MATH 308 D200, Fall 2019

12. First summary

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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We Should (Could (Might)) Know...

LP Problems

- Intuitive definition of LP problem—too vague; many problems can be formulated as an LP problem (e.g., circular disk problem) but this 'modelling' can be difficult.
- Canonical form of an LP problem—starting point to 'LP World'

$$\begin{aligned} \textbf{maximize } f(\boldsymbol{x}) &= \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d \\ \text{subject to } \boldsymbol{A}\boldsymbol{x} &\leqslant 0 \\ \boldsymbol{x} &\geqslant \mathbf{0} \end{aligned} \qquad \begin{aligned} \textbf{minimize } g(\boldsymbol{x}) &= \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d \\ \text{subject to } \boldsymbol{A}\boldsymbol{x} &\geqslant 0 \\ \boldsymbol{x} &\geqslant \mathbf{0} \end{aligned}$$

Math Behind

- Constraint set of a canonical LP problem is a polyhedral convex set—intersection of finitely many closed half-spaces. Even more—if it is non-empty then there is always an extreme point.
- Constraint set may be empty $\Rightarrow$ infeasible LP problem.
- Classification of LP problems:
  - (i) infeasible LP problems
  - (ii) unbounded LP problems
  - (iii) LP problems having bounded constraint set for which the optimal value of the objective function is attained at an extreme point
  - (iv) LP problems having unbounded constraint set for which the optimal value of the objective function is attained at an extreme point
- If there is an optimal solution for an LP problem then it is attained in some extreme point.

Thus the idea of finding an optimal solution is simple: We can focus on extreme points.

Geometric Method

- Having n variables and m main constraints we are dealing with $n+m$ hyperplanes.
- We have to decide whether the LP problem is bounded or not.
- Taking n equations at a time out of all $n+m$ we get $\binom{n+m}{n}$ systems of n linear equations with n unknowns.
- We obtain set of extreme point candidates.
- We eliminate infeasible points and get the set of all extreme points.
- By plugging then into the objective function we find an optimal solution.

There is a better way!

Simplex Algorithm for Maximization LP Problems

- Standard (equational) form, slack variables, canonical slack form, Tucker tableaux.
- Basic (dependent) variables, non-basic (independent) variables.
- Basic solution (extreme point candidate), basic feasible solution (extreme point).

Pivot Transformation on TT

- One variable enters the basis (entering variable), another leaves the basis (leaving variable).
- We require the basis to define a non-singular matrix (in fact an identity matrix) hence the pivot has to be non-zero.

$$
\begin{array}{r}
x_1 \leqslant 3 \\
-x_1 + x_2 \leqslant 2 \\
x_1 + 2x_2 \leqslant 1 \\
\hline
-x_1 + 2x_2 + 3 = f
\end{array}
\longrightarrow
\begin{array}{cc|c|l}
x_1 & x_2 & -1 & \\
\hline
1 & 0 & 3 & = -x_3 \\
-1 & 1 & 2 & = -x_4 \\
1 & 2^* & 1 & = -x_5 \\
\hline
-1 & 2 & -3 & = f \\
\hline
\end{array}
\longrightarrow
\begin{array}{cc|c|l}
x_1 & x_5 & -1 & \\
\hline
1 & 0 & 3 & = -x_3 \\
-3/2 & -1/2 & 3/2 & = -x_4 \\
1/2 & 1/2 & 1/2 & = -x_2 \\
\hline
-2 & -1 & -4 & = f \\
\hline
\end{array}
$$

And now we perform the same operation to LP in standard equational form:

x_1	x_2	x_3	x_4	x_5	b
1	0	1	0	0	3
-1	1	0	1	0	2
1	2	0	0	1	1
1	-2	0	0	0	3

Theorem

Let i, j, B, B' be as stated above. Let $k \in \{1, 2, \ldots, m\}$ be such that $a_{ki} = 1$. For B' to define an identity matrix we need to perform following elementary operations on the system $\boldsymbol{A}^s \boldsymbol{x}^s = \boldsymbol{b}$

- *multiply row k by $\frac{1}{a_{kj}}$*
- *for each row $\ell \neq k$, add $-a_{\ell j}$ multiple of (new) row k to row ℓ*

x_1	x_2	x_3	x_4	x_5	b
1	0	1	0	0	3
-1	1	0	1	0	2
1/2	1	0	0	1/2	1/2
1	-2	0	0	0	3

x_1	x_2	x_3	x_4	x_5	b
1	0	1	0	0	3
-3/2	0	0	1	-1/2	3/2
1/2	1	0	0	1/2	1/2
1	-2	0	0	0	3

x_1	x_2	x_3	x_4	x_5	b
1	0	1	0	0	3
-3/2	0	0	1	-1/2	3/2
1/2	1	0	0	1/2	1/2
2	0	0	0	1	4

SA for Maximum Tableaux

1. We have maximum Tucker tableau.
2. If $b_1, b_2, \dots, b_m \geqslant 0$, go to **Step 6.**
3. Choose $b_i < 0$ such that i is maximal.
4. If $a_{i1}, a_{i2}, \dots, a_{in} \geqslant 0 \Longrightarrow$ **STOP**; the problem is infeasible.
5. If $i = m$, choose $a_{mj} < 0$, pivot on a_{mj}, and go to **Step 1.**
   If $i < m$, choose $a_{ij} < 0$, compute

   $$\alpha = \min(\{b_i/a_{ij}\} \cup \{b_k/a_{kj} : k > i, a_{kj} > 0\}),$$

   and choose any p with $b_p/a_{pj} = \alpha$. Pivot on a_{pj} and go to **Step 1.**
6. We have MBFT $(b_1, b_2, \dots, b_m \geqslant 0)$
7. If $c_1, c_2, \dots, c_n \leqslant 0 \Longrightarrow$ **STOP**; the current basic feasible solution is optimal.
8. Choose any j with $c_j > 0$
9. If $a_{1j}, a_{2j}, \dots, a_{mj} \leqslant 0 \Longrightarrow$ **STOP**; the problem is unbounded.
10. Compute

    $$\alpha = \min_{1 \leqslant i \leqslant m} \{b_i/a_{ij} : a_{ij} > 0\}$$

    and choose any p with $b_p/a_{pj} = \alpha$. Pivot on a_{pj} and go to the **Step 6.**

To prevent cycling we can employ anticycling rules; Usually it is not necessary.

Maximize $2x_1 - x_2 + 8x_3$, subject to

$$\begin{aligned} 2x_3 &\leqslant 1 \\ 2x_1 - 4x_2 + 6x_3 &\leqslant 3 \\ -x_1 + 3x_2 + 4x_3 &\leqslant 2 \\ x_1, x_2, x_3 &\geqslant 0 \end{aligned}$$

x_1	x_2	x_3	-1	
0	0	2	1	$= -t_1$
2	-4	6	3	$= -t_2$
-1	3	4*	2	$= -t_3$
2	-1	8	0	$= f$

x_1	x_2	t_3	-1	
1/2	-3/2	-1/2	0	$= -t_1$
7/2*	-17/2	-3/2	0	$= -t_2$
-1/4	3/4	1/4	1/2	$= -x_3$
4	-7	-2	-4	$= f$

t_2	x_2	t_3	-1	
-1/2	-2/7	-2/7	0	$= -t_1$
2/7	-17/7	-3/7	0	$= -x_1$
1/14	1/7*	1/7	1/2	$= -x_3$
-8/7	19/7	-2/7	-4	$= f$

t_2	x_2	t_3	-1	
0	2	0	1	$= -t_1$
3/2	17	2	17/2	$= -x_1$
1/2	7	1	7/2	$= -x_3$
-5/2	-19	-3	-27/2	$= f$

SA for Minimum Tableaux

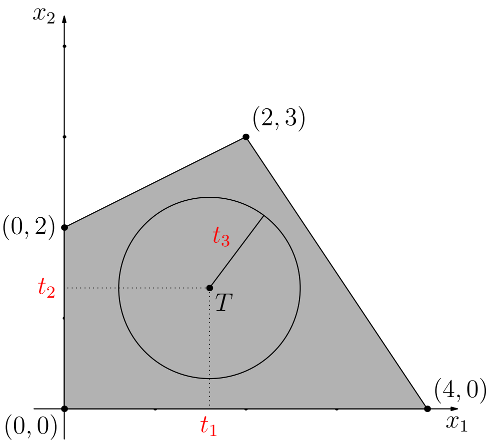
1. We have minimum Tucker tableau.
2. Take the negative transposition of the tableau to obtain a maximum tableau.
3. Apply SA for maximum tableaux.
4. $\min g = -\max(-g)$.

SA vs Geometric Method

- SA detects both infeasibility and unboundedness.
- SA is much more effective:
  For 15 main constraints, 10 variables—$\binom{25}{10} > 3\,200\,000$.
  SA would only require 13 to 50 pivot transformations.
- SA has many refinements, optimized for particular problems.
- SA is easily implemented on computers.

Nice example revisited—Exact vs. approximate solution

Consider we are given 4 points in the plane—$A = (0,0)$, $B = (4,0)$, $C = (2,3)$, $D = (0,2)$. Find the largest circular disk that fits in the quadrangle $ABCD$.



maximize $f(t_1, t_2, t_3) = t_3$, subject to

$$\begin{aligned} 3t_1 + 2t_2 &\leqslant 12 \\ -t_1 + 2t_2 &\leqslant 4 \\ -t_1 + t_3 &\leqslant 0 \\ -t_2 + t_3 &\leqslant 0 \\ 3t_1 + 2t_2 + \sqrt{13}t_3 &\leqslant 12 \\ -t_1 + 2t_2 + \sqrt{5}t_3 &\leqslant 4 \\ t_1, t_2, t_3 &\geqslant 0 \end{aligned}$$

only approximation:

$$\sqrt{13} \approx 3.605551275$$

$$\sqrt{5} \approx 2.236067977$$

Another Pivot rule—Largest increase of the objective function

x_1	x_2	x_3	-1	
1	2	1	4	$=-x_4$
2	1	5	5	$=-x_5$
3	2	0	6	$=-x_6$
1	2	3	0	$=f$

x_6	x_2	x_3	-1	
-1/3	4/3	1	2	$=-x_4$
-2/3	-1/3	5	1	$=-x_5$
1/3	2/3	0	2	$=-x_1$
-1/3	4/3	3	-2	$=f$

x_1	x_4	x_3	-1	
1/2	1/2	1/2	2	$=-x_2$
3/2	-1/2	9/2	3	$=-x_5$
2	-1	-1	2	$=-x_6$
0	-1	2	-4	$=f$

x_1	x_2	x_5	-1	
3/5	9/5	-1/5	3	$=-x_4$
2/5	1/5	1/5	1	$=-x_3$
3	2	0	6	$=-x_6$
-1/5	7/5	-3/5	-3	$=f$