MATH 308 D200, Fall 2019

16. Duality equation and complementary slackness

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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Reading the primal dual solutions from the final primal tableau

$$\begin{array}{cccc|c|l}
x_1 & x_3 & t_1 & t_2 & -1 & \\
\hline
2 & -4 & -5 & 3 & 8 & = -x_2 \\
1 & 1 & -2 & 1 & 5 & = -x_4 \\
5 & -1 & 4 & -2 & 3 & = -t_3 \\
\hline
-1 & -2 & -5 & -4 & -10 & = f \\
\hline
\end{array}$$

Reading the primal dual solutions from the final primal tableau

indep. variables

$$\begin{array}{|cccc|c|l} \multicolumn{1}{c}{x_1} & x_3 & t_1 & \multicolumn{1}{c}{t_2} & \multicolumn{1}{c}{-1} & \\ \hline 2 & -4 & -5 & 3 & 8 & = -x_2 \\ 1 & 1 & -2 & 1 & 5 & = -x_4 \\ 5 & -1 & 4 & -2 & 3 & = -t_3 \\ \hline -1 & -2 & -5 & -4 & -10 & = f \\ \hline \end{array}$$

dep. variables

Primal: Identify the independent variables
and the primal variables x_j
and the primal slack variables t_i

Reading the primal dual solutions from the final primal tableau

indep. variables

$$\begin{array}{cccc|c|l} x_1 & x_3 & t_1 & t_2 & -1 & \\ \hline 2 & -4 & -5 & 3 & 8 & = -x_2 \\ 1 & 1 & -2 & 1 & 5 & = -x_4 \\ 5 & -1 & 4 & -2 & 3 & = -t_3 \\ \hline -1 & -2 & -5 & -4 & -10 & = f \\ \hline \end{array}$$

dep. variables

Primal: Identify the independent variables
and the primal variables x_j
and the primal slack variables t_i

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 5 \end{bmatrix}, \quad \boldsymbol{t} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Reading the primal dual solutions from the final primal tableau

indep. variables

indep. dual variables

$$\begin{array}{c|cccc|c|l}
 & x_1 & x_3 & t_1 & t_2 & -1 & \\ \hline
s_2 & 2 & -4 & -5 & 3 & 8 & = -x_2 \\
s_4 & 1 & 1 & -2 & 1 & 5 & = -x_4 \\
y_3 & 5 & -1 & 4 & -2 & 3 & = -t_3 \\ \hline
-1 & -1 & -2 & -5 & -4 & -10 & = f \\ \hline
 & = s_1 & = s_3 & = y_1 & = y_2 & = g &
\end{array}$$

dep. variables

dep. dual variables

Primal: Identify the independent variables
and the primal variables x_j
and the primal slack variables t_i

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 5 \end{bmatrix}, \quad \boldsymbol{t} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Dual: Identify the independent dual variables
and the dual variables y_i $\quad \boldsymbol{t} \leftrightarrow \boldsymbol{y}$
and the dual slack variables s_j $\quad \boldsymbol{x} \leftrightarrow \boldsymbol{s}$

Reading the primal dual solutions from the final primal tableau

indep. variables

indep. dual variables

$$\begin{array}{c|cccc|c|l} & x_1 & x_3 & t_1 & t_2 & -1 & \\ \hline s_2 & 2 & -4 & -5 & 3 & 8 & = -x_2 \\ s_4 & 1 & 1 & -2 & 1 & 5 & = -x_4 \\ y_3 & 5 & -1 & 4 & -2 & 3 & = -t_3 \\ \hline -1 & -1 & -2 & -5 & -4 & -10 & = f \\ \hline & = s_1 & = s_3 & = y_1 & = y_2 & = g & \end{array}$$

dep. variables

dep. dual variables

Primal: Identify the independent variables
and the primal variables x_j
and the primal slack variables t_i

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 5 \end{bmatrix}, \quad \boldsymbol{t} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Dual: Identify the independent dual variables
and the dual variables y_i $\boldsymbol{t} \leftrightarrow \boldsymbol{y}$
and the dual slack variables s_j $\boldsymbol{x} \leftrightarrow \boldsymbol{s}$

$$\boldsymbol{y} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}, \quad \boldsymbol{s} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

The Duality Equation

Initial Tableau:

$$\begin{array}{c|cccc|c|l}
 & x_1 & x_2 & \dots & x_n & -1 & \\ \hline
y_1 & a_{11} & a_{12} & \dots & a_{1n} & b_1 & = -t_1 \\
y_2 & a_{21} & a_{22} & \dots & a_{2n} & b_2 & = -t_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
y_m & a_{m1} & a_{m2} & \dots & a_{mn} & b_m & = -t_m \\ \hline
-1 & c_1 & c_2 & \dots & c_n & d & = f \\ \hline
 & = s_1 & = s_2 & \dots & = s_n & = g &
\end{array}$$

Primal LP (P)

$$\begin{aligned} \text{Maximize } & f(\boldsymbol{x}) = \boldsymbol{c}^\intercal \boldsymbol{x} - d \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b} \\ & \boldsymbol{x} \geqslant \boldsymbol{0} \end{aligned}$$

Dual LP (D)

$$\begin{aligned} \text{Minimize } & g(\boldsymbol{x}) = \boldsymbol{b}^\intercal \boldsymbol{y} - d \\ \text{subject to } & \boldsymbol{A}^\intercal \boldsymbol{y} \geqslant \boldsymbol{c} \\ & \boldsymbol{y} \geqslant \boldsymbol{0} \end{aligned}$$

The Duality Equation

Initial Tableau:

$$\begin{array}{c|cccc|c|l}
 & x_1 & x_2 & \dots & x_n & -1 & \\ \hline
y_1 & a_{11} & a_{12} & \dots & a_{1n} & b_1 & = -t_1 \\
y_2 & a_{21} & a_{22} & \dots & a_{2n} & b_2 & = -t_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
y_m & a_{m1} & a_{m2} & \dots & a_{mn} & b_m & = -t_m \\ \hline
-1 & c_1 & c_2 & \dots & c_n & d & = f \\ \hline
 & = s_1 & = s_2 & \dots & = s_n & = g &
\end{array}$$

Primal LP (P)

$$\begin{aligned} \text{Maximize } f(\boldsymbol{x}) &= \boldsymbol{c}^\top \boldsymbol{x} - d \\ \text{subject to } \boldsymbol{A}\boldsymbol{x} &\leqslant \boldsymbol{b} \\ \boldsymbol{x} &\geqslant \boldsymbol{0} \end{aligned}$$

Dual LP (D)

$$\begin{aligned} \text{Minimize } g(\boldsymbol{x}) &= \boldsymbol{b}^\top \boldsymbol{y} - d \\ \text{subject to } \boldsymbol{A}^\top \boldsymbol{y} &\geqslant \boldsymbol{c} \\ \boldsymbol{y} &\geqslant \boldsymbol{0} \end{aligned}$$

$$\begin{aligned} \text{Maximize } f(\boldsymbol{x}) &= \boldsymbol{c}^\top \boldsymbol{x} - d \\ \text{subject to } -\boldsymbol{t} &= \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{t} &\geqslant \boldsymbol{0} \end{aligned}$$

$$\begin{aligned} \text{Minimize } g(\boldsymbol{x}) &= \boldsymbol{y}^\top \boldsymbol{b} - d \\ \text{subject to } \boldsymbol{s}^\top &= \boldsymbol{y}^\top \boldsymbol{A} - \boldsymbol{c}^\top \\ \boldsymbol{y}, \boldsymbol{s} &\geqslant \boldsymbol{0} \end{aligned}$$

$$\begin{aligned} &\text{Maximize } f(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - d \\ &\text{subject to } \; -\boldsymbol{t} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \\ &\qquad\qquad \boldsymbol{x}, \boldsymbol{t} \geqslant \boldsymbol{0} \end{aligned}$$

$$\begin{aligned} &\text{Minimize } g(\boldsymbol{x}) = \boldsymbol{y}^{\mathsf{T}}\boldsymbol{b} - d \\ &\text{subject to } \boldsymbol{s}^{\mathsf{T}} = \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} - \boldsymbol{c}^{\mathsf{T}} \\ &\qquad\qquad \boldsymbol{y}, \boldsymbol{s} \geqslant \boldsymbol{0} \end{aligned}$$

Theorem (Tucker Duality Equation)

For any feasible solution $(\boldsymbol{x}, \boldsymbol{t})$ of the primal slack LP problem and any feasible solution $(\boldsymbol{y}, \boldsymbol{s})$ of the dual slack LP problem we have

$$g(\boldsymbol{y}) - f(\boldsymbol{x}) = \boldsymbol{s}^{\mathsf{T}}\boldsymbol{x} + \boldsymbol{y}^{\mathsf{T}}\boldsymbol{t} \quad .$$

Proof.

□

$$\begin{aligned} \text{Maximize } & f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d \\ \text{subject to } & -\boldsymbol{t} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{t} \geqslant \boldsymbol{0} \end{aligned}$$

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Theorem (Tucker Duality Equation)

For any feasible solution $(\boldsymbol{x}, \boldsymbol{t})$ *of the primal slack LP problem and any feasible solution* $(\boldsymbol{y}, \boldsymbol{s})$ *of the dual slack LP problem we have*

$$g(\boldsymbol{y}) - f(\boldsymbol{x}) = \boldsymbol{s}^\mathsf{T}\boldsymbol{x} + \boldsymbol{y}^\mathsf{T}\boldsymbol{t} \ .$$

Proof.

Since $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{t}$ and $\boldsymbol{c}^\mathsf{T} = \boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{s}^\mathsf{T}$ we have

$$\begin{aligned} g(\boldsymbol{y}) - f(\boldsymbol{x}) &= (\boldsymbol{y}^\mathsf{T}\boldsymbol{b} - d) - (\boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d) \\ &= \boldsymbol{y}^\mathsf{T}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{t}) - d - (\boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{s}^\mathsf{T})\boldsymbol{x} + d \end{aligned}$$

□

$$\begin{aligned} \text{Maximize } & f(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - d \\ \text{subject to } & -\boldsymbol{t} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{t} \geqslant \boldsymbol{0} \end{aligned}$$

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Proof.

Since $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{t}$ and $\boldsymbol{c}^{\mathsf{T}} = \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} - \boldsymbol{s}^{\mathsf{T}}$ we have

$$\begin{aligned} g(\boldsymbol{y}) - f(\boldsymbol{x}) &= (\boldsymbol{y}^{\mathsf{T}}\boldsymbol{b} - d) - (\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} - d) \\ &= \boldsymbol{y}^{\mathsf{T}}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{t}) - d - (\boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} - \boldsymbol{s}^{\mathsf{T}})\boldsymbol{x} + d \\ &= \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{y}^{\mathsf{T}}\boldsymbol{t} - \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{s}^{\mathsf{T}}\boldsymbol{x} \end{aligned}$$

□

$$\begin{aligned} \text{Maximize } & f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d \\ \text{subject to } & -\boldsymbol{t} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{t} \geqslant \boldsymbol{0} \end{aligned} \qquad \begin{aligned} \text{Minimize } & g(\boldsymbol{x}) = \boldsymbol{y}^\mathsf{T}\boldsymbol{b} - d \\ \text{subject to } & \boldsymbol{s}^\mathsf{T} = \boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{c}^\mathsf{T} \\ & \boldsymbol{y}, \boldsymbol{s} \geqslant \boldsymbol{0} \end{aligned}$$

Theorem (Tucker Duality Equation)

For any feasible solution $(\boldsymbol{x}, \boldsymbol{t})$ *of the primal slack LP problem and any feasible solution* $(\boldsymbol{y}, \boldsymbol{s})$ *of the dual slack LP problem we have*

$$g(\boldsymbol{y}) - f(\boldsymbol{x}) = \boldsymbol{s}^\mathsf{T}\boldsymbol{x} + \boldsymbol{y}^\mathsf{T}\boldsymbol{t} \ .$$

Proof.

Since $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{t}$ and $\boldsymbol{c}^\mathsf{T} = \boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{s}^\mathsf{T}$ we have

$$\begin{aligned} g(\boldsymbol{y}) - f(\boldsymbol{x}) &= (\boldsymbol{y}^\mathsf{T}\boldsymbol{b} - d) - (\boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d) \\ &= \boldsymbol{y}^\mathsf{T}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{t}) - d - (\boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{s}^\mathsf{T})\boldsymbol{x} + d \\ &= \boldsymbol{y}^\mathsf{T}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{y}^\mathsf{T}\boldsymbol{t} - \boldsymbol{y}^\mathsf{T}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{s}^\mathsf{T}\boldsymbol{x} \\ &= \boldsymbol{y}^\mathsf{T}\boldsymbol{t} + \boldsymbol{s}^\mathsf{T}\boldsymbol{x}. \end{aligned}$$

☐

Application of Duality Equation

Estimate by how much the current value of f (resp. g) differs from an optimal value in the current dual tableau.

$$\begin{array}{c|cc|c|l}
\multicolumn{1}{c}{} & x_1 & x_2 & \multicolumn{1}{c}{-1} & \\ \cline{2-4}
y_1 & 1 & 2 & 3 & = -t_1 \\
y_2 & 4 & 5 & 6 & = -t_2 \\ \cline{2-4}
-1 & 7 & 8 & 9 & = f \\ \cline{2-4}
\multicolumn{1}{c}{} & = s_1 & = s_2 & \multicolumn{1}{c}{= g} &
\end{array}$$

Application of Duality Equation

Note

Duality equation can also be used for infeasible solutions that satisfy ALL main constraints but fail to satisfy some of the non-negativity constraints, as its proof uses only the objective function and the main constraints.

Check that solutions $\vec{x} = (1, -1)$ and $\vec{y} = (1, 2)$ satisfy the duality equation in the tableau below. Notice, $\vec{x}$ is not a feasible solution.

$$
\begin{array}{c|cc|c|l}
 & x_1 & x_2 & -1 & \\
\cline{2-4}
y_1 & 1 & 2 & 3 & = -t_1 \\
y_2 & 4 & 5 & 6 & = -t_2 \\
\cline{2-4}
-1 & 7 & 8 & 9 & = f \\
\cline{2-4}
 & = s_1 & = s_2 & = g &
\end{array}
$$

Complementary Slackness

Definition (Complementary Slackness)

Any pair of feasible solutions $\boldsymbol{x}^*, \boldsymbol{y}^*$ of the dual canonical LP problems for which

(i) $x_j \neq 0 \Rightarrow s_j = 0$, for every $j = 1, 2, \ldots, n$, and

(ii) $y_i \neq 0 \Rightarrow t_i = 0$, for every $i = 1, 2, \ldots, m$

are said to exhibit *complementary slackness*.

Alternative form:

Theorem

A pair of feasible solutions $\boldsymbol{x}^*, \boldsymbol{y}^*$ of the dual canonical LP problems exhibit complementary slackness if and only if they are optimal solutions.

Complementary Slackness

Applications of Complementary Slackness

How is previous theorem useful?

$$\begin{array}{c|cc|c|l}
 & x_1 & x_2 & -1 & \\ \hline
y_1 & 1 & 1 & 2 & = -t_1 \\
y_2 & 1 & 2 & 3 & = -t_2 \\ \hline
-1 & 3 & 4 & 0 & = f \\ \hline
 & = s_1 & = s_2 & = g &
\end{array}$$

$$\begin{aligned}
\text{maximize} \quad & f(x_1, x_2) = 3x_1 + 4x_2 \\
\text{subject to} \quad & x_1 + x_2 \leqslant 2 \\
& x_1 + 2x_2 \leqslant 3 \\
& x_1, x_2 \geqslant 0
\end{aligned}$$

Is $\boldsymbol{x} = (1, 0)$ an optimal solution of this problem? Try to avoid using SA...

Applications of Complementary Slackness

How is previous theorem useful?

$$\begin{array}{c|cc|c|l}
 & x_1 & x_2 & -1 & \\
\hline
y_1 & 1 & 1 & 2 & = -t_1 \\
y_2 & 1 & 2 & 3 & = -t_2 \\
\hline
-1 & 3 & 4 & 0 & = f \\
\hline
 & = s_1 & = s_2 & = g &
\end{array}$$

$$\begin{array}{rl}
\text{maximize} & f(x_1, x_2) = 3x_1 + 4x_2 \\
\text{subject to} & x_1 + x_2 \leqslant 2 \\
 & x_1 + 2x_2 \leqslant 3 \\
 & x_1, x_2 \geqslant 0
\end{array}$$

Is $\boldsymbol{x} = (1, 1)$ an optimal solution of this problem? Try to avoid using SA...

Applications of Complementary Slackness

Verification of optimality without the optimal tableau:

Suppose you get vectors $(\vec{x}, \vec{t}) = (1, 1, 0, 0)$ and $(\vec{y}, \vec{s}) = (2, 1, 0, 0)$ as an optimal solution to

$$\begin{array}{cc|c|l} x_1 & x_2 & -1 & \\ \hline 1 & 1 & 2 & = -t_1 \\ 1 & 2 & 3 & = -t_2 \\ \hline 3 & 4 & 0 & = f \\ \hline \end{array}$$

How do you verify that those are indeed optimal solutions without solving the LP problem?

SA verification:

$$\begin{array}{cc|c|l} x_1 & x_2 & -1 & \\ \hline 1^* & 1 & 2 & = -t_1 \\ 1 & 2 & 3 & = -t_2 \\ \hline 3 & 4 & 0 & = f \\ \hline \end{array} \qquad \begin{array}{cc|c|l} t_1 & x_2 & -1 & \\ \hline 1 & 1 & 2 & = -x_1 \\ -1 & 1^* & 1 & = -t_2 \\ \hline -3 & 1 & -6 & = f \\ \hline \end{array} \qquad \begin{array}{cc|c|l} t_1 & t_2 & -1 & \\ \hline 2 & -1 & 1 & = -x_1 \\ -1 & 1 & 1 & = -x_2 \\ \hline -2 & -1 & -7 & = f \\ \hline \end{array}$$