Midterm 1 Review Problems

1. List all strings of length 2 over the alphabet $\{X, Y, Z\}$.

2. Find all permutations over $\{P, Q, R\}$

3. Find all strings over the alphabet $\{a, b, c, d\}$ with length 2 and no repeated letter.

4. Find all strings over the alphabet $\{A, B, C\}$ with length 3 and no AB, BC, or CA substring.

5. Find all strings over the alphabet $\{0, 1, 2\}$ of length 4 that have 22 as a substring

6. Draw the graph $G = (V, E)$ given by

$$V = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad E = \{\{1, 3\}, \{1, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{5, 6\}\}.$$

7. Draw the directed graph $G = (V, E)$ given by

$$V = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad E = \{(1, 3), (1, 5), (2, 1), (3, 4), (5, 1), (5, 3), (6, 1), (6, 4)\}$$

8. Find all graphs with vertex set $\{1, 2, 3, 4\}$ that have $\{1, 2\}$ and $\{3, 4\}$ as edges, and have exactly two more edges in addition to these.

9. A restaurant has a special deal on a 3-course dinner consisting of an appetizer, an entree, and a dessert. If there are 3 possible appetizers, 4 entrees, and 2 desserts, how many ways can you choose a different dinner?

10. Consider the word TUMULTUOUS.
    (a) How many ways are there to arrange these letters?
    (b) How many arrangements have the substring SUM?
    (c) How many arrangements have the substring TOT?
    (d) How many have both SUM and TOT?

11. Determine the coefficient of $x^{10}y^{40}$ the expansion of $(2x - 3y)^{40}$.

12. Consider the word HOMEOMORPHISM.

(a) How many ways can you arrange these letters?

(b) If we omit all three copies of M, leaving HOEOORPHIS, how many ways are there to arrange these remaining letters?

(c) How many ways can you arrange the letters so that no two M's are consecutive? Hint: start by arranging the letters other than M using your answer to part (12b), then choose the positions in which to insert the three M's

13. There are n balls labelled $1, \ldots, n$ and k bins labelled $1, \ldots, k$.

    (a) How many ways are there to put the balls into the bins?

    (b) How many are there if each bin must receive at most one ball?

14. How many strings over the alphabet $\{A, B, C, D\}$ of length $2n$ have the property that D does not appear in any odd position?

15. How many permutations over $\{A, B, C, D, E, F, G, H\}$ start with A and do not end with H? How many permutations over this alphabet start with a letter other than A and end with a letter other than H?

16. A club with 16 people must select a leadership committee with a president, secretary, and treasurer. Assuming no person can have more than one such position, how many ways are there to select this committee?

17. At a summer camp, the campers are divided into 10 cabins with 8 campers in each cabin. How many ways are there to choose a set of 13 campers S with the property that 2 cabins have exactly 3 campers in S, 3 cabins have exactly 2 campers in S, and 1 cabin has exactly 1 camper in S?

18. How many ways are there to put 9 indistinguishable balls into four (distinguishable) containers numbered $1, 2, 3, 4$ with the following added constraint:

    (a) Every container has at least one ball.

    (b) Container 1 has an odd number of balls.

    (c) Exactly two containers have 0 balls.

19. A bookcase has four shelves and we wish to put 30 (distinguishable) books on these four shelves. For this problem we consider two arrangements of the books to be the same only when they have exactly the same books on each shelf in exactly the same order. Under these assumptions, how many arrangements of the books are possible?

20. A binary string has k *runs* if it can be divided into k (but not fewer) substrings so that each substring either has all entries 0 or all entries 1. For instance, the string 0011110100010111 has 8 runs since it divides into the substrings

$$00 \quad 1111 \quad 0 \quad 1 \quad 000 \quad 1 \quad 0 \quad 111$$

How many binary strings of length n have exactly k runs?

21. How many binary strings of length n have exactly one occurrence of the substring 10?

22. Consider the following equation where $x_1, \ldots, x_5$ are integer variables

$$x_1 + x_2 + x_3 + x_4 + x_5 = 30.$$

For each part below, determine how many solutions there are to the above equation satisfying the additional constraint.

(a) $x_i \geq 0$ for all $1 \leq i \leq 5$

(b) $x_i \geq -2$ for all $1 \leq i \leq 5$

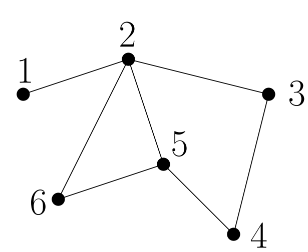
(c) $x_i \geq i$ for $1 \leq i \leq 5$

23. Consider the graph in the figure below.

(a) Find all subgraphs that are cycles.

(b) Find the subgraph induced by the vertices $\{1, 3, 4, 6\}$.

(c) Find all spanning subgraphs that are paths.

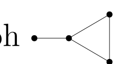


24. In this problem we fix a vertex v in the graph K_n and consider walks starting at v. Let w_k be the number of walks starting at v of length k (i.e. k edges) and let w_k° be the number of walks of length k both starting and ending at v.

    (a) Find a formula for w_k.

    (b) Suppose that W is a walk of length $k-1$ starting at v and ending at a vertex u. What condition on the vertex u permits us to extend W by one edge (and one new vertex) to get a walk of length k from v to itself?

    (c) Show that $w_k^\circ = w_{k-1} - w_{k-1}^\circ$

25. How many subgraphs of K_n are isomorphic to the graph  ?

26. How many subgraphs of K_n are isomorphic to $K_{a,b}$?

27. Up to isomorphism, find all graphs meeting each description.

    (a) 5 vertices and 3 edges

    (b) 5 vertices and 6 edges containing a cycle of length 4.

    (c) 6 vertices and 8 edges containing a cycle of length 6.

28. Let $n \geq 3$ and let P_n be a graph with vertex set $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ and the following edges:

    - $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\}$,
    - $\{y_1, y_2\}, \{y_2, y_3\}, \ldots, \{y_{n-1}, y_n\}, \{y_n, y_1\}$, and
    - $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}$

    (a) Draw the graph P_3

    (b) How many edges are in P_n?

    (c) How many paths of length 2 are in the graph P_n?

    (d) For $n \geq 5$ how many cycles of length 4 are in the graph P_n?

29. Let $n \geq 4$ and let R_n be a graph with vertex set $\{x_1, \ldots, x_n\} \cup \{y, y'\}$ and the following edges:

- $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\}$, and
- $\{x_i, y\}$ and $\{x_i, y'\}$ for every $1 \leq i \leq n$

(a) Draw the graph R_4.

(b) How many edges are in R_n?

(c) How many subgraphs of R_n are isomorphic to K_4^-?

(d) How many paths of length 3 are there in R_n?

30. Let G be a graph with m edges and the property that every vertex is adjacent to d other vertices. If G has no cycle of length 3, how many 4 vertex paths does G have? Hint: try choosing the middle edge.

31. For positive integers a, b we let $G_{a,b}$ be a graph with vertex set consisting of all ordered pairs of integers (x, y) with $1 \leq x \leq a$ and $1 \leq y \leq b$. Draw the graph $G_{3,4}$. Determine for all a, b how many cycles of length 6 there are in $G_{a,b}$.

32. Let $2 \leq b \leq a \leq n$. Define K_a^{-b} to be a graph obtained from K_a by choosing b vertices and deleting all edges between chosen vertices. How many subgraphs of K_n are isomorphic to K_a^{-b}?

33. A person is going to climb n steps and can take either 1 or 2 stairs at the same time. Let a_n denote the number of ways this can be done. Determine a_1, a_2, a_3, a_4 and then find a recurrence for a_n.

34. How many strings of length 10 over the alphabet $\{A, B, C, D, E, F\}$ have at least one copy of each letter? Use inclusion-exclusion to solve, and clearly state the conditions you will use and the formula.

35. How many ways can you rearrange the letters of $SUSPENSEFULL$ so that none of the substrings $LENS$, $LESS$, PEN, PUS appear? Use inclusion-exclusion to solve, and clearly state the conditions you will use and the formula.

36. In this exercise we will consider ordered rooted trees satisfying the following property

$$(*) \text{ every internal vertex has either 2 or 4 children}$$

(a) Find all ordered rooted trees satisfying $(*)$ with height at most 1

(b) Find all ordered rooted trees satisfying $(*)$ with height at most 2 and the added condition that the root vertex has 2 children.

(c) Let ℓ_n be the number of ordered rooted trees of height at most n satisfying $(*)$. Use part (a) to determine ℓ_0 and ℓ_1. Then find a recurrence for ℓ_n.

37. In this exercise we will consider rooted trees satisfying the following property

$$(\star) \text{ every internal vertex has 2 or 3 children}$$

(a) Find all rooted trees of height at most 2 satisfying $(\star)$

(b) Find all rooted trees with height 2 satisfying $(\star)$ with the added condition that the root vertex has 2 children.

(c) Let r_n denote the number of rooted trees of height at most n satisfying $(\star)$. Use part (a) to determine r_0 and r_1. Then find a recurrence for r_n.