MATH 308 D200, Fall 2019

12. Non-canonical LP problems - unconstrained variables

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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Non-canonical LP Problems

Simplex algorithm solves **only** canonical maximization and canonical minimization problems:

- all initial (real) variables (variables that come from an LP formulation) are non-negative
- all constraints have inequality form

If we find a way how to tackle these restrictions we will be able to solve a broader class of problems:

- problems with unconstrained initial variables
- problems with equations as main constraints

Unconstrained Variables

Definition (unconstrained variable)

A real variable in an LP problem is said to be *unconstrained* if there is no non-negativity constraint on the variable.

Definition refers to the initial variables of the problem; all **slack variables are required to be non-negative**.

Method I: Replacing an unconstrained variable by two constrained variables.

Example 1

Maximize $f(x, y) = x + 3y$, subject to

$$\begin{aligned} x + 2y &\leqslant 10 \\ -3x - y &\leqslant -15 \end{aligned}$$

Both variables are unconstrained!

$$\begin{aligned} x &= x^+ - x^-, \quad x^+, x^- \geqslant 0 \\ y &= y^+ - y^-, \quad y^+, y^- \geqslant 0 \end{aligned}$$

We get new LP problem (equivalent to the original one) which is a canonical maximization LP problem.
Replace every occurrence of x by $x^+ - x^-$ and every occurrence of y by $y^+ - y^-$:

Maximize $f(x^+, x^-, y^+, y^-) = x^+ - x^- + 3y^+ - 3y^-$, subject to

$$x^+ - x^- + 2y^+ - 2y^- \leqslant 10$$
$$-3x^+ + 3x^- - y^+ + y^- \leqslant -15$$
$$x^+, x^-, y^+, y^- \geqslant 0$$

x^+	x^-	y^+	y^-	-1	
1	−1	2	−2	10	$= -t_1$
−3	3	**−1**	1	−15	$= -t_2$
1	−1	3	−3	0	$= f$

$\longrightarrow$

x^+	x^-	t_2	y^-	-1	
−5	5	2	0	−20	$= -t_1$
3	−3	−1	−1	15	$= -y^+$
−8	8	3	0	−45	$= f$

t_1	x^-	t_2	y^-	-1	
−1/5	−1	−2/5	0	4	$= -x^+$
3/5	0	1/5	−1	3	$= -y^+$
−8/5	0	−1/5	0	−13	$= f$

$\rightarrow$

optimal solution

$x^- = y^- = 0$
$x^+ = 4$
$y^+ = 3$
$f = 13$

$\rightarrow$

optimal solution

$x = 4$
$y = 3$
$f = 13$

Example 2

Maximize $f(x, y) = x + 3y$
subject to

$$x + 2y \leqslant 10$$
$$3x + y \leqslant 15$$

Both variables are unconstrained. Use the substitutions

$$x = x^+ - x^-, \quad x^+, x^- \geqslant 0$$
$$y = y^+ - y^-, \quad y^+, y^- \geqslant 0$$

to the the canonical maximization LP.

Maximize $f(x, y) = x^+ - x^+ + 3y^+ - 3y^-$
subject to

$$x^+ - x^- + 2y^+ - 2y^- \leqslant 10$$
$$3x^+ - 3x^- + y^+ - y^- \leqslant 15$$
$$x^+, x^-, y^+, y^- \geq 0$$

Apply simplex algorithm.

x^+	x^-	y^+	y^-	-1	
1	-1	2	-2	10	$= -t_1$
3^*	-3	1	-1	15	$= -t_2$
1	-1	3	-3	0	$= f$

$\Rightarrow$

t_2	x^-	y^+	y^-	-1	
-1/3	0	$5/3^*$	-5/3	5	$= -t_1$
1/3	-1	1/3	-1/3	5	$= -x^+$
-1/3	0	8/3	-8/3	-5	$= f$

t_2	x^-	t_1	y^-	-1	
-1/5	0	3/5	-1	3	$= -y^+$
$2/5^*$	-1	-1/5	0	4	$= -x^+$
1/5	0	-8/5	0	-13	$= f$

$\Rightarrow$

x^+	x^-	t_1	y^-	-1	
1/2	-1/2	1/2	-1	5	$= -y^+$
5/2	-5/2	-1/2	0	10	$= -t_2$
-1/2	1/2	-3/2	0	-15	$= f$

The optimal solution is $(x^+, x^-, y^+, y^-) = (0, 0, 5, 0)$, which corresponds to $(x, y) = (0, 5)$.

Method II: Pivoting unconstrained variables into the basis.

Example 1′

Maximize $f(x, y) = x + 3y$, subject to

$$x + 2y \leqslant 10$$
$$-3x - y \leqslant -15$$

1) Write a TT, but **circle every unconstrained variable.**
2) Pivot down unconstrained variable x

$$\begin{array}{cc|c|l} x & y & -1 & \\ \hline 1^* & 2 & 10 & = -t_1 \\ -3 & -1 & -15 & = -t_2 \\ \hline 1 & 3 & 0 & = f \end{array} \longrightarrow \begin{array}{cc|c|l} t_1 & y & -1 & \\ \hline 1 & 2 & 10 & = -x \\ 3 & 5 & 15 & = -t_2 \\ \hline -1 & 1 & -10 & = f \end{array}$$

Record the equation $t_1 + 2y - 10 = -x$, then delete the row from the TT.

3) Pivot down unconstrained variable y

$$\begin{array}{cc|c|l} t_1 & y & -1 & \\ \hline 3 & 5^* & 15 & = -t_2 \\ \hline -1 & 1 & -10 & = f \end{array} \longrightarrow \begin{array}{cc|c|l} t_1 & t_2 & -1 & \\ \hline 3/5 & 1/5 & 3 & = -y \\ \hline -8/5 & -1/5 & -13 & = f \end{array}$$

Record equation $\frac{3}{5}t_1 + \frac{1}{5}t_2 - 3 = -y$, then delete the row.

4) Run the SA on the resulting TT:

$$\begin{array}{cc|c|l} t_1 & t_2 & -1 & \\ \hline -8/5 & -1/5 & -13 & = f \end{array}$$

Here there is nothing to do. We have no main constraints, and an optimum has be reached. Solve for x and y, by setting the independent variables t_1, t_2 to zero in the recorded equations: $0 + 2y - 10 = -x$, $0 + 0 - 3 = -y \implies (x, y) = (0, 5)$.

Example 2′

Maximize $f(x,y) = x + 3y$, subject to

$$x + 2y \leqslant 10$$
$$3x + y \leqslant 15$$

1)
$$\begin{array}{ccc|c|l} & x & y & -1 & \\ \hline & 1 & 2 & 10 & = -t_1 \\ & 3 & 1^* & 15 & = -t_2 \\ \hline & 1 & 3 & 0 & = f \end{array} \longrightarrow \begin{array}{cc|c|l} x & t_2 & -1 & \\ \hline -5 & -2 & -20 & = -t_1 \\ 3 & 1 & 15 & = -y \\ \hline -8 & -3 & -45 & = f \end{array}$$

Record $3x + t_2 - 15 = -y$, delete row.

2)
$$\begin{array}{cc|c|l} x & t_2 & -1 & \\ \hline -5^* & -2 & -20 & = -t_1 \\ \hline -8 & -3 & -45 & = f \end{array} \longrightarrow \begin{array}{cc|c|l} t_1 & t_2 & -1 & \\ \hline -1/5 & 2/5 & 4 & = -x \\ \hline -8/5 & 1/5 & -13 & = f \end{array}$$

Record $-\frac{1}{5}t_1 + \frac{2}{5}t_2 - 4 = -x$, delete row to get $\begin{array}{cc|c|l} t_1 & t_2 & -1 & \\ \hline -8/5 & 1/5 & -13 & = f. \end{array}$

3) We now run the SA. Again, there are no main constraints so the SA is finished. Here, the t_2 entry is positive, so the LP is unbounded!

Example 2′

Maximize $f(x,y) = x + 3y$, subject to

$$x + 2y \leqslant 10$$
$$3x + y \leqslant 15$$

1)
$$\begin{array}{ccc|c}
 & x & y & -1 \\
\hline
 & 1 & 2 & 10 \\
 & 3 & 1^* & 15 \\
\hline
 & 1 & 3 & 0
\end{array}
\begin{array}{l} \\ = -t_1 \\ = -t_2 \\ = f \end{array}
\longrightarrow
\begin{array}{ccc|c}
 & x & t_2 & -1 \\
\hline
 & -5 & -2 & -20 \\
 & 3 & 1 & 15 \\
\hline
 & -8 & -3 & -45
\end{array}
\begin{array}{l} \\ = -t_1 \\ = -y \\ = f \end{array}$$

Record $3x + t_2 - 15 = -y$, delete row.

2)
$$\begin{array}{cc|c}
x & t_2 & -1 \\
\hline
-5^* & -2 & -20 \\
-8 & -3 & -45
\end{array}
\begin{array}{l} \\ = -t_1 \\ = f \end{array}
\longrightarrow
\begin{array}{cc|c}
t_1 & t_2 & -1 \\
\hline
-1/5 & 2/5 & 4 \\
-8/5 & 1/5 & -13
\end{array}
\begin{array}{l} \\ = -x \\ = f \end{array}$$

Record $-\frac{1}{5}t_1 + \frac{2}{5}t_2 - 4 = -x$, delete row to get
$$\begin{array}{cc|c}
t_1 & t_2 & -1 \\
\hline
-8/5 & 1/5 & -13
\end{array} = f.$$

3) We now run the SA. Again, there are no main constraints so the SA is finished.
Here, the t_2 entry is positive, so the LP is unbounded!
For $t_1 = 0$ and any number $t_2 \geq 0$ we solve the recorded equations to find $x = 4 - \frac{2}{5}t_2$ and

$$y = 15 - t_2 - 3x = 15 - t_2 - 3\left(4 - \frac{2}{5}t_2\right) = 3 - \frac{1}{5}t_2.$$

As $t_2 \to \infty$ the point $(x,y) = (4 - \frac{2}{5}t_2, 3 - \frac{1}{5}t_2)$ remains feasible, with unbounded objective

$$f(x,y) = \left(4 - \frac{2}{5}t_2\right) + 3\left(3 - \frac{1}{5}t_2\right) = \frac{1}{5}t_2 + 13 \to \infty$$