MACM 201 - Discrete Mathematics

6. Inclusion-exclusion

Department of Mathematics

Simon Fraser University

Lecture outline: inclusion-exclusion

Inclusion/exclusion is a relatively simple counting technique that allows to count the number of elements of a set that satisfy none of a list of given properties.

It is a very general technique, that has applications in many fields, from theoretical ones (computing the permanent of a matrix) to applied ones (estimating the reliability of a communication network).

The precise topics we will study are:

- The Inclusion/Exclusion: motivation, notations, statement, proof, examples.
- The Inclusion/Exclusion: proof and generalization
- The example of derangements.

Introductory example

Example. Count the number of integers in $[1, 30]$ that are not divisible by 2 and not divisible by 3 and not divisible by 5.

The inclusion-exclusion theorem

1. Let S be a set of N elements, i.e., $|S| = N$.
2. Let $C_1, C_2, \ldots, C_t$ be t properties that the elements of this set can satisfy.
3. We denote the complement of a property C_i by $\overline{C_i}$: an element of S satisfies property $\overline{C_i}$ if it does not satisfy property C_i.
4. For a given subset $\{i_1, i_2, \ldots, i_k\}$ of $[t]$, we denote by

$$N(C_{i_1} C_{i_2} \cdots C_{i_k})$$

the number of elements of S that satisfy **all of properties** $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$.
5. We denote by $\overline{N}$ the number

$$N(\overline{C_1}\,\overline{C_2} \cdots \overline{C_t}),$$

i.e., the number of elements of S that satisfy **none of the properties** $C_1, C_2, \ldots, C_t$.

Theorem (Inclusion-Exclusion)

$$\overline{N} = \sum_{k=0}^{t} (-1)^k S_k,$$

where $S_0 = N$, *and for* $k > 0$

$$S_k = \sum_{\{i_1, i_2, \ldots, i_k\} \subseteq [t]} N(C_{i_1} C_{i_2} \cdots C_{i_k}).$$

The inclusion-exclusion theorem

Note

The number S_k in previous theorem is not the number of elements from S that do satisfy at least k properties as some elements that satisfy more that k properties are counted more than once.

Corollary

The number of elements of S satisfying at least one of the properties is

$$N - N(\overline{C_1}\,\overline{C_2}\cdots\overline{C_t}).$$

Counting constrained solutions

Problem. Find the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 27$$

satisfying $0 \leq x_i \leq 10$ for $1 \leq i \leq 5$.

Strings with forbidden substrings

Problem. How many permutations of the letters of $CTAGCGAAAT$ do not contain either of the substrings CAC, GG, $ATTA$?