CENG 251: ELECTRIC CIRCUITS ANALYSIS AND DESIGN

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Content

- Basic Concepts
- Basic Laws
- Methods of Analysis
- Circuit Theorems
- Capacitors and Inductors
- First – Order Circuits
- Second – Order Circuits
- Laplace Transforms
- Operational Amplifiers
- Passive and Active Filters

Assessment

- Continuous assessment – 40 marks
  - Weekly quizzes and assignments - 20 marks
  - Mid-semester examination – 20 marks

- End of semester examination – 60mks
  - Multiple choice questions – 20 marks
  - Written examination – 40 marks

Further study

- Fundamentals of Electric Circuits (5th Edition): Charles K. Alexander, Matthew N. O. Sadiku, 2013.

Click here for lecture materials

Basic Concepts

Basic Electrical Quantities

Quantity	Unit	Basic equation
Electric current (I): the time rate of change of charge.	amperes (A)	$i \triangleq \frac{dq}{dt}$ q – charge (C) t – time (s)
Voltage / potential difference (V): the energy required to move a unit charge through an element.	volts (V)	$v_{ab} \triangleq \frac{dw}{dq}$ w – energy (J) q – charge (C)
Power (P): the time rate of expending or absorbing energy.	watts (W)	$P = V.I$
Energy (E) the capacity to do work.	joules (J)	$w = \int_{t_0}^{t} p\,dt = \int_{t_0}^{t} vi\,dt$
Resistance (R): the physical property of an element or device that impedes the flow of current.	ohms (Ω)	$R = \frac{V}{I}$

Circuit Elements

- An element is the basic building block of a circuit
- **Circuit analysis** is the process of determining voltages across (or the currents through) the elements of the circuit.
- There are two types of elements, namely (1) passive element and (2) active element
- An active element generates energy, while a passive element does not generate energy.
- The most important active elements are voltage or current sources that generally deliver power to the circuit connected to them.
- There are two kinds of sources: independent and dependent sources.
- An **ideal independent source** provides a specified voltage or current that is completely independent of other circuit elements.
- An ideal dependent (or controlled) source is an active element in which the source quantity is controlled by another voltage or current.
- Dependent sources are usually designated by diamond-shaped symbols as shown in Fig. 1.

- There are four possible types of dependent sources, namely:
  1. A voltage-controlled voltage source (VCVS)
  2. A current-controlled voltage source (CCVS)
  3. A voltage-controlled current source (VCCS)
  4. A current-controlled current source (CCCS)

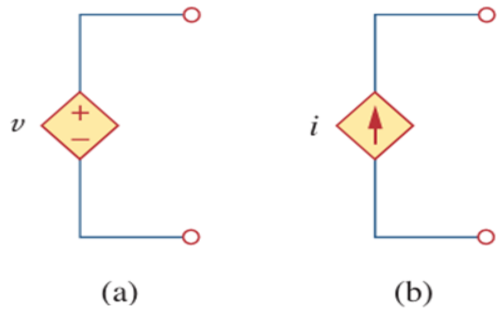
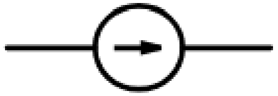


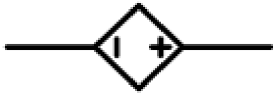
Figure 1
Symbols for: (a) dependent voltage source, (b) dependent current source.

Circuit Elements

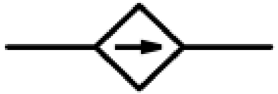
Independent voltage source



Independent current source



Dependent voltage source



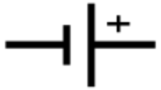
Dependent current source

Diode

Capacitor

Inductor

Resistor



DC voltage source

AC voltage source

Variable resistor

Basic Laws

Ohm's Law

- **Ohm's law** states that the voltage v across a resistor is directly proportional to the current i flowing through the resistor.

$$v \propto i$$
$$v = iR$$

- The resistance of any material with a uniform cross-sectional area A, depends on A and its length l, as shown in Fig. 2.

$$R = \rho \frac{\ell}{A}$$

- Where ρ is the resistivity of the material in ohm-meters.
- Conductance G is a measure of how well an element will conduct electricity, measured in mho (ohm spelled backwards).
- It's the reciprocal of R. $G = \frac{1}{R}$, unit ℧, SI unit is siemens (S).

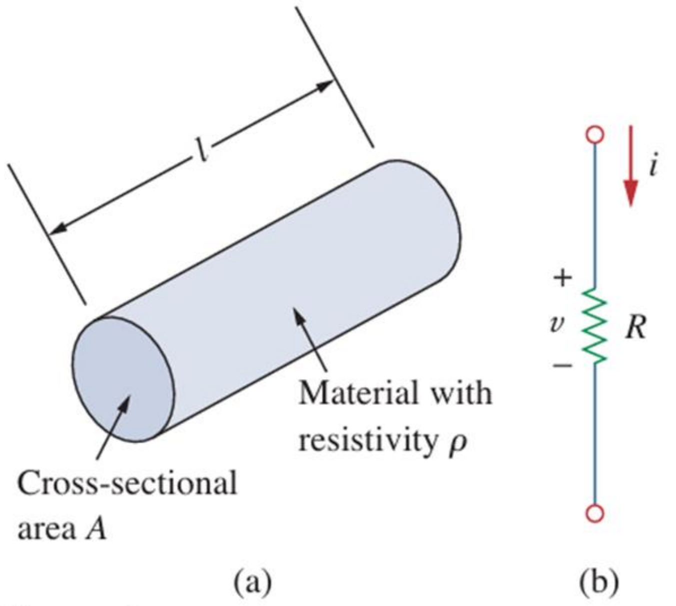
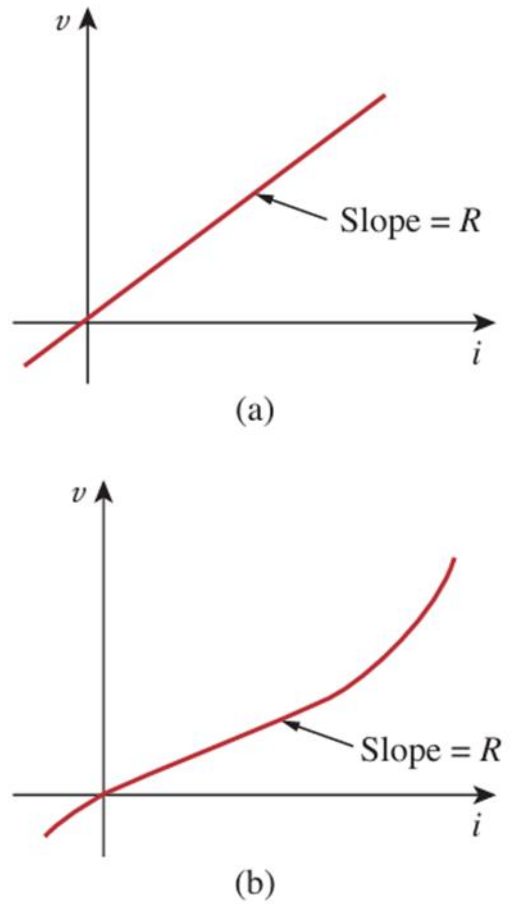


Figure 2
(a) Resistor, (b) Circuit symbol for resistance.

Ohm's Law

- It should be noted that not all resistors obey Ohm's law.
- A resistor that obeys Ohm's law is known as a linear resistor and has i-v characteristics as shown in Figure (a).
- A nonlinear resistor does not obey Ohm's law. Its resistance varies with current and its i-v characteristic is typically shown in Figure (b).



The i-v characteristic of: (a) a linear resistor, (b) a nonlinear resistor.

Branches, Nodes, and Loops

- A **branch** represents a single element.
- A **node** is the point of connection between two or more branches.
- A **loop** is any closed path in a circuit.
- Fig. 3 can be redrawn as Fig. 4 by combining the nodes connected by conducting wires.
- Two or more elements are in **series** if they exclusively share a single node and consequently carry the same current.
- Two or more elements are in **parallel** if they are connected to the same two nodes and consequently have the same voltage across them.

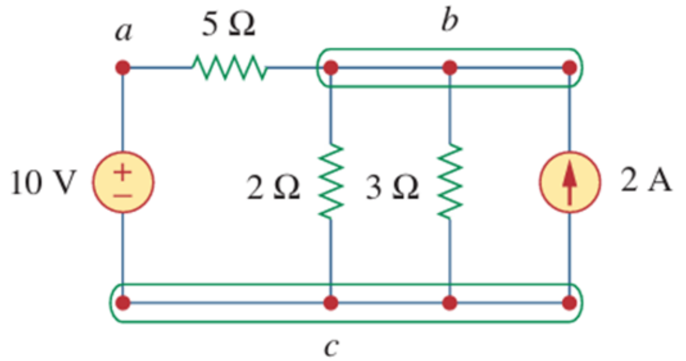


Figure 3
Nodes, branches, and loops.

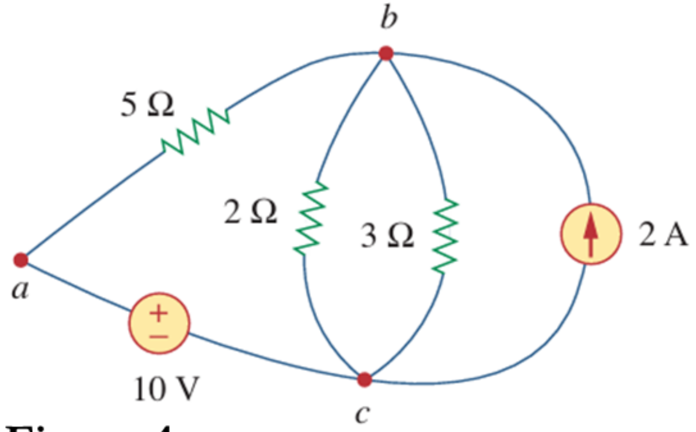


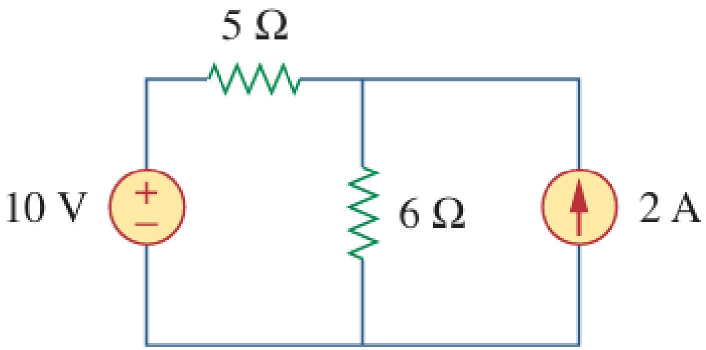
Figure 4
The three-node circuit of Fig. 2.10 is redrawn.

Branches, Nodes, and Loops

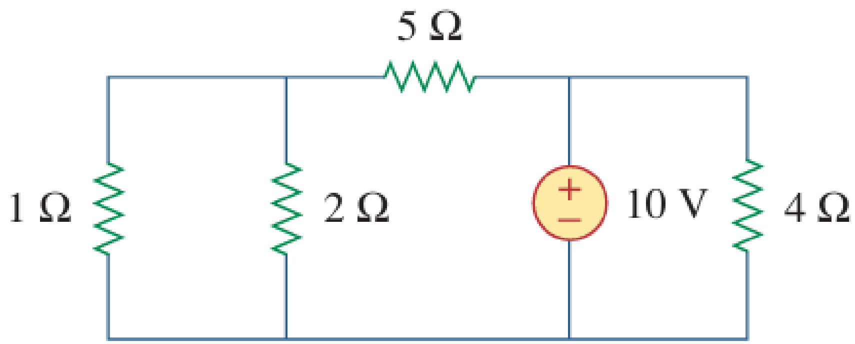
Problem

- Determine the number of branches and nodes in the circuits shown below. Identify which elements are in series and which are in parallel.

(a)



(b)



Kirchhoff's Laws

- Kirchhoff's current law (KCL) states that the algebraic sum of currents entering a node (or a closed boundary) is zero.
- The sum of the currents entering a node is equal to the sum of the currents leaving the node.

$$i_1 + i_3 + i_4 = i_2 + i_5$$

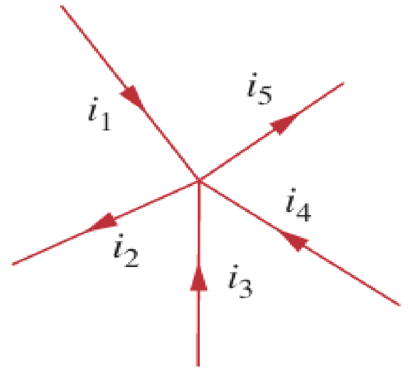


Figure 2.16
Currents at a node illustrating KCL.

- In Fig. 2.17, the total current entering the closed surface is equal to the total current leaving the surface.

- KCL can be applied to current sources in parallel such that

$$I_T = I_1 - I_2 + I_3$$

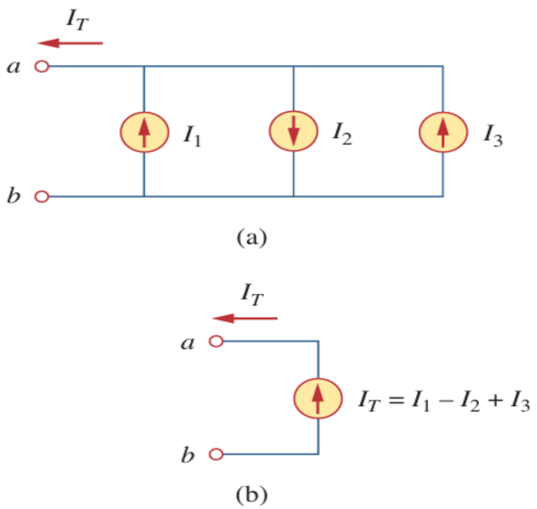


Figure 2.18
Current sources in parallel: (a) original circuit, (b) equivalent circuit.

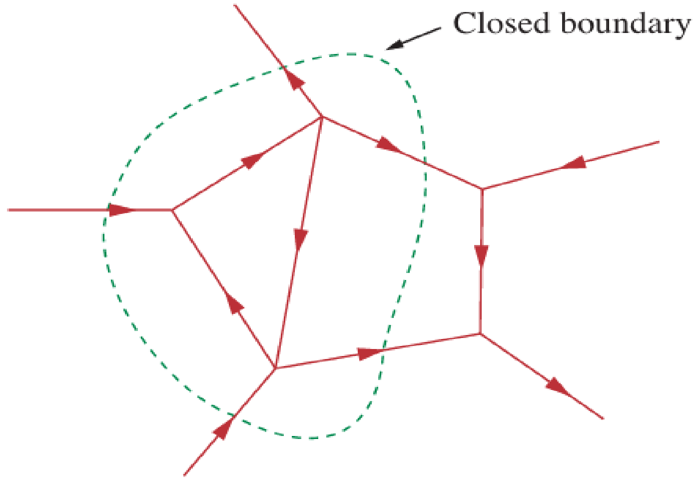


Figure 2.17
Applying KCL to a closed boundary.

Kirchhoff’s Laws

- Kirchhoff’s voltage law (KVL) states that the algebraic sum of all voltages around a closed path (or loop) is zero.
- Consider the circuit in Fig. 2.19. The sign on each voltage is the polarity of the terminal encountered first as we travel around the loop.

$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$
$$v_1 + v_4 = v_2 + v_3 + v_5$$

- In other words,

$$Sum\ of\ voltage\ drops\ =\ Sum\ of\ voltage\ rises$$

- In Fig. 2.20, KVL can be applied to voltage sources in series such

$$V_{ab} = V_1 + V_2 - V_3$$

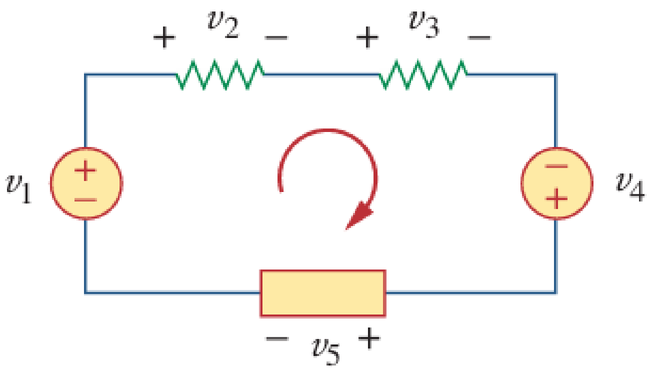


Figure 2.19
A single-loop circuit illustrating KVL.

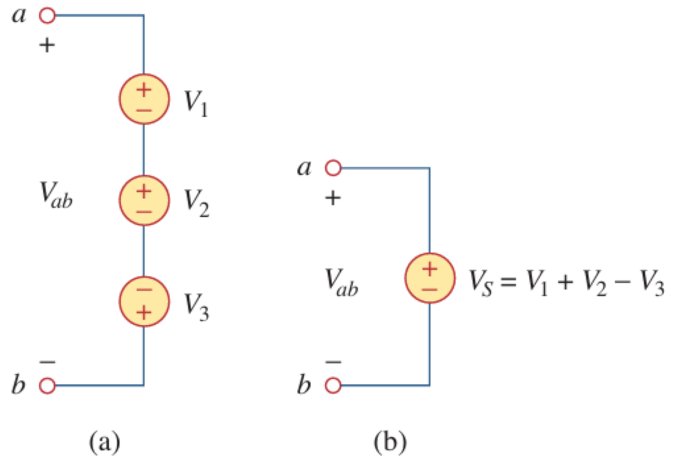
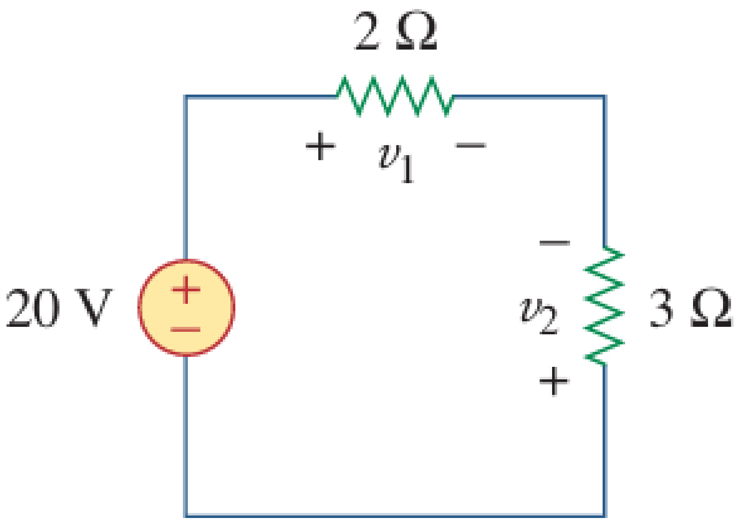


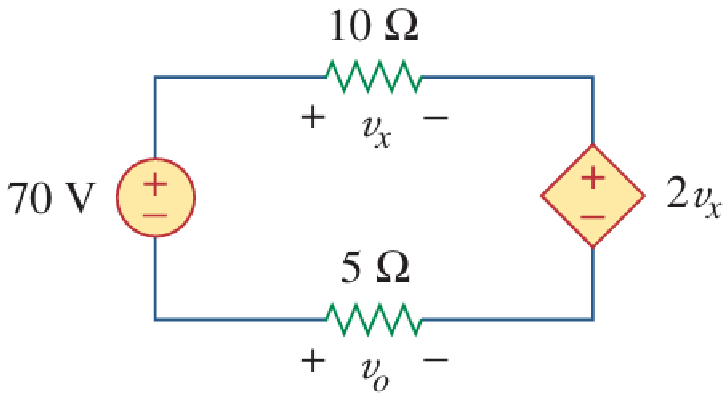
Figure 2.20
Voltage sources in series: (a) original circuit, (b) equivalent circuit.

Problems

For the circuits below, find voltage v_1 and v_2.

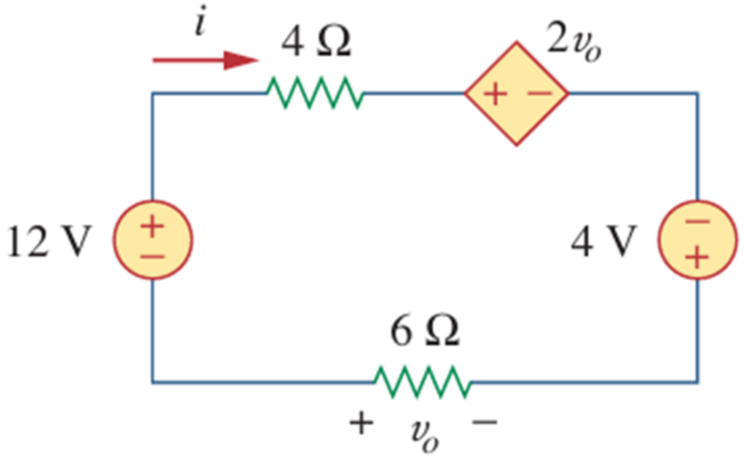


(a)



For the circuits below, find voltage v_x and v_o.

Determine v_o and i in the circuit below



(b)

Series Resistors and Voltage Division

- For two resistors in series as shown in Fig. 2.29

$$R_{eq} = R_1 + R_2$$

- The equivalent resistance of any number of resistors connected in series is the sum of the individual resistances.
- Per the voltage divider rule, the source voltage v is divided among the resistors in direct proportion to their resistances.

$$v_1 = \frac{R_1}{R_1 + R_2} v$$

$$v_2 = \frac{R_2}{R_1 + R_2} v$$

- In general, if a voltage divider has N resistors (R_1, R_2, R_3 . . . , R_N) in series with the source voltage v, the nth resistor (R_n) will have a voltage drop of

$$v_n = \frac{R_n}{R_1 + R_2 + R_3 \ldots + R_N} v$$

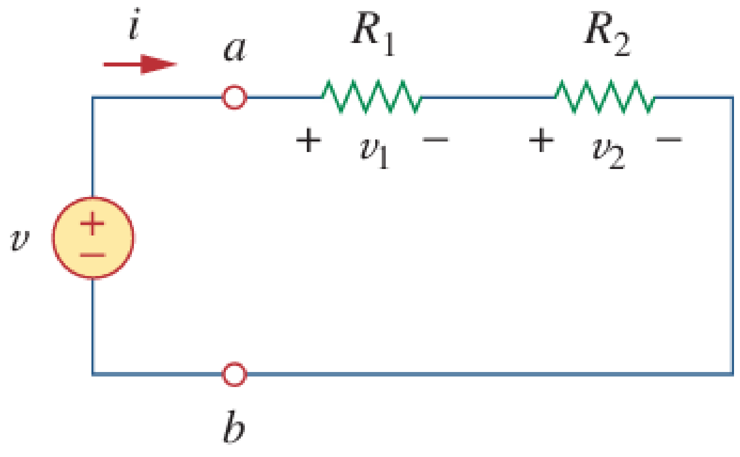


Figure 2.29
A single-loop circuit with two resistors in series.

Parallel Resistors and Current Division

- For two resistors in parallel as shown in Fig. 2.31.

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

- In the general case of a circuit with N resistors in parallel. The equivalent resistance is

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}$$

- Per the current divider rule,

$$i_1 = \frac{R_2}{R_1 + R_2} i$$

$$i_2 = \frac{R_1}{R_1 + R_2} i$$

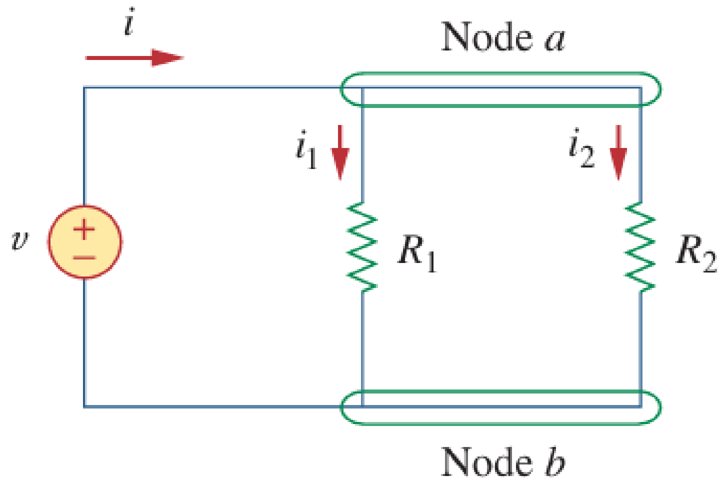
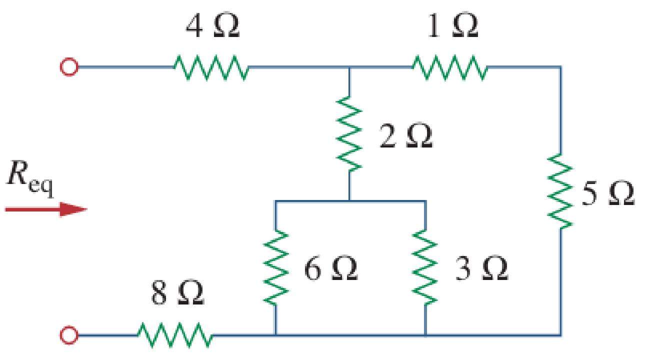
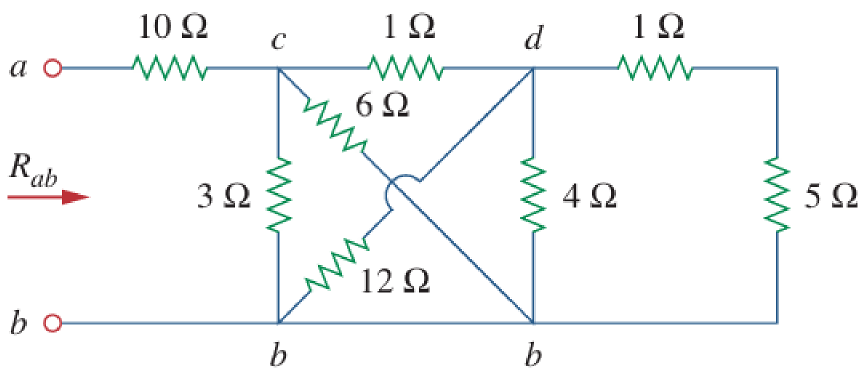


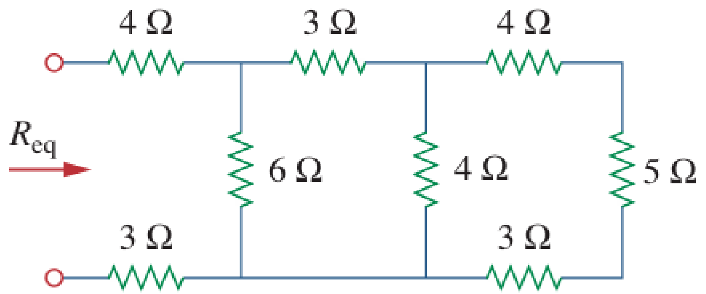
Figure 2.31
Two resistors in parallel.

Problems

- Find the equivalent resistances of the circuits shown below







Methods of Analysis

- **<u>Nodal Analysis with only current sources</u>**

Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining $n-1$ nodes. The voltages are referenced with respect to the reference node.
2. Apply KCL to each of the $n-1$ nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

- Current flows from a higher potential to a lower potential in a resistor.

$$\boxed{i = \frac{v_{\text{higher}} - v_{\text{lower}}}{R}}$$

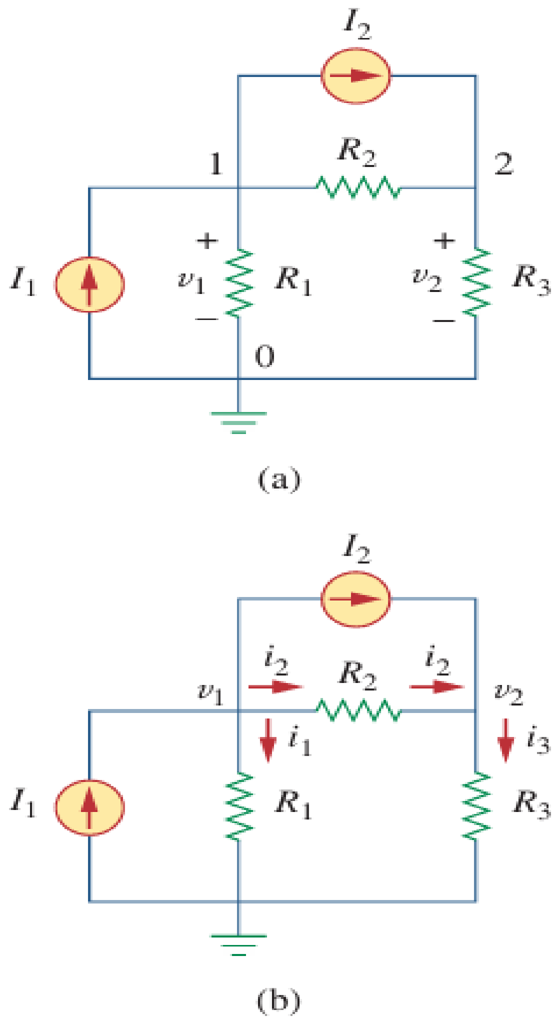


Figure 3.2
Typical circuit for nodal analysis.

At node 1,

$$I_1 = I_2 + i_1 + I_2$$

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}$$

At node 2,

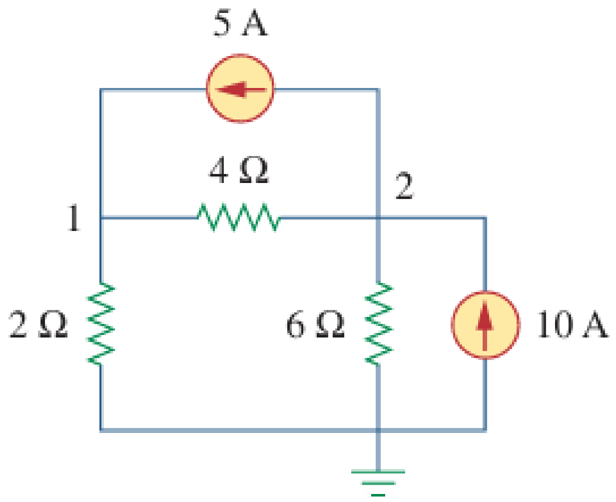
$$I_2 = i_3 + i_3$$

$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

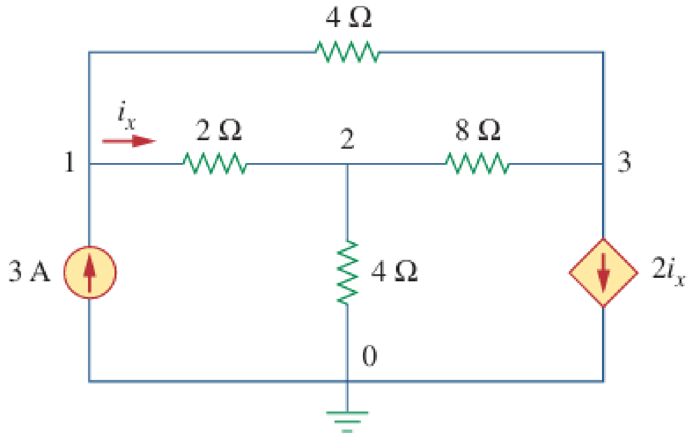
Nodal Analysis

- **Problems**

Calculate the node voltages in the circuits (a) and (b)



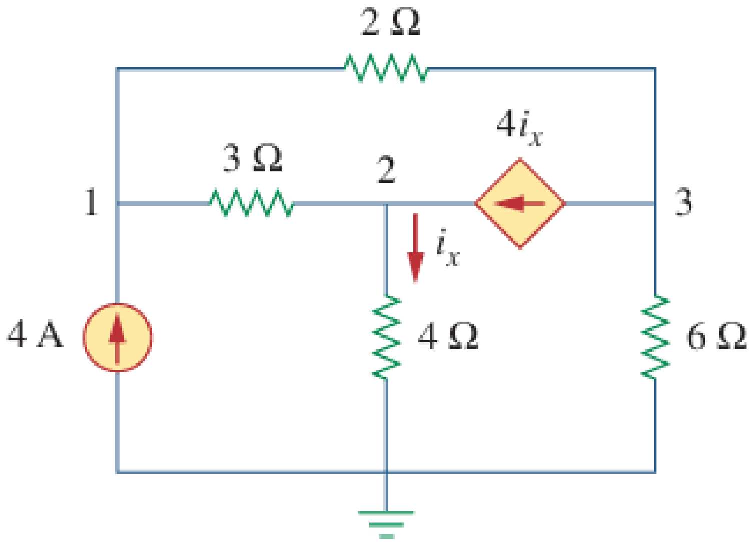
(a)



(b)

- **Assignment 1** – (Deadline: Thursday Jan 16, 2025, 23:59)

Determine the voltage at the three non-reference nodes



Nodal Analysis with Voltage Source

- **<u>Nodal Analysis with voltage sources</u>**
- CASE1: If a voltage source is connected between the reference node and a nonreference node, simply set the voltage at the non reference node equal to the voltage of the voltage source.
- In Fig. 3.7, for example, $v_1 = 10\ V$.
- CASE 2: If the voltage source (dependent or independent) is connected between two non-reference nodes, the two non-reference nodes form a **supernode**, apply both KCL and KVL to determine the node voltages.

A supernode is formed by enclosing a (dependent or independent) voltage source connected between two nonreference nodes and any elements connected in parallel with it.

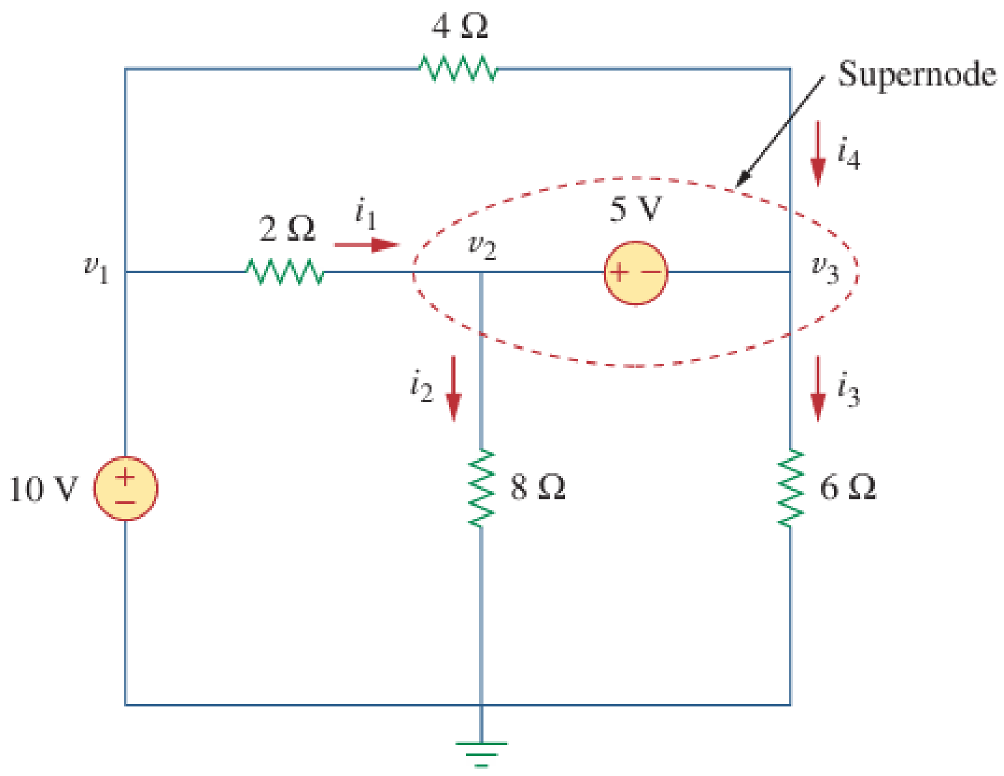


Figure 3.7
A circuit with a supernode.

Nodal Analysis with Voltage Source

- At the supernode,

$$i_1 + i_4 = i_2 + i_3$$

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$$

- Applying KVL to the supernode,

$$-v_2 + 5 + v_3 = 0 \quad \Rightarrow \quad v_2 - v_3 = 5$$

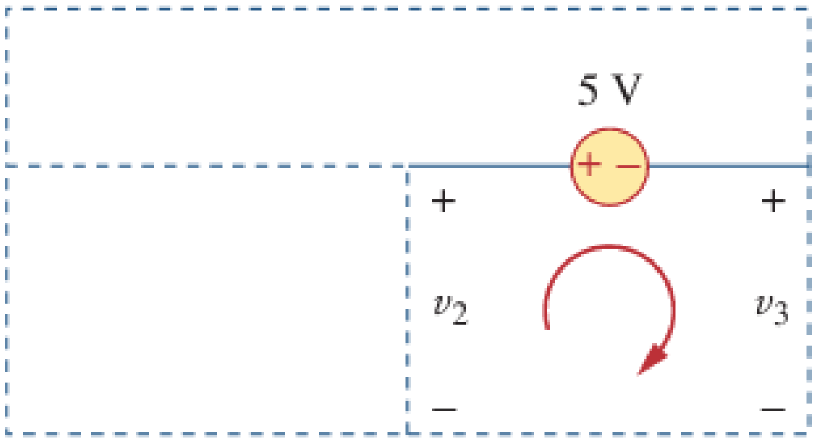
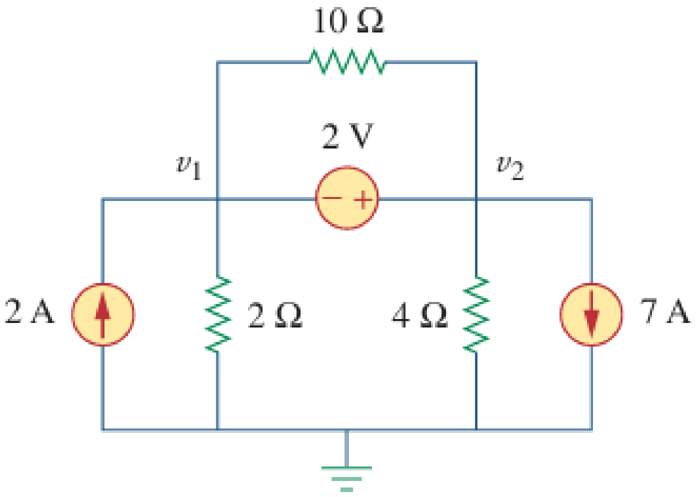


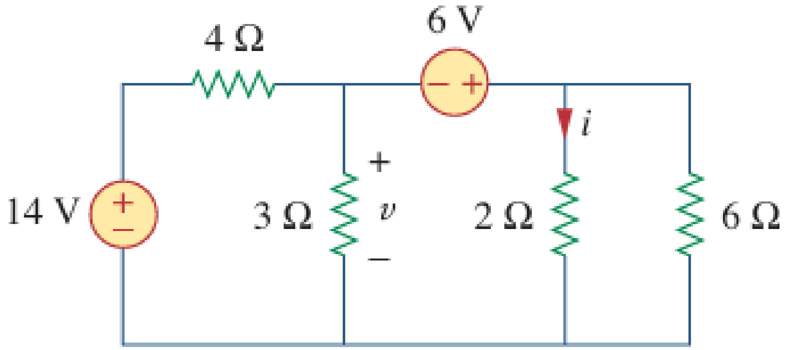
Figure 3.8
Applying KVL to a supernode.

Nodal Analysis with Voltage Source

Problems

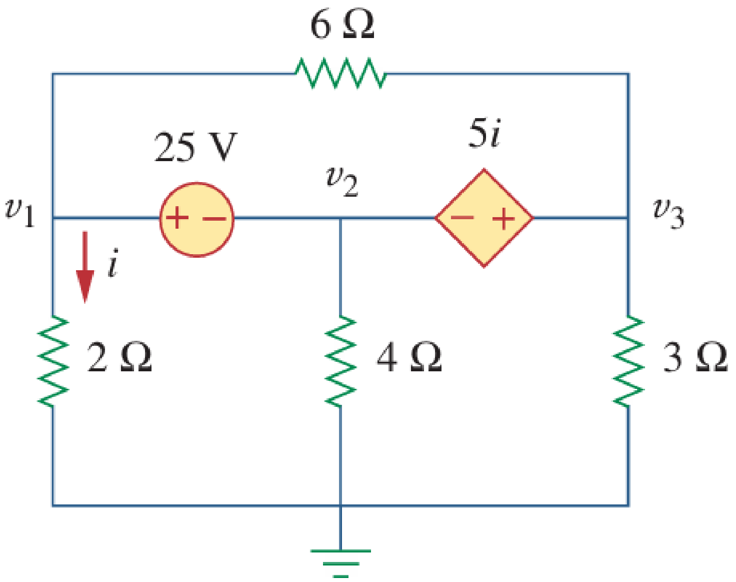
Find the node voltages in the circuit below





Find v and i.

Assignment 2 – (Deadline: Sunday, Jan 19, 2025, 20:00)



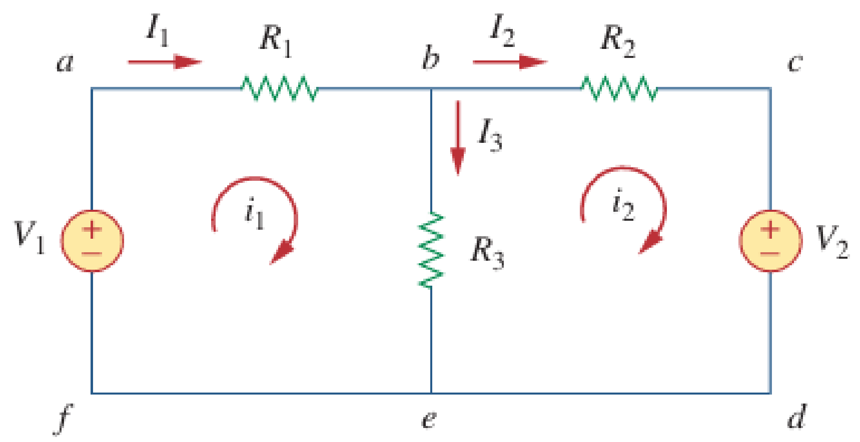
Using nodal analysis, find v_1, v_2, and v_3 in the circuit above

Mesh Analysis

- A mesh is a loop which does not contain any other loops within it.

Steps to Determine Mesh Currents:

1. Assign mesh currents $i_1, i_2, \ldots, i_n$ to the n meshes.
2. Apply KVL to each of the n meshes. Use Ohm's law to express the voltages in terms of the mesh currents.
3. Solve the resulting n simultaneous equations to get the mesh currents.



Applying KVL to mesh 1

$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0$$

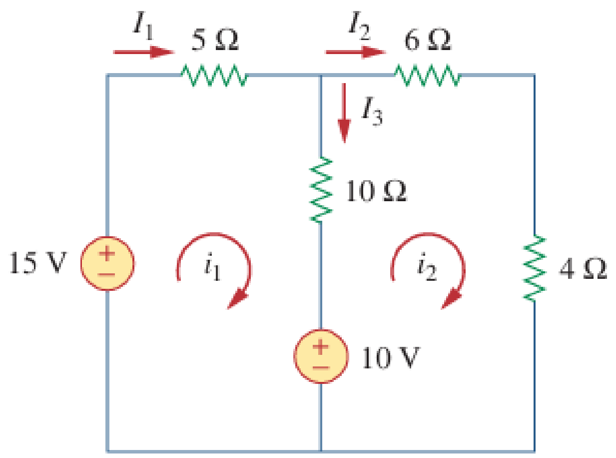
$$(R_1 + R_3)i_1 - R_3 i_2 = V_1$$

Applying KVL to mesh 2

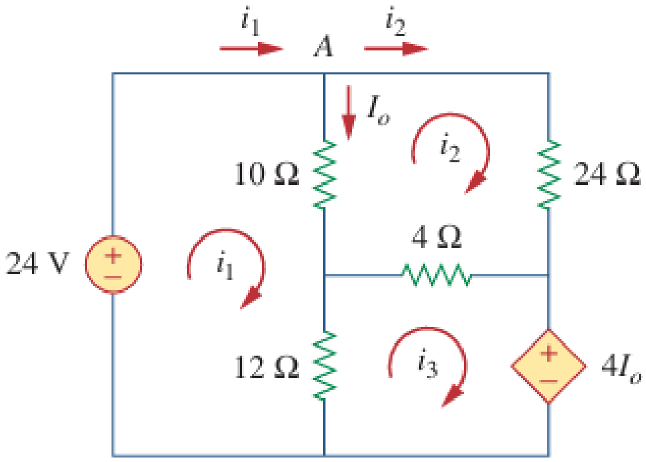
$$R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0$$

$$-R_3 i_1 + (R_2 + R_3)i_2 = -V_2$$

Problems



Find the branch currents using mesh analysis.



Use mesh analysis to find the mesh currents in the circuit

Mesh Analysis with Current Sources

- CASE 1: When a current source exists only in one mesh: Consider the circuit in Fig. 3.22 for example, set $i_2 = -5\,A$ and write a mesh equation for the other mesh in the usual way.

$$-10 + 4i_1 + 6(i_1 - i_2) = 0 \quad \Rightarrow \quad i_1 = -2\text{ A}$$

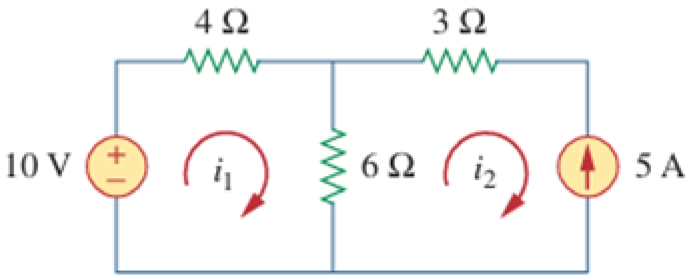


Figure 3.22
A circuit with a current source.

- CASE 2: When a current source exists between two meshes: Consider the circuit in Fig. 3.23(a), for example. Create a supermesh by excluding the current source and any elements connected in series with it, as shown in Fig. 3.23(b).

A supermesh results when two meshes share a current source (dependent or independent).

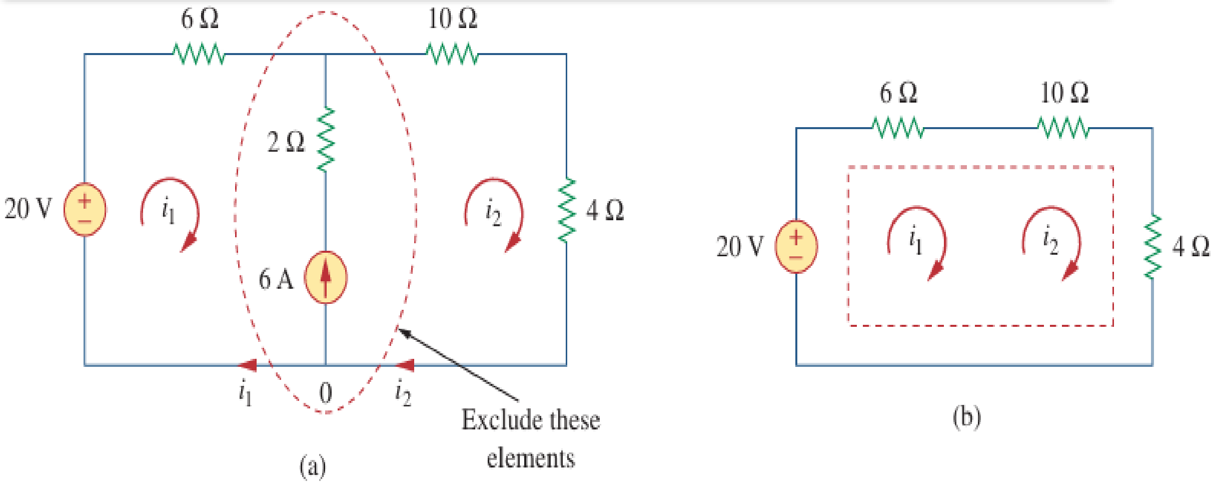


Figure 3.23
(a) Two meshes having a current source in common, (b) a supermesh, created by excluding the current source.

$$-20 + 6i_1 + 10i_2 + 4i_2 = 0$$

$$6i_1 + 14i_2 = 20 \qquad (1)$$

apply KCL to the node where the two meshes meet.

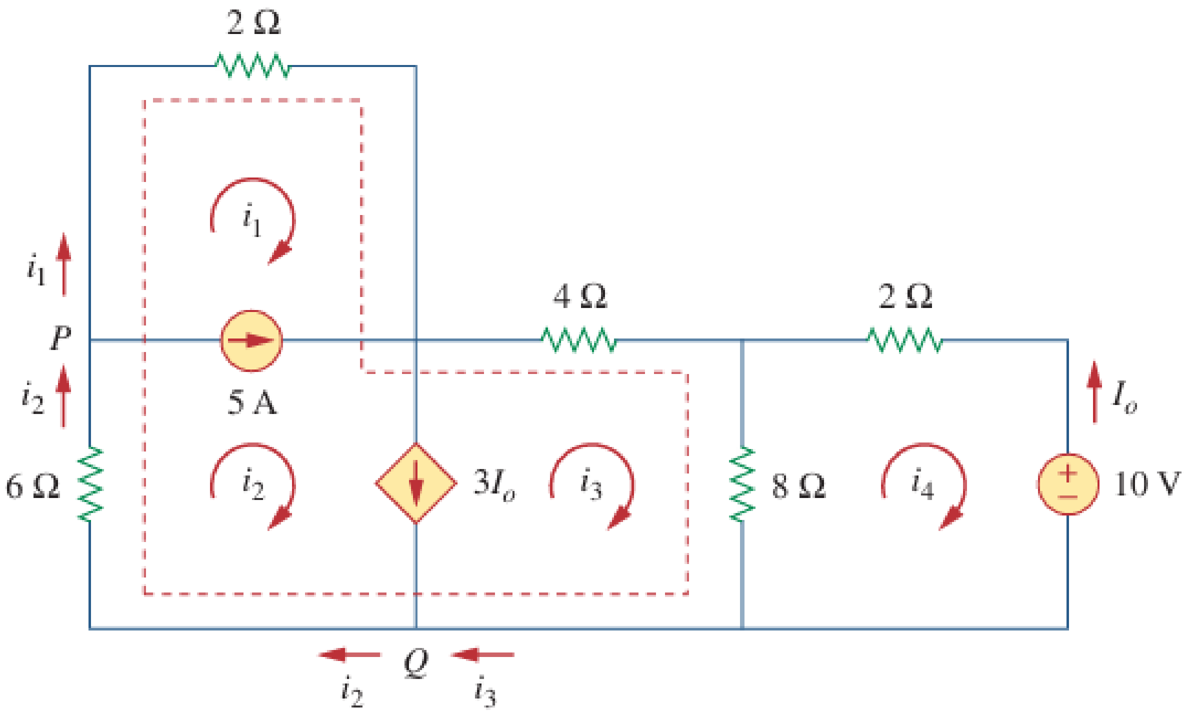
$$i_2 = i_1 + 6 \qquad \textbf{OR} \qquad \boldsymbol{i_2 - i_1 = 6} \qquad (2)$$

Solving equation (1) and (2) simultaneously,

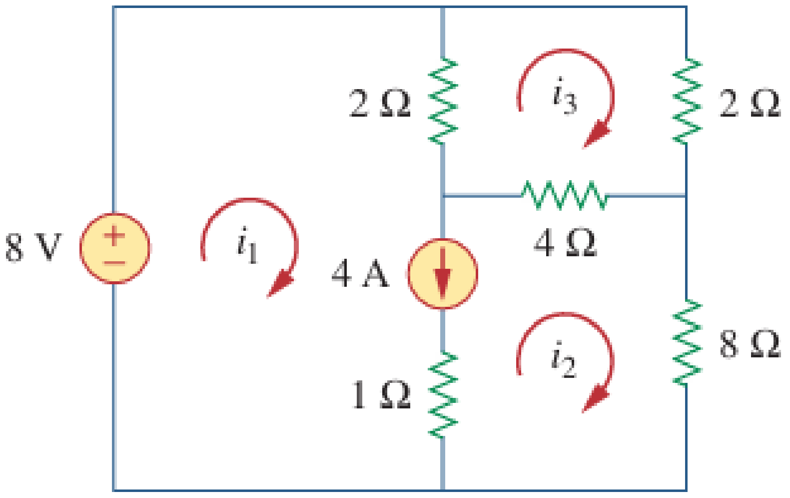
$$\boldsymbol{i_1 = -3.2\,A} \qquad \boldsymbol{i_2 = 2.8\,A}$$

Problems

Use mesh analysis to determine i_1, i_2, i_3 and i_4 in the circuit below.



Assignment 3 – (Deadline: Tuesday, Jan 28, 2025, 23:59)



Use mesh analysis to determine i_1, i_2 and i_3 in the circuit above.

Nodal Analysis vs. Mesh Analysis

Difference between modal analysis and mesh analysis

Nodal analysis	Mesh analysis
Used to find node voltages	Used to find branch and mesh currents
Can solve non-planar circuits	Cannot solve non-planar circuit
Primarily depends on KCL	Primarily depends on KVL
A supernode is when a voltage source is found in between two non-reference nodes	A supermesh is when a current source is shared by two meshes
Supernode is secondarily solved using KVL	Supermesh is secondarily solved using KCL

Circuit Theorems

Linearity Property of Circuits

- The linearity property is a combination of both the **homogeneity (scaling) property** and the **additivity property**.
- The homogeneity property requires that if the input is multiplied by a constant, then the output is multiplied by the same constant.
- The additivity property requires that the response to a sum of inputs is the sum of the responses to each input applied separately.

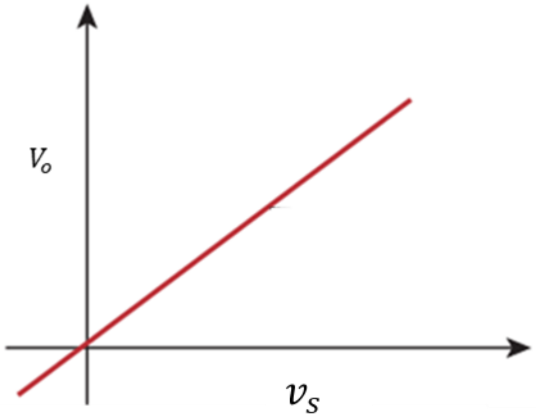


Fig: *An illustrative diagram showing how the output voltage (V_o) linear relates to the source voltage (V_s) in a linear circuit*

A linear circuit is one whose output is linearly related (or directly proportional) to its input.

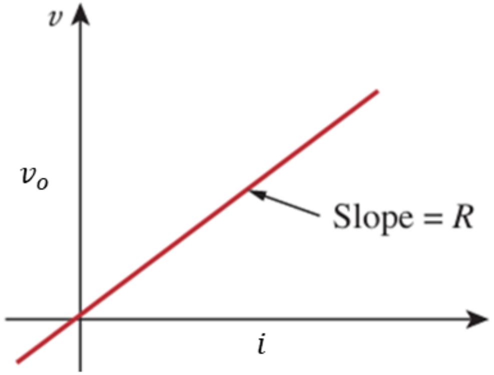


Fig: *An illustrative diagram showing how current linearly relates to voltage*

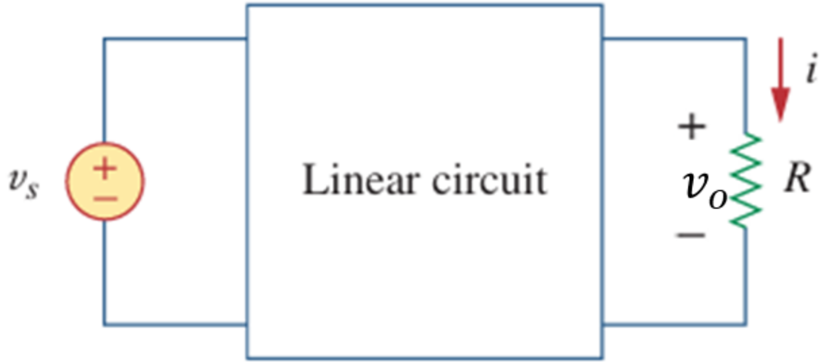


Fig: *A linear circuit with input v_s and output $v_o - i$.*

Problems

Assume $I_o = 1$ A and use linearity to find the actual value of I_o in the circuit of Fig. 4.4.

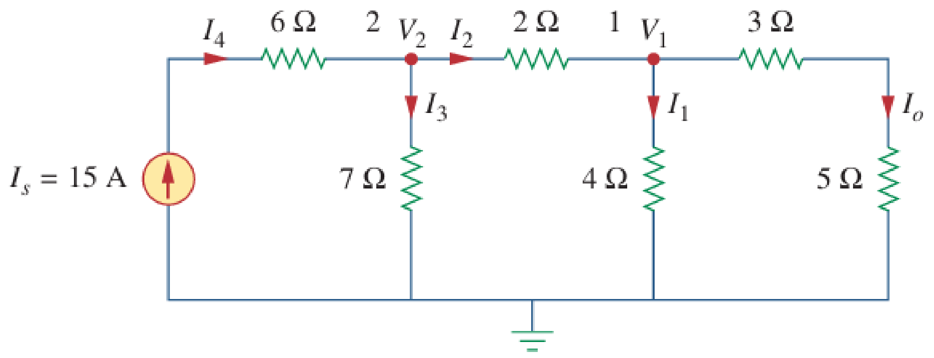


Figure 4.4
For Example 4.2.

Assume that $V_o = 1$ V and use linearity to calculate the actual value of V_o in the circuit of Fig. 4.5.

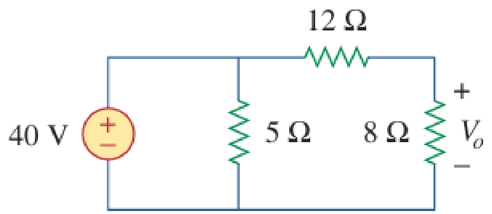
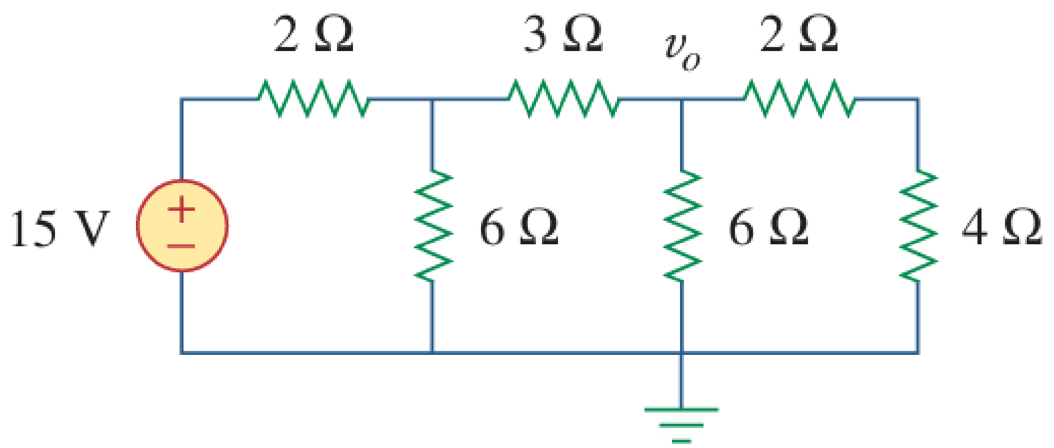


Figure 4.5
For Practice Prob. 4.2.

Assignment 4

In the circuit below, V_o is the 5V at a certain value of the source voltage, determine the actual value of V_o.

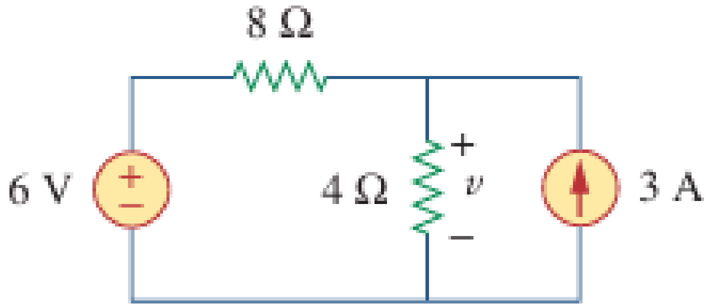


Superposition

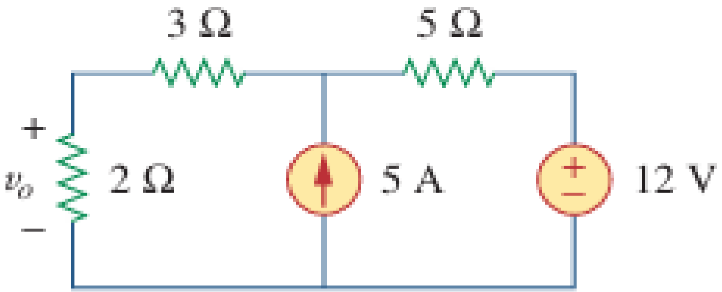
- The **superposition principle** states that the voltage across (or current through) an element in a ***linear circuit*** is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

Steps to Apply Superposition Principle:

1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using the techniques covered in Chapters 2 and 3.
2. Repeat step 1 for each of the other independent sources.
3. Find the total contribution by adding algebraically all the contributions due to the independent sources.



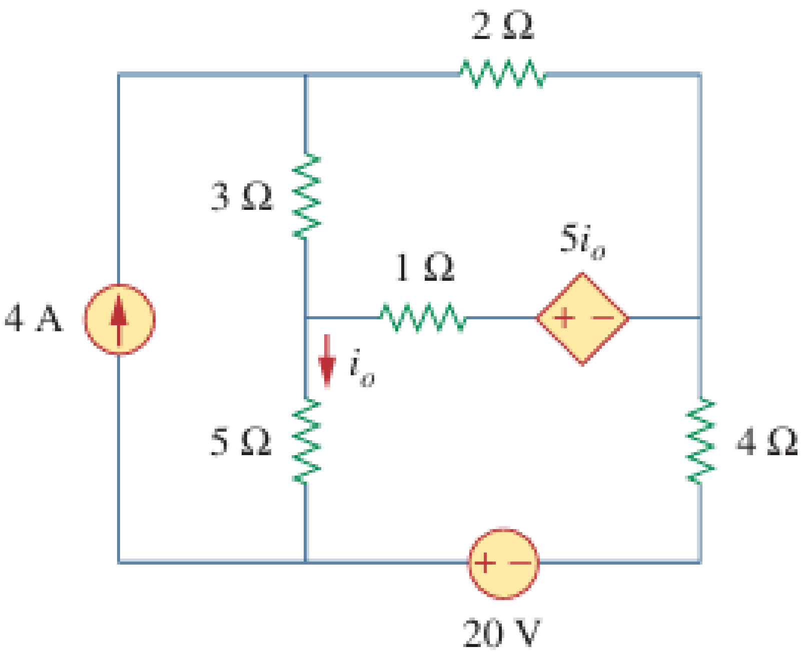
Use the superposition theorem to find v in the circuit above



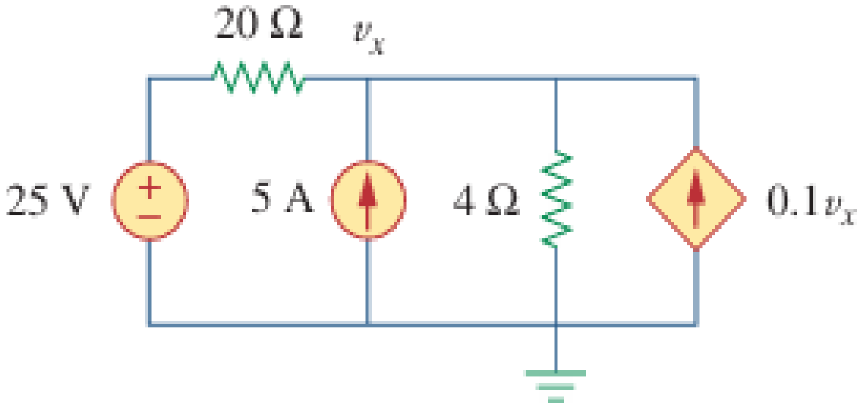
Using the superposition theorem, find v_o in the circuit

Problems

(1) Find i_o in the circuit below

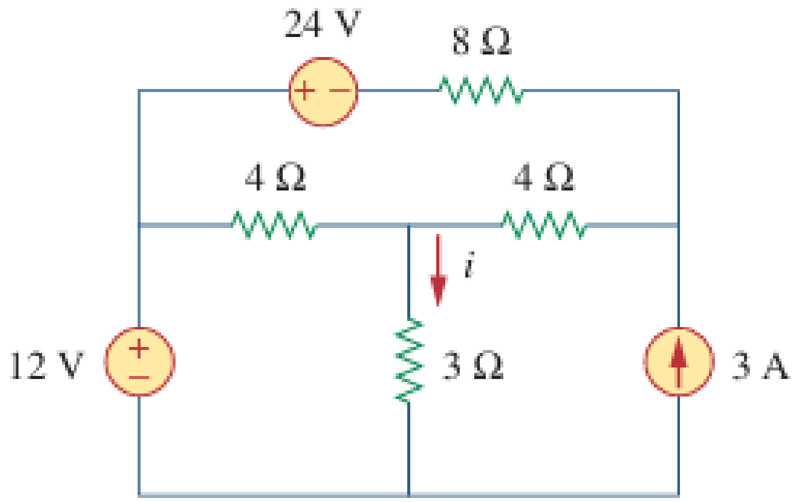


(2) Find v_x using superposition



Assignment 5

Find i using the superposition theorem



Source Transformation

A source transformation is the process of replacing a voltage source v_s in series with a resistor R with a current source i_s in parallel with a resistor R, or vice versa.

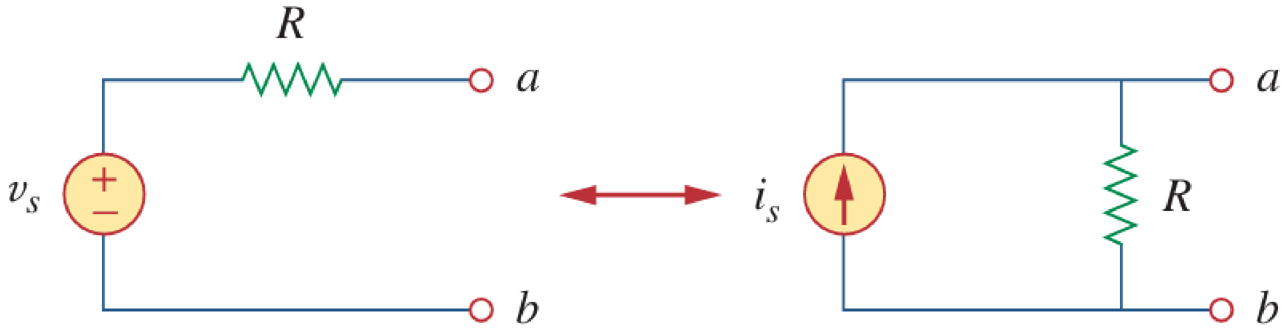


Fig: *Transformation of independent sources*

$$v_s = i_s R \text{ – for current source transformation}$$

$$i_s = \frac{v_s}{R} \text{ – for voltage source transformation}$$

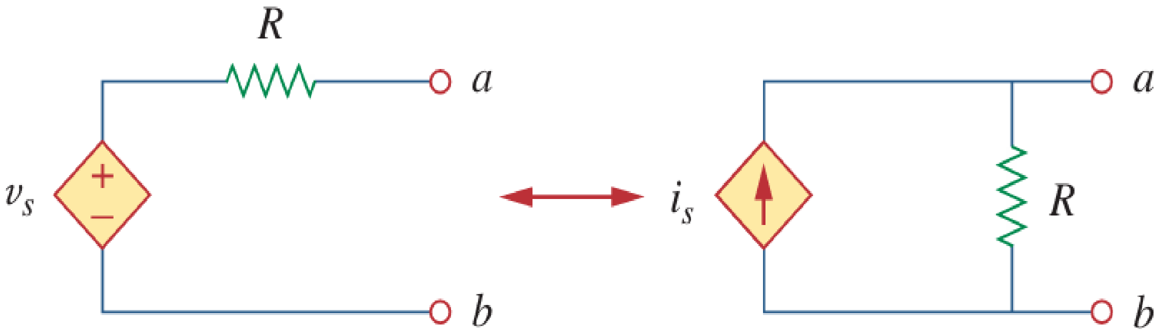


Fig: *Transformation of dependent sources*

Problem

Find i_o in the circuit of Fig. 4.19 using source transformation.

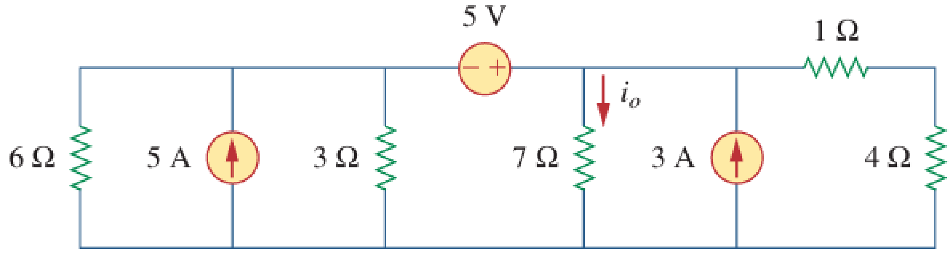


Figure 4.19
For Practice Prob. 4.6.

Thevenin's Theorem

- Thevenin's theorem states that a ***linear two-terminal circuit*** can be replaced by an equivalent circuit consisting of a **voltage source V_{TH}** in series with a **resistor R_{TH}**, where V_{TH} is the open-circuit voltage at the terminals and R_{TH} is the input or equivalent resistance at the terminals when the independent sources are turned off.

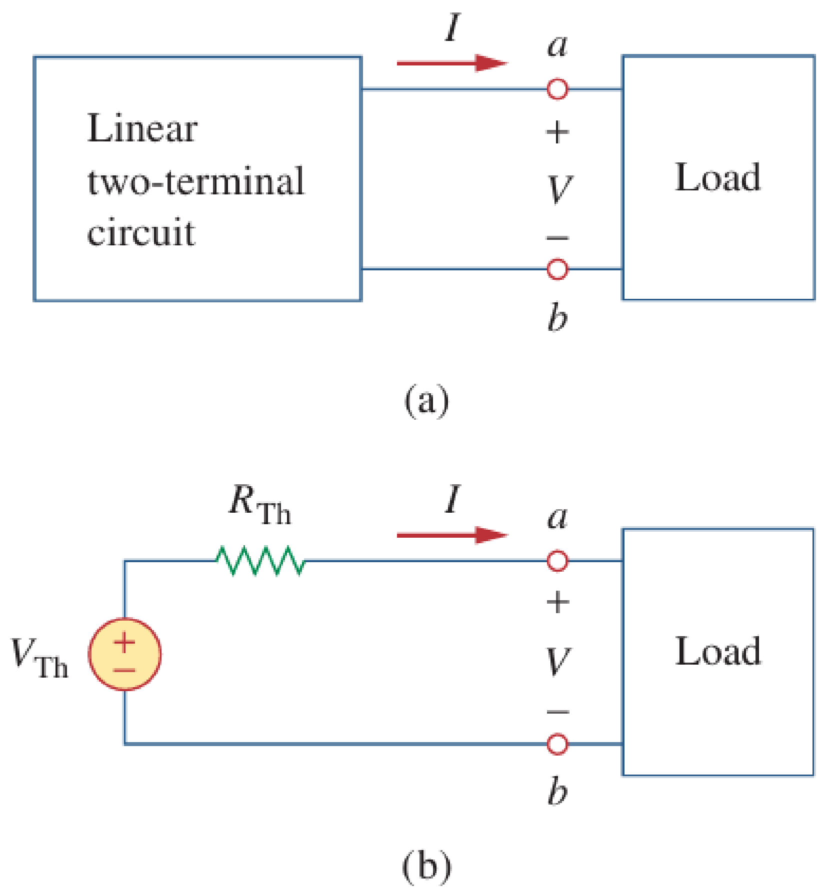


Figure 4.23
Replacing a linear two-terminal circuit by its Thevenin equivalent: (a) original circuit, (b) the Thevenin equivalent circuit.

Thevenin Equivalent Circuit

- Thevenin's theorem helps to simplify a circuit.
- By Thevenin's theorem, a large circuit may be replaced by a single independent voltage source V_{TH} and a single resistor R_{TH}.
- Consider a linear circuit terminated by a load R_L, as shown in Fig. 4.26(a). The current through the load and the voltage across the load is easily determined once the Thevenin equivalent circuit of the circuit at the load's terminals is obtained, as shown in Fig. 4.26 (b). From Fig. 4.26(b), we obtain

$$I_L = \frac{V_{Th}}{R_{Th} + R_L}$$

$$V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th}$$

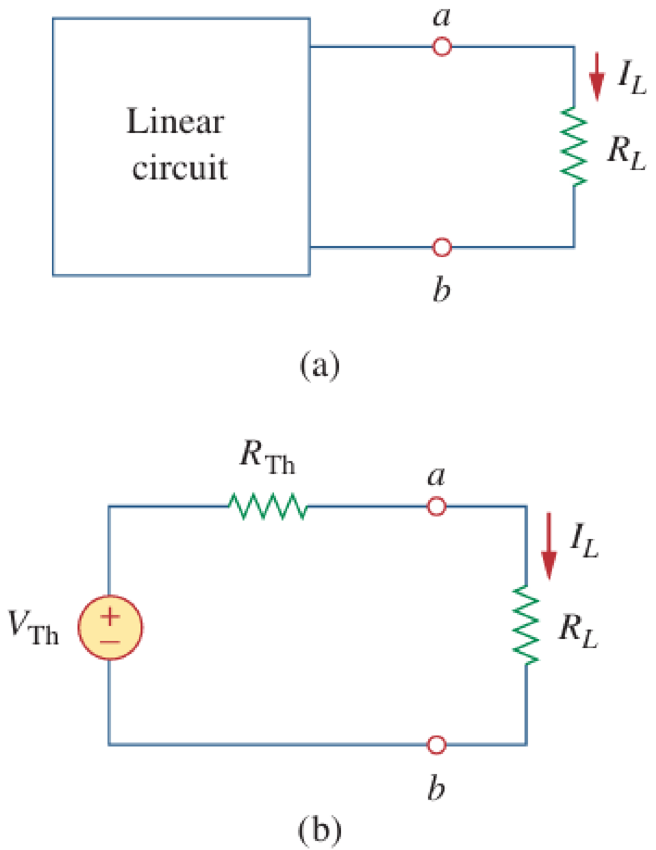


Figure 4.26
A circuit with a load: (a) original circuit, (b) Thevenin equivalent.

Thevenin Equivalent Circuit – CASE 1

- **CASE 1:** If the network has no dependent sources, turn off all independent sources. R_{TH} is the input resistance of the network looking between terminals a and b, as shown in Fig. 4.24(b).

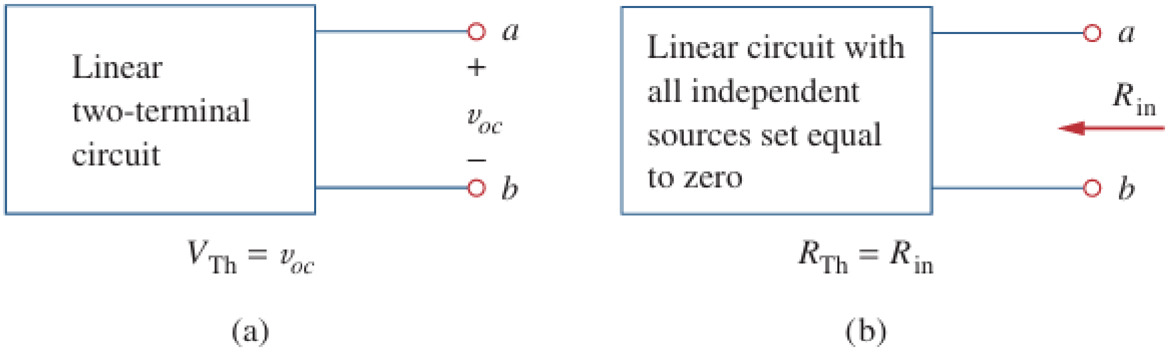
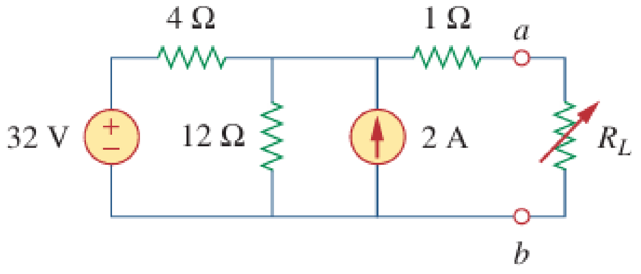


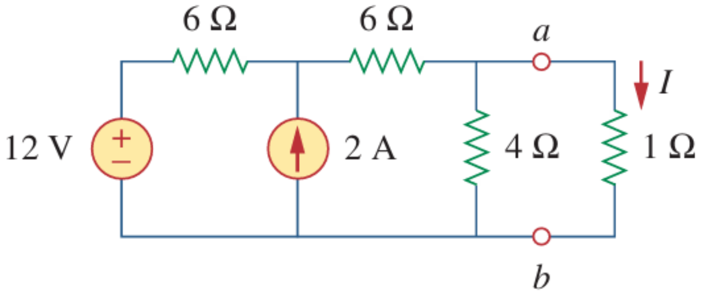
Figure 4.24
Finding V_{Th} and R_{Th}.

Problems

(1) Find the Thevenin equivalent circuit of the circuit below, to the left of the terminals Then find the current through $R_L = 6\ \Omega$, $16\ \Omega$ and $36\ \Omega$.



(2) Using Thevenin's theorem, find the equivalent circuit to the left of the terminals in the circuit below. Then find I.



CASE 2

- If the network has dependent sources, turn off all independent sources. As with superposition, dependent sources are not to be turned off because they are controlled by circuit variables.
- Apply a voltage source v_o at terminals a and b and determine the resulting current i_o. Then $R_{TH} = v_o/i_o$, as shown in Fig. 4.25(a).
- Alternatively, insert a current source at terminals $a - b$ as shown in Fig. 4.25(b) and find the terminal voltage. Again $R_{TH} = v_o/i_o$.
- Either of the two approaches will give the same result. In either approach, assuming any value of v_o and i_o would yield the same results.
- For this course, always assume $v_o = 1$ V or $i_o = 1$ A.

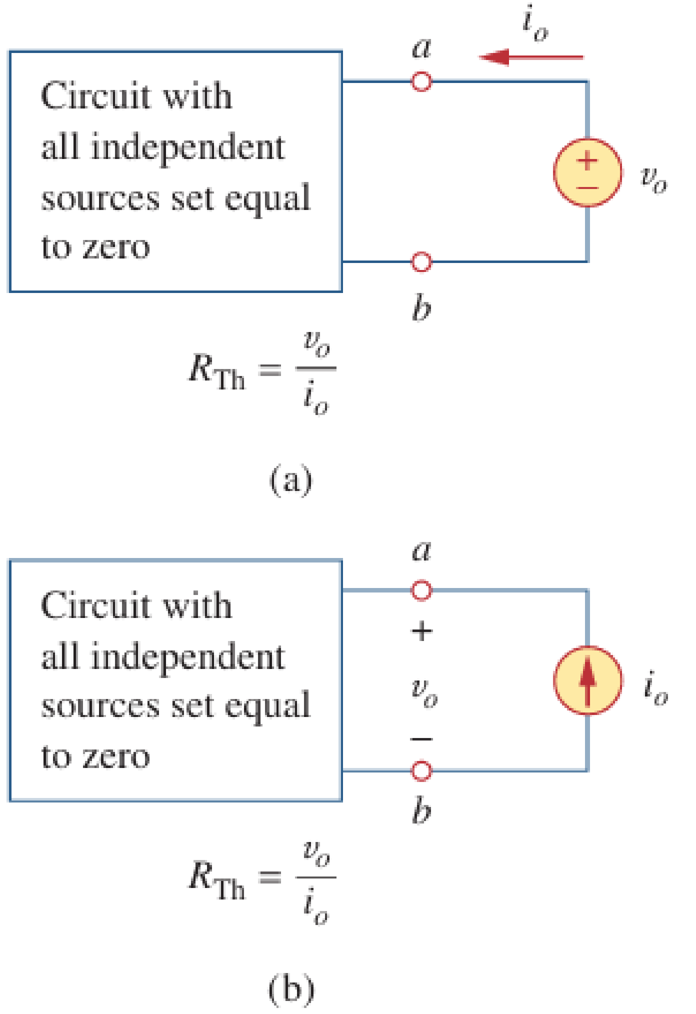
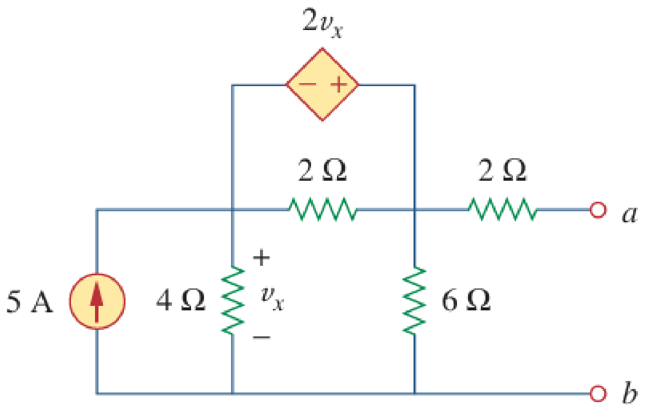


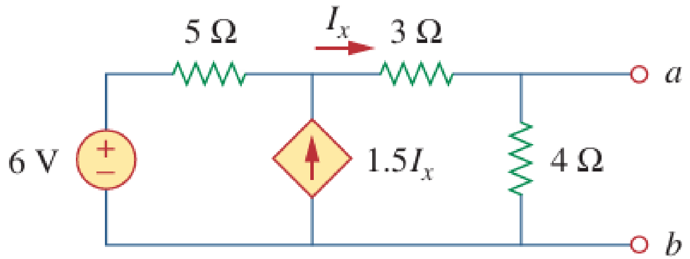
Figure 4.25
Finding R_{Th} when circuit has dependent sources.

Problems

Find the Thevenin equivalent circuit of the circuit below at terminals a-b.



Find the Thevenin equivalent circuit of the circuit below to the left of the terminals.



Assignment 6

Find the Thevenin equivalent circuit of the circuit below to the left of the terminals a-b.

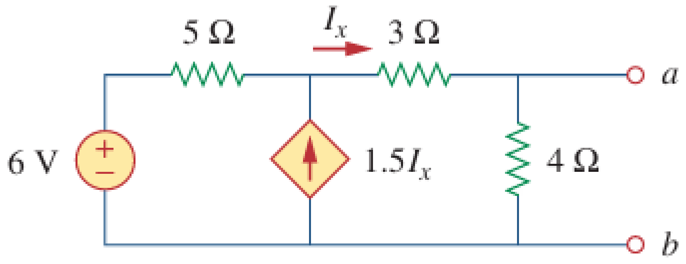
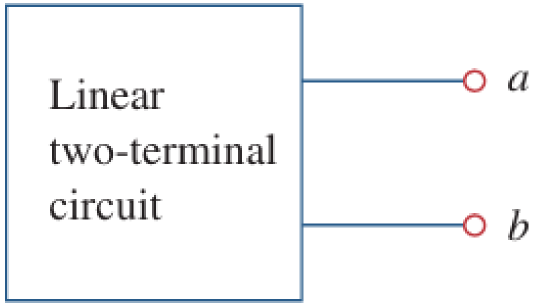
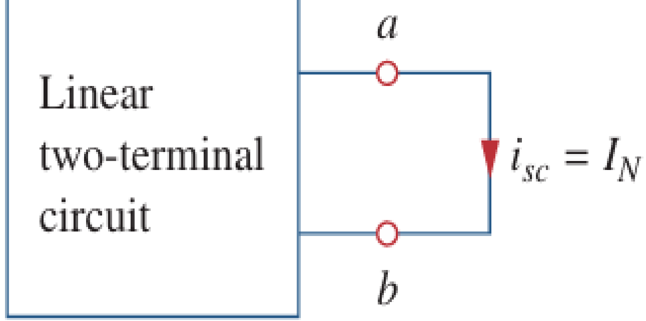
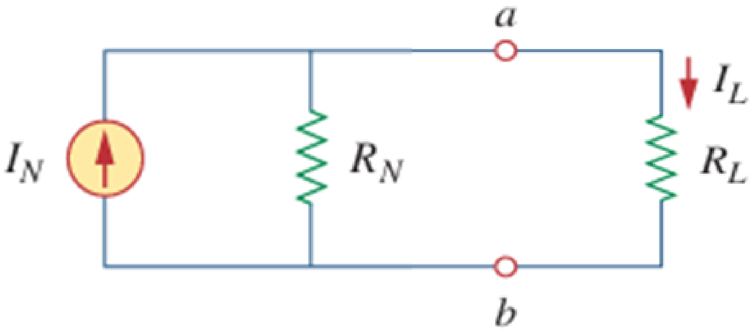


Figure 4.34
For Practice Prob. 4.9.

Norton’s Theorem







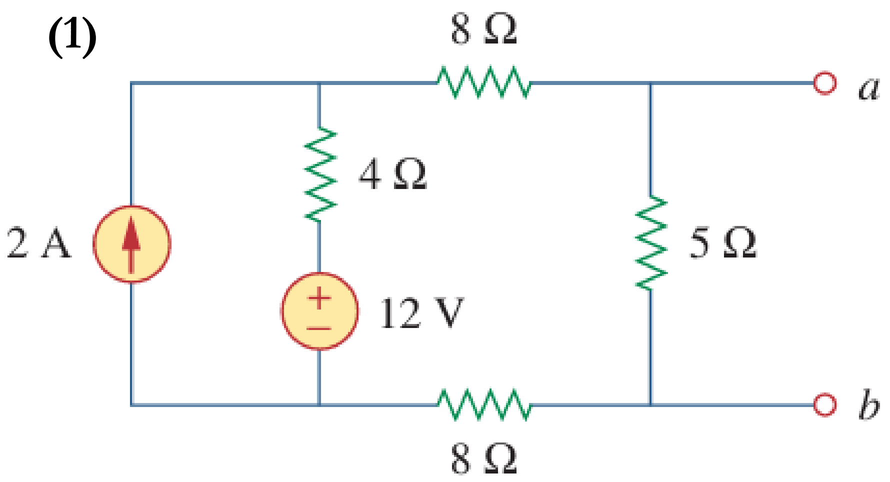
- Norton’s theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a **current source I_N** in parallel with a **resistor R_N** , where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.
- To find the Norton current I_N, determine the short-circuit current flowing from terminal a to b in both circuits.

I_L

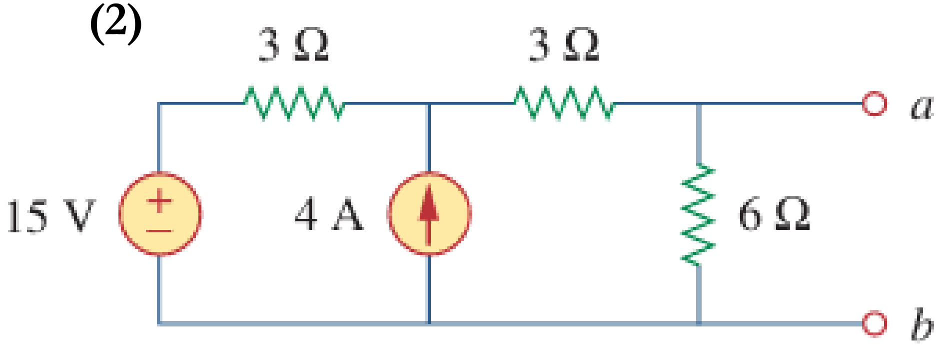
Problems

Find the Norton equivalent circuit of the circuit above at terminals a-b.

(1)



(2)



Capacitors and Inductors

Capacitors

- A capacitor is a passive element designed to store energy in its electric field.
- Capacitors are used extensively in electronics, communications, computers, and power systems.

A capacitor consists of two conducting plates separated by an insulator (or dielectric).

- In many practical applications, the plates may be aluminum foil while the dielectric may be air, ceramic, paper, or mica.

- When a voltage source v is connected to the capacitor, as in Fig. 6.2, the source deposits a positive charge q on one plate and a negative charge q on the other.

- The amount of charge stored is directly proportional to the applied voltage.

$$q = Cv$$

Where C, the constant of proportionality, is known as the capacitance of the capacitorbbb.

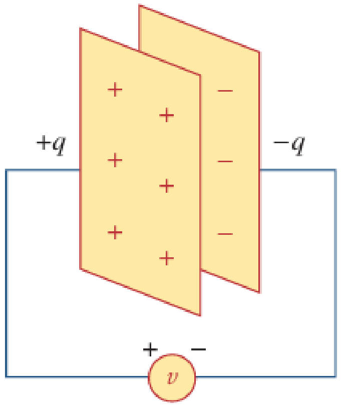


Figure 6.2
A capacitor with applied voltage v.

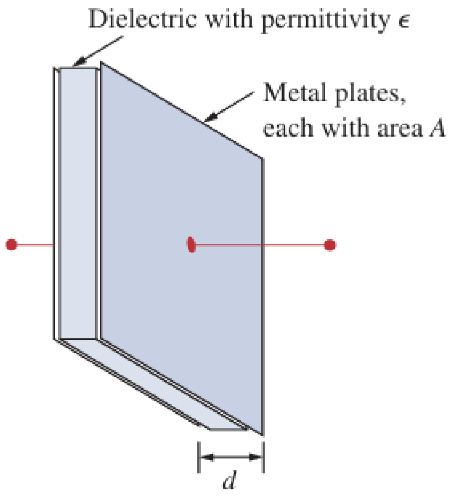


Figure 6.1
A typical capacitor.

Capacitance of a Capacitor

Capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).

- Although the capacitance C of a capacitor is the ratio of the charge q per plate to the applied voltage v, it does not depend on q or v.
- It depends on the physical dimensions of the capacitor. For example, for the parallel-plate capacitor shown in Fig. 6.1, the capacitance is given by

$$C = \frac{\epsilon A}{d}$$

Where,
A - the surface area of each plate
d - the distance between the plates
ϵ- the permittivity of the dielectric material between the plates.

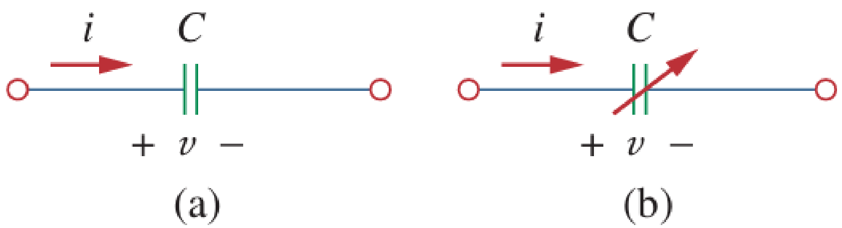


Figure 6.3
Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

- Figure 6.3 shows the circuit symbols for fixed and variable capacitors.
- Note that according to the passive sign convention, if $v > 0$ and $i > 0$ or if $v < 0$ and $i < 0$, the capacitor is being charged, and if $v.i < 0$ then the capacitor is discharging.

Current-Voltage Relationship of a Capacitor

$$q = Cv$$

- To obtain the current-voltage relationship of the capacitor, take the derivative of both sides of the equation above.

$$\frac{dq}{dt} = C\frac{dv}{dt}$$

Where, $\frac{dq}{dt}$ is current i

Hence, $\boldsymbol{i = C\frac{dv}{dt}}$,

the voltage current relationship is obtained by integrating the equation above

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

$$\boxed{v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)}$$

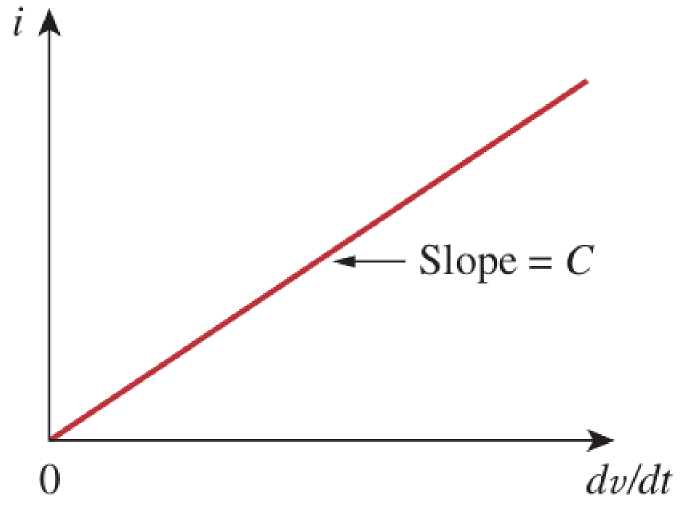


Figure 6.6
Current-voltage relationship of a capacitor.

Where $v(t_0) = q(t_0)/C$ is the voltage across the capacitor at time t_0

Power in a Capacitor

- The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{dv}{dt}$$

- The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p(\tau)\,d\tau = C\int_{-\infty}^{t} v\frac{dv}{d\tau}\,d\tau = C\int_{v(-\infty)}^{v(t)} v\,dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)}$$

- Note that v(-∞) = 0 because the capacitor was uncharged at time t = -∞

$$\boxed{w = \frac{1}{2}Cv^2}$$

$$w = \frac{q^2}{2C}$$

Properties of a Capacitor

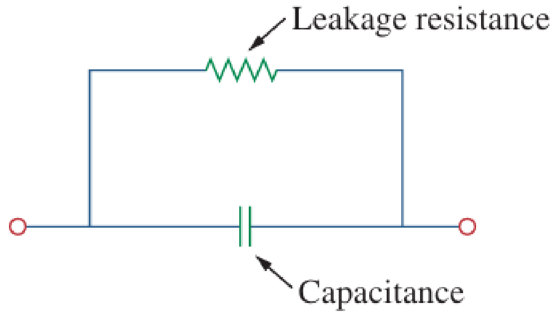
- When the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus,

A capacitor is an open circuit to dc.

- The voltage on the capacitor must be continuous.

The voltage on a capacitor cannot change abruptly.

- The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.
- A real, nonideal capacitor has a parallel-model leakage resistance, as shown in the figure. The leakage resistance may be as high as 100 MΩ and can be neglected for most practical applications. For this reason, we will assume ideal capacitors in this course.



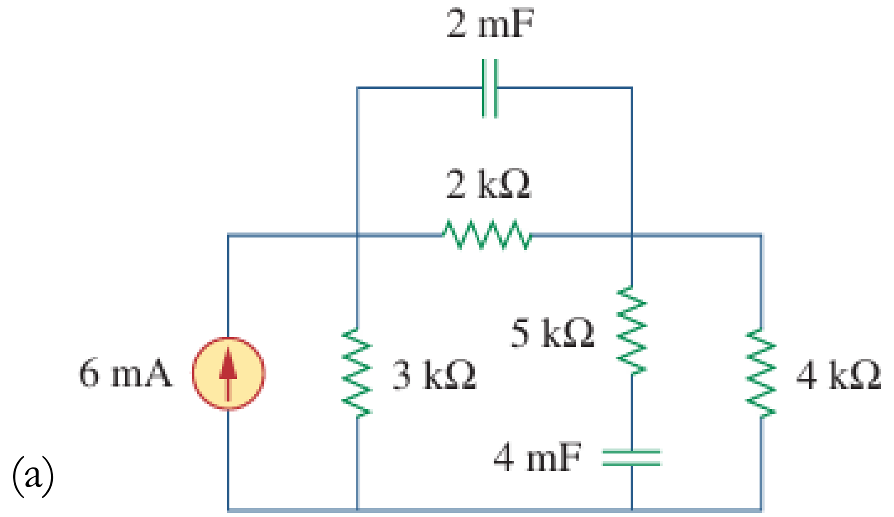
Circuit model of a nonideal capacitor.

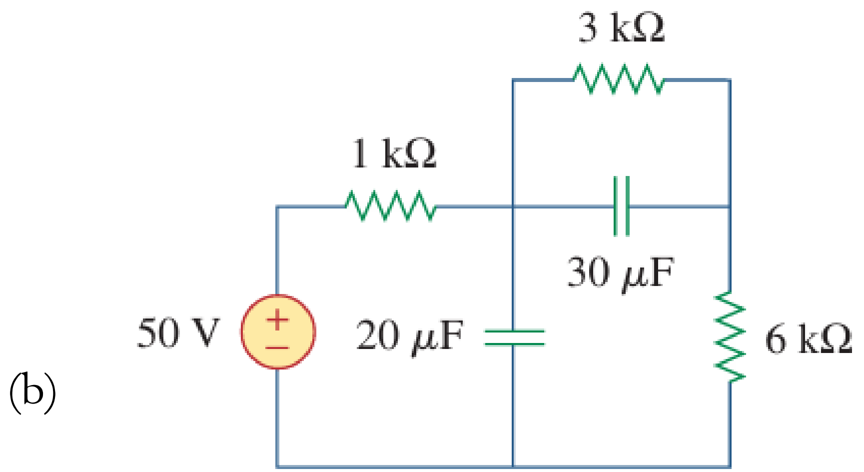
Problems

1. What is the voltage across a 4.5-mF capacitor if the charge on one plate is 0.12 mC? How much energy is stored?

2. The voltage across a 5-mF capacitor is $v(t) = 10\cos 6000t$ V Calculate the current through it.

- Obtain the energy stored in each capacitor in the circuit under dc conditions.





Parallel Capacitors

Applying KCL,

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

$$i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt} + \cdots + C_N\frac{dv}{dt}$$

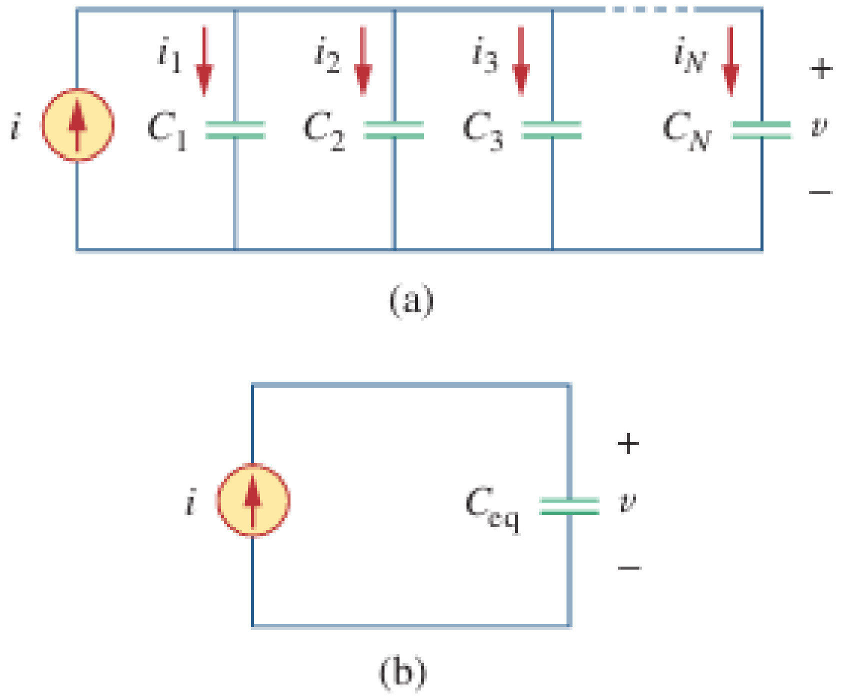
$$i = (C_1 + C_2 + C_3 + \cdots + C_N)\frac{dv}{dt}$$

$$i = C_{eq}\frac{dv}{dt}$$

Hence,

$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N$$

The **equivalent capacitance** of N parallel-connected capacitors is the sum of the individual capacitances.



(a) Parallel-connected N capacitors, (b) equivalent circuit for the parallel capacitors.

Series Capacitors

Applying KVL,

$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

$$v = \frac{1}{C_1}\int_{t_0}^{t} i(\tau)d\tau + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(\tau)d\tau + v_2(t_0)$$

$$+ \cdots + \frac{1}{C_N}\int_{t_0}^{t} i(\tau)d\tau + v_N(t_0)$$

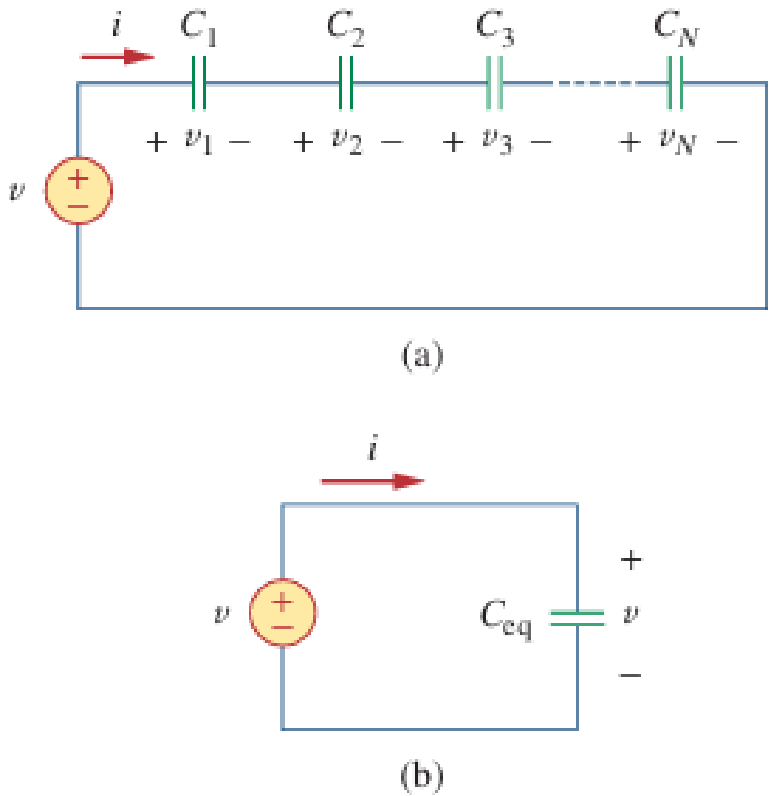
$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(\tau)d\tau + v_1(t_0) + v_2(t_0)$$

$$+ \cdots + v_N(t_0)$$

$$= \frac{1}{C_{eq}}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$$

Hence,

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}$$

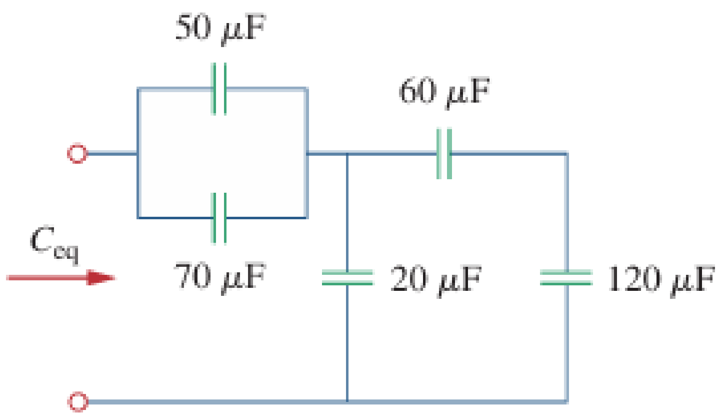


(a) Series-connected N capacitors, (b) equivalent circuit for the series capacitor.

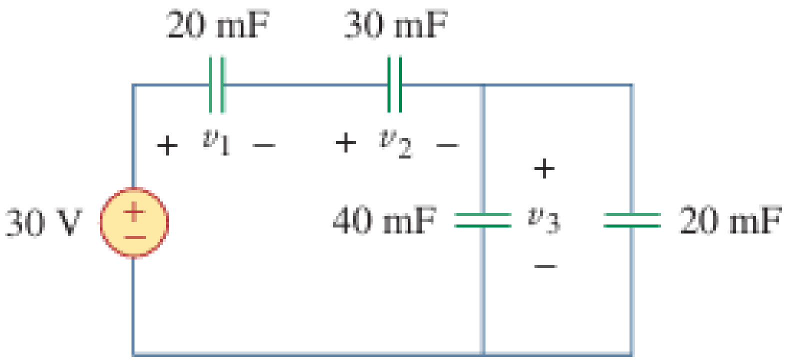
The **equivalent capacitance** of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

Problems

Find the equivalent capacitance seen at the terminals of the circuit in the circuit below



Find the voltage across each capacitor in the circuit below



Inductors

- An inductor is a passive element designed to store energy in its magnetic field.
- They are used in power supplies, transformers, radios, TVs, radars, and electric motors.

An inductor consists of a coil of conducting wire.

- Any conductor of electric current has inductive properties and may be regarded as an inductor.
- But in order to enhance the inductive effect, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire, as seen in the Fig. 6.21.

- The voltage across the inductor is directly proportional to the time rate of change of the current.

$$v = L\frac{di}{dt}$$

Where L, the constant of proportionality, is known as the inductance of the inductor.

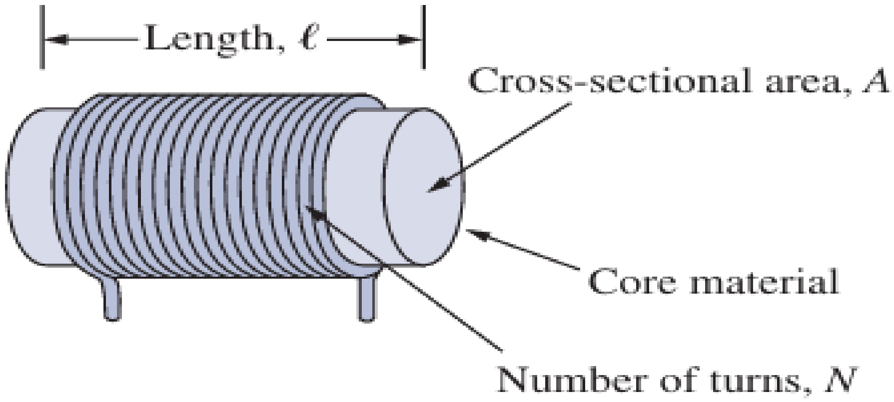


Figure 6.21
Typical form of an inductor.

Inductance of an Inductor

Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).

- The inductance of an inductor depends on its physical dimension and construction.
- It depends on the physical dimensions of the inductor. For the solenoid shown in Fig. 6.1, the inductance is given by

$$L = \frac{N^2 \mu A}{l}$$

Where,
N - the number of turns,
l - the length,
A - the cross-sectional area
μ - the permeability of the core

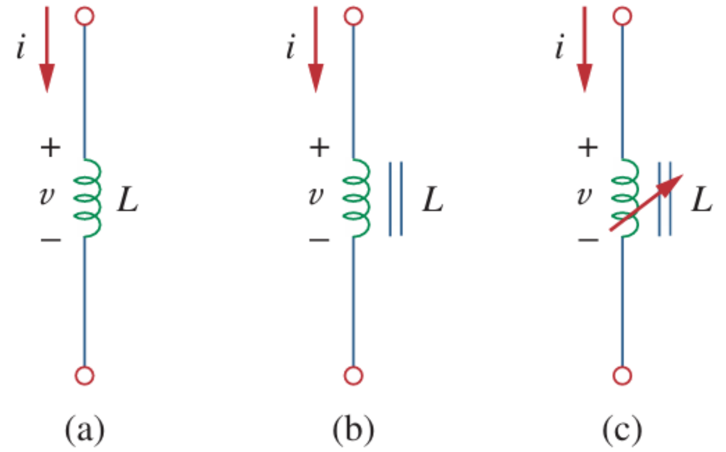


Figure 6.23
Circuit symbols for inductors: (a) air-core, (b) iron-core, (c) variable iron-core.

- The circuit symbols for inductors are shown in Fig.6.23, following the passive sign convention.

Current-Voltage Relationship of a Inductor

$$v = L\frac{di}{dt}$$

- The current-voltage relationship of the inductor is obtained from the equation above,

$$di = \frac{1}{L}v$$

$$i = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau$$

$$\boxed{i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0)}$$

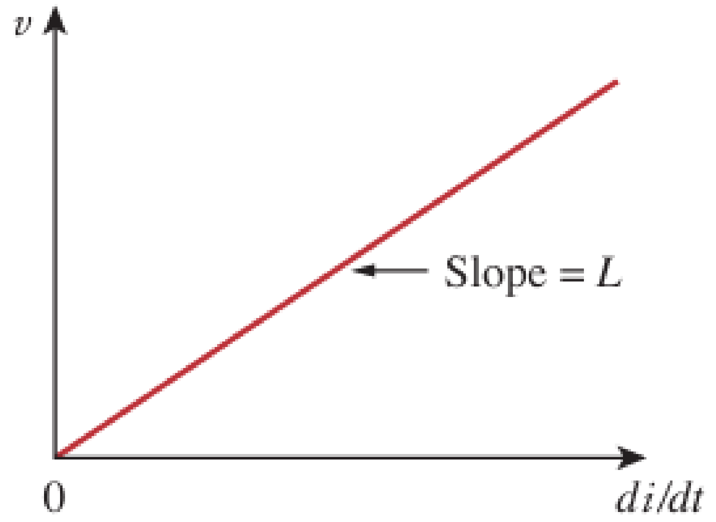


Figure 6.24
Voltage-current relationship of an inductor.

Where $i(t_0)$ is the total current for $-\infty < t < t_0$ and $i(-\infty) = 0$, which represents a time in the past when there was no current in the inductor.

Power in a Inductor

- The instantaneous power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right)i$$

- The energy stored in the inductor is therefore

$$w = \int_{-\infty}^{t} p(\tau)\,d\tau = L\int_{-\infty}^{t} \frac{di}{d\tau}\, i d\tau$$

$$= L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty)$$

- Since i(-∞) = 0

$$\boxed{w = \frac{1}{2}Li^2}$$

Properties of an inductor

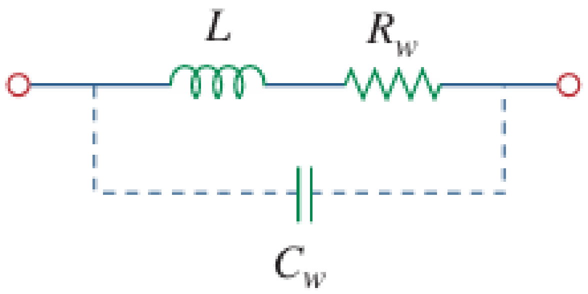
- The voltage across an inductor is zero when the current is constant. Thus,

An inductor acts like a short circuit to dc.

- An important property of the inductor is its opposition to the change in current flowing through it.

The current through an inductor cannot change instantaneously.

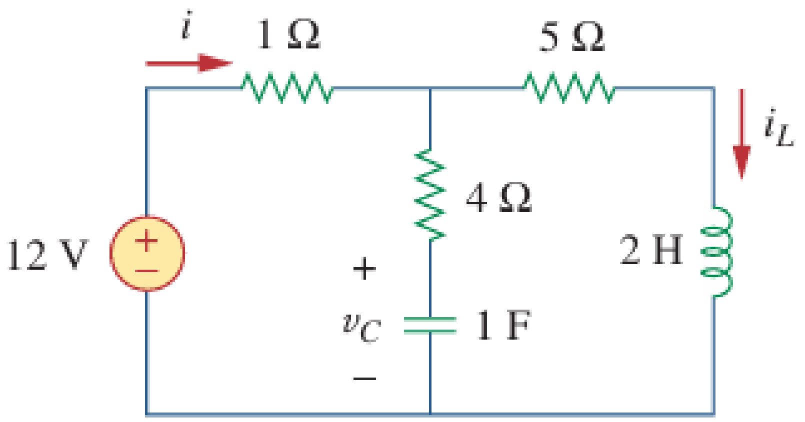
- Like the ideal capacitor, the ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.
- A practical, nonideal inductor has a significant resistive component called th winding resistance R_w , and it appears in series with the inductance of th inductor, making a real inductor also an energy dissipation device. Since R_w i usually very small, it is ignored in most cases. The nonideal inductor also has winding capacitance due to the capacitive coupling between the conductin coils. We will assume ideal inductors in this course.



Circuit model for a practical inductor.

Problems

1. Consider the circuit below. Under dc conditions, find: (a) i, v_C and i_L and (b) the energy stored in the capacitor and inductor.



Series Inductors

Applying KVL,

$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

Substituting $v_k = L_k\, di/dt$ results in

$$v = L_1\frac{di}{dt} + L_2\frac{di}{dt} + L_3\frac{di}{dt} + \cdots + L_N\frac{di}{dt}$$

$$= (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt}$$

$$= \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{eq}\frac{di}{dt}$$

where

$$\boxed{L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N}$$

The **equivalent inductance** of series-connected inductors is the sum of the individual inductances.

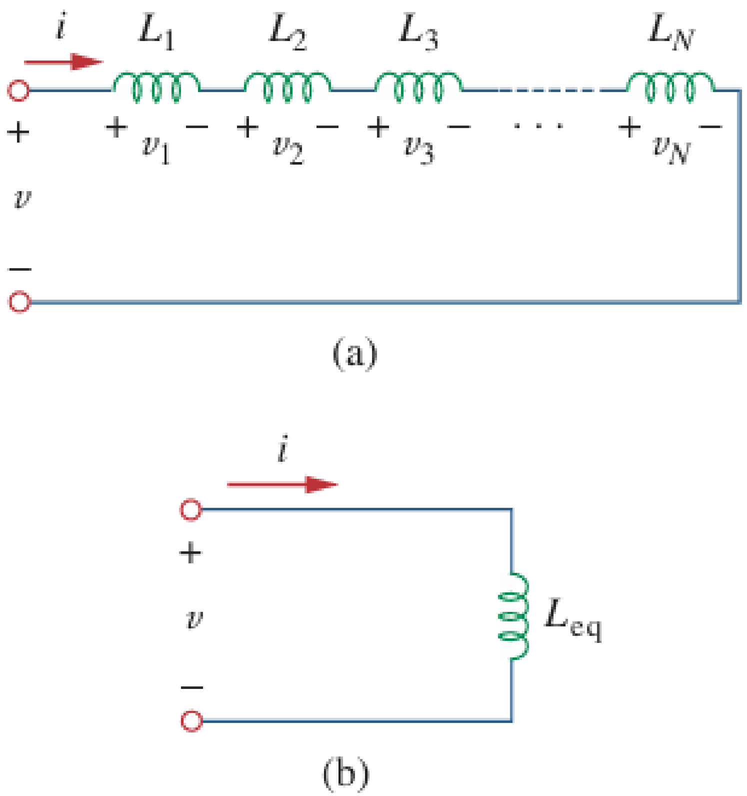


Figure 6.29
(a) A series connection of N inductors, (b) equivalent circuit for the series inductors.

Parallel Inductors

Applying KCL,

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

$$i = \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0)$$

$$+ \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N(t_0)$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1(t_0) + i_2(t_0)$$

$$+ \cdots + i_N(t_0)$$

$$= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} v\,dt + i(t_0)$$

where

$$\boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}}$$

The **equivalent inductance** of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances.

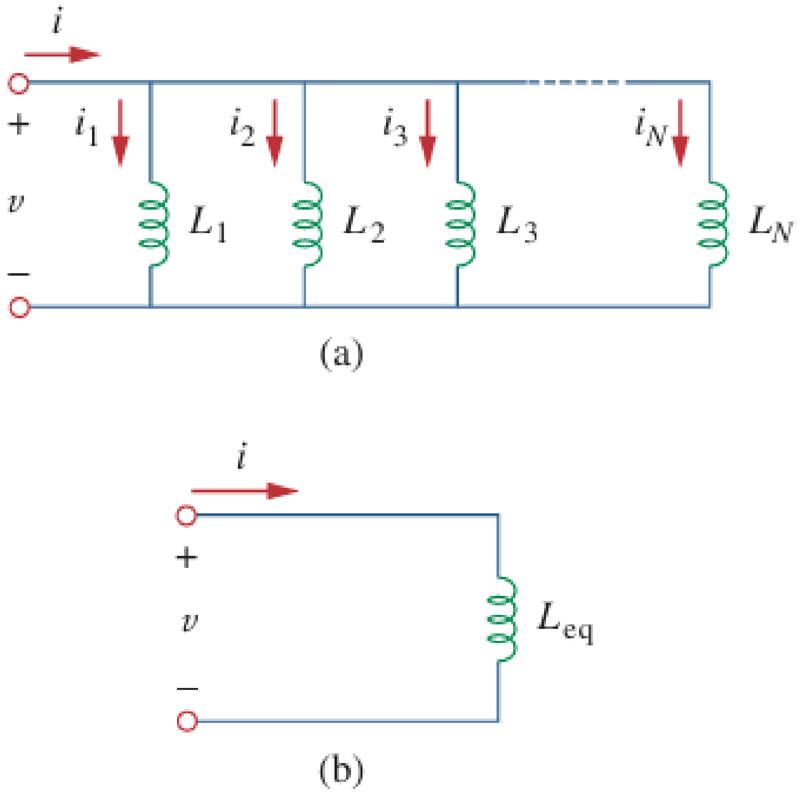
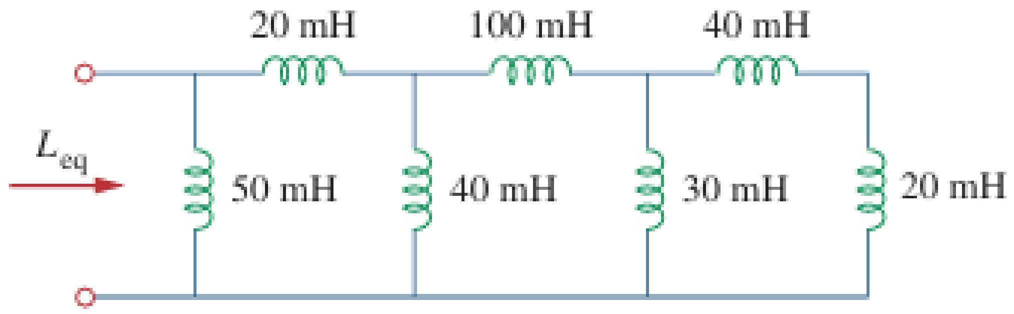


Figure 6.30
(a) A parallel connection of N inductors, (b) equivalent circuit for the parallel inductors.

Series and Parallel Inductors

Problems

Find the equivalent inductance seen at the terminals of the circuit in the circuit below



First-Order Circuits

Intro – First-Order Circuits

- A first-order circuit is characterized by a first-order differential equation.
- Two types of simple circuits make up a first-order circuit:
  1. a circuit comprising a resistor and capacitor, called an RC circuit and
  2. a circuit comprising a resistor and an inductor, called an RL circuit.
- RC and RL circuits are analysed by applying Kirchhoff's laws, as is done in resistive circuits.
- Applying Kirchoff's laws in resistive circuits yields algebraic equations, while applying the laws in RC and RL circuits yields differential equations.
- There are two ways of exciting first-order circuits:
  1. by initial conditions of the storage elements – *source-free circuits*.
  2. by independent sources

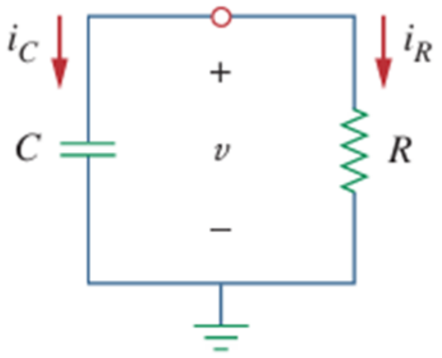
Source-Free RC Circuits

- A source-free RC circuit occurs when its DC source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.
- To determine the voltage response (*which is the voltage* $\boldsymbol{v(t)}$ *across the capacitor*), it is assumed that the capacitor is initially charged, with energy $\boldsymbol{w(0) = \frac{1}{2}CV_0^2}$, where V_0 is the initial voltage $v(0)$ at time $t = 0$
- Applying KCL to the node, $i_C + i_R = 0$

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

- This is a first-order differential equation.



A source-free RC circuit

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

$$\ln v = -\frac{t}{RC} + \ln A$$

where $\ln A$ is the integration constant.

$$\ln\frac{v}{A} = -\frac{t}{RC}$$

$$v(t) = Ae^{-t/RC}$$

From initial conditions $v(0) = A = V_0$

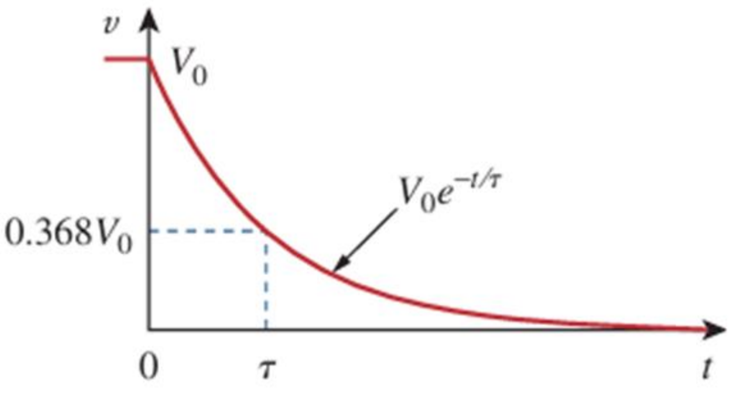
$$\boldsymbol{v(t) = V_0 e^{-t/RC}}$$

- It can be seen that the voltage response of the RC circuit is an exponential decay of the initial voltage.
- The response is due to the initial energy stored and the physical characteristics of the circuit hence it is termed as the ***natural response of the circuit***.

Time constant of a Source-Free RC Circuit

- The natural response is illustrated in the graph below.
- As time increases, the voltage decreases towards zero.
- The rate at which the voltage decays is characterized by the time constant τ.

The time constant of a circuit is the time required for the response to decay to a factor of $\mathbf{1/e}$ or 36.8 % of its initial value.



The voltage response of the RC circuit

- The time constant is the product of the resistance and capacitance, hence $\boldsymbol{\tau = RC}$
- At $t = \tau$

$$v(\tau) = V_0e^{-\tau/RC} = V_0e^{-\tau/\tau} = V_0e^{-1} = 0.368V_0$$

- It is evident from the table below that the voltage is less than 1 % of it's initial value V_0 after 5τ.
- Thus, it is customary to assume that the capacitor is fully discharged (or charged) after five times the time constant.

Table: Values of $\boldsymbol{v(\tau) = V_0e^{-t/\tau}}$

t	$v(t)$
τ	$0.36788\,V_0$
2τ	$0.13534V_0$
3τ	$0.04979\,V_0$
4τ	$0.01832\,V_0$
5τ	$0.00674\,V_0$

Power dissipation in a Source-Free RC Circuit

- Power dissipation in a source-free RC circuit happens in the resistor
- Power dissipated in the resistor,

$$p(t) = v\, i_R = V_0 e^{-t/\tau} \times \frac{V_0}{R} e^{-t/\tau}$$

$$= \frac{V_0^2}{R} e^{-2t/\tau}$$

- The energy absorbed by the resistor up to time t is

$$w_R(t) = \int_0^t p(\lambda)d\lambda = \int_0^t \frac{V_0^2}{R} e^{-2\lambda/\tau} d\lambda$$

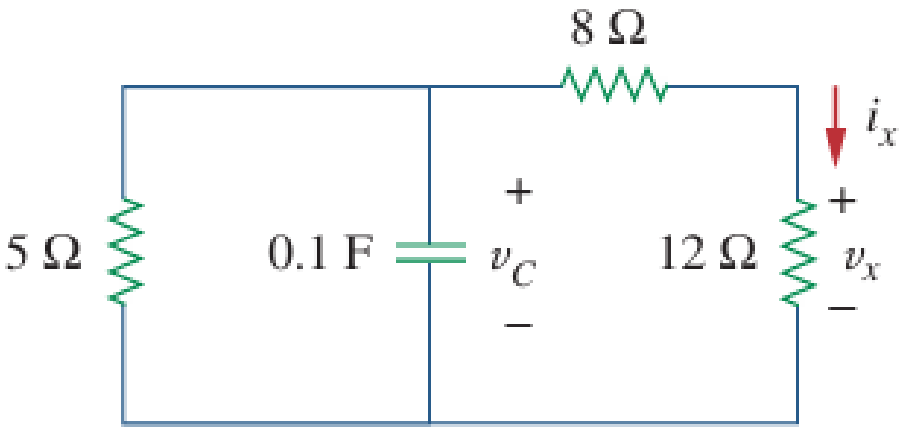
$$= -\frac{\tau V_0^2}{2R} e^{-2\lambda/\tau}\Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}),$$

The Key to Working with a Source-Free *RC* Circuit Is Finding:

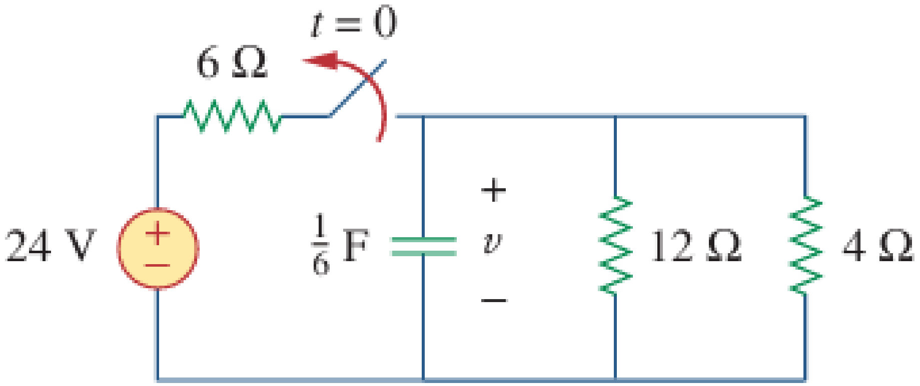
1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ.

Problems – Source-Free RC Circuits

In the circuit below, $v_c(0) = 15$ V. Find v_C, v_x, and i_x for t > 0.



If the switch in the figure below opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.



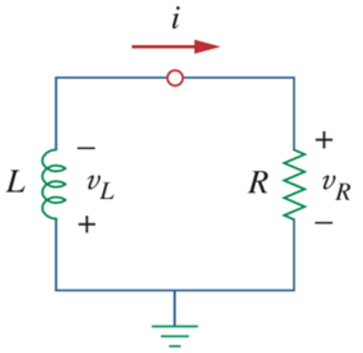
Source-Free RL Circuit

- A source-free RL circuit occurs when its DC source is suddenly disconnected. The energy already stored in the inductor is released to the resistors.
- To determine the current response (*which is the current $i(t)$ across through the inductor*), it is assumed that the inductor is initially charged, with energy $w(0) = \frac{1}{2}LI_0^2$, where I_0 is the initial current $i(0)$ at time $t = 0$
- Applying KVL to the loop, $v_L + v_R = 0$

$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

- This is a first-order differential equation.



A source-free RL circuit

$$\frac{di}{i} = -\frac{R}{L}dt$$

Integrating both sides, $\ln i = -\frac{Rt}{L} + \ln A$

where $\ln A$ is the integration constant.

$$\ln\frac{i}{A} = -\frac{Rt}{L}$$

$$i(t) = Ae^{-Rt/L}$$

From initial conditions $i(0) = A = I_0$

$$v(t) = I_0e^{-Rt/L}$$

- It can be seen that the voltage response of the RC circuit is an exponential decay of the initial current.

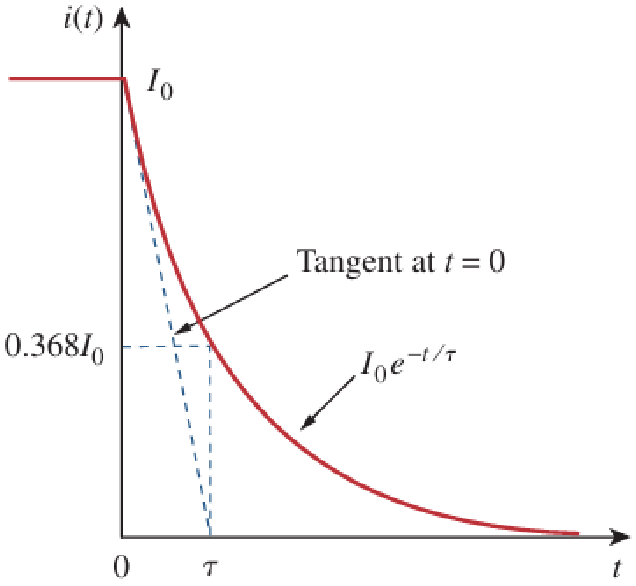
Time constant of a Source-Free RL Circuit

- The natural response is illustrated in the graph below.
- As time increases, the current decreases towards zero.

$$\tau = \frac{L}{R}$$

- At $t = \tau$

$$i(\tau) = I_0 e^{-R\tau/L} = I_0 e^{-\tau/\tau} = I_0 e^{-1} = 0.368 I_0$$



The current response of the RL circuit

Table: Values of $i(\tau) = I_0 e^{-t/\tau}$

t	$v(t)$
τ	$0.36788\, I_0$
2τ	$0.13534 I_0$
3τ	$0.04979\, I_0$
4τ	$0.01832\, I_0$
5τ	$0.00674\, I_0$

Power dissipation in a Source-Free RL Circuit

- Power dissipation in a source-free RL circuit happens in the resistor
- Power dissipated in the resistor,

$$p(t) = v_R i = I_0 R e^{-t/\tau} \times I_0 e^{-t/\tau}$$
$$= I_0^2 e^{-2t/\tau}$$

- The energy absorbed by the resistor up to time t is

$$w_R(t) = \int_0^t p(\lambda) d\lambda = \int_0^t I_0^2 e^{-2\lambda/\tau} \, d\lambda = -\frac{\tau}{2} I_0^2 R e^{-2\lambda/\tau} \Big|_0^t,$$

or

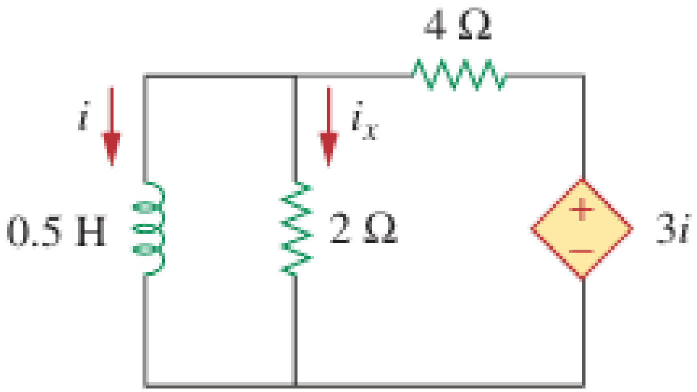
$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau})$$

The Key to Working with a Source-Free *RL* Circuit Is to Find:

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant τ of the circuit.

Problems – Source-Free RL Circuits

Find $i(t)$ and i_x in the circuit below.
Let $i(0) = 10$ A.



Find $i(t)$ for $t > 0$ in the circuit below