MACM 201 - Discrete Mathematics

2. Basic counting principles

Department of Mathematics

Simon Fraser University

Definition

Rule of Sum: If there are n ways to perform an action and m ways to perform another type of action, the total number of ways to choose one of these two actions is $n + m$.

Example. Moonbucks sells coffee and tea. There are 10 types of coffee beverages and 8 types of tea beverages to choose from. How many ways are there to choose a single drink?

Definition

Rule of Product: If there are n ways to perform an action and m ways to perform another type of action, there are nm ways to perform one action of each type.

Example. Suppose that in the previous example you plan to order one coffee and one tea each day from Moonbucks. How many days would it take to try all possible coffee & tea combinations?

Counting strings

Fact

If $\mathcal{A}$ is an alphabet of size k, the number of strings of length n over $\mathcal{A}$ is k^n.

Proof.

Definition

Strings over the alphabet $\{0, 1\}$ are called *binary* strings.

Problem. How many binary strings of length 10 are there?

Fact

If X and Y are finite sets, the number of functions from X to Y is equal to $|Y|^{|X|}$.

Proof.

Problem. How many functions are there from $\{1, 2, 3, 4\}$ to $\{A, B, C, D, E\}$?

Counting permutations

Fact

The number of permutations of a set of n *objects is* $n! = n(n-1)(n-2)\cdots 2\cdot 1$.

Proof.

Reminder

If S is a set of size n, the number of k element subsets of S is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

(we will see another proof of this later)

Definition

If $k_1, \ldots, k_r$ are nonnegative integers and $k_1 + k_2 + \ldots + k_r = n$, we define

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1!\,k_2!\,\ldots, k_r!}$$

Note

If $n = h + k$ *then* $\binom{n}{k} = \binom{n}{h} = \frac{n!}{h!\,k!} = \binom{n}{h, k}$.

Counting types of strings

Theorem

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_r\}$ be an alphabet and let $k_1, \ldots, k_r$ be nonnegative integers with $k_1 + k_2 + \ldots + k_r = n$. The number of strings of length n over $\mathcal{A}$ with exactly k_i copies of the letter a_i is equal to

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1!\, k_2! \cdots k_r!}$$

Note

If $k_1 = k_2 = \cdots = k_r = 1$ this is the same as permutations of n objects.

Proof.

Proof (continued)

Example. How many ways can the letters in the following word be arranged?

commemorative

Corollary

The number of binary strings of length n with exactly k 1's (and thus $n-k$ 0's) is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Choosing subsets

Fact

If S is a set of size n, the number of subsets of S with size k is equal to $\binom{n}{k}$.

Proof.