MACM 201 - Discrete Mathematics

4. Counting in graphs

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Bipartite graphs

Definition

A *bipartition* of a graph $G = (V, E)$ is a pair (V_1, V_2) of subsets of V satisfying:

(1) $V_1 \cap V_2 = \emptyset$

(2) $V_1 \cup V_2 = V$

(3) every edge of G is incident with one vertex in V_1 and one vertex in V_2.

If a graph G has a bipartition, we call it *bipartite*.

Example. Draw the bipartite graph given by $V_1 = \{1, 2, 3\}$, $V_2 = \{a, b, c, d\}$, and $E = \big\{\{1, b\}, \{1, d\}, \{2, b\}, \{2, c\}, \{2, d\}, \{3, d\}\big\}$.

Definition

For nonnegative integers n_1, n_2 we define the *complete bipartite graph* K_{n_1,n_2} to be a bipartite graph with bipartition (V_1, V_2) where $|V_i| = n_i$ for $i = 1, 2$ and $E = \{\{v_1, v_2\} \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}$.

Problem 1. How many graphs have vertex set $V = \{1, 2, \ldots, n\}$?

Problem 2. How many graphs have vertex set $V = \{1, 2, \ldots, n\}$ and m edges?

Problem 3. Let V_1, V_2 be disjoint sets with $|V_1| = n_1$ and $|V_2| = n_2$. How many graphs have bipartition (V_1, V_2)?

Problem 4. Let V_1, V_2 be disjoint sets with $|V_1| = n_1$ and $|V_2| = n_2$. How many graphs have bipartition (V_1, V_2) and exactly m edges?

Walks

Definition

A *walk* in a graph $G = (V, E)$ from v_1 to v_r of *length* r is a sequence $v_1, e_1, v_2, e_2, \ldots, e_r, v_{r+1}$ satisfying:

(1) $v_i \in V$ for $1 \leq i \leq r+1$

(2) $e_i \in E$ for $1 \leq i \leq r$.

(2) $e_i = \{v_i, v_{i+1}\}$ for $1 \leq i \leq r$.

Note that the length of a walk is the number of edges.

Definition

A graph $G = (V, E)$ is *connected* if for every $x, y \in V$ there is a walk from x to y.

Problem 1. How many walks in K_n have length r?

Problem 2. How many walks in K_{n_1,n_2} have length r?

Definition

Let $G = (V, E)$ be a graph. A *subgraph* of G is a graph $G' = (V', E')$ satisfying:

(1) $V' \subseteq V$

(2) $E' \subseteq E$

Note: if $\{u, v\} \in E'$ then we must have $u, v \in V'$ since G' is a graph. If $V' = V$ then we call G' a *spanning* subgraph of G

Problem. How many spanning subgraphs of K_{n_1,n_2} have exactly m edges?

Have we seen this problem before?

Problem. Let $G = (V, E)$ be a graph with m edges. How many spanning subgraphs of G have exactly m' edges?

Induced subgraphs

Definition

Let $G = (V, E)$ be a graph and let $V' \subseteq V$. The subgraph of G *induced* by V' is the subgraph $G' = (V', E')$ where

$$E' = \{\{u, v\} \in E \mid u, v \in V'\}$$

Example.

Problem. If $G = (V, E)$ is a graph with $|V| = n$, how many induced subgraphs does G have?

Paths and cycles

Definition

if $\left\{ \begin{array}{c} P \\ C \end{array} \right\}$ is a subgraph of G that is a $\left\{ \begin{array}{c} \text{path} \\ \text{cycle} \end{array} \right\}$ we call $\left\{ \begin{array}{c} P \text{ a path} \\ C \text{ a cycle} \end{array} \right\}$ of G.

Note: Paths and cycles are subgraphs of G; walks are sequences of vertices and edges!

Example

Problem 1. How many k-vertex paths does the graph K_n have?

Problem 2. How many k-vertex paths does the graph K_{n_1,n_2} have?