MACM 201 - Discrete Mathematics

7. Recurrence relations I

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Many of the counting problems we have considered so far give us infinite sequences of numbers.

Examples.

(1) If b_n is the number of binary sequences of length n, then $b_n = 2^n$.

(2) If q_n is the number of edges in K_n, then $q_n = \binom{n}{2}$.

(3) If z_n is the number of graphs on $\{1, 2, \ldots, n\}$, then $z_n = 2^{\binom{n}{2}}$.

Now we are going to change perspective and begin thinking in terms of these infinite sequences.This change offers a new way of determining these sequences by working recursively.

Example 1. As above we let b_n denote the number of binary strings of length n. To construct a binary sequence of length $n+1$ we can take an arbitrary binary sequence of length n and then append either a 0 or a 1. It follows that

$$b_{n+1} = 2b_n$$

Since the empty string is a valid binary string of length 0 we also have the initial condition

$$b_0 = 1$$

The above conditions recursively determine b_n.

$$(b_0, b_1, b_2, \ldots) = (1, 2, 4, \ldots)$$

Example 2. As above we let q_n denote the number of edges in a complete graph on n vertices. To construct the complete graph on $n+1$ vertices from the complete graph on n vertices, we add one new vertex and all n edges between this new vertex and the existing vertices. Therefore

$$q_{n+1} = q_n + n$$

Since K_1 has no edges we have

$$q_1 = 0$$

The above conditions recursively determine q_n.

$$(q_1, q_2, q_3, \ldots) = (0, 1, 3, \ldots)$$

Recurrence relations

Definition

A *recurrence relation* of *order* k for the sequence $(a_1, a_2, a_3, \ldots)$ is a formula that determines a_{n+k} given the previous k terms $a_n, a_{n+1}, \ldots, a_{n+k-1}$.

Observe that if we have a recurrence relation of order k and we know the first k terms of the sequence, this is enough information to determine the entire sequence.

Example The Fibonaci sequence $(f_0, f_1, f_2, \ldots)$ is given by the order 2 recurrence

$$f_{n+2} = f_{n+1} + f_n$$

together with the initial condition

$$f_0 = f_1 = 1$$

It is easy to generate the sequence from this information:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Setup

For a sequence of numbers $(a_1, a_2, a_3, \ldots)$ we most prefer to have a closed form expression telling us exactly what a_n is for an arbitrary n.

Example The Fibonacci sequence is also given by the closed form

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

We split the problem of finding a closed form expression into two steps:

(1) Find a recurrence relation and initial conditions.

(2) Solve the recurrence (i.e. find a closed form)

Note: we will focus only on (1) right now.

Problem. Let $\mathcal{C}_n$ be the set of all binary strings of length n with the property that every 1 is followed by a 0 (in particular 1 is not permitted to be the last letter). Define $c_n = |\mathcal{C}_n|$.

(1) List the set $\mathcal{C}_n$ for $n = 0, 1, 2, 3$

(2) Find a recurrence for c_n.

Problem. Let $\mathcal{D}_n$ be the set of all strings over the alphabet $\{A, B, C, D\}$ with the property that every A is followed by C, and every B is followed by DD. Define $d_n = |\mathcal{D}_n|$.

(1) List the set $\mathcal{D}_n$ for $n = 0, 1, 2$

(2) Find a recurrence for d_n.