MACM 201 - Discrete Mathematics

Prelude to generating functions

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We have seen the binomial coefficient $\binom{n}{k}$ associated with a number of different things:

$$\begin{aligned}\binom{n}{k} &= \text{the number of } k \text{ element subsets of } \{1, \ldots, n\} \\ &= \text{the number of binary strings of length } n \text{ with } k \text{ 1's} \\ &= \text{the coefficient of } x^k y^{n-k} \text{ in the expansion of } (x+y)^n\end{aligned}$$

Now we are going to change perspective and *use* polynomials to encode counting problems. For instance, there is a natural correspondence

$$\text{binary strings length five with two 1's} \sim \text{coefficient of } x^2 \text{ in } (1+x)^5$$

A new way of thinking

More generally, we can view the coefficients of $(1+x)^n$ as useful information. Namely, the coefficient of x^k tells us how many binary strings of length n have exactly k 1's.

Note

This is an extremely different way of thinking about polynomials than appears in calculus or earlier math courses!

Instead of viewing a polynomial as a function that takes a real number and gives you another real number, we are now treating it as a formal mathematical object. Just as seen above, we will be interested in using coefficients to encode counting questions.

What's the point? Polynomials come equipped with a natural algebraic structure. We can add them, subtract them, multiply them, and compose them. These operations have powerful applications for our counting questions.

Definition

If $P(x)$ is a polynomial and k a nonnegative integer, then we use $[x^k]P(x)$ to denote the coefficient of x^k in $P(x)$.

Counting solutions

Question. How many solutions to the equation

$$a_1 + a_2 + a_3 = 7$$

Satisfy $0 \leq a_1 \leq 3$ and $0 \leq a_2 \leq 3$ and $0 \leq a_3 \leq 3$?

We can use inclusion-exclusion to find a solution here... for the ground set, take all triples (a_1, a_2, a_3) of nonnegative integers summing to 7. Then for $i = 1, 2, 3$ let C_i be the condition that $a_i \geq 4$.

Now we are going to ignore the value of the answer, and instead focus on representing this value.

Problem. Represent the solution to the above question as the coefficient of a polynomial.

Exercise. For each equation, express the answer using the coefficient of a polynomial.

$$a_1 + a_2 + a_3 = 13 \text{ where } 2 \leq a_1 \leq 4,\ 1 \leq a_2 \leq 5, \text{ and } 3 \leq a_3 \leq 7$$

$$a_1 + a_2 + a_3 = 10 \text{ where } a_1 \geq 1 \text{ is odd, } a_2 \geq 2 \text{ is even, and } 1 \leq a_3 \leq 4$$

$$a_1 + a_2 + a_3 = 12 \text{ where } a_1 \in \{1, 3, 4, 8\},\ 1 \leq a_2 \leq 5, \text{ and } a_3 \in \{4, 5, 7, 9, 11\}$$

Making change

Question. You have 7 nickels, 5 dimes, and 4 quarters. How many ways can you select coins that add up to a dollar?

Problem. Encode this counting question using a polynomial.

Sicherman Dice

If two six-sided dice are rolled, the total will be a number between 2 and 12. There is just one way to roll a 2 (you need two 1's) and one way to roll a 12 (two 6's) but there are multiple ways of getting the other numbers in between. This is naturally encoded using the following polynomial

$$\begin{aligned}&(x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6)\\&\qquad = x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}\end{aligned}$$

As you can see, the coefficient of x^k is telling us exactly how many ways we can get a total of k when we roll our two dice.

Here is a funny idea due to George Sicherman. He factored the above polynomial and noticed that it can also be written as the following product:

$$\begin{aligned}&x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}\\=&(x+2x^2+2x^3+x^4)(x+x^3+x^4+x^5+x^6+x^8)\end{aligned}$$

This means that you can make one die with faces numbered $1, 2, 2, 3, 3, 4$ and another with faces numbered $1, 3, 4, 5, 6, 8$ and if you roll these two in combination, you get exactly the same distribution as two ordinary dice!

Question. How many solutions to the equation

$$a_1 + a_2 + a_3 = n$$

Satisfy $a_1, a_2, a_3 \geq 0$?

Recall the answer is $\binom{n+2}{2}$. This can be described using polynomials as follows:

$$[x^n]\Big((1 + x + x^2 \ldots + x^n)(1 + x + x^2 \ldots + x^n)(1 + x + x^2 \ldots + x^n)\Big)$$

Note that in modelling this question using polynomials, we added the obvious restriction $a_i \leq n$ since there is no need to consider larger values.

Now for an unusual move...

We could ignore these restrictions completely and consider an "infinite polynomial". The coefficient of x^n in the infinite polynomial product

$$(1 + x + x^2 + x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots)$$

still tells us the number of solutions to $a_1 + a_2 + a_3 = n$ where $a_1, a_2, a_3 \geq 0$. (if you want to get x^n you can't choose a higher power of x in any of the three terms). So the coefficients of this infinite polynomial are encoding some useful counting information!

Generating functions

Definition

A **generating function** is a formal expression of the form

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{i=0}^{\infty} a_i x^i$$

We view this as another way of encoding the infinite sequence $a_0, a_1, a_2, \ldots$

Note

You have seen expressions such as $1 + x + x^2 + \ldots$ in Calculus. For instance, you can define a function f with domain $0 \leq x < 1$ by the rule $f(x) = 1 + x + x^2 + \ldots = \sum_{i=0}^{\infty} x^i$. Then $f(\frac{1}{2}) = 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2$. This is not what we are doing here!

Here "formal expression" means that we are **NOT** plugging in numbers for x and computing $A(x)$. Instead $A(x)$ is just another way of encoding the sequence $a_0, a_1, a_2, \ldots$. We treating this sequence as an "infinite polynomial" $a_0 + a_1 x + a_2 x^2 + \ldots$

What's the point? Thinking about a sequence as a generating function gives rise to meaningful algebraic tools. We can add, multiply, and compose generating functions!

Terminology

Every polynomial may be viewed as a generating function where all coefficients from some point on are 0. For instance, the polynomial

$$a_0 + a_1 x + a_2 x^2 + \dots a_k x^k$$

is also the generating function

$$A(x) = \sum_{i=0}^{\infty} a_i x^i \quad \text{where we set } 0 = a_{k+1}, a_{k+2}, \dots.$$

Since we are primarily interested in the coefficients of our generating functions we will frequently call on the following.

Definition

For a generating function $B(x) = \sum_{i=0}^{\infty} b_i x^i$ we define

$$[x^k]B(x) = b_k.$$

In words, $[x^k]B(x)$ is the coefficient of x^k in $B(x)$.

Problem. Write down a generating function where the coefficient of x^n indicates how many ways there are to use nickels, dimes, and quarters to add up to a total of n cents.

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