# MACM 201 - Discrete Mathematics

## Generating functions II - Rational GF's

Department of Mathematics

Simon Fraser University

## Rational GF's

We have already seen generating functions compactly expressed using inverses:

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

The following more general class of generating functions similarly have compact representations and will be our main subject.

### Definition

A generating function $A(x)$ is called **rational** if it can be expressed as

$$A(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

Our main interests with rational GF's are:

(1) Given a sequence of numbers express it as a rational GF.

(2) Given a rational GF, find the associated sequence (coefficient extraction)

## Two useful GF's

You will need to know the following two basic GF's.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{1}{(1-x)^2}$$

Using just these two GF's together with basic arithmetic operations gives us the ability to describe many other GF's.

### Note

*Let* $C(x) = c_0 + c_1 x + c_2 x^2 + \ldots = \sum_{n=0}^{\infty} c_n x^n$.

*(1) Adding a power function, say* $ax^d$, *to* $C(x)$ *changes the coefficient of* $x^d$ *to* $(c_d + a)$.

*(2) Multiplying* $C(x)$ *by a power of* $x$, *say* $x^k$, *shifts the coefficients by* $k$

$$x^k C(x) = c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \ldots = \sum_{n=0}^{\infty} c_n x^{k+n}$$

*Problem.* Determine the sequence for each GF.

(1) $\dfrac{x^3-2}{1-x} = (x^3-2)\left(\dfrac{1}{1-x}\right) = (x^3-2)(1+x+x^2+x^3+\dots)$

$= x^3(1+x+x^2+\dots) - 2(1+x+x^2+\dots) = (x^3+x^4+x^5+\dots)+(-2-2x-2x^2\dots)$

$= -2-2x-2x^2-x^3-x^4-x^5-\dots$

$$[x^n]\,\frac{x^3-2}{1-x} = \begin{cases} -2 & \text{if } n \le 2 \\ -1 & \text{if } n \ge 3 \end{cases}$$

(2) $\dfrac{2x^2+5}{(1-x)^2} + 7x = 7x + (2x^2+5)\dfrac{1}{(1-x)^2}$

$= 7x + (2x^2+5)\sum_{n=0}^{\infty}(n+1)x^n = 7x + 2x^2\sum_{n=0}^{\infty}(n+1)x^n + 5\sum_{n=0}^{\infty}(n+1)x^n$

$= 7x + \boxed{\sum_{n=0}^{\infty} 2(n+1)x^{n+2}} + \sum_{n=0}^{\infty} 5(n+1)x^n$

$m = n+2$

$n = m-2$

$\sum_{m=2}^{\infty} 2(m-1)x^m$

$$[x^n]\,\frac{2x^2+5}{(1-x)^2} = \begin{cases} 5 & \text{if } n=0 \\ 17 & \text{if } n=1 \\ 2(n-1)+5(n+1) & \text{if } n\ge 2 \end{cases}$$

## Substitution

Many of the things we do with functions still make sense for generating functions.

**Definition**

Let $C(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots = \sum_{n=0}^{\infty} c_n x^n$ be a generating function. Then we define

$$C(kx^m) = c_0 + c_1(kx^m) + c_2(kx^m)^2 + c_3(kx^m)^3 + \ldots = \sum_{n=0}^{\infty} c_n k^n x^{mn}$$

*Example.* The GF for nickels $N(x) = 1 + x^5 + x^{10} + x^{15} + \ldots = \sum_{n=0}^{\infty} x^{5n}$ is obtained from the GF $A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$ by substitution:

$$A(x^5) = \sum_{n=0}^{\infty} (x^5)^n = \sum_{n=0}^{\infty} x^{5n} = N(x)$$

We can then use the formula $A(x) = \frac{1}{1-x}$ to deduce $N(x) = A(x^5) = \frac{1}{1-x^5}$.
(we already found this formula for $N(x)$, but this is an easier way)

## Substituting in $-x$

(1) Using the GF $\frac{1}{1-x} = 1 + x + x^2 + \dots$ and substituting in $-x$ for $x$ gives

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} \\ &= 1 + (-x) + (-x)^2 + (-x)^3 + \dots \\ &= 1 - x + x^2 - x^3 + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \end{aligned}$$

*Problem.* Express $C(x) = 1 - 2x + 4x^2 - 8x^3 + 16x^4 \dots$ as a rational function.

$$A(x) = 1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x}$$

$$C(x) = A(-2x) = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - \dots = \frac{1}{1+2x}$$

$$\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 + \dots$$

## Substituting in $-x$

(2) Using the GF $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots$ and substituting in $-x$ for $x$ gives

$$\frac{1}{(1+x)^2} = \frac{1}{(1-(-x))^2}$$

$$\frac{1}{(1-\square)^2} = 1 + 2\square + 3\square^2 + 4\square^3 + \dots$$

$$= 1 + 2(-x) + 3(-x)^2 + 4(-x)^3 + \dots$$

$$= 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$= \sum_{n=0}^{\infty} (n+1)(-1)^n x^n$$

*Problem.* Express $D(x) = -x + 2x^2 - 3x^3 + 4x^4 + \dots$ as a rational function.

$$D(x) = -x\left(1 - 2x + 3x^2 - 4x^3 + \dots\right)$$

$$= -x\left(\frac{1}{(1+x)^2}\right) = \frac{-x}{(1+x)^2}$$

## Finding Coefficients

Using substitution and our two baisc GF's, we can now determine the coefficients for any GF that can be expressed as

$$\frac{p(x)}{ax+b} \qquad \text{or} \qquad \frac{p(x)}{(ax+b)^2}$$

where $p(x)$ is a polynomial

*Problem.* Find the coefficient of $x^k$ in the GF $C(x) = \dfrac{x^2}{2x+3}$

$$C(x) = x^2\left(\frac{1}{3+2x}\right) = \frac{1}{3}x^2\left(\frac{1}{1-(-2/3\,x)}\right)$$

$$= \frac{1}{3}x^2\sum_{n=0}^{\infty}\left(-2/3\,x\right)^n = \sum_{n=0}^{\infty}(-2)^n\left(\frac{1}{3}\right)^{n+1}x^{n+2}$$

$n+2=m \qquad n=m-2$

$$C(x) = \sum_{m=2}^{\infty}(-2)^{m-2}\left(\frac{1}{3}\right)^{m-1}x^m$$

$$[x^k]\,C(x) = \begin{cases} 0 & \text{if } k \le 1 \\ (-2)^{k-2}\left(\frac{1}{3}\right)^{k-1} & \text{if } k \ge 2 \end{cases}$$

*Problem.* Find the coefficient of $x^k$ in the GF $D(x) = \dfrac{x^2+1}{(5x+2)^2}$

$$D(x) = (x^2+1)\frac{1}{(2+5x)^2} = \frac{1}{4}(x^2+1)\frac{1}{\left(1-\left(-\frac{5}{2}x\right)\right)^2}$$

$$\frac{1}{(1-\square)^2} = \sum_{n=0}^{\infty}(n+1)\square^n$$

$$= \frac{1}{4}(x^2+1)\sum_{n=0}^{\infty}(n+1)\left(-\frac{5}{2}x\right)^n$$

$m = n+2$

$n = m-2$

$$= \underbrace{\sum_{n=0}^{\infty}\frac{1}{4}(n+1)\left(-\frac{5}{2}\right)^n x^{n+2}} + \sum_{n=0}^{\infty}\frac{1}{4}(n+1)\left(-\frac{5}{2}\right)^n x^n$$

$$= \sum_{m=2}^{\infty}\frac{1}{4}(m-1)\left(-\frac{5}{2}\right)^{m-2}x^m + \sum_{n=0}^{\infty}\frac{1}{4}(n+1)\left(-\frac{5}{2}\right)^n x^n$$

$$[x^k]\,D(x) = \begin{cases}\frac{1}{4}(k+1)\left(-\frac{5}{2}\right)^k & \text{if } k\le 1\\ \frac{1}{4}(k-1)\left(-\frac{5}{2}\right)^{k-2} + \frac{1}{4}(k+1)\left(-\frac{5}{2}\right)^k & \text{if } k\ge 2\end{cases}$$

# General Form

We have been working with the two basic GF's

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n$$

There is a more general generating function as follows:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

Combining this formula with substitution allows us to determine the coefficients of any rational polynomial of the form $\frac{p(x)}{(ax+b)^k}$.

The book uses a natural generalization of binomial coefficients, called the extended binomial theorem to get these coefficients. This is definitely worth having a look at. However for our class I will not insist you know this.

# Partial fractions

Based on the previous slide, we have the information to extract the coefficients from a rational GF of the form $\frac{P(x)}{(1-x)^k}$ or more generally (using substitution) a rational GF of the form $\frac{P(x)}{(a-bx)^k}$.

Combining this method and partial fractions gives us a recipe for extracting coefficients from every rational GF.

*Problem.* Find values for $A, B, C$ so that the expression below is true, then use this to determine $[x^n]D(x)$

$$D(x) = \frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

mult by $(x-3)(x-2)^2$

$1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$

$x=2 \quad 1 = -C \quad C = -1$

$x=3 \quad 1 = A$

$1 = (x-2)^2 + B(x-3)(x-2) - (x-3)$

$x=4 \quad 1 = 4 + 2B - 1 \quad B = -1$

$$D(x) = \frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$D(x) = \frac{1}{x-3} + \frac{-1}{x-2} + \frac{-1}{(x-2)^2}$$

$$= \left(-\frac{1}{3}\right)\frac{1}{1-\frac{x}{3}} + \left(\frac{1}{2}\right)\frac{1}{1-\frac{x}{2}} + \left(\frac{-1}{4}\right)\frac{1}{\left(1-\frac{x}{2}\right)^2}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)\left(\frac{1}{3}\right)^n x^n + \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^n x^n + \sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)(n+1)\left(\frac{1}{2}\right)^n x^n$$

$$[x^n]\, D(x) = \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right)^n + \frac{1}{2}\left(\frac{1}{2}\right)^n + \left(\frac{-1}{4}\right)(n+1)\left(\frac{1}{2}\right)^n$$

$$= -\frac{1}{3^{n+1}} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}(n+1)$$

## Method

Let $q(x)$ be a polynomial which can be factored as $q(x) = (x - r_1)^{d_1}(x - r_2)^{d_2} \ldots (x - r_k)^{d_k}$ then there exist constants so that

$$\frac{1}{q(x)} = \frac{A_{1,1}}{x - r_1} + \frac{A_{1,2}}{(x - r_1)^2} + \ldots \frac{A_{1,d_1}}{(x - r_1)^{d_1}} + \ldots\ldots + \ldots \frac{A_{k,d_k}}{(x - r_k)^{d_k}}$$

*Example.* Although we will not solve it, there exist constants $A, B, C, D, E$ so that the following expression is valid:

$$\frac{1}{(x-2)^2(x-3)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3}$$

### Note

*We can use this together with the formula*

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

*To do coefficient extraction whenever we have a rational function and we have factored the denominator.*