MACM 201 - Discrete Mathematics

11. Recurrence relations IV

Department of Mathematics

Simon Fraser University

Non-homogeneous linear recurrences

We will consider recurrence relations of one of the following forms:

$$
\begin{aligned}
ax_n + bx_{n-1} &= f(n) \qquad && n \geq 1 \\
ax_n + bx_{n-1} + cx_{n-2} &= f(n) \qquad && n \geq 2
\end{aligned}
$$

Recall: When $f = 0$ the relation is **homogeneous** and we have learned how to find general solutions.

Goal: When $f \neq 0$ the relation is **nonhomogeneous** and we will develop techniques to solve such recurrences for certain functions f.

Associated homogeneous relations

Definition

For a recurrence relation of the form

(1) $ax_n + bx_{n-1} = f(n) \qquad n \geq 1$

(2) $ax_n + bx_{n-1} + cx_{n-2} = f(n) \qquad n \geq 2$

the **associated homogeneous relation** is obtained by setting f to be 0

(1) $ax_n + bx_{n-1} = 0 \qquad n \geq 1$

(2) $ax_n + bx_{n-1} + cx_{n-2} = 0 \qquad n \geq 2$

Note (on terminology)

- *A* **particular solution** *is a single sequence x_n satisfying a recurrence (without the initial condition)*
- *The* **general solution** *to a recurrence is the set of all sequences x_n satisfying it (no initial condition)*
- *A* **unique solution** *is the unique answer to a recurrence with given initial conditions.*

Example

Problem: For the recurrence $x_n - 4x_{n-1} + 3x_{n-2} = 2^n$,

(a) Find the associated homogeneous recurrence.

(b) Find the general solution to part (a)

(c) Check that $x_n = -2^n$ is a particular solution to the non-homogeneous recurrence. (i.e. this sequence satisfies the recurrence).

(d) Verify that taking $x_n = -2^n$ from (c) and adding the general solution from (b) gives a solution to the non-homogeneous recurrence.

General solutions for non-homogeneous

Theorem

The general solution to a non-homogeneous recurrence is given by one particular solution plus the general solution to the associated homogeneous equation.

Note: As before, to obtain a unique solution satisfying some initial conditions we take the general and solve for the unknown constants.

Problem. Find a closed form solution to the following recurrence relation

$$x_0 = 5, \quad x_1 = 6 \quad \text{and} \quad x_n - 4x_{n-1} + 3x_{n-2} = 2^n \quad \text{for } n \geq 2$$

General Method

Solving non-homogeneous recurrence relations

(1) Find a particular solution to the non-homogeneous recurrence.

(2) Find the general solution to the associated homogeneous recurrence.

(3) Adding (1) and (2) then gives the general solution to the non-homog. recurrence.

(4) To find a unique solution satisfying given initial conditions, take the general solution from (3) and solve for the constants.

Idea: If the function f in the nonhomogenous equation is exponential, say $f(n) = kr^n$, we look for a solution of the form $x_n = Cr^n$ by solving for the unknown coefficient C.

Problem. Find a particular solution to $x_n - 6x_{n-1} = 3^n$

Now assume $x_0 = 7$ and find a unique solution.

Finding particular solutions

Note

To find a particular solution to a non-homog. recurrence of the form

$$ax_n + bx_{n-1} = f(n) \qquad n \geq 1$$

$$ax_n + bx_{n-1} + cx_{n-2} = f(n) \qquad n \geq 2$$

(1) Exponential functions $f(n) = kr^n$

- *(a) Look for a solution of the form* $x_n = Cr^n$
- *(b) If (a) fails, try* $x_n = Cnr^n$
- *(c) If (b) fails, try* $x_n = Cn^2r^n$

(2) Power functions $f(n) = kn^d$

- *(a) Look for a solution of the form* $f(n) = a_d n^d + a_{d-1}n^{d-1} \ldots + a_1 n + a_0$
- *(b) If (a) fails, multiply the form from (a) by* n *and try again.*
- *(c) If (b) fails, multiply the form from (b) by* n *and try again.*

Problem. Find a particular solution to $x_n + x_{n-1} - 6x_{n-2} = 2^n$

Problem. Find a particular solution to $x_n - 3x_{n-1} + 2x_{n-2} = 4n$