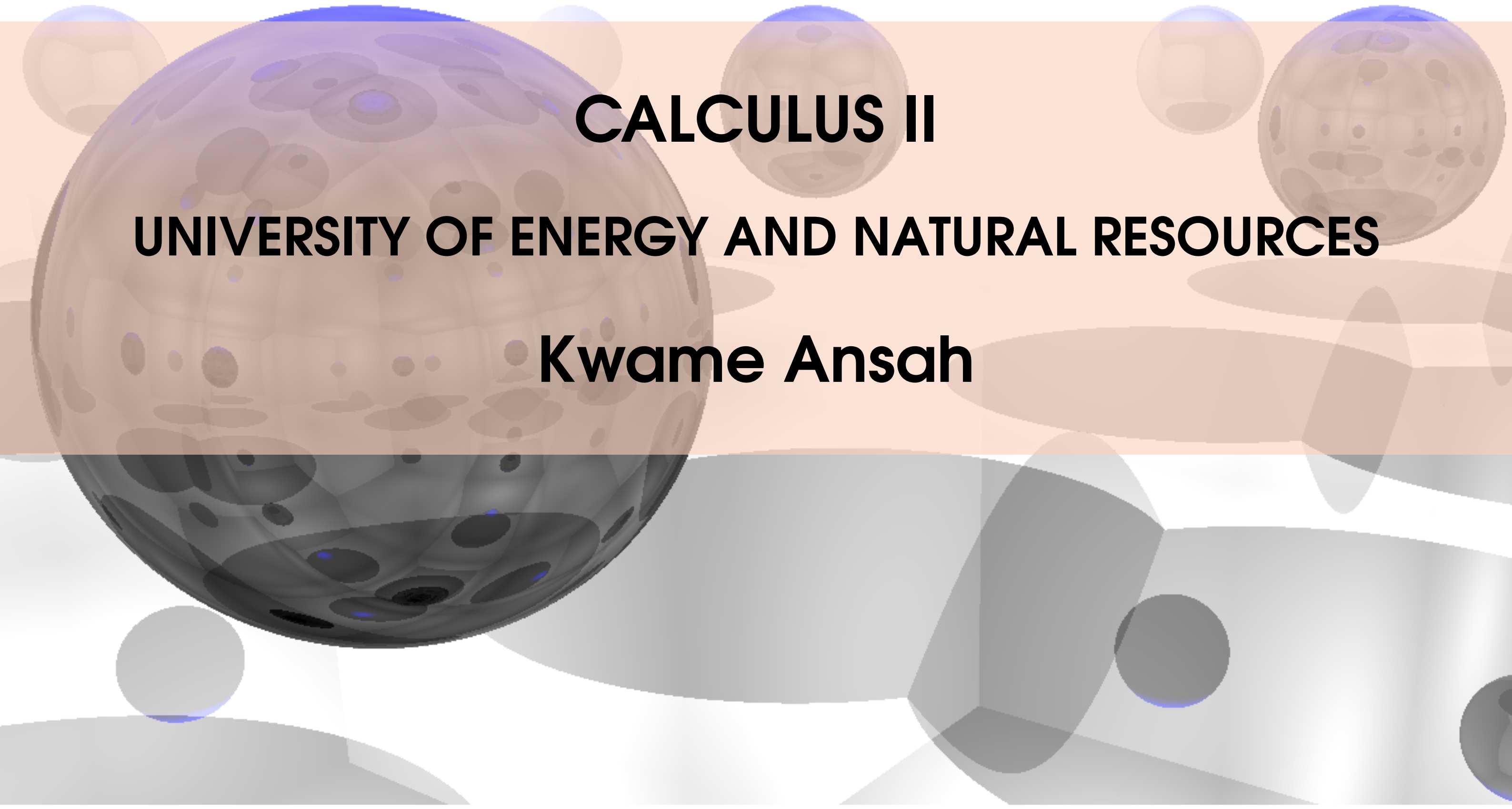
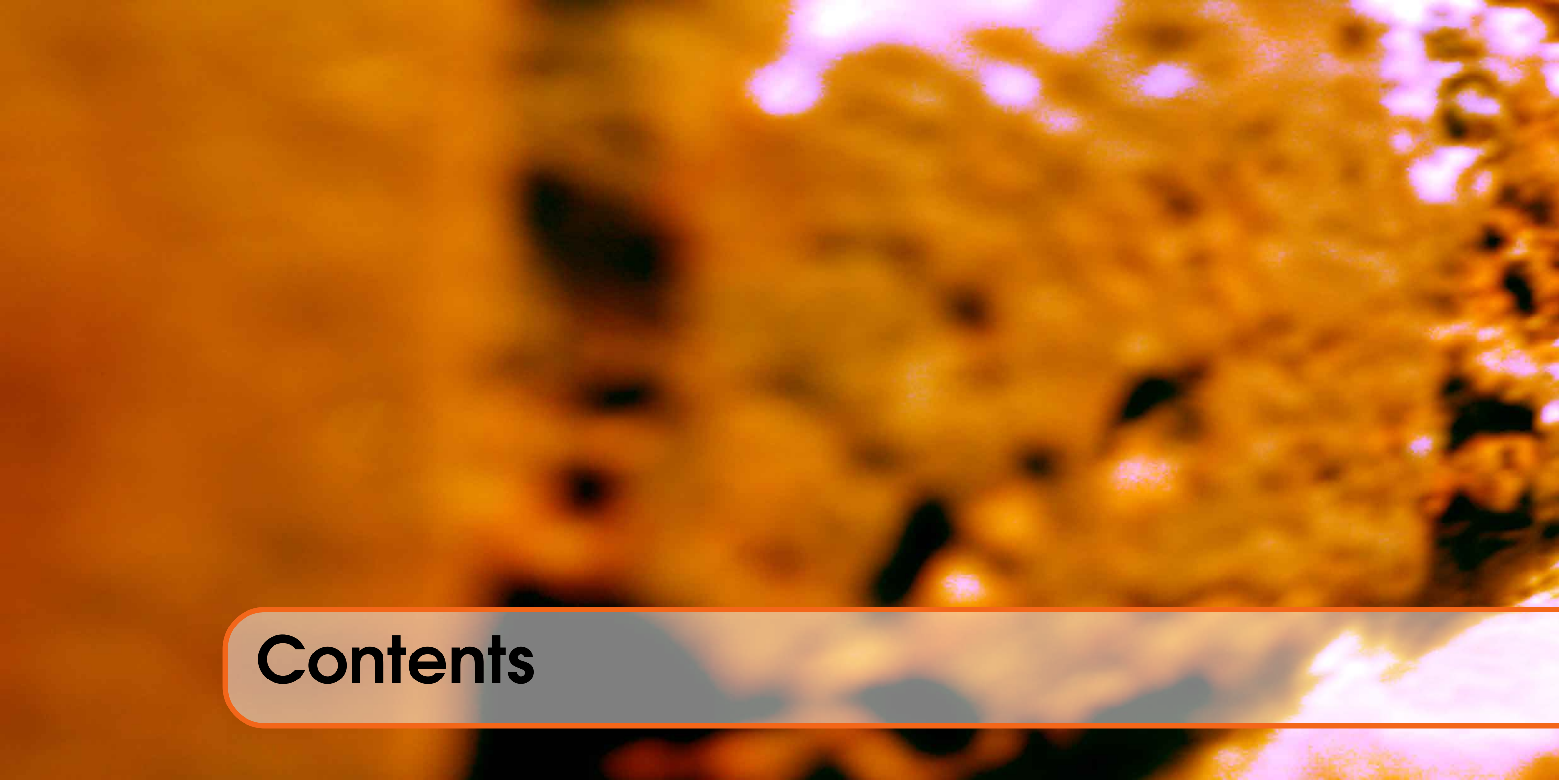


CALCULUS II

UNIVERSITY OF ENERGY AND NATURAL RESOURCES

Kwame Ansah



Contents

Part One

1. SEQUENCES

Definition 1.0.1 A sequence is an ordered list of numbers. The following is a sequence of odd numbers:

$$1, 3, 5, 7, \cdots$$

A term of a sequence is identified by its position in the sequence. In the above sequence, 1 is the first term, 3 is the second term, etc. The ellipsis symbol $(\cdots)$ indicates that the sequence continues forever. Every number in the sequence is called a **term** of the sequence.

Note 1.1 If the sequence has a last term, then it is called a **Finite Sequence**. If the number of terms is infinite, then the sequence is called an **Infinite Sequence**. The n^{th} term of the sequence will be denoted by U_n, while the sequence itself will be denoted by $\{U_n\}$.

■ **Example 1.1** $\{A_n\} = \{1, 2, \cdots, 30\}$ is a **Finite Sequence** of the counting numbers up to 30. ■

■ **Example 1.2** $\{U_n\} = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots\right\}$ is an **Infinite Sequence** with the n^{th} term $U_n = \frac{1}{n}$. ■

1.1 Types Of Sequence

1.1.1 Arithmetic Sequence:

Definition 1.1.1 A sequence $\{U_n\}$ of numbers is said to be an arithmetic sequence if

$$U_{n+1} - U_n = d \ , \forall \ n.$$

The n^{th} term of an AP is defined by:

$$U_n = a + (n-1)d$$

where $a =$ first term and $d =$common difference.

Example 1.3 Determine the 9^{th} term of and 16^{th} term of the sequence $2, 7, 12, 17, \cdots$

Solution 1.1 From the above sequence our common difference d is 5 and the first term a is 2
Determining the 9^{th} term,

$$\begin{aligned} U_n &= a + (n-1)d \\ U_9 &= 2 + (9-1)5 \\ U_9 &= 42 \end{aligned}$$

For the 16^{th} term

$$\begin{aligned} U_{16} &= 2 + (16-1)5 \\ U_{16} &= 77 \end{aligned}$$

Example 1.4 The 6^{th} term of an A.P is 17 and the 13^{th} term is 38. Determine the 19^{th} term.

Solution 1.2 We are given the value of the 6^{th} and 13^{th} term so in order to find the next term we first need to know our first term and the common difference.

$$\begin{aligned} U_n &= a + (n-1)d \\ U_6 &= a + (6-1)d = 17 \\ \\ &= a + 5d = 17 \quad \cdots\cdots\cdots\cdots (1) \\ \\ U_{13} &= a + (13-1)d = 38 \\ U_{13} &= a + 12d = 38 \\ \\ &= a + 12d = 38 \cdots\cdots\cdots\cdots (2) \end{aligned}$$

Solving eqn(1) and (2) simultaneously we have $a = 2$ and $d = 3$
Therefore

$$\begin{aligned} U_{19} &= 2 + (19-1)3 \\ &= 56 \end{aligned}$$

Definition 1.1.2 The sum of the first n terms of an Arithmetic sequence is given by :

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

or

$$S_n = \frac{n}{2}\left[a+l\right]$$

where a= first term, d=common difference and $l(\text{last term}) = a+(n-1)d$

■ **Example 1.5** Find the sum of the first 12 terms of $5, 9, 13, 17\cdots$ ■

Solution 1.3 In the above example we where given the first term and asked to find the sum to the 12^{th} term.

$$\begin{aligned} S_n &= \frac{n}{2}[2a+(n-1)d] \\ S_{12} &= \frac{12}{2}[2(5)+(12-1)4] \\ S_{12} &= 324 \end{aligned}$$

■ **Example 1.6** Find the sum of the series

$$1+3.5+6+8.5+\cdots+101.$$

■

Solution 1.4 From the above sequence our common difference d is 2.5 and the first term a is 1.

$$\begin{aligned} l &= a+(n-1)d \\ 101 &= 1+(n-1)\times 2.5 \\ 100 &= (n-1)\times 2.5 \text{ (make } n \text{ the subject)} \\ n &= 41 \\ S_n &= \frac{n}{2}[a+l] \\ S_{41} &= \frac{41}{2}[1+101] \\ S_{41} &= 2091 \end{aligned}$$

1.1.2 Geometric Sequence

Definition 1.1.3 When a sequence has a constant ratio between successive terms is called a **G.P**. The constant term is called the **common ratio.**

The n^{th} term of a G.P is given by:

$$U_n = ar^{n-1}$$

where r = common ratio

Definition 1.1.4 The sum of the first n terms of a geometric sequence is given by :

$$S_n = \frac{a(r^n - 1)}{r-1}, \ |r| > 1$$

Or

$$S_n = \frac{a(1 - r^n)}{1-r}, \ |r| < 1$$

■ **Example 1.7** Find the sum of the first 8 terms of the G.P $1, 2, 4, 8, 16 \cdots$ ■

Solution 1.5 From the above sequence $|r| = 2 > 1$ and the first term a is 1

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r-1} \\ S_8 &= \frac{1(2^8 - 1)}{2-1} \\ &= \frac{2^8 - 1}{1} \\ S_8 &= 255 \end{aligned}$$

■ **Example 1.8** Determine the 9^{th} term of the of the G.P $1, 2, 4, 8 \cdots$ ■

Solution 1.6 Using the formula $U_n = ar^{n-1}$ and deducing $a = 1, \ |r| = 2 > 1, \ n = 9$ we have

$$\begin{aligned} U_9 &= (1)(2)^{9-1} \\ &= 2^8 \\ U_9 &= 256 \end{aligned}$$

■ **Example 1.9** Determine the 10^{th} term of the series $3, 6, 12, 24, \cdots$ ■

Solution 1.7 $3, 6, 12, 24, \cdots$ is a geometric progression with a common ratio r of 2.

The n^{th} term of a GP is ar^{n-1}, where a is the first term.

Hence the 10^{th} term is: $(3)(2)^{10-1} = (3)(2)^9 = 3(512) = 1536$

Definition 1.1.5 Sum to infinity (S_∞)

$$S_\infty = \frac{a}{1-r}, \quad |r| < 1$$

Applications

■ **Example 1.10** A hire tool firm finds that their net return hiring tools is decreasing by 10 percent per annum. If the net gain on a certain tool is given is 400 cedis. The net gain form a series 400 + 400 x 0.9 + 400 x $0.9^2 + \cdots$ Find the possible total of all future profits fro this tool (assuming the tool lasts forever). ■

Solution 1.8

$$a = 400, \; r = 0.9$$

$$S_\infty = \frac{a}{1-r} = \frac{400}{1-0.9} = 4000$$

■ **Example 1.11** A drilling machine is to have 6 speeds ranging from 50 rev/min to 750 rev/min. If the speeds form a geometric progression determine their values, each correct to the nearest whole number ■

Solution 1.9 Let the GP of n terms by given by $a, ar, ar^2, \cdots ar^{n-1}$
The 1^{st} term $a = 50$ rev/min.
The 6^{th} term is given by ar^{6-1}, which is 750 rev/min,
i.e., $ar^5 = 750$
from which $r^5 = \frac{750}{a} = \frac{750}{50} = 15$
Thus the common ratio, $r = \sqrt[5]{15} = 1.7188$

The 1^{st} term is $a = 50$ rev/min.

the 2^{nd} term is $ar = (50)(1.7188) = 85.94$,
the 3^{rd} term is $ar^2 = (50)(1.7188)^2 = 147.71$,
the 4^{th} term is $ar^3 = (50)(1.7188)^3 = 253.89$,
the 5^{th} term is $ar^4 = (50)(1.7188)^4 = 436.39$,
the 6^{th} term is $ar^5 = (50)(1.7188)^5 = 750.06$
Hence, correct to the nearest whole number, the 6 speeds of the drilling machine are:
50, 86, 148, 254, 43, and 750 rev/min.

Example 1.12 Tickets for a certain show were printed bearing numbers from 1 to 100. The odd number tickets were sold by receiving cedis equal to thrice the number on the ticket while the even number tickets were issued by receiving cedis equal to twice the number on the ticket. How much was received by the issuing agency? ■

Solution 1.10 Let S_1 and S_2 be the amounts received for odd number and even number tickets respectively, then $S_1 = 3[1+3+5+\cdots+99]$ and $S_2 = 2[2+4+6+\cdots+100]$

$$\text{Thus, } S_1+S_2 = 3\times\frac{50}{2}(1+99)+2\times\frac{50}{2}(2+100)$$

Therefore there are 50 terms in each series

$$S_1+S_2 = 7500+5100 = 12600$$

.

Example 1.13 A clock strikes once when its hour hand is at 1, twice when it is at 2 and so on. How many times does the clock strike in six hours? ■

Solution 1.11 Since the clock strikes once when its hour hand is at 1, twice when it is at 2 and so on, so the sequence of strikes from 1 hour to 6 hours is 1,2,3,4,5,6. Here

$$a = 1,\ d = 2-1 = 1, n = 6$$

$$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{6}{2}[2(1)+(6-1)(1)] = 3(2+5) = 3(7) = 21$$

Example 1.14 A factory owner repays his loan of 2088000 Cedis by 20000 Cedis in the first monthly installment and then increases the payment by 1000 Cedis in every installment. How many installments will it take for him to clear his loan?

Solution 1.12 By the given conditions, we have $a = 20000,\ d = 1000,\ s_n = 2088000$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$2088000 = \frac{n}{2}\left[2\left(20000\right)+\left(n-1\right)\left(1000\right)\right]$$

$$\Rightarrow 2088 = \frac{n}{2}\left[40+n-1\right]$$

$$\Rightarrow 4176 = n^2+39n$$

$$\Rightarrow n^2+39n-4176 = 0$$

$$\Rightarrow n = \frac{-39 \pm \sqrt{\left(39\right)^2-4\left(1\right)\left(-4176\right)}}{2\left(1\right)} = \frac{-39 \pm \sqrt{1521+16704}}{2}$$

$$= \frac{-39 \pm \sqrt{18225}}{2} = \frac{-39 \pm 135}{2}$$

$$= \frac{-39+135}{2}, \frac{-39-135}{2}$$

$$\Rightarrow n = 48, -87$$

Now n, being the number of installments, cannot be negative, so $n = 48$. This shows that the factory owner will clear his loan in 48 monthly installments.

■

1.1.3 Exercise

Exercise 1.1 The sum of 7 terms of an AP is 35 and the common difference is 1.2. Determine the 1st term of the series ■

Exercise 1.2 Three numbers are in arithmetic progression. Their sum is 15 and their product is 80. Determine the three numbers. [2, 5 and 8] ■

Exercise 1.3 Find the sum of all the numbers between 0 and 207 which are exactly divisible by 3 ■

Exercise 1.4 On commencing employment a man is paid a salary of £7200 per annum and receives annual increments of £350. Determine his salary in the 9th year and calculate the total he will have received in the first 12 years.
[£10 000, £109 500] ■

Exercise 1.5 An oil company bores a hole 80m deep. Estimate the cost of boring if the cost is £30 for drilling the first metre with an increase in cost of £2 per metre for each succeeding metre. [£8720] ■

Exercise 1.6 The first term of a geometric progression is 12 and the 5th term is 55. Determine the 8th term and the 11th term ■

Exercise 1.7 In a geometric progression the 6th term is 8 times the 3rd term and the sum of the 7th and 8th terms is 192. Determine

i the common ratio
ii the 1st term and
iii the sum of the 5th to 11th term, inclusive

■

Exercise 1.8 If £100 is invested at compound interest of 8% per annum, determine

i the value after 10 years
ii the time, correct to the nearest year, it takes to reach more than £300

■

Exercise 1.9 100 g of a radioactive substance disintegrates at a rate of 3% per annum. How much of the substance is left after 11 years?[71.53 g] ■

Exercise 1.10 If the population of Great Britain is 55 million and is decreasing at 2.4% per annum, what will be the population in 5 years time? [48.71 M] ■

Exercise 1.11 A drilling machine is to have 8 speeds ranging from 100 rev/min to 1000 rev/min. If the speeds form a geometric progression determine their values, each correct to the nearest whole number[100, 139, 193, 268, 373, 518, 720, 1000 rev/min]

1.2 Limit Of Sequence

Definition 1.2.1 An infinite sequence $\{U_n\}_{n=1}^{\infty}$ is said to have a limit L as n approaches ∞, if U_n can be made as close to L as we please by choosing n sufficiently large. We then have

$$\lim_{n\to\infty} U_n = L$$

The above definition may be more rigorously written as follows :

Definition 1.2.2 For every (Epsilon) $\varepsilon >\ 0$ (sufficiently small), there exist a positive integer N (depending on ε)such that $|U_n - L| < \varepsilon, \forall\, n > N$

Note 1.2 If $\lim\limits_{n\to\infty} U_n = L$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

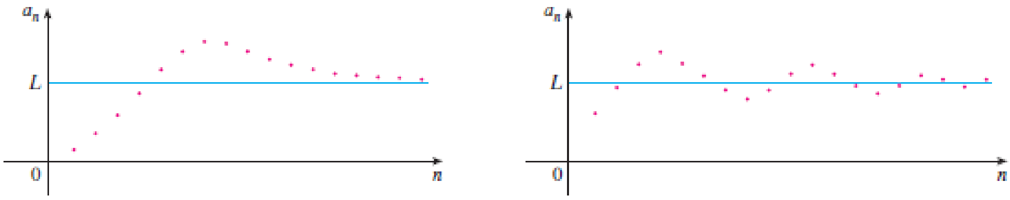


Figure 1.1: Illustrates Definition 1.2.1 by showing the graphs of two sequences that have the limit .

Another illustration of Definition 1.2.2 is given in Figure 1.3. The points on the graph of $\{U_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small is chosen, but usually a smaller requires a larger .

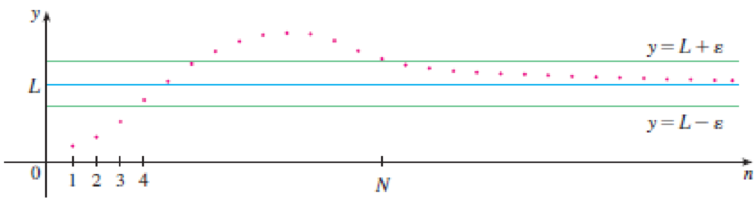


Figure 1.2

If you compare Definition 1.2.2 with Theorem 1.2.1 you will see that the only difference $\lim_{n\to\infty} U_n = L$ and $\lim_{n\to\infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem, which is illustrated by Figure 1.4.

Theorem 1.2.1 If $\lim_{n\to\infty} f(x) = L$ and $f(n) = U_n$ when n is an integer, then $\lim_{n\to\infty} U_n = L$

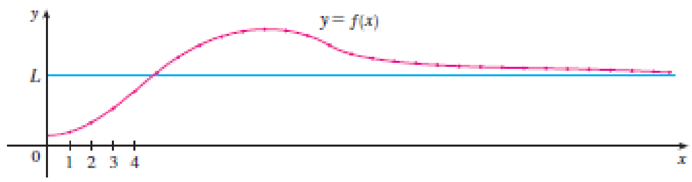


Figure 1.3

■ **Example 1.15** Evaluate $\lim_{n\to\infty} \frac{1}{x^r}$ where $r > 0$ ■

Solution 1.13 $\lim_{n\to\infty} \frac{1}{n^r} = 0$ if $r > 0$

■ **Example 1.16** Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$ using the definition of limit of a sequence. ■

Proof 1.1 This is going to be the first proof in this chapter.

$$\forall\, \varepsilon, \exists\ N(\varepsilon) > 0\ s.t\ |U_n - L| < \varepsilon, \forall\, n > N.$$

$$|U_n - L| < \varepsilon,\ U_n = \frac{1}{n},\ L = 0$$

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\left|\frac{1}{n}\right| < \varepsilon$$

$$\frac{1}{n} < \varepsilon (\text{make n the subject})$$

$$\frac{1}{\varepsilon} < n \text{ or } n > \frac{1}{\varepsilon}$$

$$but \quad n > N$$

$$\therefore N = \frac{1}{\varepsilon}$$

■ **Example 1.17** Prove that $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)=1$ using the definition of limit of a sequence. ■

Proof 1.2

$$\forall\, \varepsilon > 0, \exists\ N(\varepsilon) > 0\ s.t\ |U_n - L| < \varepsilon,\ \forall\, n > N$$

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon$$

$$\left|\frac{n}{n+1} - \frac{-1}{1}\right| < \varepsilon$$

$$\left|\frac{n-n-1}{n+1}\right| < \varepsilon$$

$$\left|\frac{-1}{n+1}\right| < \varepsilon$$

$$\left|\frac{-1}{n+1}\right| < \varepsilon$$

$$\frac{1}{n+1} < \varepsilon\ (make\ n\ the\ subject)$$

$$\frac{1}{\varepsilon} < n+1$$

$$\frac{1}{\varepsilon} - 1 < n$$

choosing

$$N = \frac{1}{\varepsilon} - 1$$

■ **Example 1.18** Show that $\lim_{n\to\infty}\frac{1}{2^n}=0$ ■

Proof 1.3

$$\forall\, \varepsilon > 0, \exists\ N(\varepsilon) > 0\ s.t\ |U_n - L| < \varepsilon,\ \forall\, n > N$$

$$\left|\frac{1}{2^n} - 0\right| < \varepsilon$$

$$\left|\frac{1}{2^n}\right| < \varepsilon$$

$$\frac{1}{2^n} < \varepsilon$$

$$\frac{1}{\varepsilon} < 2^n\ \text{[Taking Log of both sides]}$$

$$\log\left(\frac{1}{\varepsilon}\right) < \log 2^n$$

$$\frac{\log(\frac{1}{\varepsilon})}{\log 2} < \frac{n\log 2}{\log 2}$$

$$\frac{\log(\frac{1}{\varepsilon})}{\log 2} < n\,, n > N$$

Choosing

$$N = \frac{\log(\frac{1}{\varepsilon})}{\log 2}$$

Theorem 1.2.2 Let U_n *and* V_n be two sequences such that $\lim\limits_{n\to\infty} U_n = A$ *and* $\lim\limits_{n\to\infty} V_n = B$. Then the following results hold:

1. $\lim\limits_{n\to\infty} U_n \pm V_n = \lim\limits_{n\to\infty} U_n \pm \lim\limits_{n\to\infty} V_n = A \pm B$
2. $\lim\limits_{n\to\infty} U_n.V_n = \lim\limits_{n\to\infty} U_n. \lim\limits_{n\to\infty} V_n = A.B$
3. $\lim\limits_{n\to\infty}\left(\frac{U_n}{V_n}\right) = \frac{\lim\limits_{n\to\infty} U_n}{\lim\limits_{n\to\infty} V_n} = \frac{A}{B},\ B \neq 0$
   (a) If $B = 0$ and $A \neq 0$, then $\lim\limits_{n\to\infty} \frac{U_n}{V_n}$ does not exist.
   (b) If $B = 0$ and $A = 0$, then $\lim\limits_{n\to\infty} = \frac{U_n}{V_n}$, may or may not exist.
4. $\lim\limits_{n\to\infty} P^{U_n} = P^{\lim\limits_{n\to\infty} U_n} = P^A$
5. $\lim\limits_{n\to\infty} (U_n)^P = [\lim\limits_{n\to\infty} U_n]^P = A^P$

Theorem 1.2.3 If a sequence $\{U_n\}$ has a limit,then the limit is unique.

Proof 1.4 Suppose that $\{U_n\}$ has two limits l_1 and l_1. For every (Epsilon) $\varepsilon > 0$ (sufficiently small), there exist a positive integer N (depending on ε)such that $|U_n - l_1| <$

$\varepsilon, \forall\, n > N$ and $|U_n - l_2| < \varepsilon, \forall\, n > N$

Then

$$\begin{aligned} |l_1 - l_2| &= |l_1 - U_n + U_n - l_2| \\ &\leq |l_1 - U_n| + |U_n - l_2| \\ &< \varepsilon + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ (however small), it follows that $|l_1 - l_2|$ is less than every positive integer.

$$\therefore |l_1 - l_2| = 0 \implies l_1 - l_2 = 0 \implies l_1 = l_2$$

This completes the proof.

1.2.1 Evaluating limits of Sequences

Definition 1.2.3 The limit L of a sequence U_n if it exists, may be evaluated using the theorems on limits.

Note 1.3 In evaluating limits of sequences, we will divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws.

■ **Example 1.19** Evaluate the limits of

$$\lim_{n\to\infty} \frac{4-2n-3n^2}{4n^2-5n}$$

■

Solution 1.14

$$= \lim_{n\to\infty} \frac{4-2n-3n^2}{4n^2-5n}$$

$$= \lim_{n\to\infty} \frac{\frac{4}{n^2}-\frac{2n}{n^2}-\frac{3n^2}{n^2}}{\frac{4n^2}{n^2}-\frac{5n}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\frac{4}{n^2}-\frac{2}{n}-3}{4-\frac{5}{n}}$$

$$= \frac{0-0-3}{4-0} = \frac{-3}{4}$$

■ **Example 1.20**

$b.\ \lim_{n\to\infty} \sqrt{\frac{n^2(n-1)}{1+\frac{1}{2}n^3}}$ ■

Solution 1.15

$$b.\ \lim_{n\to\infty} \sqrt{\frac{n^2(n-1)}{1+\frac{1}{2}n^3}}$$

$$= \lim_{n\to\infty} \frac{n^2(n-1)}{1+\frac{1}{2}n^3}$$

$$= \lim_{n\to\infty} \frac{n^3-n^2}{1+\frac{1}{2}n^3}$$

$$= \lim_{n\to\infty} \frac{\frac{n^3}{n^3}-\frac{n^2}{n^3}}{\frac{1}{n^3}+\frac{\frac{1}{2}n^3}{n^3}}$$

$$= \lim_{n\to\infty} \frac{1-\frac{1}{n}}{\frac{1}{n^3}-\frac{1}{2}}$$

$$= \frac{1-0}{0-\frac{1}{2}} = 2$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^2(n-1)}{1+\frac{1}{2}n^3}} = \sqrt{2}$$

■ **Example 1.21** Calculate $\lim\limits_{n\to\infty} \frac{\ln n}{n}$ ■

Solution 1.16 Notice that both numerator and denominator approach infinity as $n \to \infty$. We can't apply l'Hospital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply l'Hospital's Rule to the related function $\frac{\ln x}{x}$ and obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Therefore, by Theorem 1.2.1, we have

$$\therefore \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

■ **Example 1.22** Find $\lim_{n\to\infty} \frac{3n^2-1}{10n+5n^2}$ ■

Solution 1.17 $\lim_{n\to\infty} \frac{3n^2-1}{10n+5n^2}$

$$= \lim_{n\to\infty} \frac{\frac{3n^2}{n^2} - \frac{1}{n^2}}{\frac{10n}{n^2} + \frac{5n^2}{n^2}}$$

$$= \lim_{n\to\infty} \frac{3 - \frac{1}{\infty}}{\frac{10}{\infty} + 5}$$

$$\therefore \lim_{n\to\infty} \frac{3n^2-1}{10n+5n^2} = \frac{3}{5}$$

■ **Example 1.23** Evaluate $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$ ■

Solution 1.18

$$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

■ **Example 1.24** Find $\lim_{n\to\infty} \left\{ \frac{n(n+2)}{n+1} - \frac{n^3}{n^2+1} \right\}$ ■

Solution 1.19

$$\lim_{n\to\infty} \left\{ \frac{n(n+2)}{n+1} - \frac{n^3}{n^2+1} \right\} = \lim_{n\to\infty} \left\{ \frac{n^3+n^2+2n}{(n+1)(n^2+1)} \right\} = \lim_{n\to\infty} \left\{ \frac{1+\frac{1}{n}+\frac{2}{n^2}}{(1+\frac{1}{n})(1+\frac{1}{n^2})} \right\}$$

$$= \frac{1+0+0}{(1+0)\cdot(1+0)} = 1$$

■ **Example 1.25** Evaluate $\lim\limits_{n\to\infty} \dfrac{6+7\cdot 10^n}{5+3\cdot 10^n}$ ■

Solution 1.20

$$\lim_{n\to\infty} \frac{6+7\cdot 10^n}{5+3\cdot 10^n} = \lim_{n\to\infty} \frac{6\cdot 10^{-n}+7}{5\cdot 10^{-n}+3} = \frac{7}{3}$$

■ **Example 1.26** Find $\lim\limits_{n\to\infty} \left(\dfrac{2n-3}{3n+7}\right)^4$ ■

Solution 1.21

$$\lim_{n\to\infty} \left(\frac{2n-3}{3n+7}\right)^4 = \left(\lim_{n\to\infty} \frac{2n-3}{3n+7}\right)^4 = \left(\lim_{n\to\infty} \frac{2-\frac{3}{n}}{3+\frac{7}{n}}\right)^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

■ **Example 1.27** A sequence U_n is defined by the recursion formula, $U_{n+1} = \sqrt{3U_n}$, $U_1 = 1$. Find $\lim\limits_{n\to\infty} U_n$ ■

Solution 1.22

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} U_{n+1} = a$$

$$\lim_{n\to\infty} U_{n+1} = \lim_{n\to\infty} \sqrt{3U_n}$$

$$\lim_{n\to\infty} U_{n+1} = \sqrt{3 \lim_{n\to\infty} U_n}$$

$$(a)^2 = (\sqrt{3a})^2$$

$$a^2 = 3a$$

$$a^2 - 3a = 0$$

$$a = 3 \; or \; a = 0$$

But since the sequence is increasing $a = 3$

$$\therefore \lim_{n\to\infty} U_n = 3$$

Theorem 1.2.4 If $\lim\limits_{n\to\infty} U_n = L$ and the function is continuous at L , then

$$\lim_{n\to\infty} f(U_n) = f(L)$$

■ **Example 1.28** Find $\lim\limits_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$ ■

Solution 1.23 Because the sine function is continuous at , Theorem 1.2.4 enables us to write

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin 0 = 0$$

Theorem 1.2.5 The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r .

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

■ **Example 1.29** Determine if the sequence $\{(-1)^n\}_{n=0}^{\infty}$ converge or diverge. If the sequence converges determine its limit. ■

Solution 1.24 For this theorem note that all we need to do is realize that this is the sequence in Theorem 1.4.2 above using $r = -1$. So, by Theorem 1.4.2 this sequence diverges.

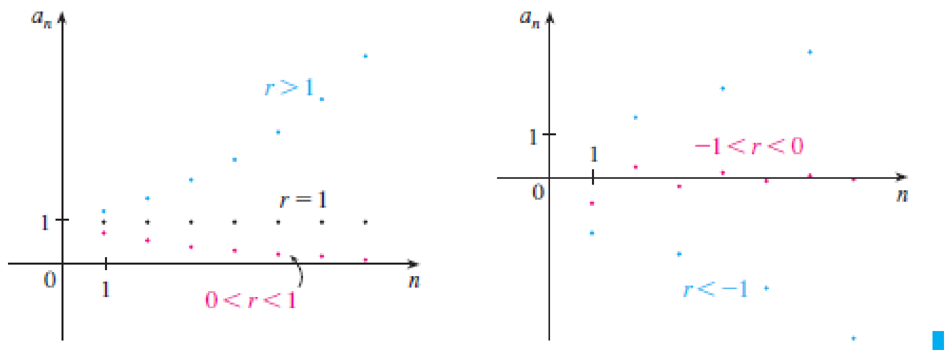


Figure 1.4: The sequence $a_n = r_n$

1.3 Bounded Sequence

Definition 1.3.1 Let A be a set. If $\exists\ M$(real) such that $x \leq M, \forall\ x \in A$,then A is said to be bounded above. M is called an **Upper bound** of set A.If $\exists\ m$(real) such that $x \geq m, \forall\ x \in A$, then A is said to be **Bounded Below** , m is called the **Lower Bound** of A.

Definition 1.3.2 Let M be an upper bounded of A.Then if for any $\varepsilon > 0, \exists$ at least element x in A that is $x > M - \varepsilon$, the number M is called the **Least Upper Bound or Supremum or sup.**

Definition 1.3.3 Let m be a lower bounded of A ,then if for any $\varepsilon > 0$ (however small). $\exists$ at least one element x in A such that $x < m + \varepsilon$,the number m is called the **Greatest Lower Bound** or **Infimum** or **inf**.

Definition 1.3.4 If $\forall\ x \in A, \exists\ m, M(\text{real})$ such that $m \leq x \leq M$, then the set A is said to be **Bounded**.

Note 1.4 If the sequence is both bounded below and bounded above we call the sequence **Bounded**.

■ **Example 1.30** Find the upper and lower bound, supremum and infimum of

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots 0\right\}$$

■

Solution 1.25 We are required to find the lower and upper bound, Least Upper Bound and Greatest Lower Bound

$$\begin{aligned} \text{Upper Bound} &= 1, 2, 3 \cdots \\ \text{Least Upper Bound} &= 1 \\ \text{Lower Bound} &= 0, 1, -2, \cdots \\ \text{Greatest Lower Bound} &= 0 \end{aligned}$$

$$0 \leq A \leq 1 \quad A \text{ is bounded}$$

■ **Example 1.31** Find the upper and lower bound, supremum and infimum of $A = [-2, 2]$
■

Solution 1.26

$$\begin{aligned} \text{Upper Bound} &= 2, 3, 4, 5 \cdots \\ \text{Least Upper Bound} &= 2 \\ \text{Lower Bound} &= -2, -3, -4, -5 \cdots \\ \text{Greatest Lower Bound} &= -2 \end{aligned}$$

$$-2 \leq A \leq 2 \quad A\ is \text{ bounded}$$

Example 1.32 Show that $\{a_n\} = \left\{\dfrac{n}{n+1}\right\}$ is bounded

Solution 1.27 First we need to determine the first few terms of the sequence $\{a_n\}$

$$\text{For } n = 1, \; a_n = \frac{1}{2}$$

$$\text{For } n = 2, \; a_n = \frac{2}{3}$$

$$\text{For } n = 3, \; a_n = \frac{4}{5}$$

Next, to determine the last term we need to find the limit of the sequence $\{a_n\}$

$$\lim_{n\to\infty} \frac{n}{n+1}$$

$$\Longrightarrow \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n} - \frac{1}{n}}$$

$$\Longrightarrow \frac{1}{1+\frac{1}{\infty}}$$

$$\Longrightarrow \frac{1}{1+0} = 1$$

$$\therefore \left\{\frac{n}{n+1}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \cdots, 1\right\}$$

$$\text{Upper Bound} = 1$$

$$\text{Lower Bound} = \frac{1}{2}$$

We can say $\{a_n\}$ is bounded because it has an upper bound =1 and lower bound $\dfrac{1}{2}$

Example 1.33 Show that $b_n = \left\{\dfrac{n+1}{n+3}\right\}$ is bounded.

Solution 1.28 First we need to determine the first few terms of the sequence $\{a_n\}$

$$\text{For } n = 1, \; b_n = \frac{2}{4}$$

$$\text{For } n = 2, \; b_n = \frac{3}{5}$$

$$\text{For } n=3,\ b_n=\frac{4}{6}$$

Next, to determine the last term we need to find the limit of the sequence $\{a_n\}$

$$\lim_{n\to\infty}\frac{n+1}{n+3}$$

$$\Longrightarrow \lim_{n\to\infty}\frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}+\frac{3}{n}}$$

$$\Longrightarrow \frac{1+\frac{1}{\infty}}{1+\frac{3}{\infty}}=1$$

$$\Longrightarrow \left\{\frac{2}{4},\frac{3}{5},\frac{4}{6},\cdots 1\right\}$$

$$\text{Upper Bound}=1$$

$$\text{Lower Bound}=\frac{1}{2}$$

We can say $\{b_n\}$ is bounded because it has an upper bound =1 and lower bound $\frac{1}{2}$

Definition 1.3.5 A set is **infinite** if it can be placed in 1-1 correspondence with a subset of itself.

■ **Example 1.34** The even natural numbers $2,4,6,8,\cdots$ is a countable set because of the 1-1 correspondence ■

1.4 Bolzano Weiestrass Theorem

Theorem 1.4.1 Every bounded infinite set has at least one limit point.

■ **Example 1.35** Verify that the sequence $\{a_n\}=\left\{\frac{1}{n},\ n=1,2,3...\right\}$ satisfies the Bolzano Weiestrass Theorem. ■

Solution 1.29 We need to show that sequence $\{a_n\}$ is bounded,infinite and has a limit.

$(i).\ \{a_n\}=\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots 0\right\}$

$$\text{Upper Bound}=1 \text{ and Lower Bound}=0 \quad \therefore \{a_n\} \text{ is bounded}$$

$$(ii).\ \{a_n\} \text{ is infinite}$$

(iii) Show that $\lim_{n\to\infty} \frac{1}{n} = 0$ using the definition of limit of a sequence that is:

$$\forall\ \varepsilon > 0, \exists\ N > 0,\ s.t\ |U_n - L| < \varepsilon, \forall\ n > N$$

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\frac{1}{n} < \varepsilon$$

$$\frac{1}{\varepsilon} < n \text{ or } n > \frac{1}{\varepsilon}$$

$$but\ n > N\ \text{ Choosing } N = \frac{1}{\varepsilon}$$

proves the existence of N and thus establishing the required result.

■ **Example 1.36** Show that $\{b_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...0\right\}$ satisfies the Bolzano Weiestrass Theorem . ■

Solution 1.30 We need to show that sequence $\{a_n\}$ is bounded,infinite and has a limit.

$$(i).\ \{b_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots 0\right\}$$

$$\text{Upper Bound} = \frac{1}{2} \text{ and Lower Bound} = 0 \qquad \therefore \{b_n\}\ is \text{ bounded.}$$

$$(ii.)\ \{b_n\}\ is\ infinite.$$

$(iii.)$ Show $that$ $\lim_{n\to\infty} \frac{1}{2^n} = 0$ using the definition of limit of a sequence that is:

$$\left|\frac{1}{2^n} - 0\right| < \varepsilon$$

$$\left|\frac{1}{2^n}\right| < \varepsilon$$

$$\frac{1}{2^n} < \varepsilon$$

$$\frac{1}{\varepsilon} < 2^n \text{ [Taking Log of both sides]}$$

$$\log\left(\frac{1}{\varepsilon}\right) < \log 2^n$$

$$\frac{\log(\frac{1}{\varepsilon})}{\log 2} < \frac{n \log 2}{\log 2}$$

$$\frac{\log(\frac{1}{\varepsilon})}{\log 2} < n \,, n > N$$

$$\text{Choosing } N = \frac{\log(\frac{1}{\varepsilon})}{\log 2}$$

proves the existence of N and thus establishing the required result.

1.5 Monotonic Sequence

Definition 1.5.1 A sequence is monotonic if it is either increasing or decreasing.

Definition 1.5.2 A sequence U_n is said to be **monotonic increasing sequence** or **non decreasing** if:

$$U_{n+1} - U_n \geq 0, \; \forall \; n.$$

A sequence U_n is called **monotonic increasing sequence** or **non decreasing** if :

$$U_{n+1} - U_n \leq 0, \forall \; n$$

.

Definition 1.5.3 We call a sequence **strictly increasing** if $U_{n+1} - U_n > 0, \forall \; n$,
If $U_{n+1} - U_n < 0, \forall n$ the sequence is said to be **strictly decreasing.**

Note 1.5 If a sequence is either monotonic increasing or monotonic decreasing it is called or said to be **Monotonic Sequence.**

Theorem 1.5.1 Every bounded monotonic sequence has a limit.

Proof 1.5 Prove as Exercise

■ **Example 1.37** Show that the following sequences are monotonic.

a. $\left\{\dfrac{n+1}{n}\right\}$

b. $\left\{\dfrac{n}{n+1}\right\}$

■

Solution 1.31

$$
\begin{aligned}
a.\ \ U_n &= \frac{n+1}{n} \\
U_n &= \frac{n+1+1}{n+1} \quad U_n+1 = \frac{n+2}{n+1} \\
U_n+1-U_n &= \frac{n+2}{n+1} - \frac{n+1}{n} \\
&= \frac{n(n+2)-(n+1)(n+1)}{(n+1)n} \\
&= \frac{n^2+2n-(n^2+2n+1)}{n^2+n} \\
&= \frac{n^2+2n-n^2-2n-1}{n^2+n} \\
&= \frac{-1}{n^2+n} \quad \text{Negative Sequence}
\end{aligned}
$$

All negatives less than zero

$$
= \frac{-1}{n^2+n} \leq 0
$$

$\therefore$ It is monotonic decreasing

$$
\begin{aligned}
b.\quad U_n &= \frac{n}{n+1} \\
U_n+1 &= \frac{n+1}{n+1+1} \quad U_n+1 = \frac{n+1}{n+2} \\
U_n+1-U_n &= \frac{n+1}{n+2} - \frac{n}{n+1} \\
&= \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} \\
&= \frac{n^2+2n+1-(n^2+2n)}{(n+2)(n+1)} \\
&= \frac{n^2-2n+1-n^2+2n}{(n+2)(n+1)} \\
&= \frac{1}{(n+2)(n+1)} \geq 0
\end{aligned}
$$

Monotonic increasing

1.6 Squeeze Theorem for Sequences

Theorem 1.6.1 If $a_n \leq b_n \leq c_n$, $\forall\, n > N$ for some N and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

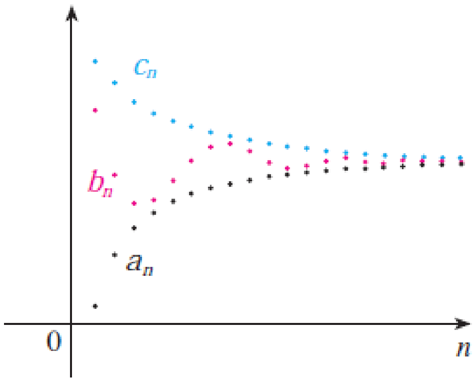


Figure 1.5: The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$

■ **Example 1.38** Find $\lim_{n\to 0} n^2\cos\left(\frac{1}{n^2}\right)$ using Squeezed Theorem. ■

Solution 1.32 We know that cosine stays between -1 and 1,so

$$-1 \leq \cos\left(\frac{1}{n^2}\right) \leq 1, \ \forall \ n \neq 0$$

Since n^2 is always positive,we can multiply by this inequality

$$-n^2 \leq n^2\cos\left(\frac{1}{n^2}\right) \leq n^2$$

So,our original sequence is bounded by $-n^2$ and n^2 .Now since

$$\lim_{n\to 0} -n^2 = \lim_{n\to 0} n^2 = 0$$

, then by the Squeezed Theorem,

$$\lim_{n\to 0} n^2\cos\left(\frac{1}{n^2}\right) = 0$$

■ **Example 1.39** Find $\lim_{n\to 0} n^2 e^{\sin\left(\frac{1}{n}\right)}$ using Squeezed Theorem. ■

Solution 1.33 Now the range of sine is also $[-1,1]$,so

$$-1 \leq \sin\left(\frac{1}{n}\right) \leq 1$$

Taking e raised to both sides of an inequality does not change the inequality,so

$$e^{-1} \leq e^{\sin\left(\frac{1}{n}\right)} \leq e^1$$

and once again we can multiply through by n^2 and get

$$n^2e^{-1} \leq n^2e^{\sin\left(\frac{1}{n}\right)} \leq n^2e^1$$

So,our original sequence is bounded by n^2e^{-1} and n^2e^1,and since

$$\lim_{n\to 0} n^2e^{-1} = \lim_{n\to 0} n^2e^1 = 0$$

, then by the Squeeze Theorem,

$$\lim_{n\to 0} n^2e^{\sin\left(\frac{1}{n}\right)} = 0$$

Theorem 1.6.2 If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

■ **Example 1.40** Find the $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n}$ ■

Solution 1.34 $\left|\frac{(-1)^{n+1}}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$

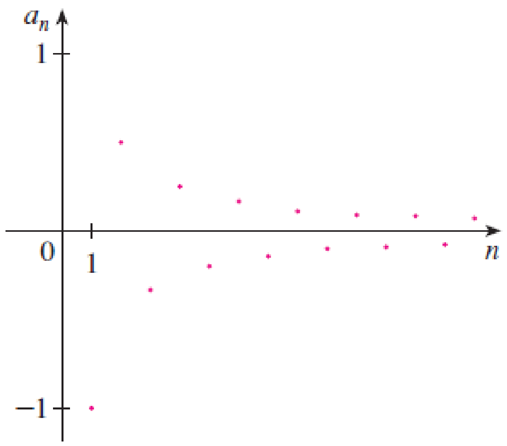


Figure 1.6

Note 1.6 Note: If the limit of the sequence doesn't exist then the sequence will diverge but if the limit exist then the sequence will converge , the sequence will also converge when you find the last term.

■ **Example 1.41** Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

a. $\left\{\frac{3n^2-1}{10n+5^2}\right\}_{n=2}^{\infty}$

b. $\left\{\frac{n^4+1}{n^3-6}\right\}_{n=1}^{\infty}$

c. $\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty}$

d. $\left\{\cos\left(\frac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ ■

Solution 1.35

$[a.]$

$$\lim_{n\to\infty} \frac{3n^2-1}{10n+5n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{3n^2}{n^2}-\frac{1}{n^2}}{\frac{10n}{n^2}+\frac{5n^2}{n^2}} = \lim_{n\to\infty} \frac{3-\frac{1}{\infty}}{\frac{10}{\infty}-5} = \frac{3}{5}$$

$$\therefore \left\{\frac{3n^2-1}{10n+5^2}\right\}_{n=2}^{\infty} \text{ Converges}$$

So the sequence converges and its limit is $= \frac{3}{5}$.

[b.]

$$\lim_{n\to\infty} \frac{n^4+1}{n^3-6}$$

$$= \lim_{n\to\infty} \frac{\frac{n^4}{n^3}+\frac{1}{n^3}}{\frac{n^3}{n^3}-\frac{6}{n^3}} = \lim_{n\to\infty} \frac{n+\frac{1}{n^3}}{1-\frac{6}{n^3}} = \frac{\infty+\frac{1}{\infty}}{1-\frac{6}{\infty}} = \frac{\infty}{1} = \infty$$

$$\left\{\frac{n^4+1}{n^3-6}\right\}_{n=1}^{\infty} \quad \text{Diverges.}$$

since the limit does not exist the sequence diverges

[c.]

$$\lim_{n\to\infty} \frac{e^{2n}}{n}$$

$$\lim_{n\to\infty} \frac{e^{2x}}{x} = \frac{e^{2(\infty)}}{\infty} = \frac{\infty}{\infty} = \text{indeterminate}$$

Next we need to use the L'hopital's rule

$$\lim_{n\to\infty} \frac{2e^{2x}}{1} = \infty$$

$$\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty} = \infty$$

since the limit does not exist the sequence diverges

[d.]

$$\lim_{n\to\infty} \cos\left\{\frac{\pi}{n}\right\}$$

$$\lim_{n\to\infty} \cos\left\{\frac{\pi}{n}\right\} = \cos\left\{\frac{\pi}{\infty}\right\} = \cos(0) = 1$$

since the limit exist the sequence is Convergent

Example 1.42 Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution 1.36 If we write out the terms of the sequence, we obtain

$$\{-1, 1, -1, 1, -1, 1, -1, \cdots\}$$

The graph of this sequence is shown in Figure 1.7. Since the terms oscillate between 1 and -1 infinitely often,a_n does not approach any number. Thus $\lim_{n\to\infty}(-1)^n$ does not exist; that is, the sequence$\{(-1)^n\}$ is divergent.

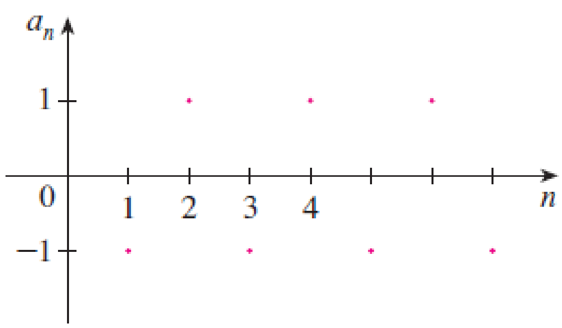


Figure 1.7

1.6.1 Exercise

Exercise 1.12 Prove that $\lim_{n\to\infty} \frac{4-2n}{3n+2} = \frac{-2}{3}$ using definition of limit.

Exercise 1.13 Evaluate the following

1. $\lim_{n\to\infty} \frac{4-2n-3n^2}{4n^2+5n}$
2. $\lim_{n\to\infty} \left\{ \sqrt{n^2+n} - n \right\}$
3. $\lim_{n\to\infty} \left\{ \frac{n}{2n^2+1} \right\}$
4. $\lim_{n\to\infty} \left\{ \sqrt{\frac{n^2(n-1)}{1+\frac{1}{2}n^3}} \right\}$
5. $\lim_{n\to\infty} (2^n+3^n)^{\frac{1}{n}}$
6. $\lim_{n\to\infty} \frac{4\cdot 10^n - 3\cdot 10^{2n}}{3\cdot 10^{n-1} + 2\cdot 10^{2n-1}}$

Exercise 1.14 Show that $\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$

Exercise 1.15 Prove that $\lim_{n\to\infty} \frac{1+2\cdot 10^n}{5+3\cdot 10^n} = \frac{2}{3}$

Exercise 1.16 Explain exactly what is meant by the statements

1. $\lim_{n\to\infty} 3^{2n-1} = \infty$
2. $\lim_{n\to\infty} (1-2n) = -\infty$

Exercise 1.17 Prove that a convergent sequence is bounded.

Exercise 1.18 Prove that the sequence $u_n = \frac{2n-7}{3n+2}$

1. is monotonic increasing,
2. is bounded above,

3. is bounded below,
4. is bounded,
5. has a limit.

■

Exercise 1.19 Let $a_n = n^2 \cos\left(\frac{1}{n}\right)$. Prove that $\lim_{n \to 0} a_n = 0$ ■

Exercise 1.20 Let $a_n = n^3 \sin\left(\frac{1}{\sqrt[3]{n}}\right)$. Use the Squeeze Theorem to find $\lim_{n \to 0} a_n = 0$ ■

Part Two

2. SERIES

Definition 2.0.1 Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n^{th} partial sum:

$$s_n = \sum_{i=0}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence s_n is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + ... + a_n + \cdots = s \quad \text{or} \sum_{n=1}^{\infty} a_n = s$$

The number is called the **sum** of the series. If the sequence s_n is divergent, then the series is called **divergent**.

Note 2.1 Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number . Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

■ **Example 2.1** Suppose we know that the sum of the first terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

■

Solution 2.1 Then the sum of the series is the limit of the sequence s_n :

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \lim s_n \\ &= \lim_{n\to\infty} \frac{2n}{3n+5} \\ &= \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}} \\ &= \frac{2}{3}\end{aligned}$$

In Example 2.1 we were given an expression for the sum of the first terms, but it's usually not easy to find such an expression

■ **Example 2.2** An important example of an infinite series is the **geometric series** ■

Solution 2.2

$$a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots=\sum_{n=1}^{\infty} ar^{n-1} \quad a\neq 0$$

Each term is obtained from the preceding one by multiplying it by the common ratio r. If $r=1$ then, $s_n=a+a+\cdots+a=na\to\pm$ Since $\lim\limits_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case.
If $r\neq 1$, we have

$$s_n=a+ar+ar^2+ar^3+\cdots+ar^{n-1}$$

and

$$rs_n=ar+ar^2+ar^3+\cdots+ar^{n-1}+ar^n$$

Subtracting these equations, we get

$$\begin{aligned}s_n-rs_n &= a-ar^n \\ s_n &= \frac{a(1-r^n)}{1-r}\end{aligned}$$

If $-1<r<1$

$$\lim_{n\to\infty} s_n=\lim_{n\to\infty}\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{a}{1-r}\lim_{n\to\infty} r^n=\frac{a}{1-r}$$

We summarize the results of Example 2.2 as follows.

2.0.2 The geometric series

Definition 2.0.2

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} \frac{a}{1-r}$$

If $|r| < 1$,the the geometric series is convergent.
If $|r| \geq 1$,the the geometric series is divergent.

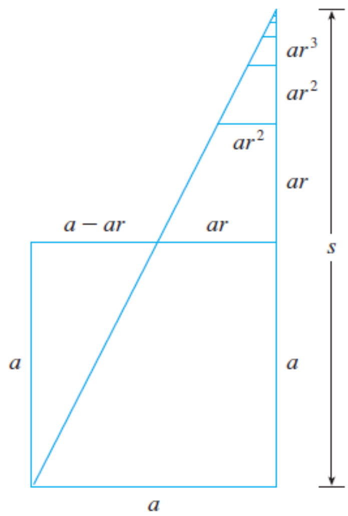


Figure 2.1

[1]

■ **Example 2.3** Find the sum to infinity of the G.P $1 + \frac{1}{2} + \frac{1}{4} + \cdots +$ ■

Solution 2.3 From the above sequence $|r| = \frac{1}{2} < 1$ and $a = 1$

[1]Figure 2.1 provides a geometric demonstration of the result in Example 2.2. If the triangles are constructed as shown and is the sum of the series, then, by similar triangles

$$
\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
&= \frac{1}{1-\frac{1}{2}} \\
&= \frac{1}{\frac{1}{2}} \\
S_\infty &= 2
\end{aligned}
$$

■ **Example 2.4** Find the n^{th} partial sum and determine if the series converges or diverges.

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots+\frac{1}{4^n}$$

■

Solution 2.4

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots+\frac{1}{4^n}=\sum_{k=1}^{n}\left(\frac{1}{4}\right)^n$$

This is a geometric series with ratio $r=\dfrac{1}{4}<1$,therefore it will converge
Now to calculate the sum for this series

$$a=\frac{1}{4},\ r=\frac{1}{4}$$

$$s_\infty=\frac{a}{1-r}=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}$$

■ **Example 2.5** Find the nth partial sum and determine if the series converges or diverges.

$$1-3+9-27+\cdots+(-1)^{n-1}(-3)^{n-1}$$

■

Solution 2.5

$$1-3+9-27+\cdots+(-1)^{n-1}(-3)^{n-1}=\sum_{k=1}^{n}(-1)^{n-1}(-3)^{n-1}$$

This is a geometric series with ratio $|r|=|(-1)(-3)|=|3|\geq 1$ therefore it will diverge.

■ **Example 2.6** Does this series converge or diverge? If it converges, find its sum.

$$\sum_{n=0}^{\infty} e^{-2n}$$

■

Solution 2.6

$$\sum_{n=0}^{\infty} e^{-2n} = \sum_{n=0}^{\infty} \left(e^{-2}\right)^n, \quad r = e^{-2} < 1$$

$$a = 1, \quad s_\infty = \frac{1}{1-e^{-2}} = \frac{e^2}{e^2-1}$$

■ **Example 2.7** Does this series converge or diverge? If it converges, find its sum

$$\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$$

■

Solution 2.7

$$\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n, \quad r = \frac{4}{3} > 1 \quad \text{diverges}$$

$$S_\infty = \frac{\frac{4}{3}}{1-\frac{4}{3}} = \frac{4}{3} \times -3 = -4$$

■ **Example 2.8** Find the values of x for which the geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-3)^n$$

■

Solution 2.8

$$r = -\frac{1}{2}(x-3)$$

$$\left|-\frac{1}{2}(x-3)\right| < 1 \implies \left|\frac{1}{2}(x-3)\right| < 1 \implies -1 < \frac{1}{2}(x-3) < 1$$

$$\implies -2 < x-3 < 2 \implies 1 < x < 5$$

This geometric series will converge for values of x that are in the interval $1 < x < 5$

Now to determine the sum

$$a = 1, s_\infty = \frac{1}{1+\frac{1}{2}(x-3)} = \frac{2}{x-1}$$

■ **Example 2.9** Find the sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$ ■

Solution 2.9 The first term is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the sum to infinity is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - \left(\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$$

■ **Example 2.10** Write the number $2.317\overline{7} = 2.31777777\cdots$ as a ratio of integers. ■

Solution 2.10 $2.31777777\cdots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$

After the first term we have a geometric series with $a = \frac{17}{10^3}$ and $r = \frac{1}{10^2}$

Therefore

$$2.31777777\cdots = 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{17}{990} = \frac{1147}{495}$$

■ **Example 2.11** Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$ ■

Solution 2.11 Notice that this series starts with $n = 0$ and so the first term is$x^0 = 1$. Thus

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with $a = 1$ and $r = x$.Since $|x| < 1$,it converges and gives

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

■ **Example 2.12** Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent? ■

Solution 2.12 Let's rewrite the n^{th} term of the series in the form ar^{n-1} :

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

We recognize this series as a geometric series with $a = 4$ and $r = \frac{4}{3}$. Since $r > 1$,the series diverges by Definition 2.0.2

■ **Example 2.13** Show that the series $\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum. ■

Solution 2.13 This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.
We can simplify this expression if we use the partial fraction decomposition

$$\begin{aligned}
\frac{1}{n(n+1)} &= \frac{A}{n} + \frac{B}{n+1} \\
1 &= A(n+1) + Bn \\
1 &= An + A + Bn \\
1 &= n(A+B) + A \\
\text{By comparison} & \\
A &= 1 \\
B &= -1
\end{aligned}$$

$$s_n = \sum_{n=0}^{\infty} \frac{1}{n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n} + \frac{-1}{n+1}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 - \frac{1}{n+1}\right)$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Note 2.2 Notice that the terms cancel in pairs. This is an example of a telescoping sum: Because of all the cancellations, the sum collapses (like a pirate's collapsing telescope) into just two terms.

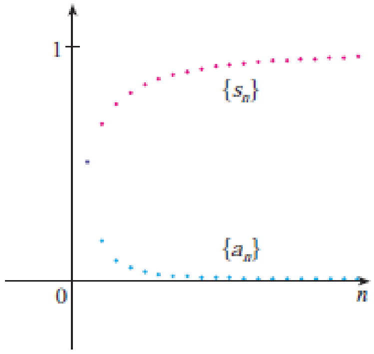


Figure 2.2

R Figure 2.2 illustrates Example 2.8 by showing the graphs of the sequence of terms $a_n = \dfrac{1}{n(n+1)}$ and the sequence s_n of partial sums. Notice that $a_n \to 0$ and $s_n \to 1$

■ **Example 2.14** Determine if the following series converges or diverges. If it converges find its value.

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$$

■

Solution 2.14 We first need the partial sums for this series

$$s_n = \frac{1}{n^2+3n+2}$$

$$\frac{1}{n^2+3n+2} = \frac{1}{(n+2)(n+1)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$1 = A(n+2) + B(n+1)$$

$$\implies \quad A = 1 \quad B = -1$$

$$\frac{1}{(n+2)(n+1)} = \frac{1}{n+1} - \frac{1}{n+2}$$

Next, let's start writing out the terms of the general partial sum for this series using the partial fraction form

$$s_n = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= 1 - \frac{1}{n+2}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{1}{n+2}\right) = 1$$

Note 2.3 Notice that every term except the first and last term canceled out. This is the origin of the name telescoping series.

Theorem 2.0.3 If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series

1. $\sum ca_n = c\sum a_n$
2. $\sum(a_n+b_n) = \sum a_n + \sum b_n$
3. $\sum(a_n-b_n) = \sum a_n - \sum b_n$

■ **Example 2.15** Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$ ■

Solution 2.15 The series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a geometric series with $a=\frac{1}{2}$ and $r=\frac{1}{2}$ so

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

In Example 2.6 we found that

$$\sum_{n=0}^{\infty}\frac{1}{n(n+1)} = 1$$

So, by Theorem 2.1.1, the given series is convergent and

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right) = 3\sum_{n=0}^{\infty}\frac{1}{n(n+1)}+\sum\frac{1}{2^n} = 3\cdot 1+1 = 4$$

Note 2.4 A finite number of terms doesn't affect the convergence or divergence of a series.

■ **Example 2.16** Write out the first few terms or the following series to show how the series starts. Then find the sum of the series.

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)$$

■

Solution 2.16

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)=(5-1)+\left(\frac{5}{2}-\frac{1}{3}\right)+\left(\frac{5}{4}-\frac{1}{9}\right)+\left(\frac{5}{8}-\frac{1}{27}\right)+\cdots+$$

Now to determine the sum of this series. To do this, I will split the original sum into a difference of two sums.

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}\right):\ a=5,\ r=\frac{1}{2},\ s_\infty=\frac{5}{1-\frac{1}{2}}=10$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{3^n}\right):\ a=1,\ r=\frac{1}{3},\ s_\infty=\left(\frac{1}{1-\frac{1}{3}}\right)=\frac{3}{2}$$

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}\right)-\sum_{n=0}^{\infty}\left(\frac{1}{3^n}\right)=10-\frac{3}{2}=\frac{17}{2}$$

2.1 Convergence of a series

2.1.1 Divergence Test

Definition 2.1.1 If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent.

■ **Example 2.17** Determine the behavior of the series $\sum_{n=0}^{\infty}\frac{4n^2-n^3}{10+2n^3}$ by using the divergence test ■

Solution 2.17

$$= \lim_{n\to\infty} \frac{4n^2 - n^3}{10 + 2n^3}$$

$$= \lim_{n\to\infty} \frac{\frac{4n^2}{n^3} - \frac{n^3}{n^3}}{\frac{10}{n^3} + \frac{2n^3}{n^3}}$$

$$= \lim_{n\to\infty} \frac{\frac{4}{n} - 1}{\frac{10}{n^3} + 2} = -\frac{1}{2} \neq 0$$

$$\text{hence} \quad \sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3} \quad \text{Diverges}$$

■ **Example 2.18** Examine the behavior of the series $\sum_{n=0}^{\infty} \ln\left(\frac{1}{n}\right)$ by using the divergence test. ■

Solution 2.18

$$\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = \ln\left(\frac{1}{\infty}\right) = \ln(0) = -\infty$$

Hence

$$\sum_{n=0}^{\infty} \ln\left(\frac{1}{n}\right) \quad \text{is} \quad \text{divergent}$$

■ **Example 2.19** Examine the behavior of the series $\sum_{n=0}^{\infty} \frac{n}{2n+1}$ by using the divergence test.

■

Solution 2.19

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{n}{2n+1} \\
&= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{2n}{n}+\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} \neq 0 \\
\text{hence } & \sum_{n=0}^{\infty} \frac{n}{2n+1} \quad is \text{ Divergent}
\end{aligned}$$

2.1.2 Polynomial Test

Definition 2.1.2 Let the series $\sum_{n=0}^{\infty} \frac{P_n}{Q_n}$ be one whose terms are positive and equal to the ratios of polynomial in such a way that $p = \text{degree}(P_n)$ and $q = \text{degree}(Q_n)$. Then the series $\sum_{n=0}^{\infty} \frac{P_n}{Q_n}$ converges if and only if $q-p>1$ and if $q-p<1$, $\sum_{n=0}^{\infty} \frac{P_n}{Q_n}$ diverges.

Note 2.5 Find the difference between the degree of q and p , if $q-p>1$ then it converges or $q-p<1$ then it diverges.

■ **Example 2.20** Examine the behavior of the series $\sum_{n=1}^{\infty} \frac{6n^3-2n+1}{5n^2-n+1}$ by using the polynomial test. ■

Solution 2.20

$$\begin{aligned}
p = 3 \quad &, q = 2 \\
q - p &= 2 - 3 \\
&= -1 < 1 \\
\sum_{n=1}^{\infty} \frac{6n^3-2n+1}{5n^2-n+1} & \quad \text{will diverge}
\end{aligned}$$

■ **Example 2.21** Examine the behavior of the series $\sum_{n=1}^{\infty} \frac{n+2}{n^4+n+2}$ by using the polynomial test.
■

Solution 2.21

$$q = 4 \quad p = 1$$
$$q - p = 4 - 1$$
$$3 > 1$$
$$\sum_{n=1}^{\infty} \frac{n+2}{n^4+n+2} \ \textit{converges}$$

2.1.3 Integral Test

Definition 2.1.3 Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:

1. If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
2. $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent

■ **Example 2.22** Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence. ■

Solution 2.22 The function $f(x) = \frac{1}{x^2+1}$ is continuous,positive and decreasing on $[1,\infty)$ so we use the Integral Test:

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\, dx = \frac{\pi}{4}$$

The integral is convergent and so the series is also convergent by the Integral Test.

Example 2.23 Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution 2.23 In this case the function we'll use is,

$$f(x) = \frac{1}{x \ln x}$$

This function is clearly positive and if we make x larger the denominator will get larger and so the function is also decreasing. Therefore, all we need to do is determine the convergence of the following integral.

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x \ln x}\, dx$$

Using method of substitution $u = \ln x$

$$\lim_{b \to \infty} [\ln(\ln x)]_2^b$$

$$\lim_{b \to \infty} ((\ln(\ln b)) - (\ln(\ln 2)))$$

$$= \infty$$

The integral is divergent and so the series is also divergent by the Integral Test.

2.1.4 P Series

Definition 2.1.4 Consider for P series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots +$$

Therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ Converges, if $p \geq 1$ and diverges if $0 < p \leq 1$

Note 2.6 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if p is negative.

Example 2.24 Examine the behavior of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution 2.24

$$p = 2 > 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ Converges}$$

■ **Example 2.25** Examine the behavior of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ ■

Solution 2.25

$$p = \frac{1}{3}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} \text{ diverges}$$

2.1.5 Ratio Test

Definition 2.1.5 Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \left(\frac{a_n + 1}{a_n} \right) = L$$

, then

1. $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$
2. $\sum_{n=1}^{\infty} a_n$ diverges if $L > 1$
3. If $L = 1$, no conclusion can be drawn.

Note 2.7 In the case of $L = 1$ the ratio test is pretty much worthless and we would need to resort to a different test to determine the convergence of the series.

■ **Example 2.26** Investigate the behaviour of the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ by using the ratio test. ■

Solution 2.26

$$a_n = \frac{1}{n!}, \quad a_{n+1} = \frac{1}{(n+1)!}$$

$$\lim_{n\to\infty} \left(\frac{1}{(n+1)!} \div \frac{1}{n!} \right)$$

$$\lim_{n\to\infty}\left(\frac{n!}{(n+1)!}\right)$$

$$\lim_{n\to\infty}\frac{n!}{(n+1)!}$$

$$\text{but } (n+1)! = (n+1)n!$$

$$\lim_{n\to\infty}\frac{n!}{(n+1)n!}$$

$$\lim_{n\to\infty}\frac{1}{n+1}$$

$$\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{n}{n}+\frac{1}{n}} = 0$$

$$L = 0 < 1$$

$$\sum_{n=1}^{\infty}\frac{1}{n!} \text{ Converges}$$

■ **Example 2.27** Investigate the behaviour of the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ by using the ratio test. ■

Solution 2.27

$$a_n = \frac{n^n}{n!}, \quad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(n+1)!}\div\frac{n^n}{n!}\right)$$

$$\lim_{n\to\infty}\left[\frac{(n+1)^n(n+1)^1}{(n+1)(n!)}\times\frac{n!}{n^n}\right]$$

$$\lim_{n\to\infty}\left(\frac{(n+1)^n}{n^n}\right)$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$\lim_{n\to\infty}\left(\frac{n}{n}+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$e = 2.71828... > 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ diverges}$$

2.1.6 Cauchy's Root Test

Definition 2.1.6 Let $\sum a_n$ denote the series of positive term and define $L = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$

1. if $L < 1$, converges.
2. if $L > 1$, diverges.
3. if $L = 1$, no conclusion can be drawn.

Example 2.28 Examine the behaviour of the series $\sum_{n=1}^{\infty} \frac{3^n}{4^n}$

Solution 2.28

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n, \; a_n = \left(\frac{3}{4}\right)^n$$

$$\lim_{n\to\infty} \left[\left(\frac{3}{4}\right)^n\right]^{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{3}{4} = \frac{3}{4}$$

$$\frac{3}{4} < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{3^n}{4^n} \quad converges$$

Example 2.29 Examine the behavior or the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

Solution 2.29

$$a_n = \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$$

$$\lim_{n\to\infty} \left[\left(\frac{1}{n} - \frac{1}{n^2}\right)\right]^{\frac{1}{n}}$$

$$\lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0$$

$$= 0 < 1$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2} \right)^n \text{ converges}$$

2.1.7 Comparison Test

Definition 2.1.7 Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \leq b_n, \forall\ n$, then $\sum a_n$ is convergent.
2. If $\sum b_n$ is divergent and $a_n \geq b_n, \forall\ n$, then $\sum a_n$ is also divergent.

■ **Example 2.30** Determine if the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+2}$ converges or diverges. ■

Solution 2.30

$$a_n = \frac{5}{2n^+4n+2}, \quad b_n = \frac{5}{2n^2}$$

$$a_n < b_n$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\text{since} \ \ p = 2 > 1 \ \ (\text{P} \ \ \text{Series})$$

$$b_n = \sum_{n=1}^{\infty} \frac{5}{2n^2} \ ;\text{will converge}$$

$$\text{then } a_n = \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+2} \text{ will also converge because } a_n < b_n$$

■ **Example 2.31** Determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^2-\sin(n)}$ converges or diverges. ■

Solution 2.31

$$a_n = \frac{n}{n^2-\sin(n)}, \quad b_n = \frac{n}{n^2} = \frac{1}{n}$$

$$b_n = \sum_{n=1}^{\infty} \frac{1}{n}, \quad p = 1 \ \ (\text{P} \ \ \text{Series})\ , b_n \text{ will diverges}$$

$$\sum b_n \text{ is divergent} \implies \sum a_n \text{ is also divergent since } a_n > b_n$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2-\sin(n)} \text{ diverges}$$

2.1.8 Alternating Series Test

Definition 2.1.8 Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. Then if,

1. $\lim\limits_{n\to\infty} b_n = 0$
2. $\{b_n\}$ is a decreasing sequence

then series $\sum a_n$ is convergent.

■ **Example 2.32** Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

■

Solution 2.32 First, identify the b_n for the test

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}, \quad b_n = \frac{1}{n}$$

Now, all that we need to do is run through the two conditions in the test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n}$$

$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$$

Both conditions are met and so by the Alternating Series Test the series must converge.

■ **Example 2.33** Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

■

Solution 2.33 The point of this problem is really just to acknowledge that it is in fact an alternating series. To see this we need to acknowledge that,

$$\cos(n\pi) = (-1)^n$$

and so the series is really,

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \implies b_n = \frac{1}{\sqrt{n}}$$

Checking the two condition gives,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}}$$

$$b_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = b_{n+1}$$

The two conditions of the test are met and so by the Alternating Series Test the series is convergent

2.1.9 Exercise

Exercise 2.1 Determine if the following series is convergent or divergent

$$\sum_{n=0}^{\infty} ne^{-n^2}$$

Exercise 2.2 Determine if the following series converges or diverges. If it converges find its value.

$$\sum_{n=0}^{\infty} \frac{1}{n^2+4n+3}$$

Answer: $\frac{5}{12}$

Exercise 2.3 Determine the value of the following series.

$$\sum_{n=0}^{\infty} \frac{4}{n^2+4n+3} - 9^{-n+2}4^{n+1}$$

Answer: $-\frac{3863}{15}$

Exercise 2.4 Use Partial fractions to find the sum of this series

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

Answer: 1

Exercise 2.5 Determine if the following series converge or diverge. If they converge give the value of the series.

1. $\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$
2. $\sum_{n=1}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

Exercise 2.6 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$$

Exercise 2.7 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$$

■

Exercise 2.8 Determine if the following series is convergent or divergent

$$\sum_{n=0}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$$

■

Exercise 2.9 Determine if the following series is convergent or divergent

$$\sum_{n=0}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$$

■

Exercise 2.10 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-12)^n}{n}$$

Hint 2.1 $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

■

Exercise 2.11 Does the following series converge or diverge.

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$$

■

Exercise 2.12 Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$$

■

2.2 Power Series

Definition 2.2.1 A power series about a point a or just power series is any series that can be written in the form:

$$\sum_{n=0}^{\infty} C_n(x-a)^n$$

where a and C_n are numbers.The $C_n{}'s$ are often called the coefficeint of the series. Power series will converge for $|x-a|<R$; and will diverge If $|x-a|>R$. R is called the radius of convergence for the series.

$$|x-a|<R$$

,

$$x-a<R,\ \ x<a+R$$

$$-(x-a)<R,\ \ x-a>-R,\ \ x>a-R$$

$$\boxed{a-R<x<a+R} \rightarrow \text{Interval Of Convergence}$$

Note 2.8 The series may or may not converge if $|x-a|=R$. The interval of all $x's$ including the endpoints if need be, for which the power series convergence is called the interval of convergences.

Example 2.34 Determine the radius of convergence and the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} n(x+3)^n$$

■

Solution 2.34

$$\text{Using the ratio test}$$

$$a_n = \frac{(-1)^n}{4^n} n(x+3)^n$$

$$a_{n+1} = \frac{(-1)^{n+1}}{4^{n+1}}(n+1)(x+3)^{n+1} = \frac{(-1)^n(-1)}{4^n.4}(n+1)(x+3)^n(x+3)$$

$$\lim_{n\to\infty}\left|\frac{(-1)^n(-1)}{4^n.4}(n+1)(x+3)^n(x+3)\times\frac{(4^n)}{(-1)^n n(x+3)}\right|$$

$$\lim_{n\to\infty}\left|\frac{(-1)(n+1)(x+3)}{4n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(n+1)}{4n}(x+3)\right|<1$$

$$\lim_{n\to\infty}\frac{n+1}{4n}|(x+3)|<1$$

$$\text{but we know that } |x-a| < R$$

$$\frac{1}{4}|x+3| < 1$$

$$|x+3| < 4, \ \ |x-(-3)| < 4$$

$$|x-a| < R$$

$$R = 4 \ \text{ radius of convergence.}$$

Now, let's get the interval of convergence. We'll get most (if not all) of the interval by solving the first inequality from above.

$$-4 < x+3 < 4$$

$$-7 < x < 1$$

So, most of the interval of validity is given by $-7 < x < 1$. All we need to do is determine if the power series will converge or diverge at the endpoints of this interval. Note that these values of x will correspond to the value of x that will give $L = 1$.
The way to determine convergence at these points is to simply plug them into the original power series and see if the series converges or diverges using any test necessary.
For $x = -7$:
In this case the series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} n(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} n(-1)^n(4)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n(-1)^n n = \sum_{n=1}^{\infty} n$$

This series is divergent by the Divergence Test since $\lim\limits_{n\to\infty} n = \infty \neq 0$
For $x = 1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} n(4)^n = \sum_{n=1}^{\infty} (-1)^n n$$

This series is also divergent by the Divergence Test since $\lim\limits_{n\to\infty} (-1)^n n = \infty$ doesn't exist.
So, in this case the power series will not converge for either endpoint. The interval of convergence is then,

$$-7 < x < 1$$

■ **Example 2.35** Determine the radius of convergence and the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

■

Solution 2.35

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$a_n = \frac{(x-3)^n}{n}$$

$$a_{n+1} = \frac{(x-3)^{n+1}}{n+1}$$

$$= \frac{(x-3)^n(x-3)}{n+1}$$

$$\left|\frac{(x-3)^n(x-3)}{n+1}\times\frac{n}{(x-3)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-3)}{n+1}n\right|$$

$$|x-3|\lim_{n\to\infty}\frac{n}{n+1}$$

$$|x-3|.1 < 1$$

$$|x-a| < R$$

$$R = 1,\ \ a = 3$$

$$|x-3| < 1$$

$$R = 1\ \ a = 3$$

$$a-R < x < a+R$$

$$3-1 < x < 3+1$$

$$2 < x < 4$$

Test for extremes

For $x = 2$:

$$\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$$

$$\sum_{n=1}^{\infty}\frac{(2-3)^n}{n}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

Using the alternating series test

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n},\ \ b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges

For $x = 4:$

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

using P-series

$$P = 1, \ \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges or because its a harmonic series it diverges.

2.2.1 Exercise

Exercise 2.13 For what values of is the series $\sum_{n=0}^{\infty} n!x^n$ convergent? ■

Exercise 2.14 For what values of does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge? ■

Exercise 2.15 Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

■

Exercise 2.16 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

■

Exercise 2.17 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

■

Exercise 2.18 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n}$$

■

Exercise 2.19 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n}$$

■

Part Three

3. CONIC SECTION

In this section we give geometric denitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called conic sections, or conics, because they result from intersecting a cone with a plane as shown in Figure 3.1.

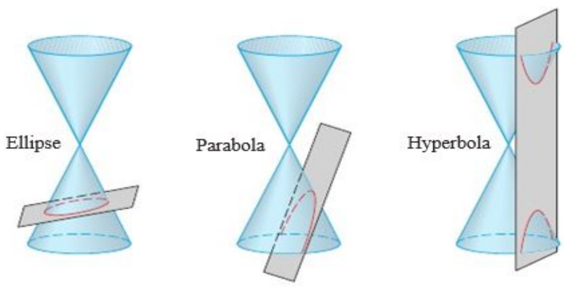


Figure 3.1

3.1 Parabolas

Definition 3.1.1 A parabola is the set of points in a plane that are equidistant from a fixed point F (called the focus) and a fixed line (called the directrix). This definition is illustrated by Figure 3.2

Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the vertex. The line through the focus perpendicular to the directrix is called the axis of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reecting telescopes, and suspension bridges.

We obtain a particularly simple equation for a parabola if we place its vertex at the origin O and its directrix parallel to the x-axis as in Figure 3.3. If the focus is the point $(0, p)$, then the directrix has the equation yp. If $P(x, y)$ is any point on the parabola, then the distance from P to the focus is

$$|PF| = \sqrt{x^2 + (y-p)^2}$$

and the distance from P to the directrix is $|y+p|$. (Figure 3.3 illustrates the case where $p > 0$.) The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y-p)^2} = |y+p|$$

We get an equivalent equation by squaring and simplifying:

$$x^2 + (y-p)^2 = |y+p| = (y+p)^2$$

$$x^2 + y^2 - 2yp + p^2 = y^2 + 2yp + p^2$$

$$x^2 = 4py$$

An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py \tag{3.1}$$

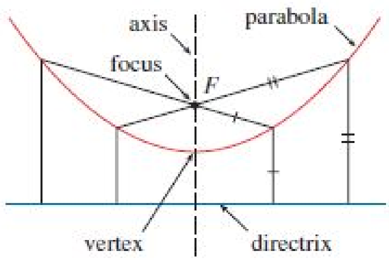


Figure 3.2

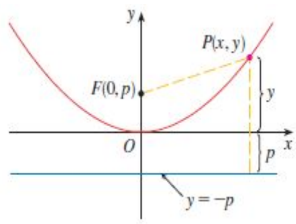


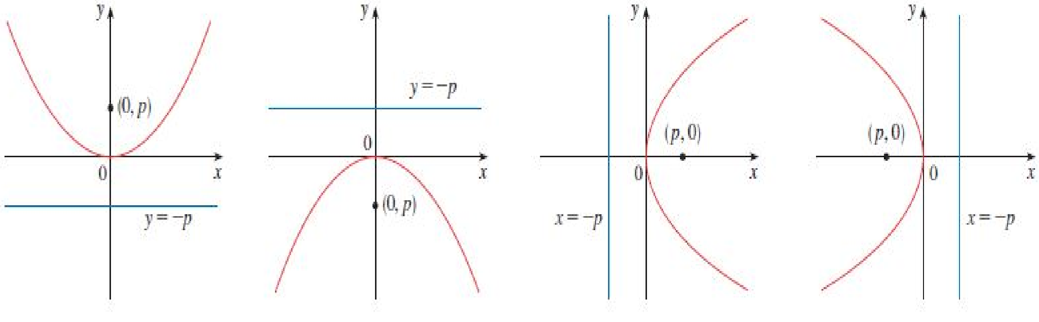
Figure 3.3

If we write $a = \frac{1}{4p}$, then the standard equation of a parabola (2.1) becomes $y = ax^2$. It opens upward if $p > 0$ and downward if $p < 0$ [see Figure 3.3, parts (a) and (b)]. The graph is symmetric with respect to the y-axis because (1) is unchanged when x is replaced by -x. If we interchange x and y in (1), we obtain

Definition 3.1.2

$$y^2 = 4px$$

which is an equation of the parabola with focus (p,0) and directrix $x = -p$. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola opens to the right if $p > 0$ and to the left if $p < 0$ [see Figure 3.4, parts (c) and (d)]. In both cases the



(a) $x^2 = 4py, p > 0$ (b) $x^2 = 4py, p < 0$ (c) $y^2 = 4px, p > 0$ (d) $y^2 = 4px, p < 0$

Figure 3.4

graph is symmetric with respect to the x-axis, which is the axis of the parabola.

■ **Example 3.1** Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph. ■

Solution 3.1 If we write the equation as $y^2 = -10x$ and compare it with equation (22), we see that $4p = -10$, so $p = \dfrac{-5}{2}$, thus the focus is $(p,0) = \left(\dfrac{-5}{2}, 0\right)$ and the directrix is $x = \dfrac{5}{2}$. The sketch is shown in Figure 3.4.

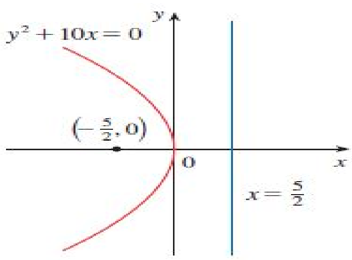


Figure 3.5

3.2 Ellipse

Definition 3.2.1 An ellipse is the set of points in a plane the sum of whose distances from two xed points F_1 and F_2 is a constant (see Figure 3.5). These two xed points are called the foci (plural of focus).

One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the Sun at one focus. In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $(-c,0)$ and $(c,0)$ as in Figure 3.6 so that the origin is halfway between the foci.

Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then $P(x,y)$ is a point on the ellipse when

$$|PF_1| + |PF_2| = 2a$$

$$\text{that is} \qquad \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

$$\text{or} \qquad \sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

Squaring both sides, we have

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to

$$a\sqrt{(x+c)^2+y^2} = a^2 + cx$$

we square again:

$$a^2(x^2+2cx+c^2+y^2) = a^4 + 2a^2cx + c^2x^2$$

which becomes

$$(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2)$$

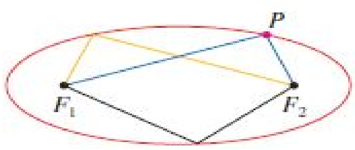


Figure 3.6

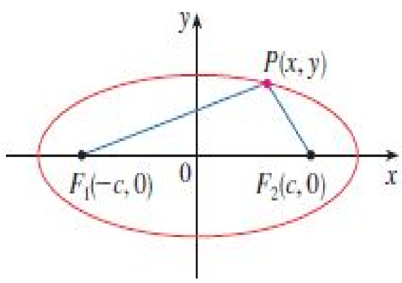


Figure 3.7

From triangle F_1F_2P in Figure 3.6 we see that $2c < 2a$, so $c < a$ and, therefore, $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Then the equation of the ellipse becomes $b^2x^2 + a^2y^2 = a^2b^2$ or, if both sides are divided by a^2b^2,

Definition 3.2.2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since $b^2 = a^2 - c^2 < a^2$ it follows that $b < a$. The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or x^2a^2 so $x = \pm a$. The corresponding points $(a,0)$ and $(-a,0)$ are called the vertices of the ellipse and the line segment joining the vertices is called the major axis. To nd the y-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$. Equation 3 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$. We summarize this discussion as follows.

Definition 3.2.3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b > 0$$

has a foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$

If the foci of an ellipse are located on the y-axis at $(0, \pm c)$,then we can find it equation by interchanging x and y in (Definition 2.3.5).

Definition 3.2.4

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b > 0$$

has a foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$

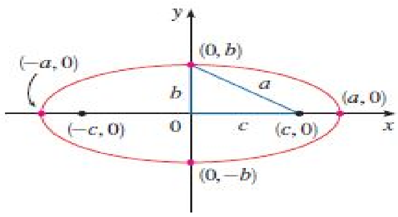


Figure 3.8

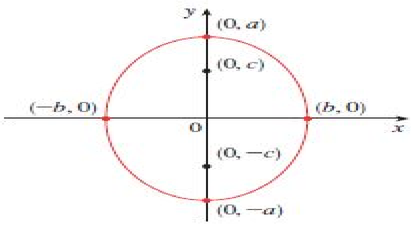


Figure 3.9

■ **Example 3.2** Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci. ■

Solution 3.2 Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16, b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4 and the y-intercepts are ± 3. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $(\sqrt{7}, 0)$. The graph is sketched in Figure 3.10.

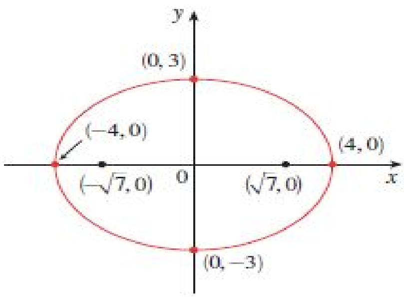


Figure 3.10

■ **Example 3.3** Find an equation of the ellipse with foci $(0, \pm 2)$and vertices $(0, \pm 3)$. ■

Solution 3.3 Using the notation of (5), we have $c = 2$ and $a = 3$. Then we obtain $b^2 = a^2 - c^2 = 9 - 4 = 5$, so the equation of the ellipse is

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Another way of writing the equation is $9x^2 + 5y^2 = 45$

Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus . This principle is used in 1 lithotripsy, a treatment for kidney stones. A reflector with elliptical cross-section is placed in such a way that the kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it without damaging surrounding tissue. The patient is spared the trauma of surgery and recovers within a few days.

3.3 Hyperbolas

Definition 3.3.1 A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in Figure 2.13

Hyperbolas occur frequently as graphs of equations in chemistry, physics, biology, and economics (Boyle's Law, Ohm's Law, supply and demand curves). A particularly significant application of hyperbolas is found in the navigation systems developed in World Wars I and II.

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. In fact, the derivation of the equation of a hyperbola is also similar to the one given earlier for an ellipse. It is left as Exercise to show that when the foci are on the x-axis at $(\pm c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{3.2}$$

where $c^2 = a^2 + b^2$. Notice that the x-intercepts are again $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the vertices of the hyperbola. But if we put $x = 0$ in Equation 2.2 we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 2.2 and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$$

This shows that $x^2 \geq a^2$, so $|x| = \sqrt{x^2} \geq a$. Therefore, we have $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its branches. When we draw a

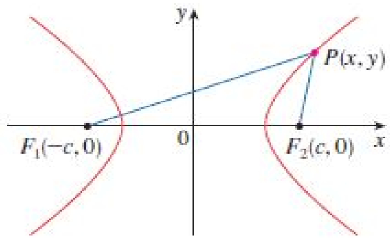


Figure 3.11

hyperbola it is useful to first draw its asymptotes, which are the dashed lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 3.11. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. The hyperbola

Definition 3.3.2

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{3.3}$$

has a foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, and vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

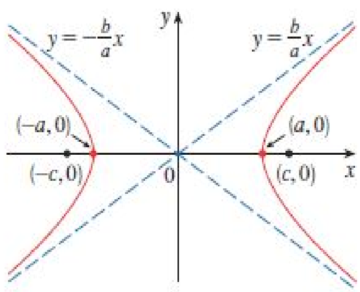


Figure 3.12

If the foci of a hyperbola are on the y-axis, then by reversing the roles of x and y we obtain the following information, which is illustrated in Figure 3.12.

Definition 3.3.3

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{3.4}$$

has a foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, and vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$

■ **Example 3.4** Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph. ■

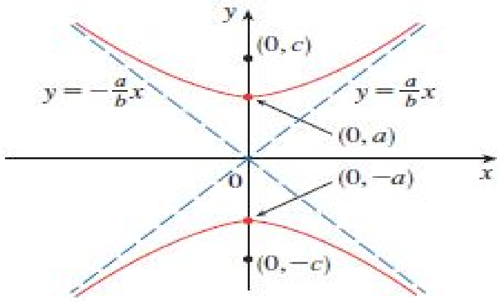


Figure 3.13

Solution 3.4 If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

which is of the form given in (27) with $a = 4$ and $b = 3$. Since $c^2 = 16+9 = 25$, the foci are$(\pm 5, 0)$ .The asymptotes are the lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The graph is shown in Figure 3.14.

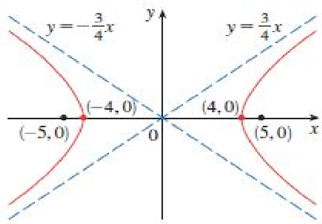


Figure 3.14

■ **Example 3.5** Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$. ■

Solution 3.5 From (28) and the given information, we see that $a = 1$ and $a/b = 2$. Thus,$b = a/2 = \frac{1}{2}$ and $c^2 = a^2 + b^2 = \frac{5}{4}$. The foci are $\left(0, \pm\frac{\sqrt{5}}{2}\right)$ and the equation of the hyperbola is

$$y^2 - 4x^2 = 1$$

3.3.1 Shifted Conics

We shift conics by taking the standard equations and replacing x and y by $x-h$ and $y-k$.

■ **Example 3.6** Find an equation of the ellipse with foci $(2, -2), (4, -2)$ and vertices $(1, -2), (5, -2)$. ■

Solution 3.6 The major axis is the line segment that joins the vertices (1,-2),(5,-2) and has length 4, so $a = 2$. The distance between the foci is 2, so $c = 1$. Thus , $b^2 = a^2 - c^2 = 3$. Since the center of the ellipse is (3,-2), we replace x and y by $x-3$ and $y+2$ to obtain

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

as the equation of the ellipse.

■ **Example 3.7** Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find the foci. ■

Solution 3.7 We complete the squares as follows:

$$4(y^2 - 2y) - 9(x^2 - 8x) = 176$$

$$4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) = 176 + 4 - 144$$
$$4(y-1)^2 - 9(x-4)^2 = 36$$
$$\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$$

Thus , $a^2 = 9$, $b^2 = 4$, and $c^2 = 13$. The hyperbola is shifted four units to the right and one unit upward. The foci are $(4, 1+\sqrt{13})$ and $(4, 1-\sqrt{13})$ and the vertices are $(4,4)$ and $(4,-2)$. The asymptotes are $y - 1 = \pm\frac{3}{2}(x-4)$. The hyperbola is sketched in Figure 3.15.

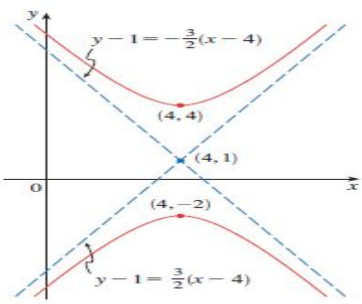


Figure 3.15

3.3.2 Modeling the Real World

■ **Example 3.8** A concrete bridge is designed with an arch in the shape of a parabola. The road over the bridge is 120 feet long and the maximum height of the arch is 50 feet. Write an equation for the parabolic arch. ■

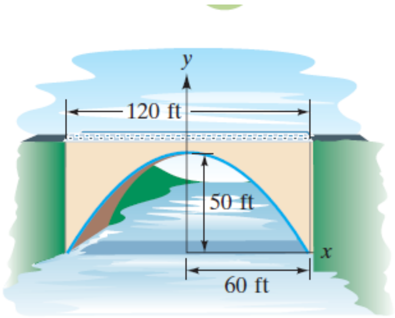


Figure 3.16

Solution 3.8 Graph the curve and label the vertex and both x-intercepts.

According to the figure, the vertex of the parabola is located at its maximum height, or $(0,50)$ .The x-intercepts are $(60,0)$ and $(-60,0)$ We need only one x-intercept to find an equation, so we will use $(60,0)$. Substitute 0 for h, 50 for k, 60 for x, and 0 for y in the equation for a vertical parabola, and solve for a.

$$y = a(x-h)^2 + k$$

$$0 = a(60-0)^2 + 50$$

$$0 = 3600a + 50$$

$$a = \frac{-1}{72}$$

We will limit our graph to a portion of an arch by restricting the domain to $-60 \leq x \leq 60$
An equation for the parabola is $y = \frac{-1}{72}(x-0)^2 + 50$

■ **Example 3.9** A satellite dish receiver is in the shape of a parabola. A cross section of the dish shows a diameter of 13 feet at a distance of 2.5 feet from the vertex of the parabola. Write an equation for the parabola. ■

Solution 3.9 Substitute 0 for h, 0 for k, 2.5 for x, and 6.5 for y in the equation for a horizontal parabola, and solve for a.

$$x = a(y-k)^2 + h$$

$$2.5 = a(6.5-0)^2 + 0$$

$$2.5 = 42.25a$$

$$a = \frac{10}{169}$$

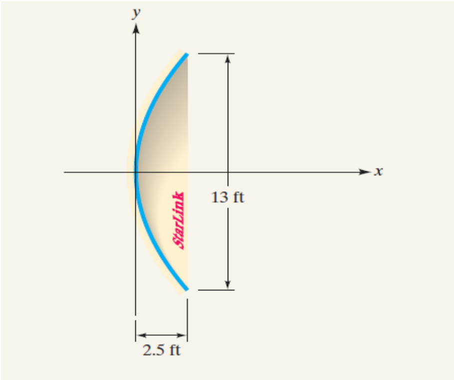


Figure 3.17

Since our object is not modeled by the complete graph, we need to limit the x-values by restricting the domain. An equation for the parabola is $x = \dfrac{10}{169}y^2$ where $-6.5 \leq x \leq 6.5$

3.4 Exercise

Exercise 3.1 Find the vertex, focus and directrix of the parabola and sketch its graph.

1. $x = 2y^2$
2. $4y + x^2 = 0$
3. $y^2 + 2y + 12x + 25 = 0$
4. $(x-2)^2 = 8(y-3)$

Exercise 3.2 An arched underpass has the shape of a parabola. A road passing under the arch is 25 feet wide, and the maximum height of the arch is 15 feet. Write an equation for the parabolic arch.

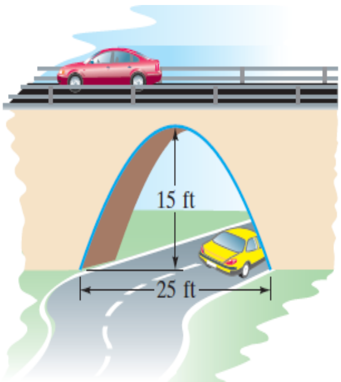


Figure 3.18

Exercise 3.3 Find an equation of the parabola. Then find the focus and directrix of Figures 3.19 and 3.20.

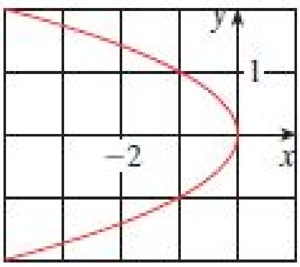


Figure 3.19

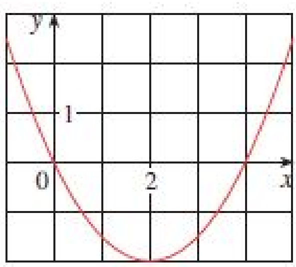


Figure 3.20

Exercise 3.4 Find the vertices and foci of the ellipse and sketch its graph.

1. $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

2. $9x^2 - 18x + 4y^2 = 27$

Exercise 3.5 Identify the conic section whose equation is given and find the vertices and foci.

1. $x^2 = y + 1$
2. $4x^2 + 4x + y^2 = 0$
3. $y^2 - 8y = 6x - 16$

Exercise 3.6 A soup bowl has a cross section with a parabolic shape, as shown in the figure. The bowl has a diameter of 8 inches and is 2.5 inches deep. Write an equation for its shape.

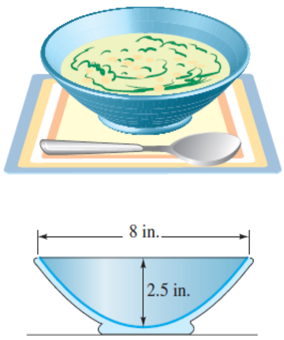


Figure 3.21

Exercise 3.7 Find an equation for the conic that satisfies the given conditions.

1. Parabola, vertex $(0,0)$, foci $(0,-2)$.
2. Parabola, focus $(-4,0)$, directrix $x = 2$
3. Ellipse, foci$(\pm 2,0)$, vertex $(\pm 5,0)$.
4. Hyperbola, foci $(0,\pm 3)$, vertices $(0,\pm 1)$.

Exercise 3.8 Many railroad viaducts are constructed in the shape of a semicircle. A stone-arch railroad viaduct at Rockville, Pennsylvania, over the Susquehanna River is made of 48 semicircular arches, each with a span of 70 feet. Use Figure 3.22 to write equations that model each of the first two arches. ■

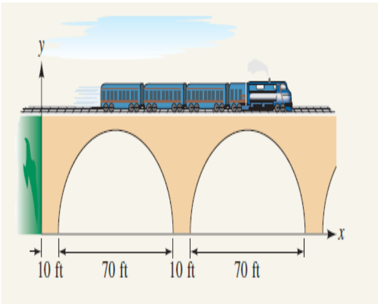
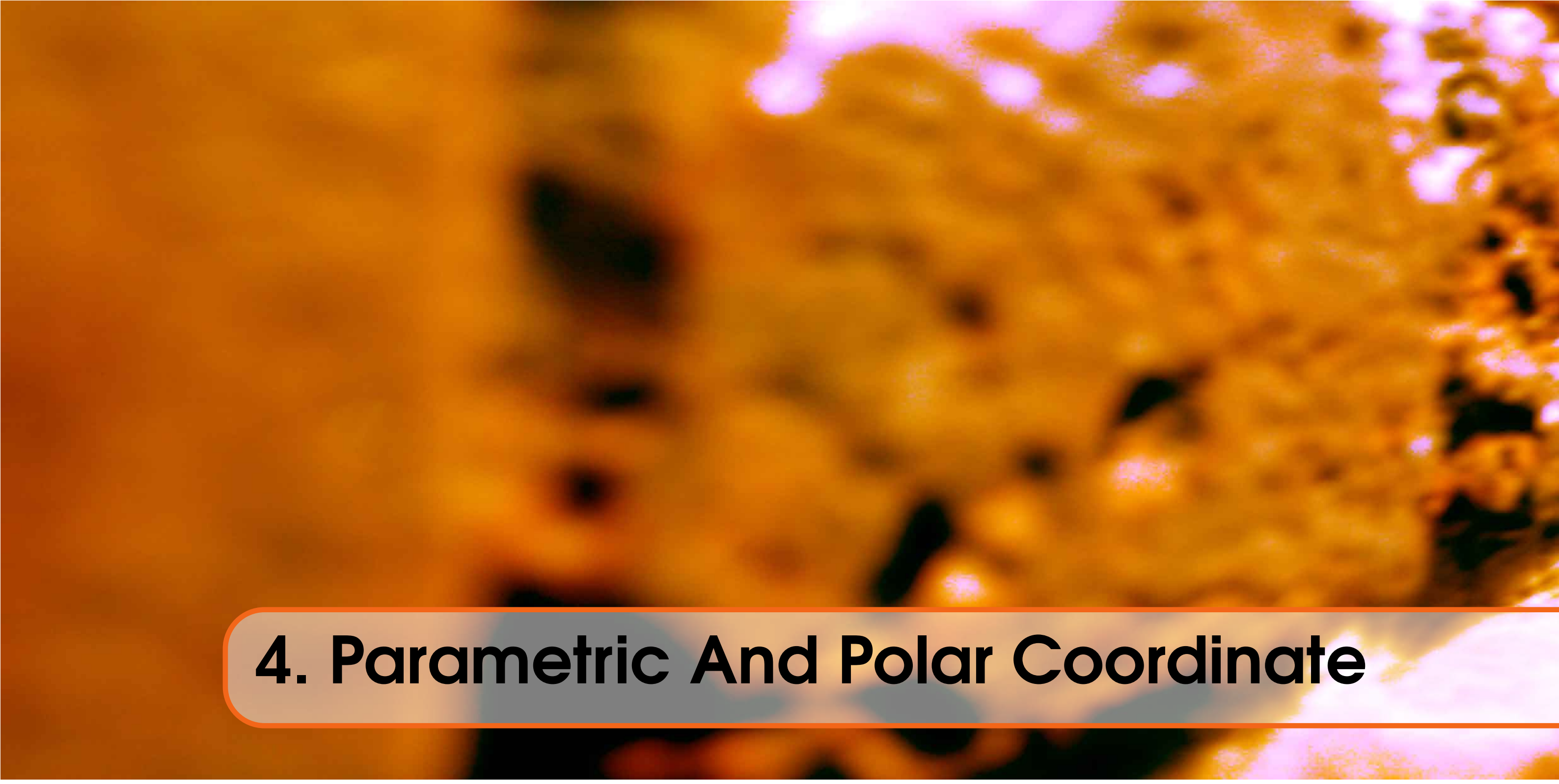


Figure 3.22



4. Parametric And Polar Coordinate

4.1 Parametric Equations

In this section we will be looking at parametric equations and polar coordinates. While the two subjects don't appear to have that much in common on the surface we will see that several of the topics in polar coordinates can be done in terms of parametric equations and so in that sense they make a good match in this chapter.

4.1.1 Parametric Equations and Curves

Definition 4.1.1 To this point we've looked almost exclusively at functions in the form $y = f(x)$ or $x = h(y)$ and almost all of the formulas that we've developed require that functions be in one of these two forms. The problem is that not all curves or equations that we'd like to look at fall easily into this form.
Take, for example, a circle. It is easy enough to write down the equation of a circle centered at the origin with radius r.

$$x^2 + y^2 = r^2$$

However, we will never be able to write the equation of a circle down as a single equation. Sure we can solve for x or y as the following two formulas show

$$y = \pm\sqrt{r^2 - x^2} \qquad\qquad x = \pm\sqrt{r^2 - y^2}$$

but there are in fact two functions in each of these. Each formula gives a portion of the circle.

$$y = \sqrt{r^2 - x^2} \quad (top) \qquad\qquad x = \sqrt{r^2 - y^2} \quad (right\ \ side)$$
$$y = -\sqrt{r^2 - x^2} \quad (bottom) \qquad\qquad x = -\sqrt{r^2 - y^2} \quad (left\ \ side)$$

Unfortunately we usually are working on the whole circle, or simply can't say that we're going to be working only on one portion of it. Even if we can narrow things down to only one of these portions the function is still often fairly unpleasant to work with. There are also a great many curves out there that we can't even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce parametric equations. Instead of defining y in terms of $x(y = f(x))$ or x in terms of $y(x = h(y))$ we define both x and y in terms of a third variable called a parameter as follows,

$$x = f(t) \qquad y = g(t)$$

This third variable is usually denoted by t (as we did here) but doesn't have to be of course. Sometimes we will restrict the values of t that we'll use and at other times we won't. This will often be dependent on the problem and just what we are attempting to do.
Each value of t defines a point $(x,y) = (f(t),g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the parametric curve.

Sketching a parametric curve is not always an easy thing to do. Let's take a look at an example to see one way of sketching a parametric curve. This example will also illustrate why this method is usually not the best

■ **Example 4.1** Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1$$

■

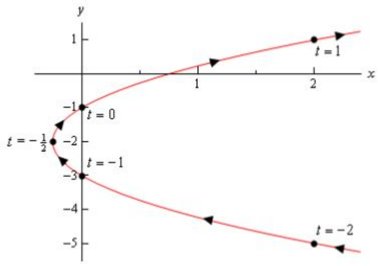
Solution 4.1 At this point our only option for sketching a parametric curve is to pick values of t, plug them into the parametric equations and then plot the points. So, let's plug in some t's.

t	x	y
-2	2	-5
-1	0	-3
$-\frac{1}{2}$	$-\frac{1}{4}$	-2
0	0	-1
1	2	1

So, it looks like we have a parabola that opens to the right.

Here is the sketch of this parametric curve.

■ **Example 4.2** Eliminate the parameter from the following set of parametric equations.
$x = t^2 + t \qquad y = 2t - 1$ ■



Solution 4.2 One of the easiest ways to eliminate the parameter is to simply solve one of the equations for the parameter (t, in this case) and substitute that into the other equation. Note that while this may be the easiest to eliminate the parameter, it's usually not the best way as we'll see soon enough.
In this case we can easily solve y for t.

$$t = \frac{1}{2}(y+1)$$

Plugging this into the equation for x gives,

$$x = \left(\frac{1}{2}(y+1)\right)^2 + \frac{1}{2}(y+1) = \frac{1}{4}y^2 + y + \frac{3}{4}$$

Sure enough from our Algebra knowledge we can see that this is a parabola that opens to the right.

4.1.2 Tangents with Parametric Equations

Definition 4.1.2 In this section we want to find the tangent lines to the parametric equations given by,

$$x = f(t) \qquad\qquad y = g(t)$$

To do this let's first recall how to find the tangent line to $y = F(x)$ at $x = a$. Here the tangent line is given by,

$$y = F(a) + m(x-a), where\ \ m = \left.\frac{dy}{dx}\right|_{x=a} = F'(a)$$

Now, notice that if we could figure out how to get the derivative $\frac{dy}{dx}$ from the parametric equations we could simply reuse this formula since we will be able to use the parametric equations to find the x and y coordinates of the point. So, just for a second let's suppose that we were able to eliminate the parameter from the parametric form and write the parametric equations in the form $y = F(x)$. Now, plug the parametric equations in for x and y. Yes, it seems silly to eliminate the parameter, then immediately put it back in, but it's what we need to do in order to get our hands on the derivative. Doing this gives,

$$g(t) = F(f(t))$$

Now, differentiate with respect to t and notice that we'll need to use the Chain Rule on the right hand side.

$$g'(t) = F'(f(t))f'(t)$$

Let's do another change in notation. We need to be careful with our derivatives here. Derivatives of the lower case function are with respect to t while derivatives of upper case functions are with respect to x. So, to make sure that we keep this straight let's rewrite things as follows.

$$\left.\frac{dy}{dt}\right| = F'(x)\frac{dx}{dt}$$

At this point we should remind ourselves just what we are after. We needed a formula for $\dfrac{dy}{dx}$ or $F'(x)$ that is in terms of the parametric formulas. Notice however that we can get that from the above equation.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad provided\ \frac{dx}{dt} \neq 0$$

Notice as well that this will be a function of t and not x.
As an aside, notice that we could also get the following formula with a similar derivation if we needed to,derivative for Parametric equations

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}, \quad provided\ \frac{dy}{dt} \neq 0$$

■ **Example 4.3** Find the tangent line(s) to the parametric curve given by
$x = t^5 - 4t^3 \qquad y = t^2$ at (0,4). ■

Solution 4.3 Note that there is apparently the potential for more than one tangent line here! We will look into this more after we're done with the example
The first thing that we should do is find the derivative so we can get the slope of the tangent line

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

At this point we've got a small problem. The derivative is in terms of t and all we've got is an $x - y$ coordinate pair. The next step then is to determine the value(s) of t which will give this point. We find these by plugging the x and y values into the parametric equations and solving for t.

$$0 = t^5 - 4t^3 = t^3(t^2 - 4) \qquad \Rightarrow \qquad t = 0, \pm 2$$

$$4 = t^2 \qquad \Rightarrow \qquad t = \pm 2$$

Any value of t which appears in both lists will give the point. So, since there are two values of t that give the point we will in fact get two tangent lines.
$t = -2$
Since we already know the x and y-coordinates of the point all that we need to do is find the slope of the tangent line.

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$$

The tangent line (at $t = -2$) is then,

$$y = 4 - \frac{1}{8}x$$

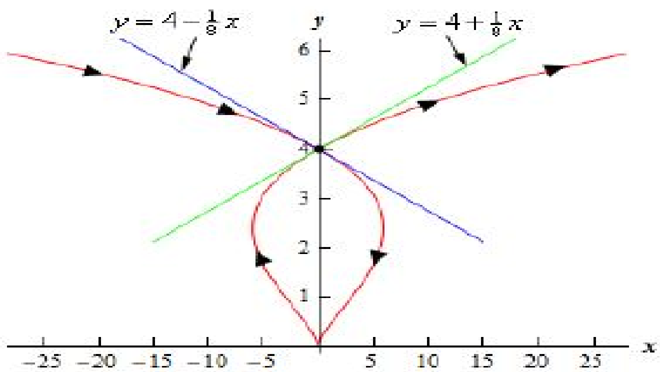
$t = 2$
Again, all we need is the slope.

$$m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$$

The tangent line (at $t = 2$) is then,

$$y = 4 + \frac{1}{8}x$$

Now, let's take a look at just how we could possibly get two tangents lines at a point. This was definitely not possible back in Calculus I where we first ran across tangent lines.
A quick graph of the parametric curve will explain what is going on here.



So, the parametric curve crosses itself! That explains how there can be more than one tangent line. There is one tangent line for each instance that the curve goes through the point.
The next topic that we need to discuss in this section is that of horizontal and vertical tangents. We can easily identify where these will occur (or at least the t's that will give

them) by looking at the derivative formula.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Horizontal tangents will occur where the derivative is zero and that means that we'll get horizontal tangent at values of t for which we have,

4.1.3 Horizontal Tangent for Parametric Equations

Definition 4.1.3

$$\frac{dy}{dt} = 0, \qquad \text{Provided } \frac{dx}{dt} \neq 0$$

Vertical tangents will occur where the derivative is not defined and so we'll get vertical tangents at values of t for which we have

4.1.4 Vertical Tangent for Parametric Equations

Definition 4.1.4

$$\frac{dx}{dt} = 0, \qquad \text{Provided } \frac{dy}{dt} \neq 0$$

Let's take a quick look at an example of this.

■ **Example 4.4** Determine the x-y coordinates of the points where the following parametric equations will have horizontal or vertical tangents.

$$x = t^3 - 3t \quad y = 3t^2 - 9$$

■

Solution 4.4 We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = 3t^2 - 3 = 3(t^2 - 1) \quad \frac{dy}{dt} = 6t$$

Horizontal Tangents: We'll have horizontal tangents where, $6t = 0 \implies t = 0$

Vertical Tangents: In this case we need to solve, where,$3(t^2 - 1) = 0 \implies \quad t = \pm 1$

The two vertical tangents will occur at the points $(2, -6)$ and $(-2, -6)$. For the sake of completeness and at least partial verification here is the sketch of the parametric curve.

Now let's move onto the final topic of this section. We would also like to know how to get the second derivative of y with respect to x.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Note that,

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$$

■ **Example 4.5** Find the second derivative for the following set of parametric equations.

$$x = t^5 - 4t^3 \qquad y = t^2$$

■

Solution 4.5 This is the set of parametric equations that we used in the first example and so we already have the following computations completed

$$\frac{dy}{dt} = 2t \qquad \frac{dx}{dt} = 5t^4 - 12t^2 \qquad \frac{dy}{dx} = \frac{2}{5t^3 - 12t}$$

We will first need the following

$$\frac{d}{dt}\left(\frac{2}{5t^3 - 12t}\right) = \frac{-2(15t^2 - 12)}{(5t^3 - 12t)^2} = \frac{24 - 30t^2}{(5t^3 - 12t)^2}$$

The second derivative is then,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{24 - 30t^2}{(5t^3 - 12t)^2}}{5t^4 - 12t^2}$$

$$= \frac{24 - 30t^2}{(5t^3 - 12t)^2(5t^4 - 12t^2)}$$

$$= \frac{24 - 30t^2}{t(5t^3 - 12t)^3}$$

4.1.5 Area With Parametric Equations

Definition 4.1.5 In this section we will find a formula for determining the area under a parametric curve given by the parametric equations,

$$x = f(t) \qquad y = g(t)$$

We will also need to further add in the assumption that the curve is traced out exactly once as t increases from α to β.
We will do this in much the same way that we found the first derivative in the previous

section. We will first recall how to find the area under $y = F(x)$ on $a \leq x \leq b$.

$$A = \int_a^b F(x)dx$$

We will now think of the parametric equation $x = f(t)$ as a substitution in the integral. We will also assume that $a = f(\alpha)$ and $b = f(\beta)$ for the purposes of this formula. There is actually no reason to assume that this will always be the case and so we'll give a corresponding formula later if it's the opposite case $(b = f(\alpha)$ and $a = f(\beta))$
So, if this is going to be a substitution we'll need,

$$dx = f'(t)dt$$

Plugging this into the area formula above and making sure to change the limits to their corresponding t values gives us,

$$A = \int_\alpha^\beta F(f(t))f'(t)dt$$

Since we don't know what F(x) is we'll use the fact that

$$y = F(x) = F(f(t)) = g(t)$$

and we arrive at the formula that we want.

4.1.6 Area Under Parametric Curve, Formula I

Definition 4.1.6

$$A = \int_\alpha^\beta g(t)f'(t)dt$$

Now, if we should happen to have $b = f(\alpha)$ and $a = f(\beta)$ the formula would be,

4.1.7 Area Under Parametric Curve, Formula II

Definition 4.1.7

$$A = \int_\beta^\alpha g(t)f'(t)dt$$

■ **Example 4.6** Determine the area under the parametric curve given by the following parametric equations.

$$x = 6(\theta - sin\theta) \qquad y = 6(1 - cos\theta) \qquad 0 \neq \theta \neq 2\pi$$

■

Solution 4.6 First, notice that we've switched the parameter to θ for this problem. This is to make sure that we don't get too locked into always having t as the parameter.

$$\frac{dx}{d\theta} = 6(1 - cos\theta)$$

The area is then,

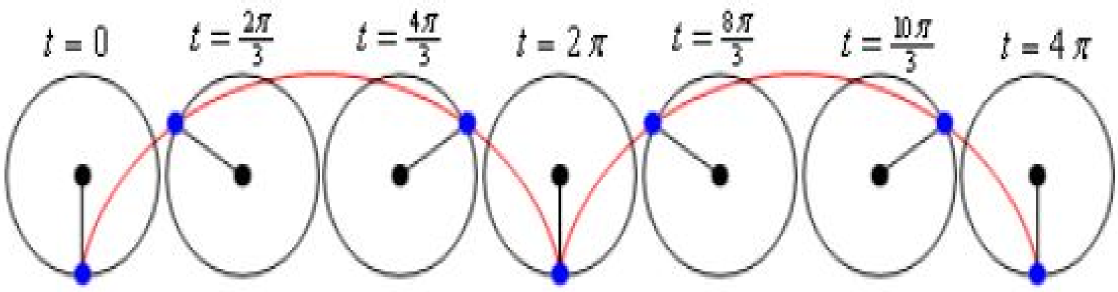
$$A = \int_0^{2\pi} 36(1 - cos\theta)^2 d\theta$$
$$= 36 \int_0^{2\pi} 1 - 2cos\theta + cos^2\theta d\theta$$
$$= 36 \int_0^{2\pi} \frac{3}{2} - 2cos\theta + \frac{1}{2}cos(2\theta)d\theta$$
$$= 36\left(\frac{3}{2}\theta - 2sin\theta + \frac{1}{4}sin(2\theta)\right)\Big|_0^{2\pi}$$
$$= 108\pi$$

The parametric curve (without the limits) we used in the previous example is called a cycloid. In its general form the cycloid is,

$$x = r(\theta - sin\theta) \qquad y = r(1 - cos\theta)$$

The cycloid represents the following situation. Consider a wheel of radius r. Let the point where the wheel touches the ground initially be called P. Then start rolling the wheel to the right. As the wheel rolls to the right trace out the path of the point P. The path that the point P traces out is called a cycloid and is given by the equations above. In these equations we can think of θ as the angle through which the point P has rotated.
Here is a cycloid sketched out with the wheel shown at various places. The blue dot is the point P on the wheel that we're using to trace out the curve.

From this sketch we can see that one arch of the cycloid is traced out in the range $0 \leq \theta \leq 2\pi$. This makes sense when you consider that the point P will be back on the ground after it has rotated through an angle of 2π.



4.1.8 Arc Length with Parametric Equations

Solution 4.7 In this section we will look at the arc length of the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \leq t \leq \beta$$

We will also be assuming that the curve is traced out exactly once as t increases from

α to β . We will also need to assume that the curve is traced out from left to right as t increases. This is equivalent to saying,

$$\frac{dx}{dt} \geq 0 \qquad\qquad for\ \ \alpha \leq t \neg \beta$$

So, let's start out the derivation by recalling the arc length formula as we first derived it in the arc length section of the Applications of Integrals chapter.

$$L = \int ds$$

Where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \qquad if y = f(x), \alpha \leq x \leq b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \qquad if x = h(y), c \leq x \leq d$$

We will use the first ds above because we have a nice formula for the derivative in terms of the parametric equations (see the Tangents with Parametric Equations section). To use this we'll also need to know that,

$$dx = f'(t)dt = \frac{dx}{dt}dt$$

The arc length formula then becomes

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \frac{dx}{dt}dt = \int_{\alpha}^{\beta} \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}} \frac{dx}{dt}dt$$

This is a particularly unpleasant formula. However, if we factor out the denominator from the square root we arrive at,

$$L = \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \frac{dx}{dt}dt$$

Now, making use of our assumption that the curve is being traced out from left to right we can drop the absolute value bars on the derivative which will allow us to cancel the two derivatives that are outside the square root and this gives,

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

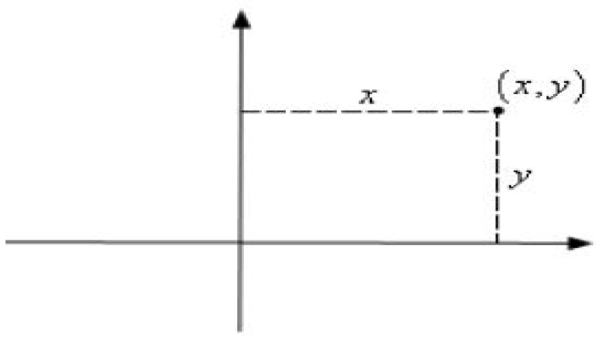
Notice that we could have used the second formula for ds above is we had assumed instead that
$$\frac{dy}{dt} \geq 0 \qquad for \;\; \alpha \leq t \leq \beta$$
If we had gone this route in the derivation we would have gotten the same formula.

■ **Example 4.7** Determine the length of the parametric curve given by the following parametric equations.
$x = 3sin(t) \qquad y = 3cos(t) \qquad 0 \leq t \leq 2\pi$ ■

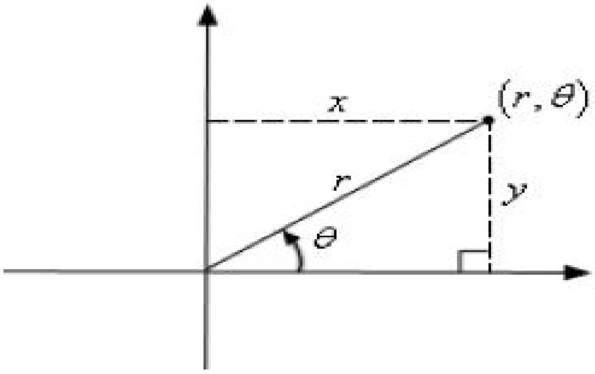
Solution 4.8 We know that this is a circle of radius 3 centered at the origin from our prior discussion about graphing parametric curves. We also know from this discussion that it will be traced out exactly once in this range.
So, we can use the formula we derived above. We'll first need the following,
$$\frac{dx}{dt} = 3cos(t) \qquad \frac{dy}{dt} = -3sin(t)$$
The length is then,
$$L = \int_0^{2\pi} \sqrt{9sin^2(t) + 9cos^2(t)}dt$$
$$= \int_0^{2\pi} 3\sqrt{sin^2(t) + cos^2(t)}dt$$
$$= 3\int_0^{2\pi} dt$$
$$= 6\pi$$

4.1.9 Polar Coordinates

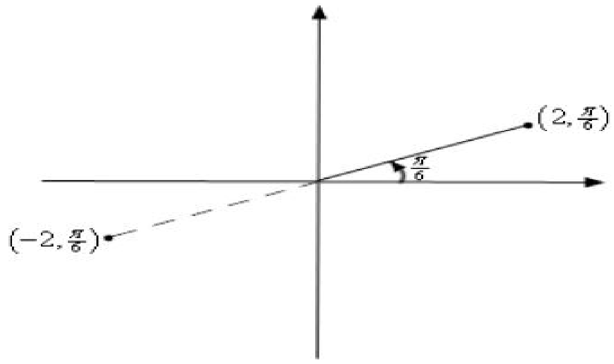
Up to this point we've dealt exclusively with the Cartesian (or Rectangular, or $x - y$) coordinate system. However, as we will see, this is not always the easiest coordinate system to work in. So, in this section we will start looking at the polar coordinate system. Coordinate systems are really nothing more than a way to define a point in space. For instance in the Cartesian coordinate system at point is given the coordinates (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically. This is shown in the sketch below.



This is not, however, the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive x-axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point. This is shown in the sketch below.



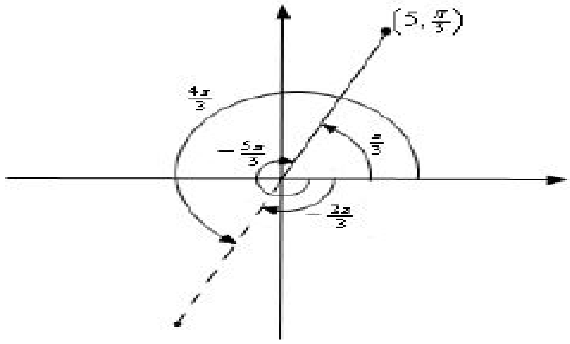
Coordinates in this form are called polar coordinates.
The above discussion may lead one to think that r must be a positive number. However, we also allow r to be negative. Below is a sketch of the two points $\left(2, \frac{\pi}{6}\right)$ and $\left(-2, \frac{\pi}{6}\right)$. From this sketch we can see that if r is positive the point will be in the same quadrant as θ.



On the other hand if r is negative the point will end up in the quadrant exactly opposite θ. Notice as well that the coordinates $\left(-2, \frac{\pi}{6}\right)$ describe the same point as the coordinates $\left(2, \frac{7\pi}{6}\right)$ do. The coordinates $\left(2, \frac{7\pi}{6}\right)$ tells us to rotate an angle of $\frac{7\pi}{6}$ from the positive x-axis, this would put us on the dashed line in the sketch above, and then move out a distance of 2.
This leads to an important difference between Cartesian coordinates and polar coordinates. In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this isn't true. In polar coordinates there is literally an infinite number of coordinates for a given point. For instance, the following four points are all coordinates for the same point.

$$\left(5, \frac{\pi}{3}\right) = \left(5, -\frac{5\pi}{3}\right) = \left(-5, \frac{4\pi}{3}\right) = \left(-5, -\frac{2\pi}{3}\right)$$

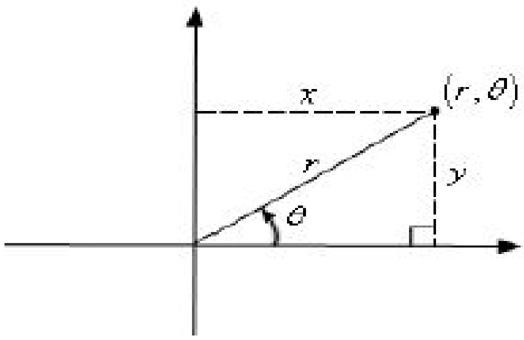
Here is a sketch of the angles used in these four sets of coordinates.



In the second coordinate pair we rotated in a clock-wise direction to get to the point. We shouldn't forget about rotating in the clock-wise direction. Sometimes it's what we have to do.
The last two coordinate pairs use the fact that if we end up in the opposite quadrant from the point we can use a negative r to get back to the point and of course there is both a counter clock-wise and a clock-wise rotation to get to the angle.
These four points only represent the coordinates of the point without rotating around the system more than once. If we allow the angle to make as many complete rotations about the axis system as we want then there are an infinite number of coordinates for the same point. In fact the point (r,θ) can be represented by any of the following coordinate pairs.

$$(r,\theta+2\pi n) \qquad (-r,\theta+(2n+1)\pi), \quad \textit{where n is any integer.}$$

Next we should talk about the origin of the coordinate system. In polar coordinates the origin is often called the pole. Because we aren't actually moving away from the origin/pole we know that $r=0$. However, we can still rotate around the system by any angle we want and so the coordinates of the origin/pole are $(0,\theta)$.
Now that we've got a grasp on polar coordinates we need to think about converting between the two coordinate systems. Well start out with the following sketch reminding us how both coordinate systems work.



Note that we've got a right triangle above and with that we can get the following equations that will convert polar coordinates into Cartesian coordinates.

4.1.10 Polar to Cartesian Conversion Formulas

Definition 4.1.8

$$x = rcos\theta \qquad y = rsin\theta$$

Converting from Cartesian is almost as easy. Let's first notice the following.

$$x^2 + y^2 = (rcos\theta)^2 + (rsin\theta)^2$$

$$= r^2cos^2\theta + r^2sin^2\theta$$

$$r^2(cos^2\theta + sin^2\theta) = r^2$$

Note that technically we should have a plus or minus in front of the root since we know that r can be either positive or negative. We will run with the convention of positive r here.

Getting an equation for θ is almost as simple. We'll start with,

$$\frac{y}{x} = \frac{rsin\theta}{rcos\theta} = tan\theta$$

Taking the inverse tangent of both sides gives,

$$\theta = tan^{-1}\left(\frac{y}{x}\right)$$

We will need to be careful with this because inverse tangents only return values in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Recall that there is a second possible angle and that the second angle is given by $\theta + \pi$.

4.1.11 Cartesian to Polar Conversion Formulas

Definition 4.1.9

$$r^2 = x^2 + y^2 \qquad r = \sqrt{x^2 + y^2}$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right)$$

■ **Example 4.8** Convert $\left(-4, \frac{2\pi}{3}\right)$ into cartesian coordinates. ■

Solution 4.9 $(2, -2\sqrt{3})$

■ **Example 4.9** Convert $(-1, -1)$ into polar coordinates. ■

Solution 4.10 $(r, \theta) = (-\sqrt{2}, \frac{\pi}{4})$

■ **Example 4.10** Convert $2x - 5x^3 = 1 + xy$ into polar coordinates. ■

Solution 4.11 $2rcos\theta - 5r^3cos^3\theta = 1 + r^2cos\theta sin\theta$

Example 4.11 Convert $r = -8cos\theta$ into cartesian coordinates. ■

Solution 4.12 $x^2 + y^2 = -8x$

4.1.12 Tangents with Polar Coordinates

Definition 4.1.10 We will start with finding tangent lines to polar curves. In this case we are going to assume that the equation is in the form $r = f(\theta)$. With the equation in this form we can actually use the equation for the derivative $\frac{dy}{dx}$ we erived when we looked at tangent lines with parametric equations. To do this however requires us to come up with a set of parametric equations to represent the curve. This is actually pretty easy to do.
From our work in the previous section we have the following set of conversion equations for going from polar coordinates to Cartesian coordinates.

$$x = rcos\theta \qquad y = rsin\theta$$

Now, we'll use the fact that we're assuming that the equation is in the form $r = f(\theta)$. Substituting this into these equations gives the following set of parametric equations (with θ as the parameter) for the curve.

$$x = f(\theta)cos\theta \qquad y = f(\theta)sin\theta$$

Now, we will need the following derivatives.

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)cos\theta - f(\theta)sin\theta & \qquad \frac{dy}{d\theta} &= f'(\theta)sin\theta + f(\theta)cos\theta \\ &= \frac{dr}{d\theta}cos\theta - rsin\theta & &= \frac{dr}{d\theta}sin\theta + rcos\theta \end{aligned}$$

The derivative $\frac{dy}{dx}$ is then,

4.1.13 Derivative with Polar Coordinates

Definition 4.1.11

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}sin\theta + rcos\theta}{\frac{dr}{d\theta}cos\theta - rsin\theta}$$

Example 4.12 Determine the equation of the tangent line to $r = 3 + 8sin\theta$ at $\theta = \frac{\pi}{6}$ ■

Solution 4.13 We'll first need the following derivative.

$$\frac{dr}{d\theta} = 8cos\theta$$

The formula for the derivative $\frac{dy}{dx}$ then becomes,

$$\frac{dy}{dx} = \frac{8cos\theta sin\theta + (3+8sin\theta)cos\theta}{8cos^2\theta - (3+8sin\theta)sin\theta} = \frac{16cos\theta sin\theta + 3cos\theta}{8cos^2\theta - 3sin\theta - 8sin^2\theta}$$

The slope of the tangent line is,

$$m = \frac{dy}{dx}\Big|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

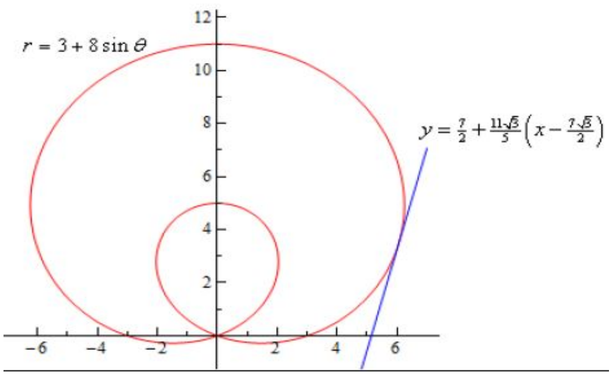
Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$. We'll need to get the corresponding $x-y$ coordinates so we can get the tangent line.

$$x = 7cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad\qquad y = 7sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

For the sake of completeness here is a graph of the curve and the tangent line.

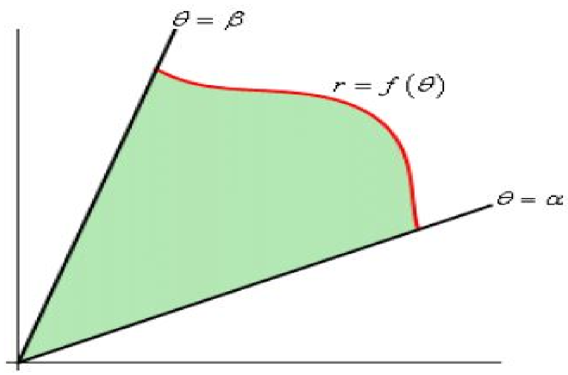


4.1.14 Area with Polar Coordinates

In this section we are going to look at areas enclosed by polar curves. Note as well that we said "enclosed by" instead of “under” as we typically have in these problems. These problems work a little differently in polar coordinates. Here is a sketch of what the area that we'll be finding in this section looks like. We'll be looking for the shaded area in the sketch above. The formula for finding this area is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

Notice that we use r in the integral instead of $f(\theta)$ so make sure and substitute accordingly when doing the integral.



■ **Example 4.13** Determine the area of the inner loop of $r = 2 + 4cos\theta$. ■

Solution 4.14 We graphed this function back when we first started looking at polar coordinates. For this problem we'll also need to know the values of where the curve goes through the origin. We can get these by setting the equation equal to zero and solving.

$$0 = 2 + 4cos\theta$$

$$cos\theta = -\frac{1}{2} \qquad \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Can you see why we needed to know the values of θ where the curve goes through the origin? These points define where the inner loop starts and ends and hence are also the limits of integration in the formula.
So, the area is then,

$$A = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(2 + 4cos\theta)^2 d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(4 + 16cos\theta + 16cos^2\theta)d\theta$$

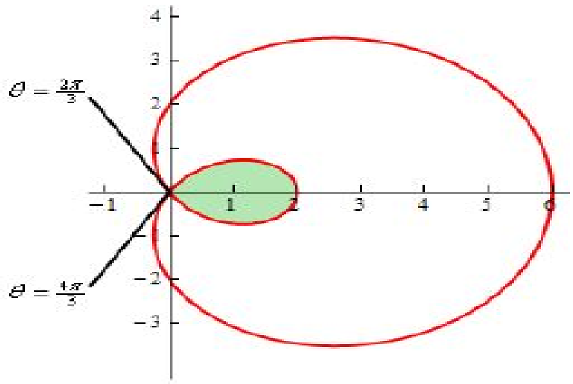
$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 2 + 8cos\theta + 4(1 + cos(2\theta))d\theta$$

$$\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} 6 + 8cos\theta + 4cos(2\theta)d\theta$$

$$= (6\theta + 8sin\theta + 2sin(2\theta))\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= 4\pi - 6\sqrt{3} = 2.174$$

Here is the sketch of this curve with the inner loop shaded in.



IV

Part four

5. Integral

Definition 5.0.12 Consider a non-negative function which is continuous over an interval $[a,b]$. To simplify the explanation and the calculations , the interval $[a,b]$ will be divided into subintervals of equal width and the sample points will correspond to the right endpoints of the subinterval.
Let the non-negative function $y = f(x)$ be continuous over $[a,b]$. We divide $[a,b]$ into n equal subintervals with $\Delta x = \dfrac{b-a}{n}$. The right endpoints of the intervals are designated $x_1, x_2, \ldots, x_n$ where $x_k = a + k\Delta x$ and $x_n = b$. For each subinterval we construct a rectangular as shown in the diagram:

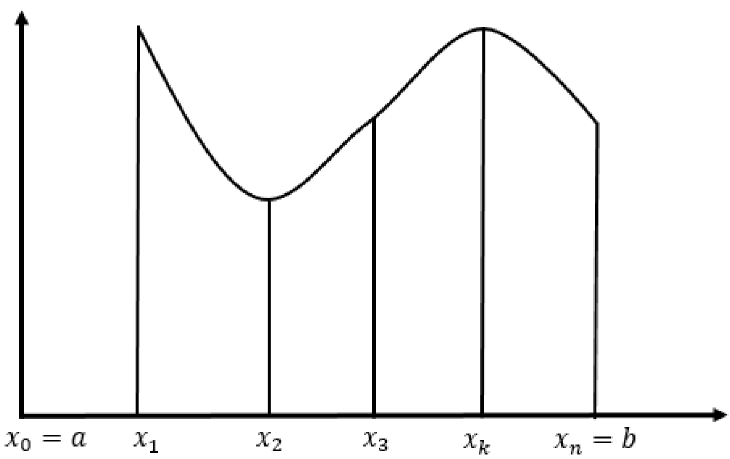


Figure 5.1

Note 5.1 The base of each rectangle is Δx The height of the rectangle k is $f(x_k)$. It follows that the area of the rectangle k is $f(x_k)\Delta x$. The sum of the areas of all n rectangles is called the **Riemann Sum** i.e $\sum_{k=1}^{n} f(x_k)\Delta x$

Definition 5.0.13 — Definite Integral. If f is continuous function defined on $[a,b]$ and if $[a,b]$ is divided into n equal subintervals of width $\Delta x = \dfrac{b-a}{n}$,and if $x_k = a + k\Delta x$ is the end point of subinterval k, then the definite integral of f from a to b is the number.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Note 5.2 — Evaluating Integrals. When we use a limit to evaluate a definite integral, we need to know how to work with sums. The following three equations give formulas for sums of powers of positive integers.

1. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
2. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

■ **Example 5.1** Use the definition of definite integral to evaluate $\displaystyle\int_0^4 (2x^2+3)dx$ ■

Solution 5.1

$$a = 0, \quad b = 4$$

$$f(x) = 2x^2 + 3$$

$$
\begin{aligned}
\Delta x &= \frac{b-a}{n} \\
&= \frac{4-0}{n} = \frac{4}{n} \\
x_k &= a + k\Delta x \\
&= 0 + k.\frac{4}{n} = \frac{4k}{n} \\
\int_a^b f(x)dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \\
x_k &= \frac{4k}{n} \\
f(x) &= 2x^2 + 3 \\
f(x_k) &= 2\left[\left(\frac{4k}{n}\right)^2\right] + 3 \\
\frac{2\times 16k^2}{n^2} 3 f(x_k) &= \frac{32k^2}{n^2} + 3
\end{aligned}
$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{32k^2}{n^2} + 3\right) \frac{4}{n}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{128k^2}{n^3} + \frac{12}{n}$$

$$\lim_{n\to\infty} \left(\frac{128}{n^3} \sum_{k=1}^{n} k^2 + \frac{12}{n} \sum_{k=1}^{n} 1\right)$$

$$\lim_{n\to\infty} \left[\frac{128}{n^3}.\frac{n(n+1)(2n+1)}{6}\right] + \frac{12}{n}.n$$

$$\lim_{n\to\infty} \frac{64}{3} \left[\frac{(n)}{n}\frac{(n+1)}{n}\frac{(2n+1)}{n}\right] \frac{12}{n}.n$$

$$\lim_{n\to\infty} \frac{64}{3} \left[(1)(1+\frac{1}{n})(2+\frac{1}{n})\right] + 12$$

$$\frac{64}{3} [(1)(1)(2)]\, 12$$

$$\frac{64\times 2}{3} + 12$$

$$= \frac{128}{3} + 12$$

$$= \frac{164}{3}$$

Alternatively;

$$\int_0^4 2x^2 + 3dx$$

$$\left[\frac{2x^3}{3} + 3x\right]_0^4 = \left(\frac{128}{3} + 12\right)$$

■ **Example 5.2** Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample point to be the right end points and $a = 0, b = 3, n = 3$. ■

Solution 5.2

$$\sigma = \sum_{k=1}^{n} f(x_k)\Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$$

$$\sum_{k=1}^{3} f(x_k)\Delta x$$

$$x_k = a + k\Delta x$$

$$\sum_{k=1}^{3} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

$$k = 1\ : x_1 = 0 + (1)(1) = 1$$

$$k = 2\ : x_2 = 0 + (2)(1) = 2$$

$$k = 3\ : x_3 = 0 + (3)(1) = 3$$

$$f(x) = x^3 - 6x$$

$$f(x_1) = f(1) = 1^3 - 6(1) = -5$$

$$f(x_2) = f(2) = 2^3 - 6(2) = -4$$

$$f(x_3) = f(3) = 3^3 - 6(3) = 9$$

$$-5(1) + -4(1) + 9(1) = 0$$

5.0.15 Properties of definite Integrals

Definition 5.0.14

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

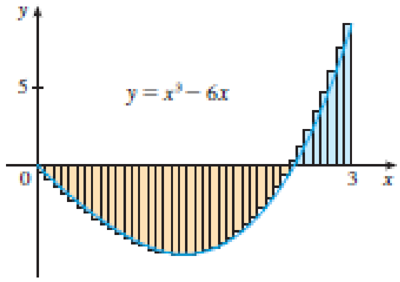


Figure 5.2

■ **Example 5.3** $\int_{1}^{2} x^2 + 1\, dx = -\int_{2}^{1} x^2 + 1\, dx$ ■

2. If $a = b$, $\Delta x = 0$,then $\int_{a}^{a} f(x) = 0$

■ **Example 5.4** $\int_{1}^{1} \cos^2 x\, dx = 0$ ■

3. If $f(x)$ is even that is $[f(x) = f(-x)]$, then $\int_{-a}^{a} f(x) = 2\int_{0}^{a} f(x)dx$

■ **Example 5.5** $\int_{-2}^{2} x^4\, dx = 2\int_{0}^{2} x^4\, dx$ ■

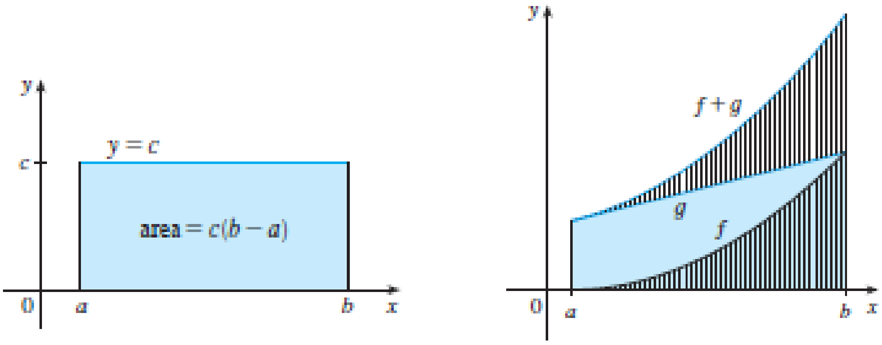
4. If $f(x)$ is odd that is $[f(x) = -f(x)]$, then $\int_{-a}^{a} f(x)dx = 0$

■ **Example 5.6** $\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4} dx = 0$ ■

5. $\int_{a}^{b} c\, dx = [cx]_a^b = cb - ca$, where c = constant

■ **Example 5.7** $\int_{-1}^{3} 3\, dx = [3x]_{-1}^{3} = 3(3) - 3(-1) = 12$ ■

6. $\int_a^b f(x) \pm g(x)\, dx = \int_{a}^{b} f(x)\, dx \pm \int_{a}^{b} g(x)\, dx$



(a) $\int_a^b c\,dx = cx = cb - ca.$ (b) $\int_a^b f(x) \ \pm \ g(x) \ dx = \int_a^b f(x) \ dx \ \pm \int_a^b g(x) \ dx$

■ **Example 5.8** $\int_2^3 \cos x \pm \tanh x\, dx = \int_2^3 \cos x\, dx \pm \int_2^3 \tanh x\, dx$ ■

7. $\int_a^b c\, f(x) dx = c \int_a^b f(x) dx,$ where

■ **Example 5.9** $\int_a^b 3\, x^8 \sqrt{x^9+1}\, dx = 3 \int_a^b x^8 \sqrt{x^9+1}\, dx$ ■

c = constant

The single most important tool used to evaluate integrals is called "The Fundamental Theo- rem of Calculus". It converts any table of derivatives into a table of integrals and vice versa. Here it is:

5.1 The Fundamental Theorem of Calculus(FTC)

Theorem 5.1.1 If $f(x)$ a continuous function on the interval $[a,b]$.

1. Part 1: If $F(x)$ on $[a,b]$ is defined by ,

$$F(x) = \int_a^x f(t)\, dt$$

, then $F'(x) = f(x)$,for all x in $[a,b]$.

2. Part 2: If $G(x)$ on $[a,b]$ satisfies $G'(x) = f(x)$,then

$$\int_a^b f(t)\,dt = G(b) - G(a)$$

We'll first do some examples illustrating the use of part 1 of the Fundamental Theorem of Calculus. Then we'll move on to part 2.

■ **Example 5.10** Find the the derivative of

$$F(x) = \int_{-1}^{x} \left(t^2+1\right)^{17}\,dt$$

■

Solution 5.3 Here $f(t) = \left(t^2+1\right)^{17}$ is a continuous function on $(-\infty,\infty)$.By the FTC,

$$F'(x) = f(x) = \left(x^2+1\right)^{17}$$

■ **Example 5.11** Find the the derivative of

$$F(x) = \int_x^{10} \sqrt{t^2+1}\,dt$$

■

Solution 5.4 Note that the variable bound is the lower bound in the definition of $F(x)$. Before applying FCT, we utilize integral properties to change the bounds into the form for which FCT is applicable.

$$F(x) = \int_x^{10} \sqrt{t^2+1}\,dt = -\int_{10}^{x} \sqrt{t^2+1}\,dt$$

Now $f(t) = -\sqrt{t^2+1}$ is a continuous function on $(-\infty,\infty)$. Apply FCT to get

$$F'(x) = f(x) = -\sqrt{x^2+1}$$

■ **Example 5.12** Find $\dfrac{d}{dx}\displaystyle\int_0^{\sin x} \sqrt{t^2+1}$ ■

Solution 5.5 Note that the variable bound is is a function instead of simply an "x". Therefore,we cannot directly apply FCT. Set $u = \sin x$. The we have a function

$$G(u) = \int_0^u \sqrt{t^2+1}\,dt$$

for which FCT is applicable with $G'(u) = \sqrt{u^2+1}$.
But what we want is $F'(x)$. As $F(x) = G(u(x))$ we can use Chain Rule $F'(x) = G'(u)u'(x)$ to get the job done:

$$F(x) = G'(u)u'(x) = \sqrt{u^2+1}\cos x = \sqrt{\sin^2 x+1}\cos x$$

Note 5.3 Part 2 of the Fundamental Theorem states that if we know an antiderivative $F(x)$ of f, then we can evaluate $\int_a^b f(x)\,dx$ simply by subtracting the values of $F(x)$ at the endpoints of the interval $[a,b]$. It's very surprising that $\int_a^b f(x)\,dx$, which was defined by a complicated procedure involving all of the values $f(x)$ of for $a \le x \le b$, can be found by knowing the values of $F(x)$ at only two points,a and b .

Definition 5.1.1 A function F is called an anti-derivative of f on the interval$[a,b]$ if one has $F'(x) = f(x)$ for all x with $a < x < b$.
For instance, $F(x) = \frac{1}{2}x^2$ is the anti-derivative of $f(x) = x$, but so is $G(x) = \frac{1}{2}x^2 + 2008$

Example 5.13 Evaluate the integral $\int_{-2}^{1} x^3\,dx$ ■

Solution 5.6 The function $f(x) = x^3$ is continuous on $[-2,1]$,now the antiderivative of f(x) is $F(x) = \frac{1}{4}x^4$ so Part 2 of the Fundamental Theorem gives

$$\int_{-2}^{1} x^3\,dx = F(1) - F(-2) = \frac{1}{4}(1)^4 - \frac{1}{4}(-2)^4 = -\frac{15}{4}$$

Example 5.14 Find the area under the parabola $y = x^2$ from 0 to 1. ■

Solution 5.7 An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$.The required area A is found using Part 2 of the Fundamental Theorem:

$$A = \int_0^1 x^2\,dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}$$

5.1.1 Indefinite Integrals

Definition 5.1.2 Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating $F(b) - F(a)$, where $F(x)$ is an antiderivative of $f(x)$.

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of and is called an **Indefinite Integral**.

Example 5.15 Integrate $\int x\,dx$ ■

Solution 5.8 By integrating

$$\begin{aligned}\int x dx &= \frac{x^{1+1}}{1+1} + c \\ &= \frac{x^2}{2} + c\end{aligned}$$

Example 5.16 $\int 2x^3\,dx$ ■

Solution 5.9

$$\begin{aligned}\int 2x^3\,dx &= \frac{2x^{3+1}}{3+1} + c \\ &= \frac{2x^4}{4} + c \\ &= \frac{x^4}{2} + c\end{aligned}$$

■ **Example 5.17** $\int (x^3 - 5x^2 + 7x - 11)\, dx$ ■

Solution 5.10

$$\int (x^3 - 5x^2 + 7x - 11) dx = \frac{x^4}{4} - \frac{5x^3}{3} + \frac{7x^2}{2} - 11x + c$$

■ **Example 5.18** Find $\int \left(2e^x + \frac{6}{x} + \ln 2\right)\, dx$ ■

Solution 5.11

$$\int \left(2e^x + \frac{6}{x} + \ln 2\right)\, dx = \left(2\int e^x\, dx + 6\int \frac{1}{x}\, dx + \ln 2 \int\, dx\right)$$
$$= 2e^x + 6\ln|x| + (\ln 2)x + c$$

■ **Example 5.19** $\int \left(\sqrt[3]{x} - \frac{1}{\sqrt[3]{x}}\right)\, dx$ ■

Solution 5.12

$$\begin{aligned}
\int \left(\sqrt[3]{x} - \frac{1}{\sqrt[3]{x}}\right) dx &= \int \left(x^{\frac{1}{3}} - \frac{1}{x^{\frac{1}{3}}}\right) dx \\
&= \int \left(x^{\frac{1}{3}} - x^{-\frac{1}{3}}\right) dx \\
&= \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} - \frac{x^{-\frac{1}{3}+1}}{\frac{2}{3}} + c \\
&= 3\frac{x^{\frac{4}{3}}}{4} - 3\frac{x^{\frac{2}{3}}}{2} + c
\end{aligned}$$

■ **Example 5.20** $\int \sqrt[3]{x^2} dx$ ■

Solution 5.13

$$\begin{aligned}
\int \sqrt[3]{x^2} dx &= \int (x^2)^{\frac{1}{3}} dx \\
\int x^{2\times\frac{1}{3}}\, dx &= \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + c \\
&= \frac{3x^{\frac{5}{3}}}{5} + c
\end{aligned}$$

5.1.2 Exponential function

Definition 5.1.3 $\int e^{ax} dx = \dfrac{e^{ax}}{a} + c$

■ **Example 5.21** $\int e^{5x} dx$ ■

Solution 5.14

$$\int e^{5x} dx = \frac{e^{5x}}{5} + c$$

■ **Example 5.22** Find $\int a^x\, dx$ ■

Solution 5.15

$$\int a^x\, dx = \int e^{\ln a^x}\, dx = \int e^{x \ln a}\, dx$$

$$= \frac{e^{x\ln a}}{\ln a} + c$$

$$\text{but } e^{x\ln a} = e^{\ln a^x} = a^x$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

■ **Example 5.23** $\int 3^{2x} dx = ?$ ■

Solution 5.16

$$\begin{aligned}
Let\ 3^{2x} &= e^{2x log_e 3} = e^{2x\ln 3}\\
and\ u &= 2x\ln 3\\
du &= 2\ln 3 dx \Rightarrow dx = \frac{du}{2\ln 3}\\
\Rightarrow \int 3^{2x} dx &= \int e^u \cdot \frac{du}{2\ln 3} = \frac{1}{2\ln 3}\int e^u du\\
&= \frac{1}{2\ln 3} e^u + c\\
&= \frac{1}{2\ln 3} e^{2x\ln 3} + c\\
&= \frac{1}{2\ln 3} 3^{2x} + c
\end{aligned}$$

■ **Example 5.24** $\int 2^x dx = ?$ ■

Solution 5.17

$$\begin{aligned}
\text{let } 2^x &= e^{x\ln 2}\\
\text{and} u &= x\ln 2\\
du &= \ln 2 dx \Rightarrow dx = \frac{du}{\ln 2}\\
\Rightarrow \int 2^x dx &= \int e^{x\ln 2} dx = \int e^u \cdot \frac{du}{\ln 2}\\
&= \frac{1}{\ln 2}\int e^u \cdot du\\
&= \frac{1}{\ln 2} \cdot e^{x\ln 2} + c\\
&= \frac{1}{\ln 2} 2^x \cdot + c
\end{aligned}$$

5.1.3 Trigonometric function

Definition 5.1.4

1. $\sin x\, dx = -\cos x + c$
2. $\cos x\, dx = \sin x + c$
3. $\sin(ax+b)\, dx = \dfrac{-\cos(ax+b)}{a} + c$
4. $\sin(3x+4)\, dx = \dfrac{-\cos(3x+4)}{3} + c$
5. $\cos(3x)\, dx = \dfrac{\sin 3x}{3} + c$

5.1.4 Standard Integrals

Definition 5.1.5

1. $\int \sec x \tan x\, dx = \sec x + c$
2. $\int cosec\, x \cot x\, dx = -cosec\, x + c$
3. $\int cosec^2 x\, dx = -\cot x + c$
4. $\int \sec^2 x\, dx = \tan x + c$
5. $\int \tan ax\, dx = \frac{1}{a} \ln |sec\ ax| + c$
6. $\displaystyle\int e^{ax} \sin x\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$
7. $\displaystyle\int e^{ax} \cos x\, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

Hyperbolic functions

Definition 5.1.6

1. $\displaystyle\int \sinh x\, dx = \cosh x + c$
2. $\displaystyle\int \cosh x\, dx = \sinh x + c$

3. $\displaystyle\int \tanh x \, dx = \ln|\cosh x| + c$

4. $\displaystyle\int \coth x \, dx = \ln|\sinh x| + c$

Inverse Trigonometric functions

Definition 5.1.7

1. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\frac{x}{a} + c$
2. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}} \, dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$
3. $\displaystyle\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
4. $\displaystyle\int -\frac{1}{\sqrt{a^2-x^2}} \, dx = \cos^{-1}\frac{x}{a} + c$

5.2 Substitution method

Definition 5.2.1 The substitution method is a trick for evaluating integrals. It is based on the following identity between differentials (where u is a function of x):

$$du = u' dx$$

Hence we can write:

$$\int f(g(x))g'(x) = \int f(u)\, du$$

■ **Example 5.25** Find $\int 2x\sqrt{1+x^2}\, dx$ ■

Solution 5.18 Using the substitution $u = 1+x^2$, $\dfrac{du}{dx} = 2x \implies \dfrac{du}{2x} = dx$, then

$$\begin{aligned}
\int 2x\sqrt{1+x^2}\, dx &= \int 2x\sqrt{u}\,\frac{du}{2x} = \int \sqrt{u}\, du \\
&= \int \sqrt{u}\, du = \frac{2}{3}u^{3/2} + c \text{ but } u = 1+x^2 \\
\int 2x\sqrt{1+x^2}\, dx &= \frac{2}{3}\left(1+x^2\right)^{3/2} + c
\end{aligned}$$

■ **Example 5.26** Find $\int 2x\cos(x^2)\, dx$ ■

Solution 5.19 Let $u = x^2$, then $\dfrac{du}{dx} = 2x \implies dx = \dfrac{du}{2x}$

$$\int 2x\cos(x^2)\, dx = \int 2x\cos(u)\,\frac{du}{2x} = \int \cos u\, du = \sin u + c$$

$$\text{but } u = x^2$$

$$\int 2x\cos(x^2)\, dx = \sin(x^2) + c$$

■ **Example 5.27** Evaluate $\int (ax+b)^n\, dx$,assuming that a and b are constants,$a \neq 0$, and n is a positive integer. ■

Solution 5.20 We let $u = ax+b$, then $\dfrac{du}{dx} = a \implies dx = \dfrac{du}{a}$.Then

$$\int (ax+b)^n\, dx = \int (u)^n\,\frac{du}{a} = \frac{1}{a}\frac{(u)^{n+1}}{n+1}$$

$$\text{but } u = ax+b$$
$$\int (ax+b)^n\, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$$

■ **Example 5.28** find $\int \frac{e^x}{e^{2x}+1}\, dx$ ■

Solution 5.21 We let $u = e^x$, then $\frac{du}{dx} = e^x \implies dx = \frac{du}{e^x}$.Then
$$\int \frac{e^x}{e^{2x+1}}\, dx = \int \frac{e^x}{u^2+1}\frac{du}{e^x} = \int \frac{1}{u^2+1}\, du$$
$$\tan^{-1} u + c$$
$$\text{but } u = e^x$$
$$\int \frac{e^x}{e^{2x}+1}\, dx = \tan^{-1}(e^x) + c$$

■ **Example 5.29** Evaluate $\int \sin(ax+b)\, dx$, assuming that a and b are constants and $a \neq 0$ ■

Solution 5.22 We let $u = ax+b$, then $\frac{du}{dx} = a \implies dx = \frac{du}{a}$.Then
$$\int \sin(ax+b)\, dx = \int \sin(u)\,\frac{du}{a} = \int \frac{1}{a}\sin u$$
$$= \frac{1}{a} - \cos u + c$$
$$\text{but } u = ax+b$$
$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos ax + b + c$$

■ **Example 5.30** Evaluate $\int xe^{x^2}\, dx$ ■

Solution 5.23 We let $u = x^2$, then $\frac{du}{dx} = 2x \implies dx = \frac{du}{2x}$.Then
$$\int xe^{x^2}\, dx = \int xe^u \frac{du}{2x} = \frac{1}{2}\int e^u\, du = \frac{1}{2}e^u + c$$
$$\text{but } u = x^2$$

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^{x^2} + c$$

Note 5.4 $\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$

■ **Example 5.31** $\int \frac{dx}{ax+b}$ ■

Solution 5.24

Note 5.5 $\int \frac{1}{x}\,dx = \ln x + c$

$$\begin{aligned}
\text{Let} \quad u &= ax+b \\
\frac{du}{dx} &= a \\
dx &= \frac{du}{a} \\
\int \frac{\frac{du}{a}}{u} &= \frac{1}{a}\int \frac{1}{u}du \\
&= \frac{1}{a}\ln u + c \\
&= \frac{1}{a}\ln|ax+b| + c
\end{aligned}$$

■ **Example 5.32** $\int \frac{1}{4x+3}dx$ ■

Solution 5.25

$$
\begin{aligned}
\text{Let} \quad u &= 4x+3 \\
\frac{du}{dx} &= 4 \\
dx &= \frac{du}{4} \\
\int \frac{\frac{du}{4}}{u} &= \frac{1}{4}\int \frac{1}{u}du \\
&= \frac{1}{4}\ln u + c \\
&= \frac{1}{4}\ln|4x+3| + c
\end{aligned}
$$

■ **Example 5.33** Find $\int \tan x \, dx$ ■

Solution 5.26 Here the idea is to write $\tan x = \dfrac{\sin x}{\cos x}$

We let $u = \cos x$, then $\dfrac{du}{dx} = -\sin x \implies dx = \dfrac{du}{-\sin x}$.Then

$$\int \tan x = \int \frac{\sin x}{\cos x}\, dx = \int \frac{\sin x}{u}\, \frac{du}{-\sin x} = -\int \frac{1}{u}$$

$$= -\ln|u| + c$$

$$\text{but } u = \cos x$$

$$\int \tan x \, dx = -\ln|\cos x| + c$$

■ **Example 5.34** $\int \dfrac{1}{x\ln x} dx$ ■

Solution 5.27

$$\begin{aligned}
&Let \quad u = \ln x \\
\int \frac{1}{x\ln x}dx &= \frac{du}{dx} = \frac{1}{x} \\
dx &= xdu \\
&= \int \frac{1}{x.u}.xdu \\
&= \frac{1}{u} + c \\
&= \ln|\ln x| + c
\end{aligned}$$

■ **Example 5.35** $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$ ■

Solution 5.28

$$\begin{aligned}
\text{Let } u &= \sqrt{x} = x^{\frac{1}{2}} \\
\frac{du}{dx} &= \frac{1}{2}x^{\frac{1}{2}} \\
dx &= \frac{du}{\frac{1}{2\sqrt{x}}} \\
du &= 2\sqrt{x}dx
\end{aligned}$$

$$\begin{aligned}
&\text{Now,} \\
\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx &= \int \frac{\cos u}{u}du.2\sqrt{x} \\
&= \int \frac{\cos u}{u}2.u\,du \\
&= 2\int \cos u.du \\
&= 2\sin u + c \\
&= 2\sin\sqrt{x} + c
\end{aligned}$$

5.3 Definite Integral

■ **Example 5.36** Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x\sin x + \cos x}{x^2+1}\,dx$$

■

Solution 5.29 There really isn't anything to do with this integral once we notice that the limits are the same.

$$\int_{130}^{130} \frac{x^3 - x\sin x + \cos x}{x^2+1}\,dx = 0$$

■ **Example 5.37** Find $\int_0^4 \sqrt{2x+1}\,dx$ ■

Solution 5.30 We let $u = 2x+1$, then $\frac{du}{dx} = 2 \implies dx = \frac{du}{2}$.Then

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int \sqrt{u}$$
$$= \frac{1}{3}u^{3/2} + c$$
$$\text{but } u = 2x+1$$
$$= \frac{1}{3}(2x+1)^{3/2} + c$$

Then we use it for computing the definite integral:

$$\int_0^4 \sqrt{2x+1}\,dx = \left[\frac{1}{3}(2x+1)^{3/2}\right]_0^4 = \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{26}{3}$$

■ **Example 5.38** Find $\int_2^{\ln\frac{1}{2}} e^t - e^{-t}\,dt$ ■

Solution 5.31

$$\int_2^{\ln\frac{1}{2}} e^t - e^{-t}\,dt = \left[e^t + e^{-t}\right]_2^{\ln\frac{1}{2}} = e^2 + e^{-2} - e^{\ln\frac{1}{2}} - e^{-\ln\frac{1}{2}}$$

$$e^2 + e^{-2} - e^{\ln\frac{1}{2}} - e^{-\ln 2} = e^2 + e^{-2} - \frac{1}{2} - 2$$

$$\int_2^{\ln\frac{1}{2}} e^t - e^{-t}\,dt = e^2 + e^{-2} - \frac{1}{2} - 2$$

Example 5.39 Evaluate $\displaystyle\int_1^2 \frac{x^2}{(x^3+1)^2}\,dx$ ■

Solution 5.32 We let $u = x^3+1$, then $\dfrac{du}{dx} = 3x^2 \implies dx = \dfrac{du}{3x^2}$.Then

$$\int_1^2 \frac{x^2}{(x^3+1)^2}\,dx = \int_1^2 \frac{x^2}{(u)^2}\frac{du}{3x^2} = \frac{1}{3\int_1^2}\frac{1}{(u)^2}\,du = -\frac{1}{3u}$$

$$\text{but } u = x^3+1$$

$$= -\frac{1}{3u} = \left[-\frac{1}{3(x^3+1)}\right]_1^2 = -\left[\frac{1}{27} - \frac{1}{6}\right] = \frac{7}{54}$$

$$\int_1^2 \frac{x^2}{(x^3+1)^2}\,dx = \frac{7}{54}$$

Example 5.40 Evaluate $\displaystyle\int_{1/2}^{1/4} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt$ ■

Solution 5.33 We let $u = \sin(\pi t)$, then $\dfrac{du}{dt} = \pi\cos(\pi t) \implies dt = \dfrac{du}{\pi\cos(\pi t)}$.

Then

$$\int_{1/2}^{1/4} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt = \int_{1/2}^{1/4} \frac{\cos(\pi t)}{u^2}\frac{du}{\pi\cos(\pi t)} = \frac{1}{\pi}\int_{1/2}^{1/4}\frac{1}{u^2}\,du = \frac{1}{\pi}\int_{1/2}^{1/4} u^{-2}\,du$$

$$= \frac{1}{\pi}\left[\frac{u^{-1}}{-1}\right]_{1/2}^{1/4} = \frac{1}{\pi}\left[-\frac{1}{u}\right]_{1/2}^{1/4}$$

$$\text{but } u = \sin(\pi t)$$

$$= \frac{1}{\pi}\left[-\frac{1}{\sin(\pi t)}\right]_{1/2}^{1/4} = -\frac{1}{\pi} + \frac{\sqrt{2}}{\pi}$$

$$\int_{1/2}^{1/4} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt = -\frac{1}{\pi} + \frac{\sqrt{2}}{\pi}$$

5.4 Powers of Sine and Cosine

Definition 5.4.1 Functions consisting of products of the sine and cosine can be integrated by using substi- tution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. Some examples will suffice to explain the approach.

■ **Example 5.41** Evaluate $\int \sin^5 x\, dx$ ■

Solution 5.34 Rewrite the function:

$$\int \sin^5 x\, dx = \int \sin x \sin^4 x\, dx = \int \sin x \left[\sin^2 x\right]^2\, dx = \int \sin x \left[1-\cos^2 x\right]^2\, dx$$

We let $u = \cos x$, then $\dfrac{du}{dx} = -\sin x \implies dx = \dfrac{du}{-\sin x}$.Then

$$\int \sin x \left[1-\cos^2 x\right]^2\, dx = \int \sin x \left[1-u^2\right]^2 \frac{du}{-\sin x} = -\int \sin x \left[1-u^2\right]^2\, du$$

$$= -\int \left[1-2u^2+u^4\right]\, du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + c$$

$$\text{but } u = \cos x$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

$$\int \sin^5 x\, dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

■ **Example 5.42** Evaluate $\int \sin^6 x\, dx$ ■

Solution 5.35 Use $\sin^2 x = \dfrac{[1-\cos 2x]}{2}$ to rewrite the function:

$$\int \sin^6 x\, dx = \int \left[\sin^2 x\right]^3\, dx = \int \frac{[1-\cos 2x]^3}{8}\, dx$$

$$= \frac{1}{8}\int \left[1 - 3\cos 2x + 3\cos^2 2x - \cos^3 3x\right]\, dx$$

Now we have four integrals to evaluate:

$$\int 1\, dx = x$$

$$\int -3\cos 2x\, dx = -\frac{3}{2}\sin 2x$$

$$\int 3\cos^2 2x\, dx = 3\frac{1+\cos 4x}{2}\, dx = \frac{3}{2}\left[x + \frac{\sin 4x}{4}\right]$$

$$\int -3\cos^3 2x\, dx = \int -\cos 2x \cos^2 2x\, dx = \int -\cos 2x \left[1-\sin^2 2x\right]\, dx$$

We let $u = \sin 2x$, then $\dfrac{du}{dx} = 2\cos 2x \implies dx = \dfrac{du}{2\cos 2x}$.Then

$$\int -\cos 2x\left[1-\sin^2 2x\right]\, dx = \int -\cos 2x\left[1-u^2\right]\frac{du}{2\cos 2x}$$

$$= \int -\cos 2x\left[1-u^2\right]\frac{du}{2\cos 2x} = -\frac{1}{2}\left[1-u^2\right]\, du = -\frac{1}{2}\left[u - \frac{u^3}{3}\right]$$

$$\text{but } u = \sin 2x$$

$$= -\frac{1}{2}\left[\sin 2x - \frac{\sin^3 2x}{3}\right]$$

$$\int -3\cos^3 2x\, dx = -\frac{1}{2}\left[\sin 2x - \frac{\sin^3 2x}{3}\right]$$

So at long last we get

$$\int \sin^6 x\, dx = \frac{1}{8}\left[x - \frac{3}{2}\sin 2x - \frac{3}{2}\left[x + \frac{\sin 4x}{4}\right] + \frac{1}{2}\left[\sin 2x - \frac{\sin^3 2x}{3}\right]\right]$$

■ **Example 5.43** ■

5.5 Trigonometric Integrals and Trigonometric Substitutions

5.5.1 Trigonometric Integrals

Here we discuss integrals of powers of trigonometric functions. To that end the following half-angle identities will be useful:

$$\sin^2 x = \frac{1}{2}(1 - cos2x),$$

$$\cos^2 x = \frac{1}{2}(1 + cos^{2x})$$

Remember also the identities:

$$\sin^2 x + \cos^2 x = 1,$$

$$\sec^2 x = 1 + \tan^2 x.$$

Integrals of Products of Sines and Cosines. We will study now integrals of the form

$$\sin^m x \cos^n x dx,$$

including cases in which $m = 0$ or $n = 0$, i.e.:

$$\int \cos^n x dx; \quad \int \sin^m x dx$$

The simplest case is when either $n = 1$ or $m = 1$, in which case the substitution $u = \sin x$ or $u = \cos x$ respectively will work.

■ **Example 5.44** Find $\int \sin^4 x \cos x dx = \cdots$ ■

Solution 5.36 Let

$$u = \sin x, \quad du = \cos x dx$$

$$= \int u^4 du = \frac{u^5}{5} + C = \frac{\sin^5}{5} + C$$

More generally if at least one exponent is odd then we can use the identity $\sin^2 x + \cos^2 x = 1$ to transform the integrand into an expression containing only one sine or one cosine.

■ **Example 5.45** Evaluate $\int \sin^2 x \cos^3 x \, dx$ ■

Solution 5.37

$$\begin{aligned} \int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x \cos^2 x \cos x dx \\ &= \int \sin^2 x (1 - \sin^2 x) \cos x dx = \cdots \end{aligned}$$

$$u = \sin x, du = \cos x dx$$

$$\begin{aligned} \cdots = \int u^2 (1 - u^2) du &= \int (u^2 - u^4) \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \end{aligned}$$

If all the exponents are even then we use the half-angle identities.

■ **Example 5.46** Find $\int \sin^2 x \cos^2 x \, dx$ ■

Solution 5.38

$$\begin{aligned}
\int \sin^2 x \cos^2 x\, dx &= \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)dx \\
&= \frac{1}{4}\int (1-\cos^2 2x)dx \\
&= \frac{1}{4}\int (1-\frac{1}{2}(1+\cos 4x))dx \\
&= \frac{1}{8}\int (1-\cos 4x)dx \\
&= \frac{x}{8}-\frac{\sin 4x}{32}+C
\end{aligned}$$

■ **Example 5.47** Find $\int \sec y\, dy$ ■

Solution 5.39

$$\int \sec y\, dy = \int \frac{\sec y}{1}\frac{(\sec y+\tan y)}{(\sec y+\tan y)}\, dy = \int \frac{\sec^2 y+\sec y\tan y}{\sec y+\tan y}\, dy$$

Next we use the following substitution. $u=\sec y+\tan y$, $\dfrac{du}{dy}=\sec y\tan y+\sec^2 y$

$\implies dy = \dfrac{du}{\sec y\tan y+\sec^2 y}$

$$= \int \frac{\sec^2 y+\sec y\tan y}{\sec y+\tan y}\, dy = \int \frac{\sec^2 y+\sec y\tan y}{u}\,\frac{du}{\sec y\tan y+\sec^2 y}$$

$$\int \sec y\, dy = \int \frac{1}{u}\, du = \ln|u|+c$$

$$\text{but}\ \ u=\sec y+\tan y$$

$$\int \sec y\, dy = \ln|\sec y+\tan y|+c$$

Analogously:

$$\int cosecx\, dx = -\ln|\csc x+\cot x|$$

More generally an integral of the form

Definition 5.5.1

$$\int \tan^m x \sec^n x\, dx$$

can be computed in the following way:

1. If m is odd, use $u=secx, du=\sec x\tan x\, dx$.

2. If n is even, use $u = \tan x, du = \sec^x\ dx$.

■ **Example 5.48** Find $\int \tan^3 x \sec^2 x\, dx$ ■

Solution 5.40 Since in this case m is odd and n is even it does not matter which method we use, so let's use the first one: Let

$$u = \sec x, \frac{du}{dx} = \sec x \tan x \implies \frac{du}{\sec x \tan x} = dx$$

$$\int \tan^3 x \sec^2 x \frac{du}{\sec x \tan x} = \int \tan^2 x\, u\, du$$

Note: $\tan^2 x = \sec^2 x - 1$

$$= \int \left(\sec^2 x - 1\right)\ u\, du = \int \left(u^2 - 1\right) u\, du$$

$$du = \int \left(u^2 - 1\right) u\, du = \frac{u^4}{4} - \frac{u^2}{2} + c$$

But $u = \sec x$

$$= \frac{\sec^4 x}{4} - \frac{\sec^2 x}{2} + c$$

$$\int \tan^3 x \sec^2 x\, dx = \frac{\sec^4 x}{4} - \frac{\sec^2 x}{2} + c$$

5.6 Trigonometric Substitutions

Definition 5.6.1 So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a "reverse" substitution, but it is really no different in principle than ordinary substitution.

■ **Example 5.49** Evaluate $\int \sqrt{1-x^2}\, dx$ ■

Solution 5.41 We let $x = \sin u$, then $\frac{dx}{du} = \cos u \implies dx = \cos u\, du$.
Then

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2 u} \cdot \cos u\, du = \int \sqrt{\cos^2 u} \cdot \cos u\, du$$

$$= \int \sqrt{\cos^2 u} \cdot \cos u \, du = \int \cos^2 u \, du = \int \frac{1+\cos 2u}{2} \, du = \frac{u}{2} + \frac{\sin 2u}{4} + c$$

but $u = \sin^{-1} x$ and $\sin 2u = 2 \sin u \cos u$

$$= \frac{u}{2} + \frac{\sin 2u}{4} + c = \frac{\sin^{-1} x}{2} + \frac{2 \sin u \cos u}{4} + c$$

$$= \frac{\sin^{-1} x}{2} + \frac{2 \sin u \cos u}{4} + c = \frac{\sin^{-1} x}{2} + \frac{2 \sin u \sqrt{1 - \sin^2 u}}{4} + c$$

but $x = \sin u$

$$\int \sqrt{1-x^2} \, dx = \frac{\sin^{-1} x}{2} + \frac{x\sqrt{1-x^2}}{2} + c$$

The following substitutions are useful in integrals containing the following expressions:

expression	substitution	identity
$a^2 - u^2$	$u = a \sin t$	$1 - \sin^2 t = \cos^2 t$
$a^2 + u^2$	$u = a \tan t$	$1 + \tan^2 t = \sec^2 t$
$u^2 - a^2$	$u = a \sec t$	$\sec^2 t - 1 = \tan^2 t$

Figure 5.3

So for instance, if an integral contains the expression $a^2 - u^2$, we may try the substitution $u = a \sin t$ and use the identity $1 - \sin^2 t = \cos^2 t$ in order to transform the original expression in this way: $a^2 - u^2 = a^2(1 - \sin^2 t) = a^2 = \cos^2 t$

■ **Example 5.50** Evaluate $\int \sqrt{4 - 9x^2} \, dx$ ■

Solution 5.42 We start by rewriting this so that it looks more like the previous example:

$$\int \sqrt{4-9x^2} \, dx = \int \sqrt{4\left(1 - \left(\frac{3x}{2}\right)^2\right)} \, dx = \int 2\sqrt{\left(1 - \left(\frac{3x}{2}\right)^2\right)} \, dx$$

Now let $\frac{3x}{2} = \sin u$ so $\frac{3}{2} \, dx = \cos u \, du \implies dx = \frac{2}{3} \cos u \, du$.

Then

$$\int 2\sqrt{\left(1-\left(\frac{3x}{2}\right)^2\right)}\,dx=\int 2\sqrt{1-\sin^2 u}\cdot\frac{2}{3}\cos u\,du=\frac{4}{3}\int \cos^2 u\,du$$

$$\frac{4}{3}\int \cos^2 u\,du=\frac{4}{3}\int\frac{1+\cos 2u}{2}\,du=\frac{4}{3}\left[\frac{u}{2}+\frac{\sin 2u}{4}\right]+c$$

$$=\frac{4u}{6}+\frac{4\sin 2u}{12}+c=\frac{2u}{3}+\frac{\sin 2u}{3}+c$$

$$\text{but}\;\; u=\sin^{-1}\frac{3x}{2}\;\;\text{and}\;\;\sin 2u=2\sin u\cos u$$

$$=\frac{2\left(\sin^{-1}\frac{3x}{2}\right)}{3}+\frac{2\sin u\cos u}{3}+c$$

$$=\frac{2\left(\sin^{-1}\frac{3x}{2}\right)}{3}+\frac{2\sin u\sqrt{1-\sin^2 u}}{3}+c$$

$$\text{but}\;\;\frac{3x}{2}=\sin u$$

$$=\frac{2\left(\sin^{-1}\frac{3x}{2}\right)}{3}+\frac{2\left(\frac{3x}{2}\right)\sqrt{1-\left(\frac{3x}{2}\right)^2}}{3}+c$$

$$=\frac{2\left(\sin^{-1}\frac{3x}{2}\right)}{3}+\frac{3x\sqrt{1-\left(\frac{3x}{2}\right)^2}}{3}+c=\frac{2\left(\sin^{-1}\frac{3x}{2}\right)}{3}+x\sqrt{1-\left(\frac{3x}{2}\right)^2}+c$$

$$\int\sqrt{4-9x^2}\,dx=\frac{2\left(\sin^{-1}\frac{3x}{2}\right)}{3}+x\sqrt{1-\left(\frac{3x}{2}\right)^2}+c$$

■ **Example 5.51** Find $\int\frac{x^3}{\sqrt{9-x^2}}\,dx$ ■

Solution 5.43 Let $x=3\sin u$, $\frac{dx}{du}=3\cos u \implies dx=3\cos u\,du$

$$\int\frac{x^3}{\sqrt{9-x^2}}\,dx=\int\frac{(3\sin u)^3}{\sqrt{9-(3\sin u)^2}}\,dx=\int\frac{27\sin^3 u}{\sqrt{9-(9\sin^2 u)}}\,dx$$

$$\text{but}\;\; dx=3\cos u\,du$$

$$= \int \frac{27\sin^3 u}{\sqrt{9-(9\sin^2 u)}}\,dx = 27\int \frac{\sin^3 u\cos u}{\sqrt{1-\sin^2 u}}\,du = 27\int \sin^3 u\,du$$

$$= 27\int (1-\cos^3 u\sin u\,du = 27\left(-\cos u + \frac{\cos^3 t}{3}\right) + c$$

$$27(-\sqrt{1-\sin^2 u} + \frac{1}{3}(1-\sin^2 u)^{3/2}) + c$$

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = -9\sqrt{9-x^2} + \frac{1}{3}(9-x^2)^{3/2} + c$$

5.7 Reduction Formulas

Assume that we want to find the following integral for a given value of $n > 0$

$$\int x^n e^x dx$$

Using integration by parts with $u = x^n$ and $dv = e^x dx$, so $v = e^x$ and $du = nx^{n-1}dx$, we get:

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx.$$

On the right hand side we get an integral similar to the original one but with x raised to $n-1$ instead of n. This kind of expression is called a reduction formula. Using this same formula several times, and taking into account that for $n = 0$ the integral becomes $/inte^x dx = e^x + C$, we can evaluate the original integral for any n. For instance:

■ **Example 5.52** $\int x^3 e^x dx$ ■

Solution 5.44

$$\begin{aligned}
\int x^3 e^x dx &= x^3e^x - 3\textstyle\int x^2 e^x dx \\
&= x^3e^x - 3(x^2e^x - 2\int xe^x dx) \\
&= x^3e^x - 3(x^2e^x - 2(xe^x - \int e^x dx)) \\
&= x^3e^x - 3(x^2e^x - 2(xe^x - e^x)) + C \\
&= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C.
\end{aligned}$$

Another,

■ **Example 5.53** $\int \sin^n x dx$ ■

Solution 5.45

$$
\begin{aligned}
\int \sin^n x dx &= \int \underbrace{\sin^{n-1} x}_{\text{u}} \underbrace{\sin x dx}_{\text{dv}} \\
&= -\sin^{n-1} x \cos x + (n-1) \int \underbrace{\cos^2 x}_{1-\sin^2 x} \sin^{n-2} dx \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} dx \\
&- (n-1) \int \sin^n x dx
\end{aligned}
$$

Adding the last term to both sides and dividing by n we get the following reduction formula:

Note 5.6

$$
\int \sin^n x dx = -\frac{\sin^{n-1} \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx
$$

5.8 Integration by parts

Definition 5.8.1

$$
\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}
$$

$$
\Longrightarrow uv = \int u\frac{dv}{dx}.dv + \int v\frac{du}{dx}.dx
$$

$$
\Longrightarrow \boxed{\int udv = uv - \int vdu}
$$

Note 5.7 Guidelines for selecting u and dv

"L-I-A-T-E" : Choose u to be the function that comes first in the list:
L = Logarithmic function
I = Inverse Trigonometric function
A = Algebraic function
T = Trigonometric function
E = Exponential function

■ **Example 5.54** $\int x^2 \sin x dx$ ■

Solution 5.46
A comes before T in L-I-A-T-E

$$\begin{aligned}
\text{So we let } u &= x^2 \\
dv &= \sin x \\
du &= 2x \\
v &= -\cos x \\
\int u dv &= -\cos x . x^2 - \int -2x\cos x \\
&= -x^2\cos x + \int 2x\cos x \\
&** \int 2x\cos x \\
u &= 2x \\
dv &= \cos x \\
du &= 2 \\
v &= \sin x
\end{aligned}$$

$$\int u dv = 2x\sin x - \int 2\ sinx$$

$$\begin{aligned}
&= 2x\sin x + 2\cos x + c \\
&= -x^2\cos x + 2x\sin x + 2x\cos x + c
\end{aligned}$$

■ **Example 5.55** $\int \ln x\, dx$ ■

Solution 5.47

$$\begin{aligned}
\text{Let } u &= \ln x \\
dv &= 1 \\
du &= \tfrac{1}{x} \\
v &= x \\
\implies \int u dv &= x.\ln x - \int \frac{1}{x}.x\, dx \\
&= x\ln x - \int dx \\
&= x\ln x - x + c
\end{aligned}$$

■ **Example 5.56** $\int x\cos(5x-1)dx$ ■

Solution 5.48

$$
\begin{aligned}
u &= x\\
dv &= \cos(5x-1)\\
du &= 1\\
v &= \frac{\sin}{5}(5x-1)
\end{aligned}
$$

$$
\begin{aligned}
\int udv &= x.\frac{\sin}{5}(5x-1) - \int \frac{\sin}{5}(5x-1).dx\\
&= x\frac{\sin}{5}(5x-1) - \left(-\frac{\cos}{25}(5x-1)\right) + c\\
&= x\frac{\sin}{5}(5x-1) + \tfrac{\cos}{25}(5x-1) + c\\
&= \frac{1}{25}\left[5x\sin(5x-1) + \cos(5x-1) + c\right]
\end{aligned}
$$

Example 5.57 Evaluate $\int e^x \sin x\, dx$

Solution 5.49 Using integration by parts $\int u\, dv = uv - \int v\, du$
Let $u = \sin x$ and $dv = e^x\, dx$

$$du = \cos x\, dx \text{ and } v = \int e^x\, dx = e^x$$

$$\int e^x \sin x = e^x \sin x - \int e^x \cos x\, dx \tag{5.1}$$

We use integration by parts again for the latter integral: Let $u = \cos x$ and $dv = e^x\, dx$

$$du = -\sin\, dx \text{ and } v = \int e^x\, dx = e^x$$

$$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx \tag{5.2}$$

Substituting (3.2) in (3.1) :

$$\int e^x \sin x\, dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x\, dx\right)$$

$$\int e^x \sin x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$$

$$\text{Thus } 2\int e^x \sin x\, dx = e^x(\sin x - \cos x) + c$$

$$\implies \int e^x \sin x\, dx = \frac{1}{2}e^x(\sin x - \cos x) + c$$

■ **Example 5.58** Find $\int \sin^{-1} x\, dx$ ■

Solution 5.50 Let $u = \sin^{-1} x$ and $dv = dx$

$$du = \frac{1}{\sqrt{1-x^2}}\, dx \text{ and } v = \int dx = x$$

$$\int \sin^{-1} x\, dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx \tag{5.3}$$

Now we have $\int \frac{x}{\sqrt{1-x^2}}\, dx$ to integrate: Let $u = 1-x^2$, $\frac{du}{dx} = -2x \implies x\, dx = \frac{du}{-2}$

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = \int \frac{1}{\sqrt{u}}\left(\frac{du}{-2}\right) = \frac{1}{-2}\int u^{-1/2}\, du = -\sqrt{u} + c$$

$$\text{but } u = \sqrt{1-x^2}$$

$$\implies \int \frac{x}{\sqrt{1-x^2}}\, dx = -\sqrt{1-x^2} \tag{5.4}$$

Substituting (3.4) in (3.2) we have :

$$\int \sin^{-1} x\, dx = x\sin^{-1} x + \left(\sqrt{1-x^2}\right) + c$$

■ **Example 5.59** Find $\int \frac{\ln x}{x^2}\, dx$ ■

Solution 5.51 Let $u = \ln x$ and $dv = \frac{1}{x^2}\, dx$

$$du = \frac{1}{x}\, dx \text{ and } v = \int x^{-2} dx = \frac{-1}{x}$$

$$\implies \int \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\, dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + c$$

$$\int \frac{\ln x}{x}\,dx = -\frac{1}{x}(\ln x + 1) + c$$

■ **Example 5.60** Find $\int \cos^{-1} x\, dx$ ■

Solution 5.52 Let $u = \cos^{-1} x$ and $dv = dx$

$$du = \frac{-1}{\sqrt{1-x^2}}\,dx \text{ and } v = \int dx = x$$

$$\int \cos^{-1} x\, dx = x\cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}}\,dx = x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\,dx \tag{5.5}$$

Now we have $\int \frac{x}{\sqrt{1-x^2}}\,dx$ to integrate: Let $u = 1-x^2$, $\frac{du}{dx} = -2x \implies x\,dx = \frac{du}{-2}$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{du}{-2}\right) = \frac{1}{-2}\int u^{-1/2}\,du = -\sqrt{u} + c$$

$$\text{but } u = \sqrt{1-x^2}$$

$$\implies \int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} \tag{5.6}$$

Putting (3.6) in (3.5) yields:

$$\int \cos^{-1} x\, dx = x\cos^{-1} x - \left(\sqrt{1-x^2}\right) + c$$

■ **Example 5.61** Find $\int \tan^{-1} x\, dx$ ■

Solution 5.53 Let $u = \tan^{-1} x$ and $dv = dx$

$$du = \frac{1}{1+x^2}\,dx \text{ and } v = \int dx = x$$

$$\int \tan^{-1} x\, dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

Now we have $\int \frac{x}{1+x^2}\,dx$ to integrate: Let $u = 1+x^2$, $\frac{du}{dx} = 2x \implies dx = \frac{du}{2x}$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{x}{u}\cdot\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c$$

$$\text{but } u = 1 + x^2$$

$$\implies \int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln|1+x^2| + c$$

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$$

■ **Example 5.62** Evaluate $\int e^x \cos x\,dx$ ■

Solution 5.54 Using integration by parts $\int u\,dv = uv - \int v\,du$
Let $u = \cos x$ and $dv = e^x\,dx$

$$du = -\sin x\,dx \text{ and } v = \int e^x\,dx = e^x$$

$$\int e^x \cos x\,dx = e^x \cos x - \left(-\int e^x \sin x\,dx\right) = e^x \cos x + \int e^x \sin x\,dx \tag{5.7}$$

We use integration by parts again for the latter integral: Let $u = \sin x$ and $dv = e^x\,dx$

$$du = \cos x\,dx \text{ and } v = \int e^x\,dx = e^x$$

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx \tag{5.8}$$

Substituting (3.8) in (3.7) :

$$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$$

$$\text{Thus } 2\int e^x \cos x\,dx = e^x(\sin x + \cos x) + c$$

$$\implies \int e^x \cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x) + c$$

5.8.1 Integration (Rational Fractions)

■ **Example 5.63** $\int \frac{dx}{x^2-9}\,dx$ ■

Solution 5.55

$$
\begin{aligned}
\frac{1}{x^2-9} &= \frac{A}{(x+3)}+\frac{B}{(x-3)} \\
1 &= A(x-3)+B(x+3) \\
\text{when} x &= 3 \\
1 &= A(3-3)+B(3+3) \\
\\
1 &= 6B \\
B &= \frac{1}{6} \\
\\
\text{when} x &= -3 \\
1 &= A(-3-3)+B(3-3) \\
1 &= -6A \\
\\
A &= -\frac{1}{6} \\
\frac{1}{x^2-9} &= -\frac{1}{6(x+3)}+\frac{1}{6(x-3)}dx \\
\text{so}: \int \frac{dx}{x^2-9} &= -\frac{1}{6}\ln|(x+3)|+\frac{1}{6}\ln|(x-3)|+c \\
\int \frac{dx}{x^2-9}\,dx &= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|+c
\end{aligned}
$$

■ **Example 5.64** Find $\int \frac{x}{(x+2)(x+3)}\,dx$ ■

Solution 5.56

$$
\begin{aligned}
\frac{x}{(x+2)(x+3)} &= \frac{A}{(x+2)}+\frac{B}{(x+3)}\\
x &= A(x+3)+B(x+2)\\
\text{when} x &= -3\\
-3 &= A(-3+3)+B(-3+2)\\
\\
-3 &= -B\\
B &= 3\\
\\
\text{when} x &= -2\\
-2 &= A(-2+3)+B(-2+2)\\
-2 &= A\\
\\
A &= -2\\
\frac{x}{(x+2)(x+3)} &= -\frac{2}{x+2}+\frac{3}{x+3}dx\\
\text{so}: \int \frac{xdx}{(x+2)(x+3)} &= -2\ln|x+2|+3\ln|x+3|+c\\
\int \frac{x}{(x+2)(x+3)}\,dx &= \ln\left|\frac{(x+3)^3}{(x+2)^2}\right|+c
\end{aligned}
$$

■ **Example 5.65** Evaluate $\int \frac{x^4-4x^2+x+1}{x^2-4}\,dx$ ■

Solution 5.57 Since the degree of the numerator is at least as great as that of the denominator,carry out the long division,

$$\frac{x^4-4x^2+x+1}{x^2-4} = x^2+\frac{x+1}{x^2-4}$$

$$\Longrightarrow \int \frac{x^4-4x^2+x+1}{x^2-4}\,dx = \int \left(x^2+\frac{x+1}{x^2-4}\right)\,dx$$

$$\text{but } \frac{x+1}{x^2-4} = \frac{x+1}{(x+2)(x-2)} =$$

$$\begin{aligned}
\frac{x}{(x+2)(x-2)} &= \frac{A}{(x+2)} + \frac{B}{(x-2)} \\
x+1 &= A(x-2)+B(x+2) \\
\text{when } x &= 2 \\
3 &= A(2-2)+B(2+2) \\
3 &= 4B \\
B &= \frac{3}{4} \\
\text{when } x &= -2 \\
-1 &= A(-2-2)+B(-2+2) \\
-1 &= -4A \\
A &= \frac{1}{4} \\
\frac{x+1}{(x+2)(x-2)} &= \frac{1}{4}\frac{1}{x+2} + \frac{3}{4}\frac{1}{x-2} \\
\text{so} : \int \frac{x+1}{(x+2)(x-2)}\,dx &= \frac{1}{4}\ln|x+2| + \frac{3}{4}\ln|x-2| + c \\
\int \frac{x^4-4x^2+x+1}{x^2-4}\,dx &= \frac{1}{4}\ln\left|\frac{(x+2)}{(x-2)^3}\right| + c
\end{aligned}$$

■ **Example 5.66** Find $\int \frac{2x^2+1}{(x-1)(x-2)(x-3)}\,dx$ ■

Solution 5.58

$$\frac{2x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)}$$

$$2x^2+1 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

$$\text{when } x=1,\ \ 3=2A \implies A=\frac{3}{2}$$

$$\text{when } x=2,\ 9=-B \implies B=-9$$

$$\text{when } x=3,\ 19=2C \implies C=\frac{19}{2}$$

$$\frac{2x^2+1}{(x-1)(x-2)(x-3)}=\frac{3}{2}\cdot\frac{1}{(x-1)}-\frac{9}{(x-2)}+\frac{19}{2}\cdot\frac{1}{(x-3)}$$

$$\text{so}\ :\int\frac{2x^2+1}{(x-1)(x-2)(x-3)}\,dx=\frac{3}{2}\ln|x-1|-9\ln|x-2|+\frac{19}{2}\ln|x-3|+c$$

$$\int\frac{2x^2+1}{(x-1)(x-2)(x-3)}\,dx=\frac{1}{2}\ln\left|\frac{(x-3)^3(x-3)^{19}}{(x-2)^{18}}\right|+c$$

5.9 Improper Integrals

Definition 5.9.1 The integral $\int_a^b f(x)\,dx$ is called improper integral if

1. $a=-\infty$ or $b=\infty$ or both
2. $f(x)$ is unbounded at one or more points of $a\leq x\leq b$. Such points are called singularities of $f(x)$.

Integrals corresponding to 1. and 2. are called improper integrals of the **First** and **Second Kind** respectively. Integrals with both conditions 1. and 2. are called improper integrals of the **Third Kind**.

1. $\int_0^\infty \sin x^2\,dx$ is an improper integral of the first kind since the upper limit of integration is infinite.
2. $\int_0^4 \frac{dx}{x-3}\,dx$ is an improper integral of the second kind because $\frac{dx}{x-3}$ is not continuous at $x=3$.
3. $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ is an improper integral of the third kind because the upper limit of integration is infinite and $\frac{e^{-x}}{\sqrt{x}}$ is not continuous at $x=0$
4. $\int_0^\infty e^{-x}\,dx$ is an improper integral of the first kind since the upper limit of integration is infinite.

Definition 5.9.2 (a) If f is integrable on $a \leq x \leq \infty$, then

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

(b) If f is integrable on $-\infty < x \leq a$, then

$$\int_{-\infty}^a f(x)dx = \lim_{a \to -\infty} \int_b^a f(x)dx$$

(c) If f is integrable on $-\infty < x < \infty$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

Where c is any number.

Note 5.8 Note as well that this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

5.10 Convergence/Divergence of improper integral (first kind)

■ **Example 5.67** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_1^\infty \frac{1}{x^2}\, dx$$

■

Solution 5.59

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty}\left[\frac{x^{-1}}{-1}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{-1}{x}\right]_1^b = \lim_{b\to\infty}\left(\frac{-1}{b}+\frac{1}{1}\right)$$

$$= \frac{-1}{\infty}+1 = 0+1 = 1$$

The limit exists and is finite and so the integral converges and the integral's value is 1

■ **Example 5.68** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} xe^{x^2}\,dx$$

■

Solution 5.60 In this case we've got infinities in both limits and so we'll need to split the integral up into two separate integrals. We can split the integral up at any point, so let's choose $a = 0$ since this will be a convenient point for the evaluation process. The integral is then,

$$\int_{-\infty}^{\infty} xe^{x^2}\,dx = \int_{-\infty}^{0} xe^{x^2}\,dx + \int_{0}^{\infty} xe^{x^2}\,dx$$

We've now got to look at each of the individual limits.

$$\int_{-\infty}^{0} xe^{x^2}\,dx = \lim_{b\to-\infty}\int_{b}^{0} xe^{x^2}\,dx$$

$$= \lim_{b\to-\infty}\left[\frac{-1}{2}e^{-x^2}\right]_b^0 = \lim_{b\to-\infty}\left[-\frac{1}{2}+\frac{1}{2}e^{-b^2}\right] = -\frac{1}{2}$$

So, the first integral is convergent. Note that this does NOT mean that the second integral will also be convergent. So, let's take a look at that one.

$$\int_{0}^{\infty} xe^{x^2}\,dx = \lim_{b\to\infty}\int_{0}^{b} xe^{x^2}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^t = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-b^2}+\frac{1}{2}\right] = \frac{1}{2}$$

This integral is convergent and so since they are both convergent the integral we were actually asked to deal with is also convergent and its value is,

$$\int_{-\infty}^{\infty} xe^{x^2}\,dx = \int_{-\infty}^{0} xe^{x^2}\,dx + \int_{0}^{\infty} xe^{x^2}\,dx = -\frac{1}{2}+\frac{1}{2} = 0$$

■ **Example 5.69** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-1}^{\infty} \cos x\,dx$$

■

Solution 5.61

$$\int_{-1}^{\infty} \cos x\,dx = \lim_{b\to\infty}\int_{-1}^{b} \cos x\,dx$$

$$= \lim_{b\to\infty}[\sin x]_{-1}^{b} = \lim_{b\to\infty}[\sin(b)-\sin(-1)]$$

This limit doesn't exist and so the integral is divergent.

5.10.1 Special Improper integrals of first Kind

Definition 5.10.1 1. Geometric/exponential integrals

$$\int_{a}^{\infty} e^{-tx}\,dx$$

If $t > 0$: converges, $t \leq 0$: diverges

2. The p integral of the first kind

$$\int_a^\infty \frac{dx}{x^p}, \text{where p is a constant and} a > 0$$

converges if $p > 1$, diverges if$p \leq 1$

3. Gamma function : Γ

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx = n!$$

■ **Example 5.70** Determine if the following integral is convergent or divergent. $\int\limits_1^\infty e^{-2x}\, dx$
■

Solution 5.62 From the geometric integral $t = 2 > 0$, $\int\limits_1^\infty e^{-2x}\, dx$: converges

■ **Example 5.71** Determine if the following integral is convergent or divergent: $\int_0^\infty x^3 e^{-x}\, dx$
■

Solution 5.63 $\int_0^\infty x^3 e^{-x}\, dx = 3! = 6$: converges

■ **Example 5.72** $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$ ■

Solution 5.64 From the p integral $p = \frac{1}{2} \leq 1$, $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$: diverges

■ **Example 5.73** $\int\limits_1^\infty e^{3x}\, dx$ ■

Solution 5.65 From the geometric integral $t = -3 < 0$, $\int\limits_1^\infty e^{3x}\, dx$: diverges

5.11 Convergence/Divergence of improper integral (second kind)

■ **Example 5.74** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^{\frac{\pi}{2}} \sec x \, dx$$

■

Solution 5.66 This is an improper integral of second kind because $\sec x$ is not continuous at $\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \sec x \, dx = \lim_{b \to \frac{\pi}{2}^-} \int_0^{b} \sec x \, dx$$

$$\lim_{b \to \frac{\pi}{2}^-} \int_0^{b} \sec x \, dx = \lim_{b \to \frac{\pi}{2}^-} \left[\ln|\sec x + \tan x|\right]_0^t = \lim_{b \to \frac{\pi}{2}^-} \ln|\sec t + \tan t| = \infty$$

This limit doesn't exist and so the integral is divergent

■ **Example 5.75** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^{3} \frac{1}{\sqrt{3-x}} \, dx$$

■

Solution 5.67 The problem point is the upper limit so we are in the first case above.

$$\int_0^{3} \frac{1}{\sqrt{3-x}} \, dx = \lim_{b \to 3^-} \int_0^{b} \frac{1}{\sqrt{3-x}} \, dx$$

$$= \lim_{b \to 3^-} \left[-2\sqrt{3-x}\right]_0^b = \lim_{b \to 3^-} \left[-2\sqrt{3-b} + 2\sqrt{3}\right] = 2\sqrt{3}$$

The limit exists and is finite and so the integral converges and the integral's value is $2\sqrt{3}$

■ **Example 5.76** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^{3} \frac{1}{x^3} \, dx$$

■

Solution 5.68 This integrand is not continuous at $x = 0$ and so we'll need to split the integral up at that point.

$$\int_{-2}^{3} \frac{1}{x^3}\, dx = \int_{-2}^{0} \frac{1}{x^3}\, dx + \int_{0}^{3} \frac{1}{x^3}\, dx$$

Now we need to look at each of these integrals and see if they are convergent.

$$\int_{-2}^{0} \frac{1}{x^3}\, dx = \lim_{b\to 0^-} \int_{-2}^{t} \frac{1}{x^3}\, dx$$

$$= \lim_{b\to 0^-} \left[-\frac{1}{2x^2}\right]_{-2}^{t} = \lim_{b\to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{8}\right) = -\infty$$

At this point we're done. One of the integrals is divergent that means the integral that we were asked to look at is divergent. We don't even need to bother with the second integral.

5.12 Convergence/Divergence of improper integral (third kind)

▪ **Example 5.77** Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{\infty} \frac{1}{x^2}\, dx$$

▪

Solution 5.69 This is an integral over an infinite interval that also contains a discontinuous integrand. To do this integral we'll need to split it up into two integrals. We can split it up anywhere, but pick a value that will be convenient for evaluation purposes.

$$\int_{0}^{\infty} \frac{1}{x^2}\, dx = \int_{0}^{1} \frac{1}{x^2}\, dx + \int_{1}^{\infty} \frac{1}{x^2}\, dx$$

In order for the integral in the example to be convergent we will need BOTH of these to be convergent. If one or both are divergent then the whole integral will also be divergent. We know that the second integral is convergent by the fact given in the infinite interval portion above. So, all we need to do is check the first integral.

$$\int_{0}^{1} \frac{1}{x^2}\, dx = \lim_{b\to 0^-} \int_{t}^{1} \frac{1}{x^2}\, dx$$

$$= \lim_{b\to 0^-} \left[-\frac{1}{x}\right]_t^1 = \lim_{b\to 0^-} \left(-1+\frac{1}{t}\right) = \infty$$

So, the first integral is divergent and so the whole integral is divergent.

5.13 Comparison Test for Improper Integrals

Sometimes an improper integral is too difficult to evaluate. One technique is to compare it with a known integral. The theorem below shows us how to do this.

Definition 5.13.1 Suppose that f and g are two continuous functions for $x \geq a$ such that $0 \leq g(x) \leq f(x)$.Then, the following is true:

1. If $\int_a^\infty f(x)\, dx$ converges then $\int_a^\infty g(x)\, dx$ also converges.
2. $\int_a^\infty g(x)\, dx$ diverges then $\int_a^\infty f(x)\, dx$ also diverges

■ **Example 5.78** Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2 x}{x^2}\, dx$$

■

Solution 5.70 Let's take a second and think about how the Comparison Test works. If this integral is convergent then we'll need to find a larger function that also converges on the same interval. Likewise, if this integral is divergent then we'll need to find a smaller function that also diverges.

So, it seems like it would be nice to have some idea as to whether the integral converges or diverges ahead of time so we will know whether we will need to look for a larger (and convergent) function or a smaller (and divergent) function.

To get the guess for this function let's notice that the numerator is nice and bounded and simply won't get too large. Therefore, it seems likely that the denominator will determine the convergence/divergence of this integral and we know that

$$\int_2^\infty \frac{1}{x^2}\, dx$$

converges since $p = 2 > 1$ (p- integral). So let's guess that this integral will converge. So we now know that

$$\frac{\cos^2 x}{x^2} < \frac{1}{x^2} = f(x)$$

so by the Comparison Test we know that

$$\int_{2}^{\infty} \frac{\cos^2 x}{x^2}\, dx$$

must also converge.

Note 5.9 $0 \leq \cos^2 x \leq 1$

■ **Example 5.79** Study the convergence of $\int_{1}^{\infty} e^{-x^2}\, dx$ ■

Solution 5.71 We need to find $f(x) = e^{-x}$,which converges because $t = -1 < 1$(geometric integral).

We also know that $e^{-x^2} < e^{-x}$ so by the Comparison Test we know that $\int_{1}^{\infty} e^{-x^2}\, dx$ will also converge.

■ **Example 5.80** Determine if the following integral is convergent or divergent.

$$\int_{1}^{\infty} \frac{1+3\sin^4(2x)}{\sqrt{x}}\, dx$$

■

Solution 5.72 First notice that as with the first example, the numerator in this function is going to be bounded since the sine is never larger than 1. Therefore, since the exponent on the denominator is less than 1 we can guess that the integral will probably diverge. We will need a smaller function that also diverges. We know that $0 \leq \sin^4(2x) \leq 1$In particular, this term is positive and so if we drop it from the numerator the numerator will get smaller. This gives,

$$\frac{1+3\sin^4(2x)}{\sqrt{x}} > \frac{1}{\sqrt{x}} = g(x)$$

$$\int_{1}^{\infty} \frac{1}{\sqrt{x}}\, dx$$

diverges so by the Comparison Test

$$\int_{1}^{\infty} \frac{1+3\sin^4(2x)}{\sqrt{x}}\, dx$$

also diverges.

■ **Example 5.81** Study the convergence of $\int\limits_0^\infty \frac{1}{e^x+1}\,dx$ ■

Solution 5.73 We need to find $f(x) = \frac{1}{e^x} = e^{-x}$,which converges because $t = -1 < 1$(geometric integral).

We also know that $\frac{1}{e^x+1} < e^{-x}$ so by the Comparison Test we know that $\int\limits_0^\infty \frac{1}{e^x+1}\,dx$ will also converge.

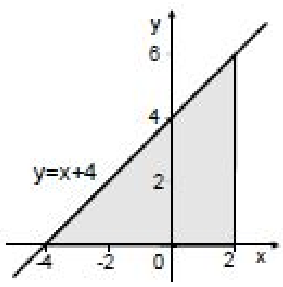
5.14 Area and Integration

In problems 1 through 5, Find the area of the region R.

■ **Example 5.82** R is the triangle with vertices (-4, 0), (2, 0) and (2, 6). ■

Solution 5.74 From the corresponding graph below you see that the region in question is bellow the line $y = x+4$ above the x-axis, and extends from $x = -4$ to $x = 2$. Hence,

$$A = \int_{-4}^{2}(x+4)dx = \left(\frac{1}{2}x^2+4x\right)\Big|_{-4}^{2} = (2+8)-(8-16) = 18.$$



■ **Example 5.83** R is the region bounded by the curve $y = e^x$, the line $x = 0$ and $x = \ln\frac{1}{2}$,and the x axis. ■

Solution 5.75 Since $\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2 \simeq -0.7$,from the corresponding graph below you see that the region in question is bellow the line $y = e^x$ above the x axis, and extends from $x = \ln\frac{1}{2}$ to $x = 0$

Hence,

$$A=\int_{\ln\frac{1}{2}}^{0} e^x dx = e^x\Big|_{\ln\frac{1}{2}}^{0} = e^0 - e^{\ln\frac{1}{2}} = 1-\frac{1}{2}=\frac{1}{2}.$$

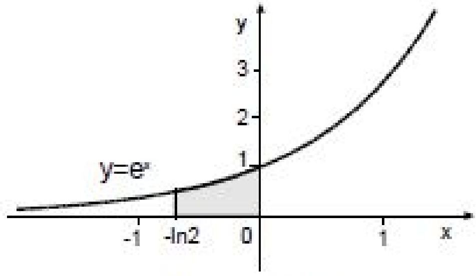


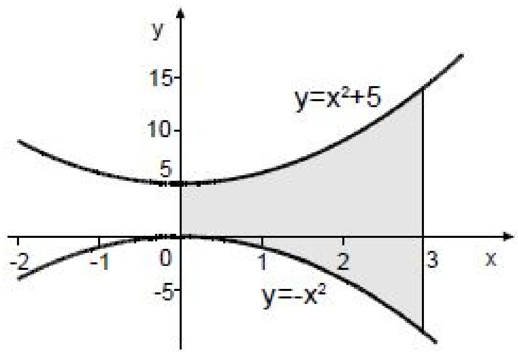
Figure 6.2.

Example 5.84 R is the region bounded by the curve $y = x^2 + 5$ and $y = -x^2$, the line $x = 3$, and the y axis.

Solution 5.76 Notice that the region in question is bounded above by the curve $y = x^2 + 5$ and below by the curve $y = -x^2$ and extends from $x = 0$ to $x = 3$. Hence,

$$A \int_0^3 [(x^2+5) - (-x^2)]dx = \int_0^3 (2x^2+5)dx = \left(\frac{2}{3}x^3 + 5x\right)\Bigg|_0^3 = 18 + 15 = 33.$$

The sketch below shows the region



Example 5.85 R is the region bounded by the curve $y = \frac{1}{x^2}$ and the lines $y = x$ and $y = \frac{x}{8}$.

Solution 5.77 First make a sketch of the region as shown below and find the points of intersection of the curve and the lines by solving the equations

$$\frac{1}{x^2} = x \qquad \frac{1}{x^2} = \frac{x}{8} \; i.e. \;\; x^3 = 1 \;\; and \;\; 8$$

to get

$$x = 1 \qquad and \qquad x = 2.$$

Then break R into two subregions,R_1 that extends from $x = 0$ to $x = 1$ and R_2 that extends from $x = 1$ to $x = 2$ as shown below.Hence, the area of the region R_1 is

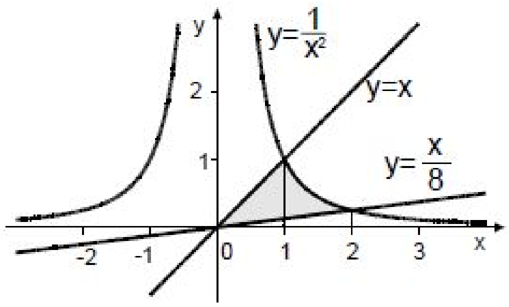
$$A_1 = \int_0^1 \left(x - \frac{x}{8}\right) dx = \int_0^1 \frac{7}{8} x dx = \frac{7}{16} x^2 \Bigg|_0^1 = \frac{7}{16}$$

and the area of the region R_2 is

$$A_2 = \left(\frac{1}{x^2} - \frac{x}{8}\right) dx = \left(-\frac{1}{x} - \frac{1}{16}x^2\right)\Bigg|_1^2 = -\frac{1}{2} - \frac{1}{4} + 1 + \frac{1}{16} = \frac{5}{16}.$$

Thus, the area of the region R is the sum

$$A = A_1 + A_2 = \frac{12}{16} = \frac{3}{4}.$$



■ **Example 5.86** R is the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$. ■

Solution 5.78 Sketch the region as shown below.Find the points of intersection by solving the equations of the two curves simultaneously to get

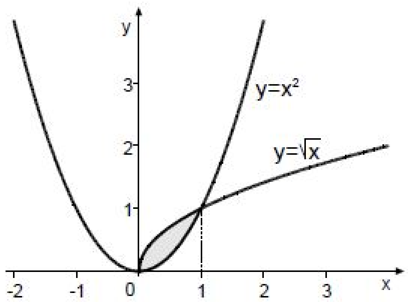
$$x^2 = \sqrt{x} \quad x^2 - \sqrt{x} = 0 \quad \sqrt{x}(x^{\frac{3}{2}} - 1) = 0$$

$$x = 0 \qquad \text{and} \qquad x = 1.$$

The corresponding points (0, 0) and (1, 1) are the points of intersection.

Notice that for $0 \le x \le 1$, the graph of $y = \sqrt{x}$ lies above that of $y = x^2$. Hence,

$$A = \int_0^1 (\sqrt{x} - x^2)dx = \left(\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right) \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$



Exercise 5.1 Solve the following integrals.

1. $\displaystyle\int \frac{3}{2x+4}\,dx$
2. $\displaystyle\int \frac{x^3}{x^4-1}\,dx$
3. $\displaystyle\int \frac{\ln x}{x}\,dx$
4. $\displaystyle\int \frac{2x+2}{x^2+2x+7}dx$
5. $\displaystyle\int \left(\frac{3}{x-1}-\frac{4}{x-2}\right)dx$
6. $\displaystyle\int (8x+4)^2 dx$
7. $\displaystyle\int \frac{1}{\sqrt{x-1}}dx$
8. $\displaystyle\int (4-2t^2)^2 t\,dt$
9. $\displaystyle\int x^2\sqrt[3]{x^3+5}\,dx$
10. $\displaystyle\int \frac{\cos 3x}{\sin^2 3x}dx$
11. $\displaystyle\int \frac{xdx}{(x+2)(x+3)}$

■

6. Vector Integration

6.1 Rules Of Vector Integration

Definition 6.1.1 Let $\dfrac{d}{dt}[\vec{f}(t)] = \vec{F}(t) \Rightarrow \vec{f}(t) = \int \vec{F}(t)dt + c$ is called vector integration.

1. $\displaystyle\int_a^b \vec{f}(t)dt = F(b) - F(a)$
2. $\displaystyle\int_a^b \vec{f}(t)dt = -\int_b^a \vec{f}(t)dt$
3. $\displaystyle\int_a^b \vec{f}(t)dt = \int_a^c \vec{f}(t)dt + \int_c^b \vec{f}(t)dt$

■ **Example 6.1** Find the value of $\displaystyle\int_0^1 (t\hat{i} + t^2\hat{j} + t^3\hat{k}) \cdot dt$ ■

Solution 6.1

$$\int_0^1 (t\hat{i} + t^2\hat{j} + t^3\hat{k}) \cdot = \left[\frac{t^2}{2}\hat{I} + \frac{t^3}{3}\hat{j} + \frac{t^4}{4}\hat{k}\right]_0^1$$
$$= \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{4}\hat{k}$$

Generally, integrals are classified as line integrals, surface integrals and volume integrals.

6.2 Line Integrals

Definition 6.2.1 We have so far integrated "over" intervals.We now investigate integration over or "along" a curve—"line integrals" are really "curve integrals". The line integral of $f(x,y)$ along c is denoted by

$$\int_c f(x,y)ds$$

We use a ds here to acknowledge the fact that we are moving along the curve, c, instead of the x-axis (denoted by dx) or the y-axis (denoted by dy). Because of the ds this is sometimes called the line integral of f with respect to arc length.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Definition 6.2.2
Here are some of the more basic curves that we'll need to know how to do as well as limits on the parameter if they are required.

CURVE	Parametric Equation
1.Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$x = a\cos t \quad 0 \le t \le 2\pi$ $y = b\sin t$
2. Circle $x^2+y^2=r^2$	$x = r\cos t \quad 0 \le t \le 2\pi$ $y = r\sin t$
3. Line segment from (x_0,y_0) to (x_1,y_1) $r(t) = (1-t) < x_0, y_0 > +t < x_1, y_1 >$	$x = (1-t)x_0 + tx_i$ $y = (1-t)y_0 + ty_i$ $0 \le t \le 1$

Let's take a look at an example of a line integral

■ **Example 6.2** Evaluate $\int_c 4x^2 dx$ where c is the line segment $(-2,-1)$ to $(1,2)$ ■

Solution 6.2

$$
\begin{aligned}
r(t) &= (1-t) < x_0, y_0 > + t < x_1, y_1 > \\
r(t) &= (1-t) < -2, -1 + t < 1, 2 > \\
&= < -2(1-t), -(1-t) > + 1 < t, 2t > \\
&= < -2 + 2t + t, \quad -1 + t + 2t \\
&= < -2 + 3t, \quad -1 + 3t > \\
ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
x &= -2 + 3t, \quad y = -1 + 3t \\
\frac{dx}{dt} &= 3 \quad \frac{dy}{dt} = 3 \\
ds &= \sqrt{(3)^2 + (3)^2} dt = \sqrt{18} dt = 3\sqrt{2} dt \\
\text{but } x &= -2 + 3t \\
\int_0^1 4(-2+3t)^3 dt &= -15\sqrt{2}
\end{aligned}
$$

■ **Example 6.3** Find the value $\int_c \vec{F} \cdot d\vec{r}$, if $F = x\hat{i} - y\hat{j}$ and c is given by $x^2 + y^2 = 4$ ■

Solution 6.3 We first need a parameterization of the circle. This is given by,

$$r = xi + yj = 2\sin\theta + 2\cos\theta$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_c F_1 dx\hat{i} + F_2 dy\hat{j}$$

$$= \int_c (xdx - ydy)$$

Put,

$$x = 2sin\theta, y = 2cos\theta, dx = 2cos\theta d\theta \text{ and } dy = -2sin\theta d\theta$$

$$\int_0^{2\pi} [2sin\theta(2cos\theta)d\theta] - [2cos\theta(-2sin\theta)d\theta]$$

$$\int_0^{2\pi} 8cos\theta sin\theta d\theta = 4\int_0^{2\pi} sin2\theta d\theta$$

$$= 4 \cdot -\frac{1}{2}[cos2\theta]_0^{2\pi} = -2[1-1] = 0$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = 0$$

Note 6.1

1. If $\vec{F}$is force then the total work done by a force is $\int_c \vec{F}\cdot d\vec{r}$.
2. If $\int_c \vec{F}\cdot d\vec{r} = 0 \Rightarrow \vec{F}$ is called conservative force.

6.3 Green's Theorem

Definition 6.3.1 In this section we are going to investigate the relationship between certain kinds of line integrals (on closed paths) and double integrals. If $M(x,y)$ and $N(x,y)$ are continuous and having continuous first order partial derivatives bounded by a closed curve 'c' then,

$$\int_c (Mdx + Ndy) = \int_{X_1}^{X_2}\int_{y_1}^{y_2}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dydx$$

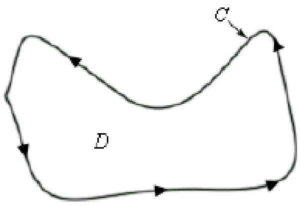


Figure 6.1

First, notice that because the curve is simple and closed there are no holes in the region D. Also notice that a direction has been put on the curve. We will use the convention here that the curve c has a positive orientation if it is traced out in a counter-clockwise direction. Another way to think of a positive orientation (that will cover much more general curves as well see later) is that as we traverse the path following the positive orientation the region D must always be on the left. .

■ **Example 6.4** Use Green's Theorem to evaluate $\int_c (3x+4y)dx + (2x-2y)dy$, where c: $x^2 + y^2 = 4$ ■

Solution 6.4 Here

$$M = 3x + 4y,\ \ N = 2x - 3y$$

By Green 's theorem

$$0 \le r \le 2,\ \ 0 \le \theta \le 2\pi,\ \ dxdy = rdrd\theta$$

$$\int_R (Mdx + Ndy) = \int_0^{2\pi} \int_0^2 \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) r dr d\theta$$
$$= -2 \int_0^{2\pi} \int_0^2 r dr d\theta = (-2)(\pi r^2) = -8\pi$$

6.4 Multiple Integrals

They are mainly classified as

1. Double Integrals
2. Triple Integrals

6.4.1 Double Integrals

Definition 6.4.1 $\int\int f(x,y)\, dxdy$ is called the double integral where $R = [a,b] \times [c,d]$. Also, we will initially assume that $f(x,y) \geq 0$ although this doesn't really have to be the case. Let's start out with the graph of the surface S given by graphing $f(x,y)$ over the rectangle R.

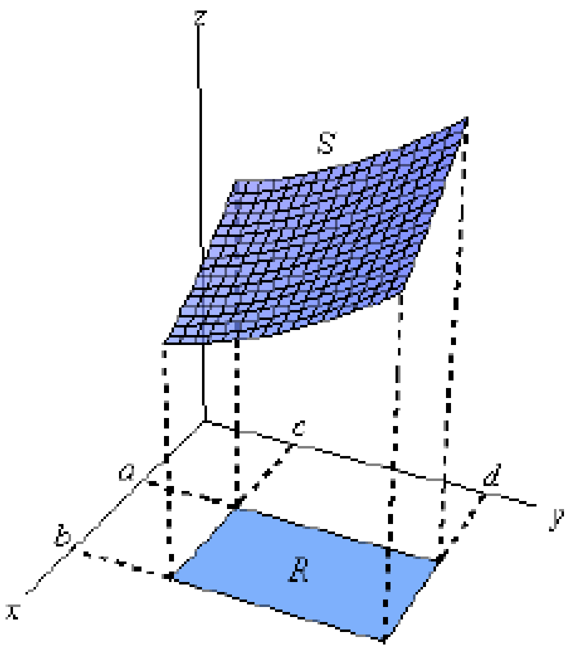


Figure 6.2

6.4.2 Methods to Evaluate Double Integrals

Theorem 6.4.1 — Fubini's Theorem. If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$ then,

$$\int\int f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dxdy=\int_a^b\int_c^d f(x,y)\,dydx$$

These integrals are called iterated integrals.

■ **Example 6.5** Find the value of $\int\int 6xy^2\,dA,\ \ R=[2,4]\times[1,2]$ ■

Solution 6.5 In this case we will integrate with respect to y first. So, the iterated integral that we need to compute is,

$$\int\int 6xy^2\,dA=\int_2^4\int_1^2 6xy^2\,dxdy$$

$$\int\int 6xy^2\,dA=\int_2^4\left[2xy^3\right]_1^2\,dx$$

$$=\int_2^4 16x-2x\,dx=\int_2^4 14x\,dx$$

Remember that we treat the x as a constant when doing the first integral and we don't do any integration with it yet. Now, we have a normal single integral so let's finish the integral by computing this.

$$\int\int 6xy^2\,dA=\left[7x^2\right]_2^4=84$$

■ **Example 6.6** Find the value of $\int_{y=0}^1\int_{x=\sqrt{y}}^y (x+y)\,dxdy$ ■

Solution 6.6

$$\int_0^1\left[\frac{x^2}{2}+xy\right]_{\sqrt{y}}^y\cdot dy=\int_0^1\left(\frac{y^2}{2}+y^2\right)-\left(\frac{y}{2}+y^{\frac{3}{2}}\right)$$

$$\left[\frac{y^3}{6}+\frac{y^3}{3}-\frac{y^2}{4}-\frac{2y^{\frac{5}{2}}}{5}\right]_0^1$$

$$=\frac{1}{6}+\frac{1}{3}-\frac{1}{4}-\frac{2}{5}$$

$$=\frac{10+20-15-24}{60}=\frac{30-39}{60}=\frac{-9}{60}=\frac{-3}{20}$$

■ **Example 6.7** Find the area bounded between the curves $y = x^2, y = x$ ■

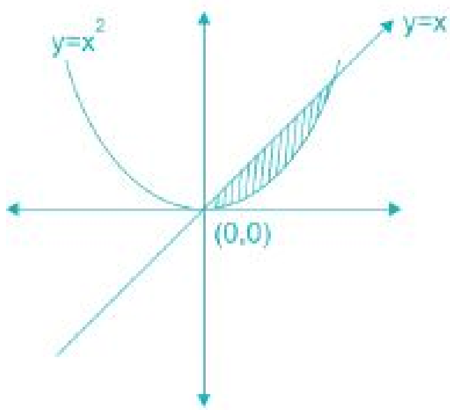


Figure 6.3

Solution 6.7

$$Area = \int_0^1 \int_{x^2}^{x} \cdot dy \cdot dx$$

$$= \int_0^1 [y]_{x^2}^{x} \cdot dx = \int_0^1 (x - x^2) \cdot dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

6.4.3 Triple Integrals

Definition 6.4.2 Triple integral is defined as:

$$\int\int\int f(x,y,z)\, dV = \int\int\int f(x,y,z)\, dxdydz$$

The order of integration :-
If z_1, z_2 are function of x and y and y_1, y_2 are function of x and x_1, x_2 re constant the order of integration is

$$\int_{x_1=a}^{x_2=b} \int_{y_1=f(x)}^{y_2=f(x)} \int_{z_1=\phi(x,y)}^{z_2=\psi(x,y)} \left[\left[\left[f(x,y,z) \quad \underset{(1)}{\underset{\downarrow}{\cdot dz}} \right] \underset{(2)}{\underset{\downarrow}{dy}} \right] \underset{(3)}{\underset{\downarrow}{dx}} \right]$$

■ **Example 6.8** Find the value of $\int_0^1 \int_0^x \int_0^y xyz\, dzdydx$ ■

Solution 6.8 $\int_0^1 \int_0^x \int_0^y xyz\, dzdydx = \int_0^1 \int_0^x xy \left[\frac{z^2}{2}\right]_0^y dy \cdot dx = \frac{1}{2}\int_0^1 \int_0^x xy^3 \cdot dy \cdot dx$

$$= \frac{1}{2}\int_0^1 x \left[\frac{y^4}{4}\right]_0^x \cdot dx$$

$$= \frac{1}{8}\left[\int_0^1 x^5 \cdot dx\right] = \left[\frac{x^6}{8(6)}\right]_0^1 = \frac{1}{48}$$

6.5 Surface Integrals

Definition 6.5.1 An integral which is to be evaluated over a surface is called surface integral.
Mathematical formula for surface integral is $= \int_s \int \vec{F} \cdot \vec{N} ds$, where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$
$\vec{N}$ = Outward unit normal vector and ds =projection of surface on to the planes

6.5.1 Method To Evaluate Surface Integral

Definition 6.5.2 1. If the surface S is on to $XY(Z = 0)$ plane then

$$\int\int_s \vec{F} \cdot \hat{N} ds = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \vec{F} \cdot \hat{N} \frac{dydx}{|\hat{N} \cdot \hat{k}|}$$

Where $\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$ if $\phi(x,y,z) = c$ is given otherwise $\hat{N} = \hat{k}$.

2. If the surface S is on to $XZ(Y = 0)$ plane then

$$\int\int_s \vec{F} \cdot \hat{N} ds = \int_{x_1}^{x_2} \int_{z_1}^{z_2} \vec{F} \cdot \hat{N} \frac{dzdx}{|\hat{N} \cdot \hat{j}|}$$

Where $\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$ if $\phi(x,y,z) = c$ is given otherwise $\hat{N} = \hat{j}$.

3. If the surface S is on to $YZ(X = 0)$ plane then

$$\int\int_s \vec{F} \cdot \hat{N} ds = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \vec{F} \cdot \hat{N} \frac{dydz}{|\hat{N} \cdot \hat{i}|}$$

Where $\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$ if $\phi(x,y,z) = c$ is given otherwise $\hat{N} = \hat{i}$.

■ **Example 6.9** The value of $\int\int_s \vec{F} \cdot \hat{N} ds$,where $\vec{F} = z\hat{i} + x\hat{j} - 3y^2\hat{k}$ and s is the surface of cylinder $x^2 + y^2 = 16$ and If the surface s is on to $YZ(X = 0)$ plane between $z = 0, z = 5$ is? ■

Solution 6.9 $\phi = x^2 + y^2 - 16$ and

$$\hat{N} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{\hat{i}(2x)+\hat{j}(2y)}{2\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{16}} = \frac{x\hat{i}+y\hat{j}}{4}$$

$$\vec{F}\cdot\hat{N} = \frac{zx+xy}{4}$$

$$\int\int_s \vec{F}\cdot\hat{N} = \int\int_s \frac{zx+xy}{4}\cdot ds$$

$x^2 + y^2 = 16$ put $x = 0 \Rightarrow y = \pm 4$
Let the surface s is projected on to YZ plane, and

$$|\hat{N}\cdot\hat{i}| = \frac{x}{4}$$

$$\int\int_s \vec{F}\cdot\hat{N}ds = \int_{z=0}^{5}\int_0^4 \frac{x}{4}(z+y)\cdot\frac{dzdy}{x}(4)$$

$$\int_{z=0}^{5}\left(zy+\frac{y^2}{2}\right)\cdot dz = \int_0^5 (4z+8)dz = [2z^2+8z]_0^5$$

$$= 50 + 40 = 90$$

6.6 Stoke's Theorem

Definition 6.6.1 According to this theorem, let S be the two sided open surface bounded by a closed curve C and $\vec{F}$ be differentiable vector function then

$$\oint_c \vec{F}\cdot d\vec{r} = \int\int_s (\vec{\nabla}\times\vec{F})\cdot\hat{N}ds$$

It gives relation between line integral and surface integral.

■ **Example 6.10** By Stoke's theorem the value of $\int\int_s r\cdot d\vec{r}$ where c is $x^2+y^2=16$, $z=0$ is where $\vec{r} = x\hat{I} + y\hat{J} + z\hat{k}$ ■

Solution 6.10

$$\int \vec{r}\cdot d\vec{r} = \int\int(\vec{\nabla}\times\vec{r})\cdot\hat{N}\cdot ds$$

$$\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{I} & \hat{J} & \hat{k} \\ \partial x & \partial y & \partial z \\ x & y & z \end{vmatrix} = \hat{I}(0)+\hat{J}(0)+\hat{k}(0) = 0$$

$$\int_c \vec{r}\cdot dr = \int\int(0)\cdot\hat{N}(ds) = 0$$

6.7 Volume Integrals

Definition 6.7.1 The integral $\int\int\int_v \vec{F}dv$ is called volume integral and it is given as

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}(F_1\hat{I}+F_2\hat{J}+F_3\hat{k})\cdot dzdydx$$

$$= \hat{i}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}F_1\ dzdydx + \hat{j}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}F_2dzdydx + \hat{k}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}F_3dzdydx$$

6.8 Gauss Divergence Theorem

It gives relation between surface integral to volume integral.

Definition 6.8.1 Let 'V' be the volume bounded by a closed surface S and $\vec{F}$ a differentiable vector function then,

$$\int\int_s(\vec{F}\cdot\hat{N})ds = \int\int_v\int(\vec{\nabla}\cdot\vec{F})dV$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_{12}}{\partial y}+\frac{\partial F_3}{\partial z}\right)dzdydx$$

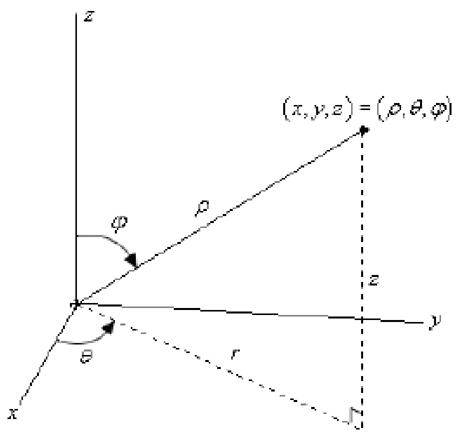


Figure 6.4

Here are the conversion formulas for spherical coordinates.

$$x = \rho sin\phi cos\theta \qquad y = \rho sin\phi sin\theta \qquad z = \rho cos\phi$$

For Cylindrical Coordinate we have

$$dV = rdzdrd\theta$$

For Spherical Coordinate we have

$$x^2+y^2+z^2=\rho^2$$

$$dV=\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

We also have the following restrictions on the coordinates.

$$\rho\geq 0 \qquad 0\leq\phi\leq\pi$$

■ **Example 6.11** By Gauss Divergence theorem find the value of $\int\int_s \vec{r}\cdot\hat{N}ds$ where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}\; dS$ is the surface of the sphere $x^2+y^2+z^2=1$ ■

Solution

Solution 6.11

$$0\leq\rho\leq 0 \qquad 0\leq\phi\leq\pi \qquad 0\leq\theta\leq 2\pi$$

$$\vec{\nabla}\cdot\vec{r}=\frac{\partial}{\partial x}(X)+\frac{\partial}{\partial y}(Y)+\frac{\partial}{\partial z}(Z)=3$$

$$\int\int_s \vec{r}\cdot\hat{N}ds=3\int_o^1\int_0^{\pi}\int_0^{2\pi}3\; dV=3\int_o^1\int_0^{\pi}\int_0^{2\pi}\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$=3\left(\frac{4\pi}{3}r^3\right)=4\pi$$