MATH 308 D200, Fall 2019

7. Tucker tableau and pivot transformation

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

Dr. Masood Masjoody

SFU Burnaby

Tucker Tableaux

Given system of linear equations $\boldsymbol{A}^s\boldsymbol{x}^s = \boldsymbol{b}$ and corresponding augmented matrix

(1)
$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n & \overbrace{x_{n+1} \quad x_{n+2} \quad \cdots \quad x_{n+m}}^{\text{basic variables}} & \Big| & -1 \end{bmatrix}$$

(2)
$$\left[\begin{array}{ccccccccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 & b_m \end{array}\right]$$

The Tucker tableau records this, and also the objective function $\boldsymbol{c}^{s\top}\boldsymbol{x}^s - db = f(\boldsymbol{x}^s)$.

(3)
$$\begin{array}{cccc|c|l} x_1 & & \cdots & x_n & -1 & \\ \hline a_{11} & a_{12} & \cdots & a_{1n} & b_1 & = -x_{n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 & = -x_{n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m & = -x_{n+m} \\ \hline c_1 & c_2 & \cdots & c_n & d & = f \\ \hline \end{array}$$

Definition (Tucker Tableau of the Canonical Slack Maximization LP Problem)

The tableau (3) is the Tucker tableau of the canonical maximization LP problem.
The variables to the North of the tableau are independent variables or non-basic variables.
The variables to the East of the tableau are dependent variables or basic variables.

Changing the basis $B \mapsto B'$ and moving to another basic solution

Matrix $\boldsymbol{A}^s$ of the system (1) has rank m and system has an obvious basic solution $\boldsymbol{x}^s = (0, \dots, 0, b_1, \dots, b_m)$ (**not necessarily feasible**) determined by the basis $B = \{n+1, n+2, \dots, n+m\}$ which defines an identity submatrix of $\boldsymbol{A}^s$.

Moving to a "nearby" basic solution: We change the basis B by removing one index $j = n + k \in B$ from B and adding other index $i \notin B$ into B.
We shall require the set $B' = (B \setminus \{i\}) \cup \{j\}$ to define an identity matrix in $\boldsymbol{A}^s$.
Take the columns of the two variables x_i and $x_j = x_{n+k}$:

$$\boldsymbol{A}^s_{\{i\}} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ki} \\ \vdots \\ a_{mi} \end{bmatrix} \qquad \boldsymbol{A}^s_{\{j\}} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{kj} \\ \vdots \\ a_{mj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \boldsymbol{e}_k$$

Changing the basis $B \mapsto B'$ and moving to another basic solution

Matrix $\boldsymbol{A}^s$ of the system (1) has rank m and system has an obvious basic solution $\boldsymbol{x}^s = (0, \ldots, 0, b_1, \ldots, b_m)$ (**not necessarily feasible**) determined by the basis $B = \{n+1, n+2, \ldots, n+m\}$ which defines an identity submatrix of $\boldsymbol{A}^s$.

Moving to a "nearby" basic solution: We change the basis B by removing one index $j = n + k \in B$ from B and adding other index $i \notin B$ into B.
We shall require the set $B' = (B \setminus \{i\}) \cup \{j\}$ to define an identity matrix in $\boldsymbol{A}^s$.
Take the columns of the two variables x_i and $x_j = x_{n+k}$:

$$\boldsymbol{A}^s_{\{i\}} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ki} \\ \vdots \\ a_{mi} \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ (} i \text{ enters the basis,)} \qquad \boldsymbol{A}^s_{\{j\}} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{kj} \\ \vdots \\ a_{mj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \boldsymbol{e}_k \mapsto \begin{bmatrix} a'_{1j} \\ \vdots \\ a'_{kj} \\ \vdots \\ a'_{mj} \end{bmatrix} \text{ (} j \text{ leaves the basis.)}$$

First, we row reduce $\boldsymbol{A}^s$, by converting $\boldsymbol{A}^s_{\{i\}} \mapsto \boldsymbol{e}_k$. Then swap columns i and j of $\boldsymbol{A}^s$.

Changing the basis $B \mapsto B'$ and moving to another basic solution

Matrix $\boldsymbol{A}^s$ of the system (1) has rank m and system has an obvious basic solution $\boldsymbol{x}^s = (0, \ldots, 0, b_1, \ldots, b_m)$ (**not necessarily feasible**) determined by the basis $B = \{n+1, n+2, \ldots, n+m\}$ which defines an identity submatrix of $\boldsymbol{A}^s$.

Moving to a "nearby" basic solution: We change the basis B by removing one index $j = n+k \in B$ from B and adding other index $i \notin B$ into B.
We shall require the set $B' = (B \setminus \{i\}) \cup \{j\}$ to define an identity matrix in $\boldsymbol{A}^s$.
Take the columns of the two variables x_i and $x_j = x_{n+k}$:

$$\boldsymbol{A}^s_{\{i\}} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ki} \\ \vdots \\ a_{mi} \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ (} i \text{ enters the basis,)} \qquad \boldsymbol{A}^s_{\{j\}} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{kj} \\ \vdots \\ a_{mj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \boldsymbol{e}_k \mapsto \begin{bmatrix} a'_{1j} \\ \vdots \\ a'_{kj} \\ \vdots \\ a'_{mj} \end{bmatrix} \text{ (} j \text{ leaves the basis.)}$$

First, we row reduce $\boldsymbol{A}^s$, by converting $\boldsymbol{A}^s_{\{i\}} \mapsto \boldsymbol{e}_k$. Then swap columns i and j of $\boldsymbol{A}^s$.
The process is called pivot transformation and it is the core of the SA.
It adds the (independent) nonbasic variable x_i to B, making it basic.
and removes the (dependent) basic variable x_j from B, making it nonbasic,
We say that x_j leaves the basis, and that x_i enters the basis.

Theorem

Let i, j, B, B' be as stated above. Let $k \in \{1, 2, \ldots, m\}$ be such that $a_{ki} \neq 0$. For B' to define an identity matrix we need to perform following elementary operations on the system $\boldsymbol{A}^s \boldsymbol{x}^s = \boldsymbol{b}$

- *multiply row k by $\frac{1}{a_{ki}}$*
- *for each row $\ell \neq k$, add $-a_{\ell i}$ multiple of (new) row k to row ℓ*
- *swap columns i and j.*

Proof.

followed by swaping columns. □

Theorem

Let i, j, B, B' be as stated above. Let $k \in \{1, 2, \ldots, m\}$ be such that $a_{ki} \neq 0$. For B' to define an identity matrix we need to perform following elementary operations on the system $\boldsymbol{A}^s \boldsymbol{x}^s = \boldsymbol{b}$

- *multiply row k by $\frac{1}{a_{ki}}$*
- *for each row $\ell \neq k$, add $-a_{\ell i}$ multiple of (new) row k to row ℓ*
- *swap columns i and j.*

Proof.

This is exactly how we apply Gaussian Elimination to convert

$$\begin{bmatrix} a_{1i} \\ \vdots \\ \\ a_{ki} \\ \\ \vdots \\ a_{mi} \end{bmatrix} \mapsto \begin{bmatrix} a_{1i} \\ \vdots \\ \\ 1 \\ \\ \vdots \\ a_{mi} \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

followed by swaping columns. $\square$

Operation described above can be performed on the Tucker tableau (3) (including the objective function row) as follows:

Algorithm (The Pivot Transformation)

(1) Choose a nonzero pivot entry p inside the main field of the tableau (pivot entries are usually noted by an asterisk *, or boxed).

(2) Interchange the variables corresponding to p's row (leaving/dependent/basic variable) and column (entering/independent/non-basic variable); leave the signs behind as they were.

(3) Replace p by $1/p$.

(4) Replace every entry r in the same row as p by r/p.

(5) Replace every entry c in the same column as c by $-c/p$.

(6) Replace every entry s not in the same row and not in the same column as p by $\frac{ps-rc}{p}$ where r is in the same row as s and same column as p and c is in the same column as s and same row as p.

With a sequence of pivots we can move from any BS, to any other BS.

Visual rule for pivoting

Summary of pivot rule

$$\begin{array}{c} x_i \\ \begin{array}{|c|c|} \hline p^* & r \\ \hline c & s \\ \hline \end{array} \end{array} = -x_j \quad \mapsto \quad \begin{array}{c} x_j \\ \begin{array}{|c|c|} \hline \frac{1}{p} & \frac{r}{p} \\ \hline -\frac{c}{p} & \frac{ps-rc}{p} \\ \hline \end{array} \end{array} = -x_i$$

First example again ... first pivot transformation directly on tableau

x	y	-1	
-1	1	1	$= -t_1$
1	6	15	$= -t_2$
4^*	-1	10	$= -t_3$
1	1	0	$= f$

$\longrightarrow$

			$=$
			$=$
			$=$
			$=$

First example again ... first pivot transformation directly on tableau

$$
\begin{array}{cc|c|l}
x & y & -1 & \\
\hline
-1 & 1 & 1 & = -t_1 \\
1 & 6 & 15 & = -t_2 \\
4^* & -1 & 10 & = -t_3 \\
\hline
1 & 1 & 0 & = f \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|c|l}
t_3 & & \\
\hline
 & & = \\
 & & = \\
 & & = -x \\
\hline
 & & = \\
\hline
\end{array}
$$

First example again ... first pivot transformation directly on tableau

x	y	-1	
-1	1	1	$= -t_1$
1	6	15	$= -t_2$
4^*	-1	10	$= -t_3$
1	1	0	$= f$

$\longrightarrow$

t_3	y	-1	
			$= -t_1$
			$= -t_2$
			$= -x$
			$= f$

First example again ... first pivot transformation directly on tableau

$$
\begin{array}{ccc|c|l}
x & y & & -1 & \\
\hline
-1 & 1 & & 1 & = -t_1 \\
1 & 6 & & 15 & = -t_2 \\
4^* & -1 & & 10 & = -t_3 \\
\hline
1 & 1 & & 0 & = f \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{cc|c|l}
t_3 & y & -1 & \\
\hline
 & & & = -t_1 \\
 & & & = -t_2 \\
1/4 & & & = -x \\
\hline
 & & & = f \\
\hline
\end{array}
$$

First example again ... first pivot transformation directly on tableau

x	y	-1	
-1	1	1	$= -t_1$
1	6	15	$= -t_2$
4^*	-1	10	$= -t_3$
1	1	0	$= f$

$\longrightarrow$

t_3	y	-1	
			$= -t_1$
			$= -t_2$
1/4	-1/4	10/4	$= -x$
			$= f$

First example again ... first pivot transformation directly on tableau

$$
\begin{array}{cc|c|l}
x & y & -1 & \\
\hline
-1 & 1 & 1 & = -t_1 \\
1 & 6 & 15 & = -t_2 \\
4^* & -1 & 10 & = -t_3 \\
\hline
1 & 1 & 0 & = f \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{cc|c|l}
t_3 & y & -1 & \\
\hline
1/4 & & & = -t_1 \\
-1/4 & & & = -t_2 \\
1/4 & -1/4 & 10/4 & = -x \\
\hline
-1/4 & & & = f \\
\hline
\end{array}
$$

First example again ... first pivot transformation directly on tableau

$$
\begin{array}{cc|c|l}
x & y & -1 & \\
\hline
-1 & 1 & 1 & = -t_1 \\
1 & 6 & 15 & = -t_2 \\
4^* & -1 & 10 & = -t_3 \\
\hline
1 & 1 & 0 & = f \\
\hline
\end{array}
\longrightarrow
\begin{array}{cc|c|l}
t_3 & y & -1 & \\
\hline
1/4 & \frac{4\cdot 1-(-1)\cdot(-1)}{4} & \frac{4\cdot 1-(-1)\cdot(10)}{4} & = -t_1 \\
-1/4 & \frac{4\cdot 6-1\cdot(-1)}{4} & \frac{4\cdot 15-1\cdot(10)}{4} & = -t_2 \\
1/4 & -1/4 & 10/4 & = -x \\
\hline
-1/4 & \frac{4\cdot 1-(-1)\cdot 1}{4} & \frac{4\cdot 0-1\cdot 10}{4} & = f \\
\hline
\end{array}
$$

First example again ... first pivot transformation directly on tableau

$$
\begin{array}{cc|c|l}
x & y & -1 & \\
\hline
-1 & 1 & 1 & = -t_1 \\
1 & 6 & 15 & = -t_2 \\
4^* & -1 & 10 & = -t_3 \\
\hline
1 & 1 & 0 & = f \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{cc|c|l}
t_3 & y & -1 & \\
\hline
1/4 & 3/4 & 14/4 & = -t_1 \\
-1/4 & 25/4 & 50/4 & = -t_2 \\
1/4 & -1/4 & 5/2 & = -x \\
\hline
-1/4 & 5/4 & -10/4 & = f \\
\hline
\end{array}
$$

$$
\begin{array}{cc|c|l}
t_3 & y & -1 & \\
\hline
1/4 & 3/4 & 7/2 & = -t_1 \\
-1/4 & 25/4^{*} & 25/2 & = -t_2 \\
1/4 & -1/4 & 5/2 & = -x \\
\hline
-1/4 & 5/4 & -5/2 & = f \\
\hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{|cc|c|l}
 & & -1 & \\
\hline
 & & & = \\
 & & & = \\
 & & & = \\
\hline
 & & & = f \\
\hline
\end{array}
$$

Important Example

Consider the following maximum Tucker tableau of a maximization LP problem:

$$
\begin{array}{|cc|c|l}
\multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{-1} & \\
\hline
2 & 1 & 14 & = -x_3 \\
0 & 1 & 4 & = -x_1 \\
-2 & 3 & 6 & = -x_5 \\
\hline
1 & 2 & 3 & = f \\
\hline
\end{array}
$$

(i) Using pivot transformations find basic solutions for the following sets of dependent (basic) variables:

$$\mathscr{B}_1 = \{x_1, x_2, x_4\}$$
$$\mathscr{B}_2 = \{x_1, x_3, x_4\}$$
$$\mathscr{B}_3 = \{x_2, x_3, x_4\}$$

(ii) For every basic solution state the complete solution of the problem as a 5-dimensional vector $(x_1, x_2, \ldots, x_5)$.

(iii) Which of these solutions are feasible and which are not?

x_2	x_4	-1	
2	1	14	$= -x_3$
0	1	4	$= -x_1$
-2	3	6	$= -x_5$
1	2	3	$= f$

	-1	
		$=$
		$=$
		$=$
		$= f$

	-1	
		$=$
		$=$
		$=$
		$= f$

	-1	
		$=$
		$=$
		$=$
		$= f$