MATH 308 D200, Fall 2019

20. Mixed strategies

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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Mixed (Probabilistic) Strategy

Many games are not strictly determined and hence lack a best pure strategy according to the Fundamental Principle of game theory. It is dangerous for either player to stick to a pure strategy. So we let players to choose their moves at random in each round of the game.

Definition (Mixed (Probabilistic) Strategy)

Let $A = [a_{ij}]$ be an $m \times n$ matrix game. A *mixed (probabilistic) strategy* for the row player is a vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_m)$ such that

$$\sum_{i=1}^{m} p_i = 1, \quad p_i \geqslant 0 \text{ for all } i \ .$$

A *mixed (probabilistic) strategy* for the column player is a vector $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$ such that

$$\sum_{j=1}^{n} q_j = 1, \quad q_j \geqslant 0 \text{ for all } j \ .$$

Any mixed strategy containing an entry of 1 is a pure strategy. e.g. $\boldsymbol{p} = \boldsymbol{e}_3 = (0, 0, 1, 0, 0, 0)$.

So in our earlier modified Rock-Paper-Scissors game, Ron can choose mixed strategy $\boldsymbol{p} = (1/6, 1/2, 1/3)$ and Coleen can choose $\boldsymbol{q} = (1/3, 1/2, 1/6)$.

Question: If Ron and Coleen use these mixed strategies, how much, on average, can Ron expect to win (or lose) on each round of the game?

Example (Original Rock-Paper-Scissors)

Suppose that Ron plays *paper* half the time, and each of the other two moves a quarter of the time, and Coleen always plays *rock*. This gives $\boldsymbol{p} = (1/4, 1/2, 1/4)$ and $\boldsymbol{q} = (1, 0, 0)$. Then Ron's expected payoff (per game) is

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Theorem

For any payoff matrix A if Ron adopts mixed strategy $\boldsymbol{p}$ and Coleen adopts mixed strategy $\boldsymbol{q}$ ($\boldsymbol{p}$ and $\boldsymbol{q}$ are column vectors), then Ron's expected payoff (per round) is $\mathbb{E}(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{p}^\intercal A \boldsymbol{q}$.

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Definition (Optimal mixed strategy)

An optimal mixed strategy is a mixed strategy that minimizes the potential loss against the opponent's best counter-strategy. For example, strategy $\boldsymbol{p}^*$ is optimal for Ron, if

$$\min_{\boldsymbol{q}} \mathbb{E}(\boldsymbol{p}^*, \boldsymbol{q}) \geq \min_{\boldsymbol{q}} \mathbb{E}(\boldsymbol{p}, \boldsymbol{q})$$

for every mixed strategy $\boldsymbol{p}$ for Ron. Finding optimal mixed strategies for both players in a matrix game is called solving the game.

Optimal Mixed Strategy

Natural Protocol:
Ron can estimate $\boldsymbol{q}$ by keeping statistics over many rounds of play. Assume that Ron knows $\boldsymbol{q}$.

Ron selects $\boldsymbol{p}$ to maximize $\mathbb{E}(\boldsymbol{p}, \boldsymbol{q})$. Then Coleen examines $\boldsymbol{p}$ and selects $\boldsymbol{q}$ to minimize $\mathbb{E}(\boldsymbol{p}, \boldsymbol{q})$. This process repeats/converges to optimal strategies $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ where

$$\mathrm{val}(A) = \max_{\boldsymbol{p}} \mathbb{E}(\boldsymbol{p}, \boldsymbol{q}^*) = \max_{\boldsymbol{p}} \min_{\boldsymbol{q}} \mathbb{E}(\boldsymbol{p}, \boldsymbol{q}) \quad \text{and}$$

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Coleen's Modified Protocol:
Coleen selects $\boldsymbol{q}$ and shows it to Ron. Then Ron examines $\boldsymbol{q}$ and chooses a pure strategy $\boldsymbol{p} = \boldsymbol{e}_i$. The expected payoff is

$$F_i(\boldsymbol{q}) := \mathbb{E}(\boldsymbol{e}_i, \boldsymbol{q}).$$

Ron selects i to maximize this quantity. Coleen selects $\boldsymbol{q}$ to minimize the expected payoff

$$\max_i F_i(\boldsymbol{q}).$$

Coleen optimal strategy, call it $\boldsymbol{q}'$, will result in an expected loss

$$v' = \max_i F_i(\boldsymbol{q}') = \min_{\boldsymbol{q}} \max_i F_i(\boldsymbol{q}).$$

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Ron's Modified Protocol:
Arguing similarly, if Ron selects strategy $\boldsymbol{p}$, and Coleen selects pure strategy $\boldsymbol{e}_j$ which minimizes

$$E_j(\boldsymbol{p}) := \mathbb{E}(\boldsymbol{p}, \boldsymbol{e}_j),$$

then Ron's optimal strategy $\boldsymbol{p}'$ is a vector which maximizes $\min_j E_j(\boldsymbol{p})$. His expected payoff is

$$u' = \max_j E_j(\boldsymbol{p}') = \max_{\boldsymbol{p}} \min_j E_j(\boldsymbol{p}).$$

Optimal Mixed Strategy

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

Coleen's modification

If Ron plays row i, the expected payoff is

$$F_i(\boldsymbol{q}) = \boldsymbol{e}_i{}^\intercal A\boldsymbol{q} = q_1 a_{i1} + \cdots + q_n a_{in}$$

Coleen chooses $\boldsymbol{q} = \boldsymbol{q}'$ in order to minimize

$$\max_{1 \leqslant i \leqslant m} F_i(\boldsymbol{q})$$

Expected payoff of Coleen's modification is

$$v' = \max_{1 \leqslant i \leqslant m} F_i(\boldsymbol{q}') = \min_{\boldsymbol{q}} \max_{1 \leqslant i \leqslant m} F_i(\boldsymbol{q})$$

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If Coleen plays column j, the expected payoff is

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$$v' \leq \mathrm{val}(A).$$

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Surprise! We have $v' = u'$ so $\mathrm{val}(A) = v' = u'$ and $(\boldsymbol{p}', \boldsymbol{q}')$ are optimal for the natural game!

Theorem

Let $A = [a_{ij}]$ be an $m \times n$ matrix game. Then the mixed strategies $\mathbf{p}$ and $\mathbf{q}$ obtained by solving the dual non-canonical LP problems with tableau

$$
\begin{array}{c|cccc|c|l}
 & v & q_1 & \dots & q_n & -1 & \\
\hline
u & 0 & -1 & \dots & -1 & -1 & = -0 \\
p_1 & -1 & & & & 0 & = -t_1 \\
\vdots & \vdots & & A & & \vdots & \vdots \\
p_m & -1 & & & & 0 & = -t_m \\
\hline
-1 & -1 & 0 & \dots & 0 & 0 & = f \\
\hline
 & = 0 & = s_1 & \dots & = s_n & = g &
\end{array}
$$

are optimal for the Ron and Coleen player respectively. Both linear programs always have an optimal solution. The dual non-canonical tableau above is called the **game tableau** *for A.*

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For any $\boldsymbol{q}$, we have

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Coleen's optimal vector $\boldsymbol{q}' = (q_1, q_2, \ldots, q_n)$ minimizes this among all probability vectors $\boldsymbol{q}$.

$$v' = \min\{v \mid v \geq F_i(\boldsymbol{q}) \ (i = 1, 2, \ldots, m) \text{ and } q_1 + q_2 + \cdots + q_n = 1 \text{ and } q_1, q_2, \ldots, q_n \geq 0\}.$$

Thus $v', q_1', q_2', \ldots, q_n'$ is the optimal solution of a linear program with variables $v, q_1, q_2, \ldots, q_n$.

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v' = \text{minimize } v & \\
v &\geq F_1(\boldsymbol{q}) \\
v &\geq F_2(\boldsymbol{q}) \\
&\vdots \\
v &\geq F_m(\boldsymbol{q}) \\
q_1 + q_2 + \ldots + q_n &= 1 \\
q_1, q_2, \ldots, q_n &\geq 0 \\
v \text{ unconstrained} &
\end{aligned}$$

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$$\begin{aligned} \text{maximize } & -v = f(v, q_1, q_2, \ldots, q_n) \\ -q_1 - q_2 - \ldots - q_n &= -1 \\ -v + F_1(\boldsymbol{q}) &\leq 0 \\ -v + F_2(\boldsymbol{q}) &\leq 0 \\ &\vdots \\ -v + F_m(\boldsymbol{q}) &\leq 0 \\ q_1, q_2, \ldots, q_n &\geq 0 \\ v \text{ unconstrained} & \end{aligned}$$

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\text{(P) maximize} & -v = f \\
-q_1 - q_2 - \ldots - q_n & = -1 \\
-v + a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n & \leq 0 \\
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Similarly Ron's optimal vector $\boldsymbol{p}' = (p_1, p_2, \ldots, p_m)$ is found by a linear program with variables $u, p_1, p_2, \ldots, p_m$.

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\begin{aligned}
u' &= \max_{\boldsymbol{p}} \min_{j} E_j(\boldsymbol{p}) \\
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&= \max\{u \mid u \leq E_j(\boldsymbol{p}),\ (j = 1, 2, \ldots, n) \text{ and } p_1 + p_2 + \cdots + p_m = 1 \text{ and } p_1, p_2, \ldots, p_m \geq 0\}
\end{aligned}
$$

$$
\begin{array}{rl}
\text{(P) maximize} & -v = f \\
& - q_1 - q_2 - \ldots - q_n = -1 \\
& -v + a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n \leq 0 \\
& -v + a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n \leq 0 \\
& \vdots \\
& -v + a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n \leq 0 \\
& q_1, q_2, \ldots, q_n \geq 0 \\
& v \text{ unconstrained}
\end{array}
$$

$$
\begin{array}{rl}
\text{(D) minimize} & -u = g \\
& - p_1 - p_2 - \ldots - p_m = -1 \\
& -u + a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq 0 \\
& -u + a_{12}p_1 + a_{23}p_2 + \cdots + a_{m2}p_m \geq 0 \\
& \vdots \\
& -u + a_{1m}p_1 + a_{2m}p_2 + \cdots + a_{mn}p_m \geq 0 \\
& p_1, p_2, \ldots, p_m \geq 0 \\
& u \text{ unconstrained}
\end{array}
$$

□

Proof.

Similarly Ron's optimal vector $\boldsymbol{p}' = (p_1, p_2, \ldots, p_m)$ is found by a linear program with variables $u, p_1, p_2, \ldots, p_m$.

$$
\begin{aligned}
u' &= \max_{\boldsymbol{p}} \min_{j} E_j(\boldsymbol{p}) \\
&= \max\{u \mid u \leq E_j(\boldsymbol{p}),\ (j = 1, 2, \ldots, n) \text{ and } p_1 + p_2 + \cdots + p_m = 1 \text{ and } p_1, p_2, \ldots, p_m \geq 0\}
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$$

(P) maximize $-v = f$

$$
\begin{array}{rcl}
- q_1 - q_2 - \ldots - q_n &=& -1 \\
-v + a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n &\leq& 0 \\
-v + a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n &\leq& 0 \\
\vdots && \\
-v + a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n &\leq& 0 \\
q_1, q_2, \ldots, q_n &\geq& 0 \\
v \text{ unconstrained} &&
\end{array}
$$

(D) minimize $-u = g$

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p_1, p_2, \ldots, p_m &\geq& 0 \\
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\end{array}
$$

These are the dual non-canonical LPs for the game tableau.
The optimal value is $f = g = -u = -v = -\text{val}(A)$.

$\square$

Proof.

Similarly Ron's optimal vector $\boldsymbol{p}' = (p_1, p_2, \ldots, p_m)$ is found by a linear program with variables $u, p_1, p_2, \ldots, p_m$.

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\begin{aligned}
u' &= \max_{\boldsymbol{p}} \ \min_{j} E_j(\boldsymbol{p}) \\
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$$
\text{(P) maximize } -v = f
$$

$$
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\vdots & & \\
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q_1, q_2, \ldots, q_n & \geq & 0 \\
v \text{ unconstrained} & &
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$$

$$
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$$
\begin{array}{rcl}
- p_1 - p_2 - \ldots - p_m & = & -1 \\
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-u + a_{12} p_1 + a_{23} p_2 + \cdots + a_{m2} p_m & \geq & 0 \\
\vdots & & \\
-u + a_{1m} p_1 + a_{2m} p_2 + \cdots + a_{mn} p_m & \geq & 0 \\
p_1, p_2, \ldots, p_m & \geq & 0 \\
u \text{ unconstrained} & &
\end{array}
$$

These are the dual non-canonical LPs for the game tableau.
The optimal value is $f = g = -u = -v = -\text{val}(A)$.
Both optima exist, since making v and $-u$ very large makes both LP's feasible.

$\square$

Von Neumann Minimax Theorem

Theorem (Von Neumann Minimax Theorem)

Let $A = [a_{ij}]$ be an $m \times n$ matrix game. Then there exist optimal mixed strategies $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$ for the row player and the column player respectively. Furthermore,

$$\min_{1 \leqslant j \leqslant n} E_j(\boldsymbol{p}^*) = \max_{\boldsymbol{p}} \min_{1 \leqslant j \leqslant n} E_j(\boldsymbol{p}) = \min_{\boldsymbol{q}} \max_{1 \leqslant i \leqslant m} F_i(\boldsymbol{q}) = \min_{1 \leqslant i \leqslant m} F_i(\boldsymbol{q}^*)$$

and this common value is said to be the **von Neumann value** *of the game.*

The von Neumann value is the amount of expected winnings of the row player and the expected losses of the column player.

A positive von Neumann value indicates that the game favours the row player, a negative von Neumann values indicates that the game favour the column player, and a von Neumann value 0 indicates that the game is fair.

Saddle Point and von Neumann Value of a Game

Recall the definition of a saddle point of a matrix game:

Definition (Saddle Point)

An entry a_{ij} of a matrix game $A \in \mathbb{R}^{m \times n}$ is called a *saddle point* if it is a minimum entry in the row $\#i$ and the maximum entry in the column $\#j$.

Theorem

If a payoff matrix A has a saddle point a_{ij}, then the von Neumann value of the game is a_{ij}.

Proof.

$\square$

Transforming Maximum Game Tableau to a Basic Feasible Game Tableau

We need at least two pivot transformations to get a Basic Feasible Tableau.

And we do not need more! We can skip Phase 1 of SA by selecting the first two pivots carefully.

Algorithm (Transforming the original game tableau into the Feasible Tableau)

1. *Find the maximum entry in each column of the matrix game A.*
2. *Choose the minimum of these maximum entries, say it is the entry a_{ij} of A.*
3. *Pivot on the -1 in the top of column $\#j$ and the -1 in the left of row $\#i$ (in either order).*

   Remove 0-column and 0-row after pivoting.

   We do not need to file equations for u and v as they can be obtained from objective functions.
4. *The resulting tableau is Feasible Tableau.*

We do not prove that this procedure works as claimed.