MACM 201 - Discrete Mathematics

9. Recurrence relations II

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Families of combinatorial objects give rise to counting sequences.

Example 1. Let $\mathcal{B}_n$ denote the set of binary strings of length n and let $b_n = |\mathcal{B}_n|$. Then $b_n = 2^n$ so the sequence of sizes of these sets is the familiar sequence of powers of two:

$$(b_0, b_1, b_2, b_3, \ldots,) = (1, 2, 4, 8, \ldots)$$

In other instances we did not immediately find a closed formula to count the number of objects in our family of size n, but we could express this number recursively.

Example 2. Let t_n denote the number of ordered rooted binary trees of height at most n. Then we have

$$t_0 = 1 \qquad \text{and} \qquad t_n = 1 + t_{n-1} + t_{n-1}^2 \quad \text{for } n \geq 1.$$

Now we will learn to take (some types of) recursive expressions and **solve** them by finding a closed form for the n^{th} term (i.e. a formula depending only on n and not on any other values of the sequence).

Recall: A recurrence relation of order k for the sequence $(x_1, x_2, x_3, \ldots)$ is a formula that determines x_n given the previous k terms $x_{n-1}, x_{n-2}, \ldots, x_{n-k}$.

The values of the first k terms of the sequence are called the **initial values**. The recurrence together with the initial values uniquely determine the sequence.

Recurrence relations can be extremely complicated to solve, so we will restrict our attention to some standard forms that can be handled with basic instructive techniques. We begin with the simplest interesting instance.

Theorem

If the sequence $(x_0, x_1, \ldots)$ satisfies the recurrence $x_n = dx_{n-1}$ and the initial value $x_0 = q$, then it is a geometric progression given by

$$x_n = qd^n$$

Proof: Induction.

Changing perspective

The previous theorem has the following equivalent statement.

Theorem

Every sequence $(x_0, x_1, \ldots)$ satisfying the recurrence $x_n = dx_{n-1}$ Has the form $x_n = Cd^n$ where C is a fixed constant.

This formulation indicates the manner in which we will solve our problems

(1) Find the **general** solution to the recurrence relation (i.e. all possible solutions). This general solution will have unknown constants.

(2) Use initial values to determine the the values of these constants, thus giving the **unique** solution to the problem.

Problem. Suppose $(x_0, x_1, \ldots)$ satisfies the recurrence $x_n = 5x_{n-1}$

(1) Find the general solution.

(2) For the initial value $x_0 = 7$, find the unique solution.

Definition

A **linear** recurrence relation of order k for a sequence $(x_0, x_1, \ldots)$ is a recurrence that can be written in the form

$$c_n x_n + c_{n-1} x_{n-1} + \ldots + c_{n-k} x_{n-k} = f(n) \qquad \text{for } n \geq k$$

where $c_n, \ldots, c_{n-k}$ are fixed constants (we assume $c_n \neq 0$).

If f is the zero function we call this recurrence **homogeneous**.

Observe that the recurrence

$$x_n = d x_{n-1}$$

that we have already investigated is a homogeneous order 1 linear recurrence with constant coefficients since we can express this as

$$x_n - d x_{n-1} = 0$$

Examples

Determine the order of each recurrence and whether it is linear with constant coefficients. If it is linear with constant coefficients, check if it is homogeneous.

(1) $x_n = x_{n-1} + x_{n-2}^2$

(2) $5x_n + 3x_{n-2} - x_{n-3} = n^3$

(3) $10x_{n+3} - 5x_{n+2} + x_n = 0$

(4) $x_n x_{n-2} - x_{n-3} = 7$

(5) $x_n = 6x_{n-1} + 10x_{n-5} - n!$

Second order

Now we turn our attention to homogeneous second order linear recurrence relations with constant coefficients. In other words, we have fixed constants a, b, c and we are interested in finding all sequences $(x_0, x_1, \ldots)$ satisfying the equation

$$ax_n + bx_{n-1} + cx_{n-2} = 0 \quad \text{for all } n \geq 2. \tag{1}$$

We have seen that the first order equation $ax_n + bx_{n-1} = 0$ will have geometric progressions as solutions. Here we make a leap by investigating the possibility of a geometric progression solving the above equation.
Suppose that $x_n = r^n$ is a solution to equation (1). In this case we have

$$ar^n + br^{n-1} + cr^{n-2} = 0 \quad \text{for all } n \geq 2. \tag{2}$$

Observe that the $n \geq 2$ condition is redundant in equation (2). If this holds for $n = 2$, then it holds for all larger values (multiplying by powers of r gives the other equations). This reduces us to a familiar equation

$$ar^2 + br + c = 0$$

Conclusion: A number r satisfies $ar^2 + br + c = 0$ if and only if $x_n = r^n$ is a solution to our recurrence.

Definition

The homogeneous second order linear recurrence relation

$$ax_n + bx_{n-1} + cx_{n-2} = 0$$

has **characteristic equation**

$$ar^2 + br + c = 0.$$

The roots of the characteristic equation are precisely those numbers r for which $x_n = r^n$ satisfies the above recurrence.

Problem. Find all real numbers r so that $x_n = r^n$ is a solution to the recurrence

$$x_n - 5x_{n-1} + 6x_{n-2} = 0$$

Linearity

Theorem

Both of the properties below hold for the recurrence relation

$$ax_n + bx_{n-1} + cx_{n-2} = 0 \tag{3}$$

(A) If $(s_0, s_1, \ldots)$ *is a solution to (3) and* C *is a real number, then*

$$C(s_0, s_1, \ldots) = (Cs_0, cs_1, \ldots) \text{ is another solution to (3)}$$

(B) If $(s_0, s_1, \ldots)$ *and* $(t_0, t_1, \ldots)$ *are solutions to (3), then*

$$(s_0, s_1, \ldots) + (t_0, t_1, \ldots) = (s_0 + t_0, s_1 + t_1, \ldots) \text{ is another solution to (3)}$$

Problem. Find infinitely many solutions to the recurrence

$$x_n - 5x_{n-1} + 6x_{n-2} = 0$$

We have seen that the recurrence

$$x_n - 5x_{n-1} + 6x_{n-2} = 0$$

has infinitely many solutions. Indeed, for any real numbers C, D the sequence

$$x_n = C2^n + D3^n$$

will be a solution. Let's use this to try and solve an initial value problem.

Problem. Find an infinite sequence $(x_0, x_1, \ldots)$ satisfying the above recurrence together with the initial values

$$x_0 = 6 \qquad \text{and} \qquad x_1 = 13$$

.

Note: there can only be one solution to this problem... once you know x_0 and x_1 the recurrence determines x_2, then x_3, and so on.

General solutions

Theorem

Let a, b, c be fixed constants with $a \neq 0$ and consider the recurrence

$$ax_n + bx_{n-1} + cx_{n-2} = 0. \tag{4}$$

If the characteristic equation,

$$ar^2 + br + c$$

has two distinct real roots, say r_1 and r_2, then every sequence satisfying this recurrence has the form

$$x_n = Cr_1^n + Dr_2^n \tag{5}$$

where C and D are fixed constants. Accordingly, we will call equation (5) the **general solution** *to the recurrence.*

Problem. The Fibonacci sequence $(f_0, f_1, \ldots)$ is defined by the recurrence

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2$$

together with the initial values

$$f_0 = 0 \qquad \text{and} \qquad f_1 = 1$$

(1) Find the general solution to the above recurrence.

(2) Find a closed form for the Fibonacci sequence. (i.e. find the unique solution for the given initial values)