MACM 201 Homework 7 - *Solutions*

1. Define the generating functions $B(x) = \sum_{n=0}^{\infty} 2^n x^n$ and $F(x) = \sum_{n=0}^{\infty} f_n x^n$ where f_n is the Fibonacci sequence determined by the recurrence relation

$$f_0 = 0 \qquad \text{and} \qquad f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2$$

Find the coefficients of the first four terms (constant up to x^3) of each GF

(a) $F(x) + B(x)$

(b) $F(x) \times B(x)$

(c) $F(x) \times F(x) \times F(x)$

Solution. to compute the first four terms for each part, we only need the first four terms of each GF: $B(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots$ and $F(x) = 0 + x + x^2 + 2x^3 + \ldots$.

(a) $F(x) + B(x) = 1 + 3x + 5x^2 + 10x^3 + \ldots$

(b) $F(x) \times B(x) = (0+x+x^2+2x^3+\ldots)(1+2x+4x^2+8x^3+\ldots) = (0+x+3x^2+8x^3+\ldots)$

(c) $F(x) \times F(x) \times F(x) = (0 + x + x^2 + 2x^3 + \ldots)^3 = (0 + 0x + 0x^2 + x^3 + \ldots)$

2. For each infinite sequence, express the associated GF in rational form.

(a) $0, 0, 1, 1, 1, 1, \ldots$

(b) $1, -1, 1, -1, 1, -1, \ldots$

(c) $0, 0, 0, a, -a, a, -a, a, \ldots$

(d) $a, 0, a, 0, a, 0, \ldots$

(e) $1, -2, 3, -4, 5, -6, \ldots$

(f) $0, 0, 0, 1, 2, 3, 4, \ldots$

(g) $0, 0, 0, 3, -6, 9, -12, 15, -18, \ldots$

(h) $0, 3, 2, 5, 4, 7, \ldots$ (Hint: this is $1-1, 2+1, 3-1, 4+1, 5-1, 6+1, \ldots$)

Solution. In each case we take the generating function associated with the given sequence and then call upon our two basic generating functions (listed below) to construct a rational representation.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$
$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$$

(a) $(x^2 + x^3 + x^4 + \dots) = x^2(1 + x + x^2 + \dots) = x^2\left(\frac{1}{1-x}\right) = \frac{x^2}{1-x}$

(b) $(1 - x + x^2 - x^3 + \dots) = (1 + (-x) + (-x)^2 + \dots) = \left(\frac{1}{1-(-x)}\right) = \frac{1}{1+x}$

(c) $(ax^3 - ax^4 + ax^5 - ax^6 + \dots) = ax^3\,(1 + (-x) + (-x)^2 + \dots) = \left(\frac{ax^3}{1-(-x)}\right) = \frac{ax^3}{1+x}$

(d) $(a + ax^2 + ax^4 + \dots) = a(1 + x^2 + x^4 + \dots) = \frac{a}{1-x^2}$

(e) $(1-2x+3x^2-4x^3+\dots) = (1+2(-x)+3(-x)^2+4(-x)^3+\dots) = \frac{1}{(1-(-x))^2} = \frac{1}{(1+x)^2}$

(f) $(x^3 + 2x^4 + 3x^5 + 4x^6 + \dots) = x^3(1 + 2x + 3x^2 + 4x^3 + \dots) = \frac{x^3}{(1-x)^2}$

(g) $(3x^3 - 6x^4 + 9x^5 - 12x^6 + \dots) = 3x^3(1 + 2(-x) + 3(-x)^2 + \dots) = \frac{3x^3}{(1-(-x))^2} = \frac{3x^3}{(1+x)^2}$

(h) We can write the generating function $\big((1-1) + (2+1)x + (3-1)x^2 \dots\big)$ as the sum $(1 + 2x + 3x^2 \dots) + (-1)(1 + (-x) + (-x)^2 + (-x)^3 + \dots) = \frac{1}{(1-x)^2} + \frac{-1}{1+x}$

3. For each generating function below, find a formula for the coefficient of x^n.

(a) $(1+2x)^3$

(b) $\frac{3x^2}{1-x}$

(c) $\frac{2x}{1-x} + \frac{3x^2}{(1-x)^2}$

(d) $\frac{x^3+1}{2-2x}$

(e) $\frac{2x}{(3+6x)^2} + 7$

Solution. As in the previous problem, we call upon our two basic GF's when useful.

(a) $(1+2x)^3 = 1 + 6x + 12x^2 + 8x^3$

(b) $\dfrac{3x^2}{1-x} = 3x^2 \sum_{m=0}^{\infty} x^m = \sum_{m=0}^{\infty} 3x^{m+2} = \sum_{n=2}^{\infty} 3x^n$ so our coefficients are given by

$$[x^n]\left(\frac{3x^2}{1-x}\right) = \begin{cases} 0 & \text{if } n \le 1 \\ 3 & \text{if } n \ge 2 \end{cases}$$

(c) We have

$$\begin{aligned}\frac{2x}{1-x}+\frac{3x^2}{(1-x)^2} &= 2x\sum_{m=0}^{\infty}x^m+3x^2\sum_{m=0}^{\infty}(m+1)x^m\\ &= \sum_{m=0}^{\infty}2x^{m+1}+\sum_{m=0}^{\infty}3(m+1)x^{m+2}\\ &= \sum_{n=1}^{\infty}2x^n+\sum_{n=2}^{\infty}3(n-1)x^n\end{aligned}$$

so our coefficients are given by

$$[x^n]\left(\frac{2x}{1-x}+\frac{3x^2}{(1-x)^2}\right)=\begin{cases}0 & \text{if } n=0\\ 2 & \text{if } n=1\\ 2+3(n-1) & \text{if } n\geq 2\end{cases}$$

(d) We have

$$\begin{aligned}\frac{x^3+1}{2-2x} &= \tfrac{1}{2}(x^3+1)\frac{1}{1-x}\\ &= \tfrac{1}{2}(x^3+1)\sum_{m=0}^{\infty}x^m\\ &= \sum_{m=0}^{\infty}\tfrac{1}{2}x^{m+3}+\sum_{m=0}^{\infty}\tfrac{1}{2}x^m\\ &= \sum_{n=3}^{\infty}\tfrac{1}{2}x^n+\sum_{n=0}^{\infty}\tfrac{1}{2}x^n.\end{aligned}$$

So

$$[x^n]\left(\frac{x^3+1}{2-2x}\right)=\begin{cases}\frac{1}{2} & \text{if } n\leq 2\\ 1 & \text{if } n\geq 3\end{cases}$$

(e) We have

$$\begin{aligned}\frac{2x}{(3+6x)^2}+7 &= \tfrac{2}{9}x\frac{1}{(1+2x)^2}+7\\ &= \sum_{m=0}^{\infty}\tfrac{2}{9}x(m+1)(-2x)^m+7\\ &= \sum_{m=0}^{\infty}-\tfrac{1}{9}(m+1)(-2)^{m+1}x^{m+1}+7\\ &= \sum_{n=1}^{\infty}-\tfrac{1}{9}n(-2)^n x^n+7.\end{aligned}$$

So

$$[x^n]\left(\frac{2x}{(3+6x)^2}+7\right)=\begin{cases} 7 & \text{if } n=0 \\ -\frac{1}{9}n(-2)^n & \text{if } n\geq 1 \end{cases}$$

4. Apply partial fractions to each GF

   (a) $A(x)=\frac{1}{(x-1)(x-2)(x-3)}$

   (b) $B(x)=\frac{1}{(x-3)^2(x-5)}$

Solution.

(a) We let A, B, C be constants and solve

$$\frac{1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

Multiply through by $(x-1)(x-2)(x-3)$ to clear the denominators giving

$$1=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

We can solve for the constants quickly by plugging in values for x. When $x=1$ we have $1=2A$ so $A=\frac{1}{2}$. When $x=2$ we have $1=-B$ so $B=-1$. Finally when $x=3$ we have $1=2C$ so $C=\frac{1}{2}$. Thus

$$A(x)=\frac{1}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}+\frac{-1}{x-2}+\frac{1}{2(x-3)}$$

(b) We let A, B, C be constants and solve

$$\frac{1}{(x-3)^2(x-5)}=\frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{x-5}$$

Multiply through by $(x-3)^2(x-5)$ to clear the denominators giving

$$1=A(x-3)(x-5)+B(x-5)+C(x-3)^2$$

Plugging in $x=3$ gives $1=-2B$ so $B=-\frac{1}{2}$. Plugging in $x=5$ gives $1=4C$ so $C=\frac{1}{4}$. To solve for A we can use these values for B, C and plug in $x=4$ (chosen since this makes $x-3$ and $x-5$ simple numbers). This gives $1=A(-1)+B(-1)+C=-A+\frac{1}{2}+\frac{1}{4}$ and we conclude $A=-\frac{1}{4}$. Thus

$$B(x)=\frac{1}{(x-3)^2(x-5)}=\frac{-1}{4(x-3)}+\frac{-1}{2(x-3)^2}+\frac{1}{4(x-5)}$$

5. For each GF in the previous exercise, find a formula for the coefficient of x^n.

Solution. We use the previous problem and our basic GF's in each case.

(a) Here we have

$$\begin{aligned} A(x) &= \frac{1}{2(x-1)} + \frac{-1}{x-2} + \frac{1}{2(x-3)} \\ &= (-\tfrac{1}{2})\frac{1}{1-x} + (\tfrac{1}{2})\frac{1}{1-(x/2)} + (-\tfrac{1}{6})\frac{1}{1-x/3} \\ &= \sum_{n=0}^{\infty} -\tfrac{1}{2}x^n + \sum_{n=0}^{\infty} \tfrac{1}{2^{n+1}}x^n + \sum_{n=0}^{\infty} -\tfrac{1}{6\cdot 3^n}x^n \end{aligned}$$

and thus our coefficients are given by

$$[x^n]A(x) = -\frac{1}{2} + \frac{1}{2^{n+1}} - \frac{1}{6\cdot 3^n}$$

(b) For this part

$$\begin{aligned} B(x) &= \frac{-1}{4(x-3)} + \frac{-1}{2(x-3)^2} + \frac{1}{4(x-5)} \\ &= (\tfrac{1}{12})\frac{1}{1-x/3} + (-\tfrac{1}{18})\frac{1}{(1-(x/3))^2} + (-\tfrac{1}{20})\frac{1}{1-(x/5)} \\ &= \sum_{n=0}^{\infty} \tfrac{1}{12\cdot 3^n}x^n + \sum_{n=0}^{\infty} -\tfrac{1}{18\cdot 3^n}(n+1)x^n + \sum_{n=0}^{\infty} -\tfrac{1}{20\cdot 5^n}x^n \end{aligned}$$

and thus we have

$$[x^n]B(x) = \frac{1}{12\cdot 3^n} - \frac{n+1}{18\cdot 3^n} - \frac{1}{20\cdot 5^n}$$

6. In each problem below you are given an infinite sequence $b_0, b_1, b_2, \ldots$ determined by a recurrence relation. Use this recurrence relation to express the GF for this sequence, $B(x) = \sum_{n=0}^{\infty} b_n x^n$, as a rational function.

(a) $b_0 = 2$, $b_1 = 3$, $b_n - 3b_{n-1} + 7b_{n-2} = 0$ for $n \geq 2$

(b) $b_0 = 1$, $b_1 = 2$, $b_n - 5b_{n-1} + 3b_{n-2} = 1$ for $n \geq 2$

(c) $b_0 = 1$, $b_1 = 0$, $b_2 = 3$, $b_n - 2b_{n-1} + b_{n-3} = n$ for $n \geq 3$

Solution. In each case we use the infinitely many equations forming the recurrence to construct an equation involving generating functions. We then rewrite the equation using $B(x)$ and then solve for $B(x)$.

(a) We multiply the recurrence $0 = b_n - 3b_{n-1} + 7b_{n-2}$ by x^n and then sum over all $n \geq 2$ to get an equation between generating functions.

$$0 = \sum_{n=2}^{\infty} \Big(b_n x^n - 3b_{n-1}x^n + 7b_{n-2}\Big)x^n.$$

Now we rewrite this equation in terms of $B(x) = \sum_{n=0}^{\infty} b_n x^n$ as follows.

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty} b_n x^n - 3x\sum_{n=2}^{\infty} b_{n-1}x^{n-1} + 7x^2\sum_{n=2}^{\infty} b_{n-2}x^{n-2} \\
&= \sum_{n=2}^{\infty} b_n x^n - 3x\sum_{m=1}^{\infty} b_m x^m + 7x^2\sum_{m=0}^{\infty} b_m x^m \\
&= \big(B(x) - b_1 x - b_0\big) - 3x\big(B(x) - b_0\big) + 7x^2 B(x) \\
&= \big(B(x) - 3x - 2\big) - 3x\big(B(x) - 2\big) + 7x^2 B(x)
\end{aligned}$$

Solving the above equation for $B(x)$ then gives our answer

$$B(x) = \frac{2 - 3x}{1 - 3x + 7x^2}.$$

(b) We multiply the recurrence $0 = b_n - 5b_{n-1} + 3b_{n-2} - 1$ by x^n and then sum over all $n \geq 2$ to get the following GF equation

$$0 = \sum_{n=2}^{\infty} \Big(b_n - 5b_{n-1} + 3b_{n-2} - 1\Big)x^n$$

Now we rewrite using $B(x)$

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty} b_n x^n - 5x\sum_{n=2}^{\infty} b_{n-1}x^{n-1} + 3x^2\sum_{n=2}^{\infty} b_{n-2}x^{n-2} - \sum_{n=2}^{\infty} x^n \\
&= \sum_{n=2}^{\infty} b_n x^n - 5x\sum_{m=1}^{\infty} b_m x^m + 3x^2\sum_{m=0}^{\infty} b_m x^m - \sum_{n=2}^{\infty} x^n \\
&= \big(B(x) - b_1 x - b_0\big) - 5x\big(B(x) - b_0\big) + 3x^2 B(x) - \big(\tfrac{1}{1-x} - x - 1\big) \\
&= \big(B(x) - 2x - 1\big) - 5x\big(B(x) - 1\big) + 3x^2 B(x) - \big(\tfrac{1}{1-x} - x - 1\big)
\end{aligned}$$

Solving the above for $B(x)$ gives the answer

$$B(x) = \frac{-4x + \frac{1}{1-x}}{1 - 5x + 3x^2} = \frac{1 - 4x + 4x^2}{1 - 6x + 9x^2 - 4x^3}$$

(c) We multiply the recurrence $0 = b_n - 2b_{n-1} + b_{n-3} - n$ by x^n and then sum over all $n \geq 3$ to get the following GF equation

$$0 = \sum_{n=3}^{\infty} \Big(b_n - 2b_{n-1} + b_{n-3} - n\Big)x^n$$

Now we rewrite using $B(x)$

$$\begin{aligned}
0 &= \sum_{n=3}^{\infty} b_n x^n - 2x\sum_{n=3}^{\infty} b_{n-1}x^{n-1} + x^3\sum_{n=3}^{\infty} b_{n-3}x^{n-3} - x\sum_{n=3}^{\infty} nx^{n-1} \\
&= \sum_{n=3}^{\infty} b_n x^n - 2x\sum_{m=2}^{\infty} b_m x^m + x^3\sum_{m=0}^{\infty} b_m x^m - x\sum_{m=2}^{\infty} (m+1)x^m \\
&= \Big(B(x) - b_2x^2 - b_1x - b_0\Big) - 2x\Big(B(x) - b_1x - b_0\Big) + x^3B(x) - x\Big(\tfrac{1}{(1-x)^2} - 2x - 1\Big) \\
&= \Big(B(x) - 3x^2 - 1\Big) - 2x\Big(B(x) - 1\Big) + x^3B(x) - x\Big(\tfrac{1}{(1-x)^2} - 2x - 1\Big)
\end{aligned}$$

This gives us the equation

$$B(x) - 2xB(x) + x^3B(x) = 1 - 3x + x^2 + \frac{x}{(1-x)^2}$$

Now solving for $B(x)$ we have

$$B(x) = \frac{1 - 3x + x^2 + \frac{x}{(1-x)^2}}{1 - 2x + x^3} = \frac{1 - 5x + 8x^2 - 5x^3 + x^4}{1 - 4x + 5x^2 - x^3 - 2x^4 + x^5}$$

7. In this problem we explore when a generating function has an inverse. (Recall that an inverse to a generating function $A(x)$ is another generating function $B(x)$ with the property that $A(x) \times B(x) = 1$.)

(a) Let $A(x) = a_0 + a_1x + a_2x^2 + \ldots$. Assuming $a_0 = 0$, show that $A(x)$ has no inverse.

(b) Suppose that $A(x)$ and $B(x)$ are inverse generating functions where

$$\begin{aligned}
A(x) &= 2 + 4x - 4x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots \\
B(x) &= b_0 + b_1x + b_2x^2 + b_3x^3 \ldots
\end{aligned}$$

(so the first three coefficients of $A(x)$ are specified, but all other coefficients are unknown constants). Determine the value of b_0. Then find b_1 and b_2.

(c) Let $A(x) = a_0 + a_1x + a_2x^2 + \ldots$ and assume $a_0 \neq 0$. Explain why there is an inverse of $A(x)$.

Solution.

(a) The coefficient of the constant term of $A(x) \times B(x)$ is the product of the constant terms of $A(x)$ and $B(x)$. Since $A(x)$ has constant term $a_0 = 0$, the GF $A(x) \times B(x)$ also has constant term 0, so it cannot be 1.

(b) $A(x)$ and $B(x)$ are inverses, so their product is equal to 1. So we have $A(x) \times B(x) = 1 + 0x + 0x^2 + 0x^3 + \ldots$.

- The constant term of $A(x) \times B(x)$ is $1 = 2b_0$ and this means $b_0 = \frac{1}{2}$.
- The coefficient of x in $A(x) \times B(x)$ is $0 = 2b_1 + 4b_0. = 2b_1 + 4\frac{1}{2}$ so $b_1 = -1$.
- The coefficient of x^2 in $A(x) \times B(x)$ is $0 = 2b_2 + 4b_1 - 4b_0 = 2b_2 - 6$ so $b_2 = 3$.

(c) We will construct a generating function $B(x) = b_0 + b_1x + b_2x^2 + \ldots$ that is an inverse to $A(x)$ one term at a time (so we first choose b_0, then b_1, and so on). We maintain the property that if $b_0, \ldots, b_k$ have been chosen, then

$$[x^i]\Big((b_0 + b_1x + \ldots + b_kx^k)A(x)\Big) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } 1 \leq i \leq k \end{cases}$$

If we continue in this manner, the full generating function $B(x)$ will then satisfy $A(x) \times B(x) = 1$, as desired. As a first step we define $b_0 = \frac{1}{a_0}$ (this is possible since $a_0 \neq 0$) and we observe that we have indeed maintained our property as $[x^0](b_0)A(x) = [x^0](b_0)(a_0 + a_1x + a_2x^2 + \ldots) = a_0b_0 = 1$. Assume now that $b_0, b_1, \ldots, b_k$ have been chosen satisfying the above property. Now we wish to choose b_{k+1} so that

$$(b_0 + b_1x + \ldots + b_{k+1}x^{k+1})A(x)$$

has the right coefficients to satisfies our property. Observe that the choice of b_{k+1} does note effect the coefficients of $x^0, x^1, \ldots, x^k$ in the above expression. So all of these coefficients will still be correct. The coefficient of x^{k+1} in $(b_0 + b_1x + \ldots + b_{k+1}x^{k+1})A(x)$ is equal to $b_0a_{k+1} + b_1a_k + \ldots + b_ka_1 + b_{k+1}a_0$ and since $a_0 \neq 0$ we may choose b_{k+1} to make this coefficient equal to 0. This gives a valid choice for b_{k+1} and by continuing this process we obtain the desired inverse generating function $B(x)$.