MACM 201 Homework 3 - *Solutions*

1. How many subgraphs of K_n are isomorphic to $K_{1,4}$?

   Solution: To choose $K_{1,4}$ we need to choose a 1 element subset of the vertices, a 4 element subset of the vertices, and another subset with the remaining vertices. This can be done in $\binom{n}{1,4,n-5}$ ways so this is the answer.

2. How many subgraphs of K_n are isomorphic to $K_{3,7}$?

   Solution: To choose $K_{3,7}$ we need to choose a 3 element subset of the vertices, a 7 element subset of the vertices, and another subset with the remaining vertices. This can be done in $\binom{n}{3,7,n-10}$ ways so this is the answer.

3. How many subgraphs of $K_{n,n}$ are isomorphic to $K_{3,7}$?

   Solution: To choose $K_{3,7}$ we can choose a 3 element set from the first set in the bipartition and a 7 element set from the second or vice versa. This gives a total of $2\binom{n}{3}\binom{n}{7}$ subgraphs.

4. Let K_5^- be a graph obtained from K_5 by deleting one edge. How many subgraphs of K_n are isomorphic to K_5^-?

   Solution: To choose a subgraph of K_n isomorphic to K_5^- we can first choose a 5 element subset of the vertices, and then choose any one of the $\binom{5}{2}$ edges to delete. This gives a total of $\binom{n}{5}\binom{5}{2} = \frac{n!}{5!(n-5)!}5!3!2! = \frac{n!}{2!3!(n-5)!}$ isomorphic subgraphs. We get the same answer by observing that in order to choose K_5^- we need to choose 3 vertices to be part of the K_5^- subgraph and not incident with the "missing edge" and 2 vertices to be part of the K_5 and incident with the "missing edge". This gives a total of $\binom{n}{2,3,n-5}$ subgraphs.

5. Let G be a graph with n vertices and the property that every vertex in G is incident with d edges. How many walks of length r are there in the graph G?

   Solution: The walk will have length r so it can be specified using a sequence of $r+1$ vertices with each consecutive pair adjacent. We have n choices for the first vertex, and for each subsequent vertex there are d possibilities (since each vertex is adjacent to d others). This gives us a total of nd^r walks of length r.

Adjacent edges. We say that two distinct edges in a graph are *adjacent* if there is a vertex incident to both (or equivalently, if $e \cap f \neq \emptyset$).

6. For the graph K_n, determine the number of sets of two edges $\{e, f\}$ with the property that e, f are adjacent. Then determine the number of those sets for which e, f are non-adjacent.

   Solution: To choose a set of two adjacent edges $\{e, f\}$ we first choose the common vertex v. This can be done in n ways. Then we have to choose a set of 2 other vertices $\{w, u\}$ to form the edges $\{v, w\}$ and $\{v, u\}$. Since $\{w, u\}$ is a subset of size 2 from an $n-1$ element set (all vertices except v) this can be done in $\binom{n-1}{2}$ ways. We deduce that there are $n\binom{n-1}{2}$ sets consisting of two adjacent edges. The total number of sets consisting of 2 edges is the size of the edge set choose 2 which is $\binom{\binom{n}{2}}{2}$. Therefore, the number of sets consisting of two non-adjacent edges equals $\binom{\binom{n}{2}}{2} - n\binom{n-1}{2}$.

7. Using the previous exercise, let W_4 be a graph obtained from K_5 by deleting two non-adjacent edges. How many subgraphs of K_n are isomorphic to W_4?

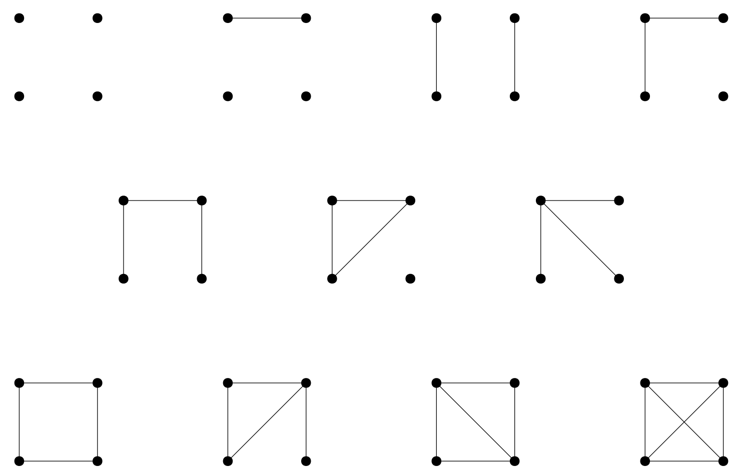
   Solution: By the previous exercise the number of sets consisting of two non-adjacent edges of K_5 is $\binom{\binom{5}{2}}{2} - 5\binom{4}{2} = 15$. To choose a W_4 we first choose a K_5 subgraph which can be done in $\binom{n}{5}$ ways, and from this we delete two non-adjacent edges which can be done in 15 ways. Therefore, the total number of subgraphs of K_n isomorphic to W_4 is equal to $\binom{n}{5}15$.

Working up to Isomorphism. In many situations, we wish to regard isomorphic graphs as the same. In these situations we say that we are operating *up to isomorphism.* For instance, up to isomorphism, there are just 4 graphs on three vertices, namely

Observe that the vertices in the above graphs have not been assigned labels. This is because the labels here don't matter. When we are working up to isomorphism, we regard two graphs with the same drawing as the same. Accordingly, when working up to isomorphism we will always use unlabelled graphs.

8. Up to isomorphism, find all graphs on 4 vertices.

   Solution:



9. Up to isomorphism, find all graphs on 5 vertices with 4 edges.

   Solution: