MATH 308 D200, Fall 2019

18. Duality of non-canonical LP problems

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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An upper bound on the optimal value of a non-canonical max LP problem

Some main constraints (I) are $\leq$-inequalities, the other constraints (E) are equalities.
Some variables (C) are constrained to be ≥ 0, the other variables (U) are unconstrained.

$$I \cup E = [1, \ldots, m], \quad I \cap E = \emptyset, \qquad C \cup U = [1, \ldots, n], \quad C \cap U = \emptyset.$$

Non-canonical primal (P): maximize $\sum_{j=1}^{n} c_j x_j$

$$\begin{aligned} \text{subject to:} \quad & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i && (\text{for } i \in I) \\ & \sum_{j=1}^{n} a_{ij} x_j = b_i && (\text{for } i \in E) \\ & x_j \geqslant 0 && (\text{for } j \in C) \\ & x_j \text{ unconst.} && (\text{for } j \in U) \end{aligned}$$

Multiply constraint i by y_i, and add them together.

(1) $\sum_{j=1}^{n} a_{ij} x_j \cdot y_i \leqslant b_i \cdot y_i, i \in I$ $\qquad$ $\sum_{j=1}^{n} a_{ij} x_j = b_i \ / \ y_i, i \in E$

(2) $\sum_{j=1}^{n} \Big(\sum_{i \in I} a_{ij} y_i \Big) x_j \leqslant \sum_{i \in I} b_i y_i$ $\qquad$ $\sum_{j=1}^{n} \Big(\sum_{i \in E} a_{ij} y_i \Big) x_j = \sum_{i \in E} b_i y_i$

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Non-canonical primal (P): maximize $\sum_{j=1}^{n} c_j x_j$

$$\begin{aligned}
\text{subject to: } & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i && (\text{for } i \in I) && (\text{Dual } C\text{-Variable: } y_i) \\
& \sum_{j=1}^{n} a_{ij} x_j = b_i && (\text{for } i \in E) && (\text{Dual } U\text{-Variable: } y_i) \\
& x_j \geqslant 0 && (\text{for } j \in C) && \\
& x_j \text{ unconst.} && (\text{for } j \in U) &&
\end{aligned}$$

Multiply constraint i by y_i, and add them together.

(1) $$\sum_{j=1}^{n} a_{ij} x_j \cdot y_i \leqslant b_i \cdot y_i, i \in I \qquad \sum_{j=1}^{n} a_{ij} x_j = b_i \ / \ y_i, i \in E$$

(2) $$\sum_{j=1}^{n} \Big(\sum_{i \in I} a_{ij} y_i\Big) x_j \leqslant \sum_{i \in I} b_i y_i \qquad \sum_{j=1}^{n} \Big(\sum_{i \in E} a_{ij} y_i\Big) x_j = \sum_{i \in E} b_i y_i$$

An upper bound on the optimal value of a non-canonical max LP problem

(3) $$f = \sum_{j=1}^{n} \boxed{c_j} x_j \leqslant \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j \leqslant \sum_{i=1}^{m} b_i y_i = g$$

Non-canonical dual LP: minimize $\sum_{i=1}^{m} b_i y_i$

subject to: $\sum_{i=1}^{m} a_{ij} y_i \geqslant c_j \quad (j \in C)$

$\sum_{i=1}^{m} a_{ij} y_i = c_j \quad (j \in U)$

$y_i \geqslant 0 \quad (i \in I)$

Example

Write the dual LP for the following non-canonical LP:

$$
\begin{aligned}
\text{maximize} \quad & x_1 - 2x_2 + x_3 \\
\text{subject to:} \quad & 2x_1 + 3x_2 - x_3 \leqslant 5 \\
& x_1 - x_2 + 3x_3 = 4 \\
& x_1 \leqslant 15 \\
& x_2 - 2x_3 \leqslant 1 \\
& x_3 \geqslant 0
\end{aligned}
$$

Definition (Dual Non-canonical Tableau)

A *dual non-canonical tableau* is a non-canonical tableau of the form

$$
\begin{array}{c|cccccc|c|l}
 & x_1 & \dots & x_l & x_{l+1} & \dots & x_n & -1 & \\
\hline
y_1 & a_{1,1} & \dots & a_{1,l} & a_{1,l+1} & \dots & a_{1,n} & b_1 & = -0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
y_k & a_{k,1} & \dots & a_{k,l} & a_{k,l+1} & \dots & a_{k,n} & b_k & = -0 \\
y_{k+1} & a_{k+1,1} & \dots & a_{k+1,l} & a_{k+1,l+1} & \dots & a_{k+1,n} & b_{k+1} & = -t_{k+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
y_m & a_{m,1} & \dots & a_{m,l} & a_{m,l+1} & \dots & a_{m,n} & b_m & = -t_m \\
\hline
-1 & c_1 & \dots & c_l & c_{l+1} & \dots & c_n & d & = f \\
\hline
 & = 0 & \dots & = 0 & = s_{l+1} & \dots & = s_n & = g &
\end{array}
$$

Solve the dual non-canonical LP problems.

$$
\begin{array}{c|ccc|c|l}
 & x_1 & x_2 & x_3 & -1 & \\
\hline
y_1 & 1 & -1 & 2 & 1 & = -0 \\
y_2 & 2 & 2 & 0 & 1 & = -t_2 \\
y_3 & 0 & 1 & 2 & -1 & = -t_3 \\
\hline
-1 & 1 & -1 & 1 & 0 & = f \\
\hline
 & = 0 & = 0 & = s_3 & = g &
\end{array}
$$

Solve the dual non-canonical LP problems.

$$
\begin{array}{c|ccc|c|l}
 & x_1 & x_2 & x_3 & -1 & \\
\hline
y_1 & 0 & -1 & -1 & -1 & = -0 \\
y_2 & -1 & -3 & 4 & 0 & = -t_2 \\
y_3 & -1 & 2 & -3 & 0 & = -t_3 \\
\hline
-1 & -1 & 0 & 0 & 0 & = f \\
\hline
 & = 0 & = s_2 & = s_3 & = g &
\end{array}
$$