MACM 201 - Discrete Mathematics

Generating functions IV - Infinite expressions

Department of Mathematics

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Strings with weights 1

Let $\mathcal{A} = \{A, B, C, D, E\}$ and declare A, B to have value 1 and C, D, E to have value 2.

Problem. Find a polynomial $P(x)$ so that $[x^k]P(x)$ is equal to the number of strings of length 99 over $\mathcal{A}$ with total value equal to k.

Strings with weights 2

Let $\mathcal{A} = \{A, B, C, D, E\}$ and declare A, B to have value 1 and C, D, E to have value 2.

Problem. Find a generating function $Q(x)$ so that $[x^k]Q(x)$ is the number of strings over $\mathcal{A}$ with weight k.

Balls and bins

Framework. We wish to count the number of ways to put k balls into n bins subject to:

(A) Balls can be numbered $1, 2, \ldots, k$ or indistinguishable

(B) Bins can be numbered $1, 2, \ldots, n$ or indistinguishable

We can also add two possible restrictions

(1) We may insist that each ball goes to a separate bin

(2) We may insist that each bin gets at least one ball.

This gives a total of $2^4 = 16$ different counting problems (many already encountered).

Case 1: Balls and bins distinguishable

Case 2: Balls indistinguishable, bins distinguishable

Partitions of a number

Definition

A **partition** of a positive integer n is an expression of the form $a_1 + a_2 + \ldots + a_k = n$ with the added constraints that $1 \leq a_1 \leq a_2 \leq \ldots \leq a_k$.

Example. Find all partitions of the number 6.

Note: Number partitions correspond to the balls and bins problem where both balls and bins are unlabelled.

GF's for number partitions

Definition

For every positive integer n we make the following definitions.

$p(n)$ is be the number of partitions of n.

$p_o(n)$ is the number of partitions of n where every term is odd.

$p_d(n)$ is the number of partitions of n where every term is distinct.

Note

All three of the above functions define counting sequences that are considered to be fundamental. No simple closed formula is known for any of these functions. However, we can operate with them as generating functions.

Definition

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n$$
$$P_o(x) = \sum_{n=0}^{\infty} p_o(n)x^n$$
$$P_d(x) = \sum_{n=0}^{\infty} p_d(n)x^n$$

So $P(x)$ is a generating function that tells us the number of partitions of n for every number n. More precisely, $[x^n]P(x)$ equals the number of partitions of n for every n.

Infinite products

Problem 1. Express $P(x)$ as an infinite product

Problem 2. Express $P_o(x)$ as an infinite product

Problem 3. Express $P_d(x)$ as an infinite product

Euler's Theorem

We begin with an easy factorization property

$$(1 - x^{2k}) = (1 - x^k)(1 + x^k) \qquad \text{so} \qquad \frac{1 - x^{2k}}{1 - x^k} = 1 + x^k$$

Now we use this to rewrite $P_d(x)$ as follows:

$P_d(x) =$