MACM 201 - Discrete Mathematics

Generating functions II - Rational GF's

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Rational GF's

We have already seen generating functions compactly expressed using inverses:

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

The following more general class of generating functions similarly have compact representations and will be our main subject.

Definition

A generating function $A(x)$ is called **rational** if it can be expressed as

$$A(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

Our main interests with rational GF's are:

(1) Given a sequence of numbers express it as a rational GF.

(2) Given a rational GF, find the associated sequence (coefficient extraction)

Two useful GF's

You will need to know the following two basic GF's.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{1}{(1-x)^2}$$

Using just these two GF's together with basic arithmetic operations gives us the ability to describe many other GF's.

Note

Let $C(x) = c_0 + c_1 x + c_2 x^2 + \ldots = \sum_{n=0}^{\infty} c_n x^n$.

(1) Adding a power function, say ax^d*, to* $C(x)$ *changes the coefficient of* x^d *to* $(c_d + a)$.

(2) Multiplying $C(x)$ *by a power of* x*, say* x^k*, shifts the coefficients by* k

$$x^k C(x) = c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \ldots = \sum_{n=0}^{\infty} c_n x^{k+n}$$

Using the two basic GF's

Problem. Determine the sequence for each GF.

(1) $\dfrac{x^3 - 2}{1 - x}$

(2) $\dfrac{2x^2 + 5}{(1 - x)^2} + 7x$

Substitution

Many of the things we do with functions still make sense for generating functions.

Definition

Let $C(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots = \sum_{n=0}^{\infty} c_n x^n$ be a generating function. Then we define

$$C(kx^m) = c_0 + c_1(kx^m) + c_2(kx^m)^2 + c_3(kx^m)^3 + \ldots = \sum_{n=0}^{\infty} c_n k^n x^{mn}$$

Example. The GF for nickels $N(x) = 1 + x^5 + x^{10} + x^{15} + \ldots = \sum_{n=0}^{\infty} x^{5n}$ is obtained from the GF $A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$ by substitution:

$$A(x^5) = \sum_{n=0}^{\infty} (x^5)^n = \sum_{n=0}^{\infty} x^{5n} = N(x)$$

We can then use the formula $A(x) = \frac{1}{1-x}$ to deduce $N(x) = A(x^5) = \frac{1}{1-x^5}$.
(we already found this formula for $N(x)$, but this is an easier way)

Substituting in $-x$

(1) Using the GF $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ and substituting in $-x$ for x gives

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} \\ &= 1 + (-x) + (-x)^2 + (-x)^3 + \ldots \\ &= 1 - x + x^2 - x^3 + \ldots \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \end{aligned}$$

Problem. Express $C(x) = 1 - 2x + 4x^2 - 8x^3 + 16x^4 \ldots$ as a rational function.

Substituting in $-x$

(2) Using the GF $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots$ and substituting in $-x$ for x gives

$$\begin{aligned}\frac{1}{(1+x)^2} &= \frac{1}{(1-(-x))^2} \\ &= 1 + 2(-x) + 3(-x)^2 + 4(-x)^3 + \ldots \\ &= 1 - 2x + 3x^2 - 4x^3 + \ldots \\ &= \sum_{n=0}^{\infty} (n+1)(-1)^n x^n\end{aligned}$$

Problem. Express $D(x) = -x + 2x^2 - 3x^3 + 4x^4 + \ldots$ as a rational function.

Finding Coefficients

Using substitution and our two baisc GF's, we can now determine the coefficients for any GF that can be expressed as

$$\frac{p(x)}{ax+b} \qquad \text{or} \qquad \frac{p(x)}{(ax+b)^2}$$

where $p(x)$ is a polynomial

Problem. Find the coefficient of x^k in the GF $C(x) = \dfrac{x^2}{2x+3}$

Problem. Find the coefficient of x^k in the GF $D(x) = \dfrac{x^2+1}{(5x+2)^2}$

We have been working with the two basic GF's

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n$$

There is a more general generating function as follows:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

Combining this formula with substitution allows us to determine the coefficients of any rational polynomial of the form $\frac{p(x)}{(ax+b)^k}$.

The book uses a natural generalization of binomial coefficients, called the extended binomial theorem to get these coefficients. This is definitely worth having a look at. However for our class I will not insist you know this.

Partial fractions

Based on the previous slide, we have the information to extract the coefficients from a rational GF of the form $\frac{P(x)}{(1-x)^k}$ or more generally (using substitution) a rational GF of the form $\frac{P(x)}{(a-bx)^k}$.

Combining this method and partial fractions gives us a recipe for extracting coefficients from every rational GF.

Problem. Find values for A, B, C so that the expression below is true, then use this to determine $[x^n]D(x)$

$$D(x) = \frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$D(x) = \frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Method

Let $q(x)$ be a polynomial which can be factored as $q(x) = (x - r_1)^{d_1}(x - r_2)^{d_2} \dots (x - r_k)^{d_k}$ then there exist constants so that

$$\frac{1}{q(x)} = \frac{A_{1,1}}{x - r_1} + \frac{A_{1,2}}{(x - r_1)^2} + \dots \frac{A_{1,d_1}}{(x - r_1)^{d_1}} + \dots\dots + \dots \frac{A_{k,d_k}}{(x - r_k)^{d_k}}$$

Example. Although we will not solve it, there exist constants A, B, C, D, E so that the following expression is valid:

$$\frac{1}{(x-2)^2(x-3)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3}$$

Note

We can use this together with the formula

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

To do coefficient extraction whenever we have a rational function and we have factored the denominator.