MATH 308 D200, Fall 2019

8. Simplex algorithm for maximum basic feasible tableau

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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So Far We Know. . . ...

- ▷ How to describe a problem as a maximization LP problem.
- ▷ How to convert the problem to canonical form.
- ▷ How to convert the canonical form to canonical slack form (slack variables).
- ▷ How to write an initial Tucker tableau for the canonical problem.
- ▷ There is a one-to-one correspondence between maximum Tucker tableaux and basic solutions of the problem.
- ▷ How to transform Tucker tableaux using pivoting and go from one basic solution to another

Lemma

Basic solution represented by a Tucker tableau is feasible if and only if $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m \geqslant 0$.

Proof.

$$\begin{array}{c|cccc|c|l}
 & \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_n & -1 & \\
\hline
 & \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} & \tilde{b}_1 & \\
 & \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 & \\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \\
 & \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mn} & \tilde{b}_m & \\
\hline
 & \tilde{c}_1 & \tilde{c}_2 & \cdots & \tilde{c}_n & \tilde{d} & = f \\
\hline
\end{array}
\quad \to \quad
\begin{array}{c}
\tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \cdots + \tilde{a}_{1n}\tilde{x}_n \leqslant \tilde{b}_1 \\
\tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + \cdots + \tilde{a}_{2n}\tilde{x}_n \leqslant \tilde{b}_2 \\
\vdots \\
\tilde{a}_{m1}\tilde{x}_1 + \tilde{a}_{m2}\tilde{x}_2 + \cdots + \tilde{a}_{mn}\tilde{x}_n \leqslant \tilde{b}_m
\end{array}$$

In basic solution, $\tilde{x}_1 = \tilde{x}_2 = \cdots = \tilde{x}_n = 0$, and hence the system is consistent. □

SA for MBFT

Algorithm (SA for MBFT)

$$
\begin{array}{cccc|c|l}
 & \textit{(ind var's)} & & & -1 & \\
\hline
a_{11} & a_{12} & \dots & a_{1n} & b_1 & \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & = -\textit{(dep var's)} \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m & \\
\hline
c_1 & c_2 & \dots & c_n & d & = f \\
\hline
\end{array}
$$

1. *We have MBFT* ($b_1, b_2, \dots, b_m \geqslant 0$)
2. *If* $c_1, c_2, \dots, c_n \leqslant 0 \Longrightarrow$ **STOP**; *the current basic feasible solution is optimal.*
3. *Choose any* j *with* $c_j > 0$
4. *If* $a_{1j}, a_{2j}, \dots, a_{mj} \leqslant 0 \Longrightarrow$ **STOP**; *the problem is unbounded.*
5. *Compute*

$$\alpha = \min_{1 \leqslant i \leqslant m} \{b_i / a_{ij} : a_{ij} > 0\}$$

and choose any p *with* $b_p / a_{pj} = \alpha$. *Pivot on* a_{pj} *and go to the* **Step 1.**

Lemma

If we pivot as in Step 5, the resulting tableau is again maximum basic feasible.

Proof.

□

Lemma

If we pivot as in Step 5, the resulting tableau is again maximum basic feasible.

Proof.

Assume that before Step 5 the basis is $B = \{n+1, n+2, \dots n+m\}$.
The corresponding BFS is $x_1 = x_2 = \cdots = x_n = 0$, and $x_{n+i} = b_i \geq 0$ for $1 \leq i \leq n$.
Assume Step 5 chooses pivot element $a_{p,j}$, so x_j enters the basis and x_p leaves the basis.

$\square$

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The corresponding BFS is $x_1 = x_2 = \cdots = x_n = 0$, and $x_{n+i} = b_i \geq 0$ for $1 \leq i \leq n$.
Assume Step 5 chooses pivot element $a_{p,j}$, so x_j enters the basis and x_p leaves the basis.

After the pivot, each of $x_1, x_2, \dots, x_n$ remains zero, except that x_j changes from 0 to

$$\alpha = b_p/a_{pj} = \min\{b_i/a_{ij} : 1 \leq i \leq m, a_{ij} > 0\} \tag{1}$$

$$\geq 0. \tag{2}$$

It remains to show that $x_{n+k} \geq 0$ for $1 \leq k \leq m$, after the pivot.

□

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It remains to show that $x_{n+k} \geq 0$ for $1 \leq k \leq m$, after the pivot.

The value of x_{n+k} is determined by row k of the tableau

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kj}x_j + \dots + a_{kn}x_n - b_k = -x_{n+k}.$$

After the pivot, the equation is

$$0 + 0 + \dots + 0 + a_{kj}\alpha + 0 + \dots + 0 - b_k = -x_{n+k}$$

so the pivot changes x_{n+k} from b_k to $b_k - a_{kj}\alpha$. We claim that $b_k - a_{kj}\alpha \geq 0$.

□

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If we pivot as in Step 5, the resulting tableau is again maximum basic feasible.

Proof.

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After the pivot, the equation is

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so the pivot changes x_{n+k} from b_k to $b_k - a_{kj}\alpha$. We claim that $b_k - a_{kj}\alpha \geq 0$.

If $a_{kj} > 0$, then $\alpha \leq b_k/a_{kj}$ by (1), so $b_k - a_{kj}\alpha \geq 0$, as claimed.
If $a_{kj} \leq 0$, then $a_{kj}\alpha \leq 0$ by (2), so $b_k - a_{kj}\alpha \geq b_k \geq 0$, as claimed. $\square$

Lemma

If the algorithm stops at Step 2., the basic solution is optimal.

Proof.

$\square$

Note

If all $c_j < 0$ $(j = 1, .., n)$, *the problem has a unique solution. However if some* $c_j = 0$, *the problem* **may have** *infinitely many solutions.*

Lemma

If the algorithm stops at Step 4., the problem is unbounded.

Proof.

SA for MBFT – used to illustrate next example

$$
\begin{array}{c}
\begin{array}{cccc|c} & \text{(ind var's)} & & & -1 \end{array} \\
\begin{array}{|cccc|c|}
\hline
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\
\hline
c_1 & c_2 & \dots & c_n & d \\
\hline
\end{array}
\begin{array}{l} \\ \\ = -\text{(dep var's)} \\ \\ = f \end{array}
\end{array}
$$

1. We have MBFT $(b_1, b_2, \dots, b_m \geqslant 0)$
2. If $c_1, c_2, \dots, c_n \leqslant 0 \Longrightarrow$ **STOP**; the current basic feasible solution is optimal.
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4. If $a_{1j}, a_{2j}, \dots, a_{mj} \leqslant 0 \Longrightarrow$ **STOP**; the problem is unbounded.
5. Compute
$$\alpha = \min_{1 \leqslant i \leqslant m} \{b_i / a_{ij} : a_{ij} > 0\}$$
and choose any p with $b_p / a_{pj} = \alpha$. Pivot on a_{pj} and go to the **Step 1.**

Example

In each tableau below, mark the next step of the SA for MBFT.

a)

1	3	2
-1	2	-1
1	-1	2
-1	5	-2

b)

1	3	2
-1	2	1
1	-1	5
-1	5	-2

c)

1	3	2
-1	2	3
-3	-1	2
0	-3	-2

d)

0	3	2
-1	2	3
-3	-1	2
1	-3	-2

e)

1	3	2
-1	2	3
3	-1	2
1	-3	-2

f)

0	3	2
-1	2	3
-3	-1	2
1	3	-2

Drawbacks of the Simplex Algorithm for MBFT

The **SA for MBFT** will only work on maximum basic feasible tableaux (MBFT).

(i) We need at least one basic **feasible** solution to start with:
maximize $f(x_1, x_2) = x_1 - 2x_2 + 3$ subject to
$x_1 + x_2 \geqslant 1$
$2x_1 + x_2 \leqslant 5$
$x_1, x_2 \geqslant 0$

(ii) Algorithm may go into infinite loop—Strayer, pp 58–59.