MACM 201 Homework 6 - *Solutions*

1. Define a polynomial $P(x)$ so that $[x^k]P(x)$ is the number of binary strings of length 100 with k 1's.

   Solution. Define the polynomial $P(x)$ as follows:

   $$P(x) = (1 + x)^{100}$$

   Now $P(x)$ is a product of 100 factors, and the coefficient of x^k in $P(x)$ is the number of ways of choosing either a term of 1 or x from each factor in such a way that x is chosen exactly k times. These choices are in correspondence with binary strings of length 100 with exactly k ones. Therefore, $[x^k]P(x)$ is the number of binary strings of length 100 with exactly k 1's

2. For each equation, express the number of solutions as the coefficient of a polynomial.

   (a) $b_1 + b_2 + b_3 = 12$ where $0 \le b_i \le 6$ holds for $i = 1, 2, 3$

   (b) $b_1 + b_2 + b_3 = 14$ where $b_1, b_2, b_3 \ge 0$ and b_1, b_2 are odd while b_3 is even.

   (c) $b_1 + b_2 + b_3 + b_4 = 17$ where $2 \le b_1 \le 5$, $b_2, b_3, b_4 \ge 0$, b_2 is even, b_3 is odd, and b_4 is a multiple of 3.

   Solution. In each case we describe a polynomial made as a product of factors, where the first factor corresponds to b_1, the second to b_2, and so on. In each case the possible values for b_i will be encoded using exponents (for example if b_2 is permitted to have value 3 then we have a term of x^3 in the second factor).

   (a) Here each b_i is permitted to have a value between 0 and 6 so the following coefficient is a solution

   $$[x^{12}]\Big((1 + x + \ldots + x^6)^3\Big)$$

   (b) Since b_1, b_2 are odd but b_3 is even the answer is

   $$[x^{14}]\Big((x + x^3 + x^5 + \ldots + x^{13})^2(1 + x^2 + x^4 + \ldots + x^{14})\Big)$$

   (c) Here our restrictions are encoded by the following expression.

   $$[x^{17}]\Big((x^2+x^3+x^4+x^5)(1+x^2+x^4+\ldots+x^{16})(x+x^3+x^5+\ldots+x^{17})(1+x^3+x^6+\ldots+x^{15})\Big)$$

3. Define a generating function $R(x)$ with the property that $[x^n]R(x)$ is the number of ways there are to use pennies, dimes, and quarters to add up to a total of n cents.

   Solution. We will describe our generating function $R(x)$ as a product of three terms, where the first term corresponds to pennies, the second to dimes, and the third to quarters. Since we can choose any number of pennies and each contributes 1 to the total value, the first term will be $(1 + x + x^2 + \ldots)$ (as usual, we are using exponents to keep track of value). We are also permitted to use any number of dimes, but each dime contributes 10 to the total value. Therefore, the term associated with dimes will be $(1 + x^{10} + x^{20} + \ldots)$. Similarly, each quarter contributes 25, so the term associated with quarters will be $(1 + x^{25} + x^{50} + \ldots)$. This gives the following generating function $R(x)$ as our solution

   $$R(x) = (1 + x + x^2 + \ldots)(1 + x^{10} + x^{20} + \ldots)(1 + x^{25} + x^{50} + \ldots)$$

   As you should check, the coefficient of x^n in $R(x)$ corresponds precisely to the number of ways of choosing pennies, dimes, and quarters to sum up to n cents.

4. In this problem we are interested in counting the number of solutions to

   $$(\star) \qquad a_1 + a_2 + a_3 + a_4 = n \qquad \text{where } a_1, a_2, a_3, a_4 \geq 0$$

   (a) Define a generating function $P_1(x)$ with the property that $[x^n]P_1(x)$ is the number of solutions to $(\star)$.

   (b) Define a generating function $P_2(x)$ with the property that $[x^n]P_2(x)$ is the number of solutions to $(\star)$ satisfying the additional condition that a_1, a_2 are even but a_3, a_4 are odd.

   (c) Define a generating function $P_3(x)$ with the property that $[x^n]P_3(x)$ is the number of solutions to $(\star)$ satisfying the additional conditions that $10 \leq a_1 \leq 100$, a_2 is even, and a_3 is a multiple of 3.

   Solution. In each case we describe a polynomial made as a product of four factors, where the first factor corresponds to a_1, the second to a_2, and so on. In each case the possible values for a_i will be encoded using exponents (for example if a_2 is permitted to have value 3 then we have a term of x^3 in the second factor).

(a) Here each a_i can have any value so the following generating function has the correct coefficients.

$$P_1(x) = (1 + x + x^2 + x^3 + \dots)^4$$

(b) adding the restriction that a_1, a_2 are even but a_3, a_4 are odd gives us the following polynomial.

$$P_2(x) = (1 + x^2 + x^4 + \dots)^2(x + x^3 + x^5 + \dots)^2$$

(c) Now our restrictions are encoded by

$$P_3(x) = (x^{10}+x^{11}+x^{12}+\dots+x^{100})(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)(1+x+x^2+\dots)$$

5. Let $\mathcal{C}_n$ denote the set of strings of length n over the alphabet $\{0, 1, 2, 3\}$.

(a) Define a polynomial $Q_1(x)$ with the property that $[x^k]Q_1(x)$ is the number of strings in $\mathcal{C}_n$ that have exactly k nonzero symbols.

Solution. Declare a symbol of 1, 2, or 3 as having a *value* of 1 since each of these contribute 1 nonzero symbol to our string, and declare a symbol of 0 to have *value* 0, since this contributes nothing to the count of nonzero symbols. With this terminology, we want our polynomial $Q_1(x)$ to have the property that the coefficient of x^k is the number of strings of length n that have a total value of k. For each of the n positions in the string, we have one way to put down a symbol with value 0 and three ways to put down a symbol with value 1. Therefore, we will associate this position of the string with a factor of the form $(1 + 3x)$. As usual here, the values are appearing in the exponent so that polynomial multiplication has the desired effect. This gives us a polynomial of

$$Q_1(x) = (1 + 3x)^n$$

Observe that each factor in Q_1 corresponds to a position in the string, and the coefficient of 1 for x^0 and 3 for x^1 correspond to the 1 way we can choose a symbol of value 0 and the 3 ways to choose a symbol of value 1. Now the coefficient of x^k will be the number of strings with exactly k nonzero symbols, as desired.

(b) Define a polynomial $Q_2(x)$ with the property that $[x^k]Q_2(x)$ is the number of strings in $\mathcal{C}_n$ that have exactly k letters that are 2 or 3 (so the remaining $n-k$ will be 0 or 1).

Solution. Declare a symbol of 0 or 1 to have *value* 0 and a symbol of 2 or 3 to have *value* 1. With this terminology, we want our polynomial $Q_2(x)$ to have the property that the coefficient of x^k is the number of strings of length n that have a total value of k. For each of the n positions in the string, we have two ways to put down a symbol with value 0 and two ways to put down a symbol with value 1. Therefore, we will associate this position of the string with a factor of the form $(2+2x)$. This gives us a polynomial of

$$Q_2(x) = (2+2x)^n$$

Observe that the coefficient of x^k will be the number of strings with exactly k symbols that are 2 or 3, as desired.

(c) Define the *weight* of a string $s \in \mathcal{C}_n$ to be $s_1 + s_2 + \ldots + s_n$ (for example, the string $s = 32102$ has weight $3+2+1+0+2 = 8$). Define a polynomial $Q_3(x)$ with the property that $[x^k]Q_3(x)$ is the number of strings in $\mathcal{C}_n$ with weight k.

Solution. Here the key parameter (that we used value for above) is just the weight. A symbol of i has a value of i. So for each position in the string, there is 1 way to put down a symbol with weight 0, 1 way for weight 1, 1 way for weight 2, and 1 way for weight 3. Accordingly, we associate each position of the string with a factor of the form $(1+x.+x^2+x^3)$. This gives us a polynomial of

$$Q_3(x) = (1+x+x^2+x^3)^n$$

Now the coefficient of x^k is the number of strings of length n with weight k.

(d) Define the *funny weight* of a string $s \in \mathcal{C}_n$ to be 7 times the number of 3's in s plus 4 times the number of 2's in s (so each 0 or 1 contributes nothing). Define a polynomial $Q_4(x)$ with the property that $[x^k]Q_4(x)$ is the number of strings in $\mathcal{C}_n$ with funny weight equal to k.

Solution. The key parameter is now the funny weight. For each position in the string, there are 2 ways to choose a symbol with funny weight 0, 1 way to choose.a symbol with funny weight 4, and 1 way to choose a symbol of funny weight 7. Accordingly, each position will be associated with a factor of the form $(2. + x^4 + x^7)$. This gives us the polynomial

$$Q_4(x) = (2 + x^4 + x^7)^n$$

Now the coefficient of x^k will give the number of strings of length n with funny weight k, as desired.