MATH 308 D200, Fall 2019

4. Extreme point method for linear programming

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

Dr. Masood Masjoody

SFU Burnaby

Definition (norm of the vector)

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The *norm* of $\boldsymbol{x}$, denoted by $\|x\|$, is

$$\|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \ .$$

- usual Euclidean distance of $\boldsymbol{x}$ from the origin

Definition (closed ball)

Let $r \geqslant 0$. The set of points $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that $\|\boldsymbol{x}\| \leqslant r$ is said to be the *closed ball of radius* r *centred at the origin*. We denote it by $\mathbb{B}^n(r)$, thus

$$\mathbb{B}^n(r) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leqslant r\} \ .$$

Definition (bounded set)

A set $S \subseteq \mathbb{R}^n$ is said to be *bounded* if there exists $r \geqslant 0$ such that

$$S \subseteq \mathbb{B}^n(r) \ .$$

Theorem (bounded constraint set)

If a constraint set S of a canonical maximization or a canonical minimization linear programming problem is bounded, then the maximum or minimum value of the objective function is attained at an extreme point of S.

Theorem (bounded objective function)

If the constraint set S of a canonical maximization (respectively minimization) linear programming problem is unbounded and the objective function f is bounded above (below) on S, then the maximum (minimum) value of the objective function is attained at an extreme point of S.

Definition (unbounded LP problem)

A canonical maximization (minimization) linear programming problem is said to be *unbounded* if its objective function is not bounded above (below) on a constraint set S.

Converting an LP into Canonical Maximization LP

Every LP can be converted to a canonical maximization LP. Here are the steps.

a) If needed, change "Minimize $g(x_1, x_2, \ldots, x_n)$" into "Maximize $-g(x_1, x_2, \ldots, x_n)$". The maximal value of $-g(x_1, x_2, \ldots, x_n)$ is the negative of the minimal value of $g(x_1, x_2, \ldots, x_n)$.

b) Change every main constraint of the form $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i$ into $-a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq -b_i$.

c) Replace every main constraint of the form $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$ into two constraints

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$
$$-a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq -b_i.$$

d) For every variable, say x, for which is **not** constrained to be non-negative, replace every instance of x in all of the constraints and in the objective function by the expression $(x^+ - x^-)$, where x^+ and x^- are new variables, and add the two new non-negativity constraints.

$$x^+, x^- \geq 0.$$

For example, the constraint $-3x + 2y \leq 4$ would be replaced by $-3x^+ + 3x^- + 2y \leq 4$.

Geometric method for LP ("Extreme Point Method")

- Convert a problem to canonical form (for now, always convert to a canonical maximization LP)

Is the constraint set S bounded?

- If yes: Then find all the extreme points $\boldsymbol{x}$ by solving systems of n equations in n variables. (at most $\binom{m+n}{n}$ extreme point candidates; some may be infeasible).
  - Perhaps $S = \emptyset$: The LP is infeasible.
  - Evaluate $f(\boldsymbol{x})$ at each extreme point, and select and output an optimum one.
- If no: Is the LP is unbounded?
  - If unbounded, then report this fact: there is no optimum solution.
  - If not, then evaluate $f(\boldsymbol{x})$ at each extreme point, and output an optimum one.

Example 1

Maximize the value $f(x,y,z) = 3x + 2y - 4z$ subject to constraints

(1) $y - 2z \leqslant 3$
(2) $x - 2y + 4z \leqslant -7$
(3) $x \geqslant 0$
(4) $y \geqslant 0$
(5) $z \geqslant 0$

Example 2

Maximize the value $f(x,y,z) = 3x + 2y - 4z$ subject to constraints

(1) $y - 2z \leqslant 3$
(2) $-x + y \leqslant 2$
(3) $x \geqslant 0$
(4) $y \geqslant 0$
(5) $z \geqslant 0$

Example 3

Maximize the value $f(x, y, z) = -3x + 2y - 4z$ subject to constraints

(1) $y - 2z \leqslant 3$
(2) $-x + y \leqslant 2$
(3) $x \geqslant 0$
(4) $y \geqslant 0$
(5) $z \geqslant 0$

3D example

Maximize function $f(x, y, z) = x - 2y - z$ subject to constraints

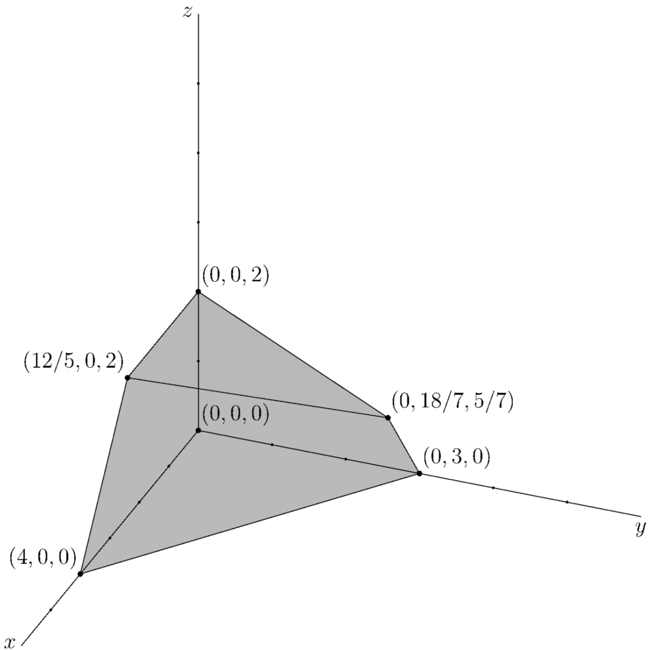
(1) $3x + 4y + \frac{12}{5}z \leqslant 12$

(2) $2y + 4z \leqslant 8$

(3) $x \geqslant 0$

(4) $y \geqslant 0$

(5) $z \geqslant 0$


z
(0, 0, 2)
(12/5, 0, 2)
(0, 0, 0)
(0, 18/7, 5/7)
(0, 3, 0)
y
(4, 0, 0)
x