CE 395 Special Topics in Machine Learning

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FOURIER TRANSFORM

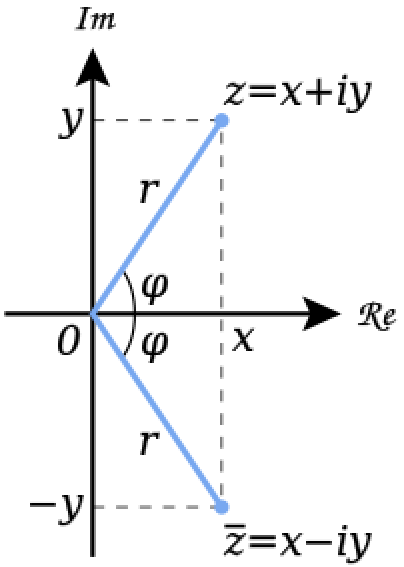
Complex numbers

- √-1=i ⟶⟶ z=a+ib

$$(a+ib)^2 = a^2 + 2aib + (ib)^2 = a^2 + 2abi - b^2$$

Practice: $i \cdot i = -1 \Rightarrow \frac{1}{i} = -i, (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i)^2 = i, (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i)^2 = -i$

complex plain →



Polar (magnitude-phase) and Cartesian (real-imaginary) representation of complex numbers:

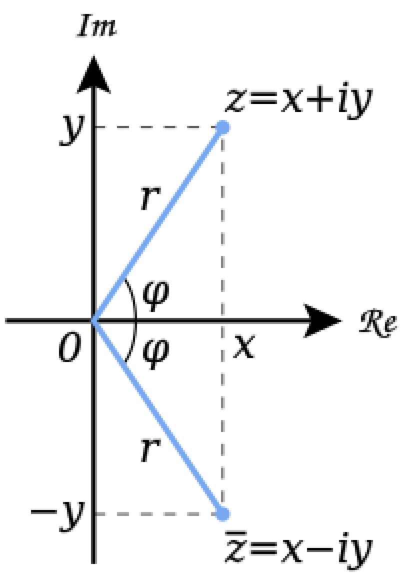
$$z = x + iy = r \cdot (\cos\varphi + i\sin\varphi)$$

$$r = |z| = \sqrt{x^2 + y^2},\ \varphi = \arg z = \tan^{-1}\frac{y}{x}$$

Complex numbers

Euler formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$



$$z = re^{i\varphi}$$

Complex numbers

Practice:

$$e^{i\phi} = \cos\phi + i\sin\phi \Rightarrow$$

$$\boxed{e^{i\pi} = -1}$$

$$\boxed{e^{2\pi i} = 1}$$

$$\boxed{e^{2\pi n i} = 1}$$

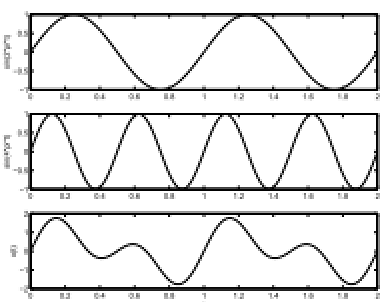
$$\boxed{z = re^{i\varphi}}$$

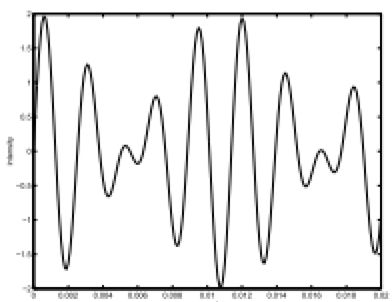
$$\boxed{z\bar{z} = r^2 e^{i\varphi} e^{-i\varphi} = r^2}$$

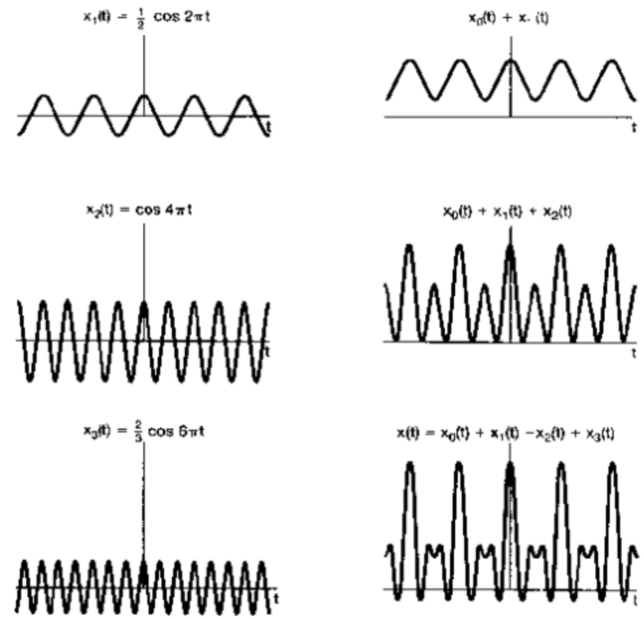
$$\boxed{z^2 = r^2 e^{2i\varphi}}$$

Signals as superpositions of harmonics

Many signals can be represented as sums of sinusoids (harmonics)

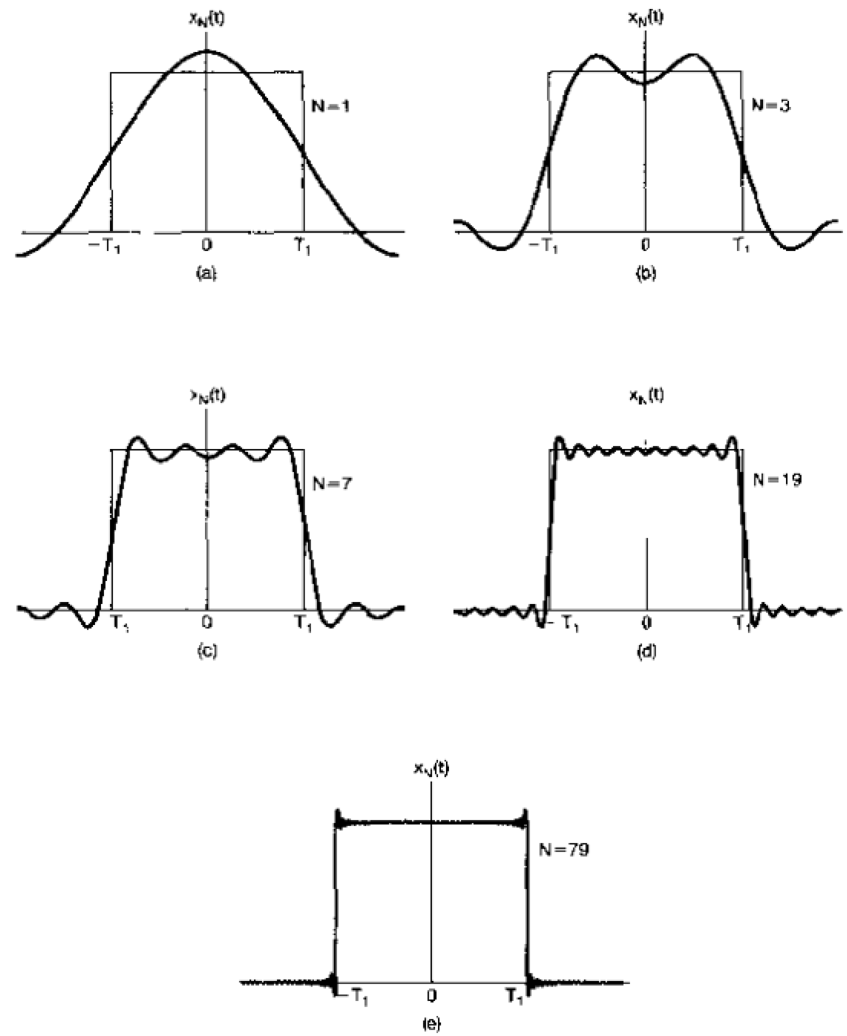






Signals as superpositions of harmonics

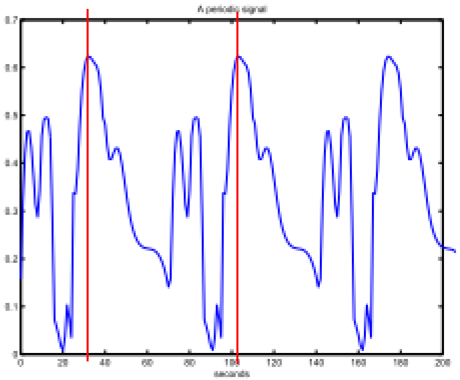
A square signal being formed from sinusoids:



Signals as superpositions of harmonics

This situation is much more general – in fact any generally smooth periodic signal can be represented as an infinite sum of sinusoids – the statement being the corner stone of the Fourier transform theory.

Fourier Transform for periodic signals



Periodic signal – any signal that repeats itself

$$x(t) = \frac{a_0}{2} + \sum_{k=-\infty}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

$$a_k = T^{-1} \int_T x(t) \cos k\omega_0 t \, dt$$

(Definition)

$$b_k = T^{-1} \int_T x(t) \sin k\omega_0 t \, dt$$

$$\omega_0 = \frac{2\pi}{T}$$

Note: complex exponents here need to be understood in terms of Euler formula as shorthand for cos and sin functions:

$$e^{\pm ik\omega_0 t} = \cos k\omega_0 t \pm i \sin k\omega_0 t$$

Definition of FT via complex Euler formula is much simpler:

$$\tilde{x}(k) = T^{-1} \int_T x(t) e^{-ik\omega_0 t} dt$$

Direct FT

$$x(t) = \sum_{k=-\infty}^{\infty} \tilde{x}(k) e^{ik\omega_0 t}$$

Inverse FT

FT for periodic signals

The reason for existence of Fourier representation is the orthogonality of complex harmonics:

$$\int_0^T \cos m\omega_0 t \cos k\omega_0 t \, dt = T\delta(m-n)$$

$$\int_0^T \sin m\omega_0 t \sin k\omega_0 t \, dt = T\delta(m-n)$$

$$\int_0^T \cos m\omega_0 t \sin k\omega_0 t \, dt = 0$$

$$\omega_0 = \frac{2\pi}{T}$$

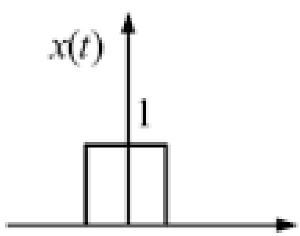
$$\int_0^T e^{im\omega_0 t} e^{-in\omega_0 t} dt = T\delta(m-n)$$

CHECK THIS AT HOME

FT for periodic signals

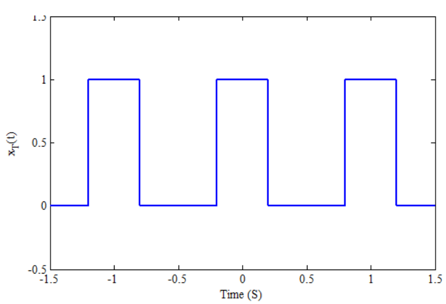
Exercise:

$$\tilde{x}(k) = T^{-1}\int_0^T x(t)\, e^{-ik\omega_0 t}\, dt$$

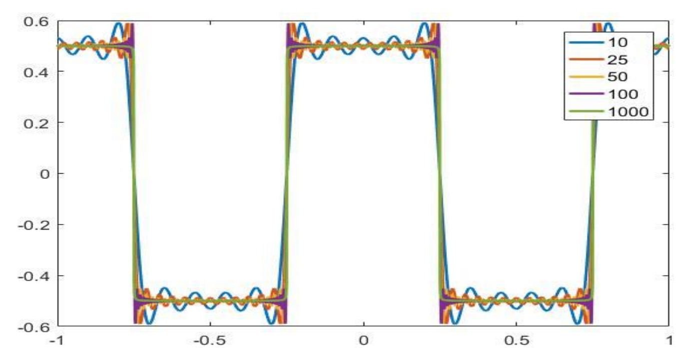


$$\tilde{x}(k) = \frac{1}{-i2\pi k}\left(e^{-ik\pi/2} - e^{ik\pi/2}\right)$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin k\pi/2}{k\pi}\, e^{ik\frac{2\pi}{T}t} = \frac{1}{2} + 2\sum_{k=1}^{\infty} \frac{\sin\frac{k\pi}{2}}{k\pi}\cos\left(k\frac{2\pi}{T}t\right)$$



$$\omega_0 = \frac{2\pi}{T}$$



FT for nonperiodic signals

Fourier image of a periodic signal exists at frequencies multiple of $\omega_0 = 2\pi/T$. As $T \rightarrow \infty$ that spacing tends to zero suggesting that FT of a nonperiodic function may be a continuous function on $(-\infty, \infty)$ – indeed a correct intuition

$$\widetilde{x}(k) = T^{-1} \int_T x(t) e^{-ik\omega_0 t} dt \qquad \omega_0 = \frac{2\pi}{T}, T \rightarrow \infty ?$$

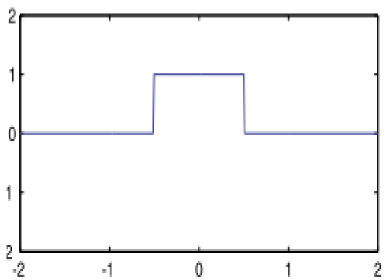
FT for nonperiodic signals

Definition of the continuous Fourier Transform

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \qquad \leftarrow \textbf{Direct FT}$$

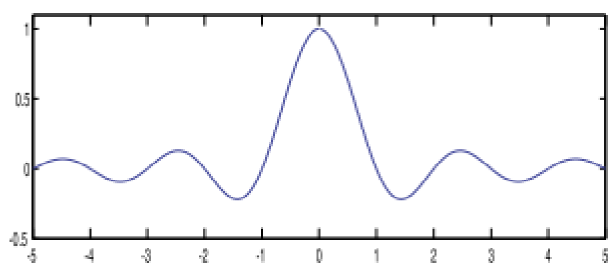
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(\omega) e^{i\omega t} d\omega \qquad \leftarrow \textbf{Inverse FT}$$

Example:



$$\tilde{x}(\omega) = \int_{-1/2}^{1/2} e^{-i\omega t} dt$$

$$= \frac{2\sin\left(\frac{\omega}{2}\right)}{\omega}$$



FT for nonperiodic signals

Orthogonality of harmonics on ±∞

$$\int e^{i\omega t} e^{-i\omega' t} dt = 2\pi\delta(\omega - \omega')$$

Continuous delta function:

$$\delta(x) = \begin{cases} 0, x \neq 0 \\ \infty, x = 0 \end{cases}$$

Main property of continuous delta function

$$\int f(x)\delta(x-a)dx = f(a)$$

$$\int e^{i\omega t} dt = 2\pi\delta(\omega)$$

$$\int e^{i\omega t} d\omega = 2\pi\delta(t)$$

Main FT properties

- Existence $\int |x(t)|^2\, dt < \infty \quad \left(\int (x(t) - \sum_{k=-\infty}^{N} \tilde{x}(k) e^{ik\omega_0 t})^2 dt \to 0 \right)$
- Linearity $F[ax+by] = aF[x] + bF[y]$
- Time shift $F[x(t+t_0)](\omega) = e^{-i\omega t_0} F[x(t)](\omega)$
- Time reversal $F[x(-t)](\omega) = F[x(t)](-\omega)$
- Frequency shift $F[e^{i\omega' t} x(t)](\omega) = F[x(t)](\omega - \omega')$
- Parseval's relation $\int |x(t)|^2\, dt = \frac{1}{2\pi} \int |x(\omega)|^2\, d\omega$

More interesting FT properties

- **Convolution** $F[x(t) * y(t)] = F[x(t)] \cdot F[y(t)]$
- **Differentiation** $F[\frac{dx(t)}{dt}] = i\omega F[x(t)]$
- **Integration** $F[\int x(t)] = (i\omega)^{-1} F[x(t)]$
- For real signals $F[x(t)](-\omega) = F[x(t)](\omega)^*$
- Multiplication (dual to convolution) $F[x(t) \cdot y(t)] = F[x(t)] * F[y(t)]$

TRY TO SHOW THESE

Some basic FT pairs

Some Fourier Transform Pairs	
Signal	Fourier Transform
$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$x(t) = 1$	$2\pi\delta(\omega)$
$\frac{sin(Wt)}{\pi t}$	$X(\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$
$step(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$tri(t)$	$sinc^2(\omega/(2\pi))$

If you need more - use a reference book!

LTI filters & FT

- Since **LTI systems/filters** ⇔ **convolution**, and **convolution** ⇔ **multiplication** in **Fourier Space**, action of LTI filters in Fourier space reduces to multiplication of the Fourier image of the input signal x~(ω) with the Fourier image of the response function h~(ω) (see the **convolution property** above)

$$F_h[x(t)] = \sum_{t'} x(t')h(t-t') = h(t) * x(t) \Rightarrow$$

$$F_h[\tilde{x}(\omega)] = \tilde{h}(\omega) \cdot \tilde{x}(\omega)$$

LTI filters & FT

- This property of Fourier space dramatically simplifies LTI filters
- The Fourier image of the response function h~(ω) is called the **transfer function**, the **frequency response,** or the **frequency characteristic** of LTI filter

$$\tilde{h}(\omega) = \mathrm{FT}[h(t)]$$

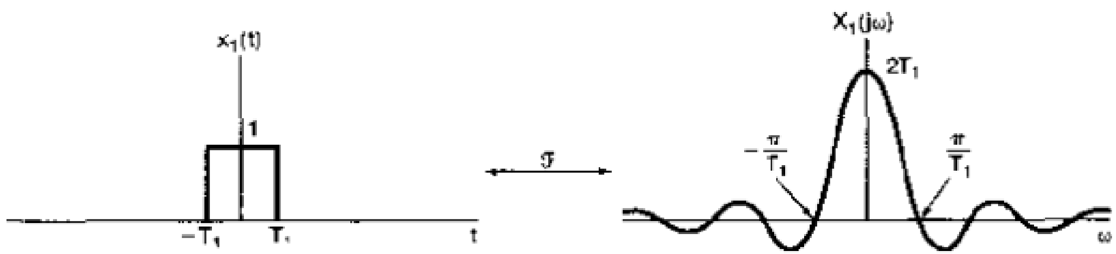
LTI filters & FT

- Given a transfer function h~(ω), the Fourier image of the filtered signal y~(ω) can be obtained simply by multiplying the Fourier transform of the input signal x~(ω) with h~(ω)

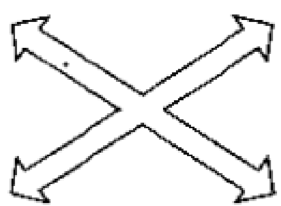
$$\tilde{x}(\omega) \rightarrow F_h[\tilde{x}(\omega)] = \tilde{h}(\omega) \cdot \tilde{x}(\omega) = \tilde{y}(\omega)$$

FT duality

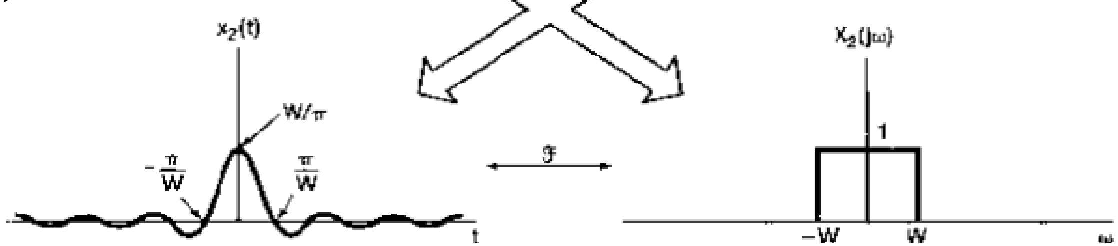
- Because of the similarity of the direct and inverse Fourier transforms, calculating Fourier transform of what looks like a Fourier image of a function essentially gives back that function! (For example, FT[box(t)] = sinc(ω) and FT[sinc(t)]=box(ω))



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(\omega) e^{i\omega t} d\omega$$



$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$



DFT

- Discrete Fourier Transform (DFT) is FT applied to discrete signals such as digitized audio or video
- Since we deal with digital signals in computing, practically always in DSP we have to deal with DFT and not the analytical FT

DFT

DFT definition:

$$\tilde{x}(k) = \sum_{n=0}^{N-1} x(n) e^{-ik\omega_0 n}, k = 0,1,\ldots, N-1$$ **← Direct DFT**

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}(k) e^{ik\omega_0 n}, n = 0,1,\ldots, N-1$$ **← Inverse DFT**

$$\omega_0 = 2\pi / N$$

DFT

Note: result of DFT is a signal labeled by discrete index k. The real frequencies corresponding to each k can be calculated via following relationship:

$$k \to k\omega_0 = 2\pi k / N, k = 0,1,\ldots, N-1$$

for

$$\tilde{x}(k) = \sum_{n=0}^{N-1} x(n) e^{-ik\omega_0 n}, k = 0,1,\ldots, N-1$$

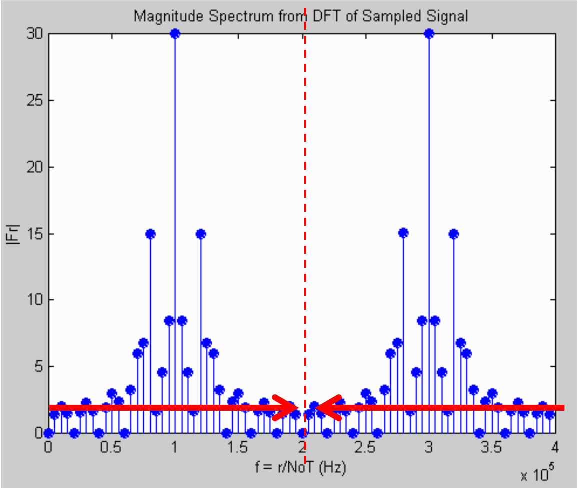
DFT

Important - frequency wrapping in DFT:

- Digital software typically outputs DFT as a signal x~(k) labeled k=0,..,N-1
- One must remember that that the high frequency half of x~(N-k) can be also seen equivalently as a negative frequency components x~(-k) by discarding not essential factors of $e^{2\pi i}=1$ (see below)

(*In fact, DFT result is periodic in N and what you get from usual DFT software is one period of that result shown over the range 0..N-1, while if that suits you better you can also view the same result shown on the range –N/2..N/2, see* *Advanced section*.)

$$\omega_k \Rightarrow 0,1,\ldots,N-1 \Leftrightarrow -N/2,\ldots,N/2$$



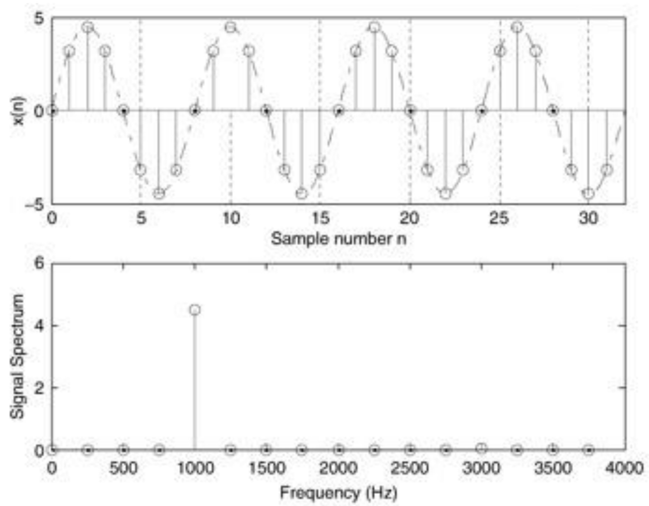
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{x}(k)e^{i\omega_k n}$$

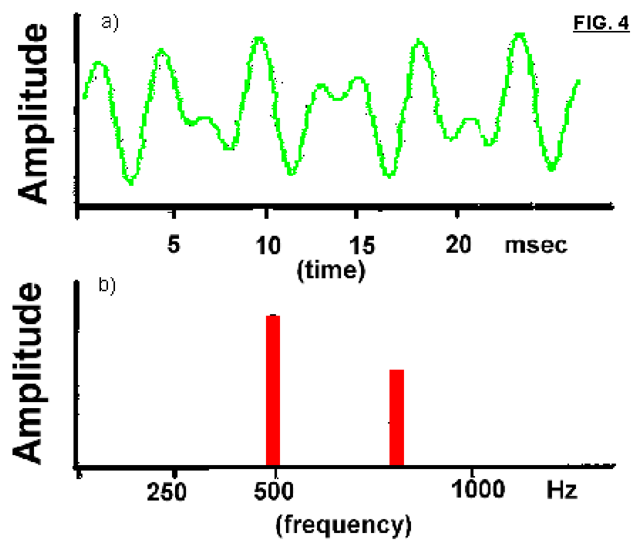
$$e^{i\omega_{N-k}t} = e^{2\pi i(N-k)/Nt} =$$

$$= e^{2\pi i t - 2\pi i k t/N} = e^{-i\omega_k t}$$

Spectrum

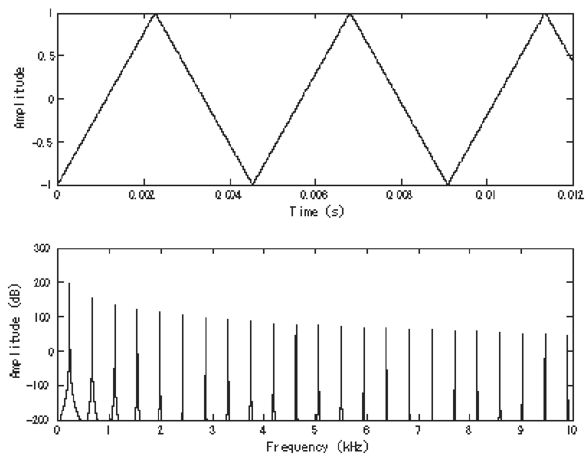
Spectrum refers generally to the Fourier (sin-waves) decomposition of a signal x~(ω)

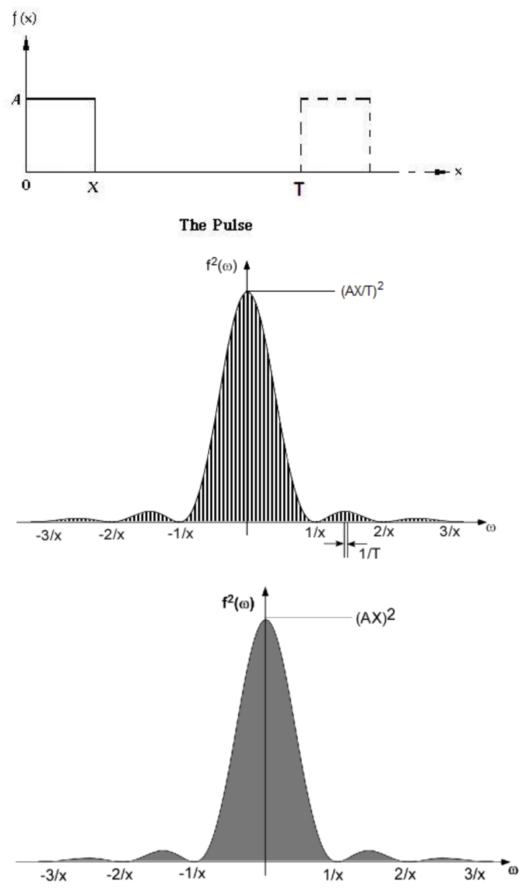




Spectrum

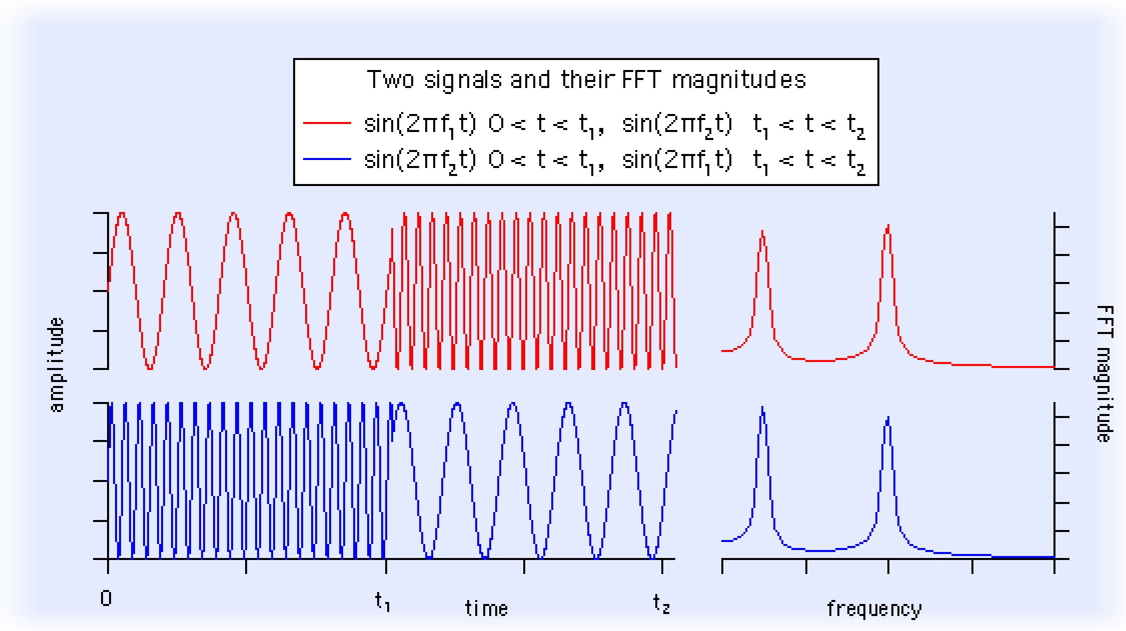
More examples





Spectrum

Yet more examples



TRY TO EXPLAIN WHY

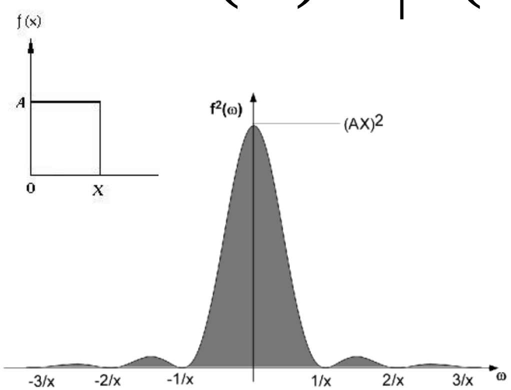
Hint: remember that

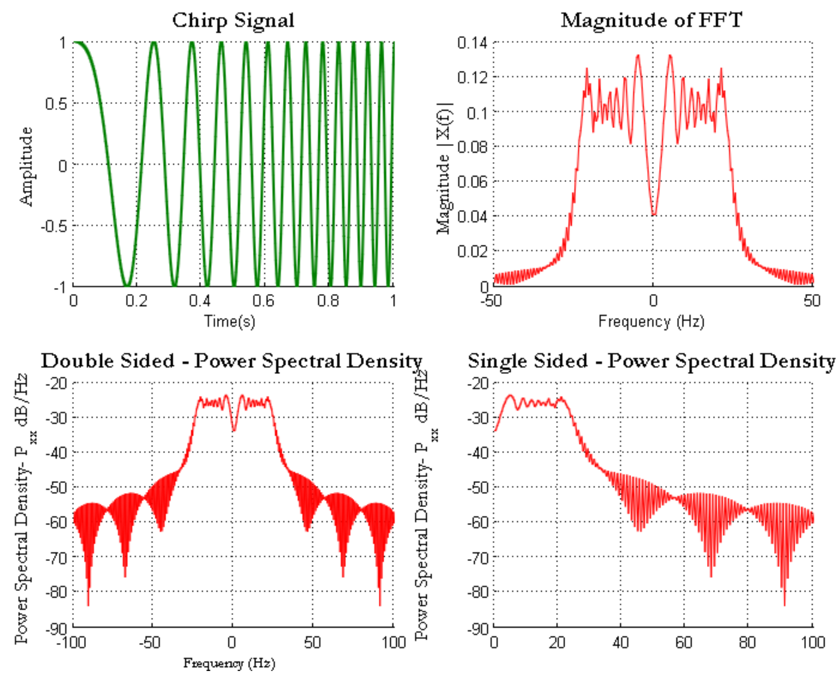
$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

Spectrum

Definition: Power spectral density (PSD) is defined as the absolute square of the FT spectrum

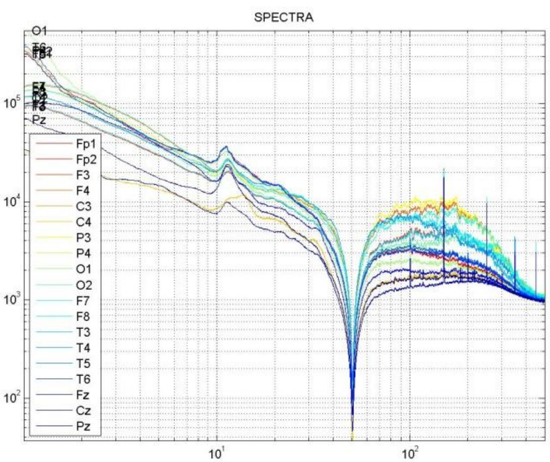
$$S(\omega) = \left|\tilde{x}(\omega)\right|^2$$

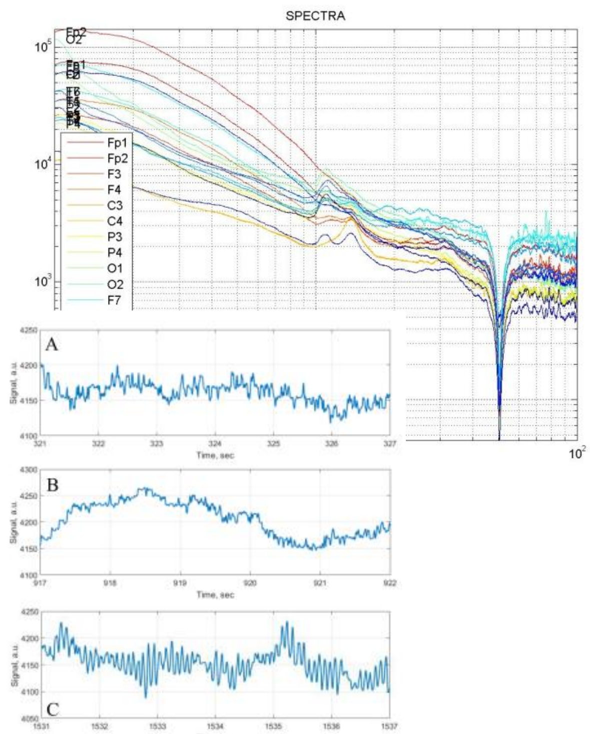


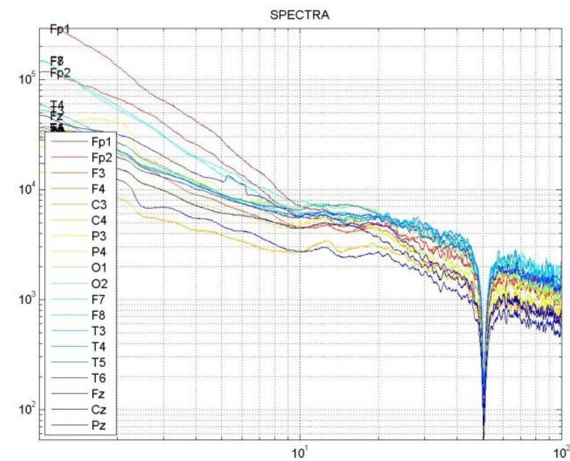


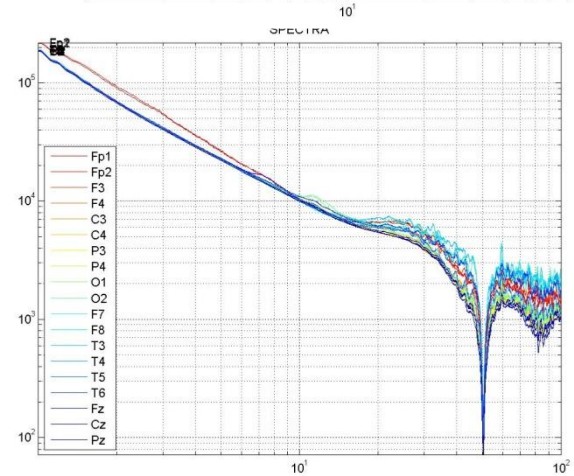
Spectrum

Spectrum of human EEG signal recorded for left and right hand movements





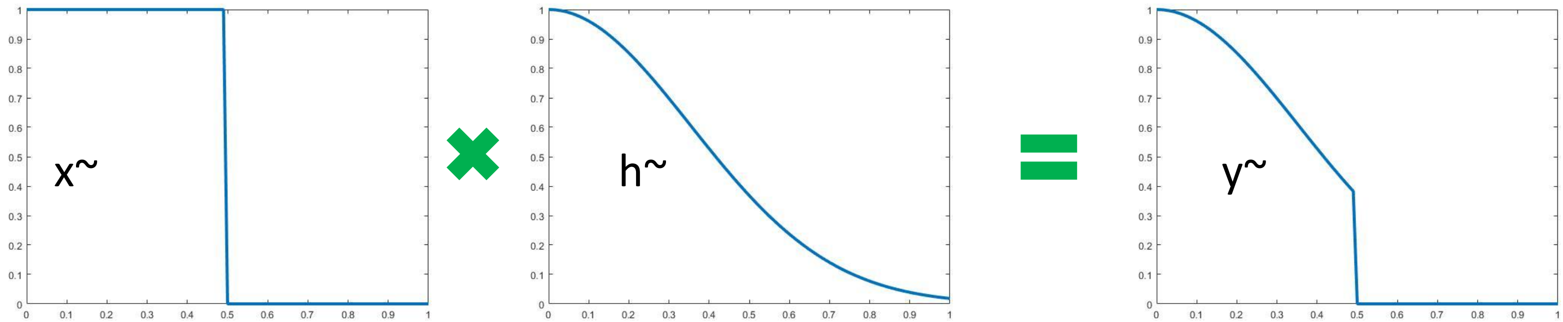




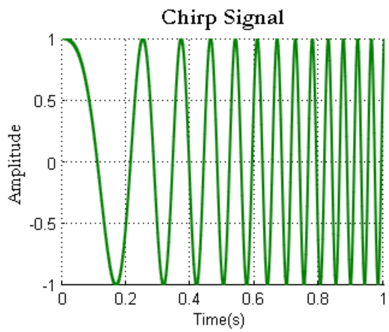
LTI filters and spectrum

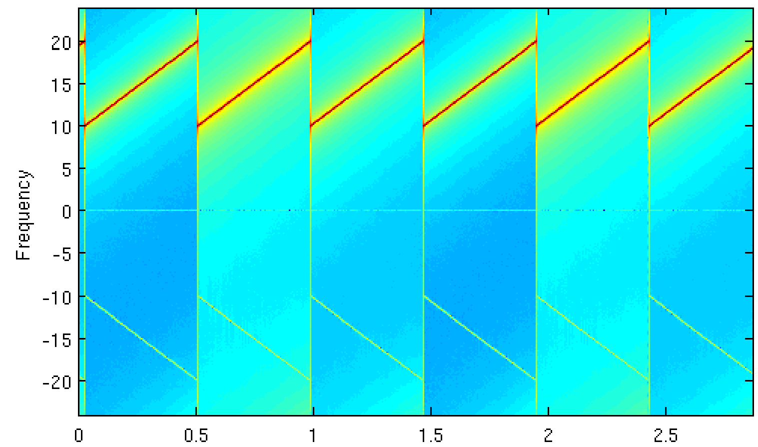
- By the earlier property, the spectrum of a filtered signal is simply the product of the filter's transfer function with the input signal's spectrum

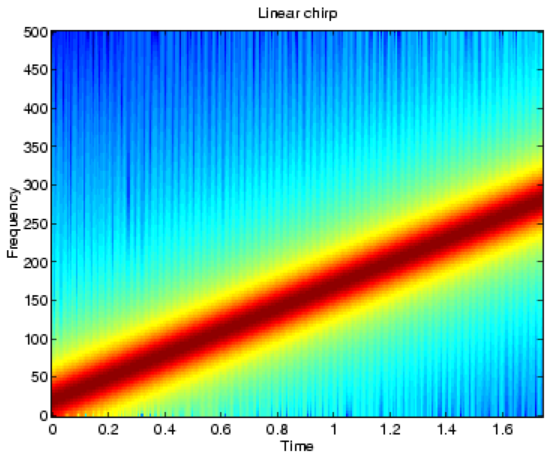
$\tilde{x}(\omega)$ $\qquad$ $\tilde{h}(\omega)$ $\qquad$ $\tilde{y}(\omega) = \tilde{h}(\omega) \cdot \tilde{x}(\omega)$

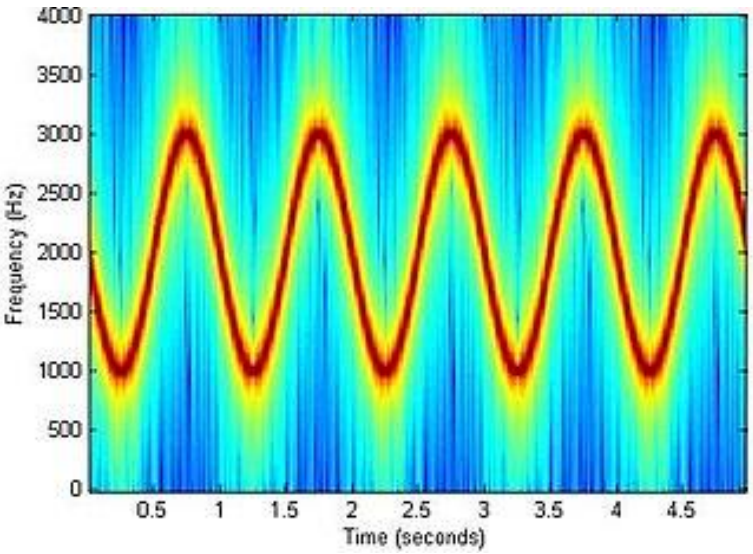


Spectrogram

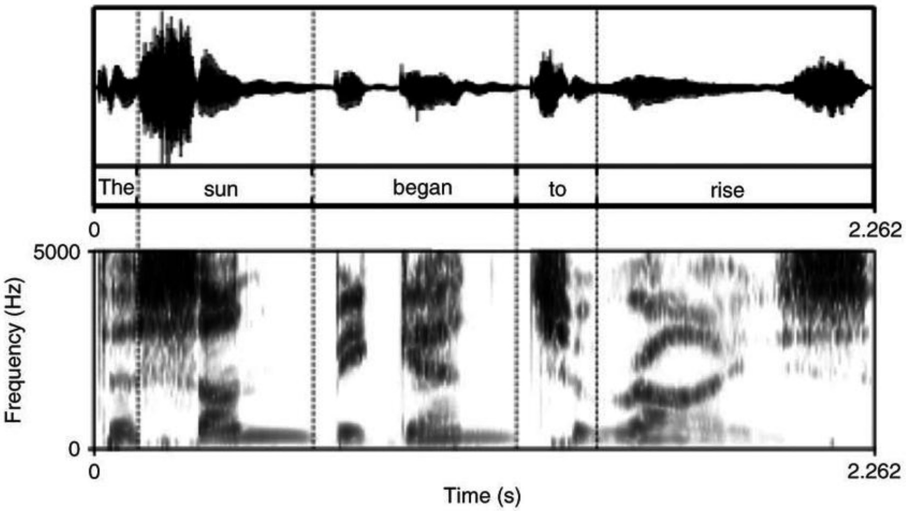


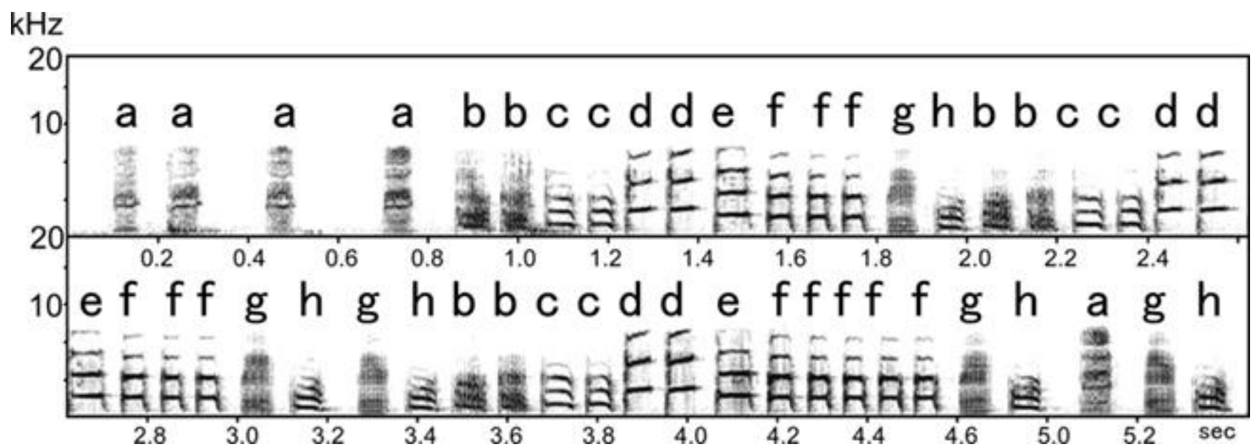






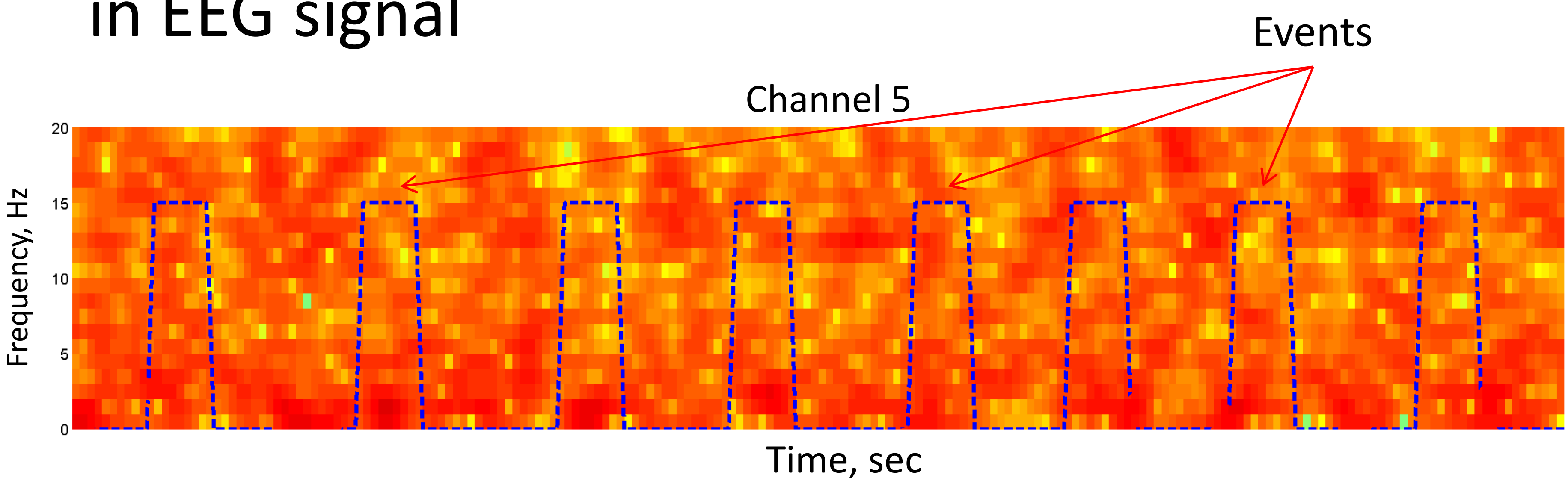
Spectrogram

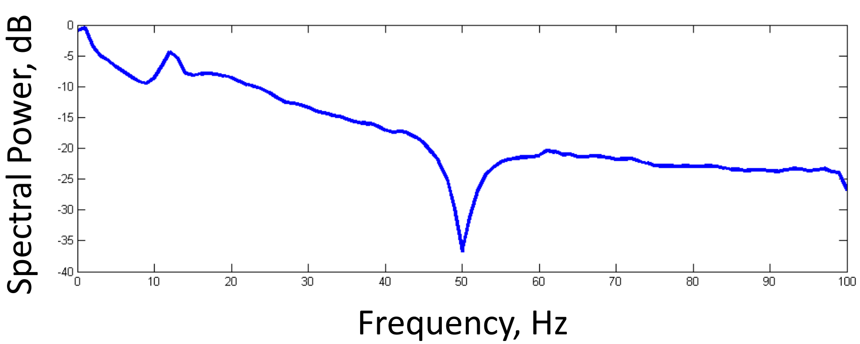


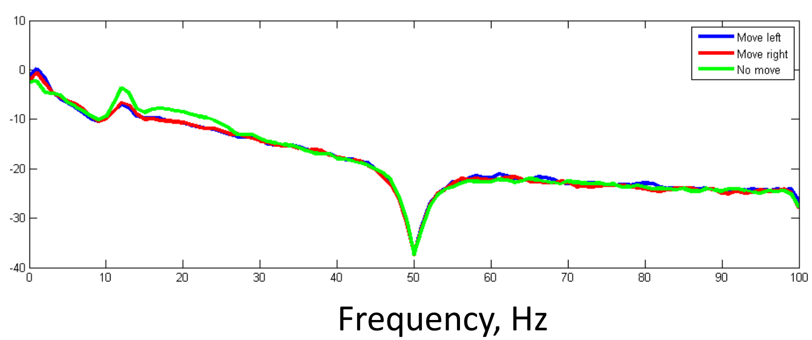


Spectrogram

Spectrogram of left and right hand movements in EEG signal



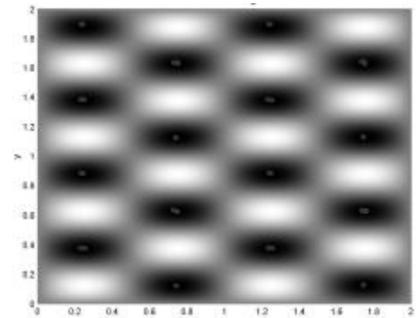


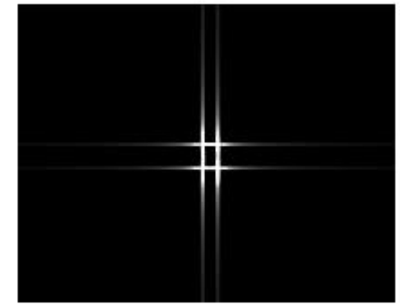


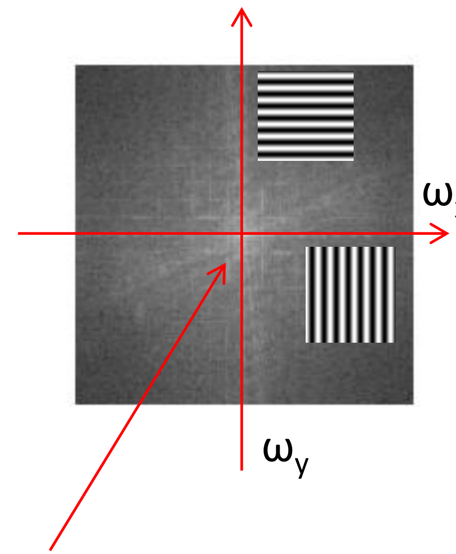
2D (image) FT

$$f(x,y)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(\omega_x,\omega_y)e^{i\omega_x x+i\omega_y y}d\omega_x d\omega_y$$

$$\tilde{f}(\omega_x,\omega_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-i\omega_x x-i\omega_y y}dxdy$$

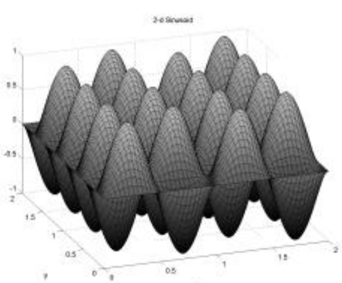


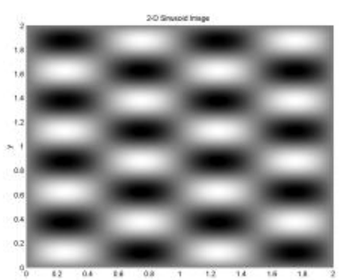




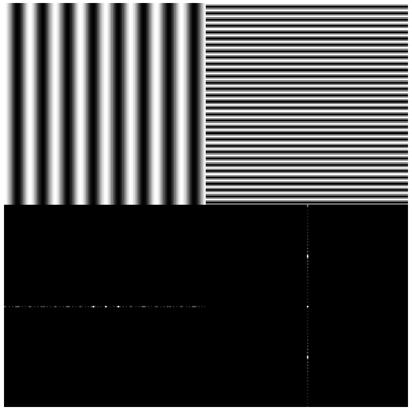
Zero frequency !

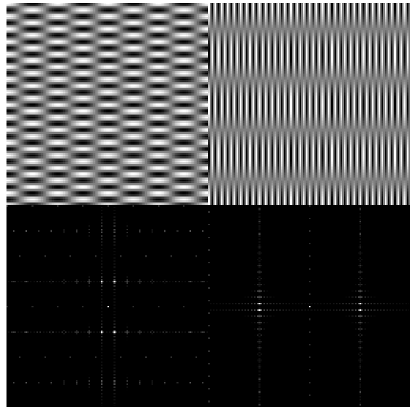
$$e^{i\omega_x x+i\omega_y y} \sim \sin(\omega_x x)\sin(\omega_y y)$$

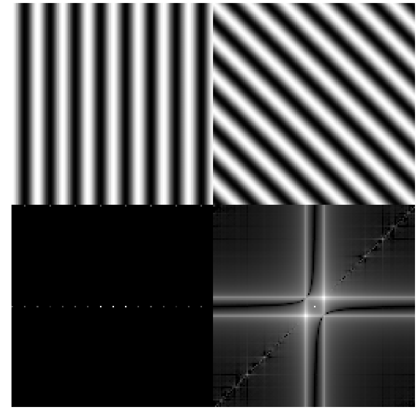


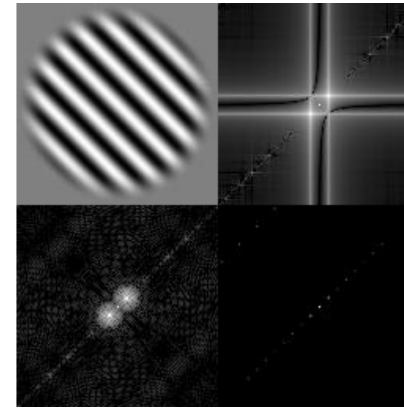


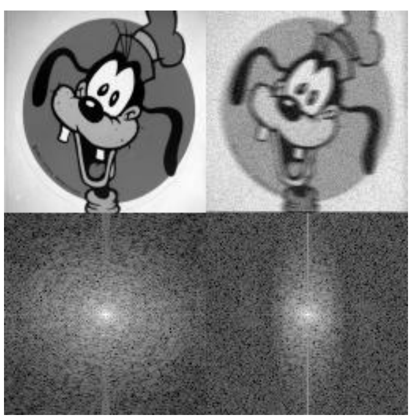
2D (image) FT

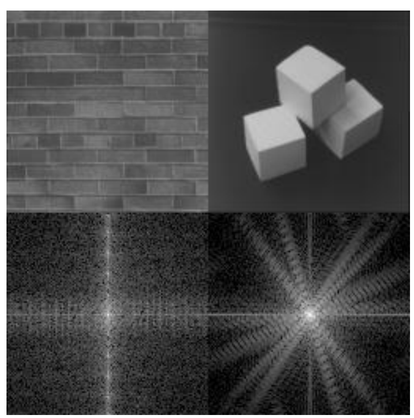


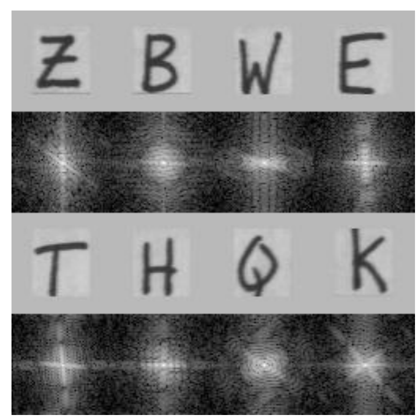


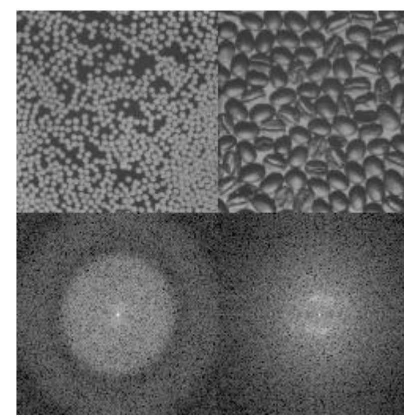












QUESTIONS FOR SELF-CONTROL

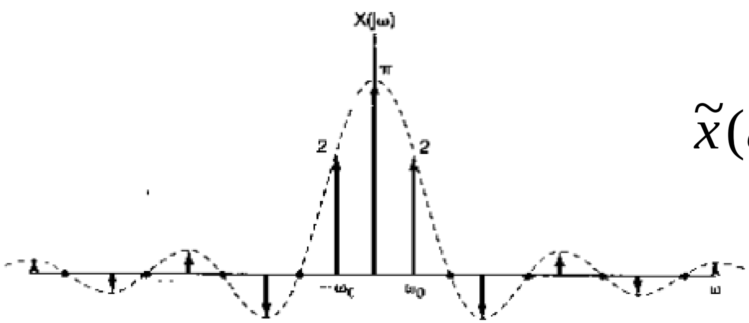
- What is Euler identity and why it is interesting?
- Express cos(φ) and sin(φ) through $e^{i\varphi}$ and $e^{-i\varphi}$ by using Euler identity.
- Find $\sqrt{i}$ and $i^{3/4}$?
- Define Fourier transform in real form and complex form.
- Prove orthogonality of $e^{in\omega t}$ and $e^{-im\omega t}$ on $[0,2\pi/\omega]$ for $n\neq m$.
- Calculate the periodic Fourier transform of *tri*(x) signal that equals to |1-x| for $-1\leq x\leq 1$ and is periodically repeated from [-1,1].
- Calculate the periodic Fourier transform of *pulse*(x) signal that equals 1 for $0\leq x\leq 1$ and is periodically repeated on [0,2]. How does your result compare with that obtained in the class? How does it compare considering now the **time-shift** property of FT?
- Calculate the nonperiodic Fourier transform of isolated *tri*(x) signal that equals|1-x| for $-1\leq x\leq 1$ and zero everywhere else. How does your result compare with the periodic FT calculation of the same?
- Calculate the nonperiodic Fourier transform for isolated *pulse*(x)=1 for $0\leq x\leq 1$ and zero everywhere else? How does your result compare with the periodic FT calculations?
- What is the effect of taking signal derivative and antiderivative on the Fourier transform?
- How does convolution appear in Fourier domain?
- Consider two functions $x(t)=\sin(10\pi t)+0.5\cos(5\pi t)$ and $y(t)=0.5\sin(15\pi t)+0.25\cos(5\pi t)$. Take advantage of the convolution property of the Fourier transform to quickly find the convolution $x(t)*y(t)$.
- Prove the convolution property of FT.
- Prove the integration property of FT.
- Define discrete and continuous delta functions.

- Consider $\int dx e^{3x+x^2} \cos(5\pi x)\delta(x-1)$. What is the value of this integral?
- Consider $\sum_k \sin(5k\pi/2)/(k+1)^2 \delta(k-10)$. What is the value of this sum?
- Define DFT.
- Explain why DFT spectrum duplicates itself as $\omega_k=2\pi k/N \rightarrow \omega_k+2\pi m$?
- Explain why spectrum of a signal sampled with time-step ΔT (that is sampling frequency $\omega_s=2\pi/\Delta T$) duplicates as $\omega_k \rightarrow \omega_k+2\pi m/\Delta T$? How is this related to the frequency aliasing during down-sampling?
- Prove using DFT definition formulas that series application of DFT-inverse DFT, $x(n) \rightarrow x\tilde{}(k) \rightarrow x(n)$, gives back the original signal x(n). (Take advantage of the formula for the sum of geometric series to find that DFT harmonics $e^{ik\omega_0 n}$ and $e^{-ik\omega_0 n'}$ are orthogonal under sums over k=0..N-1).
- Describe what is spectrum.
- Define power spectral density.
- Consider $x(t)=\sin(10\pi t)+0.5\cos(5\pi t)$. What is the spectrum and the PSD of this signal?
- Calculate the PSD of chirp signal.
- Describe what is spectrogram.
- How does the spectrogram of chirp signal look like?
- Draw an example spectrogram of a frequency modulated (FM) signal.
- What kind of pattern $e^{in\omega_x x}e^{im\omega_y y}$ produces in xy-image? What is the Fourier transform of that pattern? What is that pattern for n=0 or m=0?
- Give examples of simple 2D image patterns and their 2D FT transforms.
- There are also questions throughout the slides marked in **red** – review and answer them too.

ADVANCED

Periodic FT in continuous space

FT of periodic signals can be treated as continuous FT resulting in a train of delta function-pulses at placed multiples of $\omega_0 = 2\pi/T$



$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt = \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \tilde{x}(k\omega_0)$$

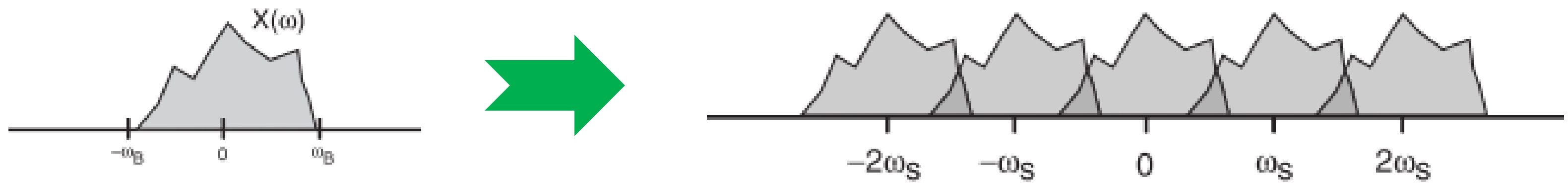
Spectrum duplication in DFT

DFT result is periodic with period N which can be easily seen from defining formulas:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{x}(k) e^{-i\omega_k n} \left(n = 0, 1, \ldots, N-1\right)$$

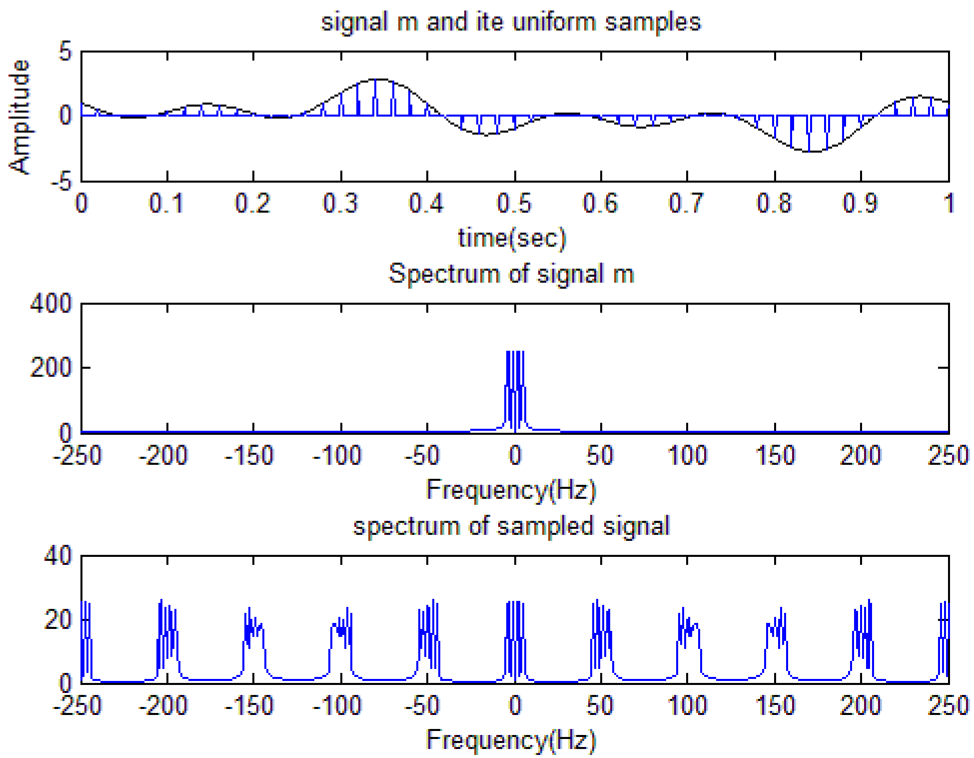
$$k \to k \pm mN \Rightarrow \omega_k \to \omega_k + 2\pi m$$

This property is responsible for the phenomenon of frequency aliasing that in the 1st lecture



Spectrum duplication in DFT

Since sampled signals are discrete, this is exactly what we have there:



$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

$$\tilde{x}(\omega_m) = \sum_k x(k\Delta T) e^{-i(\omega + \frac{2\pi}{\Delta T} m) k\Delta T}$$

Fast Fourier Transform

Fast Fourier Transform is the standard algorithm for computing DFT

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} nk}$$

$$X_k = \underbrace{\sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N}(2m)k}}_{E_k} + \underbrace{\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k}}_{O_k}$$

$$X_k = \begin{cases} E_k + e^{-\frac{2\pi i}{N}k} O_k & \text{for } 0 \le k < N/2 \\ E_{k-N/2} + e^{-\frac{2\pi i}{N}k} O_{k-N/2} & \text{for } N/2 \le k < N. \end{cases}$$

$$\begin{aligned} X_k &= E_k + e^{-\frac{2\pi i}{N}k} O_k \\ X_{k+\frac{N}{2}} &= E_k - e^{-\frac{2\pi i}{N}k} O_k \end{aligned}$$

Fast Fourier Transform

FFT is a classical example of Divide and Conquer algorithm – it works by recursively subdividing the DFT problem of size N into two smaller problems of size N/2 – $\boldsymbol{E}$ and $\boldsymbol{O}$ – corresponding to the FFT of all-evens and all-odds samples in the original signal

$$\begin{aligned} X_k &= E_k + e^{-\frac{2\pi i}{N}k} O_k \\ X_{k+\frac{N}{2}} &= E_k - e^{-\frac{2\pi i}{N}k} O_k \end{aligned}$$

Fast Fourier Transform

- Original DFT's complexity is $O(n^2)$, however by using Divide & Conquer FFT achieves computational complexity of $O(n \log n)$, very fast, similar to the same result in **sorting**

Fast Fourier Transform

- FFT is fastest if length of input easily sub-divides into a hierarchy of 2s – that is 2^k, and may **slow down dramatically** if $n \neq 2^k$, even if n is much much smaller. For that reason, signals as a common practice need to be **padded with zeros** to the nearest greater 2^k if that is the case (in fact most modern algorithms will do that automatically for you)