle 0|f[a]a_k^\dagger a_{k'} f[a^\dagger]|0\right\rangle$ and $\left\langle 0|g[b]b_{-k'}^\dagger b_{-k} g[b^\dagger]|0\right\rangle$ can be independently set to any positive definite hermitian linear integral operators so that

$$\begin{aligned}\left\langle T|a_k^\dagger a_{k'} + b_{-k'}^\dagger b_{-k}|T\right\rangle &= \mathcal{M} + \left\langle 0|\kappa[ab](a_k^\dagger a_{k'} + b_{-k'}^\dagger b_{-k})\kappa[a^\dagger b^\dagger]|0\right\rangle; \\ \left\langle T|b_k a_{k'} + a_k^\dagger b_{-k'}^\dagger|T\right\rangle &= \left\langle 0|\kappa[ab](b_k a_{k'} + a_k^\dagger b_{-k'}^\dagger)\kappa[a^\dagger b^\dagger]|0\right\rangle.\end{aligned} \tag{4.59}$$

QGP is sufficient to reproduce any $B = \left\langle \eta|b_k a_{k'} + a_k^\dagger b_{-k'}^\dagger|\eta\right\rangle$ by setting $\theta = \sinh^{-1}(B/2)$,

then we find that Eq.(4.52) and Eq.(4.57) are sufficient to generate any pair of hermitian (A_η, B_η) such that[1]

$$\begin{gathered} A_\eta = \mathcal{M} + \sqrt{1 + B_\eta^2} - 1, \\ \mathcal{M} \succ 0. \end{gathered} \tag{4.60}$$

In principle, this constraint is sufficient condition stating that, if Eq.(4.60) is satisfied, there exist an appropriate quantum state to generate (A_η, B_η). The question arises if this relation is also necessary?

One way to provide the answer would be to show that QGP is the minimum of $A_\eta[B_\eta]$ in the sense that for any $|\eta\rangle$: $A_\eta \succ A_\theta = \sqrt{1 + B_\eta^2} - 1$, as is indicated by Eq.(4.60). While we will concentrate on rigorous treatment of this statement later, let us now give some arguments that made us believe in that Eq.(4.60) is, indeed, an exact constraint. For that we shall prove following points:

1. QGP is an extremum of A_η with fixed B_η, although we cannot show that it is a global minimum;

2. Eq.(4.60) has correct asymptotic behavior for small and large B_η.

That Eq.(4.60) has correct asymptotic behavior plus that QGP is an extremal point of $A_\eta[B_\eta]$ gives a strong hint that form of Eq.(4.60) is correct. We will start with finding asymptotic solutions of symmetric decomposition problem.

If B_η is large, Eq.(4.60) reduces to Eq.(4.51) since

$$A_\eta \succ \sqrt{1 + B_\eta^2} - 1 \approx |B_\eta| - 1. \tag{4.61}$$

Thus, in this limit Eq.(4.60) gives the necessary condition Eq.(4.51). In the limit of small B_η the best way to represent B_η is by a state with the lowest possible excitation, i.e.

$$|T\rangle = |0\rangle + \frac{1}{2} \int dk dk' B(k', k) a_k^\dagger b_{-k'}^\dagger |0\rangle. \tag{4.62}$$

[1] note that $B_\eta^2 = \langle\eta|B|\eta\rangle\langle\eta|B|\eta\rangle \neq \langle\eta|B^2|\eta\rangle$, where $B = a^\dagger b^\dagger + ab$

In the case of small B we may forget about proper normalization of state, which would contribute only in the next order, so that

$$\begin{array}{c} \langle T|b_{-k'}a_k|T\rangle = \left\langle T|a^\dagger_{k'}b^\dagger_{-k}|T\right\rangle = B_{k'k}/2, \\ \left\langle T|b_{-k}a_{k'} + a^\dagger_k b^\dagger_{-k'}|T\right\rangle = B_{kk'}. \end{array} \tag{4.63}$$

Furthermore,

$$\begin{array}{c} \left\langle T|a^\dagger_k a_{k'}|T\right\rangle = \left\langle T|b^\dagger_{-k'}b_{-k}|T\right\rangle = \frac{1}{4}\int dt B^*(t,k)B(t,k') = \frac{1}{4}(B^2)_{kk'}, \\ \left\langle T|a^\dagger_k a_{k'} + b^\dagger_{-k'}b_{-k}|T\right\rangle = (B^2)_{kk'}/2 \end{array} \tag{4.64}$$

in agreement with $\sqrt{1+B^2} - 1 \approx B^2/2$ for small B. We see, thus, that Eq.(4.60) gives correct asymptotic behavior for the necessary constraint $A_\eta[B_\eta]$.

Let us now see if QGP is also an extremum of $A_\eta[B_\eta]$. For simplicity we work in the basis in which B is diagonal. We also use the QGP Theorem and treat QGP-expectations as expectations over the vacuum state for modified ladder operators. Note that in this case

$$\begin{array}{c} A = \left\langle \eta|(\cosh(\theta)_k a^\dagger_k + \sinh(\theta)_k b_{-k})(\cosh(\theta)_{k'}a_{k'} + \sinh(\theta)_{k'}b^\dagger_{-k'})|\eta\right\rangle + \\ \left\langle \eta|(\cosh(\theta)_{k'}b^\dagger_{-k'} + \sinh(\theta)_{k'}a_{k'})(\cosh(\theta)_k b_{-k} + \sinh(\theta)_k a^\dagger_k)|\eta\right\rangle; \end{array} \tag{4.65}$$

$$\begin{array}{c} B = \left\langle \eta|(\cosh(\theta)_k a_k + \sinh(\theta)_k b^\dagger_{-k})(\cosh(\theta)_{k'}b_{-k'} + \sinh(\theta)_{k'}a^\dagger_{k'})|\eta\right\rangle + \\ \left\langle \eta|(\cosh(\theta)_{k'}a^\dagger_{k'} + \sinh(\theta)_{k'}b_{-k'})(\cosh(\theta)_k b^\dagger_{-k} + \sinh(\theta)_k a_k)|\eta\right\rangle. \end{array} \tag{4.66}$$

We will consider trial state in the form $|T\rangle = \sqrt{1-\langle\alpha|\alpha\rangle}|0\rangle + |\alpha\rangle$, $\langle 0|\alpha\rangle = 0$. Note that in this form $|0\rangle$ stands for QGP $|\theta\rangle$. After certain algebraic manipulations we can find that in this transformation

$$\begin{array}{c} A \to A + \sqrt{1-\langle\alpha|\alpha\rangle}\left(\langle 0|b_{-k}a_{k'}|\alpha\rangle + \left\langle \alpha|a^\dagger_k b^\dagger_{-k'}|0\right\rangle\right)\sinh(\theta_k+\theta_{k'}) + \\ \left\langle \alpha|a^\dagger_k b^\dagger_{-k'} + a_{k'}b_{-k}|\alpha\right\rangle \sinh(\theta_k+\theta_{k'}) + \left\langle \alpha|a^\dagger_k a_{k'} + b^\dagger_{-k'}b_{-k}|\alpha\right\rangle \cosh(\theta_k+\theta_{k'}), \end{array} \tag{4.67}$$

and

$$\begin{array}{c} B \to B + \sqrt{1-\langle\alpha|\alpha\rangle}\left(\langle 0|b_{-k}a_{k'}|\alpha\rangle + \left\langle \alpha|a^\dagger_k b^\dagger_{-k'}|0\right\rangle\right)\cosh(\theta_k+\theta_{k'}) + \\ \left\langle \alpha|a^\dagger_k b^\dagger_{-k'} + a_{k'}b_{-k}|\alpha\right\rangle \cosh(\theta_k+\theta_{k'}) + \left\langle \alpha|a^\dagger_k a_{k'} + b^\dagger_{-k'}b_{-k}|\alpha\right\rangle \sinh(\theta_k+\theta_{k'}). \end{array} \tag{4.68}$$

From $\delta B = 0$ in this transformation it follows that

$$\begin{aligned}\sqrt{1-\langle\alpha|\alpha\rangle}(\langle 0|b_{-k}a_{k'}|\alpha\rangle + \left\langle \alpha|a_k^\dagger b_{-k'}^\dagger|0\right\rangle) + \left\langle \alpha|a_k^\dagger b_{-k'}^\dagger + a_{k'}b_{-k}|\alpha\right\rangle = \\ = -\left\langle \alpha|a_k^\dagger a_{k'} + b_{-k'}^\dagger b_{-k}|\alpha\right\rangle \frac{\sinh(\theta_k+\theta_{k'})}{\cosh(\theta_k+\theta_{k'})},\end{aligned} \tag{4.69}$$

so that we can obtain for A

$$\delta A \to \frac{\left\langle \alpha|a_k^\dagger a_{k'} + b_{-k'}^\dagger b_{-k}|\alpha\right\rangle}{\cosh(\theta_k+\theta_{k'})}. \tag{4.70}$$

Note that this is quadratic in $|\alpha\rangle$ so that QGP is indeed an extremum for $A_\eta[B_\eta]$. However, even though denominator of Eq.(4.70) is a positive definite operator, we yet cannot guarantee positive definiteness of the whole expression. Note also that $|\alpha\rangle$ is still subject to constraint Eq.(4.69) so that we equally cannot show the converse. Nonetheless, we've shown that QGP is an extremum for A_η, which along with correct asymptotic behavior gives strong support to that Eq.(4.60) is the exact constraints on (A_η, B_η). Given that, let us finish our discussion by proving the following proposition.

Symmetric Decomposition Theorem III: Set of all possible pairs (A_η, B_η), defined by Eq.(4.49), is equal to that of hermitian linear integral operator pairs (A, B) subject to

$$A \succ \sqrt{1+B^2} - 1. \tag{4.71}$$

From the particle-antiparticle symmetry this also may be extended to:

Corollary: Set of all possible operators $\hat{g}$, specified by Eq.(4.48), is equal to that of operators $\hat{g}$ subject to

$$\begin{aligned}g_{12} = g_{21}^\dagger; g_{11} \text{ and } g_{22} \text{ hermitian, positive definite;} \\ g_{11}, g_{22} \succ (\sqrt{1+(g_{12}+g_{21})^2} - 1)/2.\end{aligned} \tag{4.72}$$

Proof: To prove this statement we reformulate the problem in a different way, i.e. we will show that, if B_η is fixed $B_\eta = B_0$, then for any function $f = f(k) = f_k$: $(f, A_\eta f) \geq (f, (\sqrt{1+B_0^2} - 1)f)$. Here inner product stands for

$$(f, Ag) = \int dkdk' f^*(k) A(k,k') g(k').$$

We will work in the basis in which B_0 is diagonal. Let $f_i(k)$ be orthonormal basis of eigenfunctions of $B_0 = B_0^\dagger$ and $B_i = (f, B_0 f_i)$ is the set of the real eigenvalues of B_0. Any function $f(k)$ can be represented as $f(k) = \sum\limits_i \alpha_i f_i(k)$, where $\sum\limits_i$ stands for summation over all discrete and integration over continuous parts of the spectrum of B_0. Then we need to show that, as long as $\langle\eta|\eta\rangle = 1$ and $(f_i, B_\eta f_i) = B_i$ is fixed for all i,

$$\sum_{i,j} \alpha_i^* \alpha_j (f_i, A_\eta f_j) \geq \sum_i |\alpha_i|^2 (\sqrt{1 + B_i^2} - 1). \tag{4.73}$$

We will look for the solution of this constrained minimization problem using the method of Lagrange multipliers.

Let λ_{ij} be Lagrange multipliers. Then for any variation $|\delta\eta\rangle$ such that $\langle\delta\eta|\eta\rangle = 0$, i.e. the norm $\langle\eta|\eta\rangle$ is conserved, it should hold

$$\begin{aligned} &\langle\delta\eta| \sum \alpha_i^* \alpha_j \left\{(f_i, A_\eta f_j) - \lambda_{ij}(f_i, B_\eta f_j)\right\} |\eta\rangle = 0 \\ &(f_i, B_\eta f_j) = \delta_{ij} B_i. \end{aligned} \tag{4.74}$$

Eq.(4.74) should be read as equation for $|\eta\rangle[\lambda_{ij}]$ such that Eq.(4.74a) is true for any norm-preserving variation $|\delta\eta\rangle$. Then Eq.(4.74b) should be read as the equation for λ_{ij} such that the constraint is satisfied.

Note that

$$\begin{aligned} (f_i, A_\eta f_j) &= \langle\eta|a_i^\dagger a_j + b_j^\dagger b_i|\eta\rangle, \\ (f_i, B_\eta f_j) &= \langle\eta|a_i^\dagger b_j^\dagger + a_j b_i|\eta\rangle, \end{aligned} \tag{4.75}$$

where

$$\begin{aligned} a_i &= \textstyle\int dk f_i(k) a_k \text{ and} \\ b_i &= \textstyle\int dk f_i(k)^* b_{-k}. \end{aligned} \tag{4.76}$$

Thus, the problem defined by Eq.(4.74) can be restated in this form: find $|\eta\rangle$ such that for any norm-preserving variation $|\delta\eta\rangle$

$$\begin{aligned} &\textstyle\sum \alpha_i^* \alpha_j \langle\delta\eta|a_i^\dagger a_j + b_j^\dagger b_i - \lambda_{ij}(a_i^\dagger b_j^\dagger + a_j b_i)|\eta\rangle = 0, \\ &\langle\eta|a_i^\dagger b_j^\dagger + a_j b_i|\eta\rangle = \delta_{ij} B_i. \end{aligned} \tag{4.77}$$

Eq.(4.77) alone can be considered as a point that Eq.(4.60), indeed, is the exact constraint since the solution of Eq.(4.77a) is obviously a QGP. Still, let us prove this

statement explicitly and, if $|\eta\rangle$ and λ_{ij} is the solution of Eq.(4.77), introduce a new basis (c_i, d_i) such that

$$\begin{aligned} a_i &= \cosh\theta_i c_i + \sinh\theta_i d_i^\dagger \\ b_i &= \cosh\theta_i d_i + \sinh\theta_i c_i^\dagger, \end{aligned} \tag{4.78}$$

where $\sinh 2\theta_i = \lambda_{ii}$. After straightforward calculations one can show that

$$\begin{aligned} A_\eta: &\ \cosh(\theta_i-\theta_j)(c_i^\dagger c_j + d_j^\dagger d_i) + \sinh(\theta_i+\theta_j)(c_i^\dagger d_j^\dagger + c_j d_i) + 2\delta_{ij}\sinh\theta_i\sinh\theta_j \\ B_\eta: &\ \cosh(\theta_i-\theta_j)(c_i^\dagger d_j^\dagger + c_j d_i) + \sinh(\theta_j-\theta_i)(c_i^\dagger c_j + d_j^\dagger d_i) + 2\delta_{ij}\cosh\theta_i\sinh\theta_j \end{aligned} \tag{4.79}$$

so that conditions (4.77) are rewritten in this basis as

$$\begin{aligned} \sum \alpha_i^*\alpha_j\langle\delta\eta| &\ \cosh(\theta_i-\theta_j)(c_i^\dagger c_j + d_j^\dagger d_i) + \sinh(\theta_i+\theta_j)(c_i^\dagger d_j^\dagger + c_j d_i) \\ &\ -\lambda_{ij}\big[\cosh(\theta_i-\theta_j)(c_i^\dagger d_j^\dagger + c_j d_i) + \sinh(\theta_j-\theta_i)(c_i^\dagger c_j + d_j^\dagger d_i)\big]|\eta\rangle = 0 \\ \langle\eta| &\ \cosh(\theta_i-\theta_j)(c_i^\dagger d_j^\dagger + c_j d_i) + \sinh(\theta_j-\theta_i)(c_i^\dagger c_j + d_j^\dagger d_i) + \\ &\ +2\delta_{ij}\cosh\theta_i\sinh\theta_j|\eta\rangle = \delta_{ij}B_i, \end{aligned} \tag{4.80}$$

where we explicitly had taken into account that $\langle\delta\eta|\eta\rangle = 0$ and thus the terms of the form $\delta_{ij}\sinh\theta_i\sinh\theta_j$ had dropped out from Eq.(4.80a).

Since Eq.(4.80) is meant to hold for any values of α_i, choosing $\alpha_i = 0$, $i \neq l$ and $\alpha_l = 1$ for some l yields

$$\langle\delta\eta|(c_l^\dagger c_l + d_l^\dagger d_l) + \sinh 2\theta_l(c_l^\dagger d_l^\dagger + c_l d_l) - \lambda_{ll}(c_l^\dagger d_l^\dagger + c_l d_l)|\eta\rangle = 0, \tag{4.81}$$

from which it follows, given the above condition $\sinh 2\theta_i = \lambda_{ii}$, that necessarily $c_l|\eta\rangle = 0$ and $d_l|\eta\rangle = 0$ for all l. Using this result we reduce Eq.(4.80) to

$$\begin{aligned} &\sum \alpha_i^*\alpha_j\langle\delta\eta|\big[\sinh(\theta_i+\theta_j) - \lambda_{ij}\cosh(\theta_i-\theta_j)\big]c_i^\dagger d_j^\dagger|\eta\rangle = 0, \\ &\Longrightarrow \lambda_{ij} = \tfrac{\sinh(\theta_i+\theta_j)}{\cosh(\theta_i-\theta_j)} \ \&\ \lambda_{ii} = \sinh 2\theta_i. \end{aligned} \tag{4.82}$$

Finally, from the constraint $B_\eta = B_0$ we derive

$$\langle\eta|2\cosh\theta_i\sinh\theta_j\delta_{ij}|\eta\rangle = \delta_{ij}B_i \tag{4.83}$$

so that $\sinh 2\theta_i = B_i$. We, therefore, obtained here the complete solution of the problem defined by Eq.(4.77)

$$\begin{array}{l} |\eta\rangle \text{ and } \lambda_{ij} \text{ such that:} \\ \left\{ \begin{array}{l} c_i|\eta\rangle = 0 \text{ and } d_i|\eta\rangle = 0 \text{ for all } i \\ \sinh 2\theta_i = B_i \\ \lambda_{ij} = \frac{\sinh(\theta_i+\theta_j)}{\cosh(\theta_i-\theta_j)}. \end{array} \right. \end{array} \tag{4.84}$$

This state is a QGP and elementary calculation yields

$$(f_i, A_\eta f_j) = 2\sinh^2\theta_i\delta_{ij} = (\sqrt{1+\sinh^2 2\theta_i} - 1)\delta_{ij} = (\sqrt{1+B_i^2} - 1)\delta_{ij}. \tag{4.85}$$

This completes the proof.

4.4 Application of Symmetric Decomposition Problem in a Variant of Nonlinear Sigma Model

We now are in position to answer the question formulated in the beginning of this chapter. In our terms the ground state of the variant of nonlinear σ-model where constraint is enforced on average is defined by

$$\left\{ \begin{array}{c} E = \left\langle N[\int d\vec{x} \sum\limits_i (\pi_i^2 + (\vec{\partial}\phi_i)^2)] \right\rangle \to \min, \\ \left\langle N[\sum\limits_i \phi_i^2] \right\rangle = \Delta. \end{array} \right. \tag{4.86}$$

If we denote with $\epsilon = \{\epsilon_k\}$ and $\epsilon^{-1} = \{\epsilon_k^{-1}\}$, then this can be recast in the terms of Symmetric Decomposition Problem:

$$\left\{ \begin{array}{l} E = \mathrm{tr}[\sum\limits_i \epsilon \cdot A_{i,\eta}] \to \min, \\ \mathrm{tr}[\sum\limits_i \epsilon^{-1} \cdot (A_{i,\eta} + B_{i,\eta})] = 2\Delta, \end{array} \right. \tag{4.87}$$

where tr stands for integration over momentum $\int dk$. We further will solve this problem within the Symmetric Decomposition Theorem III.

Instead of $(A_{i,\eta}, B_{i,\eta})$ we shall introduce new variables $(Q_i = A_{i,\eta} + B_{i,\eta}, B_i = B_{i,\eta})$ which are constrained accordingly by

$$Q_i \succ \sqrt{1 + B_i^2} + B_i - 1. \tag{4.88}$$

In these variables, the problem can be formulated as follows:

$$\begin{gathered} \sum_i \mathrm{tr}(\epsilon \cdot Q_i - \epsilon \cdot B_i) \to \min, \sum_i \mathrm{tr}(\epsilon^{-1} \cdot Q_i) = 2\Delta \\ Q_i \geq \sqrt{1 + B_i^2} + B_i - 1. \end{gathered} \tag{4.89}$$

Since Q and B appear in Eq.(4.89) independently for given Q, to minimize the energy clearly one shall try to maximize B. This leave us with the following problem

$$\sum_i \mathrm{tr}[\epsilon \cdot (\sqrt{1 + B_i^2} - 1)] \to \min, \sum_i \mathrm{tr}[\epsilon^{-1} \cdot (\sqrt{1 + B_i^2} + B_i - 1)] = 2\Delta. \tag{4.90}$$

Note that $N[\varphi^2(x)] = \Delta^2$ also requires

$$\begin{gathered} \mathrm{tr}[\epsilon^{-1/2} D_q \epsilon^{-1/2} \cdot (\sqrt{1 + B_i^2} + B_i - 1)] = 0 \Leftrightarrow \\ \sum_i \int dk \frac{1}{\sqrt{\epsilon_k \epsilon_{k+q}}} [\sqrt{1 + B_i^2} + B_i - 1](k, k+q) = 0 \end{gathered} \tag{4.91}$$

for all $q \neq 0$ and $D_q = \{\delta_{k,k+q}\}$, however, as we shall show below, this condition is superficial since solution to Eq.(4.90) is diagonal, for which the constraint is satisfied automatically.

Let us now show that solution must be diagonal. Assume that solution B is not diagonal. Note that we effectively are left with problem

$$\begin{gathered} \mathrm{tr}(\epsilon \cdot A) \to \min \\ \mathrm{tr}(\epsilon^{-1} \cdot (A + B)) = 2\Delta \end{gathered} \tag{4.92}$$

where $A = \sqrt{1 + B^2} - 1$. Let us vary $B \to B + \delta B$ so that δB is off-diagonal. Then $\delta \mathrm{tr}(\epsilon^{-1} B) = 0$ and we have

$$\begin{gathered} \delta\text{find } \mathrm{tr}(\epsilon \cdot A), \\ \delta \mathrm{tr}(\epsilon^{-1} \cdot A) = 0, \delta A = \frac{B}{\sqrt{1+B^2}} \cdot \delta B. \end{gathered} \tag{4.93}$$

Choose off-diagonal entry $p \neq q$ such that $(\frac{B}{\sqrt{1+B^2}})_{pq} \neq 0$ and let only δB_{pq} and δB_{qp} (required since B is hermitian) be real nonzero. Then Eq.(4.93) reads

$$\begin{gathered}\delta\mathrm{tr}(\epsilon \cdot A) = \epsilon_p(\tfrac{B}{\sqrt{1+B^2}})_{pq}\delta B_{qp} + \epsilon_q(\tfrac{B}{\sqrt{1+B^2}})_{qp}\delta B_{pq} \\ \epsilon_p^{-1}(\tfrac{B}{\sqrt{1+B^2}})_{pq}\delta B_{qp} + \epsilon_q^{-1}(\tfrac{B}{\sqrt{1+B^2}})_{qp}\delta B_{pq} = 0.\end{gathered} \tag{4.94}$$

Excluding not essential degeneration in $\vec{p}$, $\vec{q}$ due to rotational symmetry, we choose

$$\begin{gathered}\delta B_{qp} = -\tfrac{\epsilon_p}{\epsilon_q}\tfrac{(\frac{B}{\sqrt{1+B^2}})_{qp}}{(\frac{B}{\sqrt{1+B^2}})_{pq}}\delta B_{pq}, \\ \delta\mathrm{tr}(\epsilon \cdot A) = (1 - \tfrac{\epsilon_p^2}{\epsilon_q^2})\epsilon_q(\tfrac{B}{\sqrt{1+B^2}})_{qp}\delta B_{pq}.\end{gathered} \tag{4.95}$$

Thus, $\delta E = \delta\mathrm{tr}(\epsilon \cdot A)$ can be made negative with appropriate sign of δB_{pq}. We've shown that non-diagonal B cannot be solution of the problem Eq.(4.93).

We will show later that, if B_i are diagonal, there exist minimizing solution so that for any δB_{pq} diagonal or off-diagonal, $\delta\mathrm{tr}(\epsilon \cdot A) = 0$ as needed. From now on we shall assume that Q_i and B_i are diagonal. Then we can solve explicitly constraint for B_i, which gives us the problem formulated in terms of Q only:

$$\frac{1}{2}\sum_{i,k}\frac{Q_{i,k}^2}{1+Q_{i,k}}\epsilon_k \to \min, \sum_{i,k} Q_{i,k}\epsilon_k^{-1} = 2\Delta. \tag{4.96}$$

With the method of Lagrange multipliers this can be solved from

$$\delta_{Q_{i,k}}\sum_k\left(\frac{1}{2}\frac{Q_k^2}{1+Q_k}\epsilon_k - \frac{\lambda}{2}Q_k\epsilon_k^{-1}\right) = 0 \tag{4.97}$$

which leads to following results. If N is the number of components in $\vec{\phi}$,

$$\begin{gathered}Q_{i,k} = \tfrac{\epsilon_k}{\sqrt{\epsilon_k^2-\lambda(\Delta)}} - 1, \\ N\int\tfrac{d\mathbf{k}}{(2\pi)^{d-1}}\tfrac{1}{2\epsilon_k}\left[\tfrac{\epsilon_k}{\sqrt{\epsilon_k^2-\lambda(\Delta)}} - 1\right] = \Delta.\end{gathered} \tag{4.98}$$

In d=2+1 the integral can be calculated explicitly yielding

$$N\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{1}{2\epsilon_k}\left[\frac{\epsilon_k}{\sqrt{\epsilon_k^2-\lambda(\Delta)}} - 1\right] = \frac{N\mu}{4\pi}(1-\sqrt{1-\lambda(\Delta)}) = \Delta. \tag{4.99}$$

We see that solution to Eq.(4.99) exists only as long as $\frac{N\mu}{4\pi} \geq \Delta$, once Δ exceeds $\Delta_{cr} = \frac{N\mu}{4\pi}$ solution cannot be entirely cast into the above form indicating presence of the 2nd order phase transition or higher in the theory.

To clarify physics behind this statement it is instructive to consider Bose condensation in this problem. For that we separate explicitly (dropping not essential here factors of 2π) 0-mode contribution

$$\begin{aligned} Q_0 &= \mu(2\Delta - \int\limits_{k\neq 0} d\mathbf{k}\frac{Q_k}{\epsilon_k}), \\ E &= \mu\frac{Q_0^2[Q_k]}{2+2Q_0[Q_k]} + \int\limits_{k\neq 0} d\mathbf{k}\epsilon_k\frac{Q_k^2}{2+2Q_k} \end{aligned} \tag{4.100}$$

so that

$$\delta_{Q_k}E = -\frac{Q_0[Q_k](2+Q_0[Q_k])}{2(1+Q_0[Q_k])^2}\frac{\mu^2}{\epsilon_k} + \frac{Q_k(2+Q_k)}{2(1+Q_k)^2}\epsilon_k. \tag{4.101}$$

Suppose at initial time the system created in the condensed state $Q_k = 0$, $Q_0 = 2\mu\Delta$. From Eq.(4.101) we see that in that case $\delta_{Q_k}E < 0$ for all k so that system begins to "evaporate" from 0-condensate to higher energy levels $k \neq 0$. In this process Q_k grows and Q_0 decreases. This process goes on until equilibrium is reached, i.e.

$$\frac{Q_0[Q_k](2+Q_0[Q_k])}{(1+Q_0[Q_k])^2}\mu^2 = \frac{Q_k(2+Q_k)}{(1+Q_k)^2}\epsilon_k^2 = \lambda. \tag{4.102}$$

Note, however, that since integral in Eq.(4.100) is finite, the "vapor" phase can only accommodate finite amount of the states so that, once $\lambda = \mu^2$ is reached, vaporization stops and further injection of states into the system (increased Δ) leads to their condensation in 0-mode, density of which becomes infinite indicating the presence of Bose-condensation. The order parameter, which in this case related to Bose-condensation, changes continuously from zero for small Δ to nonzero for large Δ so that the phase transition is at least of the second type.

Given Symmetric Decomposition Theorems, our solution is exact for the formulation of the nonlinear σ-model stated in the beginning of this chapter. We see that the result closely parallels the ORM result, which is not surprising since QGPs represent in this case relevantly all of the dynamical domain of the theory. Besides

recovering the critical point, Symmetric Decomposition also helped to gain better understanding of the physics behind the phase transition, i.e. generation of nontrivial Bose-condensation at 0-mode caused by finite "capacity" of the stable Q-distribution. Since in this variant of nonlinear sigma model the constraint is enforced softly (on average), our formulation is not identical to the conventional nonlinear sigma model in which $\phi^2(x)|\eta\rangle = R^2|\eta\rangle$ exactly and thus our result, although exact solution for this "soft" variant of nonlinear sigma model, may be considered only as an approximation for the conventional form. This explains why our answer coincides with an approximate large N expansion result.

We also shall comment on the trivial case μ strictly equal to zero, in which case symmetric phase in our solution is absent. This, at first glance surprising result, is not totally unexpected since $\mu = 0$ means that the theory is formulated for massless bosons. In this case it cost system no energy to condense arbitrary number of bosons in 0-mode and, thus, satisfy the constraint $\langle\phi^2\rangle = R^2$. The ground state, in this case, is degenerate and all constant solutions $\phi(x) = R$ have, in fact, the same energy 0.

We point out that our method, although exact here, has all limitation of a variational calculation. In particular, the integral expressions are sensible only in 1+1 and 2+1 dimensions, where no renormalization beyond normal ordering is needed. In the other cases renormalization effects have to be incorporated either in μ^2, Δ or in the Fock space treatment, which is yet an open question.

4.5 Summary

In this chapter we introduced and considered in details method of Symmetric Decomposition Problem. In this problem, one tries to answer the following question: given quantum operators $\{A, B, C, \dots\}$, what may be the expectation values of these operators $\{\langle A\rangle, \langle B\rangle, \langle C\rangle, \dots\}$ and what are the relations among these on a symmetrized

Fock space $\mathcal{F}$? We obtained a complete solution for Symmetric Decomposition Problem for quadratic operators

$$A(k,k') = a_k^\dagger a_{k'} + b_{-k'}^\dagger b_{-k},$$
$$B(k,k') = a_k^\dagger b_{-k'}^\dagger + a_{k'} b_{-k}, \text{ etc.}$$

We found that any pair of kernels for hermitian linear integral operators that satisfy a specific constraint,

$$1 + A \succ \sqrt{1 + B^2},$$

can be represented on a symmetrized Fock space and vice versa. We then applied the Symmetric Decomposition Problem to a variational study of the ground state in a variant of nonlinear σ model. In this model the free scalar quantum field theory is constrained to the subset of the symmetrized Fock space on which nonlinear σ model constraint is enforced on average,

$$\langle \vec{\phi}^2 \rangle = R^2.$$

The advantage of this model for our theory is in that it can be completely formulated in terms of the expectation values of quadratic quantum operators. Our complete knowledge of the Fock space image in terms of these expectation values then allows to straightforwardly obtain the ground state of such problem. As we have shown, although extremely simple in formulation, this model exhibit very nontrivial behavior and, in particular, has a critical point at which a phase transition of at least second kind occurs.

Symmetric Decomposition Problem can be seen as a novel variational approach and a curious mathematical result. One may try to formulate variational problem not in terms of the trial states from the Fock space, which may take extremely complicated form in QFT, but directly in terms of the expectation values of the relevant operators and relations between them. One then obtains a constrained minimization problem which can be solved entirely in terms of the relevant expectation values. Even without exact boundaries for expectation values of quantum operators, the method

of Symmetric Decomposition Problem is capable of providing a variational estimate without referring to underlying Fock space. Extension of our knowledge of the Fock space in terms of the relations between the expectation values of different quantum operators appears to be interesting from mathematical point of view. While we realize that it may not be an easy task to extend such relations to higher order polynomial quantum operators, this work definitely presents the first step toward the theoretical development of significant mathematical and applied interest.

Chapter 5

Conclusions

It is hard to underestimate the importance of the apparatus of Canonical Transformations in fundamental and applied problems of Classical Mechanics. In Quantum Physics, however, this tool appears to be severely impaired. Yet, the machinery of CTs remains to be of great advantage and of great importance in study of nonperturbative phenomena in field theories. Particularly, quantum linear canonical transformation, more commonly referred as Bogoliubov Transformation, claimed many important contributions in condensed matter physics as well as in particle physics and QCD. In this dissertation, we considered two recent applications of Bogoliubov Transformation in field theory of critical phenomena and quantum mixing phenomenology.

In particle phenomenology the phenomenon of quantum mixing takes by right one of the central places due to its sensitivity to the Standard Model parameters and New Physics. Even though few would object that the ultimate theory of mixing should be field-theoretic, the conventional treatment of quantum mixing usually have dealt with quantum-mechanical ideas. The extension from Quantum Mechanics to QFT turned out to be not trivial and offered a few surprises. As was shown in the literature, in field theory serious difficulties exist with the definition of the flavor states. This provided a serious blow to simple wave-packet approach in Field Theory and Quantum Mechanics. It was suggested by some authors that a version of Bogoliubov Transformation can be used to diagonalize the mixing Hamiltonian and, thus, to introduce a

Fock space for flavor eigenstates. Such theory was first introduced by Blasone and Vitiello in 1995 and further researched by other authors.

As part of this dissertation, we described our contribution along this direction. Following the review of the original idea of Blasone and Vitiello, we presented the application of BV theory to the field-theoretical mixing in the system of two scalar mesons and demonstrated some dramatic deviations from Quantum Mechanics due to nontrivial interactions with flavor condensate in the vacuum. We furthermore improved and generalized our understanding of field-theoretic mixing by introducing a general formulation of the field theory of mixing in which we have shown from general principles that mixing in QFT involves two sectors - particle-particle mixing and particle-antiparticle cross-mixing. Using this formulation, we have obtained a number of explicit results for the flavor Fock space, quantum fields time dynamics and observables oscillations. Particularly, we have obtained full solution of the field-theoretical mixing problem with arbitrary number of fields and boson or fermion statistics. We investigated the time dynamics of quantum fields by calculating non-equal time commutators. We found an explicit solution for the Fock space of the interacting fields and the corresponding vacuum structure that turned out to be a generalized coherent state. We showed unitary nonequivalence between the flavor Fock space and the free-field Fock space in the infinite volume limit. We considered applications of this approach to mixing of 2 and 3 scalar or vector mesons ($S = 0$, $S = 1$) and fermions ($S = 1/2$). We found that the scalar/pseudoscalar ($S = 0$) boson mixing is the same as the mixing of transverse components of the vector fields, while for the longitudinal component of the vector fields we found richer momentum dependence than in the spin-zero case. We presented a new algebraic method for finding flavor vacuum state which is distinct from the conventional method of using a differential equation for $\mathcal{Z}$. We found an intrinsic difference between the fermion and boson cases in that the normalization factor $\mathcal{Z}$ for the bosons has a singularity on the complex plane while the corresponding result for the fermions doesn't have any singularity. This results in divergence of the perturbative expansions in powers

of the mixing angle θ. We also addressed the problem of time-to-space conversion in field theory of mixing using canonical quantization generalization of the wave-packet method in QFT. That allowed us to retain explicitly the effect from nontrivial flavor vacuum which otherwise is lost in quantum-mechanical or perturbative treatment. We also considered phenomenological aspects of the field theoretical mixing formalism and estimated the magnitude of the field-theoretical effect in known mixed systems. We found that in many systems both for mesons and neutrinos this effect is very negligible. Only in strongly mixed systems, such as $\eta - \eta'$ or $\omega - \phi$, or for very low-energy neutrinos the corrections may be as large as 10% and thus may need additional attention should these systems be used in oscillation experiment.

As another application of Bogoliubov Transformation in QFT, we considered its use in the Oscillator Representation Method which is concerned with the study of critical phenomena and phase transitions in field-theoretic models. The ORM uses Bogoliubov Transformation as an integral part to define a set of alternative degrees of freedom in terms of which the phase transition can be considered and described in details. Nontrivial vacuum condensation and dynamical mass generation had been successfully described using this approach as well as phenomenon of duality had been observed. In these settings a strong-interacting field-theory with light elementary particles can be converted into a weak-interacting model with heavy particles which mass is dynamically generated. These features of the ORM are encouraging for the application to the dynamical mass generation of constituent quarks and the duality between current and constituent quarks in QCD. Although simple in the applications and transparent in the results, this method enjoyed little attention in the literature relative to other comparable techniques as Hartree approximation or GEP method.

In this dissertation we fully reviewed the ORM and considered its applications to the phase transition in nonlinear σ-model and other field-theoretic models. In nonlinear σ-model, we found that the ORM gives results reproducing large N limit. Second order phase transition is recovered. In the case that a magnetic field is added, the second order phase transition is completely destroyed in agreement with general

arguments. In the form of nonlinear σ model, in which the constraint is algebraically resolved and the symmetry is broken, the ORM ran into terminal difficulties with its estimate of the ground state energy being a divergent series. In this situation, the ORM could not be applied other than using RGE. Here, however, the ORM relies entirely on the use of RG functions and gives only some additional insight into the dynamics of the phase transition. Its conclusions also depend significantly on the quality with which RG functions are known and change appreciably from order to order.

We also applied the ORM to ϕ^4 in 1+1 dimensions with magnetic field and found that, although qualitatively correct in the case without magnetic field, the ORM in this case contradicts to the strict theorems of the Constructive QFT that claim disappearance of the phase transition in 1+1 ϕ^4 theory for whatever small perturbations. In fact, we found that, according to the ORM, the phase transition of the first kind should persist when a small perturbation is added. We found that such peculiar behavior was a general result of the 1st order phase transition obtained in the ORM in the absence of magnetic field. We also computed first perturbative corrections to the vacuum energies in Symmetric and Broken Symmetry phases of ϕ^4 theory. We found that, although these corrections were small in S-phase for g small and in BS-phase for g large, in the region of the phase transition perturbative effects were significant and substantially affected the nature and location of the phase transition point. For large g, after the main quantum contributions to the vacuum energy were accounted for by going to BS representation, the BS-phase became weakly coupled and hence the perturbative approach could be successful for increasingly large g.

We also considered different possibilities to improve the ORM. We investigated various extensions including reformulation of the "correct" form to higher orders of RGE, taking into account finite pieces of RGE, interpreting the "correct" form as the "best" perturbative representation, variational extensions and QEA extensions. Unfortunately, no reliable and unambiguously successful extension was found as such

attempts have been hindered by inconsistencies in higher orders of perturbation theory and the loss of the original conceptual content of the ORM. In this respect, variational extension where the trial state description is extended from single "mass" parameter, such as in the ORM or GEP, to a form-factor presents clear advantage both in the sense of nonperturbative character and in representation of appropriate degrees of freedom. This additional flexibility may be crucial to account for higher order renormalizations of the theory where simple "mass" parametrization fails to grasp the nature of the true ground state. Another interesting approach may be that of QEA extension of the ORM where study had shown certain improvements when higher order corrections were taken into account using Borel Summation. While such improvements are certainly welcomed, one should be cautious about them given low reliability of perturbative expansion in the massive phase in the near critical region of a field-theory.

In Chapter 4, we introduced and considered in details the ORM inspired approach of Symmetric Decomposition Problem. In this problem, one tries to answer the following question: given quantum operators $\{A, B, C, \dots\}$, what may be the expectation values of these operators $\{\langle A\rangle, \langle B\rangle, \langle C\rangle, \dots\}$ and what are the relations among these on a symmetrized Fock space $\mathcal{F}$? In Chapter 4, we obtained a complete solution for Symmetric Decomposition Problem for quadratic operators $A = a_k^\dagger a_{k'} + b_{-k'}^\dagger b_{-k}$, $B = a_k^\dagger b_{-k'}^\dagger + a_{k'} b_{-k}$ etc which allowed us to apply the Symmetric Decomposition Problem to a variational study of the ground state in nonlinear σ model. Symmetric Decomposition Problem can be seen as a novel variational approach and a curious mathematical result. Here one tries to formulate variational problem not in terms of the trial states, which may take extremely complicated form in QFT, but directly in terms of the expectation values of the relevant operators and relations between them. One then obtains a constrained minimization problem in terms of the relevant expectation values.

The topics investigated in this dissertation are quite diverse. Still they share the same underlying idea of Canonical Transformation and degrees of freedom change.

Even though significant progress has been achieved here both in field theory of Flavor Oscillations and the Oscillator Representation Method, many interesting problems remain unanswered.

In Blasone-Vitiello theory of flavor oscillations, in particular, one of the major drawbacks is explicit breakdown of Lorentz invariance since the theory is formulated in terms of canonical quantization. Research aiming at giving BV theory a covariant formulation may be important for better understanding of the fundamental physics involved in this theory. Even more puzzling is the problem of mass parametrization. While this problem is more of a conceptual rather than mathematical character, it does need to be resolved in order to establish the proper place of BV theory in QFT.

In the method of Oscillator Representation, many directions remain open for further exploration as well. As we have shown in this dissertation, the ORM is essentially zeroth order approach which gives erroneous results in, e.g., ϕ^4 theory in 1+1 dimension. Contrary to the simplicity in formulation and applications, the ORM proved to be quite difficult to be generalized and improved. While we considered such extensions of the ORM as "best" perturbative representation, variational and QEA analogs, these may need a significant further attention. Especially, our variational extension of the ORM offers greater flexibility through momentum dependence of the $|\alpha\rangle$ state form-factors and may be crucial in order to accommodate higher order mass renormalizations, which otherwise lead to the ORM breakdown. Method of QEA, supplemented with the ORM mass self-consistency condition and nonperturbative resummation technique such as Borel Summation, also has shown some signs of promise. Still, more work is needed to firmly establish this progress by going to higher orders to which the QEA is being computed.

Finally, the method of Symmetric Decomposition Problem appears to be rather interesting and new undertaking in the realm of variational principle. Even without exact boundaries for expectation values of quantum operators, this method is capable of providing a variational estimate without referring to underlying Fock space. Extension of our knowledge of the Fock space in terms of the relations between the

expectation values of different quantum operators appears to be interesting from mathematical point of view. While we realize that it may not be an easy task to extend such relations to higher order polynomial quantum operators, this work definitely presents the first step toward the theoretical development of significant mathematical and applied interest.

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Appendix A

Essential Cases of Mixing Parameters H and h

Here, we list H and h parameters for the most essential cases in modern particle physics (scalar/pseudoscalar spin 0, vector spin 1 bosons and spin 1/2 fermions). For these cases mixing theory parameters are explicitly derived from QFT [31, 41]. We have for scalar/pseudoscalar fields (spin 0):

$$u_{\vec{k},0} = v_{\vec{k},0} = 1, \tag{A.1}$$

for vector fields (spin 1):

$$\begin{aligned} u_{\vec{k},0} = v_{\vec{k},0} &= \left(\tfrac{k}{m}, i\tfrac{\epsilon(k)}{m}\vec{n}\right), \\ u_{\vec{k},\pm1} = v_{\vec{k},\pm1} &= \left(0, i\vec{n}_{\pm}\right), \end{aligned} \tag{A.2}$$

where $\vec{n} = \frac{\vec{k}}{k} = \vec{e}_z$ and $\vec{n}_{\pm} = \mp\frac{1}{\sqrt{2}}(\vec{e}_x \pm i\vec{e}_y)$ form a spherical basis. For bi-spinor fields (spin 1/2), we use the standard representation of the γ-matrices

$$\gamma^0 = \begin{pmatrix} \hat{I} & 0 \\ 0 & -\hat{I} \end{pmatrix}, \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{A.3}$$

and the corresponding representations of spinors:

$$\begin{aligned} u_{\vec{k},\sigma} &= (\sqrt{\epsilon\left(k\right)+m}\omega_\sigma, \sqrt{\epsilon\left(k\right)-m}\left(\vec{n}\vec{\sigma}\right)\omega_\sigma), \\ v_{-\vec{k},\sigma} &= (-\sqrt{\epsilon\left(k\right)-m}\left(\vec{n}\cdot\vec{\sigma}\right)\omega_{-\sigma}, \sqrt{\epsilon\left(k\right)+m}\omega_{-\sigma}), \end{aligned} \tag{A.4}$$

where ω_σ is spinor satisfying $(\vec{n}\cdot\vec{\sigma})\,\omega_\sigma = \sigma\cdot\omega_\sigma$ and σ takes values ± 1.

The H and h parameters are then for scalar case:

$$H^{\mu j} = h^{\mu j} = 1, \tag{A.5}$$

for spin 1:

$$\begin{array}{c} \left\{ \begin{array}{r} H^{\mu j}_{\vec{k},0} = \frac{\epsilon_\mu(k)\epsilon_j(k)-k^2}{m_\mu m_j}, \\ h^{\mu j}_{\vec{k},0} = \frac{\epsilon_\mu(k)\epsilon_j(k)+k^2}{m_\mu m_j} \end{array} \right. , \sigma = 0 \\ H^{\mu j}_{\vec{k},\pm} = h^{\mu j}_{\vec{k},\pm} = 1, \sigma = \pm 1; \end{array} \tag{A.6}$$

and for spin 1/2:

$$\begin{array}{r} H^{\mu j}_{\vec{k},\sigma} = \sqrt{(\epsilon_\mu(k)+m_\mu)(\epsilon_j(k)+m_j)} + \sqrt{(\epsilon_\mu(k)-m_\mu)(\epsilon_j(k)-m_j)}, \\ h^{\mu j}_{\vec{k},\sigma} = \sigma\left(\sqrt{(\epsilon_\mu(k)-m_\mu)(\epsilon_j(k)+m_j)} - \sqrt{(\epsilon_\mu(k)+m_\mu)(\epsilon_j(k)-m_j)}\right). \end{array} \tag{A.7}$$

Appendix B

The ORM functions reference formulas

In this section we present reference formulas for δ and Δ functions in the ORM.

$\delta_\mu(m) = \delta(\mu, m)$ is defined as $\delta_{\alpha,\mu}(m) = N_m[(\partial_\nu\phi)^2] - N_\mu[(\partial_\nu\phi)^2]$. In dimensions $2 \leq d \leq 4$ it is given by

$$\delta_{\alpha,\mu}(m) = \begin{cases} d = 2, \frac{m^2-\mu^2}{4\pi}, \\ d = 3, \frac{m^3-\mu^3}{12\pi}, \\ d = 4, \frac{(m^2-\mu^2)\ln 2\Lambda - 3/4(m^4-\mu^4) - m^4 \ln m - \mu^4 \ln \mu}{16\pi}. \end{cases} \tag{B.1}$$

$\Delta_\mu(m) = \Delta(\mu, m)$ is defined as $\Delta_{\alpha,\mu}(m) = N_m[\phi^2] - N_\mu[\phi^2]$ and in dimensions $2 \leq d \leq 4$ is given by

$$\Delta_{\alpha,\mu}(m) = \begin{cases} d = 2, \frac{\ln(\mu/m)}{2\pi}, \\ d = 3, \frac{\mu-m}{4\pi}, \\ d = 4, \frac{(\mu^2-m^2)\ln 2\Lambda + (m^2-\mu^2)/2 + m^2 \ln m - \mu^2 \ln \mu}{8\pi}. \end{cases} \tag{B.2}$$

$d_\mu(m) = d(\mu, m)$ is defined as $\frac{1}{2}[\delta_\mu(m) - \Delta_\mu(m)]$ and can be obtained easily from the above expressions.

As one can see in $d = 4$ cutoff parameter $\Lambda = \alpha^{-1}$ did not entirely disappear. Moreover, the power form of the $\delta_\mu(m) = \mu^d \delta_1(m/\mu)$, $\Delta_\mu(m) = \mu^{d-2}\Delta_1(m/\mu)$, prescribed by the power counting, is also violated. This indicates that N_m should be renormalized by additional counter-terms beyond simple subtraction at point μ. This is where the original formulation of the ORM fails.

Appendix C

Unitary nonequivalence of the ORM states

In this appendix we explicitly show that the transformations of the mass change and field shift generate a family of unitary inequivalent vacua. As discussed in the text, the change of mass $m \to M$ transformation is given by

$$\begin{aligned} \tilde{a}_{\mathbf{k}} &= \frac{\sqrt{\frac{\omega_m(\mathbf{k})}{\omega_M(\mathbf{k})}}+\sqrt{\frac{\omega_M(\mathbf{k})}{\omega_m(\mathbf{k})}}}{2} a_{\mathbf{k}} - \frac{\sqrt{\frac{\omega_m(\mathbf{k})}{\omega_M(\mathbf{k})}}-\sqrt{\frac{\omega_M(\mathbf{k})}{\omega_m(\mathbf{k})}}}{2} a^{\dagger}_{-\mathbf{k}}, \\ \tilde{a}^{\dagger}_{\mathbf{k}} &= \frac{\sqrt{\frac{\omega_m(\mathbf{k})}{\omega_M(\mathbf{k})}}+\sqrt{\frac{\omega_M(\mathbf{k})}{\omega_m(\mathbf{k})}}}{2} a^{\dagger}_{\mathbf{k}} - \frac{\sqrt{\frac{\omega_m(\mathbf{k})}{\omega_M(\mathbf{k})}}-\sqrt{\frac{\omega_M(\mathbf{k})}{\omega_m(\mathbf{k})}}}{2} a_{-\mathbf{k}} \end{aligned} \tag{C.1}$$

and the quantization point shift is given by

$$\begin{aligned} \phi(x) &\mapsto \phi(x) + b, \\ a_{\mathbf{k}} &\to a_{\mathbf{k}} - 2\pi m b \delta(\mathbf{k}). \end{aligned} \tag{C.2}$$

The transformation Eq.(C.1,C.2) can be generated with following operators

$$\begin{aligned} U_1 &= \exp\left\{-2\pi m b (a_{\mathbf{0}} - a^{\dagger}_{\mathbf{0}})\right\} \\ U_2 &= \exp\left\{\tfrac{1}{2}\int d\mathbf{k}\lambda(\mathbf{k})(a_{-\mathbf{k}}a_{\mathbf{k}} - a^{\dagger}_{\mathbf{k}}a^{\dagger}_{-\mathbf{k}})\right\} \end{aligned} \tag{C.3}$$

with $\lambda(\mathbf{k}) = \frac{1}{2}\ln\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}(t)}$ and $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + t^2 m^2}$.

Despite operators Eq.(C.3) are apparently unitary, their action on the Fock space of the free field with mass m is not unitary. In fact, two vacua associated with (b_1, t_1)

and (b_2, t_2) are orthogonal. To show that we use transformation given by Eqs.(C.3) with $b = b_2 - b_1$ and $\lambda(\mathbf{k}) = \frac{1}{2}\ln\frac{\omega_{\mathbf{k}}(t_1)}{\omega_{\mathbf{k}}(t_2)}$ to explicitly relate these states. Straightforward algebraic manipulation yields

$$\begin{array}{c} U_1 = e^{-\frac{1}{2}(2\pi m b)^2[a_{\mathbf{0}}, a_{\mathbf{0}}^\dagger]} N_m[U_1], \\ U_2 = e^{-\frac{V}{(2\pi)^3}\int d\mathbf{k}\ln(\cosh[\lambda(\mathbf{k})])} N_m[U_2]. \end{array} \tag{C.4}$$

where $[a_{\mathbf{0}}, a_{\mathbf{0}}^\dagger] = \delta^{(3)}(\mathbf{0}) = \frac{V}{(2\pi)^3}$. Note also that transformation U_1 affects only $\mathbf{k} = \mathbf{0}$ mode and therefore

$$N_m[U_2 \circ U_1] = \left(\prod_{\mathbf{k}\neq 0} N_m[U_2(\mathbf{k})]\right) N_m[U_2(\mathbf{0}) \circ U_1(\mathbf{0})]. \tag{C.5}$$

For the inner product of the vacuum associated with (b_1, t_1) and the vacuum associated with (b_2, t_2) one obtains

$$\begin{array}{c} {}_1\langle 0|0\rangle_2 = \prod {}_{1,\mathbf{k}}\langle 0|0\rangle_{2,\mathbf{k}} \leq \prod_{\mathbf{k}\neq 0} {}_{1,\mathbf{k}}\langle 0|0\rangle_{2,\mathbf{k}} = \\ e^{-\frac{V}{(2\pi)^3}\int\limits_{\mathbf{k}\neq 0} d\mathbf{k}\ln(\cosh[\lambda(\mathbf{k})])} \to 0,\ V \to \infty. \end{array} \tag{C.6}$$

Two vacua are said to be unitary inequivalent in the infinite volume limit if their t parameters are different. Analogously, if $t_1 = t_2$ but $b_1 \neq b_2$ then as well

$${}_1\langle 0|0\rangle_2 = \prod {}_{1,\mathbf{k}}\langle 0|0\rangle_{2,\mathbf{k}} = {}_{1,\mathbf{k}=\mathbf{0}}\langle 0|0\rangle_{2,\mathbf{k}=\mathbf{0}} = e^{-\frac{1}{2}(2\pi m b)^2\frac{V}{(2\pi)^3}} \to 0, V \to 0, \tag{C.7}$$

so that the transformation given by Eq.(C.1,C.2) produces a family of unitary inequivalent vacua which can be associated with possible distinct phases of the system [88,95].