MACM 201 - Discrete Mathematics

Graph Theory 5 - planarity

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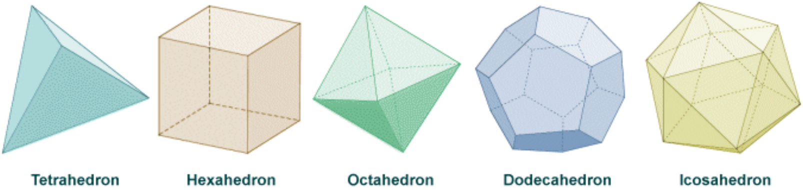
Planar graphs

Definition

A graph G is **planar** if G has a drawing (in the plane) so that the edges intersect only at the endpoints. Such a drawing is called an **embedding** of G in the plane.

Examples

Platonic solids



Two nonplanar graphs

Observation

The graph $K_{3,3}$ is not planar.

Proof sketch

Observation

The graph K_5 is not planar.

Proof sketch

Subdivision

Definition

Let e be an edge of G incident with the vertices v_1, v_2. To **subdivide** e is to delete the edge e, add a new vertex w and then add two new edges, e_1 incident to v_1, w and e_2 incident to v_2, w.

Example:

Definition

Let H be a multigraph. Any multigraph obtained from H by a sequence of subdivisions is called a **subdivision** of H.

Example:

Subdivisions and planarity

Observation

If H is a subdivision of the graph G, then H is planar if and only if G is planar.

Proof (sketch).

Note

This means that every subdivision of $K_{3,3}$ and every subdivision of K_5 is nonplanar.

Characterizing planarity

Definition

Let G and H be multigraphs. We say that G **contains a subdivision** of H if there exists a subgraph of G isomorphic to some subdivision of H.

Theorem (Kuratowski-Wagner)

A multigraph G is planar if and only if G does not contain a subdivision of $K_{3,3}$ or a subdivision of K_5.

Proof of "only if" direction.

Faces

Definition

Let G be a planar graph embedded in the plane. The embedding breaks the plane into connected regions called **faces**. There is one unbounded face called the **infinite face**, all other faces are **internal faces**.

Example

Definition

Every face in an embedding of a graph in the plane has vertices and edges on its boundary. They form a closed walk called a **facial walk**

Euler's formula

Theorem

If $G = (V, E)$ is a connected multigraph embedded in the plane and F is the set of faces, then

$$|V| - |E| + |F| = 2.$$

In particular, all embeddings of a planar graph have the same number of faces.

Proof.

Face degrees

Definition

Let $G = (V, E)$ be a multigraph embedded in the plane. We define the **degree** of a face f of this embedding to be the number of edges in a facial walk of f. We denote this by $\deg(f)$.

Example

Summing face degrees

Theorem

If $G = (V, E)$ is a multigraph embedded in the plane with faces $f_1, \ldots, f_k$, then

$$\sum_{i=1}^{k} \deg(f_i) = 2|E|.$$

Proof.

Bounding the number of edges 1

Theorem

If $G = (V, E)$ is a connected planar graph with no loops or parallel edges and $|V| \geq 3$, then

$$|E| \leq 3|V| - 6.$$

Proof.

Corollary

The graph K_5 is not planar.

Bounding the number of edges 2

Theorem

If $G = (V, E)$ is a connected planar graph with no cycle of length 3 or less and $|V| \geq 3$, then

$$|E| \leq 2|V| - 4.$$

Proof.

Corollary

The graph $K_{3,3}$ is not planar.

Duality

Definition

Let G be a multigraph embedded in the plane. To construct a **dual** multigraph G^*, put one vertex of G^* in each face of G, then for each edge $e \in E(G)$, if e lies on the boundary of faces f and f' (in the embedding of G), make an edge e^* in the dual graph G^* between the vertices corresponding to f and f' (this may be done so that e^* crosses e and G^* also ends up embedded in the plane).

Examples

Features of duals

Note

(1) Duals only exist for embedded planar graphs.

(2) If G^* is a dual of G, then G is a dual of G^*!

(3) The degree of a vertex in G^* is the degree of the corresponding face of G, and vice-versa.