MACM 201 - Discrete Mathematics

1. Fundamental combinatorial objects

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Lecture overview

Discrete mathematics differs from continuous mathematics (calculus) in that it is the mathematics of objects composed of a **finite** set of elements arranged into a specific **structure**.

So combinatorial objects are defined by:

What are the elements (atoms) that compose them?

How are they structured?

A family of combinatorial objects is defined by these two characteristics. We will see four main kinds of objects:

sets and subsets, strings and permutations, graphs, trees

The notion of an atom naturally leads to a notion of its **size**, defined as the (integer) **number of atoms** the object contains, e.g., number of letters in a string.

Definition

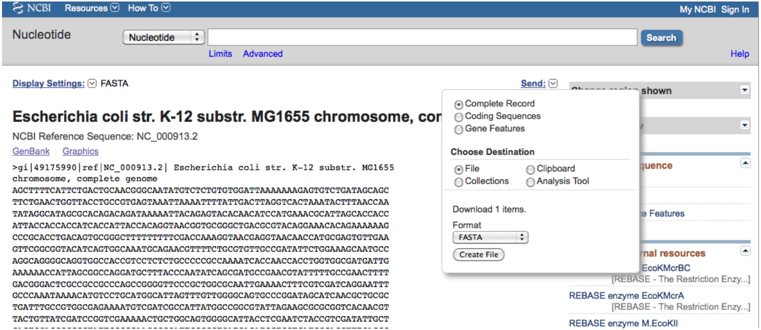
Let $\mathcal{A}$ be a set that we will refer to as an **alphabet**. A **string** S, of size n, over $\mathcal{A}$ is a totally ordered list of n elements, called **letters**, of $\mathcal{A}$. Note that a string over $\mathcal{A}$ may contain a letter of A many times or zero times.

Note: The atoms of a string are the letters (taken from $\mathcal{A}$) and the structure is a total order: there is a first letter, a second letter, ..., a last letter.

Examples.

$$\mathcal{A} = \{0, 1\} \qquad\qquad \mathcal{A} = \{A, C, G, T\}$$

Applications. Strings are used in many applications, from computer code analysis (binary strings) to data streams analysis in big data, through genomics (DNA and protein strings).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Definition

Let S be a string of length n over the alphabet $\mathcal{A}$. For every $1 \leq i \leq n$ we let $S[i]$ denote the i^{th} letter of S. If $1 \leq i \leq j \leq n$ then we define the string

$$S[i,j] = S[i], S[i+1], \ldots S[j]$$

and we call any such string a **substring** of S. Note that a the letters of a substring of S must appear consecutively in S.

Examples.

$S = 11001100$ $\quad$ $S = ABCDEF$

Problem. Find all strings of length 6 over $\{0,1\}$ that do not have 10 a substring.

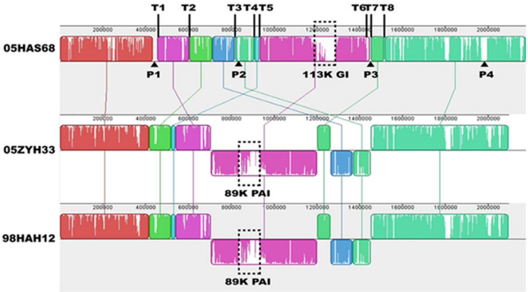
Permutations

Definition

A **permutation** P over an alphabet $\mathcal{A}$ is a string over $\mathcal{A}$ where every symbol of $\mathcal{A}$ occurs exactly once.

Problem. For $\mathcal{A} = \{1, 2, 3\}$, find all permutations over $\mathcal{A}$

Applications. Permutations are useful to model the various orders of **distinguishable atoms**.



Graphs

Definition

A **graph** $G = (V, E)$ is composed of a set V of **vertices** and E of **edges**, defined as unordered pairs of vertices. If $e \in E$ and $e = \{i, j\}$ then we say that e is **incident** with the vertices i and j. Two distinct vertices $i, j \in V$ are **adjacent** if $\{i, j\} \in E$. The **size** of a graph is its number of vertices.

Example.

$$V = \{1, 2, 3, 4, 5, 6\}, \;\; E = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{4, 5\}, \{4, 6\}\}$$

Applications. Graphs are perfectly adapted at modelling binary relations (the edges) between a finite set of elements (the atoms): social networks, biological networks, . . .

Basic Families of Graphs

Definition

A graph $G = (V, E)$ is **complete** if for every $i, j \in V$ with $i \neq j$ we have $\{i, j\} \in E$. We let K_n denote a complete graph of size n.

Example. K_5.

Definition

A graph $G = (V, E)$ is called a **path** if V may be ordered $v_1, v_2, \ldots, v_n$ so that $E = \Big\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\Big\}$.

Example: A path of size 5

Definition

A graph $G = (V, E)$ is called a **cycle** if V may be ordered $v_1, v_2, \ldots, v_n$ so that $E = \Big\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\Big\}$.

Example: A cycle of size 5

Directed Graphs

Definition

A **directed graph** $G = (V, E)$ is composed of a set V of **vertices** and E of **edges**, defined as ordered pairs of vertices. If $e \in E$ and $e = (i, j)$ then e goes from the vertex i to the vertex j. The **size** of a directed graph is its number of vertices.

Example.

$$V = \{1, 2, 3, 4, 5, 6\}, \;\; E = \{(1, 2), (2, 1), (1, 5), (3, 2), (2, 5), (4, 3), (6, 4)\}$$

Rooted Trees

Definition

A **rooted tree** T is a tree with a distinguished vertex called the **root**. (we will define tree more precisely later).

Here is another equivalent definition for which all terms have been defined.

Definition

A **rooted tree** $T = (V, E)$ is a graph with a distinguished vertex called the **root** with the added property that every vertex $v \in V$ has an associated **level** so that the following conditions are satisfied:

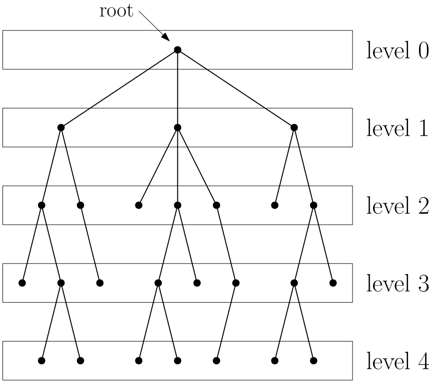
(1) The level of every vertex is a nonnegative integer.

(2) The root is the unique vertex of level 0.

(3) For every edge $\{i, j\}$ the levels of i and j differ by exactly 1.

(4) For every non-root vertex $v \in V$ with level i there is exactly one vertex of level $i - 1$ adjacent to v.

Example.



Applications. Rooted trees are well suited to model elements with a hierarchical structure: evolutionary trees, decision trees, . . .