MATH 308 D200, Fall 2019

13. Non-canonical LP problems - equations of constraints

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

Dr. Masood Masjoody

SFU Burnaby

Equations as Constraints

Method I: Replacing an equality constraint by two inequality constraints.

Example 3

Maximize $f(x, y, z) = 2x + y - 2z$
subject to

$$\begin{aligned} x + y + z &\leqslant 1 \\ y + 4z &= 2 \\ x, y, z &\geqslant 0 \end{aligned}$$

Equations as Constraints

Method I: Replacing an equality constraint by two inequality constraints.

Example 3

Maximize $f(x, y, z) = 2x + y - 2z$
subject to

$$\begin{aligned} x + y + z &\leqslant 1 \\ y + 4z &\leq 2 \\ -y - 4z &\leq -2 \\ x, y, z &\geqslant 0 \end{aligned}$$

Method I: Replacing an equality constraint by two inequality constraints.

Example 3

Maximize $f(x, y, z) = 2x + y - 2z$
subject to

$$\begin{aligned} x + y + z &\leqslant 1 \\ y + 4z &\leq 2 \\ -y - 4z &\leq -2 \\ x, y, z &\geqslant 0 \end{aligned}$$

$$\begin{array}{cccc|c|l} & x & y & z & -1 & \\ \hline & 1 & 1 & 1 & 1 & -t_1 \\ & 0 & 1 & 4 & 2 & = -t_2 \\ & 0 & -1^* & -4 & -2 & = -t_3 \\ \hline & 2 & 1 & -2 & 0 & = P \\ \hline \end{array} \Rightarrow \begin{array}{cccc|c|l} & x & t_3 & z & -1 & \\ \hline & 1 & 1 & -3^* & -1 & = -t_1 \\ & 0 & 1 & 0 & 0 & = -t_2 \\ & 0 & -1 & 4 & 2 & = -y \\ \hline & 2 & 1 & -6 & -2 & = P \\ \hline \end{array} \Rightarrow$$

$$\begin{array}{cccc|c|l} & x & t_2 & t_1 & -1 & \\ \hline & -1/3 & -1/3 & -1/3 & 1/3 & = -z \\ & 0 & 1 & 0 & 0 & = -t_1 \\ & 4/3 & 1/3 & 4/3 & 2/3 & = -y \\ \hline & 0 & -1 & -2 & 0 & = P^* \\ \hline \end{array}$$

Optimum Solution: $(x, y, z) = (0, 2/3, 1/3)$

Equations as Constraints

Method I: Replacing an equality constraint by two inequality constraints.

Example 3

Maximize $f(x, y, z) = 2x + y - 2z$
subject to

$$
\begin{aligned}
x + y + z &\leqslant 1 \\
y + 4z &\leq 2 \\
-y - 4z &\leq -2 \\
x, y, z &\geqslant 0
\end{aligned}
$$

$$
\begin{array}{cccc|c|l}
 & x & y & z & -1 & \\
\cline{2-5}
 & 1 & 1 & 1 & 1 & -t_1 \\
 & 0 & 1 & 4 & 2 & = -t_2 \\
 & 0 & -1^* & -4 & -2 & = -t_3 \\
\cline{2-5}
 & 2 & 1 & -2 & 0 & = P \\
\cline{2-5}
\end{array}
\Rightarrow
\begin{array}{cccc|c|l}
 & x & t_3 & z & -1 & \\
\cline{2-5}
 & 1 & 1 & -3^* & -1 & = -t_1 \\
 & 0 & 1 & 0 & 0 & = -t_2 \\
 & 0 & -1 & 4 & 2 & = -y \\
\cline{2-5}
 & 2 & 1 & -6 & -2 & = P \\
\cline{2-5}
\end{array}
\Rightarrow
$$

$$
\begin{array}{cccc|c|l}
 & x & t_2 & t_1 & -1 & \\
\cline{2-5}
 & -1/3 & -1/3 & -1/3 & 1/3 & = -z \\
 & 0 & 1 & 0 & 0 & = -t_1 \\
 & 4/3 & 1/3 & 4/3 & 2/3 & = -y \\
\cline{2-5}
 & 0 & -1 & -2 & 0 & = P^* \\
\cline{2-5}
\end{array}
$$

Optimum Solution: $(x, y, z) = (0, 2/3, 1/3)$

There may be infinitely many optimum solutions

Aside: recognizing an infinite number of optimal solutions

Example 3 has infinitely many solutions.

x	t_3	t_1	-1	
-1/3	-1/3	-1/3	1/3	$= -z$
0	1	0	0	$= -t_2$
4/3	1/3	4/3	2/3	$= -y$
0	-1	-2	0	$= f$

Aside: recognizing an infinite number of optimal solutions

Example 3 has infinitely many solutions.

x	t_3	t_1	-1	
-1/3	-1/3	-1/3	1/3	$= -z$
0	1	0	0	$= -t_2$
4/3	1/3	4/3	2/3	$= -y$
0	-1	-2	0	$= f$

Increasing x while the equations hold at $t_3 = t_1 = 0$ will not affect the objective value f.

Aside: recognizing an infinite number of optimal solutions

Example 3 has infinitely many solutions.

$$
\begin{array}{cccc|cl}
x & t_3 & t_1 & -1 & & \\
\hline
-1/3 & -1/3 & -1/3 & 1/3 & = -z \\
0 & 1 & 0 & 0 & = -t_2 \\
4/3 & 1/3 & 4/3 & 2/3 & = -y \\
\hline
0 & -1 & -2 & 0 & = f
\end{array}
$$

$$-\frac{1}{3}x - \frac{1}{3} = -z \leq 0$$

$$0 = -t_2$$

$$\frac{4}{3}x - \frac{2}{3} = -y \leq 0$$

Increasing x while the equations hold at $t_3 = t_1 = 0$ will not affect the objective value f.

Example 3 has infinitely many solutions.

$$
\begin{array}{cccc|cl}
x & t_3 & t_1 & -1 & & \\
\hline
-1/3 & -1/3 & -1/3 & 1/3 & = -z \\
0 & 1 & 0 & 0 & = -t_2 \\
4/3 & 1/3 & 4/3 & 2/3 & = -y \\
\hline
0 & -1 & -2 & 0 & = f
\end{array}
$$

$$-\frac{1}{3}x - \frac{1}{3} = -z \leq 0 \implies x \geq 1$$

$$0 = -t_2 \implies t_2 = 0$$

$$\frac{4}{3}x - \frac{2}{3} = -y \leq 0 \implies x \leq \frac{1}{2}$$

Increasing x while the equations hold at $t_3 = t_1 = 0$ will not affect the objective value f.

Aside: recognizing an infinite number of optimal solutions

Example 3 has infinitely many solutions.

x	t_3	t_1	-1	
-1/3	-1/3	-1/3	1/3	$= -z$
0	1	0	0	$= -t_2$
4/3	1/3	4/3	2/3	$= -y$
0	-1	-2	0	$= f$

$$-\frac{1}{3}x - \frac{1}{3} = -z \leq 0 \implies x \geq 1$$

$$0 = -t_2 \implies t_2 = 0$$

$$\frac{4}{3}x - \frac{2}{3} = -y \leq 0 \implies x \leq \frac{1}{2}$$

Increasing x while the equations hold at $t_3 = t_1 = 0$ will not affect the objective value f.
We may increase x until z, t_2 or y becomes negative.

Aside: recognizing an infinite number of optimal solutions

Example 3 has infinitely many solutions.

x	t_3	t_1	-1	
-1/3	-1/3	-1/3	1/3	$= -z$
0	1	0	0	$= -t_2$
4/3	1/3	4/3	2/3	$= -y$
0	-1	-2	0	$= f$

$$-\frac{1}{3}x - \frac{1}{3} = -z \leq 0 \implies x \geq 1$$

$$0 = -t_2 \implies t_2 = 0$$

$$\frac{4}{3}x - \frac{2}{3} = -y \leq 0 \implies x \leq \frac{1}{2}$$

Increasing x while the equations hold at $t_3 = t_1 = 0$ will not affect the objective value f. We may increase x until z, t_2 or y becomes negative. (Here $y \geq 0$ implies $x \leq \frac{1}{2}$)

Aside: recognizing an infinite number of optimal solutions

Example 3 has infinitely many solutions.

x	t_3	t_1	-1		
-1/3	-1/3	-1/3	1/3	$= -z$	N/A
0	1	0	0	$= -t_2$	N/A
4/3	1/3	4/3	2/3	$= -y$	$\frac{2/3}{4/3}$
0	-1	-2	0	$= f$	

$$-\frac{1}{3}x - \frac{1}{3} = -z \leq 0 \quad \implies x \geq 1$$

$$0 = -t_2 \quad \implies t_2 = 0$$

$$\frac{4}{3}x - \frac{2}{3} = -y \leq 0 \quad \implies x \leq \frac{1}{2}$$

Increasing x while the equations hold at $t_3 = t_1 = 0$ will not affect the objective value f.
We may increase x until z, t_2 or y becomes negative. (Here $y \geq 0$ implies $x \leq \frac{1}{2}$)
Shortcut: Compute ratios as if we were pivoting column x.

Example 3 has infinitely many solutions.

x	t_3	t_1	-1				
-1/3	-1/3	-1/3	1/3	$= -z$	N/A	$-\frac{1}{3}x - \frac{1}{3} = -z \leq 0$	$\implies x \geq 1$
0	1	0	0	$= -t_2$	N/A	$0 = -t_2$	$\implies t_2 = 0$
4/3	1/3	4/3	2/3	$= -y$	$\frac{2/3}{4/3}$	$\frac{4}{3}x - \frac{2}{3} = -y \leq 0$	$\implies x \leq \frac{1}{2}$
0	-1	-2	0	$= f$			

Increasing x while the equations hold at $t_3 = t_1 = 0$ will not affect the objective value f.
We may increase x until z, t_2 or y becomes negative. (Here $y \geq 0$ implies $x \leq \frac{1}{2}$)
Shortcut: Compute ratios as if we were pivoting column x.

The optimal solutions form a line segment in $\mathbb{R}^3$ parameterized by x

$$\left\{ \left(x,\ \frac{2}{3} - \frac{4}{3}x,\ \frac{1}{3}x + \frac{1}{3} \right) : 0 \leq x \leq \frac{1}{2} \right\}$$

Example 3 has infinitely many solutions.

x	t_3	t_1	-1		
-1/3	-1/3	-1/3	1/3	$= -z$	N/A
0	1	0	0	$= -t_2$	N/A
4/3	1/3	4/3	2/3	$= -y$	$\frac{2/3}{4/3}$
0	-1	-2	0	$= f$	

$$-\frac{1}{3}x - \frac{1}{3} = -z \leq 0 \implies x \geq 1$$

$$0 = -t_2 \implies t_2 = 0$$

$$\frac{4}{3}x - \frac{2}{3} = -y \leq 0 \implies x \leq \frac{1}{2}$$

Increasing x while the equations hold at $t_3 = t_1 = 0$ will not affect the objective value f.
We may increase x until z, t_2 or y becomes negative. (Here $y \geq 0$ implies $x \leq \frac{1}{2}$)
Shortcut: Compute ratios as if we were pivoting column x.

The optimal solutions form a line segment in $\mathbb{R}^3$ parameterized by x

$$\left\{\left(x,\ \frac{2}{3} - \frac{4}{3}x,\ \frac{1}{3}x + \frac{1}{3}\right) : 0 \leq x \leq \frac{1}{2}\right\}$$

$$= \overline{\boldsymbol{ab}}, \quad \text{where } \boldsymbol{a} = \left(0, \frac{2}{3}, \frac{1}{3}\right), \quad \boldsymbol{b} = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

Aside: recognizing an infinite number of optimal solutions

Example 3 has infinitely many solutions.

x	t_3	t_1	-1		
-1/3	-1/3	-1/3	1/3	$= -z$	N/A
0	1	0	0	$= -t_2$	N/A
4/3	1/3	4/3	2/3	$= -y$	$\frac{2/3}{4/3}$
0	-1	-2	0	$= f$	

$$-\frac{1}{3}x - \frac{1}{3} = -z \leq 0 \quad \Longrightarrow \quad x \geq 1$$
$$0 = -t_2 \quad \Longrightarrow \quad t_2 = 0$$
$$\frac{4}{3}x - \frac{2}{3} = -y \leq 0 \quad \Longrightarrow \quad x \leq \frac{1}{2}$$

Increasing x while the equations hold at $t_3 = t_1 = 0$ will not affect the objective value f.
We may increase x until z, t_2 or y becomes negative. (Here $y \geq 0$ implies $x \leq \frac{1}{2}$)
Shortcut: Compute ratios as if we were pivoting column x.

The optimal solutions form a line segment in $\mathbb{R}^3$ parameterized by x

$$\left\{ \left(x,\ \frac{2}{3} - \frac{4}{3}x,\ \frac{1}{3}x + \frac{1}{3} \right) : 0 \leq x \leq \frac{1}{2} \right\}$$
$$= \overline{\boldsymbol{ab}}, \quad \text{where } \boldsymbol{a} = \left(0, \frac{2}{3}, \frac{1}{3}\right), \quad \boldsymbol{b} = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$
$$= \left\{ t\left(0, \frac{2}{3}, \frac{1}{3}\right) + (1-t)\left(\frac{1}{2}, 0, \frac{1}{2}\right) \mid 0 \leq t \leq 1 \right\}$$

Method II: Pivot up the zero row, then eliminate the column.

Example $3'$

Maximize $f(x,y,z) = 2x + y - 2z$, subject to

$$\begin{aligned} x + y + z &\leqslant 1 \\ y + 4z &= 2 \\ x, y, z &\geqslant 0 \end{aligned}$$

Record second slack variable as "0"

$$\begin{array}{ccc|c|l} x & y & z & -1 & \\ \hline 1 & 1 & 1 & 1 & = -t_1 \\ 0 & 1 & 4 & 2 & = -0 \\ \hline 2 & 1 & -2 & 0 & = f \end{array} \rightarrow$$

Method II: Pivot up the zero row, then eliminate the column.

Example 3′

Maximize $f(x, y, z) = 2x + y - 2z$, subject to

$$\begin{aligned} x + y + z &\leqslant 1 \\ y + 4z &= 2 \\ x, y, z &\geqslant 0 \end{aligned}$$

Record second slack variable as "0"
Find non-zero entry in 0-row (e.g. column y)

$$\begin{array}{ccc|c|l} x & y & z & -1 & \\ \hline 1 & 1 & 1 & 1 & = -t_1 \\ 0 & 1^* & 4 & 2 & = -0 \\ \hline 2 & 1 & -2 & 0 & = f \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{ccc|c|l} & & & -1 & \\ \hline & & & & = \\ & & & & = \\ \hline & & & & = f \\ \hline \end{array}$$

Method II: Pivot up the zero row, then eliminate the column.

Example $3'$

Maximize $f(x,y,z) = 2x + y - 2z$, subject to

$$x + y + z \leqslant 1$$
$$y + 4z = 2$$
$$x, y, z \geqslant 0$$

Record second slack variable as "0"
Find non-zero entry in 0-row (e.g. column y) and pivot.

$$\begin{array}{ccc|c} x & y & z & -1 \\ \hline 1 & 1 & 1 & 1 \\ 0 & 1^* & 4 & 2 \\ \hline 2 & 1 & -2 & 0 \end{array} \begin{array}{l} \\ = -t_1 \\ = -0 \\ = f \end{array} \rightarrow \begin{array}{ccc|c} x & 0 & z & -1 \\ \hline 1 & -1 & -3 & -1 \\ 0 & 1 & 4 & 2 \\ \hline 2 & -1 & -6 & -2 \end{array} \begin{array}{l} \\ = -t_1 \\ = -y \\ = f \end{array}$$

Method II: Pivot up the zero row, then eliminate the column.

Example 3′

Maximize $f(x,y,z) = 2x + y - 2z$, subject to

$$\begin{aligned} x + y + z &\leqslant 1 \\ y + 4z &= 2 \\ x, y, z &\geqslant 0 \end{aligned}$$

Record second slack variable as "0"
Find non-zero entry in 0-row (e.g. column y) and pivot.
Delete 0-column.

$$\begin{array}{cccc|c} x & y & z & -1 & \\ \hline 1 & 1 & 1 & 1 & = -t_1 \\ 0 & 1^* & 4 & 2 & = -0 \\ \hline 2 & 1 & -2 & 0 & = f \end{array} \rightarrow \begin{array}{cccc|c} x & 0 & z & -1 & \\ \hline 1 & -1 & -3 & -1 & = -t_1 \\ 0 & 1 & 4 & 2 & = -y \\ \hline 2 & -1 & -6 & -2 & = f \end{array} \rightarrow \begin{array}{ccc|c} x & z & -1 & \\ \hline 1 & -3 & -1 & = -t_1 \\ 0 & 4 & 2 & = -y \\ \hline 2 & -6 & -2 & = f \end{array}$$

Method II: Pivot up the zero row, then eliminate the column.

Example 3′

Maximize $f(x, y, z) = 2x + y - 2z$, subject to

$$\begin{aligned} x + y + z &\leqslant 1 \\ y + 4z &= 2 \\ x, y, z &\geqslant 0 \end{aligned}$$

Record second slack variable as "0"
Find non-zero entry in 0-row (e.g. column y) and pivot.
Delete 0-column.
Run simplex algorithm.

$$\begin{array}{ccc|c|l} x & y & z & -1 & \\ \hline 1 & 1 & 1 & 1 & = -t_1 \\ 0 & 1^* & 4 & 2 & = -0 \\ \hline 2 & 1 & -2 & 0 & = f \end{array} \rightarrow \begin{array}{ccc|c|l} x & 0 & z & -1 & \\ \hline 1 & -1 & -3 & -1 & = -t_1 \\ 0 & 1 & 4 & 2 & = -y \\ \hline 2 & -1 & -6 & -2 & = f \end{array} \rightarrow \begin{array}{cc|c|l} x & z & -1 & \\ \hline 1 & -3 & -1 & = -t_1 \\ 0 & 4 & 2 & = -y \\ \hline 2 & -6 & -2 & = f \end{array}$$

$$\begin{array}{cc|c|l} x & z & -1 & \\ \hline 1 & -3^* & -1 & = -t_1 \\ 0 & 4 & 2 & = -y \\ \hline 2 & -6 & -2 & = f \end{array} \rightarrow \begin{array}{cc|c|l} x & t_1 & -1 & \\ \hline -1/3 & -1/3 & 1/3 & = -z \\ 4/3 & 4/3 & 2/3 & = -y \\ \hline 0 & -2 & 0 & = f \end{array}$$

Method II: Pivot up the zero row, then eliminate the column.

Example 3′

Maximize $f(x,y,z) = 2x + y - 2z$, subject to

$$\begin{aligned} x + y + z &\leqslant 1 \\ y + 4z &= 2 \\ x, y, z &\geqslant 0 \end{aligned}$$

Record second slack variable as "0"
Find non-zero entry in 0-row (e.g. column y) and pivot.
Delete 0-column.
Run simplex algorithm.

$$\begin{array}{ccc|c|l} x & y & z & -1 & \\ \hline 1 & 1 & 1 & 1 & = -t_1 \\ 0 & 1^* & 4 & 2 & = -0 \\ \hline 2 & 1 & -2 & 0 & = f \\ \hline \end{array} \rightarrow \begin{array}{ccc|c|l} x & 0 & z & -1 & \\ \hline 1 & -1 & -3 & -1 & = -t_1 \\ 0 & 1 & 4 & 2 & = -y \\ \hline 2 & -1 & -6 & -2 & = f \\ \hline \end{array} \rightarrow \begin{array}{cc|c|l} x & z & -1 & \\ \hline 1 & -3 & -1 & = -t_1 \\ 0 & 4 & 2 & = -y \\ \hline 2 & -6 & -2 & = f \\ \hline \end{array}$$

$$\begin{array}{cc|c|l} x & z & -1 & \\ \hline 1 & -3^* & -1 & = -t_1 \\ 0 & 4 & 2 & = -y \\ \hline 2 & -6 & -2 & = f \\ \hline \end{array} \rightarrow \begin{array}{cc|c|ll} x & t_1 & -1 & & \\ \hline -1/3 & -1/3 & 1/3 & = -z & \text{N/A} \\ 4/3 & 4/3 & 2/3 & = -y & \frac{2/3}{4/3} = \frac{1}{2} \\ \hline 0 & -2 & 0 & = f & \\ \hline \end{array}$$

Again we have infinitely many solutions, for $0 \leq x \leq \frac{1}{2}$

Example 4′

Maximize $f(x, y, z) = x + 2y - 3z$, subject to

$$\begin{aligned} x - y + z &= 3 \\ x + 2y - z &\leqslant 4 \\ x - z &\leqslant 6 \\ x, y, z &\geqslant 0 \end{aligned}$$

$$\begin{array}{cccc|l} x & y & z & -1 & \\ \hline 1 & -1 & 1 & 3 & = -0 \\ 1 & 2 & -1 & 4 & = -t_1 \quad \rightarrow \\ 1 & 0 & -1 & 6 & = -t_2 \\ \hline 1 & 2 & -3 & 0 & = f \end{array}$$

Example $4'$

Maximize $f(x,y,z) = x + 2y - 3z$, subject to

$$\begin{aligned} x - y + z &= 3 \\ x + 2y - z &\leqslant 4 \\ x - z &\leqslant 6 \\ x, y, z &\geqslant 0 \end{aligned}$$

$$\begin{array}{cccc|c|l} & x & y & z & -1 & \\ \hline & 1^* & -1 & 1 & 3 & = -0 \\ & 1 & 2 & -1 & 4 & = -t_1 \\ & 1 & 0 & -1 & 6 & = -t_2 \\ \hline & 1 & 2 & -3 & 0 & = f \end{array} \quad \rightarrow \quad \begin{array}{c|c|l} 0 & -1 & \\ \hline & & = -x \\ & & = \\ & & = \\ \hline & & = f \end{array}$$

Example $4'$

Maximize $f(x,y,z) = x + 2y - 3z$, subject to

$$\begin{aligned} x - y + z &= 3 \\ x + 2y - z &\leqslant 4 \\ x - z &\leqslant 6 \\ x, y, z &\geqslant 0 \end{aligned}$$

$$\begin{array}{cccc|c|l} x & y & z & & -1 & \\ \hline 1^* & -1 & 1 & & 3 & = -0 \\ 1 & 2 & -1 & & 4 & = -t_1 \\ 1 & 0 & -1 & & 6 & = -t_2 \\ \hline 1 & 2 & -3 & & 0 & = f \end{array} \quad \rightarrow \quad \begin{array}{ccc|c|l} 0 & y & z & -1 & \\ \hline 1 & -1 & 1 & 3 & = -x \\ -1 & 3 & -2 & 1 & = -t_1 \\ -1 & 1 & -2 & 3 & = -t_2 \\ \hline -1 & 3 & -4 & -3 & = f \end{array}$$

Example 4′

Maximize $f(x,y,z) = x + 2y - 3z$, subject to

$$\begin{aligned} x - y + z &= 3 \\ x + 2y - z &\leqslant 4 \\ x - z &\leqslant 6 \\ x, y, z &\geqslant 0 \end{aligned}$$

$$\begin{array}{cccc|c|l} x & y & z & & -1 & \\ \hline 1^* & -1 & 1 & & 3 & = -0 \\ 1 & 2 & -1 & & 4 & = -t_1 \\ 1 & 0 & -1 & & 6 & = -t_2 \\ \hline 1 & 2 & -3 & & 0 & = f \end{array} \quad \to \quad \begin{array}{ccc|c|l} 0 & y & z & -1 & \\ \hline 1 & -1 & 1 & 3 & = -x \\ -1 & 3 & -2 & 1 & = -t_1 \\ -1 & 1 & -2 & 3 & = -t_2 \\ \hline -1 & 3 & -4 & -3 & = f \end{array} \quad \to$$

$$\begin{array}{cc|c|l} y & z & -1 & \\ \hline -1 & 1 & 3 & = -x \\ 3 & -2 & 1 & = -t_1 \\ 1 & -2 & 3 & = -t_2 \\ \hline 3 & -4 & -3 & = f \end{array}$$

Example 4′

Maximize $f(x,y,z) = x + 2y - 3z$, subject to

$$
\begin{aligned}
x - y + z &= 3 \\
x + 2y - z &\leqslant 4 \\
x - z &\leqslant 6 \\
x, y, z &\geqslant 0
\end{aligned}
$$

$$
\begin{array}{ccc|c|l}
x & y & z & -1 & \\
\hline
1^* & -1 & 1 & 3 & = -\color{red}{0} \\
1 & 2 & -1 & 4 & = -t_1 \\
1 & 0 & -1 & 6 & = -t_2 \\
\hline
1 & 2 & -3 & 0 & = f
\end{array}
\quad \to \quad
\begin{array}{ccc|c|l}
\color{red}{0} & y & z & -1 & \\
\hline
1 & -1 & 1 & 3 & = -x \\
-1 & 3 & -2 & 1 & = -t_1 \\
-1 & 1 & -2 & 3 & = -t_2 \\
\hline
-1 & 3 & -4 & -3 & = f
\end{array}
\quad \to
$$

$$
\begin{array}{cc|c|l}
y & z & -1 & \\
\hline
-1 & 1 & 3 & = -x \\
3^* & -2 & 1 & = -t_1 \\
1 & -2 & 3 & = -t_2 \\
\hline
3 & -4 & -3 & = f
\end{array}
$$

Example 4′

Maximize $f(x, y, z) = x + 2y - 3z$, subject to

$$
\begin{aligned}
x - y + z &= 3 \\
x + 2y - z &\leqslant 4 \\
x - z &\leqslant 6 \\
x, y, z &\geqslant 0
\end{aligned}
$$

$$
\begin{array}{cccc|c|l}
 & x & y & z & -1 & \\
\hline
 & 1^* & -1 & 1 & 3 & = -0 \\
 & 1 & 2 & -1 & 4 & = -t_1 \\
 & 1 & 0 & -1 & 6 & = -t_2 \\
\hline
 & 1 & 2 & -3 & 0 & = f
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc|c|l}
0 & y & z & -1 & \\
\hline
1 & -1 & 1 & 3 & = -x \\
-1 & 3 & -2 & 1 & = -t_1 \\
-1 & 1 & -2 & 3 & = -t_2 \\
\hline
-1 & 3 & -4 & -3 & = f
\end{array}
\quad \rightarrow
$$

$$
\begin{array}{cc|c|l}
y & z & -1 & \\
\hline
-1 & 1 & 3 & = -x \\
3^* & -2 & 1 & = -t_1 \\
1 & -2 & 3 & = -t_2 \\
\hline
3 & -4 & -3 & = f
\end{array}
\quad \rightarrow \quad
\begin{array}{cc|c|l}
t_1 & z & -1 & \\
\hline
1/3 & 1/3 & 10/3 & = -x \\
1/3 & -2/3 & 1/3 & = -y \\
-1/3 & -4/3 & 8/3 & = -t_2 \\
\hline
-1 & -2 & -4 & = f
\end{array}
$$

Example 5′

Maximize $f(x, y, z) = x + 2y + z$
subject to

$$\begin{aligned} x - y + z &= 6 \\ x + y &\leqslant 1 \\ x, z &\geqslant 0 \end{aligned}$$

This has both an unrestricted variable and an equality constraint.

Example $5'$

Maximize $f(x, y, z) = x + 2y + z$
subject to

$$\begin{aligned} x - y + z &= 6 \\ x + y &\leqslant 1 \\ x, z &\geqslant 0 \end{aligned}$$

This has both an unrestricted variable and an equality constraint.
We can do both at once!

$$\begin{array}{ccc|c|l} x & y & z & -1 & \\ \hline 1 & -1 & 1 & 6 & = -0 \\ 1 & 1 & 0 & 1 & = -t_1 \\ \hline 1 & 2 & 1 & 0 & = f \\ \hline \end{array}$$

Example 5′

Maximize $f(x, y, z) = x + 2y + z$
subject to

$$\begin{aligned} x - y + z &= 6 \\ x + y &\leqslant 1 \\ x, z &\geqslant 0 \end{aligned}$$

This has both an unrestricted variable and an equality constraint.
We can do both at once!

$$\begin{array}{ccc|c|l} x & y & z & -1 & \\ \hline 1 & -1^* & 1 & 6 & = -0 \\ 1 & 1 & 0 & 1 & = -t_1 \\ \hline 1 & 2 & 1 & 0 & = f \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{c|c|l} 0 & -1 & \\ \hline & & = -y \\ & & = \\ \hline & & = f \\ \hline \end{array}$$

Example $5'$

Maximize $f(x, y, z) = x + 2y + z$
subject to

$$\begin{aligned} x - y + z &= 6 \\ x + y &\leqslant 1 \\ x, z &\geqslant 0 \end{aligned}$$

This has both an unrestricted variable and an equality constraint.
We can do both at once!

$$\begin{array}{ccc|c|l} x & y & z & -1 & \\ \hline 1 & -1^* & 1 & 6 & = -0 \\ 1 & 1 & 0 & 1 & = -t_1 \\ \hline 1 & 2 & 1 & 0 & = f \\ \hline \end{array} \quad \to \quad \begin{array}{ccc|c|l} x & 0 & z & -1 & \\ \hline -1 & -1 & -1 & -6 & = -y \\ 2 & 1 & 1 & 7 & = -t_1 \\ \hline 3 & 2 & 3 & 12 & = f \\ \hline \end{array}$$

Example 5′

Maximize $f(x, y, z) = x + 2y + z$
subject to

$$\begin{aligned} x - y + z &= 6 \\ x + y &\leqslant 1 \\ x, z &\geqslant 0 \end{aligned}$$

This has both an unrestricted variable and an equality constraint.
We can do both at once!

$$\begin{array}{ccc|c|l} x & y & z & -1 & \\ \hline 1 & -1^* & 1 & 6 & = -0 \\ 1 & 1 & 0 & 1 & = -t_1 \\ \hline 1 & 2 & 1 & 0 & = f \end{array} \quad \rightarrow \quad \begin{array}{ccc|c|l} x & 0 & z & -1 & \\ \hline -1 & -1 & -1 & -6 & = -y \\ 2 & 1 & 1 & 7 & = -t_1 \\ \hline 3 & 2 & 3 & 12 & = f \end{array}$$

Delete row and delete column, record $-x - z + 6 = -y$.

$$\begin{array}{cc|c|l} x & z & -1 & \\ \hline 2^* & 1 & 7 & = -t_1 \\ \hline 3 & 3 & 12 & = f \end{array}$$

Example 5′

Maximize $f(x, y, z) = x + 2y + z$
subject to

$$\begin{aligned} x - y + z &= 6 \\ x + y &\leqslant 1 \\ x, z &\geqslant 0 \end{aligned}$$

This has both an unrestricted variable and an equality constraint.
We can do both at once!

$$\begin{array}{cccc|c} x & y & z & -1 & \\ \hline 1 & -1^* & 1 & 6 & = -0 \\ 1 & 1 & 0 & 1 & = -t_1 \\ \hline 1 & 2 & 1 & 0 & = f \end{array} \quad \rightarrow \quad \begin{array}{cccc|c} x & 0 & z & -1 & \\ \hline -1 & -1 & -1 & -6 & = -y \\ 2 & 1 & 1 & 7 & = -t_1 \\ \hline 3 & 2 & 3 & 12 & = f \end{array}$$

Delete row and delete column, record $-x - z + 6 = -y$.

$$\begin{array}{ccc|c} x & z & -1 & \\ \hline 2^* & 1 & 7 & = -t_1 \\ 3 & 3 & 12 & = f \end{array} \rightarrow \begin{array}{ccc|c} t_1 & z & -1 & \\ \hline 1/2 & 1/2^* & 7/2 & = -x \\ -3/2 & 3/2 & 3/2 & = f \end{array} \rightarrow \begin{array}{ccc|c} t_1 & x & -1 & \\ \hline 1 & 2 & 7 & = -z \\ -3 & -3 & -9 & = f \end{array}$$

Run SA as usual.

Example 5′

Maximize $f(x, y, z) = x + 2y + z$
subject to

$$\begin{aligned} x - y + z &= 6 \\ x + y &\leqslant 1 \\ x, z &\geqslant 0 \end{aligned}$$

This has both an unrestricted variable and an equality constraint.
We can do both at once!

$$\begin{array}{ccc|c|l} x & y & z & -1 & \\ \hline 1 & -1^* & 1 & 6 & = -0 \\ 1 & 1 & 0 & 1 & = -t_1 \\ \hline 1 & 2 & 1 & 0 & = f \\ \hline \end{array} \quad \to \quad \begin{array}{ccc|c|l} x & 0 & z & -1 & \\ \hline -1 & -1 & -1 & -6 & = -y \\ 2 & 1 & 1 & 7 & = -t_1 \\ \hline 3 & 2 & 3 & 12 & = f \\ \hline \end{array}$$

Delete row and delete column, record $-x - z + 6 = -y$.

$$\begin{array}{cc|c|l} x & z & -1 & \\ \hline 2^* & 1 & 7 & = -t_1 \\ \hline 3 & 3 & 12 & = f \\ \hline \end{array} \quad \to \quad \begin{array}{cc|c|l} t_1 & z & -1 & \\ \hline 1/2 & 1/2^* & 7/2 & = -x \\ \hline -3/2 & 3/2 & 3/2 & = f \\ \hline \end{array} \quad \to \quad \begin{array}{cc|c|l} t_1 & x & -1 & \\ \hline 1 & 2 & 7 & = -z \\ \hline -3 & -3 & -9 & = f \\ \hline \end{array}$$

Run SA as usual.
The optimum is at $x = 0$, $z = 7$, so $y = 0 + 7 - 6 = 1$.