MATH 308 D200, Fall 2019

3. Geometric background of linear programming

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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Definition (hyperplane, closed half-space)

Let $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{R}^n$ be a non-zero vector. The set of points $\boldsymbol{x} \in \mathbb{R}^n$ satisfying an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is called a **hyperplane** of $\mathbb{R}^n$. The set of points $\boldsymbol{x} \in \mathbb{R}^n$ satisfying an inequality

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leqslant b$$

or

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geqslant b$$

is called a **closed half-space** of $\mathbb{R}^n$.

Alternative notations:

- $\sum_{i=1}^n a_ix_i = b$, $\sum_{i=1}^n a_ix_i \leqslant b$, $\sum_{i=1}^n a_ix_i \geqslant b$
- $\boldsymbol{a} \cdot \boldsymbol{x} = b$, $\boldsymbol{a} \cdot \boldsymbol{x} \leqslant b$, $\boldsymbol{a} \cdot \boldsymbol{x} \geqslant b$ (dot product of the vectors $\boldsymbol{a}, \boldsymbol{x}$)
- $\boldsymbol{a}^\mathsf{T}\boldsymbol{x} = b$, $\boldsymbol{a}^\mathsf{T}\boldsymbol{x} \leqslant b$, $\boldsymbol{a}^\mathsf{T}\boldsymbol{x} \geqslant b$ (matrix product)

Here, vectors are viewed as $n \times 1$ matrices: $\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Definition (line segment)

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n), \boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then the set

$$\overline{\boldsymbol{x}\boldsymbol{y}} = \{t\boldsymbol{x} + (1-t)\boldsymbol{y} : 0 \leqslant t \leqslant 1\}$$

is said to be the **line segment** between $\boldsymbol{x}$ and $\boldsymbol{y}$ inclusive.

Definition (convex set)

Let S be a subset of $\mathbb{R}^n$. S is said to be **convex** if

$$\overline{\boldsymbol{x}\boldsymbol{y}} \subseteq S \text{ for any } \boldsymbol{x}, \boldsymbol{y} \in S \ .$$

The following facts hold for real numbers a, b, t and for $n \times 1$ matrices $\boldsymbol{a}$, $\boldsymbol{x}$ and $\boldsymbol{y}$.

(1) We have $a \leq b$ if and only if $-a \geq -b$

(2) If $a \leq b$ and $t \geq 0$, then $ta \leq tb$

(3) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$

(4) The statement $0 \leq x \leq b$ means $0 \leq x$ and $x \leq b$.

(5) $t(\boldsymbol{a}^\intercal \boldsymbol{x}) = (t\boldsymbol{a}^\intercal)\boldsymbol{x} = \boldsymbol{a}^\intercal(t\boldsymbol{x})$

(6) $\boldsymbol{a}^\intercal \boldsymbol{x} + \boldsymbol{a}^\intercal \boldsymbol{y} = \boldsymbol{a}^\intercal(\boldsymbol{x} + \boldsymbol{y})$

Theorem

The intersection of arbitrary collection of convex sets in $\mathbb{R}^n$ is convex.

Proof.

$\square$

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Every closed half-space of $\mathbb{R}^n$ is a convex set.

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Let $\{S_i : i \in I\}$ ($I \neq \varnothing$) be a collection of convex sets and let $T = \bigcap_i S_i$ be their intersection. Let $\boldsymbol{x}, \boldsymbol{y} \in T$. Each of the sets S_i is convex, so $\overline{\boldsymbol{xy}} \subseteq S_i$. This holds for every set S_i, so $\overline{\boldsymbol{xy}} \subseteq T$. Therefore T is also convex. □

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Theorem

Every closed half-space of $\mathbb{R}^n$ is a convex set.

Proof.

Let H be a half-space, so $H = \{\boldsymbol{z} : \boldsymbol{a}^\intercal \boldsymbol{z} \leq b\}$ for some $\boldsymbol{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Let $\boldsymbol{x}, \boldsymbol{y} \in H$, and consider any number t where $0 \leq t \leq 1$.

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Therefore $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in H$. We have shown that $\overline{\boldsymbol{x}\boldsymbol{y}} \subseteq H$, so H is convex.

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Theorem

Every closed half-space of $\mathbb{R}^n$ is a convex set.

Proof.

Let H be a half-space, so $H = \{\boldsymbol{z} : \boldsymbol{a}^\top \boldsymbol{z} \leq b\}$ for some $\boldsymbol{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Let $\boldsymbol{x}, \boldsymbol{y} \in H$, and consider any number t where $0 \leq t \leq 1$.
Then $\boldsymbol{a}^\top \boldsymbol{x} \leq b$ and $\boldsymbol{a}^\top \boldsymbol{y} \leq b$. Since t and $1-t$ are non-negative, these imply $t(\boldsymbol{a}^\top \boldsymbol{x}) \leq tb$ and $(1-t)(\boldsymbol{a}^\top \boldsymbol{y}) \leq (1-t)b$. Adding and simplifying,

$$t(\boldsymbol{a}^\top \boldsymbol{x}) + (1-t)(\boldsymbol{a}^\top \boldsymbol{y}) \leq tb + (1-t)b$$

$$\boldsymbol{a}^\top (t\boldsymbol{x} + (1-t)\boldsymbol{y}) \leq b.$$

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The intersection of arbitrary collection of convex sets in $\mathbb{R}^n$ is convex.

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Let $\{S_i : i \in I\}$ ($I \neq \varnothing$) be a collection of convex sets and let $T = \bigcap_i S_i$ be their intersection. Let $\boldsymbol{x}, \boldsymbol{y} \in T$. Each of the sets S_i is convex, so $\overline{\boldsymbol{x}\boldsymbol{y}} \subseteq S_i$. This holds for every set S_i, so $\overline{\boldsymbol{x}\boldsymbol{y}} \subseteq T$. Therefore T is also convex. □

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Then $\boldsymbol{a}^\intercal \boldsymbol{x} \leq b$ and $\boldsymbol{a}^\intercal \boldsymbol{y} \leq b$. Since t and $1-t$ are non-negative, these imply $t(\boldsymbol{a}^\intercal \boldsymbol{x}) \leq tb$ and $(1-t)(\boldsymbol{a}^\intercal \boldsymbol{y}) \leq (1-t)b$. Adding and simplifying,

$$
\begin{aligned}
t(\boldsymbol{a}^\intercal \boldsymbol{x}) + (1-t)(\boldsymbol{a}^\intercal \boldsymbol{y}) &\leq tb + (1-t)b \\
\boldsymbol{a}^\intercal (t\boldsymbol{x} + (1-t)\boldsymbol{y}) &\leq b.
\end{aligned}
$$

Therefore $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in H$. We have shown that $\overline{\boldsymbol{x}\boldsymbol{y}} \subseteq H$, so H is convex. □

Definition (Canonical Maximization LP Problem)

The problem

$$\textbf{Maximize } f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots c_nx_n - d$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leqslant b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leqslant b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leqslant b_m \\ x_1, x_2, \ldots, x_n &\geqslant 0 \end{aligned}$$

is said to be a **canonical maximization linear programming problem**.

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Definition (Canonical Maximization LP Problem (Vector Notation))

Writing $\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\boldsymbol{a}_i^{\mathsf{T}} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$, $i = 1 \dots m$.

$$
\begin{aligned}
\textbf{Maximize } & f(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - d \\
\text{subject to } \quad & \boldsymbol{a}_1^{\mathsf{T}}\boldsymbol{x} \leqslant b_1 \\
& \boldsymbol{a}_2^{\mathsf{T}}\boldsymbol{x} \leqslant b_2 \\
& \vdots \\
& \boldsymbol{a}_m^{\mathsf{T}}\boldsymbol{x} \leqslant b_m \\
& x_1, x_2, \dots, x_n \geqslant 0,\ j = 1, 2, \dots, n
\end{aligned}
$$

Definition (Canonical Maximization LP Problem (Matrix Notation))

Writing $\boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ and $A = \begin{bmatrix} \boldsymbol{a}_1{}^{\mathsf{T}} \\ \boldsymbol{a}_2{}^{\mathsf{T}} \\ \vdots \\ \boldsymbol{a}_m{}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$.

$$\begin{aligned} &\textbf{Maximize } f(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - d \\ &\text{subject to} \quad A\boldsymbol{x} \leqslant \boldsymbol{b} \\ &\qquad\qquad\quad \boldsymbol{x} \geqslant \boldsymbol{0} \end{aligned}$$

Here $A\boldsymbol{x} \leqslant \boldsymbol{b}$ means $\begin{cases} \boldsymbol{a}_1{}^{\mathsf{T}}\boldsymbol{x} \leqslant b_1 \\ \boldsymbol{a}_2{}^{\mathsf{T}}\boldsymbol{x} \leqslant b_2 \\ \vdots \\ \boldsymbol{a}_m{}^{\mathsf{T}}\boldsymbol{x} \leqslant b_m \end{cases}$

Matrix Notation for Linear Programming Problems

Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^n$, $d \in \mathbb{R}$.

Definition (canonical maximization LP problem)

$$\begin{aligned} \textbf{Maximize} \quad & f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d && \text{(objective function)} \\ \text{s.t.} \quad & A\boldsymbol{x} \leqslant \boldsymbol{b} && (m \text{ main constraints}) \\ & \boldsymbol{x} \geqslant \boldsymbol{0} && (n \text{ non-negativaty constraints}) \end{aligned}$$

Definition (canonical minimization LP problem)

$$\begin{aligned} \textbf{Miminize} \quad & f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d && \text{(objective function)} \\ \text{s.t.} \quad & A\boldsymbol{x} \geqslant \boldsymbol{b} && (m \text{ main constraints}) \\ & \boldsymbol{x} \geqslant \boldsymbol{0} && (n \text{ non-negativaty constraints}) \end{aligned}$$

Definition (objective function)

The linear functions f in definitions above are called **objective functions**.

Definition (constraint set, feasible points)

The set of all points $\boldsymbol{x} \in \mathbb{R}^n$ satisfying the $m + n$ constraints of a canonical LP problem is said to be **constraint set** or **set of feasible points** of the problem.

Definition

Any feasible solution of a canonical maximization (respectively minimization) LP problem which maximizes (minimizes) the objective function is said to be an **optimal solution**.

Example

Show that the following LP is infeasible.

$$\begin{aligned} \textbf{Maximize } f(\boldsymbol{x}) &= 2x + 3y \\ \text{subject to} \quad x + y &\geq 2 \\ -x + y &\leq 0 \\ 3x - y &\leq 1 \end{aligned}$$

Example of using inequality rules to prove infeasibility

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Solution: Suppose that a point (x, y) satisfies all three inequalities. We aim for a contradiction using the facts on slide 4. We multiply the first equation by -1 and multiply the second equation by 2 to get

$$\begin{aligned} -x - y &\leq -2 \\ -2x + 2y &\leq 0 \end{aligned}$$

Adding these inequalities we get $-3x + y \leq -2$. Adding the last inequality now gives $0 \leq -1$. This absurdity contradicts our assumption that (x, y) exists. Therefore the constraint set is empty and the LP is infeasible.

Definition (polyhedral convex set)

Intersection of finitely many closed half-spaces is called a **polyhedral convex set**.

Proposition

The constraint set of a canonical LP problem is a polyhedral convex set.

Example

Draw the constraint set corresponding to constraints

$$\begin{aligned} 2x + 5y &\geqslant 10 \\ x - 2y &\leqslant 8 \\ x, y &\geqslant 0 \end{aligned}$$

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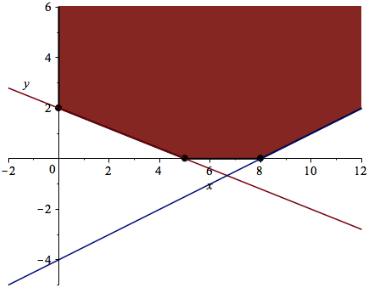


Note: This region is unbounded.

Definition (extreme point)

Let $S \subseteq \mathbb{R}^n$ be a convex set. Point $\boldsymbol{e} \in S$ is said to be an **extreme point** of S if there do not exist points $\boldsymbol{x}, \boldsymbol{y} \in S$ and $0 < t < 1$ such that $\boldsymbol{e} = t\boldsymbol{x} + (1-t)\boldsymbol{y}$.

Theorem

A point $\boldsymbol{x} \in \mathbb{R}^n$ is an extreme point of a constraint set

$$\boldsymbol{a}_i{}^{\mathsf{T}}\boldsymbol{x} \leqslant b_i, \ (\text{for } i = 1, 2, \ldots, m), \tag{1}$$

$$\boldsymbol{x} \geqslant \boldsymbol{0} \tag{2}$$

if and only if each of the following hold.

a. $\boldsymbol{x}$ *is feasible. That is,* $\boldsymbol{x}$ *satisfies* (1) *and* (2).

b. $\boldsymbol{x}$ *satisfies* n *independent constraints with equality. That is, there exist index sets* $B_M \subseteq \{1, 2, \ldots, m\}$, $B_N \subseteq \{1, 2, \ldots, n\}$ *such that* $|B_M| + |B_N| = n$ *and*

(i)

$$\boldsymbol{a}_i{}^{\mathsf{T}}\boldsymbol{x} = b_i, \ (\text{for } i \in B_M)$$
$$x_j = 0, (\text{for } j \in B_N)$$

(ii) *the set of vectors* $\{\boldsymbol{a}_i : i \in B_M\} \cup \{\boldsymbol{e}_j : j \in B_N\}$ *is linearly independent, where* $\boldsymbol{e}_i = [\,0\ 0\ \ldots\ 0\ 1\ 0\ \ldots\ 0\ 0\,]^{\mathsf{T}}$ *has a* 1 *in the* i*th entry.*

Example:

$$x_1 + 2x_2 + 4x_3 \geq 8 \qquad (1)$$

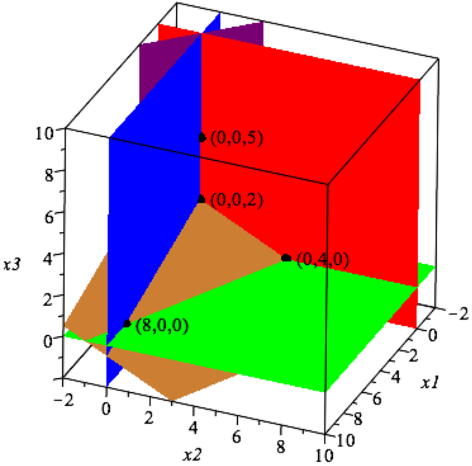
$$x_1 + x_2 \geq 0 \qquad (2)$$

$$x_1 \geq 0 \qquad (3)$$

$$x_2 \geq 0 \qquad (4)$$

$$x_3 \geq 0 \qquad (5)$$

Point	Tight constraints M	N
(8,0,0)	(1),	(4), (5)
(0,4,0)	(1),	(3), (5)
(0,0,2)	(1), (2)	(3), (4)
(0,0,5)	(2),	(3), (4)
(0,0,0)		(3), (4), (5)



Note: (0,0,5) is not an extreme point, even though there are 3 tight constraints, because $\boldsymbol{a}_2 = [1,1,0]^\mathsf{T}, \boldsymbol{e}_1 = [1,0,0]^\mathsf{T}, \boldsymbol{e}_2 = [0,1,0]^\mathsf{T}$ are not linearly independent!

(0,0,0) is not an extreme point because it is not feasible!