PABLO

MATH 114 : ENGINEERING MATHS II

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Overview

- Types Of Sequence
- Limit Of Sequence
- Bounded Sequence
- Monotonic Sequence
- Squeeze Theorem for Sequences

2 SERIES

- The geometric series
- Convergence of a series
- Power Series

Definition

A sequence is an ordered list of numbers. The following is a sequence of odd numbers:

$$1, 3, 5, 7, \cdots$$

A term of a sequence is identified by its position in the sequence. In the above sequence, 1 is the first term, 3 is the second term, etc. The ellipsis symbol ($\cdots$) indicates that the sequence continues forever. Every number in the sequence is called a **term** of the sequence.

Note

If the sequence has a last term, then it is called a **Finite Sequence** If the number of terms is infinite, then the sequence is called an **Infinite Sequence**. The n^{th} term of the sequence will be denoted by U_n, while the sequence itself will be denoted by $\{U_n\}$.

Example

$\{A_n\} = \{1, 2, \cdots, 30\}$ is a **Finite Sequence** of the counting numbers up to 30.

Example

$\{U_n\} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$ is an **Infinite Sequence** with the n^{th} term $U_n = \frac{1}{n}$.

There are two types of Sequence namely:

- Arithmetic Sequence or Arithmetic Progression
- Geometric Sequence or Geometric Progression

Arithmetic Sequence or Arithmetic Progression

Definition

A sequence $\{U_n\}$ of numbers is said to be an arithmetic sequence if

$$U_{n+1} - U_n = d, \ \ \forall n.$$

The n^{th} term of an **A.P** is defined by:

$$U_n = a + (n-1)d$$

where $a =$ first term and d =common difference.

Example

The 6^{th} term of an A.P is 17 and the 13^{th} term is 38. Determine the 19^{th} term.

Solution

We are given the value of the 6^{th} *and* 13^{th} *term so in order to find the next term we first need to know our first term and the common difference.*

$$\begin{aligned}
U_n &= a + (n-1)d \\
U_6 &= a + (6-1)d = 17 \\
\\
&= a + 5d = 17 \quad \cdots\cdots\cdots\cdots\cdots(1) \\
\\
U_{13} &= a + (13-1)d = 38 \\
U_{13} &= a + 12d = 38 \\
\\
&= a + 12d = 38 \cdots\cdots\cdots\cdots\cdots(2)
\end{aligned}$$

Solving equations (1) and (2) simultaneously we have $a = 2$ *and* $d = 3$
Therefore

$$\begin{aligned}
U_{19} &= 2 + (19-1)3 \\
&= 56
\end{aligned}$$

Definition

The sum of the first n terms of an Arithmetic sequence is given by :

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

or

$$S_n = \frac{n}{2}\left[a + l\right]$$

where a= first term, d=common difference and
l(last term) $= a + (n-1)d$

Example

Find the sum of the first 12 terms of $5, 9, 13, 17 \cdots$

Solution

In the above example we where given the first term and asked to find the sum to the 12^{th} *term.*

$$
\begin{aligned}
S_n &= \frac{n}{2}[2a + (n-1)d] \\
S_{12} &= \frac{12}{2}[2(5) + (12-1)4] \\
S_{12} &= 324
\end{aligned}
$$

Example

Find the sum of the sequence

$$1, 3.5, 6, 8.5, \cdots, 101.$$

Solution

From the above sequence our common difference d is 2.5 and the first term a is 1.

$$\begin{aligned}
l &= a + (n-1)d \\
101 &= 1 + (n-1) \times 2.5 \\
100 &= (n-1) \times 2.5 \ (\textit{make n the subject}) \\
n &= 41 \\
S_n &= \frac{n}{2}[a + l] \\
S_{41} &= \frac{41}{2}[1 + 101] \\
S_{41} &= 2091
\end{aligned}$$

Geometric Sequence

Definition

When a sequence has a constant ratio between successive terms is called a Geometric Progression **G.P**. The constant term is called the **common ratio.** The n^{th} term of a G.P is given by:

$$U_n = ar^{n-1}$$

where r = common ratio and a = first term

Definition

The sum of the first n terms of a geometric sequence is given by :

$$S_n = \frac{a(r^n - 1)}{r - 1}, \ |r| > 1$$

Or

$$S_n = \frac{a(1 - r^n)}{1 - r}, \ |r| < 1$$

Example

Determine the 10^{th} term of the series $3, 6, 12, 24, \cdots$

Solution

$3, 6, 12, 24, \cdots$ *is a geometric progression with a common ratio r of 2.*
The n^{th} term of a G.P is ar^{n-1}, where a is the first term.
Hence the 10^{th} term is:

$$(3)(2)^{10-1} = (3)(2)^9 = 3(512) = 1536$$

Example

Find the sum of the first 8 terms of the G.P $1, 2, 4, 8, 16 \cdots$

Solution

From the above sequence $|r| = 2 > 1$ *and the first term* a *is 1*

$$
\begin{aligned}
S_n &= \frac{a(r^n - 1)}{r - 1} \\
S_8 &= \frac{1(2^8 - 1)}{2 - 1} \\
&= \frac{2^8 - 1}{1} \\
S_8 &= 255
\end{aligned}
$$

Definition

Sum to infinity (S_∞)

$$S_\infty = \frac{a}{1-r}, \quad |r| < 1$$

Example

Find the sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

Solution

The first term is $a = 5$ *and the common ratio is* $r = -\frac{2}{3}$. *Since* $|r| = \frac{2}{3} < 1$, *the sum to infinity is*

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - \quad -\frac{2}{3}} = \frac{5}{\frac{5}{3}} = 3$$

Example

Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

Solution

Notice that this series starts with $n = 0$ *and so the first term is* $x^0 = 1$. *Thus*

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with $a = 1$ *and* $r = x$. *Since* $|x| < 1$, *it converges and gives*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Example

A drilling machine is to have 6 speeds ranging from 50 rev/min to 750 rev/min. If the speeds form a geometric progression determine their values, each correct to the nearest whole number

Solution: Let the GP of n terms by given by $a, ar, ar^2, \cdots, ar^{n-1}$ The 1st term a = 50 rev/min.

The 6th term is given by ar^{6-1}, which is 750 rev/min, i.e., $ar^5 = 750$

from which $r^5 = \dfrac{750}{a} = \dfrac{750}{50} = 15$

Thus the common ratio, $r = \sqrt[5]{15} = 1.7188$

The 1st term is $a = 50$ rev/min.

the 2nd term is $ar = (50)(1.7188) = 85.94$,

the 3rd term is $ar^2 = (50)(1.7188)^2 = 147.71$, the 4th term is $ar^3 = (50)(1.7188)^3 = 253.89$, the 5th term is $ar^4 = (50)(1.7188)^4 = 436.39$,

the 6th term is $ar^5 = (50)(1.7188)^5 = 750.06$

Hence, correct to the nearest whole number, the 6 speeds of the drilling machine are: 50, 86, 148, 254, 436, and 750 rev/min.

Example

A hire tool firm finds that their net return hiring tools is decreasing by 10% per annum. If the net gain on a certain tool this year is 400 cedis. The net gain forms a series $400 + 400 \times 0.9 + 400 \times 0.9^2 + \cdots$. Find the possible total of all future profits for this tool (assuming the tool lasts forever).

Solution

The net gain forms a series:

$$400 + (400 \times 0.9) + (400 \times 0.9^2) + \cdots$$

$$a = 400, \quad r = 0.9$$

The sum to infinity,

$$S_\infty = \frac{a}{1-r} = \frac{400}{1-0.9} = 4000 \text{ } cedis$$

Limit Of Sequence

There are sequences which are neither arithmetic nor geometric. In what follows general properties of sequences will be discussed.

Definition

An infinite sequence $\{U_n\}_{n=1}^{\infty}$ is said to have a limit L as n approaches ∞, if U_n can be made as close to L as we please by choosing n sufficiently large. We then have

$$\lim_{n\to\infty} U_n = L$$

The above definition may be more rigorously written as follows :

Definition of Limit

For every $>$ 0 sufficiently small), there exist a positive integer$N$ (depending on) such that $|U_n - L| <$, $\forall n > N$.

Note

If $\lim_{n \to \infty} U_n = L$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Example

Show that $\lim_{n\to\infty} \frac{1}{2^n} = 0$ using the definition of limit of a sequence.

Proof.

$$\forall, \quad \exists\ N(\ ) > 0 \text{ such that } |U_n - L| < , \quad \forall n > N.$$

$$|U_n - L| < , \quad U_n = \frac{1}{2^n}, \quad L = 0$$

$$\Longrightarrow \quad \frac{1}{2^n} - 0 \ <$$

$$\Longrightarrow \quad \frac{1}{2^n} \ <$$

$$\Longrightarrow \quad \frac{1}{2^n} <$$

$\square$

cont'd.

$$\Longrightarrow \quad \frac{1}{\;} < 2^n \text{ [Taking Log of both sides]}$$

$$\Longrightarrow \quad \log \frac{1}{\;} < \log 2^n$$

$$\Longrightarrow \quad \frac{\log \frac{1}{\;}}{\log 2} < \frac{n \log 2}{\log 2}$$

$$\Longrightarrow \quad \frac{\log \frac{1}{\;}}{\log 2} < n \,, n > N$$

Choosing $N = \dfrac{\log \frac{1}{\;}}{\log 2}$, we see that $\dfrac{1}{2^n} - 0 < \,, \quad \forall n > N$

Proving the existence of N and thus establishing the required result. $\square$

Theorem (1.1)

If $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = U_n$ *when n is an integer, then* $\lim_{n\to\infty} U_n = L$

Note

$n\to\infty$

Definition

$\lim_{n\to\infty} U_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{If } n > N \text{ then } U_n > M.$$

If U_n becomes large as n becomes large, we use $\lim U_n = \infty$.

Note If lim $U_n = \infty$ then the sequence $\{U_n\}$ is divergent.

Example

Evaluate $\lim_{n\to\infty} \frac{1}{x^r}$, where $r > 0$

Solution

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \ if \ r > 0$$

$n\to\infty$

Theorem (Limit Laws for Sequences)

Let U_n and V_n be two sequences such that $\lim\limits_{n\to\infty} U_n = A$ and $\lim\limits_{n\to\infty} V_n = B$ and c a constant, Then the following results hold:

1. $\lim\limits_{n\to\infty} (U_n \pm V_n) = \lim\limits_{n\to\infty} U_n \pm \lim\limits_{n\to\infty} V_n = A \pm B$
2. $\lim\limits_{n\to\infty} (U_n \cdot V_n) = \lim\limits_{n\to\infty} U_n \cdot \lim\limits_{n\to\infty} V_n = A.B$
3. $\lim\limits_{n\to\infty} cU_n = c \lim\limits_{n\to\infty} U_n$, *where* $\lim\limits_{n\to\infty} c = c$
4. $\lim\limits_{n\to\infty} \dfrac{U_n}{V_n} = \dfrac{\lim\limits_{n\to\infty} U_n}{\lim\limits_{n\to\infty} V_n} = \dfrac{A}{B}$, *if* $\lim\limits_{n\to\infty} V_n = 0$
5. $\lim\limits_{n\to\infty} P^{U_n} = P^{\left[\lim\limits_{n\to\infty} U_n\right]} = P^A$
6. $\lim\limits_{n\to\infty} (U_n)^P = [\lim\limits_{n\to\infty} U_n]^P = A^P$

Theorem (1.2)

If a sequence $\{U_n\}$ has a limit, then the limit is unique.

Proof.

Suppose that $\{U_n\}$ has two limits l_1 and l_1. For every $>\quad 0$ sufficiently small), there exist a positive integer N (depending on) such that $|U_n - l_1| < , \quad \forall n > N$ and $|U_n - l_2| < , \quad \forall n > N$ Then

$$|l_1 - l_2| = |l_1 - U_n + U_n - l_2|$$

$$\leq |l_1 - U_n| + |U_n - l_2|$$

$$< \quad +$$

Since $>\quad 0$ (however small), it follows that $|l_1 - l_2|$ is less than every positive integer.

$$\therefore |l_1 - l_2| = 0 \implies l_1 - l_2 = 0 \implies l_1 = l_2$$

This completes the proof. □

Evaluating limits of Sequences

Definition

The limit L of a sequence U_n if it exists, may be evaluated using the theorems on limits.

Note

In evaluating limits of sequences, we will divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws.

Example

Find $\lim_{n \to \infty} \dfrac{n}{n+1}$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)=\lim_{n\to\infty}\left(\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)=1$$

Example

Evaluate $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$

Solution

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$$

$$=\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$=\frac{1}{\sqrt{n+1}+\sqrt{n}}=0$$

Example

Calculate $\lim_{n \to \infty} \frac{\ln n}{n}$

Solution

It is realized that both the numerator and the denominator approach infinity as $n \to \infty$. *We can't apply L'Hospital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply L'Hospital's Rule to the related function* $\frac{\ln x}{x}$ *and obtain*

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$$

Therefore, by Theorem 1.2.1, we have

$$\lim_{n \to \infty} \frac{\ln n}{n} = 0$$

If $\lim_{n\to\infty} U_n = L$ *and the function is continuous at* L *, then*

$$\lim_{n\to\infty} f(U_n) = f(L)$$

Example

Find $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$

Solution

Because the sine function is continuous at 0 , Theorem 1.3 enables us to write

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin 0 = 0$$

Theorem (1.4)

The sequence $\{r^n\}$ *is convergent if* $-1 < r \leq 1$ *and divergent for all other values of* r *.*

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Example

Determine if the sequence $\{(-1)^n\}_{n=0}^{\infty}$ converge or diverge. If the sequence converges determine its limit.

Solution

From Theorem 1.4 above $r = -1$*. So, by Theorem 1.4 this sequence diverges.*

Bounded Sequence

Definition

Let U_n be a sequence. If $U_n \leq M$ for $n = 1,2,3\cdots$, where M is a constant (independent of n), we say that the sequence $\{U_n\}$ is **Bounded above** and M is called an **Upper bound**. If $U_n \geq m$, the sequence is **Bounded Below** and m is called the **Lower Bound** .

Definition

Let M be an upper bounded of U_n, then if for any $> 0, \exists$ at least element x in U_n that is $x > M -$, the number M is called the **Least Upper Bound or Supremum or sup.**

Definition

Let m be a lower bounded of U_n, then if for any $> \quad 0$ (however small). $\exists$ at least one element x in U_n such that $x < m + \quad$,the number m is called the **Greatest Lower Bound** or **Infimum** or **inf**.

Definition

If $\forall\ x \in U_n,\ \exists\ m, M$(real) such that $m \leq x \leq M$, then the sequence U_n is said to be **Bounded**.

Note: If the sequence is both **bounded below** and **bounded above** we call the sequence **Bounded**.

Example

Find the upper and lower bound, supremum and infimum of $A = [-2, 2]$

Solution

$$\begin{aligned} \textit{Upper Bound} &= 2, 3, 4, 5 \cdots \\ \textit{Least Upper Bound} &= 2 \\ \textit{Lower Bound} &= -2, -3, -4, -5 \cdots \\ \textit{Greatest Lower Bound} &= -2 \end{aligned}$$

$$-2 \leq A \leq 2, \quad A \text{ is bounded}$$

Example

Show that $b_n = \dfrac{n+1}{n+3}$ is bounded.

Solution

First we need to determine the first few terms of the sequence $\{a_n\}$

$$For \;\; n = 1, \;\; b_1 = \frac{2}{4}$$

$$For \;\; n = 2, \;\; b_2 = \frac{3}{5}$$

$$For \;\; n = 3, \;\; b_3 = \frac{4}{6}$$

Solution (cont'd)

Next, to determine the last term we need to find the limit of the sequence $\{a_n\}$

$$\lim_{n\to\infty} \frac{n+1}{n+3} = \lim_{n\to\infty} \left(\frac{\frac{n}{n}+\frac{1}{n}}{\frac{n}{n}+\frac{3}{n}} \right) = \left(\frac{1+\frac{1}{\infty}}{1+\frac{3}{\infty}} \right) = 1$$

$$\therefore \quad b_n = \frac{n+1}{n+3} = \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \cdots, 1$$

$$Upper\ Bound = 1, \quad Lower\ Bound = \frac{1}{2}$$

We can say $\{b_n\}$ *is bounded because it has an upper bound =1 and lower bound=* $\frac{1}{2}$

Monotonic Sequence

Definition

A sequence is **monotonic** if it is either increasing or decreasing.

Definition

A sequence U_n is said to be **monotonic increasing sequence** or **non decreasing** if:

$$U_{n+1} - U_n \geq 0, \quad \forall\ n \geq 1.$$

A sequence U_n is called **monotonic decreasing sequence** or **non increasing** if :

$$U_{n+1} - U_n \leq 0, \quad \forall\ n \geq 1.$$

Definition

We call a sequence **strictly increasing** if $U_{n+1} - U_n > 0, \ \forall\ n \geq 1$,
If $U_{n+1} - U_n < 0, \ \forall\ n \geq 1$ the sequence is said to be **strictly decreasing.**

Theorem

Every bounded monotonic sequence has a limit.

Solution

Prove as exercise

Example

Show that the following sequences are monotonic.

a. $\dfrac{n+1}{n}$ and b. $\dfrac{n}{n+1}$

Solution

$$a.\quad U_n = \frac{n+1}{n}, \quad U_{n+1} = \frac{n+2}{n+1}$$

$$U_{n+1} - U_n = \frac{n+2}{n+1} - \frac{n+1}{n}$$

$$= \frac{n(n+2) - (n+1)(n+1)}{(n+1)n}$$

$$= \frac{n^2 + 2n - (n^2 + 2n + 1)}{n^2 + n}$$

Solution

$$= \frac{n^2 + 2n - n^2 - 2n - 1}{n^2 + n}$$

$$= \frac{-1}{n^2 + n} \quad \textit{Negative Sequence}$$

All negatives less than zero

$$= \frac{-1}{n^2 + n} \leq 0$$

$\therefore$ *It is monotonic decreasing*

Solution

b.

$$U_n = \frac{n}{n+1} \qquad U_{n+1} = \frac{n+1}{n+2}$$

$$U_{n+1} - U_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$= \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}$$

$$= n^2 + 2n + 1 - (n^2 + 2n)$$

$$(n+2)(n+1)$$

$$= \frac{n^2 - 2n + 1 - n^2 + 2n}{(n+2)(n+1)}$$

$$= \frac{1}{(n+2)(n+1)} \geq 0$$

$\therefore$ it is Monotonic increasing

Squeeze Theorem for Sequences

Theorem (1.5)

If $a_n \leq b_n \leq c_n$, $\forall n > N$ for some N and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Example

Find $\lim_{n\to 0} n^2 \cos\left(\frac{1}{n^2}\right)$ using Squeezed Theorem.

Solution

Solution: *We know that cosine stays between* -1 *and* 1*, so*

$$-1 \leq \cos\left(\frac{1}{n^2}\right) \leq 1, \quad \forall\, n = 0$$

Since n^2 *is always positive, we can multiply by this inequality*

$$-n^2 \leq n^2 \cos\left(\frac{1}{n^2}\right) \leq n^2$$

So,our original sequence is bounded by $-n^2$ *and* n^2*. Now since*

$$\lim_{n\to 0} -n^2 = \lim_{n\to 0} n^2 = 0$$

then by the Squeezed Theorem,

$$\lim_{n\to 0} n^2 \cos\left(\frac{1}{n^2}\right) = 0$$

Example

Find $\lim_{n\to 0} n^2 e^{\sin\left(\frac{1}{n}\right)}$ using Squeezed Theorem.

Solution: The range of sine is $[-1,1]$, so

$$-1 \leq \sin\left(\frac{1}{n}\right) \leq 1$$

Taking e raised to both sides of an inequality does not change the inequality, so

$$e^{-1} \leq e^{\sin\left(\frac{1}{n}\right)} \leq e^{1}$$

we can multiply through by n^2 and get

$$n^2 e^{-1} \leq n^2 e^{\sin\left(\frac{1}{n}\right)} \leq n^2 e^{1}$$

So,our original sequence is bounded by n^2e^{-1} and n^2e^1,and since

$$\lim_{n\to 0} n^2e^{-1} = \lim_{n\to 0} n^2e^{1} = 0$$

then by the Squeeze Theorem,

$$\lim_{n\to 0} n^2e^{\sin(\frac{1}{n})} = 0$$

Theorem (1.6)

If $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$

Note: Note: If the limit of the sequence doesn't exist then the sequence will diverge but if the limit exist then the sequence will converge , the sequence will also converge when you find the last term.

Example

Find the $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n}$

Solution

$$\frac{(-1)^{n+1}}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty}\left(\frac{3n^2-1}{10n+5n^2}\right) = \lim_{n\to\infty}\left(\frac{\frac{3n^2}{n^2}-\frac{1}{n^2}}{\frac{10n}{n^2}+\frac{5n^2}{n^2}}\right)$$

$$\left(\quad 1 \right)$$

Example

Determine if the sequence $\left\{ \dfrac{3n^2 - 1}{10n + 5^2} \right\}_{n=2}^{\infty}$ converge or diverge. If the sequence converges determine its limit.

Solution

3 –

$$= \lim_{n\to\infty} \frac{\quad}{10 \quad n} = \frac{3}{5}$$

– 5

n

$\therefore \left\{ \dfrac{3n^2 - 1}{\quad 2} \right\}^{\infty}$ Converges and its limit is =. $\dfrac{3}{5}$

$10n + 5$

$n=2$

Example

Determine if the sequence $\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty}$ converge or diverge. If the sequence converges determine its limit.

Solution

$$\lim_{n\to\infty}\left(\frac{e^{2n}}{n}\right) = \lim_{x\to\infty}\left(\frac{e^{2x}}{x}\right) = \frac{e^{2(\infty)}}{\infty} = \frac{\infty}{\infty} = indeterminate$$

Next we need to use the L'hopital's rule

$$\lim_{n\to\infty}\frac{2e^{2x}}{1} = \infty$$

$$\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty} = \infty$$

since the limit does not exist the sequence diverges

Example

Determine if the sequence $\left\{\cos\left(\frac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ converge or diverge. If the sequence converges determine its limit.

Solution

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$$

$$= \cos\left(\frac{\pi}{\infty}\right) = \cos(0) = 1$$

since the limit exist the sequence is Convergent and its value is 1

Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n^{th} partial sum:

$$s_n = \sum_{i=0}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence s_n is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + ... + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number is called the **sum** of the series. If the sequence s_n is divergent, then the series is called **divergent**

Note

Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number . Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Example

Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Solution

Then the sum of the series is the limit of the sequence s_n :

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \left(\frac{2}{3+\frac{5}{n}} \right) = \frac{2}{3}$$

The geometric series

Definition

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| < 1$,the the geometric series is **convergent**
If $|r| \geq 1$,the the geometric series is **divergent**

Example

Find the n^{th} partial sum and determine if the series converges or diverges.

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots + \frac{1}{4^n}$$

Solution

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots + \frac{1}{4^n} = \sum_{k=1}^{n} \frac{1}{4}^{\,n}$$

This is a geometric series with ratio $r = \frac{1}{4} < 1$, therefore it will **converge**.
Now to calculate the sum for this series

$$a = \frac{1}{4}, \; r = \frac{1}{4}$$

$$s_\infty = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example

Find the n^{th} partial sum and determine if the series converges or diverges.

$$1 - 3 + 9 - 27 + \cdots + (-1)^{n-1}(-3)^{n-1}$$

Solution

$$1 - 3 + 9 - 27 + \cdots + (-1)^{n-1}(-3)^{n-1}$$

$$= \sum_{k=1}^{n}(-1)^{n-1}(-3)^{n-1}$$

This is a geometric series with ratio $|r| = |(-1)(-3)| = |3| \geq 1$ *therefore it will diverge.*

Example

Find the values of x for which the geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

$$\sum_{n=0}^{\infty} \frac{-1}{2}^{\,n} (x-3)^n$$

Solution

$$r = -\frac{1}{2}(x-3)$$

$$-\frac{1}{2}(x-3) < 1 \implies \frac{1}{2}(x-3) < 1 \implies 1 < \frac{1}{2}(x-3) < 1$$

$$\implies 2 < x-3 < 2 \implies 1 < x < 5$$

Solution (cont'd)

This geometric series will converge for values of x that are in the interval $1 < x < 5$

Now to determine the sum

$$a = 1, \; s_\infty = \frac{1}{1 + \frac{1}{2}(x-3)} = \frac{2}{x-1}$$

Convergence of a series

Example

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

Solution

This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.
We can simplify this expression if we use the partial fraction decomposition

$$
\begin{aligned}
\frac{1}{n(n+1)} &= \frac{A}{n} + \frac{B}{n+1} \\
1 &= A(n+1) + Bn \\
1 &= An + A + Bn \\
1 &= n(A+B) + A
\end{aligned}
$$

By comparison

$$
\begin{aligned}
A &= 1 \\
B &= -1
\end{aligned}
$$

Solution (cont'd)

$$s_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} + \frac{-1}{n+1}$$

$$= \quad 1 - \frac{1}{2} \quad + \quad \frac{1}{2} - \frac{1}{3} \quad + \quad \frac{1}{3} - \frac{1}{4} \quad + \cdots + \quad \frac{1}{n} - \frac{1}{n+1}$$

$$= \quad 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \quad 1 - \frac{1}{n+1} \quad = 1 - 0 = 1$$

Note: *Notice that the terms cancel in pairs. This is an example of a telescoping sum: Because of all the cancellations, the sum collapses (like a pirate's collapsing telescope) into just two terms.*

Divergence Test

Definition

If $\lim_{n \to \infty} a_n = 0$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent.

Example

Determine the behavior of the series $\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$ by using the divergence test.

Solution

$$\lim_{n\to\infty} \left(\frac{4n^2 - n^3}{10 + 2n^3} \right) = \lim_{n\to\infty} \left(\frac{\frac{4n^2}{n^3} - \frac{n^3}{n^3}}{\frac{10}{n^3} + \frac{2n^3}{n^3}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{\frac{4}{n} - 1}{\frac{10}{n^3} + 2} \right) = -\frac{1}{2} = 0$$

Hence

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3} \quad \textit{is divergent}$$

Example

Examine the behavior of the series $\sum_{n=0}^{\infty} \frac{n}{2n+1}$ by using the divergence test.

Solution

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \left(\frac{\frac{n}{n}}{\frac{2n}{n} + \frac{1}{n}} \right) = \lim_{n\to\infty} \left(\frac{1}{2 + \frac{1}{n}} \right) = \frac{1}{2} = 0$$

Hence

$$\sum_{n=0}^{\infty} \frac{n}{2n+1} \quad \textit{is divergent}$$

Polynomial Test

Definition

Let the series $\sum_{n=0}^{\infty} \frac{P_n}{Q_n}$ be one whose terms are positive and equal to the ratios of polynomial in such a way that $p = \text{degree}(P_n)$ and $q = \text{degree}(Q_n)$. Then the series $\sum_{n=0}^{\infty} \frac{P_n}{Q_n}$ converges if and only if $q - p > 1$ and if $q - p < 1$, $\sum_{n=0}^{\infty} \frac{P_n}{Q_n}$ diverges.

Note

Find the difference between the degree of q and p, if $q - p > 1$ then it converges or $q - p < 1$ then it diverges.

Example

Examine the behavior of the series $\sum_{n=1}^{\infty} \frac{6n^3 - 2n + 1}{5n^2 - n + 1}$ by using the polynomial test.

Solution

$$p = 3, \quad q = 2$$

$$q - p = 2 - 3 = -1 < 1$$

Hence

$$\sum_{n=1}^{\infty} \frac{6n^3 - 2n + 1}{5n^2 - n + 1} \qquad \textit{diverges}$$

Example

Examine the behavior of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^4+n+2}$ by using the polynomial test.

Solution

$$q = 4, \quad p = 1$$

$$q - p = 4 - 1 = 3 > 1$$

Hence

$$\sum_{n=1}^{\infty} \frac{n+2}{n^4+n+2} \quad \textit{converges}$$

The Integral Test

Definition

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

1. If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
2. $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

Solution

The function $f(x) = \frac{1}{x^2+1}$ *is continuous,positive and decreasing on* $[1, \infty)$ *so we use the Integral Test:*

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\,dx = \frac{\pi}{4}$$

The integral is convergent and so the series is also convergent by the Integral Test.

Example

Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution

In this case the function we'll use is,

$$f(x) = \frac{1}{x \ln x}$$

This function is clearly positive and if we make x larger the denominator will get larger and so the function is also decreasing. Therefore, all we need to do is determine the convergence of the following integral.

$$\int_{2}^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x \ln x}\, dx$$

Solution (cont'd)

Using method of substitution $u = \ln x$

$$\lim_{b\to\infty} [\ln(\ln x)]_{2}^{b}$$

$$\lim_{b\to\infty} ((\ln(\ln b)) - (\ln(\ln 2))) = \infty$$

The integral is divergent and so the series is also divergent by the Integral Test.

Note

∞

X 1

P Series

Definition

Consider for P series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots +$$

Therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ Converges, if $p > 1$ and diverges if $0 < p \leq 1$

$\underline{\quad}_p$ diverges if p is negative.

n n=1

Example

Examine the behavior of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Solution

$$p = 3 > 1$$
$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ Converges}$$

Example

Examine the behavior of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

Solution

$$p = \frac{1}{3}$$

$$\therefore \sum^{\infty} \frac{1}{\quad} \sqrt{\ } \_$$

diverges $3n$ $n=1$

Ratio Test

Definition

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

, then

1. $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$
2. $\sum_{n=1}^{\infty} a_n$ diverges if $L > 1$
3. If $L = 1$, no conclusion can be drawn.

a different test to determine the convergence of the series.

Example

Investigate the behaviour of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ by using the ratio test.

Solution

$$a_n = \frac{n^n}{n!}, \quad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \div \frac{n^n}{n!} \right)$$

$$\lim_{n\to\infty} \left[\frac{(n+1)^n (n+1)^1}{(n+1)(n!)} \times \frac{n!}{n^n} \right]$$

Solution (cont'd)

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n}{n} + \frac{1}{n}\right)^n$$

$$L = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e = 2.71828\cdots > 1$$

Hence

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} \quad \textit{diverges}$$

Example

Investigate the behaviour of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ by using the ratio test.

Solution

$$a_n = \frac{1}{n!}, \quad a_{n+1} = \frac{1}{(n+1)!}$$

$$L = \lim_{n\to\infty} \frac{1}{(n+1)!} \div \frac{1}{n!}$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

Solution (cont'd)

$$but\ (n+1)\,! = (n+1)n!$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)n!}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$L = \lim_{n\to\infty} \left(\frac{\frac{1}{n}}{\frac{n}{n} + \frac{1}{n}} \right) = 0 = L < 1$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n!}\ \ converges$$

Cauchy’s Root Test

Definition

Let $\sum a_n$ denote the series of positive term and define $L = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$

1. if $L < 1$, converges.
2. if $L > 1$, diverges.
3. if $L = 1$, no conclusion can be drawn.

Example

Verify whether the series $\sum_{n=1}^{\infty} \frac{5^n}{3^n}$ converges or diverges.

Solution

$$\sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^n, \quad a_n = \left(\frac{5}{3}\right)^n$$

$$L = \lim_{n\to\infty} \left(\left(\frac{5}{3}\right)^n\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{5}{3} = \frac{5}{3} < 1$$

Hence

$$\sum_{n=1}^{\infty} \frac{5^n}{3^n} \quad \textit{diverges}$$

Example

Examine the behavior of the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2} \right)^n$.

Solution

$$a_n = \left(\frac{1}{n} - \frac{1}{n^2} \right)^n$$

$$L = \lim_{n\to\infty} \left(\left(\frac{1}{n} - \frac{1}{n^2} \right)^n \right)^{\frac{1}{n}}$$

$$\lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2} \right) = 0 < 1$$

Hence

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2} \right)^n \quad \textit{converges}$$

Comparison Test

Definition

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \leq b_n$, $\forall\ n$, then $\sum a_n$ is convergent.
2. If $\sum b_n$ is divergent and $a_n \geq b_n$, $\forall\ n$, then $\sum a_n$ is also divergent.

Example

Determine if the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 2}$ converges or diverges.

Solution

$$a_n = \frac{5}{2n^+ 4n + 2}, \quad b_n = \frac{5}{2n^2} \qquad where \;\; a_n < b_n$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad since \;\; p = 2 > 1 \;\; (P \;\; Series)$$

$$b_n = \sum_{n=1}^{\infty} \frac{5}{2n^2} \; ; will \; converge$$

$$then \; a_n = \sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 2} \; will \; also \; converge \; because \; a_n < b_n$$

Example

Determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^2 - \sin(n)}$ converges or diverges.

Solution

$$a_n = \frac{n}{n^2 - \sin(n)}, \quad b_n = \frac{n}{n^2} = \frac{1}{n}$$

$$b_n = \sum_{n=1}^{\infty} \frac{1}{n}, \quad p = 1 \ \textit{(P Series)}, b_n \textit{ will diverges}$$

$$\sum b_n \textit{ is divergent } \Longrightarrow \sum a_n \textit{ is also divergent since } a_n > b_n$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2 - \sin(n)} \textit{ diverges}$$

Alternating Series Test

Definition

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. Then if,

1. $\lim_{n\to\infty} b_n = 0$
2. $\{b_n\}$ is a decreasing sequence, then series $\sum a_n$ is convergent.

Example

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Solution: First, we identify the b_n for the test

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}, \qquad b_n = \frac{1}{n}$$

Now, we test the various conditions

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$$

Both conditions are met and so by the Alternating Series Test the series must converge.

Definition

A power series about a point a or just power series is any series that can be written in the form:

$$\sum_{n=0}^{\infty} C_n(x-a)^n$$

where a and C_n are numbers. The $C_n^{\prime}s$ are often called the coefficeint of the series. R is called the radius of convergence for the series.
Power series will converge for $|x-a|<R$ and will diverge If $|x-a|>R$.
The interval of convergence must then contain the interval

$$\boxed{a}\boxed{-}\boxed{R}\boxed{<}\boxed{x}\boxed{<}\boxed{a}\boxed{+}\boxed{R}$$

.

Example

Determine the radius of convergence and the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} \cdot n(x+3)^n$$

Solution

$$\textit{Using the ratio test}$$

$$a_n = \frac{(-1)^n}{4^n} \cdot n(x+3)^n$$

$$a_{n+1} = \frac{(-1)^{n+1}}{4^{n+1}}(n+1)(x+3)^{n+1} = \frac{(-1)^n(-1)}{4^n.4}(n+1)(x+3)^n(x+3)$$

$$\lim_{n\to\infty} \frac{(-1)^n(-1)}{4^n.4}(n+1)(x+3)^n(x+3) \times \frac{(4^n)}{(-1)^n n(x+3)}$$

$$\lim_{n\to\infty} \frac{(-1)(n+1)(x+3)}{4n}$$

Solution

$$\lim_{n\to\infty} \frac{(n+1)}{4n}(x+3) \ < 1$$

$$\lim_{n\to\infty} \frac{n+1}{4n} \ |(x+3)| < 1$$

$$but \ we \quad know \quad that \ |x-a| < R$$

$$\frac{1}{4}|x+3| < 1$$

$$|x+3| < 4, \quad |x-(-3)| < 4$$

$$|x-a| < R$$

$$R = 4 \ \ radius \ of \ convergence.$$

Solution (cont'd)

Now, let's get the interval of convergence. We'll get most (if not all) of the interval by solving the first inequality from above.

$$-4 < x + 3 < 4$$

$$-7 < x < 1$$

The way to determine convergence at these points is to simply plug them into the original power series and see if the series converges or diverges using any test necessary.

Solution

Test for extremes

For $x = -7$:

In this case the series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} n(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} n(-1)^n(4)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n(-1)^n n = \sum_{n=1}^{\infty} n$$

This series is divergent by the Divergence Test since $\lim_{n\to\infty} n = \infty \neq 0$

Solution (cont'd)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} n(4)^n = \sum_{n=1}^{\infty} (-1)^n n$$

This series is also divergent by the Divergence Test since $\lim_{n\to\infty} (-1)^n n = \infty$ *doesn't exist.*

So, in this case the power series will not converge for either endpoint. The interval of convergence is then,

$$-7 < x < 1$$

ThankYou