MATH 308 D200, Fall 2019

22. The transportation problem

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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The Transportation Problem

- Another traditional example of LP problems.
- Solvable directly by using general "LP techniques".
- Due to the special shape of these problems we can develop more efficient algorithms.

The Balanced Transportation Problem

A manufacturer of cellphones owns three warehouses W_1, W_2, W_3 and sells to three markets M_1, M_2, M_3. The supply of each warehouse, the demand of each market and the shipping costs per 100 cellphones are depicted in the following table

	M_1	M_2	M_3	supply
W_1	\$30	\$20	\$10	4000
W_2	\$15	\$30	\$25	3000
W_3	\$30	\$20	\$15	3000
demand	4500	3000	2500	

How should the manufacturer ship the cellphones if they want to meet all demand/supply requirements and minimize total transportation cost?

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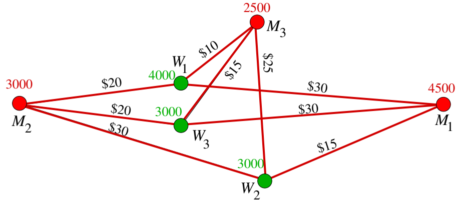

First make the units consistent:

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W_1	\$30	\$20	\$10	4000
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W_3	\$30	\$20	\$15	3000
demand	4500	3000	2500	

$\mapsto$

	M_1	M_2	M_3	supply
W_1				
W_2				
W_3				
demand				

Demand is # of phones.
Transport cost is per 100 phones.

Demand is # of units of 100 phones.
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$\mapsto$

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W_2	15	30	25	30
W_3	30	20	15	30
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Let x_{ij} be the number of hundreds of cellphones shipped from W_i to M_j, $1 \leq i, j \leq 3$.

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$$\text{minimize } g = 30x_{11} + 20x_{12} + 10x_{13} + 15x_{21} + 30x_{22} + 25x_{23} + 30x_{31} + 20x_{32} + 15x_{33}$$

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$$
\begin{array}{ll}
\text{minimize} & g = 30x_{11} + 20x_{12} + 10x_{13} + 15x_{21} + 30x_{22} + 25x_{23} + 30x_{31} + 20x_{32} + 15x_{33} \\
\text{subject to} & x_{11} + x_{12} + x_{13} = 40 \quad W_1 \text{ supply constraint} \\
& x_{21} + x_{22} + x_{23} = 30 \quad W_2 \text{ supply constraint} \\
& x_{31} + x_{32} + x_{33} = 30 \quad W_3 \text{ supply constraint}
\end{array}
$$

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W_3	30	20	15	30
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$$
\begin{array}{lll}
\text{minimize} & g = 30x_{11} + 20x_{12} + 10x_{13} + 15x_{21} + 30x_{22} + 25x_{23} + 30x_{31} + 20x_{32} + 15x_{33} & \\
\text{subject to} & x_{11} + x_{12} + x_{13} = 40 & W_1 \text{ supply constraint} \\
& x_{21} + x_{22} + x_{23} = 30 & W_2 \text{ supply constraint} \\
& x_{31} + x_{32} + x_{33} = 30 & W_3 \text{ supply constraint} \\
& x_{11} + x_{21} + x_{31} = 45 & M_1 \text{ demand constraint} \\
& x_{12} + x_{22} + x_{32} = 30 & M_2 \text{ demand constraint} \\
& x_{31} + x_{23} + x_{33} = 25 & M_3 \text{ demand constraint} \\
& x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \geq 0 &
\end{array}
$$

The initial (minimization) Tucker tableau is noncanonical.

x_{11}	1	0	0	1	0	0	30
x_{12}	1	0	0	0	1	0	20
x_{13}	1	0	0	0	0	1	10
x_{21}	0	1	0	1	0	0	15
x_{22}	0	1	0	0	1	0	30
x_{23}	0	1	0	0	0	1	25
x_{31}	0	0	1	1	0	0	30
x_{32}	0	0	1	0	1	0	20
x_{33}	0	0	1	0	0	1	15
-1	40	30	30	45	30	25	0
	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$	$= g$

Noncannonical max tableau (negative transpose):

x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	-1	
-1^*	-1	-1	0	0	0	0	0	0	-40	$= 0$
0	0	0	-1	-1	-1	0	0	0	-30	$= 0$
0	0	0	0	0	0	-1	-1	-1	-30	$= 0$
-1	0	0	-1	0	0	-1	0	0	-45	$= 0$
0	-1	0	0	-1	0	0	-1	0	-30	$= 0$
0	0	-1	0	0	-1	0	0	-1	-25	$= 0$
-30	-20	-10	-15	-30	-5	-30	-20	-15	0	$= f$

$\rightarrow$

0	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	-1	
-1	1	1	0	0	0	0	0	0	40	$= -x_{11}$
0	0	0	-1	-1	-1	0	0	0	-30	$= 0$
0	0	0	0	0	0	-1	-1	-1	-30	$= 0$
-1	1	1	-1	0	0	-1	0	0	-5	$= 0$
0	-1	0	0	-1	0	0	-1	0	-30	$= 0$
0	0	-1	0	0	-1	0	0	-1	-25	$= 0$
-30	10	20	-15	-30	-25	-30	-20	-15	1200	$= f$

$\rightarrow$ Delete column

	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	-1	
	1	1	0	0	0	0	0	0	40	$= -x_{11}$
	0	0	-1^*	-1	-1	0	0	0	-30	$= 0$
$\rightarrow$	0	0	0	0	0	-1	-1	-1	-30	$= 0$
	1	1	-1	0	0	-1	0	0	-5	$= 0$
	-1	0	0	-1	0	0	-1	0	-30	$= 0$
	0	-1	0	0	-1	0	0	-1	-25	$= 0$
	10	20	-15	-30	-25	-30	-20	-15	1200	$= f$

	x_{12}	x_{13}	0	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	-1	
	1	1	0	0	0	0	0	0	40	$= -x_{11}$
	0	0	-1	1	1	0	0	0	30	$= -x_{21}$
$\rightarrow$	0	0	0	0	0	-1	-1	-1	-30	$= 0$
	1	1	-1	1	1	-1	0	0	25	$= 0$
	-1	0	0	-1	0	0	-1	0	-30	$= 0$
	0	-1	0	0	-1	0	0	-1	-25	$= 0$
	10	20	-15	-15	-10	-30	-20	-15	1650	$= f$

$\rightarrow$ Delete column

$$
\rightarrow\quad
\begin{array}{c|ccccccc|c|l}
 & x_{12} & x_{13} & x_{22} & x_{23} & x_{31} & x_{32} & x_{33} & -1 & \\
\hline
 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 40 & = -x_{11} \\
 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 30 & = -x_{21} \\
 & 0 & 0 & 0 & 0 & -1^{*} & -1 & -1 & -30 & = 0 \\
 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 25 & = 0 \\
 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -30 & = 0 \\
 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & -25 & = 0 \\
\hline
 & 10 & 20 & -15 & -10 & -30 & -20 & -15 & 1650 & = f \\
\hline
\end{array}
$$

$$
\rightarrow\quad
\begin{array}{c|ccccccc|c|l}
 & x_{12} & x_{13} & x_{22} & x_{23} & 0 & x_{32} & x_{33} & -1 & \\
\hline
 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 40 & = -x_{11} \\
 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 30 & = -x_{21} \\
 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 30 & = -x_{31} \\
 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 55 & = 0 \\
 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -30 & = 0 \\
 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & -25 & = 0 \\
\hline
 & 10 & 20 & -15 & -10 & -30 & 10 & 15 & 2550 & = f \\
\hline
\end{array}
$$

$\rightarrow$ Delete column

$$\begin{array}{c|cccccc|c|l}
 & x_{12} & x_{13} & x_{22} & x_{23} & x_{32} & x_{33} & -1 & \\
\hline
 & 1 & 1 & 0 & 0 & 0 & 0 & 40 & = -x_{11} \\
 & 0 & 0 & 1 & 1 & 0 & 0 & 30 & = -x_{21} \\
\rightarrow & 0 & 0 & 0 & 0 & 1 & 1 & 30 & = -x_{31} \\
 & 1^* & 1 & 1 & 1 & 1 & 1 & 55 & = 0 \\
 & -1 & 0 & -1 & 0 & -1 & 0 & -30 & = 0 \\
 & 0 & -1 & 0 & -1 & 0 & -1 & -25 & = 0 \\
\hline
 & 10 & 20 & -15 & -10 & 10 & 15 & 2550 & = f \\
\hline
\end{array}$$

$$\begin{array}{c|cccccc|c|l}
 & 0 & x_{13} & x_{22} & x_{23} & x_{32} & x_{33} & -1 & \\
\hline
 & -1 & 0 & -1 & -1 & -1 & -1 & -15 & = -x_{11} \\
 & 0 & 0 & 1 & 1 & 0 & 0 & 30 & = -x_{21} \\
\rightarrow & 0 & 0 & 0 & 0 & 1 & 1 & 30 & = -x_{31} \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 55 & = -x_{12} \\
 & 1^* & 1 & 0 & 1 & 0 & 1 & 25 & = 0 \\
 & 0 & -1 & 0 & -1 & 0 & -1 & -25 & = 0 \\
\hline
 & -10 & 10 & -25 & -20 & 0 & 5 & 2000 & = f \\
\hline
\end{array}$$

$\rightarrow$ Delete column

$$
\rightarrow \quad
\begin{array}{ccccc|c|l}
x_{13} & x_{22} & x_{23} & x_{32} & x_{33} & -1 & \\
\hline
0 & -1 & -1 & -1 & -1 & -15 & = -x_{11} \\
0 & 1 & 1 & 0 & 0 & 30 & = -x_{21} \\
0 & 0 & 0 & 1 & 1 & 30 & = -x_{31} \\
1 & 1 & 1 & 1 & 1 & 55 & = -x_{12} \\
1^* & 0 & 1 & 0 & 1 & 25 & = 0 \\
-1 & 0 & -1 & 0 & -1 & -25 & = 0 \\
\hline
10 & -25 & -20 & 0 & 0 & 2000 & = f \\
\hline
\end{array}
$$

$$
\rightarrow \quad
\begin{array}{ccccc|c|l}
0 & x_{22} & x_{23} & x_{32} & x_{33} & -1 & \\
\hline
0 & -1 & -1 & -1 & -1 & -15 & = -x_{11} \\
0 & 1 & 1 & 0 & 0 & 30 & = -x_{21} \\
0 & 0 & 0 & 1 & 1 & 30 & = -x_{31} \\
-1 & 1 & 0 & 1 & 0 & 30 & = -x_{12} \\
1 & 0 & 1 & 0 & 1 & 25 & = -x_{13} \\
1 & 0 & 0 & 0 & 0 & 0 & = 0 \\
\hline
-10 & -25 & -30 & 0 & -5 & 1750 & = f \\
\hline
\end{array}
$$

$\rightarrow$ Delete column

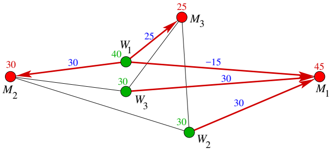
$\rightarrow$

x_{22}	x_{23}	x_{32}	x_{33}	-1	
-1	-1	-1	-1	-15	$= -x_{11}$
1	1	0	0	30	$= -x_{21}$
0	0	1	1	30	$= -x_{31}$
1	0	1	0	30	$= -x_{12}$
0	1	0	1	25	$= -x_{13}$
0	0	0	0	0	$= 0$ Delete Row! (Why did this happen?)
-25	-30	0	-5	1750	$= f$

$\rightarrow$

x_{22}	x_{23}	x_{32}	x_{33}	-1	
-1^*	-1	-1	-1	-15	$= -x_{11}$
1	1	0	0	30	$= -x_{21}$
0	0	1	1	30	$= -x_{31}$
1	0	1	0	30	$= -x_{12}$
0	1	0	1	25	$= -x_{13}$
-25	-30	0	-5	1750	$= f$

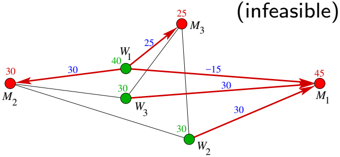
Tableau is canonical (but infeasible), ready for the simplex algorithm.



For basic solutions, the basic edges form a spanning forest.

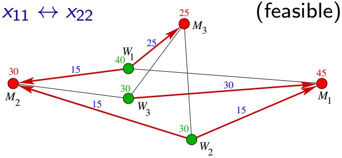
Basic tableau $\rightarrow$ Phase 1

(infeasible)



	x_{11}	x_{23}	x_{32}	x_{33}	-1	
	-1	1	1^*	1	15	$=-x_{22}$
	1	0	-1	-1	15	$=-x_{21}$
$\rightarrow$	0	0	1	1	30	$=-x_{31}$
	1	-1	0	-1	15	$=-x_{12}$
	0	1	0	1	25	$=-x_{13}$
	-25	-5	25	20	2125	$=f$

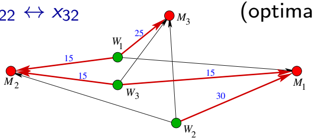
$x_{11} \leftrightarrow x_{22}$ (feasible)



Feasible tableau $\rightarrow$ Phase 2

	x_{11}	x_{23}	x_{22}	x_{33}	-1	
	-1	1	1	1	15	$=-x_{32}$
	0	1	1	0	30	$=-x_{21}$
$\rightarrow$	1^*	-1	-1	0	15	$=-x_{31}$
	1	-1	0	-1	15	$=-x_{12}$
	0	1	0	1	25	$=-x_{13}$
	0	-30	-25	-5	1750	$=f$

$x_{22} \leftrightarrow x_{32}$ (optimal)



(Multiple optimal solutions) $\rightarrow$ Pivot on 1^* to get a second optimal BFS

x_{11}	x_{23}	x_{22}	x_{33}	-1	
-1	1	1	1	15	$= -x_{32}$
0	1	1	0	30	$= -x_{21}$
1^*	-1	-1	0	15	$= -x_{31}$
1	-1	0	-1	15	$= -x_{12}$
0	1	0	1	25	$= -x_{13}$
0	-30	-25	-5	1750	$= f$

$\rightarrow$

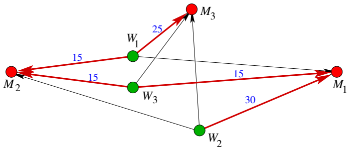
x_{12}	x_{23}	x_{22}	x_{33}	-1	
1	0	1	0	30	$= -x_{32}$
0	1	1	0	30	$= -x_{21}$
1	-1	0	-1	15	$= -x_{11}$
-1	0	-1	1	0	$= -x_{31}$
0	1	0	1	25	$= -x_{13}$
0	-30	-25	-5	1750	$= f^*$

First optimal solution: $\mathbf{x}^1$

Second optimal solution: $\mathbf{x}^2$

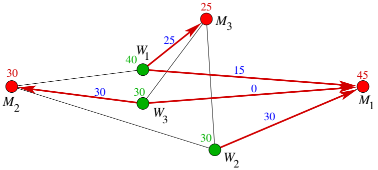
Optimal solution set is the line segment $\overline{\mathbf{x}^1\mathbf{x}^2}$ where

$$
\begin{array}{rl}
 & (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{13}, x_{31}, x_{32}, x_{33}) \\
\mathbf{x}^1 = & (\ 0,\ 15,\ 25,\ 30,\ 0,\ 0,\ 15,\ 15,\ 0\ ) \\
\mathbf{x}^2 = & (\ 15,\ 0,\ 25,\ 30,\ 0,\ 0,\ 0,\ 30,\ 0\ )
\end{array}
$$



$\mathbf{x}^1$

$x_{13} \leftrightarrow x_{11}$



$\mathbf{x}^2$

Red arcs are basic.

Notice: Pivoting only makes changes on a **single graph** cycle.

General Balanced Transportation Problem

- warehouses $W_1, W_2, \ldots, W_m$
- supplies $s_1, s_2, \ldots, s_m$
- markets $M_1, M_2, \ldots, M_n$
- demands $d_1, d_2, \ldots, d_n$
- *balanced transportation tableau* with unit shipping cost c_{ij} from W_i to M_j
- let x_{ij} be # of units shipped from W_i to M_j

$$
\begin{array}{c|cccc|l}
 & M_1 & M_2 & \ldots & M_n & \\
\hline
W_1 & c_{11} & c_{12} & \ldots & c_{1n} & s_1 \\
W_2 & c_{21} & c_{22} & \ldots & c_{2n} & s_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
W_m & c_{m1} & c_{m2} & \ldots & c_{mn} & s_m \\
\hline
 & d_1 & d_2 & \ldots & d_n & \displaystyle\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j
\end{array}
$$

$$
\text{total supply} = \sum_{i=1}^{m} s_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} d_j = \text{total demand}
$$

LP formulation ...

$$\text{Minimize} \qquad C = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} c_{ij}$$

$$\text{subject to} \qquad \left.\sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \ldots, m)\right\} \text{Warehouse constraints}$$

$$\left.\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \ldots, n)\right\} \text{Market constraints}$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

Lemma

Every balanced transportation problem has an optimal solution.

Proof.

LP formulation ...

$$\text{Minimize} \qquad C = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

$$\text{subject to} \qquad \left.\sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \ldots, m)\right\} \text{Warehouse constraints}$$

$$\left.\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \ldots, n)\right\} \text{Market constraints}$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

Lemma

Every balanced transportation problem has an optimal solution.

Proof.

Let $T =$ total supply $=$ total demand.

The LP is **feasible**:

LP formulation ...

Minimize $$C = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} c_{ij}$$

subject to $$\left.\sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \ldots, m)\right\} \text{ Warehouse constraints}$$

$$\left.\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \ldots, n)\right\} \text{ Market constraints}$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

Lemma

Every balanced transportation problem has an optimal solution.

Proof.

Let $T =$ total supply $=$ total demand.

The LP is **feasible**: Let $x_{ij} = \frac{s_i d_j}{T}$ for each i, j. Then $x_{ij} \geq 0$, and

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{s_i d_j}{T} = \frac{s_i}{T} \sum_{j=1}^{n} d_j = s_i, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = \sum_{1=1}^{m} \frac{s_i d_j}{T} = \frac{d_j}{T} \sum_{i=1}^{n} s_i = d_j, \quad j = 1, 2, \ldots, n.$$

So (x_{ij}) is a feasible solution. □

LP formulation ...

$$\text{Minimize} \qquad C = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

$$\text{subject to} \qquad \left.\sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \ldots, m)\right\} \text{Warehouse constraints}$$

$$\left.\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \ldots, n)\right\} \text{Market constraints}$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

Lemma

Every balanced transportation problem has an optimal solution.

Proof.

Let $T =$ total supply $=$ total demand.

The LP is **feasible**:
The LP is **bounded**:

LP formulation . . .

$$\text{Minimize} \qquad C = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} c_{ij}$$

$$\text{subject to} \qquad \left.\sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \ldots, m)\right\} \text{ Warehouse constraints}$$

$$\left.\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \ldots, n)\right\} \text{ Market constraints}$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

Lemma

Every balanced transportation problem has an optimal solution.

Proof.

Let $T =$ total supply $=$ total demand.

The LP is **feasible**:

The LP is **bounded**: Let $c_{\max} = \max\limits_{i,j} c_{ij}$. If $\mathbf{x} = (x_{ij})$ is feasible, then

$$\begin{aligned} C(\mathbf{x}) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} &\leq \sum_{i=1}^{m}\sum_{j=1}^{n} c_{\max} x_{ij} \quad (\text{since each } x_{ij} \geq 0) \\ &= c_{\max} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \\ &= c_{\max} T. \end{aligned}$$

$\square$

LP formulation ...

$$\text{Minimize} \qquad C = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

$$\text{subject to} \qquad \left.\sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \ldots, m)\right\} \text{Warehouse constraints}$$

$$\left.\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \ldots, n)\right\} \text{Market constraints}$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

Lemma

Every balanced transportation problem has an optimal solution.

Proof.

Let $T =$ total supply $=$ total demand.

The LP is **feasible**:
The LP is **bounded**:

Therefore the LP has an optimal solution.

□

Dual Problem to the Balanced Transportation Problem

$$\text{(P)}\ \min\ C = 30x_{11} + 20x_{12} + 10x_{13} + 15x_{21} + 30x_{22} + 25x_{23} + 30x_{31} + 20x_{32} + 15x_{33}$$

$$\begin{aligned}
\text{subject to}\ x_{11} + x_{12} + x_{13} &= 40 \\
x_{21} + x_{22} + x_{23} &= 30 \\
x_{31} + x_{32} + x_{33} &= 30 \\
x_{11} + x_{21} + x_{31} &= 45 \\
x_{12} + x_{22} + x_{32} &= 30 \\
x_{13} + x_{23} + x_{33} &= 25 \\
x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} &\geq 0
\end{aligned}$$

$$
\begin{array}{llccccccccccl}
\text{(P)} & \min\ g = 30x_{11} + 20x_{12} + 10x_{13} + 15x_{21} + 30x_{22} + 25x_{23} + 30x_{31} + 20x_{32} + 15x_{33} \\
& \text{s. t. } x_{11} + x_{12} + x_{13} & & & = 40 \\
& & x_{21} + x_{22} + x_{23} & & = 30 \\
& & & x_{31} + x_{32} + x_{33} & = 30 \\
& x_{11} & + x_{21} & + x_{31} & = 45 \\
& x_{12} & + x_{22} & + x_{32} & = 30 \\
& x_{31} & + x_{23} & + x_{33} & = 25 \\
& & & x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{13}, x_{33}, x_{32}, x_{33} & \geq 0
\end{array}
$$

$$
\begin{array}{lll}
\text{(P)} \ \min & g = 30x_{11} + 20x_{12} + 10x_{13} + 15x_{21} + 30x_{22} + 25x_{23} + 30x_{31} + 20x_{32} + 15x_{33} & \\
\text{s. t.} & x_{11} + x_{12} + x_{13} = 40 & (a_1) \\
& x_{21} + x_{22} + x_{23} = 30 & (a_2) \\
& x_{31} + x_{32} + x_{33} = 30 & (a_3) \\
& x_{11} + x_{21} + x_{31} = 45 & (b_1) \\
& x_{12} + x_{22} + x_{32} = 30 & (b_2) \\
& x_{31} + x_{23} + x_{33} = 25 & (b_3) \\
& x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{13}, x_{33}, x_{32}, x_{33} \geq 0 &
\end{array}
$$

Dual Problem to the Balanced Transportation Problem

$$
\begin{array}{llr}
\text{(P) min} & g = 30x_{11} + 20x_{12} + 10x_{13} + 15x_{21} + 30x_{22} + 25x_{23} + 30x_{31} + 20x_{32} + 15x_{33} & \\
\text{s. t.} & x_{11} + x_{12} + x_{13} = 40 & (a_1) \\
 & x_{21} + x_{22} + x_{23} = 30 & (a_2) \\
 & x_{31} + x_{32} + x_{33} = 30 & (a_3) \\
 & x_{11} + x_{21} + x_{31} = 45 & (b_1) \\
 & x_{12} + x_{22} + x_{32} = 30 & (b_2) \\
 & x_{31} + x_{23} + x_{33} = 25 & (b_3) \\
 & x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{13}, x_{33}, x_{32}, x_{33} \geq 0 &
\end{array}
$$

$$
\text{(D) max } P = 40a_1 + 30a_2 + 30a_3 + 45b_1 + 30b_2 + 25b_3
$$

$$
\begin{array}{lll}
\text{subject to} & a_1 + b_1 \leq 30 & (x_{11}) \\
 & a_1 + b_2 \leq 20 & (x_{12}) \\
 & a_1 + b_3 \leq 10 & (x_{13}) \\
 & a_2 + b_1 \leq 15 & (x_{21}) \\
 & a_2 + b_2 \leq 30 & (x_{22}) \\
 & a_2 + b_3 \leq 25 & (x_{23}) \\
 & a_3 + b_1 \leq 30 & (x_{31}) \\
 & a_3 + b_2 \leq 20 & (x_{32}) \\
 & a_3 + b_3 \leq 15 & (x_{33}) \\
 & a_1, a_2, a_3, b_1, b_2, b_3 \text{ unrestricted} &
\end{array}
$$

Dual Problem to the Balanced Transportation Problem

$$\text{(P) min} \quad C = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

$$\text{s. t.} \quad \sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \dots, m)$$

$$\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \dots, n)$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

Dual Problem to the Balanced Transportation Problem

$$(\text{P}) \min \quad C = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$$

$$\text{s. t.} \quad \sum_{j=1}^{n} x_{ij} = s_i, \quad (i = 1, 2, \ldots, m) \quad (a_i)$$

$$\sum_{i=1}^{m} x_{ij} = d_j, \quad (j = 1, 2, \ldots, n) \quad (b_j)$$

$$x_{ij} \geqslant 0, \quad \text{for all } i, j$$

Dual Problem to the Balanced Transportation Problem

(P) min $\quad C = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$

s. t. $\quad \sum_{j=1}^{n} x_{ij} = s_i, \ \ (i = 1, 2, \ldots, m) \quad (a_i)$

$\sum_{i=1}^{m} x_{ij} = d_j, \ \ (j = 1, 2, \ldots, n) \quad (b_j)$

$x_{ij} \geqslant 0, \quad$ for all i, j

(D) max $\quad P = \sum_{i=1}^{m} s_i a_i + \sum_{j=1}^{n} d_j b_j$

s. t. $\quad a_i + b_j \leq c_{ij}$ for all $i, j \quad (x_{ij})$

a_i, b_j unrestricted

Dual Problem to the Balanced Transportation Problem

(P) min $\quad C = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}$

s. t. $\quad \sum_{j=1}^{n} x_{ij} = s_i, \ \ (i = 1, 2, \ldots, m) \quad (a_i)$

$\quad \sum_{i=1}^{m} x_{ij} = d_j, \ \ (j = 1, 2, \ldots, n) \quad (b_j)$

$\quad x_{ij} \geqslant 0, \quad$ for all i, j

(D) max $\quad P = \sum_{i=1}^{m} s_i a_i + \sum_{j=1}^{n} d_j b_j$

s. t. $\quad a_i + b_j \leq c_{ij}$ for all $i, j \quad (x_{ij})$

$\quad a_i, b_j$ unrestricted

Interpretation:
Each warehouse W_i (and market M_j) is assigned a "node price", a_i (resp. b_j), that they will contribute toward the cost of transporting one item. Complementary slackness stipulates that $x_{ij} > 0$ implies $a_i + b_j = c_{ij}$, i. e. the transportation cost of every item shipped along edge ij is exactly covered by W_i and M_j.

$$
\begin{array}{c|cccc|l}
 & M_1 & M_2 & \dots & M_n & \\
\hline
W_1 & c_{11} & c_{12} & \dots & c_{1n} & s_1 \\
W_2 & c_{21} & c_{22} & \dots & c_{2n} & s_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
W_m & c_{m1} & c_{m2} & \dots & c_{mn} & s_m \\
\hline
 & d_1 & d_2 & \dots & d_n & \displaystyle\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j
\end{array}
$$

- the entries of the *transportation tableau* are called *cells*
- **not a Tucker tableau**.
- corresponding Tucker tableau is much larger.
- we will learn how to implement the simplex algorithm directly on the transportation tableau.
- the algorithm two parts:
  i) transform the transportation tableau into **basic feasible** transp. tableau using **VAM**.
  ii) transform the feasible tableau into an **optimal** one using the **transportation algorithm**.

The Vogel Advanced-Start Method

Using simplex required three steps: Making the tableau Canonical, then Phase 1, then Phase 2. **VAM** does the first two steps directly on the simpler **transportation tableau**.

1. Compute the difference of **two smallest entries** in every row and column of the tableau. Write this difference opposite the row or column. (If there is only one entry, just write that entry.)
2. Select the row or column with the **largest difference**.
3. Either empty a warehouse or fill a market demand using **smallest-cost cell** chosen from the selected row or column. (If there is a tie in step **2**, choose the cell with the smaller cost.)
4. In the chosen cell, **circle** the cost used and **write** above the circle the amount shipped by that route. **Reduce** the supply and demand in the row and column containing the cell.
5. **Delete** the row or column of the emptied warehouse or fully supplied market. If both happen simultaneously, delete the row unless it is the only row remaining in which case delete the column.
6. If all tableau entries are deleted, **STOP**; otherwise go to **Step 1.**

Fact: VAM always outputs a feasible basic solution.

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
10	15	30	25	30
5	30	20	15	30
	45	30	25	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

$\downarrow$ largest diff among the 6

diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
10	15	30	25	30
5	30	20	15	30
	45	30	25	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
10	15	30	25	30
5	30	20	15	30
	45	30	25	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
10	15 ^{30}	30	25	~~30~~ 0
5	30	20	15	30
	~~45~~ 15	30	25	

Transfer 30 units from W_2 to M_1

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5		
10	30	20	10	40	
~~10~~	~~15~~ 30	~~30~~	~~25~~	~~30 0~~	satisfied
5	30	20	15	30	
	~~45~~ 15	30	25		

W_2 is depleted, cross out the row.

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15~~ 30	~~30~~	~~25~~	~~30~~ ~~0~~
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
10	30	20	10	40
~~10~~	~~15~~ 30	~~30~~	~~25~~	~~0~~
5	30	20	15	30
	15	30	25	

Recompute diffs ignoring crossed out entries.

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15~~ 30	~~30~~	~~25~~	~~30 0~~
5	30	20	15	30
	~~45~~ 15	30	25	

		0	0	5	
→	10	30	20	10	40
	~~10~~	~~15~~ 30	~~30~~	~~25~~	~~0~~
	5	30	20	15	30
		15	30	25	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15~~ ^{30}	~~30~~	~~25~~	~~30~~ ~~0~~
5	30	20	15	30
	~~45~~ 15	30	25	

		0	0	5	
→	10	30	20	10	40
	~~10~~	~~15~~ ^{30}	~~30~~	~~25~~	~~0~~
	5	30	20	15	30
		15	30	25	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~30 0~~
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10^{25}	~~40~~ 15
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~0~~
5	30	20	15	30
	15	30	~~25~~ 0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~30 0~~
5	30	20	15	30
	~~45~~ 15	30	25	

		0	0	~~5~~	
→	10	30	20	~~10^{25}~~	~~40~~ 15
	~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~0~~
	5	30	20	~~15~~	30
		15	30	~~25~~ ~~0~~	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15~~ (30)	~~30~~	~~25~~	~~30~~ ~~0~~
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10 (25)	~~40~~ 15
~~10~~	~~15~~ (30)	~~30~~	~~25~~	~~0~~
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	5	
10	30	20	10 (25)	15
~~10~~	~~15~~ (30)	~~30~~	~~25~~	~~0~~
10	30	20	15	30
	15	30	0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15~~ ^{30}	~~30~~	~~25~~	~~30~~ ~~0~~
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	~~5~~	
→ 10	30	20	~~10~~ ^{25}	~~40~~ 15
~~10~~	~~15~~ ^{30}	~~30~~	~~25~~	~~0~~
5	30	20	~~15~~	30
	15	30	~~25~~ ~~0~~	

	0	0	~~5~~	
→ 10	30	20	~~10~~ ^{25}	15
~~10~~	~~15~~ ^{30}	~~30~~	~~25~~	~~0~~
→ 10	30	20	~~15~~	30
	15	30	~~0~~	

Choose either row, since $20 = 20$.

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15~~ ^{30}	~~30~~	~~25~~	~~30~~ ~~0~~
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	~~5~~	
→ 10	30	20	~~10~~ ^{25}	~~40~~ 15
~~10~~	~~15~~ ^{30}	~~30~~	~~25~~	~~0~~
5	30	20	~~15~~	30
	15	30	~~25~~ 0	

	0	0	~~5~~	
→ 10	30	20	~~10~~ ^{25}	15
~~10~~	~~15~~ ^{30}	~~30~~	~~25~~	~~0~~
10	30	20	~~15~~	30
	15	30	~~0~~	

Choose either row, since $20 = 20$.

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~30 0~~
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	~~5~~	
→ 10	30	20	~~10^{25}~~	~~40~~ 15
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~0~~
5	30	20	~~15~~	30
	15	30	~~25~~ 0	

	0	0	~~5~~	
→ 10	30	20^{15}	~~10^{25}~~	~~15~~ 0
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~0~~
10	30	20	~~15~~	30
	15	~~30~~ 15	~~0~~	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	15^{30}	30	25	30 0
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10^{25}	~~40~~ 15
~~10~~	15^{30}	30	25	0
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	5	
→ ~~10~~	30	20^{15}	10^{25}	~~15~~ 0
~~10~~	15^{30}	30	25	0
10	30	20	15	30
	15	~~30~~ 15	0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	15^{30}	30	25	~~30~~ 0
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10^{25}	~~40~~ 15
~~10~~	15^{30}	30	25	0
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	5	
→ ~~10~~	30	20^{15}	10^{25}	~~15~~ 0
~~10~~	15^{30}	30	25	0
10	30	20	15	30
	15	~~30~~ 15	0	

	30	20	5	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
10	30	20	15	30
	15	15	0	

When computing diffs, write down the entry value (e.g. 30 and 20) if it is the only cell in the column.

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
10	15^{30}	30	25	30 0
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10^{25}	~~40~~ 15
10	15^{30}	30	25	0
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	5	
→ 10	30	20^{15}	10^{25}	~~15~~ 0
10	15^{30}	30	25	0
10	30	20	15	30
	15	~~30~~ 15	0	

	↓ 30	20	5	
10	30	20^{15}	10^{25}	0
10	15^{30}	30	25	0
10	30	20	15	30
	15	15	0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
10	15^{30}	30	25	30 0
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10^{25}	~~40~~ 15
10	15^{30}	30	25	0
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	5	
→ 10	30	20^{15}	10^{25}	~~15~~ 0
10	15^{30}	30	25	0
10	30	20	15	30
	15	~~30~~ 15	0	

	↓ 30	20	5	
10	30	20^{15}	10^{25}	0
10	15^{30}	30	25	0
10	30	20	15	30
	15	15	0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	15^{30}	30	25	~~30~~ 0
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10^{25}	~~40~~ 15
~~10~~	15^{30}	30	25	0
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	5	
→ ~~10~~	30	20^{15}	10^{25}	~~15~~ 0
~~10~~	15^{30}	30	25	0
10	30	20	15	30
	15	~~30~~ 15	0	

	↓ 30	20	5	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
10	30^{15}	20	15	~~30~~ 15
	~~15~~ 0	15	0	

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	15 [30]	~~30~~	~~25~~	~~30~~ 0
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10 [25]	~~40~~ 15
~~10~~	15 [30]	30	25	0
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	5	
→ ~~10~~	30	20 [15]	10 [25]	~~15~~ 0
~~10~~	15 [30]	30	25	0
10	30	20	15	30
	15	~~30~~ 15	0	

	↓ 30	20	5	
~~10~~	30	20 [15]	10 [25]	0
~~10~~	15 [30]	30	25	0
10	30 [15]	20	15	~~30~~ 15
	~~15~~ 0	15	0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	15^{30}	30	25	~~30~~ 0
5	30	20	15	30
	~~45~~ 15	30	25	

		0	0	5	
→	10	30	20	10^{25}	~~40~~ 15
	~~10~~	15^{30}	30	25	0
	5	30	20	15	30
		15	30	~~25~~ 0	

		0	0	5	
→	~~10~~	30	20^{15}	10^{25}	~~15~~ 0
	~~10~~	15^{30}	30	25	0
	10	30	20	15	30
		15	~~30~~ 15	0	

	↓			
	30	20	5	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
10	30^{15}	20	15	~~30~~ 15
	~~15~~ 0	15	0	

	30	20	5	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
20	30^{15}	20	15	15
	0	15	0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	15^{30}	30	25	30 0
5	30	20	15	30
	~~45~~ 15	30	25	

		0	0	5	
→	10	30	20	10^{25}	~~40~~ 15
	~~10~~	15^{30}	30	25	0
	5	30	20	15	30
		15	30	~~25~~ 0	

		0	0	5	
→	~~10~~	30	20^{15}	10^{25}	~~15~~ 0
	~~10~~	15^{30}	30	25	0
	10	30	20	15	30
		15	~~30~~ 15	0	

	↓			
	30	20	5	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
10	30^{15}	20	15	~~30~~ 15
	~~15~~ 0	15	0	

		↓		
	30	20	5	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
20	30^{15}	20	15	15
	0	15	0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~30~~ 0
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	~~5~~	
→ 10	30	20	10^{25}	~~40~~ 15
~~10~~	15^{30}	30	25	0
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	~~5~~	
→ ~~10~~	30	20^{15}	10^{25}	~~15~~ 0
~~10~~	15^{30}	30	25	0
10	30	20	15	30
	15	~~30~~ 15	0	

	↓ ~~30~~	20	~~5~~	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
10	30^{15}	20	15	~~30~~ 15
	~~15~~ 0	15	0	

	~~30~~	↓ 20	~~5~~	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
20	30^{15}	20	15	15
	0	15	0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~30 0~~
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10^{25}	~~40~~ 15
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~0~~
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	5	
→ ~~10~~	~~30~~	~~20^{15}~~	~~10^{25}~~	~~15~~ 0
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~0~~
10	30	20	15	30
	15	~~30~~ 15	0	

	↓ 30	20	5	
~~10~~	~~30~~	~~20^{15}~~	~~10^{25}~~	~~0~~
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~0~~
10	30^{15}	20	15	~~30~~ 15
	~~15~~ 0	15	0	

	30	↓ 20	5	
~~10~~	~~30~~	~~20^{15}~~	~~10^{25}~~	~~0~~
~~10~~	~~15^{30}~~	~~30~~	~~25~~	~~0~~
20	30^{15}	20^{15}	15	~~15~~ 0
	0	~~15~~ 0	0	

Apply VAM to the cellphone problem

	M_1	M_2	M_3	
W_1	30	20	10	40
W_2	15	30	25	30
W_3	30	20	15	30
	45	30	25	100

smallest entry in row/col with
↓ largest diff among the 6
diffs of smallest 2 entries

	15	0	5	
10	30	20	10	40
~~10~~	15^{30}	30	25	~~30~~ 0
5	30	20	15	30
	~~45~~ 15	30	25	

	0	0	5	
→ 10	30	20	10^{25}	~~40~~ 15
~~10~~	15^{30}	30	25	0
5	30	20	15	30
	15	30	~~25~~ 0	

	0	0	5	
→ ~~10~~	30	20^{15}	10^{25}	~~15~~ 0
~~10~~	15^{30}	30	25	0
10	30	20	15	30
	15	~~30~~ 15	0	

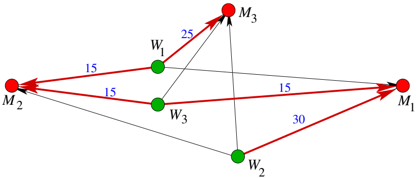
	↓ 30	20	5	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
10	30^{15}	20	15	~~30~~ 15
	~~15~~ 0	15	0	

	30	↓ 20	5	
~~10~~	30	20^{15}	10^{25}	0
~~10~~	15^{30}	30	25	0
~~20~~	30^{15}	20^{15}	15	~~15~~ 0
	0	~~15~~ 0	0	

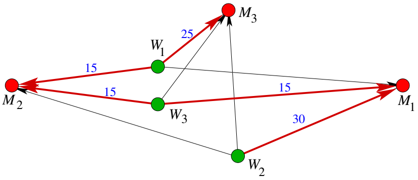
	30	20	5	
30	30	20^{15}	10^{25}	0
10	15^{30}	30	25	0
20	30^{15}	20^{15}	15	0
	0	0	0	

	M_1	M_2	M_3		
W_1	30	20^{15}	10^{25}	0	40
W_2	15^{30}	30	25	0	30
W_3	30^{15}	20^{15}	15	0	30
	0	0	0		
	45	30	25		

	M_1	M_2	M_3		
W_1	30	20 ^{15}	10 ^{25}	0	40
W_2	15 ^{30}	30	25	0	30
W_3	30 ^{15}	20 ^{15}	15	0	30
	0	0	0		
	45	30	25		

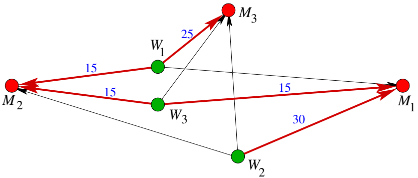


	M_1	M_2	M_3		
W_1	30	20^{15}	10^{25}	0	40
W_2	15^{30}	30	25	0	30
W_3	30^{15}	20^{15}	15	0	30
	0	0	0		
	45	30	25		



Here, VAM actually resulted in an optimal solution, but we do not yet know this fact!

	M_1	M_2	M_3		
W_1	30	20^{15}	10^{25}	0	40
W_2	15^{30}	30	25	0	30
W_3	30^{15}	20^{15}	15	0	30
	0	0	0		
	45	30	25		



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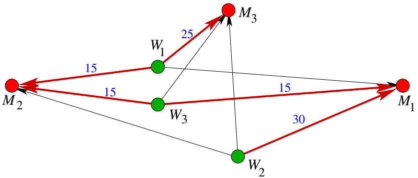
In general, it gives only an *initial basic feasible solution*. Which we learn how to solve in the next lecture.

	M_1	M_2	M_3		
W_1	30	20^{15}	10^{25}	0	40
W_2	15^{30}	30	25	0	30
W_3	30^{15}	20^{15}	15	0	30
	0	0	0		
	45	30	25		

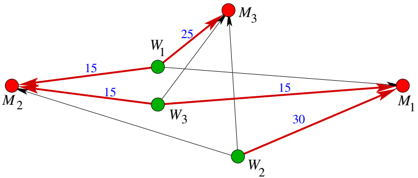


Here, VAM actually resulted in an optimal solution, but we do not yet know this fact!
In general, it gives only an *initial basic feasible solution*. Which we learn how to solve in the next lecture.

Notes:

- All non-circled cells correspond to x_{ij}-value 0
- A transportation tableau corresponds to a **basic solution** of the LP iff the circled entries correspond to a spanning forest of the graph. The circled cells constitute a **basis** for the solution.
- A basic solution is **feasible** if no flow value is negative.

	M_1	M_2	M_3		
W_1	30	20^{15}	10^{25}	0	40
W_2	15^{30}	30	25	0	30
W_3	30^{15}	20^{15}	15	0	30
	0	0	0		
	45	30	25		



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Notes:

- All non-circled cells correspond to x_{ij}-value 0
- A transportation tableau corresponds to a **basic solution** of the LP iff the circled entries correspond to a spanning forest of the graph. The circled cells constitute a **basis** for the solution.
- A basic solution is **feasible** if no flow value is negative.

Theorem

The VAM produces a basic feasible solution for any balanced transportation problem.