MACM 201 - Discrete Mathematics

10. Recurrence relations III

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Solving second order homogeneous linear recurrences

Suppose the sequence $(x_0, x_1, \ldots)$ satisfies the recurrence

$$ax_n + bx_{n-1} + cx_{n-2} \qquad \text{for all } n \geq 2.$$

Then we have an associated characteristic equation

$$ar^2 + br + c = 0.$$

and we know exactly how to proceed when this quadratic equation two distinct roots. Namely, if r_1 and r_2 satisfy the above equation and $r_1 \neq r_2$ then the general solution to our recurrence is

$$x_n = Cr_1^n + Dr_2^n$$

where C and D are constants.

But this is only part of the story... what if our quadratic equation has a repeated root, or no real number solutions?

Repeated root

Let's construct a simple example to study the case when there is a repeated root. Consider the polynomial

$$(r-3)^2 = r^2 - 6r + 9$$

The above polynomial has 3 as a repeated root since it factors into two $(r-3)$ terms. Let's see what happens when we make a recurrence based on this equation. Suppose that $(x_0, x_1, \dots)$ is an infinite sequence satisfying the recurrence

$$x_n - 6x_{n-1} + 9x_{n-2} = 0$$

We already know that $x_n = 3^n$ is a solution to this equation, and more generally $C3^n$ will be a solution for any C. However there are still more solutions. We claim that $x_n = n3^n$ is also a solution. Let's check to see that it works by plugging this in to our recurrence.

$$\begin{aligned} n3^n - 6(n-1)3^{n-1} + 9(n-2)3^{n-2} =& n\left(3^n - 6 \cdot 3^{n-1} + 9 \cdot 3^{n-2}\right) \\ &+ 6 \cdot 3^{n-1} - 18 \cdot 3^{n-2} \end{aligned}$$

The right hand side evaluates to 0, so we have indeed found another solution.

Theorem

Let a, b, c be fixed constants with $a \neq 0$ and consider the recurrence

$$ax_n + bx_{n-1} + cx_{n-2} = 0. \tag{1}$$

If the characteristic equation,

$$ar^2 + br + c$$

has r as a repeated root, then every sequence satisfying this recurrence has the form

$$x_n = Cr^n + Dnr^n \tag{2}$$

where C and D are fixed constants. Equation (4) is the **general solution** *to the recurrence.*

Problem Solve the following recurrence for a sequence $(x_0, x_1, \ldots)$

$$x_n - 6x_{n-1} + 9x_{n-2} = 0 \qquad \text{and} \qquad x_0 = 2, x_1 = 4$$

The quadratic formula

When solving quadratic equations

$$ar^2 + br + c = 0$$

we have the quadratic formula to tell us the answers

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The term $b^2 - 4ac$ appearing under the square root of our quadratic equation is called the **discriminant** because it determines the nature of the solutions.

(+) If $b^2 - 4ac > 0$ then there are two distinct real solutions r_1, r_2.

(0) If $b^2 - 4ac = 0$ there is just one solution r and it is a repeated root.

(-) If $b^2 - 4ac < 0$ there are no real number solutions.

How can we solve our recurrences when there are no real number solutions to our characteristic equation?

An imaginary number

We don't need to make complicated quadratics to arrange for no real number solutions. For instance, the equation below obviously has no real numbers satisfying it.

$$x^2 = -1$$

In the 16th century the Italian mathematician Gerolamo Cardano was studying polynomial equations and decided to introduce a special new quantity, denoted i for imaginary, that would satisfy

$$i^2 = -1$$

This quantity was not treated as a full-fledged number, but more as a bookkeeping device. It allowed expressions such as

$$3 + 4i$$

and these proved to be extremely useful in understanding equations. For starters, expressions of the above type can handle describe any outcome from the quadratic equation

$$\frac{6 \pm \sqrt{-81}}{3} = \frac{6 \pm 9\sqrt{-1}}{3} = 2 + 3\sqrt{-1} = 2 + 3i$$

Over time it was revealed that these expressions of the form

$$a + bi$$

Form an extremely important type of number.

Complex numbers

Definition

We will continue to use the symbol i for a special number satisfying $i^2 = -1$. A **complex number** is an expression of the form

$$a + bi$$

where both a and b are real numbers.

Definition

The sum of two complex numbers is given by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

When multiplying complex numbers we will use the fact that $i^2 = -1$ to keep our expression in the same form.

Definition

The product of two complex numbers is given by

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Theorem

Let a, b, c be fixed constants with $a \neq 0$ and consider the recurrence

$$ax_n + bx_{n-1} + cx_{n-2} = 0. \tag{3}$$

If the characteristic equation,

$$ar^2 + br + c$$

has no real roots (because $b^2 - 4ac < 0$, then it has two complex roots ρ_1, ρ_2. And every complex valued sequence satisfying the recurrence has the form

$$x_n = \alpha\rho_1^n + \beta n\rho_2^n \tag{4}$$

where α and β are fixed complex constants. Equation (4) is the **general solution** *to the recurrence.*

Key point: even if we only care about real number solutions, we can use the above formula and work in the setting of complex numbers to find them.

Problem. Solve the following recurrence

$$\begin{cases} a_0 & = & 1 \\ a_1 & = & 2 \\ a_n & = & 2a_{n-1} - 2a_{n-2} \quad n \geq 2 \end{cases}$$