MACM 201 Homework 6 (Quiz Oct. 23)

1. Define a polynomial $P(x)$ so that $[x^k]P(x)$ is the number of binary strings of length 100 with k 1's.

2. For each equation, express the number of solutions as the coefficient of a polynomial.

   (a) $b_1 + b_2 + b_3 = 12$ where $0 \leq b_i \leq 6$ holds for $i = 1, 2, 3$

   (b) $b_1 + b_2 + b_3 = 14$ where $b_1, b_2, b_3 \geq 0$ and b_1, b_2 are odd while b_3 is even.

   (c) $b_1 + b_2 + b_3 + b_4 = 17$ where $2 \leq b_1 \leq 5$, $b_2, b_3, b_4 \geq 0$, b_2 is even, b_3 is odd, and b_4 is a multiple of 3.

3. Define a generating function $R(x)$ with the property that $[x^n]R(x)$ is the number of ways there are to use pennies, dimes, and quarters to add up to a total of n cents.

4. In this problem we are interested in counting the number of solutions to

$$(\star) \qquad a_1 + a_2 + a_3 + a_4 = n \qquad \text{where } a_1, a_2, a_3, a_4 \geq 0$$

   (a) Define a generating function $P_1(x)$ with the property that $[x^n]P_1(x)$ is the number of solutions to $(\star)$.

   (b) Define a generating function $P_2(x)$ with the property that $[x^n]P_2(x)$ is the number of solutions to $(\star)$ satisfying the additional condition that a_1, a_2 are even but a_3, a_4 are odd.

   (c) Define a generating function $P_2(x)$ with the property that $[x^n]P_2(x)$ is the number of solutions to $(\star)$ satisfying the additional conditions that $10 \leq a_1 \leq 100$, a_2 is even, and a_3 is a multiple of 3.

5. Let $\mathcal{C}_n$ denote the set of strings of length n over the alphabet $\{0, 1, 2, 3\}$.

   (a) Define a polynomial $Q_1(x)$ with the property that $[x^k]Q_1(x)$ is the number of strings in $\mathcal{C}_n$ that have exactly k nonzero symbols.

   (b) Define a polynomial $Q_2(x)$ with the property that $[x^k]Q_2(x)$ is the number of strings in $\mathcal{C}_n$ that have exactly k letters that are 2 or 3 (so the remaining $n - k$ will be 0 or 1).

(c) Define the *weight* of a string $s \in \mathcal{C}_n$ to be $s_1 + s_2 + \ldots + s_n$ (for example, the string $s = 32102$ has weight $3 + 2 + 1 + 0 + 2 = 8$). Define a polynomial $Q_3(x)$ with the property that $[x^k]Q_3(x)$ is the number of strings in $\mathcal{C}_n$ with weight k.

(d) Define the *funny weight* of a string $s \in \mathcal{C}_n$ to be 7 times the number of 3's in s plus 4 times the number of 2's in s (so each 0 or 1 contributes nothing). Define a polynomial $Q_4(x)$ with the property that $[x^k]Q_4(x)$ is the number of strings in $\mathcal{C}_n$ with funny weight equal to k.