MATH 308 D200, Fall 2019

23. The transportation algorithm

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

Dr. Masood Masjoody

SFU Burnaby

The goal is to convert a **feasible basic transportation tableau**, such as is produced by VAM, into an **optimal basic transportation tableau**. The **transportation algorithm** of the textbook mimics Phase 2 of the dual simplex algorithm.

How can we tell if a transportation tableau is optimal?

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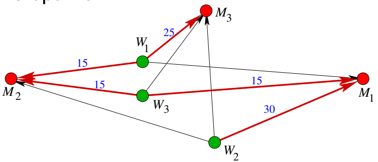

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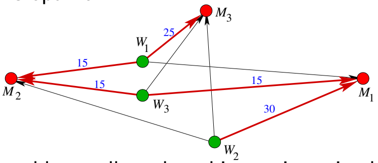

The **dual slack variables**, s_{ij}, of the simplex tableau tell us that this one is optimal.

x_{11}	x_{23}	x_{22}	x_{33}	-1	
-1	1	1	1	15	$= -x_{32}$
0	1	1	0	30	$= -x_{21}$
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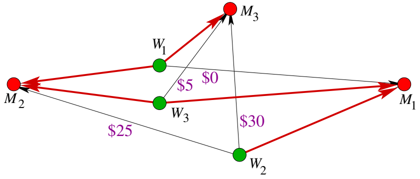

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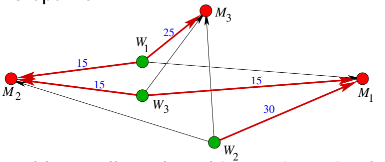

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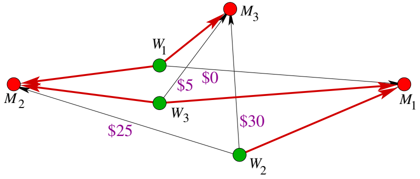
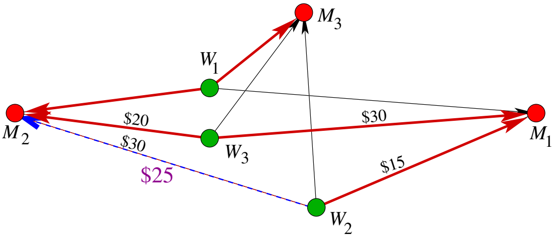


We need to compute s_{ij} for every non-basic edge, and check whether $s_{ij} \geq 0$.

Computing dual slack values

Think of s_{ij} as a price differential or reduced cost: it is the extra cost of rerouting an item through edge ij instead of through the basic edges in the tree.



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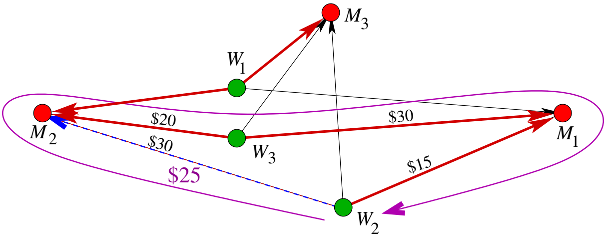

To compute s_{22}:

Naive method: Find the cycle using edge (W_2, M_2) and some tree edges. Then take an alternating sum of their costs.

$$s_{22} = 30 - 20 + 30 - 15 = \$25.$$

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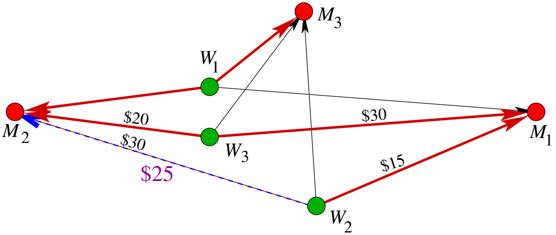

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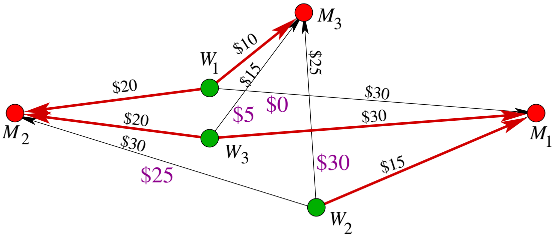

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Repeat this for every nonbasic edge.

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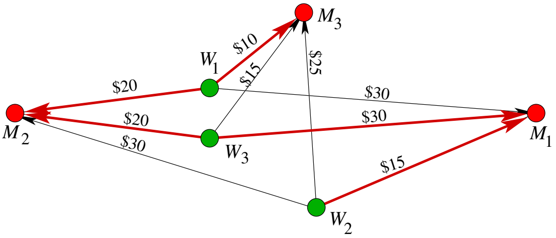

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Better method: Compute *node prices* $a_1, a_2, a_3, b_1, b_3, b_3$ for every node:

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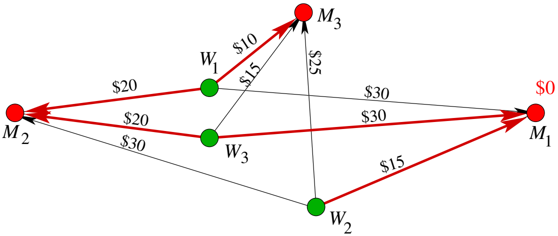

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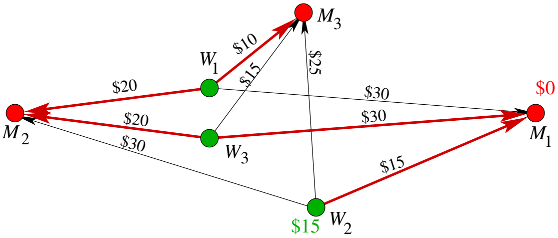

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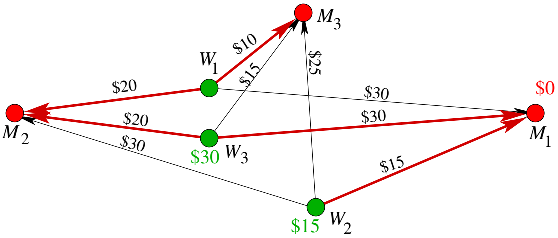

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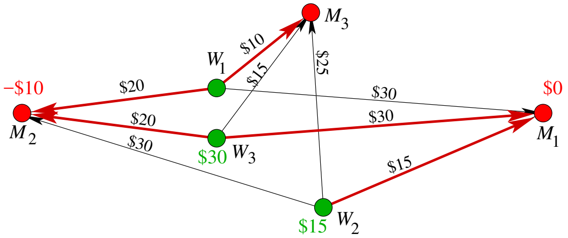

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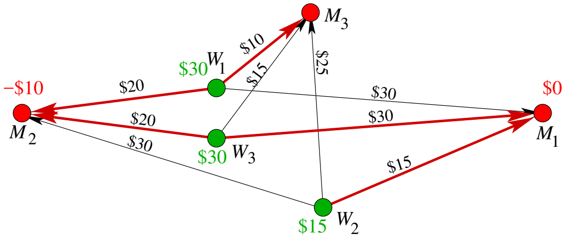

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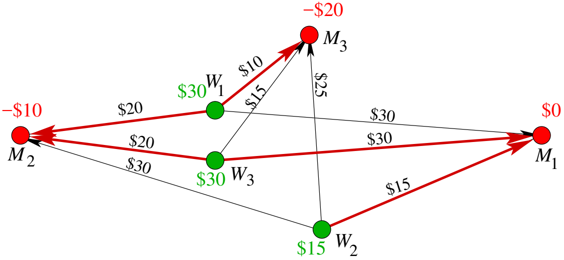

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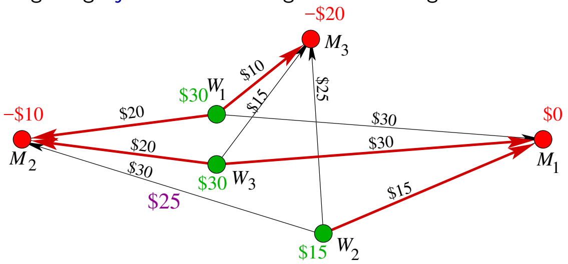

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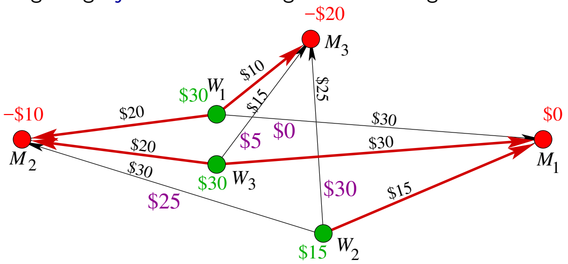

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Big Advantage: Fewer calculations for each nonbasic edge.
Small Disadvantage: The reduced costs a_i, b_j must be updated after each pivot.

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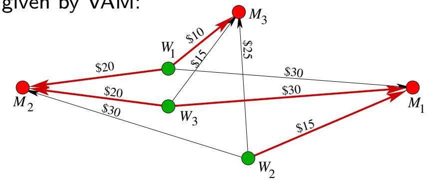

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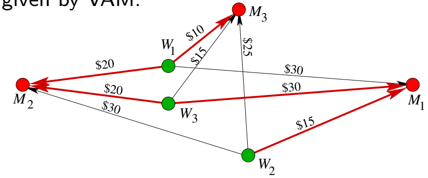

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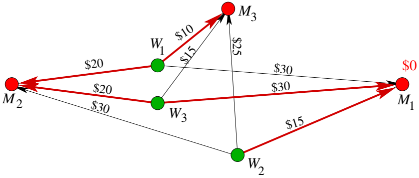

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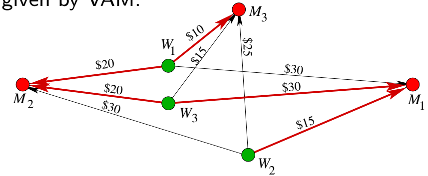

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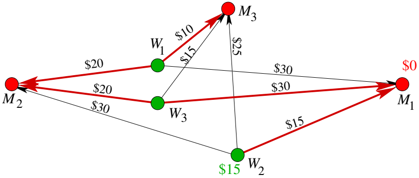

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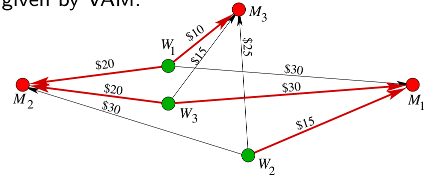

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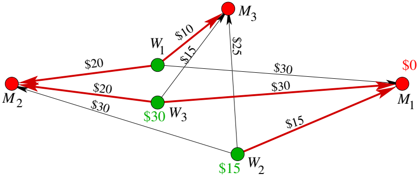

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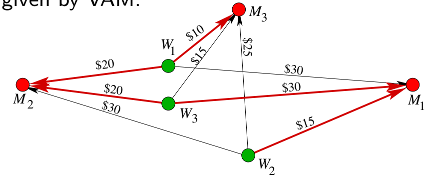

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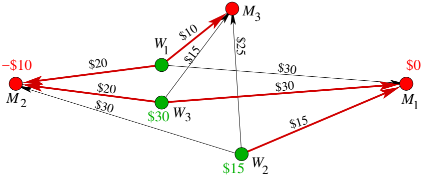

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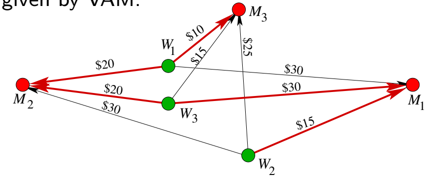

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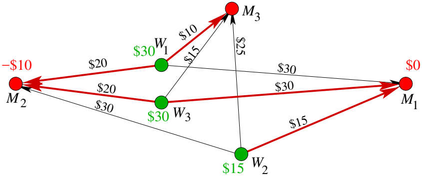

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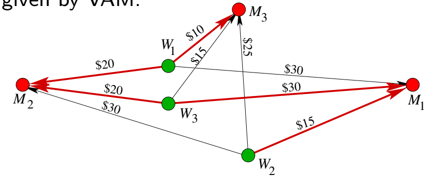

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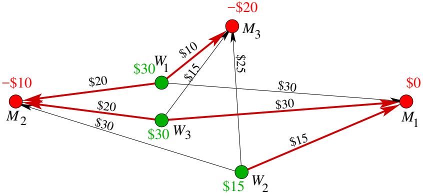

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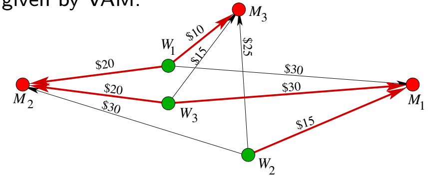

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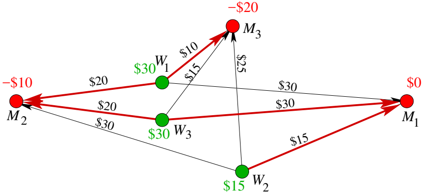
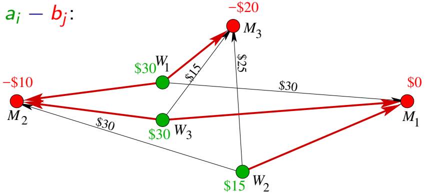


Find reduced prices, using $s_{ij} = c_{ij} - a_i - b_j$:

	0	-10	-20
30	30	0	0
15	0	30	25
30	0	0	15



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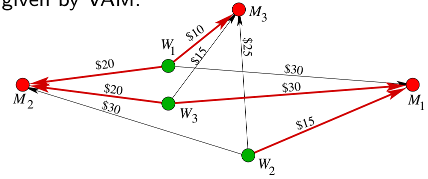

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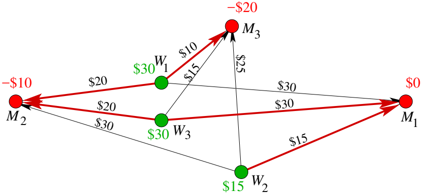

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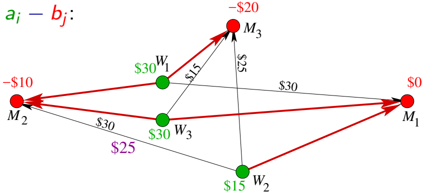
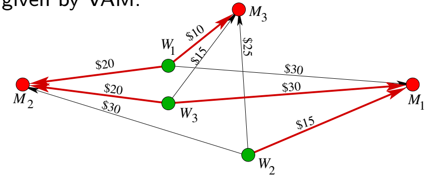


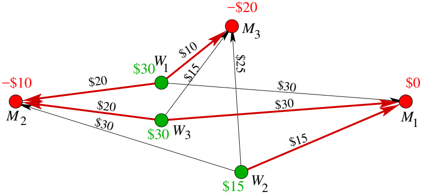
Using the feasible transportation tableau given by VAM:

	M_1	M_2	M_3	
W_1	30	20^{15}	10^{25}	40
W_2	15^{30}	30	25	30
W_3	30^{15}	20^{15}	15	30
	45	30	25	



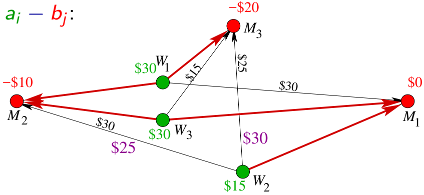
Find node prices, using $c_{ij} = a_i + b_j$:

	0	-10	-20
30	30	20	10
15	15	30	25
30	30	20	15



Find reduced prices, using $s_{ij} = c_{ij} - a_i - b_j$:

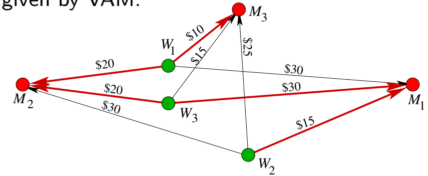
	0	-10	-20
30	30	0	0
15	0	25	30
30	0	0	15



Transportation Algorithm

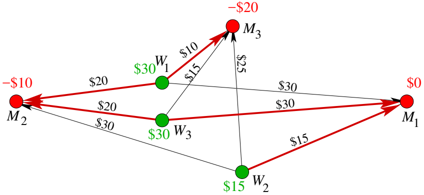
Using the feasible transportation tableau given by VAM:

	M_1	M_2	M_3	
W_1	30	20^{15}	10^{25}	40
W_2	15^{30}	30	25	30
W_3	30^{15}	20^{15}	15	30
	45	30	25	



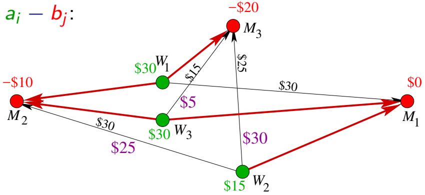
Find node prices, using $c_{ij} = a_i + b_j$:

	0	-10	-20
30	30	20	10
15	15	30	25
30	30	20	15



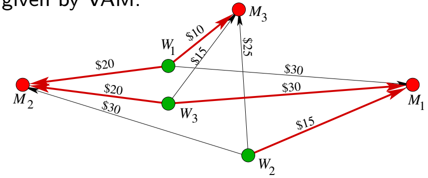
Find reduced prices, using $s_{ij} = c_{ij} - a_i - b_j$:

	0	-10	-20
30	30	0	0
15	0	25	30
30	0	0	5



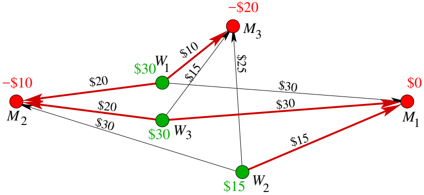
Using the feasible transportation tableau given by VAM:

	M_1	M_2	M_3	
W_1	30	20^{15}	10^{25}	40
W_2	15^{30}	30	25	30
W_3	30^{15}	20^{15}	15	30
	45	30	25	



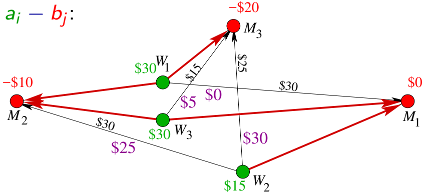
Find node prices, using $c_{ij} = a_i + b_j$:

	0	-10	-20
30	30	20	10
15	15	30	25
30	30	20	15



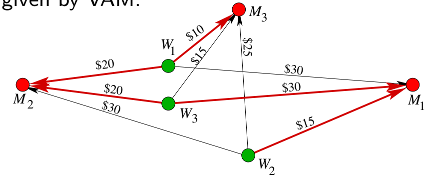
Find reduced prices, using $s_{ij} = c_{ij} - a_i - b_j$:

	0	-10	-20
30	0	0	0
15	0	25	30
30	0	0	5



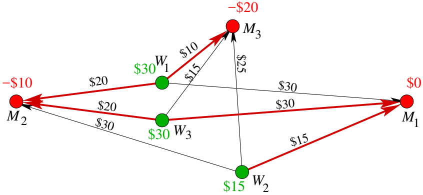
Using the feasible transportation tableau given by VAM:

	M_1	M_2	M_3	
W_1	30	20^{15}	10^{25}	40
W_2	15^{30}	30	25	30
W_3	30^{15}	20^{15}	15	30
	45	30	25	



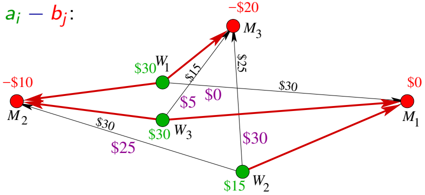
Find node prices, using $c_{ij} = a_i + b_j$:

	0	-10	-20
30	30	20	10
15	15	30	25
30	30	20	15



Find reduced prices, using $s_{ij} = c_{ij} - a_i - b_j$:

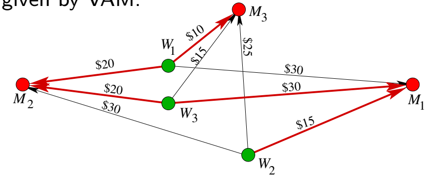
	0	-10	-20
30	0	0	0
15	0	25	30
30	0	0	5



This is **optimal**, so **STOP**.

Using the feasible transportation tableau given by VAM:

	M_1	M_2	M_3	
W_1	30	20 (15)	10 (25)	40
W_2	15 (30)	30	25	30
W_3	30 (15)	20 (15)	15	30
	45	30	25	

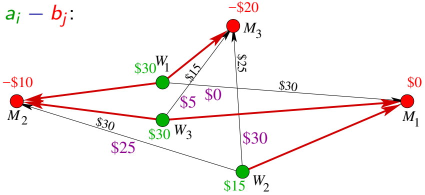


The initial node and price can be chosen, since $c_{ij} - (a_i + t) - (b_j + t) = c_{ij} - a_i - b_j$.

15		
30	20	10
15	30	25
30	20	15

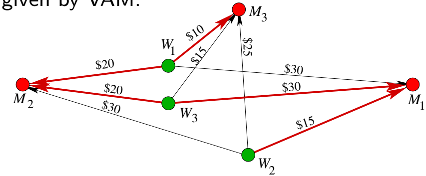
Find reduced prices, using $s_{ij} = c_{ij} - a_i - b_j$:

	0	-10	-20
30	0	0	0
15	0	25	30
30	0	0	5



Using the feasible transportation tableau given by VAM:

	M_1	M_2	M_3	
W_1	30	20^{15}	10^{25}	40
W_2	15^{30}	30	25	30
W_3	30^{15}	20^{15}	15	30
	45	30	25	

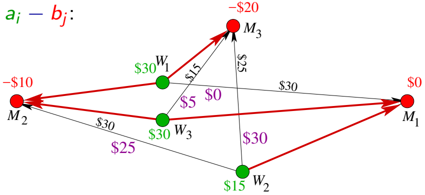


The initial node and price can be chosen, since $c_{ij} - (a_i + t) - (b_j + t) = c_{ij} - a_i - b_j$.

	15	5	-5
15	30	20	10
0	15	30	25
15	30	20	15

Find reduced prices, using $s_{ij} = c_{ij} - a_i - b_j$:

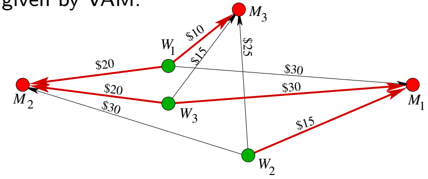
	0	-10	-20
30	0	0	0
15	0	25	30
30	0	0	5



Transportation Algorithm

Using the feasible transportation tableau given by VAM:

	M_1	M_2	M_3	
W_1	30	20^{15}	10^{25}	40
W_2	15^{30}	30	25	30
W_3	30^{15}	20^{15}	15	30
	45	30	25	

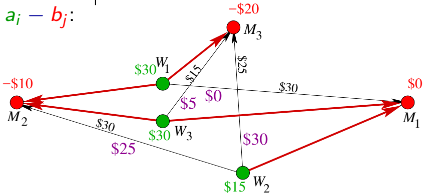


The initial node and price can be chosen, since $c_{ij} - (a_i + t) - (b_j + t) = c_{ij} - a_i - b_j$.

	$0+15$	$-10+15$	$-20+15$
$30-15$	30	20	10
$15-15$	15	30	25
$30-15$	30	20	15

Find reduced prices, using $s_{ij} = c_{ij} - a_i - b_j$:

	0	-10	-20
30	0	0	0
15	0	25	30
30	0	0	5



Transportation Algorithm

Recall a_i, b_j are just dual variables for the original LP:

$$\text{(P) min} \quad C = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

$$\text{s. t.} \quad \sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \ldots, m)$$

$$\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \ldots, n)$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

Recall a_i, b_j are just dual variables for the original LP:

$$
\begin{aligned}
\text{(P) min} \quad & C = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij} \\
\text{s. t.} \quad & \sum_{j=1}^{n} x_{ij} = s_i, \ \ (i = 1, 2, \ldots, m) \quad (a_i) \\
& \sum_{i=1}^{m} x_{ij} = d_j, \ \ (j = 1, 2, \ldots, n) \quad (b_j) \\
& x_{ij} \geqslant 0, \quad \text{for all } i, j
\end{aligned}
$$

Recall a_i, b_j are just dual variables for the original LP:

(P) min $\quad C = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} c_{ij}$

s. t.

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \dots, m) \quad (a_i)$$

$$\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \dots, n) \quad (b_j)$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

(D) max $\quad P = \sum_{i=1}^{m} s_i a_i + \sum_{j=1}^{n} d_j b_j$

s. t.

$$a_i + b_j \leq c_{ij} \ \text{ for all } i, j \quad (x_{ij})$$

a_i, b_j unrestricted

Definition (Cycle in a Tableau)

A *cycle* C in a balanced transportation tableau T is a collection of cells of T such that each row and each column of T contains exactly zero or two cells of C.

Only horizontal and vertical movement is allowed to connect the cells...

The Transportation Algorithm TA

0. Given an initial balanced transportation tableau.

1. Apply VAM to obtain a basic feasible solution and a corresponding basis.

2. Let $b_1 = 0$ (or any number). Determine $a_1, a_2, \ldots, a_m$ and $b_2, b_3, \ldots, b_n$ uniquely such that $a_i + b_j = c_{ij}$ for all **basis** cells.

3. For each i, j, replace cell costs c_{ij} by reduced costs $s_{ij} = c_{ij} - a_i - b_j$; the textbook calls s_{ij} the "new cell costs c_{ij}".)

4. If $s_{ij} \geqslant 0$ for all i, j, then replace all cells with their original costs c_{ij}; the current basic feasible solution is optimal. Otherwise, continue.

5. Choose $s_{ij} < 0$. To break ties use Bland's anti-cycling rule: Choose $s_{ij} < 0$ with smallest i and with respect to this with the smallest j (the Northwest-most negative cell). Label the "getter" cell with ($\square^+$). Find the unique cycle C determined by the getter cell and some of the basis cells. Label cells of C starting from $\square$ alternately "getter"$(+)$ and "giver"$(-)$. Choose the "giver"$(-)$ cell associated with the smallest flow of goods; break ties arbitrarily.

6. Adjust the flows x_{ij}: Add the squared cell of **Step 5.** to the basis, i.e., circle it in a new tableau. Remove the chosen "giver" from the basis, i.e., do not circle it in a new tableau. Add the amount of goods of this "giver" to amount of goods of all "getters" in C and subtract from the amount of goods of all "givers" in C. Go to **Step 2.**

Apply TA to the following (balanced) transportation problem

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

2^{20}	1	2^{20}	40
9^{10}	4^{50}	7	60
1^{10}	2	9	10
40	50	20	110

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

0			
2^{20}	1	2^{20}	40
9^{10}	4^{50}	7	60
1^{10}	2	9	10
40	50	20	110

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0			
2	2^{20}	1	2^{20}	40
	9^{10}	4^{50}	7	60
	1^{10}	2	9	10
	40	50	20	110

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0			
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
	1^{10}	2	9	10
	40	50	20	110

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0			
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	−5		
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	−5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$. Compute reduced costs $s_{ij} = c_{ij} - a_i - b_j$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	−5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	−5	0
2	$\bigcirc^{20}$		$\bigcirc^{20}$
9	$\bigcirc^{10}$	$\bigcirc^{50}$	
1	$\bigcirc^{10}$		

Reduced costs s_{ij}

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$. Compute reduced costs $s_{ij} = c_{ij} - a_i - b_j$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	-5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	-5	0
2	0^{20}		0^{20}
9	0^{10}	0^{50}	
1	0^{10}		

Reduced costs s_{ij}

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$. Compute reduced costs $s_{ij} = c_{ij} - a_i - b_j$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	-5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	-5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	
1	0^{10}		

Reduced costs s_{ij}

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$. Compute reduced costs $s_{ij} = c_{ij} - a_i - b_j$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	−5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	−5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	−2
1	0^{10}		

Reduced costs s_{ij}

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$. Compute reduced costs $s_{ij} = c_{ij} - a_i - b_j$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	−5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	−5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	−2
1	0^{10}	6	

Reduced costs s_{ij}

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$. Compute reduced costs $s_{ij} = c_{ij} - a_i - b_j$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	−5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	−5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	−2
1	0^{10}	6	8

Reduced costs s_{ij}

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$. Compute reduced costs $s_{ij} = c_{ij} - a_i - b_j$.
3. Not yet optimal. Select a "getter" cell $s_{ij} < 0$ to enter basis, and mark it with $\boxed{}^{+}$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	−5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	−5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	−2
1	0^{10}	6	8

Reduced costs s_{ij}

$\rightarrow$

	0	−5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	-2
1	0^{10}	6	8

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$. Compute reduced costs $s_{ij} = c_{ij} - a_i - b_j$.
3. Not yet optimal. Select a "getter" cell $s_{ij} < 0$ to enter basis, and mark it with $\boxed{}^{+}$.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	-5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	-5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	-2
1	0^{10}	6	8

Reduced costs s_{ij}

$\rightarrow$

	0	-5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	$\boxed{-2}$
1	0^{10}	6	8

Apply TA to the following (balanced) transportation problem

1. Find a BFS by applying VAM. Costs are c_{ij}. Supply/Demands are blue.
2. Compute node values, start with, say, $b_1 = 0$. Compute reduced costs $s_{ij} = c_{ij} - a_i - b_j$.
3. Not yet optimal. Select a "getter" cell $s_{ij} < 0$ to enter basis, and mark it with $\boxed{}^{+}$.
4. Find the cycle of basic cells containing the getter cell.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	−5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	−5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	-2
1	0^{10}	6	8

Reduced costs s_{ij}

$\rightarrow$

	0	−5	0
2	0^{20}	4	0^{20}
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1	0^{10}	6	8

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Reduced costs s_{ij}

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$\rightarrow$

	0	-5	0
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$\rightarrow$

	0	-5	0
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1	0^{10}	6	8

Reduced costs s_{ij}

$\rightarrow$

	0	-5	0
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$\rightarrow$

0		
2^{30}	1	2^{10}
9	4^{50}	7^{10}
1^{10}	2	9

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Reduced costs s_{ij}

$\rightarrow$

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$\rightarrow$

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$\rightarrow$

	0	-5	0
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1	0^{10}	6	8

Reduced costs s_{ij}

$\rightarrow$

	0	-5	0
2	0^{20+10}	4	0^{20-10}
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$\rightarrow$

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9	4	7	60
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$\xrightarrow{\text{VAM}}$

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	40	50	20	110

$\rightarrow$

	0	-5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	-2
1	0^{10}	6	8

Reduced costs s_{ij}

$\rightarrow$

	0	-5	0
2	0^{20+10}	4	0^{20-10}
9	0^{10-10}	0^{50}	$\boxed{-2}^{+10}$
1	0^{10}	6	8

$\rightarrow$

	0	-3	0
2	2^{30}	1	2^{10}
7	9	4^{50}	7^{10}
1	1^{10}	2	9

Apply TA to the following (balanced) transportation problem

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9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	-5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	-5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	-2
1	0^{10}	6	8

Reduced costs s_{ij}

$\rightarrow$

	0	-5	0
2	0^{20+10}	4	0^{20-10}
9	0^{10-10}	0^{50}	$\boxed{-2}^{+10}$
1	0^{10}	6	8

$\rightarrow$

	0	-3	0
2	2^{30}	1	2^{10}
7	9	4^{50}	7^{10}
1	1^{10}	2	9

$\rightarrow$

	0	-3	0
2	0^{30}	2	0^{10}
7	2	0^{50}	0^{10}
1	0^{10}	4	8

Apply TA to the following (balanced) transportation problem

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6. The giver with the smallest flow will leave the basis. Adjust the flows.
7. Update node numbers and reduced costs. Some of them will change by ± 2.
8. All no negative reduced costs. Flow is optimal.

2	1	2	40
9	4	7	60
1	2	9	10
40	50	20	110

$\xrightarrow{\text{VAM}}$

	0	−5	0	
2	2^{20}	1	2^{20}	40
9	9^{10}	4^{50}	7	60
1	1^{10}	2	9	10
	40	50	20	110

$\rightarrow$

	0	−5	0
2	0^{20}	4	0^{20}
9	0^{10}	0^{50}	-2
1	0^{10}	6	8

Reduced costs s_{ij}

$\rightarrow$

	0	−5	0
2	0^{20+10}	4	0^{20-10}
9	0^{10-10}	0^{50}	$\boxed{-2}^{+10}$
1	0^{10}	6	8

$\rightarrow$

	0	−3	0
2	2^{30}	1	2^{10}
7	9	4^{50}	7^{10}
1	1^{10}	2	9

Optimal

$\rightarrow$

	0	−3	0
2	0^{30}	2	0^{10}
7	2	0^{50}	0^{10}
1	0^{10}	4	8

Solve the following BTP

5	12	8	50	26
11	4	10	8	20
14	50	1	9	30
15	20	26	15	

5^{15}	12^{0}	8	50^{11}	26
11	4^{20}	10	8	20
14	50	1^{26}	9^{4}	30
15	20	26	15	

5	12	8	50	26
11	4	10	8	20
14	50	1	9	30
15	20	26	15	

Variations of the Transportation Problem

Negative costs

Maximization

Forbidden routes

Unbalanced Transportation

Negative costs

Add a big constant M, such as $M = -\min_{i,j} -c_{ij}$, to *all* the edge costs. Now solve the new problem. Notice the new objective function C' is just a shift of the old one C, so both problemas have the same optimal solutions (x_{ij}).

$$C' = \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij} + M)x_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}c_{ij} + M\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = C + M\sum_{1\leq i\leq m} s_i.$$

Maximization

Forbidden routes

Unbalanced Transportation

Negative costs
Add a big constant M, such as $M = -\min_{i,j} -c_{ij}$, to *all* the edge costs. Now solve the new problem. Notice the new objective function C' is just a shift of the old one C, so both problemas have the same optimal solutions (x_{ij}).

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Maximization
Convert to a minimization problem by replacing each cost $c'_{ij} = -c_{ij}$.

Forbidden routes

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Maximization

Convert to a minimization problem by replacing each cost $c'_{ij} = -c_{ij}$.

Forbidden routes

If the route from W_i to M_j is forbidden, then put a prohibitively high cost $c_{ij} = \infty$.

Unbalanced Transportation

Variations of the Transportation Problem

Negative costs

Add a big constant M, such as $M = -\min_{i,j} -c_{ij}$, to *all* the edge costs. Now solve the new problem. Notice the new objective function C' is just a shift of the old one C, so both problemas have the same optimal solutions (x_{ij}).

$$C' = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + M) x_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij} + M \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = C + M \sum_{1 \le i \le m} s_i .$$

Maximization

Convert to a minimization problem by replacing each cost $c'_{ij} = -c_{ij}$.

Forbidden routes

If the route from W_i to M_j is forbidden, then put a prohibitively high cost $c_{ij} = \infty$.

Unbalanced Transportation

If total supply exceeds demand

$$\sum_{1 \le i \le m} s_i > \sum_{1 \le j \le n} d_j,$$

then add a "dummy" market M_{n+1} with edge costs

$$c_{i,m+1} = 0 \quad \text{for } i = 1, 2, \ldots, m$$

and demand

$$d_{m+1} = \sum_{1 \le i \le m} s_i = \sum_{1 \le j \le n} d_j .$$

Unbalanced Transportation Problems

$$
\begin{array}{c|cccc|c}
 & M_1 & M_2 & \dots & M_n & \\
\hline
W_1 & c_{11} & c_{12} & \dots & c_{1n} & s_1 \\
W_2 & c_{21} & c_{22} & \dots & c_{2n} & s_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
W_m & c_{m1} & c_{m2} & \dots & c_{mn} & s_m \\
\hline
 & d_1 & d_2 & \dots & d_n & \displaystyle\sum_{i=1}^{m} s_i \neq \sum_{j=1}^{n} d_j
\end{array}
$$

Transform to balanced problem ...

Case I: demand exceeds supply

$$\sum_{i=1}^{m} s_i < \sum_{j=1}^{n} d_j$$

Solution: We introduce a fictitious warehouse W_{m+1} which supplies the excess demand.

- Set $c_{m+1,j} = 0$ for all $j = 1, 2, \ldots, n$.
- In reality we may use different costs—loss in sale, alternative supply, . . .
- Interpretation—demand of some markets is not fully satisfied.

2	1	2	40
9	4	7	60
1	2	9	10
50	60	30	

Case II: supply exceeds demand

$$\sum_{i=1}^{m} s_i > \sum_{j=1}^{n} d_j$$

Solution: We introduce a fictitious market M_{n+1} which demands the excess supply.

- Set $c_{i,n+1} = 0$ for all $i = 1, 2, \ldots, m$.
- In reality we may use different costs—spoilage costs, storage costs, ...
- Interpretation—goods "shipped" to the fictitious market retain in their respective warehouses.

2	1	2	50
9	4	7	70
1	2	9	20
40	50	20	