MACM 201 - Discrete Mathematics

Generating functions III - Recurrences

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Recurrences and generating functions

We first saw recurrences appear when counting families of strings and rooted trees.

We have since introduced the framework of generating functions.

(1) They have natural application as a framework for counting.

(2) Rational GF's have compact representations that are efficient to work with.

In fact, rational GF's are very closely related to recurrence relations. We will see how to use these GF's in a natural way to solve recurrences.

Using GF's to solve recurrences

Generating functions provide a framework for solving recurrences that has many advantages. It allows us to handle the homogeneous and non-homogeneous variants and unifies the treatment of the different types of roots to the characteristic equation.

Note

- *For solving recurrences of order 2, the methods we already have seen are easiest.*
- *For general algorithms to be used by a computer, generating functions are best.*

Method for using GF's to solve recurrences. Let $A(x) = a_0 + a_1 x + a_2 x^2 + \dots$ be a generating function where $a_0, a_1, a_2 \dots$ satisfies a recurrence.

(1) Use the recurrence to find a rational expression for the GF.
(2) Extract the coefficients from the rational expression (for this we factor the denominator, use partial fractions, and then apply a formula)

We will focus on (1) in this part of the notes since you have already seen how to extract coefficients from a rational GF.

Finding a rational expression (example)

Problem. Consider the infinite sequence given by

$$a_0 = 1 \qquad \text{and} \qquad a_n - 3a_{n-1} = n \quad \text{for } n \geq 1$$

Define the GF $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ and then use the recurrence to find a rational expression for $A(x)$.

Key: The recurrence relation is an infinite number of equations, but generating functions naturally encode this (if two GF's satisfy $A(x) = B(x)$, all coefficients match).

The recurrence relation is an infinite number of equations:

Multiply each equation by an appropriate power of x:

Add all equations together:

Finding a rational expression (example)

Now we express the equation in terms of $A(x)$

Then solve for $A(x)$.

Method

Let $a_0, a_1, a_2 \ldots$ be a sequence satisfying a recurrence. Here is how we find a rational expression for the GF $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

(1) Multiply the recurrence through by x^n (or similar)
(2) Sum up the terms from (1) to get an equation for GF's
(3) Rewrite your GF equation using $A(x)$
(4) Solve for $A(x)$.

Problem. Consider the sequence defined by

$$a_0 = 0 \qquad a_1 = 1 \qquad \text{and} \qquad a_n - 5a_{n-1} + 6a_{n-2} = 0 \quad \text{for } n \geq 2$$

Find a rational expression for $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$a_0 = 0 \qquad a_1 = 1 \qquad \text{and} \qquad a_n - 5a_{n-1} + 6a_{n-2} = 0 \quad \text{for } n \geq 2$$

Recurrences and generating functions: general case

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = g(n), \;\; n \geq 2$$

$$\left(c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2}\right) x^n = g(n) x^n$$

$$\sum_{n \geq 2} \left(c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2}\right) x^n = \sum_{n \geq 2} g(n) x^n$$

$$\left(\sum_{n \geq 2} c_n a_n x^n\right) + \left(\sum_{n \geq 2} c_{n-1} a_{n-1} x^n\right) + \left(\sum_{n \geq 2} c_{n-2} a_{n-2} x^n\right) = G(x)$$

$$c_n \left(\sum_{n \geq 2} a_n x^n\right) + c_{n-1} \left(\sum_{n \geq 2} a_{n-1} x^n\right) + c_{n-2} \left(\sum_{n \geq 2} a_{n-2} x^n\right) = G(x)$$

$$c_n \left(A(x) - a_0 - a_1 x\right) + c_{n-1} \left(\sum_{n \geq 2} a_{n-1} x^n\right) + c_{n-2} \left(\sum_{n \geq 2} a_{n-2} x^n\right) = G(x)$$

$$c_n \left(A(x) - a_0 - a_1 x\right) + x c_{n-1} \left(\sum_{n \geq 2} a_{n-1} x^{n-1}\right) + x^2 c_{n-2} \left(\sum_{n \geq 2} a_{n-2} x^{n-2}\right) = G(x)$$

Recurrences and generating functions: general case

$$c_n\left(A(x)-a_0-a_1x\right)+xc_{n-1}\left(\sum_{n\geq 1}a_nx^n\right)+x^2c_{n-2}\left(\sum_{n\geq 0}a_nx^n\right)=G(x)$$

$$c_n\left(A(x)-a_0-a_1x\right)+xc_{n-1}\left(A(x)-a_0\right)+x^2c_{n-2}A(x)=G(x)$$

$$A(x)\left(c_n+c_{n-1}x+c_{n-2}x^2\right)=G(x)+c_na_0+(c_na_1+c_{n-1}a_0)x$$

$$A(x)=\frac{G(x)+c_na_0+(c_na_1+c_{n-1}a_0)x}{(c_n+c_{n-1}x+c_{n-2}x^2)}$$

and so

$$a_n=[x^n]\frac{G(x)+c_na_0+(c_na_1+c_{n-1}a_0)x}{(c_n+c_{n-1}x+c_{n-2}x^2)}$$

Note

The key point here is that we can take a sequence satisfying a second order linear recurrence and construct a rational expression for the associated GF

Link with the characteristic polynomial

We can finally understand why the roots r_1 and r_2 of the characteristic polynomial play such a role in solving recurrences.

Fact

If

$$c_n r^2 + c_{n-1} r + c_{n-2} = (r - r_1)(r - r_2)$$

then divide by $1/r^2$ *on both sides and substitute* x *for* $1/r$ *and you obtain*

$$c_n + c_{n-1} x + c_{n-2} x^2 = (1 - r_1 x)(1 - r_2 x)$$

So for $r_1 \neq r_2$, $A(x)$ *can be rewritten as*

$$A(x) = \frac{G(x) + c_n a_0 + (c_n a_1 + c_{n-1} a_0)x}{(1 - r_1 x)(1 - r_2 x)} = \frac{A_1(x)}{(1 - r_1 x)} + \frac{A_2(x)}{(1 - r_2 x)}$$

for some $A_1(x)$ *and* $A_2(x)$ *(see the partial fraction decomposition part of notes).*

Fact

By the coefficient extraction techniques we see that

$$[x^n]\frac{1}{1 - \alpha x} = \alpha^n \qquad \textit{and} \qquad [x^n]\frac{1}{(1 - \alpha x)^2} = n\alpha^n$$

Link with the characteristic polynomial

Putting both facts together, we understand where the terms r_1^n and r_2^n stem from.

For the case where $r_1 = r_2$, we have

$$A(x) = \frac{G(x) + c_n a_0 + (c_n a_1 + c_{n-1} a_0)x}{(1 - r_1 x)^2} = \frac{A_1(x)}{(1 - r_1 x)} + \frac{A_2(x)}{(1 - r_1 x)^2}$$

and coefficient extraction explains us why we see the $n r_1^n$ factor appear.