MACM 201 - Discrete Mathematics

Graph Theory I - multigraphs, degree, connectivity

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A more general kind of graph

Definition

A **multigraph** $G = (V, E)$ consists of a set V of vertices, a set E of edges, and a relation called **incidence** so that every edge is incident with either one or two vertices. If $v \in V$ is incident with $e \in E$ we may write this as $v \sim e$.

An edge e that is incident with just one vertex, say v, is called a **loop** and we think of e as having two "ends" that are both at the vertex v. If $f, f' \in E$ are distinct edges that are both incident with the same vertices, we call them **parallel**

Example: $G = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$, $E = \{a, b, c, d, e, f, g, h\}$ and $a \sim 1$, $a \sim 2$, $b \sim 1$, $b \sim 2$, $c \sim 2$, $c \sim 3$, $d \sim 3$, $e \sim 3$, $e \sim 4$, $f \sim 4$, $f \sim 1$, $g \sim 4$, $g \sim 5$, $h \sim 5$, $h \sim 1$.

Graphs are Multigraphs

Every graph may be viewed as a multigraph where the edge $\{u, v\}$ is incident with u and v. Graphs are just multigraphs that have no loops and no parallel edges.

Note

Many concepts that we defined for graph also make sense for multigraphs. Notably:

- *subgraph (induced, spanning)*
- *path*
- *walk*

However we will (slightly) expand our concept of cycle.

Definition

We define a multigraph $G = (V, E)$ to be a **cycle** if there is an ordering of the vertices $v_1, v_2, \ldots, v_n$ and an ordering of the edges $e_1, \ldots, e_n$ so that for $1 \leq i \leq n-1$ the edge e_i has ends v_i and v_{i+1} and the edge e_n has ends v_n and v_1. Note: we permit $n = 1, 2, \ldots$

Examples:

Examples

Note

Going forward we will be interested in structural properties of multigraphs, not in questions of isomorphism and counting.

Degree

Definition

If $G = (V, E)$ is a multigraph and $v \in V$, the **degree** of v, denoted $\deg(v)$ is the number of non-loop edges incident to v plus twice the number of loop edges incident with v.

Example

Theorem

Every multigraph $G = (V, E)$ *satisfies* $\sum_{v \in V} \deg(v) = 2|E|$

Connectivity

Definition

A multigraph $G = (V, E)$ is **connected** if for every $u, v \in V$ there is a walk from u to v.

Examples

Connected components

Let $G = (V, E)$ be a multigraph and define a relation on V by the rule $u \to v$ if there is a walk from u to v. The following properties of $\to$ are straightforward to verify.

(1) *Reflexive* $v \to v$ for every $v \in V$
(2) *Symmetric* If $u \to v$, then $v \to u$ (this holds for all $u, v \in V$)
(3) *Transitive* If $u \to v$ and $v \to w$, then $u \to w$ (this holds for all $u, v, w \in V$)

Recall. A relation satisfying the reflexive, symmetric, and transitive properties is called an *equivalence* relation.

Since $\to$ is an equivalence relation on V, there is a partition of V, say $\{V_1, V_2, \ldots, V_k\}$ so that $u, v \in V$ satisfy $u \to v$ if and only u and v are in the same block of the partition (i.e. $u, v \in V_i$ for some $1 \leq i \leq k$). For $1 \leq i \leq k$ let G_i be the subgraph of G induced by V_i. We call $G_1, \ldots, G_k$ the **connected components** of G.

Example