MATH 308 D200, Fall 2019

21. Examples of matrix games

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

Dr. Masood Masjoody

SFU Burnaby

Game 1: Modified Rock, Paper, Scissors (Texbook Example 1, p. 117)

Find the optimal mixed strategies for the following matrix game

Two players, say Ron ("odd" player) and Coleen ("even" player), each secretly choose an integer from $\{1, 2, 3\}$. Both players reveal their numbers simultaneously.

∘ If the sum of the numbers is odd, Coleen wins \$3 from Ron.

∘ If the sum of the numbers is even, Ron wins \$$x$ from Coleen where:

- x equals to the difference of the numbers, provided they are different numbers.
- x equals the sum of the numbers, provided they are the same number

Game 1: Modified Rock, Paper, Scissors (Texbook Example 1, p. 117)

Find the optimal mixed strategies for the following matrix game

Two players, say Ron ("odd" player) and Coleen ("even" player), each secretly choose an integer from $\{1, 2, 3\}$. Both players reveal their numbers simultaneously.

◦ If the sum of the numbers is odd, Coleen wins \$3 from Ron.

◦ If the sum of the numbers is even, Ron wins \$$x$ from Coleen where:

- x equals to the difference of the numbers, provided they are different numbers.
- x equals the sum of the numbers, provided they are the same number

Six step solution:

Game 1: Modified Rock, Paper, Scissors (Texbook Example 1, p. 117)

Find the optimal mixed strategies for the following matrix game
Two players, say Ron ("odd" player) and Coleen ("even" player), each secretly choose an integer from $\{1, 2, 3\}$. Both players reveal their numbers simultaneously.

- If the sum of the numbers is odd, Coleen wins \$3 from Ron.
- If the sum of the numbers is even, Ron wins \$$x$ from Coleen where:
  - x equals to the difference of the numbers, provided they are different numbers.
  - x equals the sum of the numbers, provided they are the same number

Six step solution:

1. Set up the payoff matrix in terms of Ron's winnings.

		Coleen (even) 1	2	3
Ron (odd)	1	2	-3	2
	2	-3	4	-3
	3	2	-3	6

Game 1: Modified Rock, Paper, Scissors (Texbook Example 1, p. 117)

Find the optimal mixed strategies for the following matrix game

Two players, say Ron ("odd" player) and Coleen ("even" player), each secretly choose an integer from $\{1, 2, 3\}$. Both players reveal their numbers simultaneously.

◦ If the sum of the numbers is odd, Coleen wins $3 from Ron.

◦ If the sum of the numbers is even, Ron wins \$$x$ from Coleen where:

- x equals to the difference of the numbers, provided they are different numbers.
- x equals the sum of the numbers, provided they are the same number

Six step solution:

1. Set up the payoff matrix in terms of Ron's winnings.
2. Domination reduce the payoff matrix (optional).

		Coleen (even) 1	2	3
Ron (odd)	1	2	-3	2
	2	-3	4	-3
	3	2	-3	6

Game 1: Modified Rock, Paper, Scissors (Texbook Example 1, p. 117)

Find the optimal mixed strategies for the following matrix game

Two players, say Ron ("odd" player) and Coleen ("even" player), each secretly choose an integer from $\{1, 2, 3\}$. Both players reveal their numbers simultaneously.

◦ If the sum of the numbers is odd, Coleen wins \$3 from Ron.

◦ If the sum of the numbers is even, Ron wins \$$x$ from Coleen where:

- x equals to the difference of the numbers, provided they are different numbers.
- x equals the sum of the numbers, provided they are the same number

Six step solution:

1. Set up the payoff matrix in terms of Ron's winnings.
2. Domination reduce the payoff matrix (optional).

$$\begin{array}{cc|c|c|c|} & & \multicolumn{3}{c}{\text{Coleen (even)}} \\ & & 1 & 2 & 3 \\ \hline & 1 & 2 & -3 & 2 \\ \hline \text{Ron (odd)} & 2 & -3 & 4 & -3 \\ \hline & 3 & 2 & -3 & 6 \\ \hline \end{array} \mapsto \begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Coleen (even)}} \\ & & 1 & 2 \\ \hline & 1 & 2 & -3 \\ \hline \text{Ron (odd)} & 2 & -3 & 4 \\ \hline & 3 & 2 & -3 \\ \hline \end{array}$$

Game 1: Modified Rock, Paper, Scissors (Texbook Example 1, p. 117)

Find the optimal mixed strategies for the following matrix game

Two players, say Ron ("odd" player) and Coleen ("even" player), each secretly choose an integer from $\{1, 2, 3\}$. Both players reveal their numbers simultaneously.

◦ If the sum of the numbers is odd, Coleen wins \$3 from Ron.

◦ If the sum of the numbers is even, Ron wins \$$x$ from Coleen where:

- x equals to the difference of the numbers, provided they are different numbers.
- x equals the sum of the numbers, provided they are the same number

Six step solution:

1. Set up the payoff matrix in terms of Ron's winnings.
2. Domination reduce the payoff matrix (optional).

Ron (odd) \ Coleen (even)	1	2	3
1	2	-3	2
2	-3	4	-3
3	2	-3	6

$\mapsto$

Ron (odd) \ Coleen (even)	1	2
1	2	-3
2	-3	4
3	2	-3

Game 1: Modified Rock, Paper, Scissors (Texbook Example 1, p. 117)

Find the optimal mixed strategies for the following matrix game

Two players, say Ron ("odd" player) and Coleen ("even" player), each secretly choose an integer from $\{1, 2, 3\}$. Both players reveal their numbers simultaneously.

- If the sum of the numbers is odd, Coleen wins \$3 from Ron.
- If the sum of the numbers is even, Ron wins \$$x$ from Coleen where:
  - x equals to the difference of the numbers, provided they are different numbers.
  - x equals the sum of the numbers, provided they are the same number

Six step solution:

1. Set up the payoff matrix in terms of Ron's winnings.
2. Domination reduce the payoff matrix (optional).

$$\begin{array}{cc} & \text{Coleen (even)} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{Ron (odd)} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{array}{|c|c|c|} \hline 2 & -3 & 2 \\ \hline -3 & 4 & -3 \\ \hline 2 & -3 & 6 \\ \hline \end{array} \end{array} \mapsto \begin{array}{cc} & \text{Coleen (even)} \\ & \begin{array}{cc} 1 & 2 \end{array} \\ \text{Ron (odd)} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{array}{|c|c|} \hline 2 & -3 \\ \hline -3 & 4 \\ \hline 2 & -3 \\ \hline \end{array} \end{array} \mapsto \begin{array}{cc} & \text{Coleen (even)} \\ & \begin{array}{cc} 1 & 2 \end{array} \\ \text{Ron (odd)} \begin{array}{c} 2 \\ 3 \end{array} & \begin{array}{|c|c|} \hline -3 & -4 \\ \hline 2 & -3 \\ \hline \end{array} \end{array}$$

Game 1: Modified Rock, Paper, Scissors (Texbook Example 1, p. 117)

Find the optimal mixed strategies for the following matrix game

Two players, say Ron ("odd" player) and Coleen ("even" player), each secretly choose an integer from $\{1, 2, 3\}$. Both players reveal their numbers simultaneously.

∘ If the sum of the numbers is odd, Coleen wins \$3 from Ron.

∘ If the sum of the numbers is even, Ron wins \$$x$ from Coleen where:

- x equals to the difference of the numbers, provided they are different numbers.
- x equals the sum of the numbers, provided they are the same number

Six step solution:

1. Set up the payoff matrix in terms of Ron's winnings.
2. Domination reduce the payoff matrix (optional).

Coleen (even)

Ron (odd)	1	2	3
1	2	-3	2
2	-3	4	-3
3	2	-3	6

$\mapsto$

Coleen (even)

Ron (odd)	1	2
1	2	-3
2	-3	4
3	2	-3

$\mapsto$

Coleen (even)

Ron (odd)	1	2
2	-3	-4
3	2	-3

3. Set up the game tableau:

$$
\begin{array}{c|ccc|c|l}
 & v & q_1 & q_2 & -1 & \\
\hline
u & 0 & -1 & -1 & -1 & = -0 \\
p_2 & -1 & -3 & 4 & 0 & = -t_2 \\
p_3 & -1 & 2 & -3 & 0 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & = s_2 & = s_3 & = g &
\end{array}
$$

$$
\begin{array}{c|ccc|c|l}
 & v & q_1 & q_2 & -1 & \\
\hline
u & 0 & -1 & -1 & -1 & = -0 \\
p_2 & -1 & -3 & 4 & 0 & = -t_2 \\
p_3 & -1 & 2^\circ & -3 & 0 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & = s_2 & = s_3 & = g &
\end{array}
$$

4. Find first two pivots:
   The 2 and 4 are largest in their column.
   The 2° is the smaller of these.
   Pivot and delete twice.

$$\begin{array}{c|ccc|c|l}
 & \textcircled{v} & q_1 & q_2 & -1 & \\
\hline
\textcircled{u} & 0 & -1^{*} & -1 & -1 & = -0 \\
p_2 & -1 & -3 & 4 & 0 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -3 & 0 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & = s_2 & = s_3 & = g &
\end{array}$$

4. Find first two pivots:
   The 2 and 4 are largest in their column.
   The 2° is the smaller of these.
   Pivot and delete twice.

$$
\begin{array}{c|ccc|c|l}
 & v & q_1 & q_2 & -1 & \\
\hline
u & 0 & -1^{*} & -1 & -1 & = -0 \\
p_2 & -1 & -3 & 4 & 0 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -3 & 0 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & = s_2 & = s_3 & = g &
\end{array}
$$

4. Find first two pivots:
   The 2 and 4 are largest in their column.
   The 2° is the smaller of these.
   Pivot and delete twice.

$$
\begin{array}{c|ccc|c|l}
 & v & 0 & q_2 & -1 & \\
\hline
s_2 & 0 & -1 & 1 & 1 & = -q_1 \\
p_2 & -1 & -3 & 7 & 3 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -5 & -2 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & u & = s_3 & = g &
\end{array}
$$

	v	q_1	q_2	-1	
u	0	-1*	-1	-1	$= -0$
p_2	-1	-3	4	0	$= -t_2$
p_3	-1**	2°	-3	0	$= -t_3$
-1	-1	0	0	0	$= f$
	$= 0$	$= s_2$	$= s_3$	$= g$	

4. Find first two pivots:
   The 2 and 4 are largest in their column.
   The 2° is the smaller of these.
   Pivot and delete twice.

	v	0	q_2	-1	
s_2	0	-1	1	1	$= -q_1$
p_2	-1	-3	7	3	$= -t_2$
p_3	-1**	2°	-5	-2	$= -t_3$
-1	-1	0	0	0	$= f$
	$= 0$	u	$= s_3$	$= g$	

$$
\begin{array}{c|ccc|c|l}
 & v & q_1 & q_2 & -1 & \\
\hline
u & 0 & -1^{*} & -1 & -1 & = -0 \\
p_2 & -1 & -3 & 4 & 0 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -3 & 0 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & = s_2 & = s_3 & = g &
\end{array}
$$

4. Find first two pivots:
   The 2 and 4 are largest in their column.
   The 2° is the smaller of these.
   Pivot and delete twice.

$$
\begin{array}{c|ccc|c|l}
 & v & 0 & q_2 & -1 & \\
\hline
s_2 & 0 & -1 & 1 & 1 & = -q_1 \\
p_2 & -1 & -3 & 7 & 3 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -5 & -2 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & u & = s_3 & = g &
\end{array}
\longrightarrow
\begin{array}{c|cc|c|l}
 & v & q_2 & -1 & \\
\hline
s_2 & 0 & 1 & 1 & = -q_1 \\
p_2 & -1 & 7 & 3 & = -t_2 \\
p_3 & -1^{**} & -5 & -2 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & = f \\
\hline
 & = 0 & = s_3 & = g &
\end{array}
$$

	v	q_1	q_2	-1	
u	0	-1*	-1	-1	$=-0$
p_2	-1	-3	4	0	$=-t_2$
p_3	-1**	2°	-3	0	$=-t_3$
-1	-1	0	0	0	$=f$
	$=0$	$=s_2$	$=s_3$	$=g$	

4. Find first two pivots:
   The 2 and 4 are largest in their column.
   The 2° is the smaller of these.
   Pivot and delete twice.

	v	0	q_2	-1	
s_2	0	-1	1	1	$=-q_1$
p_2	-1	-3	7	3	$=-t_2$
p_3	-1**	2°	-5	-2	$=-t_3$
-1	-1	0	0	0	$=f$
	$=0$	u	$=s_3$	$=g$	

$\longrightarrow$

	v	q_2	-1	
s_2	0	1	1	$=-q_1$
p_2	-1	7	3	$=-t_2$
p_3	-1**	-5	-2	$=-t_3$
-1	-1	0	0	$=f$
	$=0$	$=s_3$	$=g$	

$\longrightarrow$

	t_3	q_2	-1	
s_2	0	1	1	$=-q_1$
p_2	-1	12	5	$=-t_2$
0	-1	5	2	$=-v$
-1	-1	5	2	$=f$
	$=p_3$	$=s_3$	$=g$	

$$
\begin{array}{c|ccc|c|l}
 & v & q_1 & q_2 & -1 & \\
\hline
u & 0 & -1^{*} & -1 & -1 & = -0 \\
p_2 & -1 & -3 & 4 & 0 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -3 & 0 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & = s_2 & = s_3 & = g &
\end{array}
$$

4. Find first two pivots:
The 2 and 4 are largest in their column.
The 2° is the smaller of these.
Pivot and delete twice.

$$
\begin{array}{c|ccc|c|l}
 & v & 0 & q_2 & -1 & \\
\hline
s_2 & 0 & -1 & 1 & 1 & = -q_1 \\
p_2 & -1 & -3 & 7 & 3 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -5 & -2 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & u & = s_3 & = g &
\end{array}
\longrightarrow
\begin{array}{c|cc|c|l}
 & v & q_2 & -1 & \\
\hline
s_2 & 0 & 1 & 1 & = -q_1 \\
p_2 & -1 & 7 & 3 & = -t_2 \\
p_3 & -1^{**} & -5 & -2 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & = f \\
\hline
 & = 0 & = s_3 & = g &
\end{array}
\longrightarrow
$$

$$
\begin{array}{c|cc|c|l}
 & t_3 & q_2 & -1 & \\
\hline
s_2 & 0 & 1 & 1 & = -q_1 \\
p_2 & -1 & 12 & 5 & = -t_2 \\
0 & -1 & 5 & 2 & = -v \\
\hline
-1 & -1 & 5 & 2 & = f \\
\hline
 & = p_3 & = s_3 & = g &
\end{array}
$$

$$
\begin{array}{c|ccc|c|l}
 & v & q_1 & q_2 & -1 & \\
\hline
u & 0 & -1^{*} & -1 & -1 & = -0 \\
p_2 & -1 & -3 & 4 & 0 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -3 & 0 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & = s_2 & = s_3 & = g &
\end{array}
$$

4. Find first two pivots:
   The 2 and 4 are largest in their column.
   The 2° is the smaller of these.
   Pivot and delete twice.

$$
\begin{array}{c|ccc|c|l}
 & v & 0 & q_2 & -1 & \\
\hline
s_2 & 0 & -1 & 1 & 1 & = -q_1 \\
p_2 & -1 & -3 & 7 & 3 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -5 & -2 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & u & = s_3 & = g &
\end{array}
\longrightarrow
\begin{array}{c|cc|c|l}
 & v & q_2 & -1 & \\
\hline
s_2 & 0 & 1 & 1 & = -q_1 \\
p_2 & -1 & 7 & 3 & = -t_2 \\
p_3 & -1^{**} & -5 & -2 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & = f \\
\hline
 & = 0 & = s_3 & = g &
\end{array}
\longrightarrow
$$

$$
\begin{array}{c|cc|c|l}
 & t_3 & q_2 & -1 & \\
\hline
s_2 & 0 & 1 & 1 & = -q_1 \\
p_2 & -1 & 12 & 5 & = -t_2 \\
0 & -1 & 5 & 2 & = -v \\
\hline
-1 & -1 & 5 & 2 & = f \\
\hline
 & = p_3 & = s_3 & = g &
\end{array}
\longrightarrow
\begin{array}{c|cc|c|l}
 & t_3 & q_2 & -1 & \\
\hline
s_2 & 0 & 1 & 1 & = -q_1 \\
p_2 & -1 & 12 & 5 & = -t_2 \\
\hline
-1 & -1 & 5 & 2 & = f \\
\hline
 & = p_3 & = s_3 & = g &
\end{array}
$$

$$
\begin{array}{c|ccc|c|l}
 & v & q_1 & q_2 & -1 & \\
\hline
u & 0 & -1^{*} & -1 & -1 & = -0 \\
p_2 & -1 & -3 & 4 & 0 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -3 & 0 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & = s_2 & = s_3 & = g &
\end{array}
$$

4. Find first two pivots:
The 2 and 4 are largest in their column.
The 2° is the smaller of these.
Pivot and delete twice.

$$
\begin{array}{c|ccc|c|l}
 & v & 0 & q_2 & -1 & \\
\hline
s_2 & 0 & -1 & 1 & 1 & = -q_1 \\
p_2 & -1 & -3 & 7 & 3 & = -t_2 \\
p_3 & -1^{**} & 2^{\circ} & -5 & -2 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & u & = s_3 & = g &
\end{array}
\longrightarrow
\begin{array}{c|cc|c|l}
 & v & q_2 & -1 & \\
\hline
s_2 & 0 & 1 & 1 & = -q_1 \\
p_2 & -1 & 7 & 3 & = -t_2 \\
p_3 & -1^{**} & -5 & -2 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & = f \\
\hline
 & = 0 & = s_3 & = g &
\end{array}
\longrightarrow
$$

$$
\begin{array}{c|cc|c|l}
 & t_3 & q_2 & -1 & \\
\hline
s_2 & 0 & 1 & 1 & = -q_1 \\
p_2 & -1 & 12 & 5 & = -t_2 \\
0 & -1 & 5 & 2 & = -v \\
\hline
-1 & -1 & 5 & 2 & = f \\
\hline
 & = p_3 & = s_3 & = g &
\end{array}
\longrightarrow
\begin{array}{c|cc|c|l}
 & t_3 & q_2 & -1 & \\
\hline
s_2 & 0 & 1 & 1 & = -q_1 \\
p_2 & -1 & 12 & 5 & = -t_2 \\
\hline
-1 & -1 & 5 & 2 & = f \\
\hline
 & = p_3 & = s_3 & = g &
\end{array}
$$

5. Apply SA Phase 2 to the feasible tableau.

	v	q_1	q_2	-1	
u	0	-1*	-1	-1	$= -0$
p_2	-1	-3	4	0	$= -t_2$
p_3	-1**	2°	-3	0	$= -t_3$
-1	-1	0	0	0	$= f$
	$= 0$	$= s_2$	$= s_3$	$= g$	

4. Find first two pivots:
The 2 and 4 are largest in their column.
The 2° is the smaller of these.
Pivot and delete twice.

	v	0	q_2	-1	
s_2	0	-1	1	1	$= -q_1$
p_2	-1	-3	7	3	$= -t_2$
p_3	-1**	2°	-5	-2	$= -t_3$
-1	-1	0	0	0	$= f$
	$= 0$	u	$= s_3$	$= g$	

$\longrightarrow$

	v	q_2	-1	
s_2	0	1	1	$= -q_1$
p_2	-1	7	3	$= -t_2$
p_3	-1**	-5	-2	$= -t_3$
-1	-1	0	0	$= f$
	$= 0$	$= s_3$	$= g$	

$\longrightarrow$

	t_3	q_2	-1	
s_2	0	1	1	$= -q_1$
p_2	-1	12	5	$= -t_2$
0	-1	5	2	$= -v$
-1	-1	5	2	$= f$
	$= p_3$	$= s_3$	$= g$	

$\longrightarrow$

	t_3	q_2	-1	
s_2	0	1	1	$= -q_1$
p_2	-1	12*	5	$= -t_2$
-1	-1	5	2	$= f$
	$= p_3$	$= s_3$	$= g$	

$\longrightarrow$

5. Apply SA Phase 2 to the feasible tableau.

$$
\begin{array}{c|cc|c|l}
 & t_3 & t_2 & -1 & \\
\hline
s_2 & 1/12 & -1/12 & 7/12 & = -q_1 \\
s_3 & -1/12 & 1/12 & 5/12 & = -q_2 \\
\hline
-1 & -7/12 & -5/12 & -1/12 & = f \\
\hline
 & = p_3 & = p_2 & = g &
\end{array}
$$

Optimal Tableau!

$$\begin{array}{c|cc|c|l} & t_3 & t_2 & -1 & \\ \hline s_2 & 1/12 & -1/12 & 7/12 & = -q_1 \\ s_3 & -1/12 & 1/12 & 5/12 & = -q_2 \\ \hline -1 & -7/12 & -5/12 & -1/12 & = f \\ \hline & = p_3 & = p_2 & = g & \end{array}$$

Optimal Tableau!

6. Summarize the optimal strategies and the Von Neumann value:

Ron chooses 2 with probability 5/12, chooses 3 with probability 7/12, and never chooses 3. Coleen chooses 1 with probability 7/12, chooses 2 with probability 5/12, and never chooses 3. Coleen expects an average payoff of 1/12 per game.

Game 2: Guess the Number:

Find the optimal mixed strategies for the following matrix game

Ron and Coleen simultaneously writes either the number 2 or 3. Each player also writes a number which they believe the opponents has written down.

- If neither player guesses right, Ron wins \$3 from Coleen.
- If both players guess right, Coleen wins \$3 from Ron.
- Otherwise, the player who guesses correctly wins (from the other player) the sum (in \$) of the (first) numbers written by two players.

Game 2: Guess the Number:

Find the optimal mixed strategies for the following matrix game

Ron and Coleen simultaneously writes either the number 2 or 3. Each player also writes a number which they believe the opponents has written down.

- If neither player guesses right, Ron wins \$3 from Coleen.
- If both players guess right, Coleen wins \$3 from Ron.
- Otherwise, the player who guesses correctly wins (from the other player) the sum (in \$) of the (first) numbers written by two players.

Notation: The pair (a, b) denotes a player's choice of writing a and guessing b, for $a, b \in \{2, 3\}$.

		Coleen			
		$(2,2)$	$(2,3)$	$(3,2)$	$(3,3)$
	$(2,2)$	-3	4	-5	3
Ron	$(2,3)$	-4	3	-3	5
	$(3,2)$	5	-3	3	-6
	$(3,3)$	3	-5	6	-3

(Already domination reduced)

Coleen

		$(2,2)$	$(2,3)$	$(3,2)$	$(3,3)$
	$(2,2)$	-3	4	-5	3
Ron	$(2,3)$	-4	3	-3	5
	$(3,2)$	5	-3	3	-6
	$(3,3)$	3	-5	6	-3

(Already domination reduced)

	v	q_{22}	q_{23}	q_{32}	q_{33}	-1	
u	0	-1	-1	-1	-1	-1	$= -0$
p_{22}	-1	-3	4°	-5	3	0	$= -t_{22}$
p_{23}	-1	-4	3	-3	5	0	$= -t_{23}$
p_{32}	-1	5	-3	3	-6	0	$= -t_{32}$
p_{33}	-1	3	-5	6	-3	0	$= -t_{33}$
-1	-1	0	0	0	0	0	$= f$
	$=0$	$= s_{22}$	$= s_{23}$	$= s_{32}$	$= s_{33}$	$= g$	

(Using natural variable names.)

Coleen

		$(2,2)$	$(2,3)$	$(3,2)$	$(3,3)$
	$(2,2)$	-3	4	-5	3
Ron	$(2,3)$	-4	3	-3	5
	$(3,2)$	5	-3	3	-6
	$(3,3)$	3	-5	6	-3

(Already domination reduced)

	v	q_{22}	q_{23}	q_{32}	q_{33}	-1	
u	0	-1	-1	-1	-1	-1	$= -0$
p_{22}	-1	-3	4°	-5	3	0	$= -t_{22}$
p_{23}	-1	-4	3	-3	5	0	$= -t_{23}$
p_{32}	-1	5	-3	3	-6	0	$= -t_{32}$
p_{33}	-1	3	-5	6	-3	0	$= -t_{33}$
-1	-1	0	0	0	0	0	$= f$
	$= 0$	$= s_{22}$	$= s_{23}$	$= s_{32}$	$= s_{33}$	$= g$	

(Using natural variable names.)

	t_{33}	t_{32}	t_{23}	q_{33}	-1	
s_{23}	-1/25	-2/75	1/15	17/15	13/25	$= -q_{23}$
s_{32}	1/5	-1/5	0	1	2/5	$= -q_{32}$
s_{22}	-4/25	17/75	-1/15	-17/15	2/25	$= -q_{22}$
p_{22}	3/5	-3/5	-1	0	1/5	$= -t_{22}$
-1	-2/25	-29/75	-8/15	-1/15	1/25	$= f$
	$= p_{33}$	$= p_{32}$	$= p_{23}$	$= s_{33}$	$= g$	

(I used computer to solve this)

		Coleen			
		(2, 2)	(2, 3)	(3, 2)	(3, 3)
	(2, 2)	-3	4	-5	3
Ron	(2, 3)	-4	3	-3	5
	(3, 2)	5	-3	3	-6
	(3, 3)	3	-5	6	-3

(Already domination reduced)

	v	q_{22}	q_{23}	q_{32}	q_{33}	-1	
u	0	-1	-1	-1	-1	-1	$= -0$
p_{22}	-1	-3	4°	-5	3	0	$= -t_{22}$
p_{23}	-1	-4	3	-3	5	0	$= -t_{23}$
p_{32}	-1	5	-3	3	-6	0	$= -t_{32}$
p_{33}	-1	3	-5	6	-3	0	$= -t_{33}$
-1	-1	0	0	0	0	0	$= f$
	$= 0$	$= s_{22}$	$= s_{23}$	$= s_{32}$	$= s_{33}$	$= g$	

(Using natural variable names.)

	t_{33}	t_{32}	t_{23}	q_{33}	-1	
s_{23}	-1/25	-2/75	1/15	17/15	13/25	$= -q_{23}$
s_{32}	1/5	-1/5	0	1	2/5	$= -q_{32}$
s_{22}	-4/25	17/75	-1/15	-17/15	2/25	$= -q_{22}$
p_{22}	3/5	-3/5	-1	0	1/5	$= -t_{22}$
-1	-2/25	-29/75	-8/15	-1/15	1/25	$= f$
	$= p_{33}$	$= p_{32}$	$= p_{23}$	$= s_{33}$	$= g$	

(I used computer to solve this)

Ron wins 4 cents per game on average. Optimal strategies:
For Ron $(p_{22}, p_{23}, p_{32}, p_{33}) = (0, \frac{8}{15}, \frac{29}{75}, \frac{2}{25})$. For Coleen $(q_{22}, q_{23}, q_{32}, q_{33}) = (\frac{2}{25}, \frac{13}{25}, \frac{2}{5}, 0)$.

Game 3. Guess an Adjacent Number:

$\boxed{1}$ A $\boxed{2}$ B $\boxed{3}$ C $\boxed{4}$ D $\boxed{5}$

Player P_1 secretly chooses one of four letters A–D, Player P_2 secretly chooses one of the five boxes 1–5. They reveal their choices

- If P_1 chooses a letter that is adjacent to the P_2's box, P_1 wins \$1 from P_2.
- Otherwise, P_2 wins \$1 from P_1.

Find

a) Payoff matrix for P_1.

b) Simplify it by dominance.

c) Suppose P_2 plays mixed strategy $(1/2, 0, 1/4, 0, 1/4)$. With respect to this strategy find a mixed strategy for P_1 that will guarantee a fair game.

a)

b)

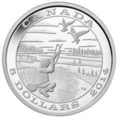
c)

Game 4: Little Casino

Ron antes \$5 and Coleen antes \$3 to the pot. Ron then secretly tosses a coin and checks the result. Ron now has two options:

FOLD—Coleen wins the pot

BET—Ron adds \$4 to the pot



If Ron bets, Coleen has two options

FOLD—Ron wins the pot

SEE—Coleen adds \$8 to the pot

If Coleen sees, the coin is revealed. If the coin is Heads, Ron wins the pot. Otherwise, Coleen wins the pot. Find the von Neumann value and the optimal strategies for this game.

- Compute payoff matrix on cell at a time. For example, suppose Ron selects BF and Coleen selects S. If the coin is Heads, then Ron will Bet and add \$4 to the pot, Coleen will See and add \$8. When Ron reveals that the coin was Heads, he collects the $3 + 8$ dollars that Coleen had put in the pot. If the coin is Tails, then Ron will Fold and lose \$5.

- Compute payoff matrix on cell at a time. For example, suppose Ron selects BF and Coleen selects S. If the coin is Heads, then Ron will Bet and add \$4 to the pot, Coleen will See and add \$8. When Ron reveals that the coin was Heads, he collects the $3+8$ dollars that Coleen had put in the pot. If the coin is Tails, then Ron will Fold and lose \$5. Each of these events has probability $\frac{1}{2}$ so we record the game matrix entry

$$a_{BF,S} = \frac{1}{2} \cdot (3+8) + \frac{1}{2} \cdot (-5) = 3.$$

- Compute payoff matrix on cell at a time. For example, suppose Ron selects BF and Coleen selects S. If the coin is Heads, then Ron will Bet and add \$4 to the pot, Coleen will See and add \$8. When Ron reveals that the coin was Heads, he collects the $3+8$ dollars that Coleen had put in the pot. If the coin is Tails, then Ron will Fold and lose \$5. Each of these events has probability $\frac{1}{2}$ so we record the game matrix entry

$$a_{BF,S} = \frac{1}{2} \cdot (3+8) + \frac{1}{2} \cdot (-5) = 3.$$

If Ron selects BB and Coleen selects S, then Ron wins $3+8$ dollars with Heads, but loses $5+4$ dollars with Tails, so $a_{BB,S} = \frac{(3+8)-(5+4)}{2} = 1$. We obtain the following game matrix.

- Compute payoff matrix on cell at a time. For example, suppose Ron selects BF and Coleen selects S. If the coin is Heads, then Ron will Bet and add \$4 to the pot, Coleen will See and add \$8. When Ron reveals that the coin was Heads, he collects the $3+8$ dollars that Coleen had put in the pot. If the coin is Tails, then Ron will Fold and lose \$5. Each of these events has probability $\frac{1}{2}$ so we record the game matrix entry

$$a_{BF,S} = \frac{1}{2}\cdot(3+8) + \frac{1}{2}\cdot(-5) = 3.$$

If Ron selects BB and Coleen selects S, then Ron wins $3+8$ dollars with Heads, but loses $5+4$ dollars with Tails, so $a_{BB,S} = \frac{(3+8)-(5+4)}{2} = 1$. We obtain the following game matrix.

$$A = \begin{array}{c} \\ FF \\ FB \\ BF \\ BB \end{array} \begin{array}{c} \begin{array}{cc} F & S \end{array} \\ \begin{bmatrix} \frac{-5-5}{2} & \frac{-5-5}{2} \\ \frac{-5+3}{2} & \frac{-5-(5+4)}{2} \\ \frac{3-5}{2} & \frac{(3+8)-5}{2} \\ \frac{3+3}{2} & \frac{(3+8)-(5+4)}{2} \end{bmatrix} \end{array} = \begin{array}{c} \\ FF \\ FB \\ BF \\ BB \end{array} \begin{array}{c} \begin{array}{cc} F & S \end{array} \\ \begin{bmatrix} -5 & -5 \\ -1 & -7 \\ -1 & 3 \\ 3 & 1 \end{bmatrix} \end{array}.$$

Game 5: Little Poker (Textbook Example 5, p. 127)

Ron and Coleen play with three cards, J, Q, and K which are face-down. They each ante 25¢ each and select a card. Ron looks at his card, then chooses to either:

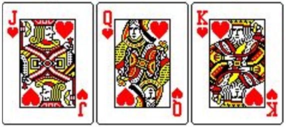
Fold (F) — here Coleen wins the pot, or
Bet (B) — here Ron adds 10¢ to the pot

If Ron bets, then Coleen looks at her card, a chooses to either:

Fold (F) — here Ron wins the pot, or
See — here Coleen also adds 10¢ to the pot

If Coleen sees, then both cards are revealed and the player with the higher card (where $J < Q < K$*) wins the pot.*

Find the von Neumann value and the optimal strategies for this game.



$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} \mathrm{FFF} & \mathrm{FFS} & \mathrm{FSF} & \mathrm{SFF} & \mathrm{FSS} & \mathrm{SFS} & \mathrm{SSF} & \mathrm{SSS} \end{array} \\
\begin{array}{c} \mathrm{FFF} \\ \mathrm{FFB} \\ \mathrm{FBF} \\ \mathrm{BFF} \\ \mathrm{FBB} \\ \mathrm{BFB} \\ \mathrm{BBF} \\ \mathrm{BBB} \end{array} &
\left[\begin{array}{cccccccc}
-25 & -25 & -25 & -25 & -25 & -25 & -25 & -25 \\
-25/3 & -25/3 & -20/3 & -20/3 & -20/3 & -20/3 & -5 & -5 \\
-25/3 & -55/3 & -25/3 & -20/3 & -55/3 & -50/3 & -20/3 & -50/3 \\
-25/3 & -55/3 & -55/3 & -25/3 & -85/3 & -55/3 & -55/3 & -85/3 \\
25/3 & -5/3 & 10 & 35/3 & 0 & 5/3 & 40/3 & 10/3 \\
25/3 & -5/3 & 0 & 10 & -10 & 0 & 5/3 & -25/3 \\
25/3 & -35/3 & -5/3 & 10 & -65/3 & -10 & 0 & -20 \\
25 & 5 & 50/3 & 85/3 & -10/3 & 25/3 & 20 & 0
\end{array}\right]
\end{array}
$$