MACM 201 - Discrete Mathematics

3. Combinations

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Reminder

For every nonnegative integer n and every $0 \leq k \leq n$ we have that

$$\begin{aligned}\binom{n}{k} &= \textit{The number of } k \textit{ element subsets of an } n \textit{ element set.} \\ &= \textit{The number of binary strings of length } n \textit{ with exactly } k \textit{ 1's}\end{aligned}$$

Fact

It holds that

$$\binom{n}{k} = \binom{n}{n-k}.$$

Binomial coefficients

Thanks to some of the counting we did in the last section we have a nice interpretation of the following identity.

Theorem

For every nonnegative integer n we have

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. Let S be the set of all binary strings of length n and for every $0 \le k \le n$ let S_k be the set of binary strings of length n with exactly k 1's. Now we have:

Expanding products of polynomials

The distributive rule gives the following (unsimplified) expression:

$$(x + y)(x + y) = x(x + y) + y(x + y) = x^2 + xy + yx + y^2$$

Note: we can obtain the four monomials appearing on the right hand side in the above equation by choosing either the x or y from the first $(x + y)$ factor on the left, and then choosing either the x or y from the second $(x + y)$ factor.

Using $xy = yx$ (commutativity) we have $(x + y)(x + y) = x^2 + 2xy + y^2$ and we can understand the coefficients $1, 2, 1$ of these monomials using the same logic as above: There is one way to choose two x's to get x^2 (similarly for y^2), while there are two ways to choose one x and one y to get xy.

Example Use the above reasoning to express $(x + y)^3$ as a sum of monomials (with like terms collected).

Binomial theorem

Theorem

If x and y are two variables and n a positive integer, then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof.

Using the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Plugging $x = y = 1$ into the above equation gives another proof of our identity

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}.$$

Plugging $x = -1$ and $y = 1$ gives the identity

$$0 = (-1+1)^n = \sum_{k=0}^{n} (-1)^n \binom{n}{k}.$$

Problem. Find the coefficient of $x^5 y^{95}$ in $(3x - y)^{100}$.

Multinomial theorem

Theorem

If $x_1, x_2, \ldots, x_m$ are variables and n a positive integer, then,

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x^{k_1} x^{k_2} \cdots x^{k_m}$$

The proof is a straightforward generalization as that for the Binomial theorem.

SNE's

Definition

If $s = (s_1, s_2, \ldots, s_k)$ is a sequence of nonnegative integers (abbreviated SNE), we define the *length* of s to be k and the *sum* of s to be $s_1 + s_2 + \ldots + s_k$. We let $SNE_{k,n}$ denote the set of all integer sequences with length k and sum n.

Examples

$(2, 1, 1)$ is a SNE with length 3 and sum 4.

$(2, 4, 1)$ is a SNE with length 3 and sum 7.

$(4, 0, 1, 2, 1, 0)$ is a SNE with length 6 and sum 8.

Theorem

If n, k are nonnegative integers, $|SNE_{k,n}| = \binom{n+k-1}{k}$.

Proof.

Problem. How many integer solutions are there to

$$x_1 + x_2 + \cdots + x_5 = 10, \quad x_i \geq 0 \ ?$$

Example. How many integer solutions are there to

$$x_1 + x_2 < 7, \quad x_1, x_2 \geq 0 \ ?$$

Example. How many ways are there to distribute 5 apples, 3 oranges, and 6 pears among 3 people such that each person receives at least one pear?