MATH 308 D200, Fall 2019

10. Simplex algorithm for minimum tableau

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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Canonical Forms for Minimization LP Problem

Canonical Minimization LP

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n - d = g(x_1, x_2, \ldots, x_n) \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geqslant b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geqslant b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geqslant b_m \\ & x_1, x_2, \ldots, x_n \geqslant 0 \end{aligned}$$

Add Slack variables

$$\begin{aligned} \text{minimize} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n - d = g(x_1, x_2, \ldots, x_n) \\ \text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 + t_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 + t_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m + t_m \\ & x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m \geqslant 0 \end{aligned}$$

Definition (Canonical Slack Form)

The linear programming problem

$$
\begin{array}{ll}
\text{minimize} & g(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n - d \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 = t_1 \\
 & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 = t_2 \\
 & \vdots \\
 & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m = t_m \\
 & x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m \geqslant 0
\end{array}
$$

is said to be *canonical slack minimization linear programming problem*. The variables $t_1, t_2, \ldots, t_m$ are said to be *slack variables*.

Tucker Tableau of the Canonical Slack Minimization LP Problem

Given the canonical slack minimization LP

$$
\begin{aligned}
\text{minimize} \quad & g(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots + c_nx_n - d \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 = t_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 = t_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m = t_m \\
& x_1, x_2, \dots, x_n, t_1, t_2, \dots, t_m \geqslant 0
\end{aligned}
$$

Its Minimum Tucker tableau is

$$
\begin{array}{c|cccc|c}
x_1 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\hline
-1 & b_1 & b_2 & \dots & b_m & d \\
\hline
 & = t_1 & = t_2 & \dots & = t_m & = g
\end{array}
$$

The independent (nonbasic) variables are listed to the West. The dependent (basic) variables are listed to the South.

Tucker Tableau of the Canonical Slack Minimization LP Problem

Given the canonical slack minimization LP

$$
\begin{array}{ll}
\text{minimize} & g(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n - d \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 = t_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 = t_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m = t_m \\
& x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m \geqslant 0
\end{array}
$$

Its Minimum Tucker tableau is

$$
\begin{array}{c|cccc|c|}
\cline{2-6}
x_1 & a_{11} & a_{21} & \ldots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \ldots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \ldots & a_{mn} & c_n \\
\cline{2-6}
-1 & b_1 & b_2 & \ldots & b_m & d \\
\cline{2-6}
 & = t_1 & = t_2 & \ldots & = t_m & = g
\end{array}
\qquad
\begin{array}{cc}
\boxed{\boldsymbol{x}^\mathsf{T}} & \boxed{A^\mathsf{T}} \\
- & \boxed{\boldsymbol{b}^\mathsf{T}} \\
= & \boxed{\boldsymbol{t}^\mathsf{T}}
\end{array}
\qquad
\min \boxed{\boldsymbol{x}^\mathsf{T}}\boxed{\boldsymbol{c}} - d
$$

The independent (nonbasic) variables are listed to the West. The dependent (basic) variables are listed to the South.

Negative Transposition

Definition (Negative Transposition)

The negative transposition of the minimum tableau

$$
\begin{array}{r|cccc|c|}
\cline{2-6}
x_1 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\cline{2-6}
-1 & b_1 & b_2 & \dots & b_m & d \\
\cline{2-6}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{= t_1} & \multicolumn{1}{c}{= t_2} & \multicolumn{1}{c}{\dots} & \multicolumn{1}{c}{= t_m} & \multicolumn{1}{c}{= g}
\end{array}
$$

Definition (Negative Transposition)

The negative transposition of the minimum tableau

$$
\begin{array}{r|cccc|c|}
\cline{2-6}
x_1 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\cline{2-6}
-1 & b_1 & b_2 & \dots & b_m & d \\
\cline{2-6}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{= t_1} & \multicolumn{1}{c}{= t_2} & \multicolumn{1}{c}{\dots} & \multicolumn{1}{c}{= t_m} & \multicolumn{1}{c}{= g}
\end{array}
$$

is the maximum tableau

$$
\begin{array}{|cccc|c|l}
\multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{\dots} & \multicolumn{1}{c}{x_n} & \multicolumn{1}{c}{-1} & \\
\hline
-a_{11} & -a_{12} & \dots & -a_{1n} & -b_1 & = -t_1 \\
-a_{21} & -a_{22} & \dots & -a_{2n} & -b_2 & = -t_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{m1} & -a_{m2} & \dots & -a_{mn} & -b_m & = -t_m \\
\hline
-c_1 & -c_n & \dots & -c_n & -d & = -g \\
\hline
\end{array}
$$

and vice versa.

Negative Transposition

Definition (Negative Transposition)

The negative transposition of the minimum tableau

$$
\begin{array}{c|cccc|c|}
\cline{2-6}
x_1 & a_{11} & a_{21} & \cdots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \cdots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \cdots & a_{mn} & c_n \\
\cline{2-6}
-1 & b_1 & b_2 & \cdots & b_m & d \\
\cline{2-6}
\multicolumn{1}{c}{} & = t_1 & = t_2 & \cdots & = t_m & \multicolumn{1}{c}{= g}
\end{array}
$$

$$\min \boxed{\boldsymbol{x}^\top} \boxed{\boldsymbol{c}} - d = g(x)$$

$$\boxed{\boldsymbol{x}^\top} \boxed{A^\top} - \boxed{\boldsymbol{b}^\top} = \boxed{\boldsymbol{t}^\top}$$

is the maximum tableau

$$
\begin{array}{|cccc|c|l}
\multicolumn{1}{c}{x_1} & x_2 & \cdots & \multicolumn{1}{c}{x_n} & \multicolumn{1}{c}{-1} & \\
\hline
-a_{11} & -a_{12} & \cdots & -a_{1n} & -b_1 & = -t_1 \\
-a_{21} & -a_{22} & \cdots & -a_{2n} & -b_2 & = -t_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{m1} & -a_{m2} & \cdots & -a_{mn} & -b_m & = -t_m \\
\hline
-c_1 & -c_n & \cdots & -c_n & -d & = -g \\
\hline
\end{array}
$$

$$\max \boxed{-\boldsymbol{c}^\top} \boxed{\boldsymbol{x}} - (-d) = -g(x)$$

$$\boxed{-A} \boxed{\boldsymbol{x}} - \boxed{\text{-}\boldsymbol{b}} = \boxed{\text{-}\boldsymbol{t}}$$

and vice versa.

Definition (Negative Transposition)

The negative transposition of the minimum tableau

$$
\begin{array}{c|cccc|c}
x_1 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\hline
-1 & b_1 & b_2 & \dots & b_m & d \\
\hline
 & = t_1 & = t_2 & \dots & = t_m & = g
\end{array}
\qquad
\begin{aligned}
\min\ \boldsymbol{x}^\intercal \boldsymbol{c} - d &= g(\boldsymbol{x}) \\
\boldsymbol{x}^\intercal A^\intercal - \boldsymbol{b}^\intercal &= \boldsymbol{t}^\intercal \\
\boldsymbol{x}, \boldsymbol{t} &\geq 0
\end{aligned}
$$

is the maximum tableau

$$
\begin{array}{|cccc|c|c}
x_1 & x_2 & \dots & x_n & -1 & \\
\hline
-a_{11} & -a_{12} & \dots & -a_{1n} & -b_1 & = -t_1 \\
-a_{21} & -a_{22} & \dots & -a_{2n} & -b_2 & = -t_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{m1} & -a_{m2} & \dots & -a_{mn} & -b_m & = -t_m \\
\hline
-c_1 & -c_n & \dots & -c_n & -d & = -g \\
\hline
\end{array}
\qquad
\begin{aligned}
\max\ -\boldsymbol{c}^\intercal \boldsymbol{x} - (-d) &= -g(x) \\
-A\boldsymbol{x} - (-\boldsymbol{b}) &= -\boldsymbol{t} \\
\boldsymbol{x}, \boldsymbol{t} &\geq 0
\end{aligned}
$$

and vice versa.

Definition (Negative Transposition)

The negative transposition of the minimum tableau

$$\begin{array}{c|cccc|c|}
\cline{2-6}
x_1 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\cline{2-6}
-1 & b_1 & b_2 & \dots & b_m & d \\
\cline{2-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{= t_1} & \multicolumn{1}{c}{= t_2} & \multicolumn{1}{c}{\dots} & \multicolumn{1}{c}{= t_m} & \multicolumn{1}{c}{= g}
\end{array}$$

$$\begin{aligned}
\min\ \boldsymbol{c}^\top \boldsymbol{x} - d &= g(\boldsymbol{x}) \\
\boldsymbol{x}^\top A^\top - \boldsymbol{b}^\top &= \boldsymbol{t}^\top \\
\boldsymbol{x}, \boldsymbol{t} &\geq 0
\end{aligned}$$

is the maximum tableau

$$\begin{array}{|cccc|c|c}
\multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{\dots} & \multicolumn{1}{c}{x_n} & \multicolumn{1}{c}{-1} & \\
\hline
-a_{11} & -a_{12} & \dots & -a_{1n} & -b_1 & = -t_1 \\
-a_{21} & -a_{22} & \dots & -a_{2n} & -b_2 & = -t_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{m1} & -a_{m2} & \dots & -a_{mn} & -b_m & = -t_m \\
\hline
-c_1 & -c_n & \dots & -c_n & -d & = -g \\
\hline
\end{array}$$

$$\begin{aligned}
\max\ -\boldsymbol{c}^\top \boldsymbol{x} - (-d) &= -g(x) \\
-A\boldsymbol{x} - (-\boldsymbol{b}) &= -\boldsymbol{t} \\
\boldsymbol{x}, \boldsymbol{t} &\geq 0
\end{aligned}$$

and vice versa.

Definition (Negative Transposition)

The negative transposition of the minimum tableau

$$
\begin{array}{r|cccc|c|}
\cline{2-6}
x_1 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\cline{2-6}
-1 & b_1 & b_2 & \dots & b_m & d \\
\cline{2-6}
\multicolumn{1}{r}{} & \multicolumn{1}{c}{= t_1} & \multicolumn{1}{c}{= t_2} & \multicolumn{1}{c}{\dots} & \multicolumn{1}{c}{= t_m} & \multicolumn{1}{c}{= g}
\end{array}
$$

$$
\begin{aligned}
\min\ \mathbf{c}^\intercal \mathbf{x} - d &= g(\mathbf{x}) \\
(A\mathbf{x})^\intercal - \mathbf{b}^\intercal &= \mathbf{t}^\intercal \\
\mathbf{x}, \mathbf{t} &\geq 0
\end{aligned}
$$

is the maximum tableau

$$
\begin{array}{|cccc|c|l}
\multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_2} & \multicolumn{1}{c}{\dots} & \multicolumn{1}{c}{x_n} & \multicolumn{1}{c}{-1} & \\
\hline
-a_{11} & -a_{12} & \dots & -a_{1n} & -b_1 & = -t_1 \\
-a_{21} & -a_{22} & \dots & -a_{2n} & -b_2 & = -t_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{m1} & -a_{m2} & \dots & -a_{mn} & -b_m & = -t_m \\
\hline
-c_1 & -c_n & \dots & -c_n & -d & = -g \\
\hline
\end{array}
$$

$$
\begin{aligned}
\max\ -\mathbf{c}^\intercal \mathbf{x} - (-d) &= -g(x) \\
-A\mathbf{x} - (-\mathbf{b}) &= -\mathbf{t} \\
\mathbf{x}, \mathbf{t} &\geq 0
\end{aligned}
$$

and vice versa.

Definition (Negative Transposition)

The negative transposition of the minimum tableau

$$
\begin{array}{c|cccc|c}
 & & & & & \\
x_1 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\hline
-1 & b_1 & b_2 & \dots & b_m & d \\
\hline
 & = t_1 & = t_2 & \dots & = t_m & = g
\end{array}
\qquad
\begin{aligned}
\min\ \mathbf{c}^{\mathsf{T}}\mathbf{x} - d &= g(\mathbf{x}) \\
A\mathbf{x} - \mathbf{b} &= \mathbf{t} \\
\mathbf{x}, \mathbf{t} &\geq 0
\end{aligned}
$$

is the maximum tableau

$$
\begin{array}{cccc|c|c}
x_1 & x_2 & \dots & x_n & -1 & \\
\hline
-a_{11} & -a_{12} & \dots & -a_{1n} & -b_1 & = -t_1 \\
-a_{21} & -a_{22} & \dots & -a_{2n} & -b_2 & = -t_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{m1} & -a_{m2} & \dots & -a_{mn} & -b_m & = -t_m \\
\hline
-c_1 & -c_n & \dots & -c_n & -d & = -g
\end{array}
\qquad
\begin{aligned}
\max\ -\mathbf{c}^{\mathsf{T}}\mathbf{x} - (-d) &= -g(x) \\
-A\mathbf{x} - (-\mathbf{b}) &= -\mathbf{t} \\
\mathbf{x}, \mathbf{t} &\geq 0
\end{aligned}
$$

and vice versa.

Definition (Negative Transposition)

The negative transposition of the minimum tableau

$$
\begin{array}{c|cccc|c}
x_1 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\hline
-1 & b_1 & b_2 & \dots & b_m & d \\
\hline
 & = t_1 & = t_2 & \dots & = t_m & = g
\end{array}
\qquad
\begin{aligned}
\min \ \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - d &= g(\boldsymbol{x}) \\
A\boldsymbol{x} - \boldsymbol{b} &= \boldsymbol{t} \\
\boldsymbol{x}, \boldsymbol{t} &\geq 0
\end{aligned}
$$

is the maximum tableau

$$
\begin{array}{|cccc|c|l}
x_1 & x_2 & \dots & x_n & -1 & \\
\hline
-a_{11} & -a_{12} & \dots & -a_{1n} & -b_1 & = -t_1 \\
-a_{21} & -a_{22} & \dots & -a_{2n} & -b_2 & = -t_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{m1} & -a_{m2} & \dots & -a_{mn} & -b_m & = -t_m \\
\hline
-c_1 & -c_n & \dots & -c_n & -d & = -g \\
\hline
\end{array}
\qquad
\begin{aligned}
\max \quad -\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + d &= -g(x) \\
-A\boldsymbol{x} - (-\boldsymbol{b}) &= -\boldsymbol{t} \\
\boldsymbol{x}, \boldsymbol{t} &\geq 0
\end{aligned}
$$

and vice versa.

Negative Transposition

Definition (Negative Transposition)

The negative transposition of the minimum tableau

$$
\begin{array}{r|cccc|c}
x_1 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
x_2 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
x_n & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\hline
-1 & b_1 & b_2 & \dots & b_m & d \\
\hline
\multicolumn{1}{r}{} & = t_1 & = t_2 & \dots & = t_m & \multicolumn{1}{c}{= g}
\end{array}
\qquad
\begin{aligned}
\min \ \boldsymbol{c}^\intercal \boldsymbol{x} - d &= g(\boldsymbol{x}) \\
A\boldsymbol{x} - \boldsymbol{b} &= \boldsymbol{t} \\
\boldsymbol{x}, \boldsymbol{t} &\geq 0
\end{aligned}
$$

is the maximum tableau

$$
\begin{array}{|cccc|c|l}
\multicolumn{1}{c}{x_1} & x_2 & \dots & \multicolumn{1}{c}{x_n} & \multicolumn{1}{c}{-1} & \\
\hline
-a_{11} & -a_{12} & \dots & -a_{1n} & -b_1 & = -t_1 \\
-a_{21} & -a_{22} & \dots & -a_{2n} & -b_2 & = -t_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_{m1} & -a_{m2} & \dots & -a_{mn} & -b_m & = -t_m \\
\hline
-c_1 & -c_n & \dots & -c_n & -d & = -g \\
\hline
\end{array}
\qquad
\begin{aligned}
\max \quad -\boldsymbol{c}^\intercal \boldsymbol{x} + d &= -g(x) \\
A\boldsymbol{x} - \boldsymbol{b} &= \boldsymbol{t} \\
\boldsymbol{x}, \boldsymbol{t} &\geq 0
\end{aligned}
$$

and vice versa.

Method 1: (See Chapter 2, §7 P. 54)

$$
\begin{array}{r|cccc|c|}
\cline{2-6}
 & a_{11} & a_{21} & \dots & a_{m1} & c_1 \\
 & a_{12} & a_{22} & \dots & a_{m2} & c_2 \\
\text{(ind var's)} & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\
\cline{2-6}
-1 & b_1 & b_2 & \dots & b_m & d \\
\cline{2-6}
\multicolumn{1}{r}{} & \multicolumn{4}{c}{=\text{(dep var's)}} & \multicolumn{1}{c}{=g}
\end{array}
$$

1. We have minimum Tucker tableau.
2. Take the negative transposition of the tableaux to obtain a maximum tableau.
3. Apply SA for maximum tableaux.
4. $\min g = -\max(-g)$.

Apply the simplex algorithm to the minimum tableau

x_1	20	25	300
x_2	40	20	500
-1	1000	800	0
	$= t_1$	$= t_2$	$= g$