MACM 201 - Discrete Mathematics

Generating functions I - inverses

Department of Mathematics

Simon Fraser University

Addition and multiplication

Definition

A **generating function** is a formal expression of the form

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots = \sum_{i=0}^{\infty} a_ix^i$$

This generating function may be viewed as another way of encoding the infinite sequence $a_0, a_1, a_2, \ldots$, and we say that $A(x)$ is the generating function **for** the sequence $a_0, a_1, \ldots$.

Although generating functions are infinite, we can still add and multiply them using the same rules as for polynomials.

Example. Let $A(x) = 1 + x + x^2 + x^3 \ldots$ and $B(x) = 2 + 2x + 2x^2 + 2x^3 + \ldots$.

(1) What is $A(x) + B(x)$? $= (1+x+x^2+x^3+\ldots) + (2+2x+2x^2+\ldots)$

$= 3+3x+3x^2+\ldots$

(2) What is $A(x) \times B(x)$?

$= (1+x+x^2+\ldots)(2+2x+2x^2+\ldots)$

$= 2+4x+6x^2+\ldots$

Addition and multiplication

Formally the definition of addition and multiplication are as follows

Definition

If $A(x) = a_0 + a_1 x + \dots$ and $B(x) = b_0 + b_1 x + \dots$ are generating functions,

Sum: $A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots = \sum_{n=0}^{\infty} (a_n + b_n)x^n$

Product: $A(x) \times B(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n$

Note

For any number n, deciding the coefficient of x^n in the sum $A(x) + B(x)$ or the product $A(x) \times B(x)$ only involves finitely many terms, so these are always well defined and can be computed.

In practice, we just operate with generating functions like we do with polynomials.

Coins

Define the generating functions

$$N(x) = 1 + x^5 + x^{10} + x^{15} + \dots \text{ (for nickels)}$$
$$D(x) = 1 + x^{10} + x^{20} + x^{30} + \dots \text{ (for dimes)}$$
$$Q(x) = 1 + x^{25} + x^{50} + x^{75} + \dots \text{ (for quarters)}$$

Recall that the coefficient of x^k in the product $N(x) \times D(x) \times Q(x)$ tells us how many ways you can express k cents as a sum of nickels, dimes, and quarters.

Example. Use the following expression to determine the actual number (not just an expression) of ways of expressing 30 cents as a sum of nickels, dimes, and quarters.

$$\left(1+x^5+x^{10}+x^{15}+x^{20}+x^{25}+x^{30}+\dots\right)\left(1+x^{10}+x^{20}+x^{30}+\dots\right)\left(1+x^{25}+\dots\right)$$

Issues: To take advantage of this framework we need:

(1) Efficient ways to describe generating functions like $N(x)$, $D(x)$, and $Q(x)$

(2) Tools to extract the coefficients of the generating functions we construct.

Multiplication

Multiplication of real numbers has many nice features. In addition to commutativity and associativity, we have the following.

Real numbers.

[identity] The number 1 has the property that $1 \cdot x = x$ for every x.

[inverses] Every nonzero real number x has an inverse $\frac{1}{x}$ so that $x \cdot \frac{1}{x} = 1$

Multiplication of generating functions is also commutative and associative. We define the following in analogy with real numbers.

Generating Functions.

[identity] The generating function 1 has the property that $1 \times A(x) = A(x)$ for every generating function $A(x)$.

[inverses] We say that two generating functions $A(x)$ and $B(x)$ are inverses if $A(x) \times B(x) = 1$. In this case we may write $B(x) = \frac{1}{A(x)}$

Note: Unlike numbers, not all nonzero generating functions have inverses!

Describing a GF using inverses

(1) Define the generating function $A(x) = 1 + x + x^2 + x^3 + \dots$.
Verify that $A(x) = \frac{1}{1-x}$ to check

$$(1-x)A(x) = (1-x)(1+x+x^2+x^3+x^4+\dots) = (1+x+x^2+x^3+\dots) - (x+x^2+x^3\dots) = 1$$

(2) Let $B(x) = 0 + x + 2x^2 + 3x^3 + \dots$.
Find $A(x) + B(x)$

$$= 1 + 2x + 3x^2 + 4x^3 + \dots$$

What is $x(A(x) + B(x))$?

$$xA(x) + xB(x) = x + 2x^2 + 3x^3 + 4x^4 + \dots = B(x)$$

Use the previous equation and solve for $B(x)$

$$\frac{x}{1-x} = xA(x) = (1-x)B(x)$$

$$\boxed{B(x) = \frac{x}{(1-x)^2}}$$

Describing a GF using inverses

(3) Let $C(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots = \sum_{n=0}^{\infty} 2^n x^n$

What is $2xC(x)$? $= 2x + 4x^2 + 8x^3 + \ldots$

$C(x) - 2xC(x) = 1 \qquad (1-2x)C(x) = 1 \qquad C(x) = \frac{1}{1-2x}$

Solve for $C(x)$

(4) Let $N(x) = 1 + x^5 + x^{10} + x^{15} + \ldots$ (the GF for nickels)

What is $x^5 N(x)$? $= x^5 + x^{10} + x^{15} + \ldots$

$N(x) - x^5 N(x) = 1 \qquad (1-x^5)N(x) = 1$

Solve for $N(x)$

$$N(x) = \frac{1}{1-x^5}$$

Note

Although generating functions are generally infinite objects, many important ones can be expressed very compactly using inverses.

Example. The nickel, dime, and quarter generating functions are

$$N(x) = \frac{1}{1-x^5} \qquad D(x) = \frac{1}{1-x^{10}} \qquad Q(x) = \frac{1}{1-x^{25}}$$

The product of these three is a generating function where the coefficient of x^n is the number of ways we can express n cents as a sum of nickels, dimes, and quarters. We can now express this GF as

$$N(x) \times D(x) \times Q(x) = \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{25}}$$

At this point we have a very compact and convenient description of our coin-possibility-counting generating function. However, in order to utilize it, we still need tools to extract (i.e. determine) its coefficients!