MATH 308 D200, Fall 2019

15. Primal Dual relationship

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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1. We have maximum Tucker tableau.

$$\begin{array}{cccc|c}
 & \text{(iv's)} & & -1 & \\
\hline
a_{11} & \cdots & a_{1n} & b_1 & \\
\vdots & \ddots & \vdots & \vdots & = -\text{(dv's)} \\
a_{m1} & \cdots & a_{mn} & b_m & \\
\hline
c_1 & \cdots & c_n & d & = f
\end{array}$$

2. If $b_1, b_2, \dots, b_m \geqslant 0$, go to **Step 6.**

3. Choose $b_i < 0$ such that i is maximal.

4. If $a_{i1}, a_{i2}, \dots, a_{in} \geqslant 0 \Longrightarrow$ **STOP**; the minimization problem is infeasible.

5. If $i = m$, choose $a_{mj} < 0$, pivot on a_{mj}, and go to **Step 1.**
If $i < m$, choose $a_{ij} < 0$, compute

$$\alpha = \min(\{b_i/a_{ij}\} \cup \{b_k/a_{kj} : k > i, a_{kj} > 0\}),$$

and choose any p with $b_p/a_{pj} = \alpha$. Pivot on a_{pj} and go to **Step 1.**

6. Apply the SA for MBFT.

1. We have minimum Tucker tableau.

$$\begin{array}{c|ccc|c}
 & a_{11} & \cdots & a_{1n} & b_1 \\
\text{(iv's)} & \vdots & \ddots & \vdots & \vdots \\
 & a_{m1} & \cdots & a_{mn} & b_m \\
\hline
-1 & c_1 & \cdots & c_n & d \\
\hline
 & & = \text{(dv's)} & & = g
\end{array}$$

2. If $c_1, c_2, \dots, c_n \leqslant 0$, go to **Step 6.**

3. Choose $c_j > 0$ such that j is maximal.

4. If $a_{1j}, a_{2j}, \dots, a_{mj} \leqslant 0 \Longrightarrow$ **STOP**; the minimization problem is infeasible.

5. If $j = n$, choose $a_{in} > 0$, pivot on a_{in}, and go to **Step 1.**
If $j < n$, choose $a_{ij} > 0$, compute

$$\alpha = \min(\{c_j/a_{ij}\} \cup \{c_k/a_{ik} : k > j, a_{ik} < 0\}),$$

and choose any p with $c_p/a_{ip} = \alpha$. Pivot on a_{ip} and go to **Step 1.**

6. Apply the DSA for MBFT.

6. The current tableau is maximum BFT ($b_1, b_2, \ldots, b_m \geqslant 0$)

$$
\begin{array}{cccc|c|l}
 & \multicolumn{3}{c}{\text{(iv's)}} & -1 & \\
\cline{2-5}
\multicolumn{1}{c|}{} & a_{11} & \cdots & a_{1n} & b_1 & \\
\multicolumn{1}{c|}{} & \vdots & \ddots & \vdots & \vdots & = -\text{(dv's)} \\
\multicolumn{1}{c|}{} & a_{m1} & \cdots & a_{mn} & b_m & \\
\cline{2-5}
\multicolumn{1}{c|}{} & c_1 & \cdots & c_n & d & = f \\
\cline{2-5}
\end{array}
$$

7. If $c_1, c_2, \ldots, c_n \leqslant 0 \Longrightarrow$ **STOP**; the current basic feasible solution is optimal.
8. Choose any j with $c_j > 0$
9. If $a_{1j}, a_{2j}, \ldots, a_{mj} \leqslant 0 \Longrightarrow$ **STOP**; the maximization problem is unbounded.
10. Compute

$$\alpha = \min_{1 \leqslant i \leqslant m} \{b_i / a_{ij} : a_{ij} > 0\}$$

and choose any p with $b_p / a_{pj} = \alpha$. Pivot on a_{pj} and go to the **Step 6.**

6. The current tableau is minimum BFT ($c_1, c_2, \ldots, c_n \leqslant 0$)

$$
\begin{array}{c|ccc|c|}
\cline{2-5}
 & a_{11} & \cdots & a_{1n} & b_1 \\
\text{(iv's)} & \vdots & \ddots & \vdots & \vdots \\
 & a_{m1} & \cdots & a_{mn} & b_m \\
\cline{2-5}
-1 & c_1 & \cdots & c_n & d \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{3}{c}{= \text{(dv's)}} & \multicolumn{1}{c}{= g}
\end{array}
$$

7. If $b_1, b_2, \ldots, b_m \geqslant 0 \Longrightarrow$ **STOP**; the current basic feasible solution is optimal.
8. Choose any i with $b_i < 0$
9. If $a_{i1}, a_{i2}, \ldots, a_{in} \geqslant 0 \Longrightarrow$ **STOP**; the minimization problem is unbounded.
10. Compute

$$\alpha = \min_{1 \leqslant j \leqslant n} \{c_j / a_{ij} : a_{ij} < 0\}$$

and choose any p with $c_p / a_{ip} = \alpha$. Pivot on a_{ip} and go to the **Step 6.**

Primal-Dual Relationship

Consider primal LP problem... ...and its dual LP problem

$$\begin{aligned} \text{Maximize } f(\boldsymbol{x}) &= \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d \\ \text{subject to } \boldsymbol{A}\boldsymbol{x} &\leqslant \boldsymbol{b} \\ \boldsymbol{x} &\geqslant \boldsymbol{0} \end{aligned}$$

$$\begin{aligned} \text{Minimize } g(\boldsymbol{x}) &= \boldsymbol{b}^\mathsf{T}\boldsymbol{y} - d \\ \text{subject to } \boldsymbol{A}^\mathsf{T}\boldsymbol{y} &\geqslant \boldsymbol{c} \\ \boldsymbol{y} &\geqslant \boldsymbol{0} \end{aligned}$$

$$\begin{array}{c|cccc|c|l}
 & x_1 & x_2 & \dots & x_n & -1 & \\ \hline
y_1 & a_{11} & a_{12} & \dots & a_{1n} & b_1 & = -t_1 \\
y_2 & a_{21} & a_{22} & \dots & a_{2n} & b_2 & = -t_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
y_m & a_{m1} & a_{m2} & \dots & a_{mn} & b_m & = -t_m \\ \hline
-1 & c_1 & c_2 & \dots & c_n & d & = f \\ \hline
 & = s_1 & = s_2 & \dots & = s_n & = g &
\end{array}$$

$$\text{(P)} \quad \text{Maximize } f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j - d$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, \; i = 1, 2, \ldots, m$$

$$x_1, x_2, \ldots, x_n \geqslant 0$$

$$\text{(D)} \quad \text{Minimize } g(y_1, y_2, \ldots, y_m) = \sum_{i=1}^{m} b_i y_i - d$$

$$\text{subject to } \sum_{i=1}^{m} a_{ij} y_i \geqslant c_j, \; j = 1, 2, \ldots, n$$

$$y_1, y_2, \ldots, y_m \geqslant 0$$

Theorem (Weak Duality)

For every primal feasible solution $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ *of* (P) *and every dual feasible solution* $\boldsymbol{y}^* = (y_1^*, y_2^*, \ldots, y_m^*)$ *of* (D) *we have* $f(\boldsymbol{x}^*) \leqslant g(\boldsymbol{y}^*)$.

Proof.

Here is a proof using sum-notation. Assume $\boldsymbol{x}^*$ is (P)-feasible and that $\boldsymbol{y}^*$ is (D)-feasible.

$$
\begin{aligned}
f(\boldsymbol{x}^*) + d &= \sum_{j=1}^{n} c_j x_j^* \\
&\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^* \right) x_j^* \quad \text{(since } \sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \text{ and } x_j^* \geq 0\text{)} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^* \right) y_i^* \quad \text{(change order of summation)} \\
&\leq \sum_{i=1}^{m} b_i y_i^* \qquad\qquad \text{(since } \sum_{j=1}^{n} a_{ij} x_j^* \geq c_j \text{ and } y_i^* \geq 0\text{)} \\
&= g(\boldsymbol{y}^*) + d.
\end{aligned}
$$

$\square$

Corollary

If $\boldsymbol{x}^*$ is (P)-feasible and $\boldsymbol{y}^*$ is (D)-feasible and $f(\boldsymbol{x}^*) = g(\boldsymbol{y}^*)$, then both solutions are optimal.

Proof.

Here is a proof using matrix-notation. We use the easy fact

(1) If vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ satisfy $\boldsymbol{u} \leq \boldsymbol{v}$ and $\boldsymbol{w} \geq \mathbf{0}$, then $\boldsymbol{u}^\intercal \boldsymbol{w} \leq \boldsymbol{u}^\intercal \boldsymbol{w}$ and $\boldsymbol{w}^\intercal \boldsymbol{u} \leq \boldsymbol{w}^\intercal \boldsymbol{v}$

Assume $\boldsymbol{x}$ is (P)-feasible and that $\boldsymbol{y}$ is (D)-feasible, and write the LP's in matrix form:

$$
\begin{array}{llll}
\text{(P)} & \max\ f = \boldsymbol{c}^\intercal \boldsymbol{x} - d & \qquad \text{(D)} & \min\ g = \boldsymbol{y}^\intercal \boldsymbol{b} - d \\
 & \quad A\boldsymbol{x} \leq \boldsymbol{b} & & \quad \boldsymbol{y}^\intercal A \ \geq \boldsymbol{c}^\intercal \\
 & \quad \ \ \boldsymbol{x} \geq 0 & & \quad \ \ \boldsymbol{y} \geq 0
\end{array}
$$

Then

$$
\begin{aligned}
f(\boldsymbol{x}^*) + d &= \boldsymbol{c}^\intercal \, \boldsymbol{x} \\
&\leq (\boldsymbol{y}^\intercal A) \, \boldsymbol{x} && \text{(by (1))} \\
&= \boldsymbol{y}^\intercal \, (A\boldsymbol{x}) \\
&\leq \boldsymbol{y}^\intercal \, \boldsymbol{b} && \text{(by (1))} \\
&= g(\boldsymbol{y}^*) + d
\end{aligned}
$$

□

Corollary

If $\boldsymbol{x}^*$ is (P)-feasible and $\boldsymbol{y}^*$ is (D)-feasible and $f(\boldsymbol{x}^*) = g(\boldsymbol{y}^*)$, then both solutions are optimal.

We recall the following

Theorem

Let (P) and (D) be dual LPs. Then:

dual

primal

	opt.	unb.	inf.
opt.			
unb.			
inf.			

We recall the following

Theorem

Let (P) and (D) be dual LPs. Then:
- (P) has an optimal solution if and only if (D) has an optimal solution.

primal \ dual	opt.	unb.	inf.
opt.	Y	N	N
unb.	N		
inf.	N		

We recall the following

Theorem

Let (P) and (D) be dual LPs. Then:
- (P) has an optimal solution if and only if (D) has an optimal solution.
- If (P) is unbounded, then (D) is infeasible.

dual

		opt.	unb.	inf.
primal	opt.	Y	N	N
	unb.	N	N	Y
	inf.	N		

We recall the following

Theorem

Let (P) and (D) be dual LPs. Then:
- (P) has an optimal solution if and only if (D) has an optimal solution.
- If (P) is unbounded, then (D) is infeasible.
- If (D) is unbounded, then (P) is infeasible.

primal \ dual	opt.	unb.	inf.
opt.	Y	N	N
unb.	N	N	Y
inf.	N	Y	

We recall the following

Theorem

Let (P) and (D) be dual LPs. Then:
- (P) has an optimal solution if and only if (D) has an optimal solution.
- If (P) is unbounded, then (D) is infeasible.
- If (D) is unbounded, then (P) is infeasible.

dual

primal	opt.	unb.	inf.
opt.	Y	N	N
unb.	N	N	Y
inf.	N	Y	Y

We recall the following

Theorem

Let (P) and (D) be dual LPs. Then:
- (P) has an optimal solution if and only if (D) has an optimal solution.
- If (P) is unbounded, then (D) is infeasible.
- If (D) is unbounded, then (P) is infeasible.

		dual		
		opt.	unb.	inf.
primal	opt.	Y	N	N
	unb.	N	N	Y
	inf.	N	Y	Y

Theorem (The duality theorem)

Let (P) and (D) be dual LPs. Exactly one of the following is true:

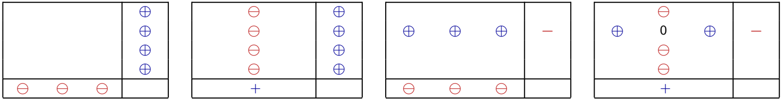
(i) Both (P) and (D) have an optimal solution, in which case $f = g$.

(ii) (P) is unbounded, and (D) is infeasible.

(iii) (D) is unbounded, and (P) is infeasible..

(iv) Both (P) and (D) are infeasible.

Final Tableau:



(i) (ii) (iii) (iv)

DSA for Maximum Tableaux

$$
\begin{array}{ccccc|l}
\multicolumn{4}{c}{\text{(ind var's)}} & -1 & \\
\hline
\multicolumn{1}{|c}{a_{11}} & a_{12} & \dots & a_{1n} & \multicolumn{1}{|c|}{b_1} & \\
\multicolumn{1}{|c}{a_{21}} & a_{22} & \dots & a_{2n} & \multicolumn{1}{|c|}{b_2} & \\
\multicolumn{1}{|c}{\vdots} & \vdots & \ddots & \vdots & \multicolumn{1}{|c|}{\vdots} & = -\text{(dep var's)} \\
\multicolumn{1}{|c}{a_{m1}} & a_{m2} & \dots & a_{mn} & \multicolumn{1}{|c|}{b_m} & \\
\hline
\multicolumn{1}{|c}{c_1} & c_2 & \dots & c_n & \multicolumn{1}{|c|}{d} & = f \\
\hline
\end{array}
$$

1. We have maximum Tucker tableau.
2. Take the negative transposition of the tableaux to obtain a minimum tableau.
3. Apply the dual simplex algorithm for minimum tableaux.
4. $\max f = -\min(-f)$.

Show all four methods to solve the following dual tableau.

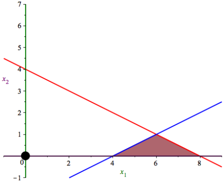
$$\begin{array}{c|cc|c|l}
 & x_1 & x_2 & -1 & \\
\hline
y_1 & 1 & 2 & 20 & = -t_1 \\
y_2 & 2 & 2 & 30 & = -t_2 \\
y_3 & 2 & 1 & 25 & = -t_3 \\
\hline
-1 & 4 & 3 & 0 & = f \\
\hline
 & = s_1 & = s_2 & = g &
\end{array}$$

$$
\text{(P)} \quad \max\ f = 2x_1 + 6x_2 \qquad
\begin{aligned}
x_1 + 2x_2 &\le \ \ 8 \quad (y_1)\\
-x_1 + 2x_2 &\le -4 \quad (y_2)\\
x_1,\ x_2 &\ge 0
\end{aligned}
$$

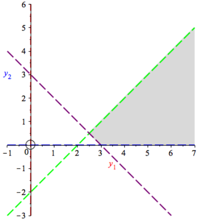
$$
\text{(D)} \quad \min\ g = 8y_1 - 4y_2 \qquad
\begin{aligned}
y_1 - \ y_2 &\ge 2 \quad (x_1)\\
2y_1 + 2y_2 &\ge 6 \quad (x_2)\\
y_1,\ y_2 &\ge 0
\end{aligned}
$$

Using: Max simplex

T_1:



$$
\begin{array}{r|rr|r|l}
 & x_1 & x_2 & -1 & \\
\hline
y_1 & 1 & 2 & 8 & = -t_1 \\
y_2 & -1^* & 2 & -4 & = -t_2 \\
\hline
-1 & 2 & 6 & 0 & = f \\
\hline
 & = s_1 & = s_2 & = g &
\end{array}
$$

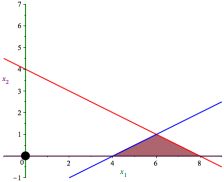


(P) $\max\ f = 2x_1 + 6x_2$
$$x_1 + 2x_2 \le 8 \quad (y_1)$$
$$-x_1 + 2x_2 \le -4 \quad (y_2)$$
$$x_1,\ x_2 \ge 0$$

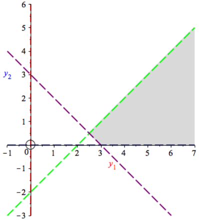
(D) $\min\ g = 8y_1 - 4y_2$
$$y_1 - y_2 \ge 2 \quad (x_1)$$
$$2y_1 + 2y_2 \ge 6 \quad (x_2)$$
$$y_1,\ y_2 \ge 0$$

Using: Max simplex

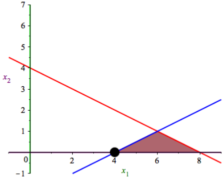
T_1:



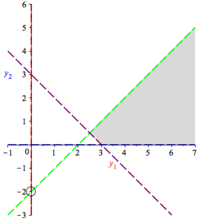
	x_1	x_2	-1	
y_1	1	2	8	$= -t_1$
y_2	-1^*	2	-4	$= -t_2$
-1	2	6	0	$= f$
	$= s_1$	$= s_2$	$= g$	



T_2:



	t_2	x_2	-1	
y_1	1^*	4	4	$= -t_1$
s_1	-1	-2	4	$= -x_1$
-1	2	10	-8	$= f$
	$= y_2$	$= s_2$	$= g$	



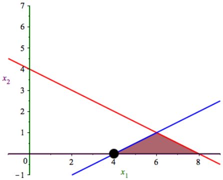
Phase 1 complete

$$\text{(P)} \quad \max\ f = 2x_1 + 6x_2 \qquad\qquad \text{(D)} \quad \min\ g = 8y_1 - 4y_2$$

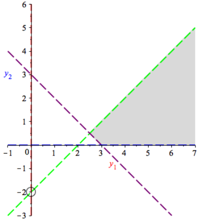
$$\begin{array}{rcrl} x_1 + 2x_2 & \le & 8 & (y_1) \\ -x_1 + 2x_2 & \le & -4 & (y_2) \\ x_1,\ x_2 & \ge & 0 & \end{array} \qquad\qquad \begin{array}{rcrl} y_1 - y_2 & \ge & 2 & (x_1) \\ 2y_1 + 2y_2 & \ge & 6 & (x_2) \\ y_1,\ y_2 & \ge & 0 & \end{array}$$

Using: Max simplex

T_2:

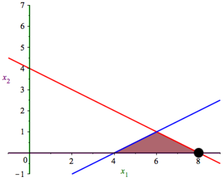


$$\begin{array}{c|cc|c|l} & t_2 & x_2 & -1 & \\ \hline y_1 & 1^* & 4 & 4 & = -t_1 \\ s_1 & -1 & -2 & 4 & = -x_1 \\ \hline -1 & 2 & 10 & -8 & = f \\ \hline & = y_2 & = s_2 & = g & \end{array}$$

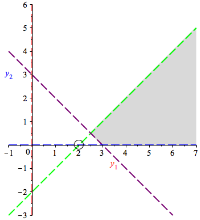


Phase 1 complete

T_3:



$$\begin{array}{c|cc|c|l} & t_1 & x_2 & -1 & \\ \hline y_2 & 1 & 4^* & 4 & = -t_2 \\ s_1 & 1 & 2 & 8 & = -x_1 \\ \hline -1 & -2 & 2 & -16 & = f \\ \hline & = y_1 & = s_2 & = g & \end{array}$$

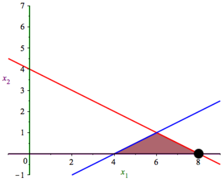


$$
\text{(P)} \quad \begin{array}{rl} \max \ f = 2x_1 + 6x_2 & \\ x_1 + 2x_2 \le \ \ 8 & (y_1) \\ -x_1 + 2x_2 \le -4 & (y_2) \\ x_1,\ x_2 \ge 0 & \end{array}
\qquad\qquad
\text{(D)} \quad \begin{array}{rl} \min \ g = 8y_1 - 4y_2 & \\ y_1 - \ y_2 \ge 2 & (x_1) \\ 2y_1 + 2y_2 \ge 6 & (x_2) \\ y_1,\ y_2 \ge 0 & \end{array}
$$

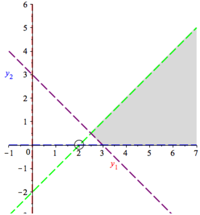
Using: Max simplex

Phase 1 complete

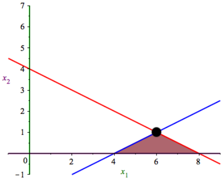
T_3:



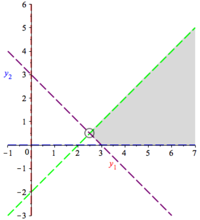
$$
\begin{array}{c|cc|c|l}
 & t_1 & x_2 & -1 & \\
\hline
y_2 & 1 & 4^* & 4 & = -t_2 \\
s_1 & 1 & 2 & 8 & = -x_1 \\
\hline
-1 & -2 & 2 & -16 & = f \\
\hline
 & = y_1 & = s_2 & = g &
\end{array}
$$



T_4:



$$
\begin{array}{c|cc|c|l}
 & t_1 & t_2 & -1 & \\
\hline
s_2 & 1/4 & 1/4 & 1 & = -x_2 \\
s_1 & 1/2 & -1/2 & 6 & = -x_1 \\
\hline
-1 & -5/2 & -1/2 & -18 & = f^* \\
\hline
 & = y_1 & = y_2 & = g &
\end{array}
$$



Phase 2 complete (optimal)

(P) $\max\ f = 2x_1 + 6x_2$

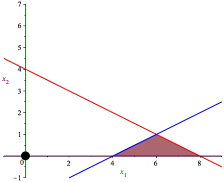
$$\begin{aligned} x_1 + 2x_2 &\le \ \ 8 \quad (y_1)\\ -x_1 + 2x_2 &\le -4 \quad (y_2)\\ x_1,\ x_2 &\ge 0 \end{aligned}$$

(D) $\min\ g = 8y_1 - 4y_2$

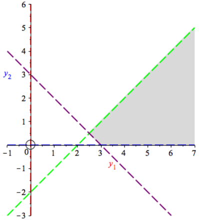
$$\begin{aligned} y_1 - \ y_2 &\ge 2 \quad (x_1)\\ 2y_1 + 2y_2 &\ge 6 \quad (x_2)\\ y_1,\ y_2 &\ge 0 \end{aligned}$$

Using: DUAL Max simplex

T_1:



$$\begin{array}{c|cc|c|l} & x_1 & x_2 & -1 & \\ \hline y_1 & 1 & 2^* & 8 & = -t_1 \\ y_2 & -1 & 2 & -4 & = -t_2 \\ \hline -1 & 2 & 6 & 0 & = f \\ \hline & = s_1 & = s_2 & = g & \end{array}$$

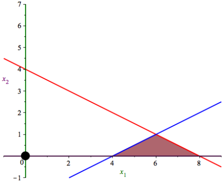


$$\text{(P)}\quad \begin{array}{rrl} \max\ f = & 2x_1 + 6x_2 & \\ & x_1 + 2x_2 \le 8 & (y_1) \\ & -x_1 + 2x_2 \le -4 & (y_2) \\ & x_1,\ x_2 \ge 0 & \end{array}$$

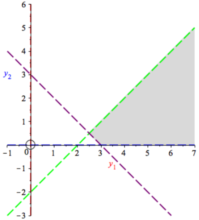
$$\text{(D)}\quad \begin{array}{rrl} \min\ g = & 8y_1 - 4y_2 & \\ & y_1 - y_2 \ge 2 & (x_1) \\ & 2y_1 + 2y_2 \ge 6 & (x_2) \\ & y_1,\ y_2 \ge 0 & \end{array}$$

Using: DUAL Max simplex

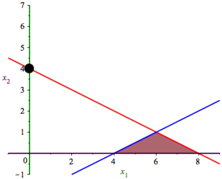
T_1:



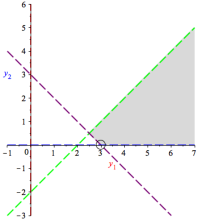
$$\begin{array}{c|cc|c|l} & x_1 & x_2 & -1 & \\ \hline y_1 & 1 & 2^* & 8 & = -t_1 \\ y_2 & -1 & 2 & -4 & = -t_2 \\ \hline -1 & 2 & 6 & 0 & = f \\ \hline & = s_1 & = s_2 & = g & \end{array}$$



T_2^*:



$$\begin{array}{c|cc|c|l} & x_1 & t_1 & -1 & \\ \hline s_2 & 1/2 & 1/2 & 4 & = -x_2 \\ y_2 & -2^* & -1 & -12 & = -t_2 \\ \hline -1 & -1 & -3 & -24 & = f \\ \hline & = s_1 & = y_1 & = g & \end{array}$$



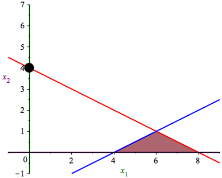
Dual phase 1 complete

$$\text{(P)}\quad \begin{aligned} \max\ f = 2x_1 + 6x_2 & \\ x_1 + 2x_2 &\le 8 \quad (y_1) \\ -x_1 + 2x_2 &\le -4 \quad (y_2) \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

$$\text{(D)}\quad \begin{aligned} \min\ g = 8y_1 - 4y_2 & \\ y_1 - y_2 &\ge 2 \quad (x_1) \\ 2y_1 + 2y_2 &\ge 6 \quad (x_2) \\ y_1,\ y_2 &\ge 0 \end{aligned}$$

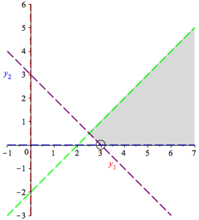
Using: DUAL Max simplex

T_2^*:

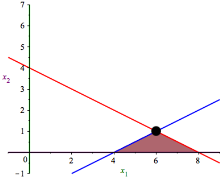


$$\begin{array}{c|cc|c|l} & x_1 & t_1 & -1 & \\ \hline s_2 & 1/2 & 1/2 & 4 & = -x_2 \\ y_2 & -2^* & -1 & -12 & = -t_2 \\ \hline -1 & -1 & -3 & -24 & = f \\ \hline & = s_1 & = y_1 & = g & \end{array}$$

Dual phase 1 complete

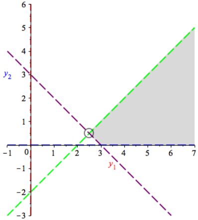


T_3^*:



$$\begin{array}{c|cc|c|l} & t_1 & t_2 & -1 & \\ \hline s_2 & 1/4 & 1/4 & 1 & = -x_2 \\ s_1 & 1/2 & -1/2 & 6 & = -x_1 \\ \hline -1 & -5/2 & -1/2 & -18 & = f \\ \hline & = y_1 & = y_2 & = g^* & \end{array}$$

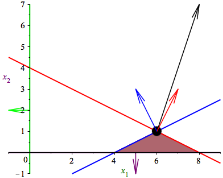
Dual phase 2 complete (optimal)



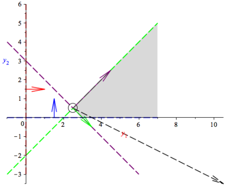
$$\text{(P)}\quad \max\ f = 2x_1 + 6x_2$$
$$\begin{aligned} x_1 + 2x_2 &\le 8 \quad (y_1)\\ -x_1 + 2x_2 &\le -4 \quad (y_2)\\ x_1,\ x_2 &\ge 0 \end{aligned}$$

$$\text{(D)}\quad \min\ g = 8y_1 - 4y_2$$
$$\begin{aligned} y_1 - y_2 &\ge 2 \quad (x_1)\\ 2y_1 + 2y_2 &\ge 6 \quad (x_2)\\ y_1,\ y_2 &\ge 0 \end{aligned}$$

At Optimality: $(x_1, x_2, t_1, t_2) = (6, 1, 0, 0)$ $\quad (y_1, y_2, s_1, s_2) = (5/2, 1/2, 0, 0)$



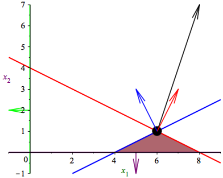
	t_1	t_2	-1	
s_2	$1/4$	$1/4$	1	$= -x_2$
s_1	$1/2$	$-1/2$	6	$= -x_1$
-1	$-5/2$	$-1/2$	-18	$= f^*$
	$= y_1$	$= y_2$	$= g^*$	



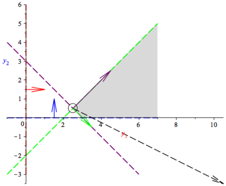
$$
\text{(P)}\quad \max\ f = 2x_1 + 6x_2 \qquad
\begin{array}{rcl}
x_1 + 2x_2 & \le & 8 \quad (y_1)\\
-x_1 + 2x_2 & \le & -4 \quad (y_2)\\
x_1,\ x_2 & \ge & 0
\end{array}
$$

$$
\text{(D)}\quad \min\ g = 8y_1 - 4y_2 \qquad
\begin{array}{rcl}
y_1 - y_2 & \ge & 2 \quad (x_1)\\
2y_1 + 2y_2 & \ge & 6 \quad (x_2)\\
y_1,\ y_2 & \ge & 0
\end{array}
$$

At Optimality: $(x_1, x_2, t_1, t_2) = (6, 1, 0, 0)$ $\qquad (y_1, y_2, s_1, s_2) = (5/2, 1/2, 0, 0)$



	t_1	t_2	-1	
s_2	$1/4$	$1/4$	1	$= -x_2$
s_1	$1/2$	$-1/2$	6	$= -x_1$
-1	$-5/2$	$-1/2$	-18	$= f^*$
	$= y_1$	$= y_2$	$= g^*$	



The dual sol'n proves f^* is maximal

$$
\begin{array}{rrcr}
5/2 \cdot (& x_1 + 2x_2 & \le & 8)\\
1/2 \cdot (& -x_1 + 2x_2 & \le & -4)\\
0 \cdot (& -x_1 \qquad & \le & 0)\\
0 \cdot (& -x_2 & \le & 0)\\
\hline
 & 2x_1 + 6x_2 & \le & 18
\end{array}
$$

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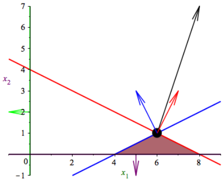
(P) $\max\ f = 2x_1 + 6x_2$

$$\begin{aligned} x_1 + 2x_2 &\le 8 \quad (y_1)\\ -x_1 + 2x_2 &\le -4 \quad (y_2)\\ x_1,\ x_2 &\ge 0 \end{aligned}$$

(D) $\min\ g = 8y_1 - 4y_2$

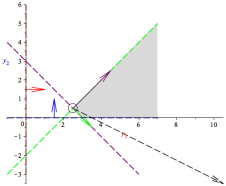
$$\begin{aligned} y_1 - y_2 &\ge 2 \quad (x_1)\\ 2y_1 + 2y_2 &\ge 6 \quad (x_2)\\ y_1,\ y_2 &\ge 0 \end{aligned}$$

At Optimality: $(x_1, x_2, t_1, t_2) = (6, 1, 0, 0)$ $(y_1, y_2, s_1, s_2) = (5/2, 1/2, 0, 0)$



$$\begin{array}{c|cc|c|l} & t_1 & t_2 & -1 & \\ \hline s_2 & 1/4 & 1/4 & 1 & = -x_2 \\ s_1 & 1/2 & -1/2 & 6 & = -x_1 \\ \hline -1 & -5/2 & -1/2 & -18 & = f^* \\ \hline & = y_1 & = y_2 & = g^* & \end{array}$$

The dual sol'n proves f^* is maximal

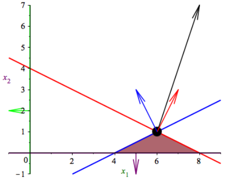


$$\begin{array}{rl} 5/2 \cdot (& x_1 + 2x_2 \le 8) \\ 1/2 \cdot (& -x_1 + 2x_2 \le -4) \\ 0 \cdot (& -x_1 \le 0) \\ 0 \cdot (& -x_2 \le 0) \\ \hline & f = 2x_1 + 6x_2 \le 18 = f^* \end{array}$$

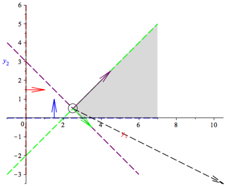
$$\text{(P)} \quad \max\ f = 2x_1 + 6x_2 \qquad\qquad \text{(D)} \quad \min\ g = 8y_1 - 4y_2$$

$$\begin{aligned} x_1 + 2x_2 &\le \ \ 8 \quad (y_1) \\ -x_1 + 2x_2 &\le -4 \quad (y_2) \\ x_1,\ x_2 &\ge 0 \end{aligned} \qquad\qquad \begin{aligned} y_1 - y_2 &\ge 2 \quad (x_1) \\ 2y_1 + 2y_2 &\ge 6 \quad (x_2) \\ y_1,\ y_2 &\ge 0 \end{aligned}$$

At Optimality: $(x_1, x_2, t_1, t_2) = (6, 1, 0, 0)$ $\quad$ $(y_1, y_2, s_1, s_2) = (5/2, 1/2, 0, 0)$

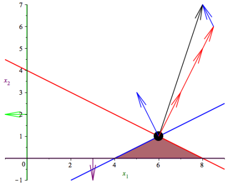


$$\begin{array}{c|cc|c|l} & t_1 & t_2 & -1 & \\ \hline s_2 & 1/4 & 1/4 & 1 & = -x_2 \\ s_1 & 1/2 & -1/2 & 6 & = -x_1 \\ \hline -1 & -5/2 & -1/2 & -18 & = f^* \\ \hline & = y_1 & = y_2 & = g^* & \end{array}$$



The dual sol'n proves f^* is maximal and expresses $\mathbf{c}$ as a positive linear combination of constraint vectors.

$$\begin{array}{r} 5/2 \cdot [\ \ 1\ ,\ \ \ 2\ ] \\ 1/2 \cdot [\ -1\ ,\ \ \ 2\ ] \\ 0 \cdot [\ -1\ ,\ \ \ 0\ ] \\ 0 \cdot [\ \ 0\ ,\ -1\ ] \\ \hline [\ \ 2\ ,\ \ \ 6\ ] = \mathbf{c}^t \end{array}$$



Pivoting and plots

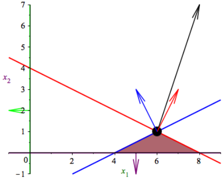
(P) $\max\ f = 2x_1 + 6x_2$

$$\begin{aligned} x_1 + 2x_2 &\le 8 \quad (y_1) \\ -x_1 + 2x_2 &\le -4 \quad (y_2) \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

(D) $\min\ g = 8y_1 - 4y_2$

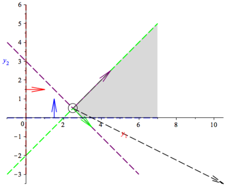
$$\begin{aligned} y_1 - y_2 &\ge 2 \quad (x_1) \\ 2y_1 + 2y_2 &\ge 6 \quad (x_2) \\ y_1,\ y_2 &\ge 0 \end{aligned}$$

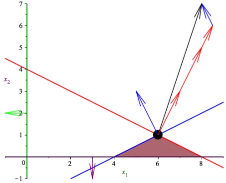
At Optimality: $(x_1, x_2, t_1, t_2) = (6, 1, 0, 0)$ $\quad (y_1, y_2, s_1, s_2) = (5/2, 1/2, 0, 0)$



	t_1	t_2	-1	
s_2	$1/4$	$1/4$	1	$= -x_2$
s_1	$1/2$	$-1/2$	6	$= -x_1$
-1	$-5/2$	$-1/2$	-18	$= f^*$
	$= y_1$	$= y_2$	$= g^*$	

The primal sol'n proves g^* is minimal





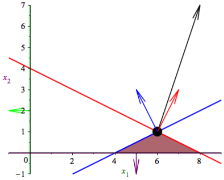
$$\begin{aligned} (\ y_1 - \ y_2 &\ge 2) \cdot 6 \\ (2y_1 + 2y_2 &\ge 6) \cdot 1 \\ (\ y_1 \qquad &\ge 0) \cdot 0 \\ (\qquad y_2 &\ge 0) \cdot 0 \\ \hline g = 8y_1 - 4y_2 &\ge 18 = g^* \end{aligned}$$

Pivoting and plots

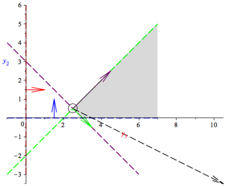
$$\text{(P)}\quad \max\ f = 2x_1 + 6x_2$$
$$x_1 + 2x_2 \le 8 \quad (y_1)$$
$$-x_1 + 2x_2 \le -4 \quad (y_2)$$
$$x_1,\ x_2 \ge 0$$

$$\text{(D)}\quad \min\ g = 8y_1 - 4y_2$$
$$y_1 - y_2 \ge 2 \quad (x_1)$$
$$2y_1 + 2y_2 \ge 6 \quad (x_2)$$
$$y_1,\ y_2 \ge 0$$

At Optimality: $(x_1, x_2, t_1, t_2) = (6, 1, 0, 0)$ $\quad$ $(y_1, y_2, s_1, s_2) = (5/2, 1/2, 0, 0)$

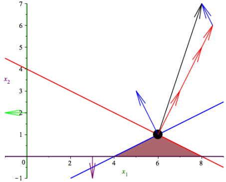


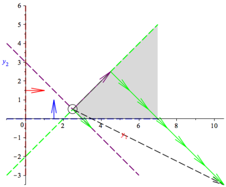
	t_1	t_2	-1	
s_2	$1/4$	$1/4$	1	$= -x_2$
s_1	$1/2$	$-1/2$	6	$= -x_1$
-1	$-5/2$	$-1/2$	-18	$= f^*$
	$= y_1$	$= y_2$	$= g^*$	



The primal sol'n proves g^* is minimal and expresses $\mathbf{b}$ as a positive linear comb. of dual constraint vectors.

$$\begin{array}{l} [\,1, -1] \cdot 6 \\ [\,2,\ \ 2] \cdot 1 \\ [\,1,\ \ 0] \cdot 0 \\ [\,0,\ \ 1] \cdot 0 \\ \hline [\,8, -4] \quad = b^t \end{array}$$





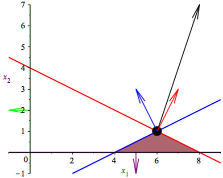
(P) $\max\ f = 2x_1 + 6x_2$

$$\begin{aligned} x_1 + 2x_2 &\le \ \ 8 \quad (y_1) \\ -x_1 + 2x_2 &\le -4 \quad (y_2) \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

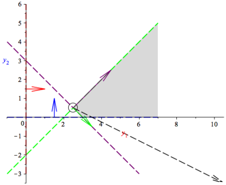
(D) $\min\ g = 8y_1 - 4y_2$

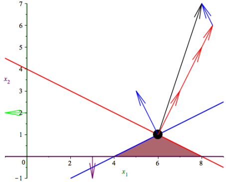
$$\begin{aligned} y_1 - \ \ y_2 &\ge 2 \quad (x_1) \\ 2y_1 + 2y_2 &\ge 6 \quad (x_2) \\ y_1,\ y_2 &\ge 0 \end{aligned}$$

At Optimality: $(x_1, x_2, t_1, t_2) = (6, 1, 0, 0)$ $\qquad (y_1, y_2, s_1, s_2) = (5/2, 1/2, 0, 0)$



	t_1	t_2	-1	
s_2	$1/4$	$1/4$	1	$= -x_2$
s_1	$1/2$	$-1/2$	6	$= -x_1$
-1	$-5/2$	$-1/2$	-18	$= f^*$
	$= y_1$	$= y_2$	$= g^*$	





Complementary slackness:

$$[x_1, x_2, t_1, t_2] = [\,6,\ \ 1,\ \ \ 0\ ,\ \ 0\,]$$
$$[s_1, s_2, y_1, y_2] = [\,0,\ \ 0, 5/2, 1/2]$$
$$g^* - f^* = \ \ 0 + 0 + 0 + 0$$