MATH 308 D200, Fall 2019

6. Canonical slack form of maximization LP
(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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The Simplex Algorithm

- SA is an effective method for solving LP problems.
- Developed in the 1940's by George B. Dantzig
- One of the most famous algorithms of the twentieth century.
- SA requires equational (yet another...) form of LP problem.
- SA based on linear algebra—systems of linear equations, linear transformations of matrices, ...
- LP problem is recorded into a tableau. SA—series of transformations of tableaux until an optimal solution is found.
- We present technique developed on in 1960's by A. W. Tucker involving Tucker tableaux, a more compact version of original tableaux used by Dantzig.

Canonical Slack Form for Maximization LP Problem

Start with a canonical maximization LP problem,

$$
\begin{aligned}
\text{maximize} \quad & f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n - d \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leqslant b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leqslant b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leqslant b_m \\
& x_1, x_2, \ldots, x_n \geqslant 0
\end{aligned}
$$

Start with a canonical maximization LP problem,
Add a *slack variable* t_i for each main constraint,

$$
\begin{aligned}
\text{maximize} \quad & f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n - d \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + t_1 = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + t_2 = b_2 \\
& \qquad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + t_m = b_m \\
& x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m \geqslant 0
\end{aligned}
$$

Canonical Slack Form for Maximization LP Problem

Start with a canonical maximization LP problem,
Add a *slack variable* t_i for each main constraint,
Put slack variables on the right hand side:

$$
\begin{array}{ll}
\text{maximize} & f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n - d \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 = -t_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 = -t_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m = -t_m \\
& x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m \geqslant 0
\end{array}
$$

Start with a canonical maximization LP problem,
Add a *slack variable* t_i for each main constraint,
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\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 = -t_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 = -t_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m = -t_m \\
& x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m \geqslant 0
\end{array}
$$

Definition

This maximization LP is said to be in *canonical slack form*.

Canonical Equational Form for Maximization LP Problem

Start with a canonical maximization LP problem,
Add a *slack variable* t_i for each main constraint,
Put slack variables on the left hand side:

$$
\begin{array}{ll}
\text{maximize} & f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n - d \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 = -t_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 = -t_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m = -t_m \\
& x_1, x_2, \dots, x_n, t_1, t_2, \dots, t_m \geqslant 0
\end{array}
$$

Canonical Equational Form for Maximization LP Problem

Start with a canonical maximization LP problem,
Add a *slack variable* t_i for each main constraint,
Put slack variables on the left hand side:

$$
\begin{array}{ll}
\text{maximize} & f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n - d \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + t_1 = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + t_2 = b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + t_m = b_m \\
& x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_m \geqslant 0
\end{array}
$$

Definition

This maximization LP is said to be in *canonical equational form*.

Canonical Equational Form for Maximization LP Problem

Start with a canonical maximization LP problem,
Add a *slack variable* t_i for each main constraint,
Put slack variables on the left hand side:
Optional: Rename the slack variables $t_i \mapsto x_{n+i}$. Slack variables are not treated specially.

$$
\begin{array}{ll}
\text{maximize} & f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n - d \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\
 & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\
 & \vdots \\
 & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\
 & x_1, x_2, \ldots, x_{n+m} \geqslant 0
\end{array}
$$

Definition

This maximization LP is said to be in *canonical equational form*.

Canonical Equational Form for Maximization LP Problem

Matrix Notation:

$$
\begin{array}{ll}
\text{maximize} & f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n - d \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m \\
& x_1, x_2, \ldots, x_{n+m} \geqslant 0
\end{array}
$$

Definition

This maximization LP is said to be in *canonical equational form*.

Canonical Equational Form for Maximization LP Problem

Matrix Notation:
Spread out new variables

$$
\begin{array}{ll}
\text{maximize} & f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n - d \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} \qquad\qquad = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + \qquad x_{n+2} \qquad = b_2 \\
& \qquad\qquad\qquad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + x_{n+m} = b_m \\
& \qquad\qquad\qquad x_1, x_2, \dots, x_{n+m} \geqslant 0
\end{array}
$$

Definition

This maximization LP is said to be in *canonical equational form*.

Canonical Equational Form for Maximization LP Problem

Matrix Notation:
Spread out new variables
Write as matrix equation

$$\begin{array}{ll} \text{maximize} & f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n - d \\ \text{subject to} & \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_{m+n} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \\ & x_1, x_2, \ldots, x_{n+m} \geqslant 0 \end{array}$$

Definition

This maximization LP is said to be in *canonical equational form*.

Matrix Notation:
Spread out new variables
Write as matrix equation

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^{s\,\mathsf{T}} \boldsymbol{x}^s - d \\ \text{subject to} \quad & \boldsymbol{A}^s \boldsymbol{x}^s = \boldsymbol{b} \\ & \boldsymbol{x}^s \geqslant \boldsymbol{0} \end{aligned}$$

Where $\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & \dots & x_n & x_{n+1} & x_{n+2} & \dots & x_{n+m} \end{bmatrix}^{\mathsf{T}} \in \mathbb{R}^{n+m}$

$$\boldsymbol{A}^s = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\boldsymbol{c}^{s\,\mathsf{T}} = \begin{bmatrix} c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 \end{bmatrix}$$

Definition

This maximization LP is said to be in *canonical equational form*.

Summary: Three canonical forms

Canonical (max) Form	Canonical Slack Form	Canonical Equational Form
$\begin{array}{ll} \max & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d \\ \text{s.t.} & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \geqslant \boldsymbol{0} \end{array}$	$\begin{array}{ll} \max & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d \\ \text{s.t.} & A\boldsymbol{x} - \boldsymbol{b} = -\boldsymbol{t} \\ & \boldsymbol{x}, \boldsymbol{t} \geqslant \boldsymbol{0} \end{array}$	$\begin{array}{ll} \max & \boldsymbol{c}^{s\mathsf{T}}\boldsymbol{x}^s - d \\ \text{s.t.} & A^s\boldsymbol{x}^s = \boldsymbol{b} \\ & \boldsymbol{x}^s \geqslant \boldsymbol{0} \end{array}$

Where

$$\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n, \qquad \boldsymbol{t} = \begin{bmatrix} t_1 & t_2 & \dots & t_m \end{bmatrix}^\mathsf{T} \in \mathbb{R}^m$$

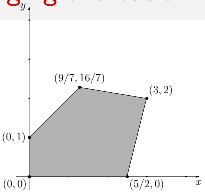
$$\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & \dots & x_n & x_{n+1} & x_{n+2} & \dots & x_{n+m} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{n+m}$$

$$A^s = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & 1 \end{bmatrix} = [\, A \; I_m \,], \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\boldsymbol{c}^{s\mathsf{T}} = \begin{bmatrix} c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n+m}$$

$$\boldsymbol{c}^\mathsf{T} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \in \mathbb{R}^n$$

$$d \in \mathbb{R}$$



Maximize $f = x + y - 0$

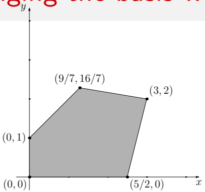
$$-x + y \leq 1$$

$$x + 6y \leq 15$$

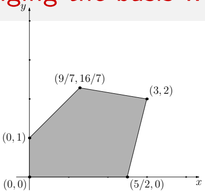
$$4x - y \leq 10$$

$$x, y \geq 0$$

Changing the basis with Gaussian elimination



$$
\begin{array}{rl}
\text{Maximize} & f = x + y - 0 \\
(t_1) & -x + y \leq 1 \\
(t_2) & x + 6y \leq 15 \\
(t_3) & 4x - y \leq 10 \\
 & x, y \geq 0
\end{array}
$$

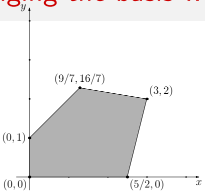


Maximize $f = x + y - 0$

$$-x + y - 1 = -t_1$$
$$x + 6y - 15 = -t_2$$
$$4x - y - 10 = -t_3$$
$$x, y, t_1, t_2, t_3 \geq 0$$

$$\begin{array}{cc|c|l} x & y & -1 & \\ \hline -1 & 1 & 1 & = -t_1 \\ 1 & 6 & 15 & = -t_2 \\ 4^* & -1 & 10 & = -t_3 \\ \hline 1 & 1 & 0 & = f \\ \hline \end{array}$$

Changing the basis with Gaussian elimination



Maximize $f = x + y - 0$

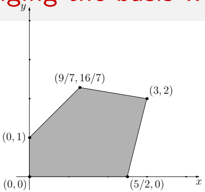
$$-x + y + t_1 = 1$$

$$x + 6y + t_2 = 15$$

$$4x - y + t_3 = 10$$

$$x, y, t_1, t_2, t_3 \geq 0$$

x	y	-1	
-1	1	1	$= -t_1$
1	6	15	$= -t_2$
4^*	-1	10	$= -t_3$
1	1	0	$= f$



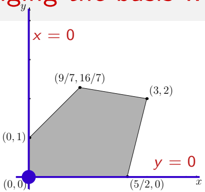
Maximize $f = x + y - 0$

$$-x + y + t_1 = 1$$
$$x + 6y + t_2 = 15$$
$$4x - y + t_3 = 10$$
$$x, y, t_1, t_2, t_3 \geq 0$$

$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right]$$

x	y	-1	
-1	1	1	$= -t_1$
1	6	15	$= -t_2$
4^*	-1	10	$= -t_3$
1	1	0	$= f$



Maximize $f = x + y - 0$

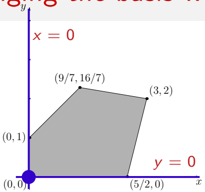
$$-x + y + t_1 = 1$$
$$x + 6y + t_2 = 15$$
$$4x - y + t_3 = 10$$
$$x, y, t_1, t_2, t_3 \geq 0$$

$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right]$$

$$BFS = (\qquad\qquad)$$

x	y	-1	
-1	1	1	$= -t_1$
1	6	15	$= -t_2$
4^*	-1	10	$= -t_3$
1	1	0	$= f$



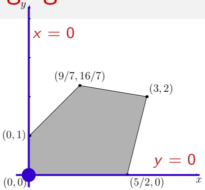
$$\begin{aligned} \text{Maximize} \quad f &= x + y - 0 \\ -x + y + t_1 &= 1 \\ x + 6y + t_2 &= 15 \\ 4x - y + t_3 &= 10 \\ x, y, t_1, t_2, t_3 &\geq 0 \end{aligned}$$

$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right]$$

$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix}$$

$$\begin{array}{cc|c|l} x & y & -1 & \\ \hline -1 & 1 & 1 & = -t_1 \\ 1 & 6 & 15 & = -t_2 \\ 4^* & -1 & 10 & = -t_3 \\ \hline 1 & 1 & 0 & = f \end{array}$$



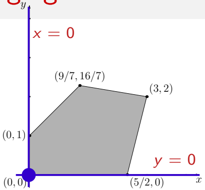
$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right]$$

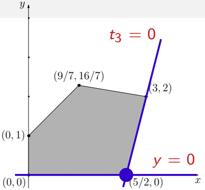
$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix}$$

x	y	-1	
-1	1	1	$= -t_1$
1	6	15	$= -t_2$
4^*	-1	10	$= -t_3$
1	1	0	$= f$

Changing the basis with Gaussian elimination



$\longrightarrow$

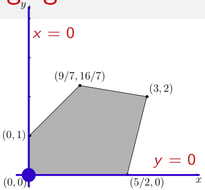


$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix} \mapsto \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

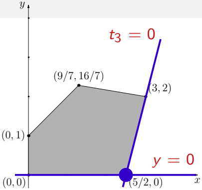
$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right]$$

$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & & & \end{pmatrix}$$

x	y	-1	
-1	1	1	$= -\,t_1$
1	6	15	$= -\,t_2$
4^*	-1	10	$= -\,t_3$
1	1	0	$= f$



$\longrightarrow$



$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix} \mapsto \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

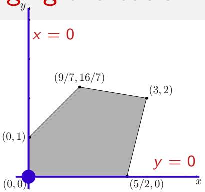
$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right]$$

$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix} \mapsto \begin{pmatrix} & 0 & & & 0 \end{pmatrix}$$

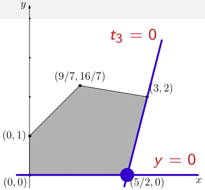
t_3 leaves basis: $t_3 = 0$

x	y	-1	
-1	1	1	$= -\ t_1$
1	6	15	$= -\ t_2$
4^*	-1	10	$= -\ t_3$
1	1	0	$= f$

Changing the basis with Gaussian elimination



$\longrightarrow$



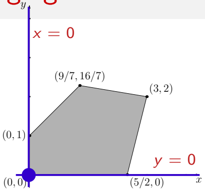
$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix} \mapsto \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 1 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{5}{2} \end{array}\right]$$

$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix} \mapsto \begin{pmatrix} & 0 & & & 0 \end{pmatrix}$$

x enters basis: Scale pivot row

x	y	-1	
-1	1	1	$= -\, t_1$
1	6	15	$= -\, t_2$
4^*	-1	10	$= -\, t_3$
1	1	0	$= f$



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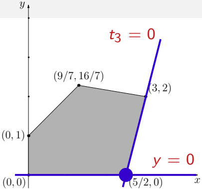


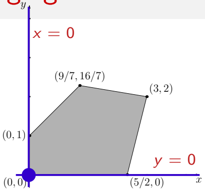
$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix} \mapsto \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} 0 & \frac{3}{4} & 1 & 0 & \frac{1}{4} & \frac{7}{2} \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 1 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{5}{2} \end{array}\right]$$

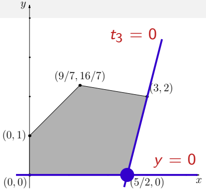
$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix} \mapsto \begin{pmatrix} & 0 & & & 0 \end{pmatrix}$$

x enters basis: Eliminate first row of x

x	y	-1	
-1	1	1	$= -\ t_1$
1	6	15	$= -\ t_2$
4^*	-1	10	$= -\ t_3$
1	1	0	$= f$



$\longrightarrow$



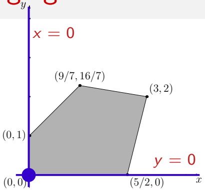
$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix} \mapsto \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} 0 & \frac{3}{4} & 1 & 0 & \frac{1}{4} & \frac{7}{2} \\ 0 & \frac{25}{4} & 0 & 1 & -\frac{1}{4} & \frac{25}{2} \\ 1 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{5}{2} \end{array}\right]$$

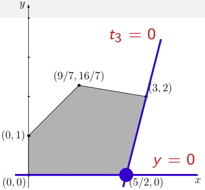
$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix} \mapsto \begin{pmatrix} & 0 & & & 0 \end{pmatrix}$$

x enters basis: Eliminate second row of x

x	y	-1	
-1	1	1	$= -\ t_1$
1	6	15	$= -\ t_2$
4^*	-1	10	$= -\ t_3$
1	1	0	$= f$



$\longrightarrow$



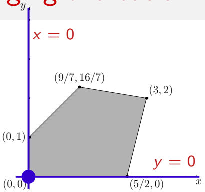
$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix} \mapsto \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} 0 & \frac{3}{4} & 1 & 0 & \frac{1}{4} & \frac{7}{2} \\ 0 & \frac{25}{4} & 0 & 1 & -\frac{1}{4} & \frac{25}{2} \\ 1 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{5}{2} \end{array}\right]$$

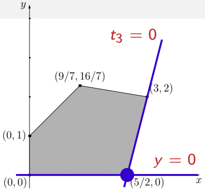
$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix} \mapsto \begin{pmatrix} \frac{5}{2} & 0 & \frac{7}{2} & \frac{25}{2} & 0 \end{pmatrix}$$

Find BFS (using the last column)

x	y	-1	
-1	1	1	$= -\ t_1$
1	6	15	$= -\ t_2$
4^*	-1	10	$= -\ t_3$
1	1	0	$= f$



$\longrightarrow$



$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix} \mapsto \begin{pmatrix} t_3 & y & t_1 & t_2 & x \end{pmatrix}$$

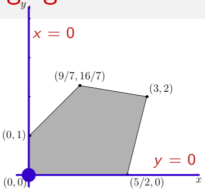
$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} \frac{1}{4} & \frac{3}{4} & 1 & 0 & 0 & \frac{7}{2} \\ -\frac{1}{4} & \frac{25}{4} & 0 & 1 & 0 & \frac{25}{2} \\ \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 1 & \frac{5}{2} \end{array}\right]$$

$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \frac{7}{2} & \frac{25}{2} & \frac{5}{2} \end{pmatrix}$$

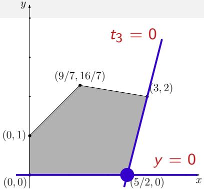
Swap columns

x	y	-1	
-1	1	1	$= -\ t_1$
1	6	15	$= -\ t_2$
4^*	-1	10	$= -\ t_3$
1	1	0	$= f$

Changing the basis with Gaussian elimination



$\longrightarrow$



$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix} \mapsto \begin{pmatrix} t_3 & y & t_1 & t_2 & x \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} \frac{1}{4} & \frac{3}{4} & 1 & 0 & 0 & \frac{7}{2} \\ -\frac{1}{4} & \frac{25}{4} & 0 & 1 & 0 & \frac{25}{2} \\ \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 1 & \frac{5}{2} \end{array}\right]$$

$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \frac{7}{2} & \frac{25}{2} & \frac{5}{2} \end{pmatrix}$$

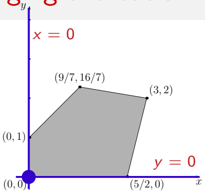
New tableau: (except last row)

x	y	-1	
-1	1	1	$= -t_1$
1	6	15	$= -t_2$
4^*	-1	10	$= -t_3$
1	1	0	$= f$

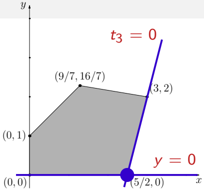
$\mapsto$

t_3	y	-1	
1/4	3/4	7/2	$= -t_1$
-1/4	25/4	25/2	$= -t_2$
1/4	-1/4	5/2	$= -x$
?	?	?	$= f$

Changing the basis with Gaussian elimination



$\longrightarrow$



$$\boldsymbol{x} = \begin{pmatrix} x & y & t_1 & t_2 & t_3 \end{pmatrix} \mapsto \begin{pmatrix} t_3 & y & t_1 & t_2 & x \end{pmatrix}$$

$$[A^s \mid \boldsymbol{b}] = \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 6 & 0 & 1 & 0 & 15 \\ 4 & -1 & 0 & 0 & 1 & 10 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} \frac{1}{4} & \frac{3}{4} & 1 & 0 & 0 & \frac{7}{2} \\ -\frac{1}{4} & \frac{25}{4} & 0 & 1 & 0 & \frac{25}{2} \\ \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 1 & \frac{5}{2} \end{array}\right]$$

$$BFS = \begin{pmatrix} 0 & 0 & 1 & 15 & 10 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \frac{7}{2} & \frac{25}{2} & \frac{5}{2} \end{pmatrix}$$

Bases: $B = \{\, t_1,\ t_2,\ t_3 \,\}$ $\qquad$ $B = \{\, t_1,\ t_2,\ x \,\}$

x	y	-1	
-1	1	1	$= -t_1$
1	6	15	$= -t_2$
4^*	-1	10	$= -t_3$
1	1	0	$= f$

$\mapsto$

t_3	y	-1	
1/4	3/4	7/2	$= -t_1$
-1/4	25/4	25/2	$= -t_2$
1/4	-1/4	5/2	$= -x$
?	?	?	$= f$

Maximize $f(\boldsymbol{x}^s) = \boldsymbol{c}^{s\intercal}\boldsymbol{x}^s - d$ subject to

$$\begin{aligned} \boldsymbol{A}^s\boldsymbol{x}^s &= \boldsymbol{b} \\ \boldsymbol{x}^s &\geqslant \boldsymbol{0} \end{aligned}$$

where

$$\boldsymbol{x}^s = \overbrace{\big(\underbrace{x_1 \quad x_2 \quad \dots \quad x_n}_{\text{Nonbasic (independent)}} \quad \underbrace{t_1 \quad t_2 \quad \dots \quad t_m}_{\text{Basic (dependent)}}\big)}^{\text{Variables}} \in \mathbb{R}^{n+m}$$

and

$$\boldsymbol{A}^s = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\boldsymbol{c}^{s\intercal} = \begin{bmatrix} c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 \end{bmatrix}$$

Standard (Equational) Form of Canonical Maximization LP problem

Maximize $f(\boldsymbol{x}^s) = \boldsymbol{c}^{s\mathsf{T}}\boldsymbol{x}^s - d$ subject to

$$\boldsymbol{A}^s\boldsymbol{x}^s = \boldsymbol{b}$$
$$\boldsymbol{x}^s \geqslant \boldsymbol{0}$$

where

$$\boldsymbol{x}^s = \overbrace{\begin{pmatrix} \underbrace{x_1 \quad x_2 \quad \dots \quad x_n}^{\text{Nonbasic (independent)}} & \underbrace{x_{n+1} \quad x_{n+2} \quad \dots \quad x_{n+m}}^{\text{Basic (dependent)}} \end{pmatrix}}^{\text{Variables}} \in \mathbb{R}^{n+m}$$

and

$$\boldsymbol{A}^s = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\boldsymbol{c}^{s\mathsf{T}} = \begin{bmatrix} c_1 & c_2 & \dots & c_n & 0 & 0 & \dots & 0 \end{bmatrix}$$

Summary:

Let $\boldsymbol{A}^s \in \mathbb{R}^{n\times(n+m)}$ be a matrix of rank n. For a subset $B \subseteq \{1, 2, \dots, n+m\}$ we denote $\boldsymbol{A}^s_B$ the matrix consisting of the columns of $\boldsymbol{A}^s$ whose indices belong to B. For instance

$$\boldsymbol{A}^s = \begin{bmatrix} 1 & 5 & 3 & 4 & 6 \\ 2 & 1 & 3 & -1 & 6 \end{bmatrix} \implies \boldsymbol{A}^s_{\{2,4\}} = \begin{bmatrix} 5 & 4 \\ 1 & -1 \end{bmatrix}$$

$$\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^\mathsf{T} \implies \boldsymbol{x}^{s\,\mathsf{T}}_{\{2,4\}} = \begin{bmatrix} x_2 & x_4 \end{bmatrix}$$

Summary:
Let $\boldsymbol{A}^s \in \mathbb{R}^{n\times(n+m)}$ be a matrix of rank n. For a subset $B \subseteq \{1,2,\ldots,n+m\}$ we denote $\boldsymbol{A}^s_B$ the matrix consisting of the columns of $\boldsymbol{A}^s$ whose indices belong to B. For instance

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$$\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^\mathsf{T} \implies {\boldsymbol{x}^s}^\mathsf{T}_{\{2,4\}} = \begin{bmatrix} x_2 & x_4 \end{bmatrix}$$

Definition (Basic Solution, Basic Feasible Solution)

Consider a max LP problem in slack form: maximize ${\boldsymbol{c}^s}^\mathsf{T}\boldsymbol{x}^s - d$ subject to $\boldsymbol{A}^s\boldsymbol{x}^s = \boldsymbol{b}, \boldsymbol{x}^s \geqslant \boldsymbol{0}$. Let $B \subseteq \{1,2,\ldots,n+m\}$ be an m-element set such that the (square) matrix $\boldsymbol{A}^s_B$ is non-singular ($\det(A^s_B) \neq 0$).

- ▷ The (unique) solution $\boldsymbol{x}^s$ of $A^s\boldsymbol{x}^s = \boldsymbol{b}$ such that $x_j = 0$ for all $j \notin B$ is a **basic solution (BS)**. To find $\boldsymbol{x}^s$, we solve the system $A^s_B({\boldsymbol{x}^s}^\mathsf{T}_B)^\mathsf{T} = \boldsymbol{b}$, to get x_j for all $j \in B$, and set $x_j = 0$ for all $j \notin B$.
- ▷ The m variables x_j with $j \in B$ are the **basic variables** (**dependent**) variables
- ▷ The n variables x_j with $j \notin B$ are called **nonbasic** (**independent**) variables.
- ▷ A basic solution satisfying $\boldsymbol{x}^s \geq 0$ is called a **basic feasible solution (BFS)**.

Summary:

Let $\boldsymbol{A}^s \in \mathbb{R}^{n\times(n+m)}$ be a matrix of rank n. For a subset $B \subseteq \{1,2,\ldots,n+m\}$ we denote $\boldsymbol{A}^s_B$ the matrix consisting of the columns of $\boldsymbol{A}^s$ whose indices belong to B. For instance

$$\boldsymbol{A}^s = \begin{bmatrix} 1 & 5 & 3 & 4 & 6 \\ 2 & 1 & 3 & -1 & 6 \end{bmatrix} \implies \boldsymbol{A}^s_{\{2,4\}} = \begin{bmatrix} 5 & 4 \\ 1 & -1 \end{bmatrix}$$

$$\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^\mathsf{T} \implies {\boldsymbol{x}^s}^\mathsf{T}_{\{2,4\}} = \begin{bmatrix} x_2 & x_4 \end{bmatrix}$$

Example:

$$\boldsymbol{A}^s\boldsymbol{x}^s = \boldsymbol{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}^s \geq \mathbf{0}$$

- When $B = \{1,2\}$: $\quad \boldsymbol{x} = \begin{bmatrix} ? & ? & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$

Summary:

Let $\boldsymbol{A}^s \in \mathbb{R}^{n\times(n+m)}$ be a matrix of rank n. For a subset $B \subseteq \{1,2,\dots,n+m\}$ we denote $\boldsymbol{A}^s_B$ the matrix consisting of the columns of $\boldsymbol{A}^s$ whose indices belong to B. For instance

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$$\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^\mathsf{T} \implies {\boldsymbol{x}^s}^\mathsf{T}_{\{2,4\}} = \begin{bmatrix} x_2 & x_4 \end{bmatrix}$$

Example:

$$\boldsymbol{A}^s \boldsymbol{x}^s = \boldsymbol{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}^s \geq \boldsymbol{0}$$

- When $B = \{1,2\}$: $\quad \boldsymbol{x} = \begin{bmatrix} ? & ? & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$

  Solve $\boldsymbol{A}^s_{\{1,2\}}({\boldsymbol{x}^s}^\mathsf{T}_{\{1,2\}})^\mathsf{T} = \boldsymbol{b}$: $\quad \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

Summary:

Let $\boldsymbol{A}^s \in \mathbb{R}^{n\times(n+m)}$ be a matrix of rank n. For a subset $B \subseteq \{1, 2, \ldots, n+m\}$ we denote $\boldsymbol{A}^s_B$ the matrix consisting of the columns of $\boldsymbol{A}^s$ whose indices belong to B. For instance

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$$\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^\mathsf{T} \implies {\boldsymbol{x}^s}^\mathsf{T}_{\{2,4\}} = \begin{bmatrix} x_2 & x_4 \end{bmatrix}$$

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- When $B = \{1, 2\}$: $\quad \boldsymbol{x} = \begin{bmatrix} ? & ? & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$

  Solve $\boldsymbol{A}^s_{\{1,2\}}({\boldsymbol{x}^s}^\mathsf{T}_{\{1,2\}})^\mathsf{T} = \boldsymbol{b}$: $\quad \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Summary:

Let $\boldsymbol{A}^s \in \mathbb{R}^{n\times(n+m)}$ be a matrix of rank n. For a subset $B \subseteq \{1, 2, \dots, n+m\}$ we denote $\boldsymbol{A}^s_B$ the matrix consisting of the columns of $\boldsymbol{A}^s$ whose indices belong to B. For instance

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$$\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^\mathsf{T} \implies {\boldsymbol{x}^s}^\mathsf{T}_{\{2,4\}} = \begin{bmatrix} x_2 & x_4 \end{bmatrix}$$

Example:

$$\boldsymbol{A}^s \boldsymbol{x}^s = \boldsymbol{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}^s \geq \boldsymbol{0}$$

- When $B = \{1, 2\}$: $\quad \boldsymbol{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$

  Solve $\boldsymbol{A}^s_{\{1,2\}} ({\boldsymbol{x}^s}^\mathsf{T}_{\{1,2\}})^\mathsf{T} = \boldsymbol{b}$: $\quad \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Summary:

Let $\boldsymbol{A}^s \in \mathbb{R}^{n\times(n+m)}$ be a matrix of rank n. For a subset $B \subseteq \{1, 2, \ldots, n+m\}$ we denote $\boldsymbol{A}^s_B$ the matrix consisting of the columns of $\boldsymbol{A}^s$ whose indices belong to B. For instance

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$$\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^\intercal \implies \boldsymbol{x}^{s\intercal}_{\{2,4\}} = \begin{bmatrix} x_2 & x_4 \end{bmatrix}$$

Example:

$$\boldsymbol{A}^s \boldsymbol{x}^s = \boldsymbol{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}^s \geq \boldsymbol{0}$$

- When $B = \{1, 2\}$: $\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^\intercal$ (this is a BFS)

  Solve $\boldsymbol{A}^s_{\{1,2\}}(\boldsymbol{x}^{s\intercal}_{\{1,2\}})^\intercal = \boldsymbol{b}$: $\begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Summary:

Let $\boldsymbol{A}^s \in \mathbb{R}^{n\times(n+m)}$ be a matrix of rank n. For a subset $B \subseteq \{1,2,\dots,n+m\}$ we denote $\boldsymbol{A}^s_B$ the matrix consisting of the columns of $\boldsymbol{A}^s$ whose indices belong to B. For instance

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$$\boldsymbol{x}^s = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^\mathsf{T} \implies {\boldsymbol{x}^s}^\mathsf{T}_{\{2,4\}} = \begin{bmatrix} x_2 & x_4 \end{bmatrix}$$

Example:

$$\boldsymbol{A}^s\boldsymbol{x}^s = \boldsymbol{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}^s \geq \boldsymbol{0}$$

- When $B = \{1,2\}$: $\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$ (this is a BFS)

  Solve $\boldsymbol{A}^s_{\{1,2\}}({\boldsymbol{x}^s}^\mathsf{T}_{\{1,2\}})^\mathsf{T} = \boldsymbol{b}$: $\begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- When $B = \{2,4\}$: $\boldsymbol{x} = \begin{bmatrix} 0 & ? & 0 & ? & 0 \end{bmatrix}^\mathsf{T}$

Summary:
Let $\boldsymbol{A}^s \in \mathbb{R}^{n\times(n+m)}$ be a matrix of rank n. For a subset $B \subseteq \{1,2,\ldots,n+m\}$ we denote $\boldsymbol{A}^s_B$ the matrix consisting of the columns of $\boldsymbol{A}^s$ whose indices belong to B. For instance

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Example:

$$\boldsymbol{A}^s\boldsymbol{x}^s = \boldsymbol{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}^s \geq \mathbf{0}$$

- When $B = \{1,2\}$: $\quad \boldsymbol{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$ (this is a BFS)

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- When $B = \{2,4\}$: $\boldsymbol{x} = \begin{bmatrix} 0 & 2 & 0 & -1 & 0 \end{bmatrix}^{\mathsf{T}}$ (this BS is not feasible)

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- When $B = \{3,5\}$: BS does not exist, since $\boldsymbol{A}^s_{\{3,5\}} = \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix}$ is not invertible.

More examples of BS, BFS

$$\mathbf{A}^s = \begin{bmatrix} 1 & 5 & 3 & 4 & 6 \\ 0 & 1 & 4 & 5 & 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 14 \\ 7 \end{bmatrix}$$

- $B = \{2, 4\}$; $\mathbf{x} = (0, 2, 0, 1, 0)$ is a BFS

- $B = \{1, 2\}$; $\mathbf{x} = (-21, 7, 0, 0, 0)$ is a BS which is not feasible

- $B = \{3, 5\}$; no solution