MACM 201 - Discrete Mathematics

Graph Theory 3 - trees

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Tree

Definition

A multigraph $G = (V, E)$ with $|V| \geq 1$ is a **tree** if G is connected and does not contain a cycle. A **forest** is a multigraph that does not contain a cycle (every connected component of a forest is a tree).

We call a vertex of degree 1 a **leaf**.

Examples:

Leaves

We previously showed that any graph with all vertices of degree ≥ 2 has a cycle. Therefore, every tree with at least two vertices must contain a leaf. (Later we will see that all such trees have at least two leaves.)

Observation

If $T = (V, E)$ is a tree and $v \in V$ is a leaf, then $T - v$ is a tree (recall that $T - v$ is the subgraph of T obtained by deleting v).

Proof.

Note

This gives us a powerful tool for proving properties of trees by induction! If you want to prove something about a tree, try going by induction on the number of vertices and then (for the inductive step) delete a leaf and apply the inductive hypothesis.

Unique paths

If P is a path with at least two vertices, then P has two leaf vertices and we call these the **ends** of the path.

Theorem

If $T = (V, E)$ is a tree and $u, v \in V$ are distinct, there is a unique path in T with ends u, v.

Proof.

Rooted Trees

We previously defined a rooted tree as follows:

Definition

A **rooted tree** $T = (V, E)$ is a graph with a distinguished vertex called the **root** with the added property that every vertex $v \in V$ has an associated **level** so that the following conditions are satisfied:

(1) The level of every vertex is a nonnegative integer.

(2) The root is the unique vertex of level 0.

(3) For every edge $\{i, j\}$ the levels of i and j differ by exactly 1.

(4) For every non-root vertex $v \in V$ with level i there is exactly one vertex of level $i - 1$ adjacent to v.

Here is a simpler but equivalent definition

Definition

A **rooted tree** T is a tree with a distinguished vertex called the **root**.

From this definition, you can easily assign levels to return to the previous setting. Let the root vertex be called u and give it level 0. Then for every non-root vertex v define the level of v to be the number of edges in the unique path of T with ends u, v.

Induction on rooted trees

There are two common ways of proving things about rooted trees by induction.

(1) If the root has children $v_1, \ldots, v_k$, apply induction to the subtrees at $v_1, \ldots, v_k$

(2) Choose a leaf vertex v of maximum level and delete v together with its siblings.

Figure

Back to unrooted trees

Lemma

If $T = (V, E)$ is a tree, then $|V| = |E| + 1$.

Proof.

Lemma

If $G = (V, E)$ is a forest with k components, then $|V| = |E| + k$

Proof.

Lemma

If $G = (V, E)$ *satisfies* $|V| = |E| + 1$*, then* G *must have a vertex of degree 0 or at least two with degree 1.*

Proof.

Lemma

Every tree $T = (V, E)$ *with* $|V| \geq 2$ *has at least two leaves.*

Proof.

Theorem

Let $G = (V, E)$ be a graph. If G satisfies any two of the conditions below, then it satisfies the third.

(1) G *is connected*

(2) G *has no cycle*

(3) $|V| = |E| + 1$.

Proof.