Quiz 6 - MACM 201 - *Solutions*

[4 pts] Define a generating function $Q(x)$ with the property that $[x^n]Q(x)$ is the number of solutions to

$$a_1 + a_2 + a_3 + a_4 = n$$

with $a_1, a_2, a_3, a_4 \geq 0$ and a_1, a_2 even.

Solution: We construct $Q(x)$ as a product of four factors where the i^{th} factor corresponds to a_i. In each case the permitted values for a_i are encoded using exponents. Since a_1, a_2 are even, they can have any value in $\{0, 2, 4, 6, \ldots\}$ while a_3, a_4 take a value in $\{0, 1, 2, 3, \ldots\}$. Therefore we may define our generating function as

$$Q(x) = \left(1 + x^2 + x^4 + x^6 + \ldots\right)^2 \left(1 + x + x^2 + x^3 + \ldots\right)^2$$

or equivalently,

$$Q(x) = \left(\sum_{n=0}^{\infty} x^{2n}\right)^2 \left(\sum_{n=0}^{\infty} x^n\right)^2$$

[4 pts] Define a polynomial $P(x)$ so that $[x^k]P(x)$ is the number of strings of length 99 over $\{0, 1, 2\}$ with k nonzero symbols.

Solution: Declare a symbol of 1 or 2 as having value 1 since this contributes 1 to the count of nonzero symbols, and declare 0 to have value 0. With this terminology we want the coefficient of x^k to be the number of strings of length 99 over $\{0, 1, 2\}$ with total value equal to k. For each of the 99 positions in our string, there are 2 ways to put a symbol with value 1 and 1 way to put a symbol of value 0. Using exponents to encode value, we associate this position with a factor of the form $(1 + 2x)$. This gives the full polynomial

$$P(x) = (1 + 2x)^{99}$$

as the answer to the problem.