MACM 201 Homework 7 (Quiz Oct. 30)

1. Define the generating functions $B(x) = \sum_{n=0}^{\infty} 2^n x^n$ and $F(x) = \sum_{n=0}^{\infty} f_n x^n$ where f_n is the Fibonacci sequence determined by the recurrence relation

$$f_0 = 0 \qquad \text{and} \qquad f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2$$

Find the coefficients of the first four terms (constant up to x^3) of each GF

(a) $F(x) + B(x)$

(b) $F(x) \times B(x)$

(c) $F(x) \times F(x) \times F(x)$

2. For each infinite sequence, express the associated GF in rational form.

(a) $0, 0, 1, 1, 1, 1, \ldots$

(b) $1, -1, 1, -1, 1, -1, \ldots$

(c) $0, 0, 0, a, -a, a, -a, a, \ldots$

(d) $a, 0, a, 0, a, 0, \ldots$

(e) $1, -2, 3, -4, 5, -6, \ldots$

(f) $0, 0, 0, 1, 2, 3, 4, \ldots$

(g) $0, 0, 0, 3, -6, 9, -12, 15, -18, \ldots$

(h) $0, 3, 2, 5, 4, 7, \ldots$ (Hint: this is $1-1, 2+1, 3-1, 4+1, 5-1, 6+1, \ldots$)

3. For each generating function below, find a formula for the coefficient of x^n.

(a) $(1 + 2x)^3$

(b) $\frac{3x^2}{1-x}$

(c) $\frac{2x}{1-x} + \frac{3x^2}{(1-x)^2}$

(d) $\frac{x^3+1}{2-2x}$

(e) $\frac{2x}{(3+6x)^2} + 7$

4. Apply partial fractions to each GF

   (a) $A(x) = \frac{1}{(x-1)(x-2)(x-3)}$

   (b) $B(x) = \frac{1}{(x-3)^2(x-5)}$

5. For each GF in the previous exercise, find a formula for the coefficient of x^n.

6. In each problem below you are given an infinite sequence $b_0, b_1, b_2, \dots$ determined by a recurrence relation. Use this recurrence relation to express the GF for this sequence, $B(x) = \sum_{n=0}^{\infty} b_n x^n$, as a rational function.

   (a) $b_0 = 2$, $b_1 = 3$, $b_n - 3b_{n-1} + 7b_{n-2} = 0$ for $n \geq 2$

   (b) $b_0 = 1$, $b_1 = 2$, $b_n - 5b_{n-1} + 3b_{n-2} = 1$ for $n \geq 2$

   (c) $b_0 = 1$, $b_1 = 0$, $b_2 = 3$, $b_n - 2b_{n-1} + b_{n-3} = n$ for $n \geq 3$

7. In this problem we explore when a generating function has an inverse. (Recall that an inverse to a generating function $A(x)$ is another generating function $B(x)$ with the property that $A(x) \times B(x) = 1$.)

   (a) Let $A(x) = a_0 + a_1 x + a_2 x^2 + \dots$. Assuming $a_0 = 0$, show that $A(x)$ has no inverse.

   (b) Suppose that $A(x)$ and $B(x)$ are inverse generating functions where

   $$A(x) = 2 + 4x - 4x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$
   $$B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \dots$$

   (so the first three coefficients of $A(x)$ are specified, but all other coefficients are unknown constants). Determine the value of b_0. Then find b_1 and b_2.

   (c) Let $A(x) = a_0 + a_1 x + a_2 x^2 + \dots$ and assume $a_0 \neq 0$. Explain why there is an inverse of $A(x)$.