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MATH 114 : ENGINEERING MATHS II

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Overview

- The Fundamental Theorem of Calculus(FTC)
- Computing Indefinite Integrals
- Substitution method
- Definite Integral
- Trigonometric Integrals and Trigonometric Substitutions
- Reduction Formulas
- Rational functions and Partial fractions
- Improper Integrals

1 Vector Integration

- Line Integrals
- Multiple Integrals
- Double Integrals
- Triple Integrals
- Surface Integrals
- Volume Integrals

Integration

Definition

Consider a non-negative function which is continuous over an interval [*a*,*b*]. The interval [*a*,*b*] will be divided into sub-intervals of equal width and the sample points will correspond to the right endpoints of the sub-interval. Let the non-negative function $y = f(x)$ be continuous over [*a*,*b*]. We divide [*a*,*b*] into *n* equal sub-intervals with $\Delta x = \frac{b-a}{n}$. The right endpoints of the intervals are designated $x_1, x_2, \ldots, x_n$ where $x_k = a + k\Delta x$ and $x_n = b$. For each sub-interval we construct a rectangle as shown in the diagram below:

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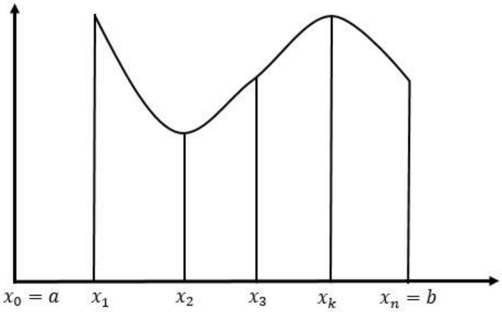

Note

The base of each rectangle is Δx and the height of the rectangle k is $f(x_k)$. It follows that the area of the rectangle k is $f(x_k)\Delta x$. The sum of the areas of all n rectangles is called the Riemann Sum i.e $\sum_{k=1}^{n} f(x_k)\Delta x$

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Definition (Definite Integral)

If f is continuous function defined on $[a, b]$ and if $[a, b]$ is divided into n equal sub-intervals of width $\Delta x = \frac{b-a}{n}$, and if $x_k = a + k\Delta x$ is the end point of sub-interval k, then the definite integral of f from a to b is the number:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Properties of definite Integrals

1. $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

example: $$\int_3^4 x^2 + 1\, dx = -\int_4^3 x^2 + 1\, dx$$

2. Given $\int_a^b f(x)\, dx$. If $a = b$, then

$b \; a \; b \; \int \int \int$

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$$\int_a f(x)\,dx = \int_a f(x)\,dx = \int_b f(x)\,dx = 0$$

example: $\displaystyle\int_4^4 \cos^2 x\,dx = 0$

3. If $f(x)$ is even that is $[f(x) = f(-x)]$, then $\displaystyle\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

example: $\displaystyle\int_{-2}^{2} x^4\,dx = 2\int_{0}^{2} x^4\,dx = \frac{64}{5}$

4. If $f(x)$ is odd that is $[f(-x) = -f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

example $\displaystyle\int^{3} \frac{\tan x}{}\,dx = 0$

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$1 + x^2 + x^4$

-3

5. $\displaystyle\int_a^b c\,dx = [cx]\Big|_a^b = cb - ca$, where c = constant

3

-1

6. $$\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

example: $\int_{-1}^{3} 3\, dx.$ $= [3x]\Big|_{-1}^{3} = 3(3) - 3(-1) = 12$

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example: $\displaystyle\int_2^b (\cos x \pm \tanh x)\,dx = \int_2^b \cos x\,dx \pm \int_2^b \tanh x\,dx$

7. $\displaystyle\int_a^b c\,f(x)dx = c\int_a^b f(x)dx$, where c = constant

example: $\displaystyle\int_a^b 3\,x^8\sqrt{x^9+1}\,dx = 3\int_a^b x^8\sqrt{x^9+1}\,dx$

The Fundamental Theorem of Calculus(FTC)

Theorem

If $f(x)$ is a continuous function on the interval $[a, b]$.

1. *Part 1: If $F(x)$ on $[a, b]$ is defined by, $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$, for all x in $[a, b]$.*

2. *Part 2: If $G(x)$ on $[a, b]$ satisfies $G'(x) = f(x)$, then*
$$\int_a^b f(t)\,dt = G(b) - G(a)$$

Example

Find the the derivative of

$$F(x) = \int_x^{10} \sqrt{t^2+1}\, dt$$

Solution

$$F(x) = \int_x^{10} \sqrt{t^2+1}\, dt = -\int_{10}^{x} \sqrt{t^2+1}\, dt$$

Now $f(t) = -\sqrt{t^2+1}$ *is a continuous function on* $(-\infty, \infty)$. *Apply FTC to get*

$$F'(x) = f(x) = -\sqrt{x^2+1}$$

Example

Find $\frac{d}{dx}\int_0^{\sin x} \sqrt{t^2+1}$

Solution

Note that the variable bound is a function instead of simply an "x". Therefore, we cannot directly apply FTC. Set $u = \sin x$. *The we have a function*

$$G(u) = \int_0^u \sqrt{t^2+1}\,dt$$

for which FTC is applicable with $G'(u) = \sqrt{u^2+1}$.
But what we want is $F'(x)$. *As* $F(x) = G(u(x))$ *we can use Chain Rule* $F'(x) = G'(u)u'(x)$ *to get the job done:*

$$F(x) = G'(u)u'(x) = \sqrt{u^2+1}\cos x = \sqrt{\sin^2 x + 1}\cos x$$

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Definition

A function F is called an anti-derivative of f on the interval $[a, b]$ if one has $F'(x) = f(x)$ for all x with $a < x < b$.

For instance, $F(x) = \frac{1}{2}x^2$ is the anti-derivative of $f(x) = x$, but so is $G(x) = \frac{1}{2}x^2 + 2008$

Example

Evaluate the integral $\int_{-2}^{1} x^3\, dx$

Solution

The function $f(x) = x^3$ is continuous on $[-2, 1]$, now the antiderivative of $f(x)$ is $F(x) = \frac{1}{4}x^4$ so Part 2 of the Fundamental Theorem gives

$$\int_{-2}^{1} x^3\, dx = F(1) - F(-2) = \frac{1}{4}(1)^4 - \frac{1}{4}(-2)^4 = -\frac{15}{4}$$

Example

Find the area under the parabola $y = x^2$ from 0 to 1.

Solution

An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$. The required area A is found using Part 2 of the Fundamental Theorem:

$$A = \int_{0}^{1} x^2\, dx = \frac{1}{3}x^3 \Big|_{0}^{1} = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}$$

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Z $$= \frac{x^{1+1}}{1+1} + c = \frac{x^2}{2} +$$

Definition

Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)\, dx$ is traditionally used for an antiderivative called an Indefinite Integral.

Example

Evaluate $\int x\, dx$

Solution

$x\ dxc$

Example

7. $\int (x^3 - 5x^2 + 7x - 11)\, dx$

Solution

$$\int (x^3 - 5x^2 + 7x - 11)dx = \frac{x^4}{4} - \frac{5x^3}{3} + \frac{7x^2}{2} - 11x + c$$

Example

Find 7. $\int \left(2e^x + \frac{6}{x} + \ln 2\right) dx$

Solution

$$\int \left(2e^x + \frac{6}{x} + \ln 2\right) \qquad dx = 2 \qquad ZZ \quad + 6 \int \frac{1}{x} \qquad e^x\, dxdx + \ln 2$$

$$dx$$

$$= 2e^x + 6\ln|x| + (\ln 2)x + c$$

Solution

Example

Evaluate 7. $\int \left(\sqrt[3]{x} - \frac{1}{\sqrt[3]{x}} \right) dx$

$$\int \left(\sqrt[3]{x} - \frac{1}{\sqrt[3]{x}} \right) dx = \int \left(x^{\frac{1}{3}} - \frac{1}{x^{\frac{1}{3}}} \right) dx$$

$$= \int \left(x^{\frac{1}{3}} - x^{-\frac{1}{3}} \right)$$

$\frac{1}{3}+1$ $-$

dx $= \frac{x^{\frac{4}{3}}}{} - \frac{x^{\frac{2}{3}}}{} + c$ $-$ $\frac{3x^{\frac{4}{3}}}{4} - \frac{3x^{\frac{2}{3}}}{2} +$

$-\frac{1}{}+1$

xx 3

Example

Evaluate 7. $\dfrac{4x^{10} - 2x^4 + 15x^2}{x^3}\, dx$

$= \overline{\frac{1}{3} + 1} \qquad \overline{\frac{2}{3}} + c \qquad \overline{\frac{4}{3}} \qquad \overline{\frac{2}{3}} \qquad = \qquad c$

Solution

$$\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\, dx = \int \frac{4x^{10}}{x^3}\, dx - \int \frac{2x^4}{x^3}\, dx + \int \frac{15x^2}{x^3}\, dx$$

Z Z

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$= \quad 4x^7\, dx - 2x\, dxdxc \quad + \int \frac{15}{x} \quad = \frac{1}{2}x^8 - x^2 + 15 \ln|x| +$

Z $\qquad e^{4x}$

$4x$

Definition

7. $e^{ax} dx = \frac{1}{a} e^{ax} = \frac{e^{ax}}{a} + c$

Example

7. $e^{5x} dx$

Solution

$$e \quad dx \quad = \quad \underline{\qquad} + c$$

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4

Example

Find $\int a^x \, dx$

Solution

$$\int a^x \, dx = \int e^{\ln a^x} \, dx = \int e^{x \ln a} \, dx$$

$$= \frac{e^{x \ln a}}{\ln a} + c$$

$$\text{but } \; e^{x \ln a} = e^{\ln a^x} = a^x$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + c$$

Example

Find $\int 3^x \, dx$

Solution

$$\int 3^x \, dx = \int e^{\ln 3^x} \, dx = \int e^{x \ln 3} \, dx$$

$$= \frac{e^{x \ln 3}}{\ln 3} + c$$

$$but \;\; e^{x \ln 3} = e^{\ln 3^x} = 3^x$$

$$\int 3^x \, dx = \frac{3^x}{\ln 3} + c$$

Definition

1. 7. $\sin x \; dx = -\cos x + c$
2. 7. $\cos x \; dx = \sin x + c$
3. 7. $\sin(ax) \; dx = \dfrac{-\cos(ax)}{a} + c$
4. 7. $\cos(ax) \; dx = \dfrac{\sin(ax)}{a} + c$
5. 7. $\sin(ax + b) \; dx = \dfrac{-\cos(ax + b)}{a} + c$
6. 7. $\cos(ax + b) \; dx = \dfrac{\sin(ax + b)}{a} + c$

Example

Given that $f'(x) = 4x^3 - 9 + 2\sin x + 7e^x$, $f(0) = 15$ determine the function $f(x)$.

Solution: $f(x) = \int f'(x)\,dx$

We integrate $f'(x)$ to determine the most general possible $f(x)$.

$$f(x) = \int 4x^3 - 9 + 2\sin x + 7e^x\,dx = x^4 - 9x - 2\cos x + 7e^x + c$$

Considering $x = 0$ where $f(0) = 15$, then we can determine the value of the constant of integration c.

$$15 = f(0) = 0^4 - 9(0) - 2\cos(0) + 7e^0 + c = -2 + 7 + c = 5 + c$$

So, $c = 10$. This means that the function is,

$$f(x) = x^4 - 9x - 2\cos x + 7e^x + 10$$

Definition

1. $\int \sec x \tan x \, dx = \sec x + c$
2. $\int \csc x \cot x \, dx = -\csc x + c$
3. $\int \csc^2 x \, dx = -\cot x + c$
4. $\int \sec^2 x \, dx = \tan x + c$
5. $\int \tan(ax) \, dx = \frac{1}{a} \ln|\sec(ax)| + c$
6. $\int e^{ax} \sin x \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + c$
7. $\int e^{ax} \cos x \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + c$

Hyperbolic functions

Definition

1. $$\int \sinh x \; dx = \cosh x + c$$
2. $$\int \cosh x \; dx = \sinh x + c$$
3. $$\int \tanh x \; dx = \ln|\cosh x| + c$$
4. $$\int \coth x \; dx = \ln|\sinh x| + c$$

Definition

1. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
2. $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$
3. $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
4. $\int -\frac{1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$

Inverse Trigonometric functions

$$\int \left(2\sec w \tan w + \frac{1}{6w}\right) dw = \int 2\sec w \tan w \, dw + \int \frac{1}{6w} \, dw$$

$$\int$$

Example

Evaluate $\int 2\sec w \tan w + \dfrac{1}{6w} \, dw$

Solution: $= \int 2\sec w \tan w \, dw + \dfrac{1}{6}\int \dfrac{1}{w} \, dw = 2\sec w + \dfrac{1}{6}\ln|w| + c$

$\int \left(\dfrac{23}{y^2+1} + 6\csc y \cot y + \dfrac{9}{y} \right) dy = 23\tan^{-1} y - 6\csc y + 9\ln|y| + c$

Example

Evaluate $\int \dfrac{23}{y^2+1} + 6\csc y \cot y + \dfrac{9}{y} \, dy$

Solution

y

Definition

The substitution method is an easier way for evaluating integrals. It is based on the following identity between differentials (where u is a function of x):

$$du = u' dx$$

Hence we can write:

$$\int f(g(x))g'(x) = \int f(u)\, du$$

Substitution method

Example

Find $\int (3x - 5)^{12}\, dx$

Solution

Using the substitution $u = 3x - 5$, *and differentiating* u *with respect to* x *we have* $du = 3dx \Rightarrow dx = \frac{du}{3}$

$$\int (3x - 5)^{12}\, dx = \int u^{12} \cdot \frac{1}{3} du = \frac{1}{3} \int u^{12} du$$

$$= \frac{1}{3} \left(\frac{1}{13} u^{13} \right) + c = \frac{1}{39} u^{13} + c$$

$$But\ u = 3x - 5$$

$$\int (3x - 5)^{12}\, dx = \frac{1}{39} (3x - 5)^{13} + c$$

Example

Find $\int x^2 \sqrt[3]{x^3 + 5}\, dx$

Solution

Using the substitution $u = x^3 + 5$, $du = 3x^2 dx \Rightarrow dx = \frac{du}{3x^2}$

$$\int x^2 \sqrt[3]{x^3 + 5}\, dx = \int \sqrt[3]{u}\, \frac{du}{3} = \frac{1}{3} \int u^{\frac{1}{3}} du$$

$$= \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) u^{\frac{4}{3}} + c = \frac{1}{4} u^{\frac{4}{3}} + c$$

But $u = x^3 + 5$

Example

$$\int x^2\sqrt[3]{x^3+5}\,dx = \frac{1}{4}(x^3+5)^{\frac{4}{3}} + c$$

$$= \frac{1}{4}\left(\sqrt[3]{x^3+5}\right)^4 +$$

Find $\int 2x\cos(x^2)\,dx$

Solution

$$Let\ u = x^2,\ then\ \frac{du}{dx} = 2x \implies dx = \frac{du}{2x}$$

$$\int 2x\cos(x^2)\,dx = \int 2x\cos(u)\,\frac{du}{2x} = \int \cos u\,du = \sin u + c$$

$$but\ \ u = x^2$$

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + c$$

Example

Evaluate $\int (ax+b)^n\ dx$, assuming that a and b are constants, $a = 0$, and n is a positive integer.

Solution

We let $u = ax + b$, *then* $\dfrac{du}{dx} = a \implies dx = \dfrac{du}{a}$, *then*

$$\int (ax+b)^n\ dx = \int (u)^n\ \frac{du}{a} = \frac{1}{a}\frac{(u)^{n+1}}{n+1} + c$$

$$but\ \ u = ax + b$$

$$\int (ax+b)^n\ dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$$

find $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$

Solution

We let $u = e^x$, *then* $\dfrac{du}{dx} = e^x \implies dx = \dfrac{du}{e^x}$. *Then*

$$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{e^x}{u^2+1}\,\frac{du}{e^x} = \int \frac{1}{u^2+1}\,du = \tan^{-1} u + c$$

$$but\ \ u = e^x$$

$$\int \frac{e^x}{e^{2x}+1}\,dx = \tan^{-1}(e^x) + c$$

Example

Evaluate $\int \sin(ax+b)\,dx$, assuming that a and b are constants and $a = 0$.

Solution

We let $u = ax + b$, *then* $\dfrac{du}{dx} = a \implies dx = \dfrac{du}{a}$. *Then*

$$\int \sin(ax+b)\,dx = \int \sin(u)\,\frac{du}{a} = \int \frac{1}{a}\sin u\,du = -\frac{1}{a}\cos u + c$$

$$\textit{but} \quad u = ax + b$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

Example

Evaluate $\int xe^{x^2}\,dx$

Solution

We let $u = x^2$, *then* $\frac{du}{dx} = 2x \implies dx = \frac{du}{2x}$. *Then*

$$\int xe^{x^2}\,dx = \int xe^{u}\,\frac{du}{2x} = \frac{1}{2}\int e^{u}\,du = \frac{1}{2}e^{u} + c$$

$$\text{but } \; u = x^2$$

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^{x^2} + c$$

Example

Evaluate $\int \dfrac{1}{ax+b}\, dx$

Solution

$$Let \quad u = ax + b, \quad \frac{du}{dx} = a, \quad dx = \frac{du}{a}$$

$$\int \frac{1}{u}\frac{du}{a} = \frac{1}{a}\int \frac{1}{u}\, du = \frac{1}{a}\ln|u| + c$$

$$but\ u = ax + b$$

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a}\ln|ax+b| + c$$

Example

Evaluate $\int \frac{1}{4x+3}\,dx$

Solution

$$Let \quad u = 4x + 3, \quad \frac{du}{dx} = 4, \quad dx = \frac{du}{4}$$

$$\int \frac{1}{u}\frac{du}{4} = \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln|u| + c$$

$$but\ u = 4x + 3$$

$$\int \frac{1}{4x+3}\,dx = \frac{1}{4}\ln|4x+3| + c$$

Example

Find $\int \tan x \, dx$

Solution

We know that $\tan x = \dfrac{\sin x}{\cos x}$. *Let* $u = \cos x$, *then*
$\dfrac{du}{dx} = -\sin x \implies dx = \dfrac{du}{-\sin x}$. *Then*

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{\sin x}{u} \frac{du}{-\sin x} = -\int \frac{1}{u} du$$

$$= -\ln|u| + c$$

$$\text{but} \;\; u = \cos x$$

$$\int \tan x \, dx = -\ln|\cos x| + c$$

Example

Find $\int \frac{1}{x \ln x}\, dx$

Solution

$$Let \quad u = \ln x, \qquad \frac{du}{dx} = \frac{1}{x}, \quad dx = x\ du$$

$$= \int \frac{1}{xu} x\ du = \frac{1}{u} + c$$

$$But\ u = \ln x$$

$$\int \frac{1}{x \ln x}\, dx = \ln|\ln x| + c$$

Example

7. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Solution

$$\text{Let} \quad u = \sqrt{x} = x^{\frac{1}{2}}, \quad \frac{du}{dx} = \frac{1}{2}\left(\frac{1}{\sqrt{x}}\right), \quad dx = 2\sqrt{x}\,du$$

$$\text{Now,} \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\cos u}{u}\,2\sqrt{x}\,du = \int \frac{\cos u}{u}2u\,du$$

$$= 2\int \cos u\,du = 2\sin u + c$$

$$\text{But } u = \sqrt{x}$$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\sin\sqrt{x} + c$$

Definite Integral

Example

Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x\sin x + \cos x}{x^2 + 1}\, dx$$

Solution

The limits are same hence,

$$\int_{130}^{130} \frac{x^3 - x\sin x + \cos x}{x^2 + 1}\, dx = 0$$

Example

Find $\int_0^4 \sqrt{2x+1}\, dx$

Solution: We let $u = 2x + 1$, then $\frac{du}{dx} = 2 \implies dx = \frac{du}{2}$. Then

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u}\, \frac{du}{2} = \frac{1}{2} \int \sqrt{u}\, du$$

$$= \frac{1}{3} u^{3/2} + c$$

but $u = 2x + 1$

$$= \frac{1}{3}(2x+1)^{3/2}$$

$$\int_0^4 \sqrt{2x+1}\, dx = \left[\frac{1}{3}(2x+1)^{3/2}\right]\Big|_0^4 = \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

Then we use it for computing the definite integral:

Example

Find $\int_{\ln\frac{1}{2}}^{2} \left(e^t - e^{-t}\right)\ dt$

Solution

$$\int_{\ln\frac{1}{2}}^{2} \left(e^t - e^{-t}\right)\ dt = \left[e^t + e^{-t}\right]\Big|_{\ln\frac{1}{2}}^{2} = e^2 + e^{-2} - \left(e^{\ln\frac{1}{2}} + e^{-\ln\frac{1}{2}}\right)$$

$$e^2 + e^{-2} - e^{\ln\frac{1}{2}} - e^{-\ln 2} = e^2 + e^{-2} - \left(\frac{1}{2} + 2\right)$$

$$\int_{\ln\frac{1}{2}}^{2} e^t - e^{-t}\ dt = e^2 + e^{-2} - \frac{5}{2}$$

Example

Evaluate $\displaystyle\int_1^2 \frac{x^2}{(x^3+1)^2}\,dx$

Solution

We let $u = x^3 + 1$, *then* $\dfrac{du}{dx} = 3x^2 \implies dx = \dfrac{du}{3x^2}$. *Then*

$$\int_1^2 \frac{x^2}{(x^3+1)^2}\,dx = \int \frac{x^2}{(u)^2}\frac{du}{3x^2} = \frac{1}{3}\int \frac{1}{(u)^2}\,du = -\frac{1}{3u}$$

$$but \;\; u = x^3 + 1$$

$$= -\left[\frac{1}{3(x^3+1)}\right]_1^2 = -\left(\frac{1}{27} - \frac{1}{6}\right) = \frac{7}{54}$$

$$\int_1^2 \frac{x^2}{(x^3+1)^2}\,dx = \frac{7}{54}$$

Example

Evaluate $\displaystyle\int_{1/2}^{1/4} \frac{\cos(\pi t)}{\sin^2(\pi t)}\, dt$

Solution

We let $u = \sin(\pi t)$, then $\dfrac{du}{dt} = \pi\cos(\pi t) \implies dt = \dfrac{du}{\pi\cos(\pi t)}$. Then

$$\int \frac{\cos(\pi t)}{\sin^2(\pi t)}\, dt = \int \frac{\cos(\pi t)}{u^2}\,\frac{du}{\pi\cos(\pi t)} = \frac{1}{\pi}\int \frac{1}{u^2}\, du = \frac{1}{\pi}\int u^{-2}\, du$$

$$= \frac{1}{\pi}\left[\frac{u^{-1}}{-1}\right] = \frac{1}{\pi}\left[-\frac{1}{u}\right]$$

but $u = \sin(\pi t)$

Therefore

$$\int_{1/2}^{1/4} \frac{\cos(\pi t)}{\sin^2(\pi t)}\, dt = \frac{1}{\pi}\left[-\frac{1}{\sin(\pi t)}\right]\Bigg|_{1/2}^{1/4} = -\frac{1}{\pi} + \frac{\sqrt{2}}{\pi}$$

Trigonometric Integrals and Trigonometric Substitutions

Trigonometric Integrals

Here we discuss certain interals in which the integrand is either a power of a trigonometric function or the product of two powers. To evaluate the integrals:

$$\int \sin^2 u \, du \text{ and } \int \cos^2 u \, du$$

the following identities will be useful:

1. $\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$
2. $\sin^2(ax) = \frac{1}{2}[1 - \cos(2ax)], \quad \cos^2(ax) = \frac{1}{2}[1 + \cos(2ax)]$

Remember also the identities:

1. $\sin^2 x + \cos^2 x = 1$, $\sec^2 x = 1 + \tan^2 x$
2. $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \csc^2 x$

Example

$$\int \sin^2 3x \, dx$$

Solution

$$\int \sin^2 3x \, dx = \int \frac{1}{2}(1 - \cos 6x) \, dx$$

$$= \frac{1}{2} \left(x - \frac{1}{6} \sin 6x \right) + C$$

$$= \frac{1}{2}x - \frac{1}{12} \sin 6x + C$$

Example

$$7. \ \cot^2 3x \ dx$$

Solution

using the substitution $u = 3x, \ du = 3dx \ , dx = \frac{1}{3}du$

$$7. \ \cot^2 3x \ dx = 7. \ (\csc^2 3x - 1) \ dx$$

$$= 7. \ (\csc^2 u - 1) \ \frac{1}{3}du$$

$$= \frac{1}{3}(\cot u - u) + C$$

$$= \frac{1}{3} \cot 3x - x + C$$

Integrals Of Product of Sines and Cosines

We will now study about integrals of the form:

$$\int \sin^m x \cos^n x \, dx \tag{1}$$

The simplest case is when either $n = 1$ or $m = 1$, in which case the substitution $u = \sin x$ or $u = \cos x$ respectively will work.

There are two cases to consider when evaluating these integrals

- Case 1: At least one of the two numbers m and n is an odd number. Thus, the other is a real number.
- Case 2: Both m and n are non-negative even numbers.

Case 1

Definition

Suppose from (1) above $m = 2t + 1$ is an odd positive integer, then we isolate $\sin x$ factor and use the identity $\sin^2 x = 1 - \cos^2 x$ to express the remaining $\sin^{m-1} x$ factor in terms of $\cos x$. Thus,

$$\int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x \cos^n x \sin x \, dx$$

$$= \int (\sin^2 x)^t \cos^n x \sin x \, dx = \int (1 - \cos^2 x)^t \cos^n x \sin x \, dx$$

Definition (cont'd)

Now the substitution $u = \cos x$, $du = -\sin x dx \implies dx = -\frac{du}{sinx}$

$$= \int (1 - u^2)^t u^n \sin x \, (-\frac{du}{sin})$$

$$= -\int (1 - u^2)^t u^n \, du$$

The exponent $t = \frac{m-1}{2}$ is a non-negative integer because m is an odd positive integer. Therefore it is easier to integrate u^n because $(1 - u^2)^t$ of the integrand is a polynomial in the variable of u.
The case 1 basically allows you to split the odd power of either sin$x$ or $\cos x$ and express it as the identity of the other.

Example

Evaluate $\int \sin^2 x \cos^3 x \, dx$

Solution:

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$$

$$= \int \sin^2 x(1 - \sin^2 x)\cos x \quad \frac{du}{\cos x} \quad dx$$

Using substitution $u = \sin x$, $du = \cos x \, dx$, $dx = \rule{2em}{0.4pt}$

$$= \int u^2(1 - u^2) \cos x \qquad du$$

cos$x$

$$= \int (u^2 - u^4)\, du = \frac{u^3}{3} - \frac{u^5}{5} + c$$

but $u = \sin x$

$$\int \sin^2 x \cos^3 x \, dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

Rewrite the integral:

$$\int \sin x \left(\sin^4 x\right)$$

Example

Evaluate $\int \sin^5 x \ dx$

Solution

$$\sin^5 x \, dx = dx$$

$$= \int \sin x \left(\sin^2 x\right)^2 dx = \int \sin x \left(1 - \cos^2 x\right)^2 dx$$

We let $u = \cos x$, then $\frac{du}{\quad} = -\sin x \quad \Longrightarrow dx = \frac{du}{\qquad}$.

dx −sin*x*

Solution[cont'd]

Then

$$\int \sin x \,(1 - \cos \;\; x) \qquad = \int \sin x \,(1 - u \;\;) \qquad 2 \;\; 2 \qquad\qquad 2\;2$$

du

dx

−sin*x*

$$= -\int \left[1 - u^2\right]^2 du$$

$$= -\int \left[1 - 2u^2 + u^4\right] du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + c$$

but $u = \cos x$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

$$\int \sin^5 x \, dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

In this case where the power of both sine and cosines are non-negative even integer, it is best to use the half-angles formulas.

Case 2

Z

Example

Find $\int \sin^2 x \cos^2 x \; dx$

Solution:

$$\int \sin^2 x \cos^2 x \; dx = \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x) \; dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x) \; dx = \frac{1}{4}\int \left(1-\frac{1}{2}(1+\cos 4x)\right) dx$$

$$= \frac{1}{8}\int (1 - \cos 4x)\, dx$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + c$$

Example

Evaluate $\displaystyle\int \sin^6 x \, dx$

Solution

Use $\sin^2 x = \dfrac{(1-\cos 2x)}{2}$ *to rewrite the function:*

$$\int \sin^6 x \, dx = \int \left[\sin^2 x\right]^3 \, dx$$

$$= \int \frac{(1-\cos 2x)^3}{8} \, dx$$

$$= \frac{1}{8} \int \left(1 - 3\cos 2x + 3\cos^2 2x - \cos^3 3x\right) \, dx$$

$$= \frac{1}{8} \left(\int 1 \, dx - \int 3\cos 2x \, dx + \int 3\cos^2 2x \, dx - \int \cos^3 3x \, dx \right)$$

Solution (cont'd)

$$\int 1\ dx = x$$

$$\int -3\cos 2x\ dx = -\frac{3}{2}\sin 2x$$

$$\int 3\cos^2 2x\ dx = 3\int\left(\frac{1+\cos 4x}{2}\right)\ dx = \frac{3}{2}\left(x + \frac{\sin 4x}{4}\right)$$

$$\int -3\cos^3 2x\ dx = \int -\cos 2x\cos^2 2x\ dx = \int -\cos 2x\left(1-\sin^2 2x\right)\ dx$$

Now we have four integrals to evaluate:

Solution (cont'd)

Evaluating the last integral: We let $u = \sin 2x$*, then*
$\dfrac{du}{dx} = 2\cos 2x \implies dx = \dfrac{du}{2\cos 2x}$.
Then

$$\int -\cos 2x\left(1-\sin^2 2x\right)\, dx = \int -\cos 2x\left(1-u^2\right)\, \frac{du}{2\cos 2x}$$

$$= \int -\cos 2x\left(1-u^2\right)\, \frac{du}{2\cos 2x} = -\frac{1}{2}\int\left(1-u^2\right)\, du = -\frac{1}{2}\left(u - \frac{u^3}{3}\right)$$

$$but \;\; u = \sin 2x$$

$$= -\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right)$$

$$\int -3\cos^3 2x\, dx = -\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right)$$

Solution (cont'd)

Finally we get

$$\int \sin^6 x \; dx$$

$$= \frac{1}{8} \quad x - \frac{3}{2}\sin 2x - \frac{3}{2} \quad x + \frac{\sin 4x}{4} \quad + \frac{1}{2} \quad \sin 2x - \frac{\sin^3 2x}{3} \qquad + c$$

Integrals Of Product of Secant and Tangent

Definition

More generally an integral of the form $\int \tan^m x \sec^n x \, dx$

can be computed in the following way:

1. If m is odd positive integer, use $u = \sec x$, $\dfrac{du}{dx} = \sec x \tan x$.
2. If n is even positive integer, use $u = \tan x$, $\dfrac{du}{dx} = \sec^2 x$.

Example

Find $\int \tan^3 x \sec^2 x \; dx$

Solution

Since in this case m is odd and n is even it does not matter which method we use, so let's use the first one:

Let

$$u = \sec x, \quad \frac{du}{dx} = \sec x \tan x \implies \frac{du}{\sec x \tan x} = dx$$

$$\int \tan^3 x \sec^2 x \frac{du}{\sec x \tan x} = \int \tan^2 x \; \sec x \; du = \int \tan^2 x \; u \; du$$

Note: $\tan^2 x = \sec^2 x - 1$

Solution (cont'd)

$$= \int \left(\sec^2 x - 1\right)\ u\ du = \int \left(u^2 - 1\right) u\ du$$

$$= \int \left(u^2 - 1\right) u\ du = \int \left(u^3 - u\right)\ du$$

$$= \frac{u^4}{4} - \frac{u^2}{2} + c$$

$$But\ \ u = \sec x$$

$$= \frac{\sec^4 x}{4} - \frac{\sec^2 x}{2} + c$$

$$\int \tan^3 x \sec^2 x\ dx = \frac{\sec^4 x}{4} - \frac{\sec^2 x}{2} + c$$

Example

7. $\tan^3 x \sec^3 x \; dx$

Solution

7. $(\sec^2 x - 1) \sec^2 x \sec x \tan x \; dx$

Let $u = \sec x, du = \sec x \tan x \; dx, \quad dx = \dfrac{1}{\sec x \tan x}$

$$7. \; (\sec^2 x - 1) \sec^2 x \sec x \tan x \cdot \frac{1}{\sec x \tan x} \; du$$

$$7. \; (\sec^2 x - 1) \sec^2 x \; du$$

$$7. \; (u^4 - u^2) \; du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + c$$

$$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + c$$

Example

Find $\int \sec y \; dy$

Solution (Method 1)

We rationalize the function using $\sec y + \tan y$

$$\int \sec y \; dy = \int \frac{\sec y}{1} \times \frac{(\sec y + \tan y)}{(\sec y + \tan y)} \, dy$$

$$= \int \frac{\sec^2 y + \sec y \tan y}{\sec y + \tan y} \, dy$$

Next we use the following substitution.

$$u = \sec y + \tan y, \quad \frac{du}{dy} = \sec y \tan y + \sec^2 y$$

$$\implies dy = \frac{du}{\sec y \tan y + \sec^2 y}$$

Solution (cont'd)

$$= \int \frac{\sec^2 y + \sec y \tan y}{\sec y + \tan y}\, dy$$

$$= \int \frac{\sec^2 y + \sec y \tan y}{u} \times \frac{du}{\sec y \tan y + \sec^2 y}$$

$$\int \sec y \; dy = \int \frac{1}{u}\, du = \ln|u| + c$$

$$but \;\; u = \sec y + \tan y$$

$$\int \sec y \; dy = \ln|\sec y + \tan y| + c$$

Solution (Method 2)

$$\sec y = \frac{1}{\cos y} = \frac{\cos y}{\cos^2 y} = \frac{\cos y}{1 - \sin^2 y}$$

Using the algebraic identity of difference of two squares

$$\frac{1}{1+t} + \frac{1}{1-t} = \frac{2}{1-t^2} \implies \frac{2\cos y}{1-\sin^2 y} = \frac{\cos y}{1+\sin y} + \frac{\cos y}{1-\sin y}$$

Therefore

$$\int \sec y \, dy = \frac{1}{2} \int \left(\frac{\cos y}{1+\sin y} + \frac{\cos y}{1-\sin y} \right) dy$$

$$= \frac{1}{2} \left(\ln|1+\sin y| - \ln|1-\sin y| \right) + c$$

Solution

We further simplify the result above

$$7.\quad \sec y = \frac{1}{2}\ln \frac{1+\sin y}{1-\sin y} + c$$

$$= \frac{1}{2}\ln \frac{(1+\sin y)^2}{1-\sin^2 y} + c$$

$$= \frac{1}{2}\ln \left(\frac{(1+\sin y)^2}{\cos^2 y}\right)^{\frac{1}{2}} + c$$

$$\frac{1}{2}\ \frac{1+\sin y}{\cos y} + c \quad = \frac{1}{2}\ln \frac{1}{\cos y} + \frac{\sin y}{\cos y} + c$$

$$= \ln|\sec y + \tan y| + c$$

Further Trigonometric Identities

Identities

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Example

$$\int \sin(3x)\cos(5x)\,dx$$

Solution

$$\int \sin(3x)\cos(5x)\,dx = \frac{1}{2}\int [\sin(3x - 5x) + \sin(3x + 5x)]\,dx$$

$$= \frac{1}{2}\left[\int \sin(3x - 5x)\,dx + \int \sin(3x + 5x)\,dx\right]$$

Solution (cont'd)

Considering the integrands

$$\int \sin(-2x)\, dx$$

Using substitution $u = -2x$, $du = -2\, dx \implies dx = \dfrac{du}{-2}$

$$\frac{-1}{2}\int \sin(u)\, du = \frac{1}{2}\cos(u)$$

but $u = -2x$ *and also note that* $\cos(-x) = \cos(x)$

$$= \frac{1}{2}\cos(-2x) = \frac{1}{2}\cos(2x) + c$$

$$\int \sin(-2x)\, dx = \frac{1}{2}\cos(2x) + c$$

Solution (cont'd)

Also

$$\int \sin(8x)\, dx$$

Using substitution $u = 8x$, $du = 8\, dx \implies dx = \frac{du}{8}$

$$\frac{1}{8}\int \sin(u)\, du = \frac{-1}{8}\cos(u)$$

but $u = 8x$

$$\int \sin(8x)\, dx = \frac{-1}{8}\cos(8x) + c$$

Therefore combining the individual integrals we have

$$\frac{1}{2}\quad \frac{1}{2}\cos(2x) - \frac{1}{8}\cos(8x) \quad + c$$

$$\int \sin(3x)\cos(5x)\, dx = \frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + c$$

Example

$$\int \sin(2x)\sin(4x)\, dx$$

Solution

$$\int \sin(2x)\sin(4x)\, dx = \frac{1}{2}\int [\cos(2x - 4x) - \cos(2x + 4x)]\, dx$$

$$\frac{1}{2}\int \cos(2x - 4x)\, dx - \int \cos(2x + 4x)\, dx$$

Solution (cont'd)

$$\int \cos(-2x)\,dx$$

Using substitution $u = -2x,\ du = -2\,dx \implies dx = \dfrac{du}{-2}$

$$\frac{-1}{2}\int \cos(u)\,du = \frac{-1}{2}\sin(u)$$

but $u = -2x$

$$= \frac{-1}{2}\sin(-2x) = \frac{1}{2}\sin(2x) + c$$

$$\int \cos(-2x)\,dx = \frac{1}{2}\sin(2x) + c$$

Solution[cont'd] Also

$$\int \cos(6x)\, dx$$

Using substitution $u = 6x$, $du = 6\, dx \Longrightarrow dx = \frac{du}{6}$

$$= \frac{1}{6} \int \cos(u)\, du = \frac{1}{6} \sin(u)$$

but $u = 6x$

$$\int \cos(6x)\ dx = \frac{-1}{6}\sin(6x) + c$$

Therefore combining the individual integrals we have

$$= \frac{1}{2}\left[\frac{1}{2}\sin(2x) - \frac{1}{6}\sin(6x)\right] + c$$

$$= \int \sin(2x)\sin(4x)\ dx = \frac{1}{4}\sin(2x) - \frac{1}{12}\sin(6x) + c$$

Example

$$\int \cos(x)\cos(4x)\; dx$$

Solution

$$\frac{1}{2}\int [\cos(x - 4x) - \cos(x + 4x)]\; dx$$

$$\frac{1}{2}\left[\int \cos(x - 4x)\, dx - \int \cos(x + 4x)\, dx\right]$$

Solution (cont'd)

$$\int \cos(-3x)\,dx$$

Using substitution $u = -3x,\ du = -3\,dx \implies dx = \dfrac{du}{-3}$

$$\frac{-1}{3}\int \cos(u)\,du = \frac{-1}{2}\sin(u)$$

but $u = -2x$

$$= \frac{-1}{3}\sin(-3x) = \frac{1}{3}\sin(3x) + c$$

$$\int \sin(-3x)\,dx = \frac{1}{3}\sin(3x) + c$$

Solution (cont'd)

Also

$$\int \cos(5x)\, dx$$

Using substitution $u = 5x$, $du = 5\, dx \implies dx = \dfrac{du}{5}$

$$= \frac{1}{5}\int \cos(u)\, du = \frac{1}{5}\sin(u)$$

but $u = 5x$

$$\int \cos(5x)\, dx = \frac{1}{5}\sin(5x) + c$$

Therefore combining the individual integrals we have

$$= \frac{1}{2}\left(\frac{1}{3}\sin(3x) - \frac{1}{5}\sin(5x)\right) + c$$

$$= \int \sin(2x)\sin(4x)\, dx = \frac{1}{6}\sin(3x) - \frac{1}{10}\sin(5x) + c$$

Trigonometric Substitutions

Definition

So far we have seen that it sometimes helps to replace a sub-expression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a "reverse" substitution, but it is really no different in principle than ordinary substitution.

The following substitutions are useful in integrals containing the following expressions:

expression	substitution	identity
$a^2 - u^2$	$u = a\sin t$	$1 - \sin^2 t = \cos^2 t$
$a^2 + u^2$	$u = a\tan t$	$1 + \tan^2 t = \sec^2 t$
$u^2 - a^2$	$u = a\sec t$	$\sec^2 t - 1 = \tan^2 t$

Example

Evaluate $\int \sqrt{1 - x^2}\, dx$

Solution

We let $x = \sin u$, *then* $\dfrac{dx}{du} = \cos u \implies dx = \cos u\, du.$

Then

$$\int \sqrt{1 - x^2}\, dx = \int \sqrt{1 - \sin^2 u} \cdot \cos u\, du$$

But $\cos^2 x = 1 - \sin^2 x$

$$= \int \sqrt{\cos^2 u} \cdot \cos u\, du$$

$$= \int \sqrt{\cos^2 u} \cdot \cos u\, du = \int \cos^2 u\, du$$

$$= \int \frac{1 + \cos 2u}{2}\, du = \frac{u}{2} + \frac{\sin 2u}{4} + c$$

Solution (cont'd)

$$but \quad u = \sin^{-1} x \quad and \quad \sin 2u = 2 \sin u \cos u$$

$$= \frac{u}{2} + \frac{\sin 2u}{4} + c = \frac{\sin^{-1} x}{2} + \frac{2 \sin u \cos u}{4} + c$$

$$= \frac{\sin^{-1} x}{2} + \frac{2 \sin u \cos u}{4} + c$$

$$= \frac{\sin^{-1} x}{2} + \frac{2 \sin u \sqrt{1 - \sin^2 u}}{4} + c$$

$$but \quad x = \sin u$$

7. $$\int \sqrt{1 - x^2}\, dx = \frac{\sin^{-1} x}{2} + \frac{x \sqrt{1 - x^2}}{2} + c$$

Example

Evaluate $\int \sqrt{4 - 9x^2}\, dx$

Solution

We start by rewriting this so that it looks more like the previous example:

$$\int \sqrt{4 - 9x^2}\, dx = \int \sqrt{4\left(1 - \left(\frac{3x}{2}\right)^2\right)}\, dx$$

$$= \int 2\sqrt{1 - \left(\frac{3x}{2}\right)^2}\, dx$$

Solution (cont'd)

Now let $\dfrac{3x}{2} = \sin u$ *so* $\dfrac{3}{2}\, dx = \cos u\ du \implies dx = \dfrac{2}{3}\cos u\ du.$

Then

$$\int \sqrt{1 - \left(\frac{3x}{2}\right)^2}\, dx = \int \sqrt{1 - \sin^2 u} \cdot \frac{2}{3}\cos u\ du$$

$$= \frac{4}{3}\int \cos^2 u\ du$$

$$\frac{4}{3}\int \cos^2 u\ du = \frac{4}{3}\int \frac{1+\cos 2u}{2}\, du = \frac{4}{3}\left(\frac{u}{2} + \frac{\sin 2u}{4}\right) + c$$

$$= \frac{4u}{6} + \frac{4\sin 2u}{12} + c = \frac{2u}{3} + \frac{\sin 2u}{3} + c$$

Solution (cont'd)

$$but \quad u = \sin^{-1}\frac{3x}{2} \qquad and \qquad \sin 2u = 2\sin u\cos u$$

$$= \frac{2\sin^{-1}\frac{3x}{2}}{3} + \frac{2\sin u\cos u}{3} + c$$

$$= \frac{2\sin^{-1}\frac{3x}{2}}{3} + \frac{2\sin u^{n}\overline{1 - \sin^2 u}}{3} + c$$

$$but \quad \frac{3x}{2} = \sin u$$

Solution (cont'd)

$$= \frac{2\sin^{-1}\left(\frac{3x}{2}\right)}{3} + \frac{2\left(\frac{3x}{2}\right)\sqrt{1-\left(\frac{3x}{2}\right)^2}}{3} + c$$

$$= \frac{2\sin^{-1}\left(\frac{3x}{2}\right)}{3} + \frac{3x\sqrt{1-\left(\frac{3x}{2}\right)^2}}{3} + c$$

$$= \frac{2\sin^{-1}\left(\frac{3x}{2}\right)}{3} + x\sqrt{1-\left(\frac{3x}{2}\right)^2} + c$$

7. $$\int \sqrt{4-9x^2}\,dx = \frac{2\sin^{-1}\left(\frac{3x}{2}\right)}{3} + x\sqrt{1-\left(\frac{3x}{2}\right)^2} + c$$

Example

Find $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

Solution

$$Let\ x = 3\sin u, \quad \frac{dx}{du} = 3\cos u \implies dx = 3\cos u\, du$$

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin u)^3}{\sqrt{9-(3\sin u)^2}}\,dx = \int \frac{27\sin^3 u}{\sqrt{9-(9\sin^2 u)}}\,dx$$

$$but\ \ dx = 3\cos u\, du$$

$$= \int \frac{27\sin^3 u}{\sqrt{9-(9\sin^2 u)}}\,dx = \int \frac{81\sin^3 u\cos u}{3\sqrt{1-\sin^2 u}}\,du$$

$$= 27\int \frac{\sin^3 u\cos u}{\sqrt{1-\sin^2 u}}\,du = 27\int \sin^3 u\, du$$

Solution (cont'd)

$$27\int \sin^2 u \sin u = 27\int (1-\cos^2 u)\sin u \, du = 27\left(-\cos u + \frac{\cos^3 u}{3}\right)$$

$$But \;\; \cos u = \sqrt{1-\sin^2 u} \;\; and \;\; \sin u = \frac{x}{3}$$

$$= 27\left(-\sqrt{1-\sin^2 u} + \frac{1}{3}(1-\sin^2 u)^{3/2}\right)$$

$$= 27\left(-\sqrt{1-\left(\frac{x}{3}\right)^2} + \frac{1}{3}\left(1-\left(\frac{x}{3}\right)^2\right)^{3/2}\right)$$

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = 27\left[-\sqrt{1-\left(\frac{x}{3}\right)^2} + \frac{1}{3}\left(1-\left(\frac{x}{3}\right)^2\right)^{3/2}\right] + c$$

Integration by Parts

Definition

Let $y = uv$, $\frac{dy}{dx} = \frac{d(uv)}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\implies uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\implies \boxed{\int udv = uv - \int vdu}$$

Guidelines for selecting u and dv

"L-I-A-T-E" : Choose u to be the function that comes first in the list:

1. L = Logarithmic function
2. I = Inverse Trigonometric function
3. A = Algebraic function
4. T = Trigonometric function
5. E = Exponential function

Example

Find $\int x^2 \sin x \; dx$

Solution

A *comes before* T *in* L-I-A-T-E

$$So \quad we \quad let \quad u = x^2, \quad dv = \sin x, \quad du = 2x \; dx, \quad v = -\cos x$$

$$\int udv = -\cos x . x^2 - \int -2x \cos x \; dx = -x^2 \cos x + \int 2x \cos x \; dx$$

Solution (cont'd)

Now integrating $\int 2x \cos x \; dx$

$$u = 2x, \quad dv = \cos x, \quad du = 2, \quad v = \sin x$$

$$\int udv = 2x \sin x - \int 2 \; sinx \; dx = 2x \sin x + 2 \cos x + c$$

$$= -x^2 \cos x + 2x \sin x + 2x \cos x + c$$

$$\int x^2 \sin x \; dx = -x^2 \cos x + 2x \sin x + 2x \cos x + c$$

Example

Find $\int \ln x \, dx$

Solution

$$Let \quad u = \ln x, \quad dv = dx, \quad du = \frac{1}{x}, \quad v = x$$

$$\Longrightarrow \int udv = x \cdot \ln x - \int \frac{1}{x} \cdot x \, dx$$

$$\int udv = x \ln x - \int dx$$

$$= x \ln x - x + c$$

$$\therefore \int \ln x \, dx = x \ln x - x + c$$

Example

Find $\int x\cos(5x-1)\,dx$

Solution

$$u = x, \quad dv = \cos(5x-1), \quad du = dx, \quad v = \frac{\sin(5x-1)}{5}$$

$$\begin{aligned}
\int udv &= \frac{x\sin(5x-1)}{5} - \int \frac{\sin(5x-1)}{5}\,dx \\
&= \frac{x\sin(5x-1)}{5} - \left(-\frac{\cos(5x-1)}{25}\right) + c \\
&= \frac{x\sin(5x-1)}{5} + \frac{\cos(5x-1)}{25} + c \\
&= \frac{1}{25}\left[5x\sin(5x-1) + \cos(5x-1) + c\right]
\end{aligned}$$

$$\therefore \int x\cos(5x-1)\,dx = \frac{1}{25}\left[5x\sin(5x-1) + \cos(5x-1) + c\right]$$

Example

Evaluate $\int e^x \sin x \, dx$

Solution

Using integration by parts $\int u \, dv = uv - \int v \, du$

Let $u = \sin x$ *and* $dv = e^x \, dx$

$$du = \cos x \, dx \quad and \quad v = \int e^x \, dx = e^x$$

$$\int e^x \sin x = e^x \sin x - \int e^x \cos x \, dx \tag{2}$$

Solution (cont'd)

We use integration by parts again for the latter integral: Let $u = \cos x$ *and* $dv = e^x\, dx$

$$du = -\sin\ dx \quad \text{and} \quad v = \int e^x\, dx = e^x$$

$$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx \tag{3}$$

Solution (cont'd)

Substituting (3) in (2) :

$$\int e^x \sin x \, dx = e^x \sin x - \quad e^x \cos x + \int e^x \sin x \, dx$$

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

Thus,

$$2\int e^x \sin x \, dx = e^x (\sin x - \cos x) + c$$

$$\Longrightarrow \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + c$$

Example

Find $\int \sin^{-1} x \, dx$

Solution

Let $u = \sin^{-1} x$ *and* $dv = dx$

$$du = \frac{1}{\sqrt{1 - x^2}} \, dx \quad and \quad v = \int dx = x$$

$$\Longrightarrow \int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, dx \tag{4}$$

Now we have $\int \frac{x}{\sqrt{1 - x^2}} \, dx$ *to integrate*

Solution (cont'd)

Let $u = 1 - x^2, \quad \dfrac{du}{dx} = -2x \implies x\,dx = \dfrac{du}{-2}$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{u}}\,\frac{du}{-2} = \frac{1}{-2}\int u^{-1/2}\,du = -\sqrt{u} + c$$

$$\text{but } u = \sqrt{1-x^2}$$

$$\implies \int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} \qquad (5)$$

Substituting (5) in (4) we have :

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + c$$

Example

Find $\int \frac{\ln x}{x^2}\, dx$

Solution

$$Let\ u = \ln x \quad and \quad dv = \frac{1}{x^2}\, dx$$

$$du = \frac{1}{x}\, dx \quad and \quad v = \int x^{-2} dx = \frac{-1}{x}$$

$$\Longrightarrow \int \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\, dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + c$$

$$\int \frac{\ln x}{x}\, dx = -\frac{1}{x}(\ln x + 1) + c$$

Example

Find $\int \cos^{-1} x \, dx$

Solution

Let $u = \cos^{-1} x$ *and* $dv = dx$

$$du = \frac{-1}{\sqrt{1-x^2}}\, dx \quad \text{and} \quad v = \int dx = x$$

$$\int \cos^{-1} x \, dx = x \cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}}\, dx$$

$$\Longrightarrow \int \cos^{-1} x \, dx = x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\, dx \tag{6}$$

Now we have $\int \frac{x}{\sqrt{1-x^2}}\, dx$ *to integrate*

Solution (cont'd)

Let $u = 1 - x^2, \ \dfrac{du}{dx} = -2x \implies x\ dx = \dfrac{du}{-2}$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{u}}\,\frac{du}{-2} = \frac{1}{-2}\int u^{-1/2}\,du = -\sqrt{u} + c$$

$$but \ \ u = \sqrt{1-x^2}$$

$$\implies \int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} \tag{7}$$

Putting (7) in (6) yields:

$$\int \cos^{-1} x\ dx = x\cos^{-1} x - \sqrt{1-x^2} + c$$

Example

Find $\int \tan^{-1} x \, dx$

Solution

Let $u = \tan^{-1} x$ *and* $dv = dx$

$$du = \frac{1}{1+x^2}\, dx \quad and \quad v = \int dx = x$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2}\, dx$$

Now we have $\int \frac{x}{1+x^2}\, dx$ *to integrate:*

Solution (cont'd)

$$Let\ u = 1 + x^2,\quad \frac{du}{dx} = 2x \implies dx = \frac{du}{2x}$$

$$\int \frac{x}{1+x^2}\, dx = \int \frac{x}{u} \cdot \frac{du}{2x} = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + c$$

$$but\ \ u = 1 + x^2$$

$$\implies \int \frac{x}{1+x^2}\, dx = \frac{1}{2}\ln|1+x^2| + c$$

$$\int \tan^{-1} x\, dx = x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$$

Example

Evaluate $\int e^x \cos x \, dx$

Solution

Using integration by parts $\int u \, dv = uv - \int v \, du$

Let $u = \cos x$ *and* $dv = e^x \, dx$

$$du = -\sin x \, dx \quad \text{and} \quad v = \int e^x \, dx = e^x$$

$$\int e^x \cos x \, dx = e^x \cos x - \left(-\int e^x \sin x \, dx \right)$$

$$\Longrightarrow \int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx \tag{8}$$

We use integration by parts again for the latter integral:

Solution (cont'd)

Let $u = \sin x$ *and* $dv = e^x\, dx$

$$du = \cos x\, dx \quad \text{and} \quad v = \int e^x\, dx = e^x$$

$$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx \tag{9}$$

Substituting (9) in (8) :

$$\int e^x \cos x\, dx = e^x \cos x + e^x \sin x - \int e^x \cos x\, dx$$

$$\text{Thus } 2\int e^x \cos x\, dx = e^x (\sin x + \cos x) + c$$

$$\Longrightarrow \int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x) + c$$

Reduction Formulas

Assume that we want to find the following integrals for a given value of $n > 0$

$$\int x^n e^x dx$$

Using integration by parts with $u = x^n$ and $dv = e^x\,dx$, so $v = e^x$ and $du = nx^{n-1}dx$, we get:

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

On the right hand side we get an integral similar to the original one but with x raised to $n - 1$ instead of n. This kind of expression is called a reduction formula. Using this same formula several times, and taking into account that for $n = 0$ the integral becomes $\int e^x\,dx = e^x + c$,

Some Reduction Formula

1. $\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx.$

$$= \frac{-1}{2}x^{n-1}e^{-x^2} + \frac{n-1}{2}\int x^{n-2}e^{-}$$

2. $\int x^{n} e^{-x^2 2x^2} dx dx$

3. $\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx x$

4. $\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$

5. $\int \sin^n x dxxdx = -\frac{\sin^{n-1} \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2}$

6. $$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

7. $$\int x^m (\ln x)^n\,dx = \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\,dx$$

Example

Find $\int x^3 e^x \, dx$

Solution

$$\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx = x^3 e^x - 3 \left(x^2 e^x - 2 \int x e^x \, dx \right)$$

$$= x^3 e^x - 3 \left(x^2 e^x - 2 \left(x e^x - \int e^x dx \right) \right)$$

$$= x^3 e^x - 3 \left(x^2 e^x - 2 \left(x e^x - e^x \right) \right) + c$$

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c.$$

Example

Find $\int \sin^n x \, dx$

Solution

$$
\begin{aligned}
\int \sin^n x dx &= \int \underbrace{\sin^{n-1} x}_{u} \underbrace{\sin x dx}_{dv} \\
&= -\sin^{n-1} x \cos x + (n-1) \int \underbrace{\cos^2 x}_{1-\sin^2 x} \sin^{n-2} dx \\
n \int \sin^n x \, dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} dx \\
&- (n-1) \int \sin^n x \, dx
\end{aligned}
$$

Adding the last term to both sides and dividing by n we get the following reduction formula:

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Reduction Formula for Secant and Tangent

Reduction formula for Secant

$$\int \sec^n x \, dx = \frac{\sec^{n-1} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

Reduction formula for Tangent

$$\int \tan^n x \, dx = \frac{\tan^{n-1}}{n-1} - \int \tan^{n-2} x \, dx$$

The reduction formula is very effective for evaluating the integrals of secant ant tangent when the n is either an even or an odd positive integer.

Example

7. $\int \tan^6 x \, dx$

Solution

$$\int \tan^6 x \, dx = \frac{1}{5}\tan^5 x - \int \tan^4 x \, dx$$

$$\frac{1}{5}\tan^5 x - \quad \frac{1}{3}\tan^3 x - \int \tan^2 x \, dx$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x$$

Example

$$\int \sec^6 2x \; dx$$

Solution

Let $u = 2x,\; du = 2\; dx, \implies \quad dx = \dfrac{du}{2}$

Thus,

$$\int \sec^6 2x \; dx = \frac{1}{2}\int \sec^6 u \; du$$

Using the reduction formula

$$\frac{1}{2}\int \sec^6 u \; du = \frac{1}{2}\left[\frac{1}{5}\sec^4 u \tan u + \frac{4}{5}\int \sec^2 u \; du\right]$$

Solution (cont'd)

$$= \frac{1}{10}\sec^4 u \tan u + \frac{2}{5} \quad \frac{1}{3}\sec^2 u \tan u + \frac{2}{3}\int \sec^2 u \, du$$

$$= \frac{1}{10}\sec^4 u \tan u + \frac{2}{15}\sec^2 u \tan u + \frac{4}{15}\tan u + c$$

$$= \frac{1}{10}\sec^4 2x \tan 2x + \frac{2}{15}\sec^2 2x \tan 2x + \frac{4}{15}\tan 2x + c$$

Rational functions and Partial fractions

Definition

Rational function $R(x)$ is a function that can be expressed as a quotient of two polynomials. That is

$$R(x) = \frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomials.

It is quite difficult to integrate a rational function so this is where the method of partial fractionswhich is an algebraic technique that decomposes $R(x)$ into a sum of terms is helpful.

Thus,

$$R(x) = \frac{P(x)}{Q(x)} = p(x) + F_1(x) + F_2(x) + \cdots + F_k(x)$$

Note: Revise on partial fraction and its evaluation

Partial Fraction decomposition

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k},\quad k=1,2,3,\ldots$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k},\quad k=1,2,3,\ldots$

Example

7. $\int \dfrac{dx}{x^2 - 9}\, dx$

Solution

$$\begin{aligned}
\frac{1}{x^2 - 9} &= \frac{A}{(x+3)} + \frac{B}{(x-3)} \\
1 &= A(x-3) + B(x+3) \\
\textit{when } x &= 3 \\
1 &= A(3-3) + B(3+3) \\
\\
1 &= 6B \\
B &= \frac{1}{6}
\end{aligned}$$

Solution

(cont'd)

$$
\begin{aligned}
\text{when } x &= -3 \\
1 &= A(-3-3)+B(3-3) \\
1 &= -6A \\
A &= -\frac{1}{6} \\
\frac{1}{x^2-9} &= -\frac{1}{6(x+3)}+\frac{1}{6(x-3)}dx \\
\text{so}: 7. \ \frac{dx}{x^2-9} &= -\frac{1}{6}\ln|(x+3)|+\frac{1}{6}\ln|(x-3)|+c \\
7. \ \frac{dx}{x^2-9}dx &= \frac{1}{6}\ln\frac{x-3}{x+3}+c
\end{aligned}
$$

Example

Find $\int \frac{x}{(x+2)(x+3)}\,dx$

Solution

$$
\begin{aligned}
\frac{x}{(x+2)(x+3)} &= \frac{A}{(x+2)} + \frac{B}{(x+3)} \\
x &= A(x+3) + B(x+2) \\
when x &= -3 \\
-3 &= A(-3+3) + B(-3+2) \\
\\
-3 &= -B \\
B &= 3
\end{aligned}
$$

Solution

(cont'd)

$$
\begin{aligned}
when x &= -2 \\
-2 &= A(-2+3) + B(-2+2) \\
-2 &= A
\end{aligned}
$$

$$
\begin{aligned}
A &= -2 \\
\frac{x}{(x+2)(x+3)} &= -\frac{2}{x+2} + \frac{3}{x+3}dx \\
so: \int \frac{xdx}{(x+2)(x+3)} &= -2\ln|x+2| + 3\ln|x+3| + c \\
\int \frac{x}{(x+2)(x+3)}dx &= \ln\frac{(x+3)^3}{(x+2)^2} + c
\end{aligned}
$$

Example

Evaluate $\int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4}\, dx$

Solution

Since the degree of the numerator is at least as great as that of the denominator,carry out the long division,

$$\frac{x^4 - 4x^2 + x + 1}{x^2 - 4} = x^2 + \frac{x + 1}{x^2 - 4}$$

$$\Longrightarrow \int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4}\, dx = \int \left(x^2 + \frac{x + 1}{x^2 - 4} \right) dx$$

$$but \ \frac{x + 1}{x^2 - 4} = \frac{x + 1}{(x + 2)(x - 2)} =$$

$$\frac{x}{(x + 2)(x - 2)} = \frac{A}{(x + 2)} + \frac{B}{(x - 2)}$$

$$x + 1 = A(x - 2) + B(x + 2)$$

Solution

(cont'd)

$$
\begin{aligned}
\text{when } x &= -2 \\
-1 &= A(-2-2) + B(-2+2) \\
-1 &= -4A \\
A &= \frac{1}{4} \\
\frac{x+1}{(x+2)(x-2)} &= \frac{1}{4}\frac{1}{x+2} + \frac{3}{4}\frac{1}{x-2} \\
so: \ 7.\ \frac{x+1}{(x+2)(x-2)}\,dx &= \frac{1}{4}\ln|x+2| + \frac{3}{4}\ln|x-2| + c \\
7.\ \frac{x^4-4x^2+x+1}{x^2-4}\,dx &= \frac{1}{4}\ln\frac{(x+2)}{(x-2)^3} + c
\end{aligned}
$$

Example

Find $7.\ \dfrac{1 - 9x^2}{x(x^2 + 9)}\, dx$

Solution

$$7.\ \frac{1 - 9x^2}{x(x^2 + 9)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 9}$$

$$1 - 9x^2 = A(x^2 + 9) + (Bx + C)x$$

$$Let\ x = 0 \implies 1 = 9A \implies A = \frac{1}{9}$$

Solution

Equating coefficients of x^2.

$$-9 = A + B \implies B = 9 - A = \frac{-8}{9}$$

Equating coefficients of x: $0 = C$

$$\implies 7. \ \frac{1-9x^2}{x(x^2+9)} = \frac{1}{9}\ \frac{1}{x} - \frac{82}{9}\ \frac{x}{x^2+9}$$

$$\implies 7. \ \frac{1-9x^2}{x(x^2+9)}\,dx = \frac{1}{9}\ln|x| - \frac{41}{9}\ln|x^2+9| + c$$

Improper Integrals

The integral $\int_a^b f(x)\,dx$ is called improper integral if

1. $a = -\infty$ or $b = \infty$ or both
2. $f(x)$ is unbounded at one or more points of $a \le x \le b$. Such points are called singularities of $f(x)$.

Integrals corresponding to 1. and 2. are called improper integrals of the First and Second Kind respectively. Integrals with both conditions 1. and 2. are called improper integrals of the Third Kind.

1. $\int_0^{\infty} \sin x^2 \, dx$ is an improper integral of the first kind since the upper limit of integration is infinite.

$\int_0^4 \frac{dx}{x-3}$

2 dx is an improper integral of the second kind because

$\frac{1}{x-3}$ dx is not continuous at $x = 3$.

3 [redacted] $\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} \, dx$ is an improper integral of the third kind because the upper limit of integration is infinite and $\frac{e^{-x}}{\sqrt{x}}$ is not continuous at x

$x = 0$ $\int$

∞

4 [redacted] $e^{-x}\,dx$ is an improper integral of the first kind since the upper

0

limit of integration is infinite.

Definition

(a) If f is integrable on $a \leq x \leq \infty$, then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)\, dx$$

(b) If f is integrable on $-\infty < x \leq a$, then

$$\int_{-\infty}^a f(x)dx = \lim_{b\to-\infty} \int_b^a f(x)\, dx$$

(c) If f is integrable on $-\infty < x < \infty$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)\, dx$$

Where c is any number.

Convergence/Divergence of improper integral (first kind)

Example

Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

Solution

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\, dx$$

$$= \lim_{b\to\infty} \int_1^{b} x^{-2}\, dx$$

Solution

$$= \lim_{b\to\infty} \left. \frac{x^{-1}}{-1} \right|_1^b$$

$$= \lim_{b\to\infty} \left. \frac{-1}{x} \right|_1^b = \lim_{b\to\infty} \frac{-1}{b} + \lim_{b\to\infty} 1 = 1$$

The limit exists and is finite and so the integral converges and the integral's value is 1.

Example

Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} xe^{x^2}\, dx$$

Solution

In this case we've got infinities in both limits and so we'll need to split the integral up into two separate integrals. We can split the integral up at any point, so let's choose $a = 0$ *since this will be a convenient point for the evaluation process. The integral is then,*

$$\int_{-\infty}^{\infty} xe^{x^2}\, dx = \int_{-\infty}^{0} xe^{x^2}\, dx + \int_{0}^{\infty} xe^{x^2}\, dx$$

Solution

We've now got to look at each of the individual limits.

$$\int_{-\infty}^{0} xe^{x^2}\,dx = \lim_{b\to-\infty} \int_{b}^{0} xe^{x^2}\,dx$$

$$= \lim_{b\to-\infty} \left. \frac{-1}{2} e^{-x^2} \right|_{b}^{0} = \lim_{b\to-\infty} \left(-\frac{1}{2} + \frac{1}{2} e^{-b^2} \right) = -\frac{1}{2}$$

So, the first integral is convergent. Note that this does NOT mean that the second integral will also be convergent. So, let's take a look at that one.

$$\int_{0}^{\infty} xe^{x^2}\,dx = \lim_{b\to\infty} \int_{0}^{b} xe^{x^2}\,dx$$

$$= \lim_{b\to\infty} \left. -\frac{1}{2} e^{-x^2} \right|_{0}^{b} = \lim_{b\to\infty} \left(-\frac{1}{2} e^{-b^2} + \frac{1}{2} \right) = \frac{1}{2}$$

Solution

This integral is convergent and so since they are both convergent the integral as required in this question is also convergent and its value is,

$$\int_{-\infty}^{\infty} xe^{x^2}\,dx = \int_{-\infty}^{0} xe^{x^2}\,dx + \int_{0}^{\infty} xe^{x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$$

Example

Determine if the integral is convergent or divergent. If it is convergent find its value.

$$\int_{-1}^{\infty} \cos x \; dx$$

Solution

$$\int_{-1}^{\infty} \cos x \; dx = \lim_{b\to\infty} \int_{-1}^{b} \cos x \; dx$$

$$= \lim_{b\to\infty} [\sin x]_{-1}^{b} = \lim_{b\to\infty} [\sin(b) - \sin(-1)] = \infty$$

This limit doesn't exist and so the integral is divergent.

Special Improper integrals of first Kind

Definition

Geometric/exponential integrals: $\int_a^{\infty} e^{-tx}\,dx,$

$$\text{if} \quad t > 0 : \text{converges}, \quad t \leq 0 : \text{diverges}$$

. The p integral of the first kind:

$$\int_a^{\infty} \frac{dx}{x^p}, \text{ where } p \text{ is a constant and } a > 0$$

$$\text{It converges if } p > 1, \text{ diverges if } p \leq 1$$

Gamma function Γ: $\Gamma(n+1) = \int_0^{\infty} x^n e^{-x}\,dx = n!, \quad \Gamma(n+1) = n\Gamma(n)$

Example

Determine if the following integral is convergent or divergent. $\int_1^{\infty} e^{-2x}\, dx$

solution: From the geometric integral $t = 2 > 0$, $\int_1^{\infty} e^{-2x}\, dx$: converges

Example

Determine if the following integral is convergent or divergent: $\int_0^{\infty} x^3 e^{-x}\, dx$

$$\int_0^{\infty} x^3 e^{-}$$

solution: $\qquad {}^{x}\, dx = 3! = 6$: converges

Example

$$\int_1^\infty \frac{1}{\sqrt{x}}\, dx$$

solution: From the p integral $p = \frac{1}{2} \leq 1$, $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$: diverges

solution: From the geometric integral t $^x\, dx$: diverges

Example

$$\int_1^\infty e^{3x}\, dx$$

$= -3 < 0$, $\int_1^\infty e^3$

Convergence/Divergence of improper integral (second kind)

Example

Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^{\frac{\pi}{2}} \sec x \, dx$$

Solution

This is an improper integral of second kind because $\sec x$ *is not continuous at* $\dfrac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \sec x \, dx = \lim_{b \to \frac{\pi}{2}^-} \int_0^{b} \sec x \, dx$$

Solution

$$\lim_{b \to \frac{\pi}{2}^-} \int_0^b \sec x \; dx = \lim_{b \to \frac{\pi}{2}^-} \left(\ln|\sec x + \tan x| \right) \Big|_0^t$$

$$= \lim_{b \to \frac{\pi}{2}^-} \ln|\sec t + \tan t| = \infty$$

This limit doesn't exist and so the integral is divergent

Example

Determine if the integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^{3} \frac{1}{x^3}\, dx$$

Solution

This integrand is not continuous at $x = 0$ *and so we'll need to split the integral up at that point.*

$$\int_{-2}^{3} \frac{1}{x^3}\, dx = \int_{-2}^{0} \frac{1}{x^3}\, dx + \int_{0}^{3} \frac{1}{x^3}\, dx$$

Solution

Now we need to look at each of these integrals and see if they are convergent.

$$\int_{-2}^{0} \frac{1}{x^3}\,dx = \lim_{t \to 0^-} \int_{-2}^{t} \frac{1}{x^3}\,dx$$

$$= \lim_{t \to 0^-} \left[-\frac{1}{2x^2} \right]_{-2}^{t} = \lim_{t \to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{8} \right) = -\infty$$

At this point we're done we don't even need to bother with the second integral because One of the integrals is divergent that means the integral that we were asked to look at is also divergent

Hence the limit does not exist.

Example

Determine if the integral is convergent or divergent. If it is convergent find its value.

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$$

Solution

The problem point is the upper limit so we are in the first case above.

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{b\to 3^-}\int_0^b \frac{1}{\sqrt{3-x}}\,dx$$

$$= \lim_{b\to 3^-}\left[-2\sqrt{3-x}\right]_0^b = \lim_{b\to 3^-}\left[-2\sqrt{3-b}+2\sqrt{3}\right] = 2\sqrt{3}$$

The limit exists and is finite and so the integral converges and the integral's value is $2\sqrt{3}$

Example

Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^{\infty} \frac{1}{x^2}\, dx$$

Solution

This is an integral over an infinite interval that also contains a discontinuous integrand. To do this integral we'll need to split it up into two integrals. We can split it up anywhere, but pick a value that will be convenient for evaluation purposes.

$$\int_0^{\infty} \frac{1}{x^2}\, dx = \int_0^{1} \frac{1}{x^2}\, dx + \int_1^{\infty} \frac{1}{x^2}\, dx$$

Solution

We know that the second integral is convergent by the fact given in the infinite interval portion above. So, all we need to do is check the first integral.

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^-}\int_t^1 \frac{1}{x^2}\,dx$$

$$= \lim_{t\to 0^-}\left(-\frac{1}{x}\right)\Bigg|_t^1 = \lim_{t\to 0^-}\left(-1+\frac{1}{t}\right) = \infty$$

So, the first integral is divergent and so the whole integral is divergent.

Comparison Test for Improper Integrals

Sometimes an improper integral is too difficult to evaluate. One technique is to compare it with a known integral. The theorem below shows us how

Suppose that f and g are two continuous functions for $x \geq a$ such that $0 \leq g(x) \leq f(x)$. Then, the following is true:

1. If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ also converges.
2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ also diverges

to do this.

Example

Determine if the integral is convergent or divergent.

$$\int_2^{\infty} \frac{\cos^2 x}{x^2}\, dx$$

Solution: $\int_2^{\infty} \frac{1}{x^2}\, dx$ converges since $p = 2 > 1$ ($p-$integral). So let's guess that this integral will converge. So we now know that

$$\frac{\cos^2 x}{x^2} < = f(x) \quad \frac{1}{x^2}$$

so by the Comparison Test we know that $\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$ must also converge.

Note: $0 \le \cos^2 x \le 1$

solution

We know that $0 \le \sin^4(2x) \le 1$. In particular, this term is positive and so if we drop it from the numerator, the numerator will get smaller. This gives,

$$1 + 3\sin^4(2x) \qquad 1$$

Example

Determine if the integral is convergent or divergent.

$$\int_1^{\infty} \frac{1 + 3\sin^4(2x)}{\sqrt{x}}\, dx$$

$$\sqrt{} > \sqrt{} = g(x)\; x \qquad x$$

Since $\int^{\infty} \frac{1}{\sqrt{x}}\, dx$ diverges so by the Comparison Test $\Longrightarrow$

1

$$\int_1^\infty \frac{1+3\sin^4(2x)}{x}$$

$\sqrt{}$ — dx also diverges.

Example

Study the convergence of $\displaystyle\int_1^{\infty} e^{-x^2}\, dx$.

Solution

We need to find $f(x) = e^{-x}$*, which converges because*
$t = 1 > 0$*(geometric integral).*
We also know that $e^{-x^2} < e^{-x}$

so by the Comparison Test we know that $\displaystyle\int_1^{\infty} e^{-x^2}\, dx$ *will also converge.*

Exercises

1. $\displaystyle\int \frac{3}{2x+4}\,dx$

2. $\displaystyle\int \frac{x^3}{x^4-1}\,dx$

3. $\displaystyle\int (8x+4)^2\,dx$

4. $\displaystyle\int \frac{1}{\sqrt{x-1}}\,dx$

5. $\displaystyle\int \frac{\cos 3x}{\sin^2 3x}\,dx$

6. $\displaystyle\int \frac{x\,dx}{(x+2)(x+3)}$

Rules Of Vector Integration

Definition

Let $\frac{d}{dt}[\vec{f}(t)] = \vec{F}(t) \Rightarrow \vec{f}(t) = \int \vec{F}(t)dt + c$ is called vector integration.

1. $$\int_a^b \vec{f}(t)dt = F(b) - F(a)$$
2. $$\int_a^b \vec{f}(t)dt = -\int_b^a \vec{f}(t)dt$$
3. $$\int_a^b \vec{f}(t)dt = \int_a^c \vec{f}(t)dt + \int_c^b \vec{f}(t)dt$$

Generally, integrals are classified as line integrals, surface integrals and volume integrals.

Example

Find the value of $\int_0^1 (t\hat{i} + t^2\hat{j} + t^3\hat{k}) \cdot dt$

Solution

$$\int_0^1 (t\hat{i} + t^2\hat{j} + t^3\hat{k}) \cdot = \left[\frac{t^2}{2}\hat{i} + \frac{t^3}{3}\hat{j} + \frac{t^4}{4}\hat{k} \right]_0^1$$

$$= \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{4}\hat{k}$$

Line Integrals

Definition

We have so far integrated "over" intervals. We now investigate integration over a curve—"line integrals" are really "curve integrals". The line integral of $f(x, y)$ along c is denoted by

$$\int_c f(x, y)\, ds$$

We use a ds here to acknowledge the fact that we are moving along the curve, c, instead of the x-axis (denoted by dx) or the y-axis (denoted by dy). Because of the ds this is sometimes called the line integral of f with respect to arc length.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Definition

Here are some basic curves and their limits on the parameter.

CURVE	Parametric Equation
1.Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos t \quad 0 \leq t \leq 2\pi$ $y = b\sin t$
2. Circle $x^2 + y^2 = r^2$	$x = r\cos t \quad 0 \leq t \leq 2\pi$ $y = r\sin t$
3. Line segment from (x_0, y_0) to (x_1, y_1) $r(t) = (1 - t) < x_0, y_0 >$ $+t < x_1, y_1 >$	$x = (1 - t)x_0 + tx_1$ $y = (1 - t)y_0 + ty_1$ $0 \leq t \leq 1$

Example

Evaluate $\int_c 4x^2\,dx$ where c is the line segment $(-2,-1)$ to $(1,2)$

Solution

$$r(t) = (1-t)\langle x_0, y_0\rangle + t\langle x_1, y_1\rangle$$

$$r(t) = (1-t)\langle -2,-1\rangle + t\langle 1, 2\rangle$$

$$= \langle -2(1-t), -(1-t)\rangle + \langle t, 2t\rangle$$

$$= \langle -2+2t+t, -1+t+2t\rangle = \langle -2+3t, -1+3t\rangle$$

Let's take a look at an example of a line integral:

Solution

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$x = -2 + 3t, \quad y = -1 + 3t$$

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 3$$

$$ds = \sqrt{(3)^2 + (3)^2}\,dt = \sqrt{18}\,dt = 3\sqrt{2}\,dt$$

$$but\ x = -2 + 3t$$

$$\int_0^1 4(-2 + 3t)^3\,dt = -15\sqrt{2}$$

Example

Find the value $\int_c \vec{F} \cdot d\vec{r}$, if $F = x\hat{i} - y\hat{j}$ and c is given by $x^2 + y^2 = 4$

Solution

We first need a parameterization of the circle. This is given by,

$$r = xi + yj = 2\sin\theta + 2\cos\theta$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_c F_1\, dx\, \hat{i} + F_2\, dy\, \hat{j} = \int_c (x\ dx - y\ dy)$$

Solution

Let $x = 2\sin\theta,\ y = 2\cos\theta,\ dx = 2\cos\theta d\theta$ *and* $dy = -2\sin\theta d\theta$

$$= \int_0^{2\pi} [2\sin\theta(2\cos\theta)d\theta] - [2\cos\theta(-2\sin\theta)d\theta]$$

$$\int_0^{2\pi} 8\cos\theta\sin\theta d\theta = 4\int_0^{2\pi} \sin 2\theta d\theta$$

$$= 4\left[-\frac{1}{2}[\cos 2\theta]\right]_0^{2\pi} = -2[1-1] = 0$$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = 0$$

Note

1. If $\vec{F}$ is force then the total work done by a force is $\int_C \vec{F} \cdot d\vec{r}$.

2. If $\int_C \vec{F} \cdot d\vec{r} = 0 \Rightarrow \vec{F}$ is called conservative force.

Multiple Integrals

They are mainly classified as

1. Double Integrals
2. Triple Integrals

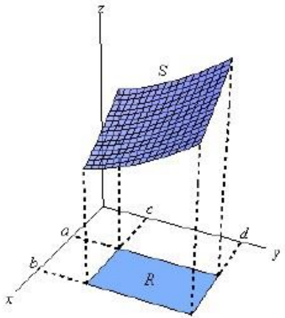
Definition

7.7.

$f(x, y)\ dxdy$ is called the double integral where $R = [a, b] \times [c, d]$.

Also, we will initially assume that $f(x, y) \geq 0$ although this doesn't really have to be the case. Let's start out with the graph of the surface S given by graphing $f(x, y)$ over the rectangle R.

Double Integr



Methods to Evaluate Double Integrals

Theorem (Fubini's Theorem)

If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$ then,

$$\iint f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dxdy = \int_a^b \int_c^d f(x, y)\, dydx$$

These integrals are called iterated integrals.

Example

Find the value of $\iint_R 6xy^2\, dA$, $R = [2, 4] \times [1, 2]$

Solution

In this case we will integrate with respect to y first. So, the iterated integral that we need to compute is,

$$\iint_R 6xy^2\, dA = \int_2^4 \int_1^2 6xy^2\, dx\,dy$$

$$\iint_R 6xy^2\, dA = \int_2^4 \left[2xy^3 \right]_1^2 dx$$

$$= \int_2^4 16x - 2x\, dx = \int_2^4 14x\, dx$$

Solution (cont'd)

Remember that we treat the x as a constant when doing the first integral and we don't do any integration with it yet. Now, we have a normal single integral so let's finish the integral by computing this.

$$\iint 6xy^2\,dA = 7x^2\Big|_2^4 = 84$$

Example

Find the value of $\displaystyle\int_{y=0}^{1}\int_{x=\sqrt{y}}^{y}(x+y)\,dxdy$

Solution

$$\int_0^1 \left[\frac{x^2}{2}+xy\right]_{\sqrt{y}}^{y} dy = \int_0^1 \left(\frac{y^2}{2}+y^2\right) - \left(\frac{y}{2}+y^{\frac{3}{2}}\right)$$

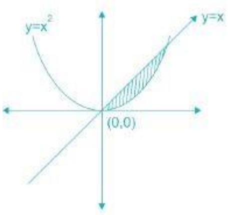
$$\left[\frac{y^3}{6}+\frac{y^3}{3}-\frac{y^2}{4}-\frac{2y^{\frac{5}{2}}}{5}\right]_0^1$$

$$=\frac{1}{6}+\frac{1}{3}-\frac{1}{4}-\frac{2}{5}$$

$$=\frac{10+20-15-24}{60}=\frac{30-39}{60}=\frac{-9}{60}=\frac{-3}{20}$$

Example

Find the area bounded between the curves $y = x^2,\ y = x$



Solution

$$Area = \int_0^1 \int_{x^2}^{x} dy\ dx$$

$$= \int_0^1 y \Big|_{x^2}^{x} dx = \int_0^1 (x - x^2)\ dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Triple Integrals

Definition

Triple integral is defined as:

$$\iiint f(x, y, z)\, dV = \iiint f(x, y, z)\, dxdydz$$

.

Example

Find the value of $\int_0^1 \int_0^x \int_0^y xyz \; dzdydx$

Solution

$$\int_0^1 \int_0^x \int_0^y xyz \; dzdydx = \int_0^1 \int_0^x xy \left[\frac{z^2}{2}\right]_0^y dy \; dx = \frac{1}{2} \int_0^1 \int_0^x xy^3 \; dy \; dx$$

$$= \frac{1}{2} \int_0^1 x \left[\frac{y^4}{4}\right]_0^x dx$$

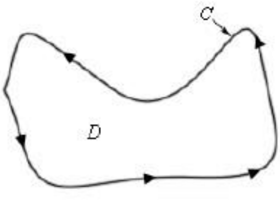
$$= \frac{1}{8} \int_0^1 x^5 \cdot dx = \left[\frac{x^6}{8(6)}\right]_0^1 = \frac{1}{48}$$

Green's Theorem

Definition

In this section we are going to investigate the relationship between certain kinds of line integrals (on closed paths) and double integrals. If $M(x,y)$ and $N(x,y)$ are continuous and having continuous first order partial derivatives bounded by a closed curve 'c' then,

$$\int_c (M\,dx + N\,dy) = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$



First, notice that because the curve is simple and closed there are no holes in the region D. Also notice that a direction has been put on the curve. We will use the convention here that the curve c has a positive orientation if it is traced out in a counter-clockwise direction. Another way to think of a positive orientation (that will cover much more general curves as well see later) is that as we traverse the path following the positive orientation the region D must always be on the left.

Example

Use Green's Theorem to evaluate $\int_c (3x + 4y)dx + (2x - 2y)dy$, where $c : x^2 + y^2 = 4$.

Solution

Here

$$M = 3x + 4y, \quad N = 2x - 3y$$

By Green 's theorem

$$0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad dxdy = rdrd\theta$$

$$\int_R (Mdx + Ndy) = \int_0^{2\pi} \int_0^2 \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) r \, drd\theta$$

$$= -2 \int_0^{2\pi} \int_0^2 rdrd\theta = (-2)(\pi r^2) = -8\pi$$

Surface Integrals

An integral which is to be evaluated over a surface is called surface integral.

Mathematical formula for surface integral is

$$= \iint \tilde{F} \cdot \tilde{N}\, ds$$

, where $\tilde{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and $\tilde{N}$ = Outward unit normal vector and ds =projection of surface on to the planes.

Method To Evaluate Surface Integral

Definition

1. If the surface S is on to $XY(Z = 0)$ plane, then

$$\iint_S \vec{F}\cdot\hat{N}\,ds = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \vec{F}\cdot\hat{N}\,\frac{dydx}{|\hat{N}\cdot\hat{k}|}$$

Where $\hat{N} = \dfrac{\nabla\varphi}{|\nabla\varphi|}$ if $\varphi(x, y, z) = c$ is given otherwise $\hat{N} = \hat{k}$.

2. If the surface S is on to $XZ(Y = 0)$ plane, then

$$\iint_S \vec{F}\cdot\hat{N}\,ds = \int_{x_1}^{x_2}\int_{z_1}^{z_2} \vec{F}\cdot\hat{N}\,\frac{dzdx}{|\hat{N}\cdot\hat{j}|}$$

Where $\hat{N} = \dfrac{\nabla\varphi}{|\nabla\varphi|}$ if $\varphi(x, y, z) = c$ is given otherwise $\hat{N} = \hat{j}$.

Definition

3. If the surface S is on to $YZ(X = 0)$ plane, then

$$\iint_S \vec{F} \cdot \hat{N}\, ds = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \vec{F} \cdot \hat{N} \frac{dydz}{|\hat{N} \cdot \hat{i}|}$$

Where $\hat{N} = \dfrac{\nabla\varphi}{|\nabla\varphi|}$ if $\varphi(x, y, z) = c$ is given otherwise $\hat{N} = \hat{i}$.

Example

The value of $\iint_s \vec{F} \cdot \hat{N}\, ds$, where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2\hat{k}$ and S is the surface of cylinder $x^2 + y^2 = 16$ and if the surface S is on to $YZ(X = 0)$ plane between $z = 0,\ z = 5$ is?

Solution

$\varphi = x^2 + y^2 - 16$ *and*

$$\hat{N} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{\hat{i}(2x) + \hat{j}(2y)}{2\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j}}{\sqrt{16}} = \frac{x\hat{i} + y\hat{j}}{4}$$

$$\vec{F} \cdot \hat{N} = \frac{zx + xy}{4}$$

$$\iint_s \vec{F} \cdot \hat{N} = \iint_s \frac{zx + xy}{4} \cdot ds$$

$x^2 + y^2 = 16$ *put* $x = 0 \Rightarrow y = \pm 4$

Solution

Let the surface S is projected on to YZ plane, and

$$|\hat{N}\cdot\hat{i}| = \frac{x}{4}$$

$$\iint_S \vec{F}\cdot\hat{N}ds = \int_{z=0}^{5}\int_{0}^{4}\frac{x}{4}(z+y)\cdot\frac{dzdy}{x}(4)$$

$$\int_{z=0}^{5}\left[zy+\frac{y^2}{2}\right]_0^4\cdot dz = \int_0^5 (4z+8)dz = [2z^2+8z]_0^5$$

$$= 50 + 40 = 90$$

Stoke's Theorem

Definition

According to this theorem, let S be the two sided open surface bounded by a closed curve c and $\vec{F}$ be differentiable vector function then

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_s (\vec{\nabla} \times \vec{F}) \cdot \hat{N}\, ds$$

It gives relation between line integral and surface integral.

Example

By Stoke's theorem find the value of $\oint_c \vec{F} \cdot d\vec{r}$ where c is $x^2 + y^2 = 16$, $z = 0$ and $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution

$$\oint \vec{F} \cdot d\vec{r} = \iint (\nabla \times \vec{r}) \cdot \hat{N} \cdot ds$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial \vec{F}}{\partial x} & \frac{\partial \vec{F}}{\partial y} & \frac{\partial \vec{F}}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}(0) + \hat{j}(0) + \hat{k}(0) = 0$$

$$\oint_c \vec{F} \cdot d\vec{r} = \iint (0) \cdot \hat{N}\, ds = 0$$

Volume Integrals

Definition

The integral $\iiint_V \vec{F} dv$ is called volume integral and it is given as

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) \, dz \, dy \, dx$$

$$= \hat{i} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F_1 \, dz \, dy \, dx + \hat{j} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F_2 \, dz \, dy \, dx$$

$$+ \hat{k} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F_3 \, dz \, dy \, dx$$

Gauss Divergence Theorem

The Gauss Divergence Theorem gives relation between surface integral to volume integral.

Definition

Let $'V'$ be the volume bounded by a closed surface S and $\vec{F}$ a differentiable vector function then

$$\iint_S (\vec{F} \cdot \hat{N})\, ds = \iiint_V (\nabla \cdot \vec{F})\, dv$$

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dz\ dy\ dx$$

Example

By Gauss Divergence theorem find the value of $\iint_s \vec{F} \cdot \hat{N}\, ds$, where $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and dS is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\iint_s \vec{F} \cdot \hat{N}\, ds = \iiint (3)\, dv$$

$$\iint_s \vec{F} \cdot \hat{N}\, ds = 3\left(\frac{4\pi}{3} r^3\right) = 4\pi$$

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ThankYou