MATH 308 D200, Fall 2019

15. Duality Theory – two linear programs in one tableau, and the strong duality theorem

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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Definition (Dual LP Problem)

Given a canonical maximization LP problem (called *primal*)

$$
\begin{aligned}
\text{(P)} \quad \text{Maximize } f(\boldsymbol{x}) &= \boldsymbol{c}^\mathsf{T}\boldsymbol{x} - d \\
\text{subject to } \boldsymbol{A}\boldsymbol{x} &\leqslant \boldsymbol{b} \\
\boldsymbol{x} &\geqslant \boldsymbol{0}
\end{aligned}
$$

its *dual LP problem* is a canonical minimization LP problem

$$
\begin{aligned}
\text{(D)} \quad \text{Minimize } g(\boldsymbol{x}) &= \boldsymbol{b}^\mathsf{T}\boldsymbol{y} - d \\
\text{subject to } \boldsymbol{A}^\mathsf{T}\boldsymbol{y} &\geqslant \boldsymbol{c} \\
\boldsymbol{y} &\geqslant \boldsymbol{0}
\end{aligned}
$$

Note

Every canonical minimization LP problem also has a dual LP problem which is a canonical maximization problem. We can think of canonical LP problems as a pair (primal and dual) of LP problems which are dual to each other. Most of the time we will assume the primal LP is a maximization problem and the dual LP is a minimization.

Dual Canonical Tableau

Any canonical Tucker tableau can be interpreted both as a canonical maximization LP problem and a canonical minimization LP problem.

$$
\begin{array}{c|cccc|c|l}
 & x_1 & x_2 & \cdots & x_n & -1 & \\
\hline
y_1 & a_{11} & a_{12} & \cdots & a_{1n} & b_1 & = -t_1 \\
y_2 & a_{21} & a_{22} & \cdots & a_{2n} & b_2 & = -t_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
y_m & a_{m1} & a_{m2} & \cdots & a_{mn} & b_m & = -t_m \\
\hline
-1 & c_1 & c_2 & \cdots & c_n & d & = f \\
\hline
 & = s_1 & = s_2 & \cdots & = s_n & = g &
\end{array}
$$

- N and E—maximization
- W and S—minimization

Definition (Dual LP Problems, Dual Canonical Tableau)

Any pair of canonical maximization and canonical minimization LP problems corresponding to the same tableau as above are said to be *dual LP problems or duals*. The tableau of dual canonical LP problems is said to be a *dual canonical tableau*.

Definition (Minimum Basic Feasible Tableau)

Let

$$
\begin{array}{r|cccc|c|}
\cline{2-6}
 & a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
 & a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
\text{(ind var's)} & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\
\cline{2-6}
-1 & c_1 & c_2 & \dots & c_n & d \\
\cline{2-6}
\multicolumn{1}{r}{} & \multicolumn{4}{c}{=\text{(dep var's)}} & \multicolumn{1}{c}{= g}
\end{array}
$$

be a tableau of a canonical minimization LP problem. The tableau is said to be *minimum basic feasible* if $c_1, c_2, \dots, c_n \leqslant 0$.

. . . corresponds to a basic feasible solution. . .

Example: solving dual LPs

(P) Maximize $f(x_1, x_2) = 4x_1 + 3x_2$
subject to

$$\begin{aligned} x_1 + 2x_2 &\leqslant 20 \\ 2x_1 + 2x_2 &\leqslant 30 \\ 2x_1 + x_2 &\leqslant 25 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

$$\begin{array}{cc|c|l} x_1 & x_2 & -1 & \\ \hline 1 & 2 & 20 & = -t_1 \\ 2 & 2 & 30 & = -t_2 \\ 2 & 1 & 25 & = -t_3 \\ \hline 4 & 3 & 0 & = f \end{array}$$

(D) Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$
subject to

$$\begin{aligned} y_1 + 2y_2 + 2y_3 &\geqslant 4 \\ 2y_1 + 2y_2 + y_3 &\geqslant 3 \\ y_1, y_2, y_3 &\geqslant 0 \end{aligned}$$

$$\begin{array}{c|cc|c} y_1 & 1 & 2 & 20 \\ y_2 & 2 & 2 & 30 \\ y_3 & 2 & 1 & 25 \\ \hline -1 & 4 & 3 & 0 \\ & = s_1 & = s_2 & = g \end{array}$$

Example: solving dual LPs

(P) Maximize $f(x_1, x_2) = 4x_1 + 3x_2$
subject to

$$\begin{aligned} x_1 + 2x_2 &\leqslant 20 \\ 2x_1 + 2x_2 &\leqslant 30 \\ 2x_1 + x_2 &\leqslant 25 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

$$\begin{array}{cc|c|l} x_1 & x_2 & -1 & \\ \hline 1 & 2 & 20 & = -t_1 \\ 2 & 2 & 30 & = -t_2 \\ 2 & 1 & 25 & = -t_3 \\ \hline 4 & 3 & 0 & = f \end{array}$$

(D) Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$
subject to

$$\begin{aligned} y_1 + 2y_2 + 2y_3 &\geqslant 4 \\ 2y_1 + 2y_2 + y_3 &\geqslant 3 \\ y_1, y_2, y_3 &\geqslant 0 \end{aligned}$$

$$\begin{array}{c|cc|c} y_1 & 1 & 2 & 20 \\ y_2 & 2 & 2 & 30 \\ y_3 & 2 & 1 & 25 \\ \hline -1 & 4 & 3 & 0 \\ & = s_1 & = s_2 & = g \end{array}$$

Recall: To solve (D) via the Minimization Simplex Algorithm:

Minimize g $\xrightarrow[\text{Tableau}]{\text{Neg. Transpose}}$ Maximize $-g$ $\xrightarrow{\text{Max SA}}$ Minimum $g = -($ Maximum $-g)$

Example: solving dual LPs

Solving (P) with Maximization SA

x_1	x_2	-1	
1	2	20	$= -t_1$
2	2	30	$= -t_2$
2^*	1	25	$= -t_3$
4	3	0	$= f$

t_3	x_2	-1	
-1/2	3/2	15/2	$= -t_1$
-1	1^*	5	$= -t_2$
1/2	1/2	25/2	$= -x_1$
-2	1	-50	$= f$

t_3	t_2	-1	
1	-3/2	0	$= -t_1$
-1	1	5	$= -x_2$
1	-1/2	10	$= -x_1$
-1	-1	-55	$= f$

Example: solving dual LPs

Solving (P) with Maximization SA

$$
\begin{array}{cc|c|l}
x_1 & x_2 & -1 & \\
\hline
1 & 2 & 20 & = -t_1 \\
2 & 2 & 30 & = -t_2 \\
2^* & 1 & 25 & = -t_3 \\
\hline
4 & 3 & 0 & = f \\
\hline
\end{array}
\qquad
\begin{array}{cc|c|l}
t_3 & x_2 & -1 & \\
\hline
-1/2 & 3/2 & 15/2 & = -t_1 \\
-1 & 1^* & 5 & = -t_2 \\
1/2 & 1/2 & 25/2 & = -x_1 \\
\hline
-2 & 1 & -50 & = f \\
\hline
\end{array}
$$

$$
\begin{array}{cc|c|l}
t_3 & t_2 & -1 & \\
\hline
1 & -3/2 & 0 & = -t_1 \\
-1 & 1 & 5 & = -x_2 \\
1 & -1/2 & 10 & = -x_1 \\
\hline
-1 & -1 & -55 & = f \\
\hline
\end{array}
$$

also solves (D)!

$$(x_1, x_2) = (10, 5), \quad (t_1, t_2, t_3) = (0, 0, 0)$$

$$(s_1, s_2) = (0, 0), \quad (y_1, y_2, y_3) = (0, 1, 1)$$

$$
\begin{array}{r|cc|c|}
\hline
y_1 & 1 & -3/2 & 0 \\
s_2 & -1 & 1 & 10 \\
s_1 & 1 & -1/2 & 5 \\
\hline
-1 & -1 & -1 & -55 \\
\hline
 & = y_3 & = y_2 & = g
\end{array}
$$

Example: solving dual LPs

Solving (P) with Maximization SA

x_1	x_2	-1	
1	2	20	$= -t_1$
2	2	30	$= -t_2$
2^*	1	25	$= -t_3$
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t_3	x_2	-1	
-1/2	3/2	15/2	$= -t_1$
-1	1^*	5	$= -t_2$
1/2	1/2	25/2	$= -x_1$
-2	1	-50	$= f$

t_3	t_2	-1	
1	-3/2	0	$= -t_1$
-1	1	5	$= -x_2$
1	-1/2	10	$= -x_1$
-1	-1	-55	$= f$

Alternative for solving (D): Dual SA (= Maximization SA on the negative transpose):

y_1	1	2	20
y_2	2	2	30
y_3	2	1	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

Example: solving dual LPs

Solving (P) with Maximization SA

x_1	x_2	-1	
1	2	20	$= -t_1$
2	2	30	$= -t_2$
2^*	1	25	$= -t_3$
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t_3	x_2	-1	
-1/2	3/2	15/2	$= -t_1$
-1	1^*	5	$= -t_2$
1/2	1/2	25/2	$= -x_1$
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t_3	t_2	-1	
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Alternative for solving (D): Dual SA (= Maximization SA on the negative transpose):

y_1	1	2	20
y_2	2	2	30
y_3	2	1	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

$\downarrow$ Negative transpose

$\uparrow$ Negative transpose

Example: solving dual LPs

Solving (P) with Maximization SA

x_1	x_2	-1	
1	2	20	$= -t_1$
2	2	30	$= -t_2$
2^*	1	25	$= -t_3$
4	3	0	$= f$

t_3	x_2	-1	
-1/2	3/2	15/2	$= -t_1$
-1	1^*	5	$= -t_2$
1/2	1/2	25/2	$= -x_1$
-2	1	-50	$= f$

t_3	t_2	-1	
1	-3/2	0	$= -t_1$
-1	1	5	$= -x_2$
1	-1/2	10	$= -x_1$
-1	-1	-55	$= f$

Alternative for solving (D): Dual SA ($=$ Maximization SA on the negative transpose):

y_1	1	2	20
y_2	2	2	30
y_3	2	1	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

$\downarrow$ Negative transpose

y_1	-3/2	1	0
s_1	-1/2	1	10
s_2	1	-1	5
-1	-1	-1	-55
	$= y_2$	$= y_3$	$= g$

$\uparrow$ Negative transpose

Example: solving dual LPs

Solving (P) with Maximization SA

x_1	x_2	-1	
1	2	20	$= -t_1$
2	2	30	$= -t_2$
2^*	1	25	$= -t_3$
4	3	0	$= f$

t_3	x_2	-1	
-1/2	3/2	15/2	$= -t_1$
-1	1^*	5	$= -t_2$
1/2	1/2	25/2	$= -x_1$
-2	1	-50	$= f$

t_3	t_2	-1	
1	-3/2	0	$= -t_1$
-1	1	5	$= -x_2$
1	-1/2	10	$= -x_1$
-1	-1	-55	$= f$

Alternative for solving (D): Dual SA (= Maximization SA on the negative transpose):

y_1	1	2	20
y_2	2	2	30
y_3	2	1^*	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

y_1	-3	-2	-30
y_2	-2^*	-2	-20
s_2	2	1	25
-1	-2	-3	-75
	$= s_1$	$= y_3$	$= g$

y_1	-3/2	1	0
s_1	-1/2	1	10
s_2	1	-1	5
-1	-1	-1	-55
	$= y_2$	$= y_3$	$= g$

Example: solving dual LPs

Solving (P) with Maximization SA

x_1	x_2	-1	
1	2	20	$= -t_1$
2	2	30	$= -t_2$
2^*	1	25	$= -t_3$
4	3	0	$= f$

t_3	x_2	-1	
-1/2	3/2	15/2	$= -t_1$
-1	1^*	5	$= -t_2$
1/2	1/2	25/2	$= -x_1$
-2	1	-50	$= f$

$\boldsymbol{x}, \boldsymbol{t}$ is (P)-feasible

$\boldsymbol{y}, \boldsymbol{s}$ is (D)-infeasible

t_3	t_2	-1	
1	-3/2	0	$= -t_1$
-1	1	5	$= -x_2$
1	-1/2	10	$= -x_1$
-1	-1	-55	$= f$

$\boldsymbol{x}, \boldsymbol{t}$ is (P)-feasible

$\boldsymbol{y}, \boldsymbol{s}$ is (D)-feasible

Alternative for solving (D): Dual SA (= Maximization SA on the negative transpose):

y_1	1	2	20
y_2	2	2	30
y_3	2	1^*	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

y_1	-3	-2	-30
y_2	-2^*	-2	-20
s_2	2	1	25
-1	-2	-3	-75
	$= s_1$	$= y_3$	$= g$

$\boldsymbol{y}, \boldsymbol{s}$ is (D)-feasible

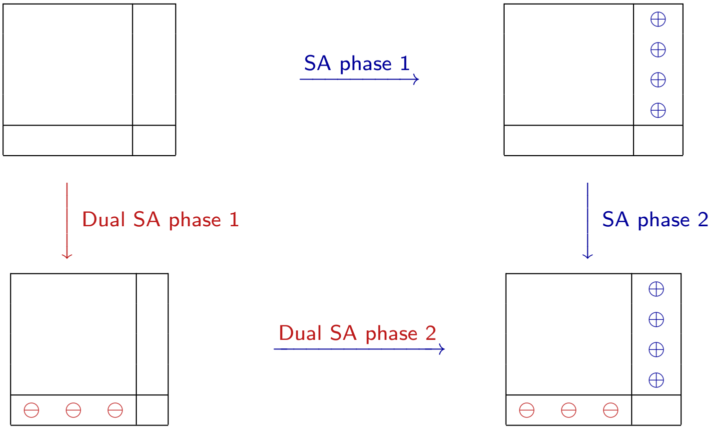
y_1	-3/2	1	0
s_1	-1/2	1	10
s_2	1	-1	5
-1	-1	-1	-55
	$= y_2$	$= y_3$	$= g$

$\boldsymbol{y}, \boldsymbol{s}$ is (D)-feasible

SA versus DSA:Different routes to the same goal

Minimum SA and Maximum SA have different sequence of pivots:

- Phase 1 of SA finds a (P)-feasible BFS
- Phase 1 of Dual SA finds a (D)-feasible BFS



The Dual SA for Minimum Tableaux

This gives the same sequence of pivots as applying the maximization SA to the negative transpose tableau.

1. We have a minimum Tucker tableau.
2. If $c_1, c_2, \ldots, c_n \leqslant 0$, go to **Step 6.** since tableau is dual feasible.
3. Choose $c_j > 0$ such that j is maximal.
4. If $a_{1j}, a_{2j}, \ldots, a_{mj} \leqslant 0 \Longrightarrow$ **STOP**; the minimization problem is infeasible.
5. If $j = n$, choose $a_{in} > 0$, pivot on a_{in}, and go to **Step 1.**
   If $j < n$, choose $a_{ij} > 0$, compute
   $$\alpha = \min(\{c_j/a_{ij}\} \cup \{c_k/a_{ik} : k > j, a_{ik} < 0\}),$$
   and choose any p with $c_p/a_{ip} = \alpha$. Pivot on a_{ip} and go to **Step 1.**
6. The current tableau is minimum BFT $(c_1, c_2, \ldots, c_n \leqslant 0)$
7. If $b_1, b_2, \ldots, b_m \geqslant 0 \Longrightarrow$ **STOP**; the current basic feasible solution is optimal.
8. Choose any i with $b_i < 0$
9. If $a_{i1}, a_{i2}, \ldots, a_{in} \geqslant 0 \Longrightarrow$ **STOP**; the minimization problem is unbounded.
10. Compute
    $$\alpha = \min_{1 \leqslant j \leqslant n} \{c_j/a_{ij} : a_{ij} < 0\}$$
    and choose any p with $c_p/a_{ip} = \alpha$. Pivot on a_{ip} and go to the **Step 6.**

Solve by using the dual SA

Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$, subject to

$$\begin{aligned} y_1 + 2y_2 + 2y_3 &\geqslant 4 \\ 2y_1 + 2y_2 + y_3 &\geqslant 3 \\ y_1, y_2, y_3 &\geqslant 0 \end{aligned}$$

Solve by using the dual SA

Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$, subject to

$$y_1 + 2y_2 + 2y_3 \geqslant 4$$
$$2y_1 + 2y_2 + y_3 \geqslant 3$$
$$y_1, y_2, y_3 \geqslant 0$$

$$\begin{array}{c|cc|c|}
\cline{2-4}
y_1 & 1 & 2 & 20 \\
y_2 & 2 & 2 & 30 \\
y_3 & 2 & 1 & 25 \\
\cline{2-4}
-1 & 4 & 3 & 0 \\
\cline{2-4}
\multicolumn{1}{c}{} & = s_1 & = s_2 & \multicolumn{1}{c}{= g}
\end{array}$$

Solve by using the dual SA

Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$, subject to

$$y_1 + 2y_2 + 2y_3 \geqslant 4$$
$$2y_1 + 2y_2 + y_3 \geqslant 3$$
$$y_1, y_2, y_3 \geqslant 0$$

Phase 1: Tableau is dual infeasible: Here $c_2 = 3$ is the rightmost positive entry so $j = 2$. This is the last column, so the pivot column is $j = 2$.

y_1	1	2	20
y_2	2	2	30
y_3	2	1	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

Solve by using the dual SA

Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$, subject to

$$
\begin{aligned}
y_1 + 2y_2 + 2y_3 &\geqslant 4 \\
2y_1 + 2y_2 + y_3 &\geqslant 3 \\
y_1, y_2, y_3 &\geqslant 0
\end{aligned}
$$

Phase 1: Tableau is dual infeasible: Here $c_2 = 3$ is the rightmost positive entry so $j = 2$.
This is the last column, so the pivot column is $j = 2$.
All three entries in column 2 are > 0, so we are free to choose i. Suppose we pick $i = 3$.

y_1	1	2	20
y_2	2	2	30
y_3	2	1^*	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

Solve by using the dual SA

Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$, subject to

$$\begin{aligned} y_1 + 2y_2 + 2y_3 &\geqslant 4 \\ 2y_1 + 2y_2 + y_3 &\geqslant 3 \\ y_1, y_2, y_3 &\geqslant 0 \end{aligned}$$

Phase 1: Tableau is dual infeasible: Here $c_2 = 3$ is the rightmost positive entry so $j = 2$.
This is the last column, so the pivot column is $j = 2$.
All three entries in column 2 are > 0, so we are free to choose i. Suppose we pick $i = 3$.

New tableau. The last row is all nonpositive, so the BFS is dually feasible.

$$\begin{array}{c|cc|c|}
\cline{2-4}
y_1 & 1 & 2 & 20 \\
y_2 & 2 & 2 & 30 \\
y_3 & 2 & 1^* & 25 \\
\cline{2-4}
-1 & 4 & 3 & 0 \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{= s_1} & \multicolumn{1}{c}{= s_2} & \multicolumn{1}{c}{= g}
\end{array}
\quad \to \quad
\begin{array}{c|cc|c|}
\cline{2-4}
y_1 & -3 & -2 & -30 \\
y_2 & -2 & -2 & -20 \\
s_2 & 2 & 1 & 25 \\
\cline{2-4}
-1 & -2 & -3 & -75 \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{= s_1} & \multicolumn{1}{c}{= y_3} & \multicolumn{1}{c}{= g}
\end{array}$$

Solve by using the dual SA

Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$, subject to

$$
\begin{aligned}
y_1 + 2y_2 + 2y_3 &\geqslant 4 \\
2y_1 + 2y_2 + y_3 &\geqslant 3 \\
y_1, y_2, y_3 &\geqslant 0
\end{aligned}
$$

Phase 1: Tableau is dual infeasible: Here $c_2 = 3$ is the rightmost positive entry so $j = 2$.
This is the last column, so the pivot column is $j = 2$.
All three entries in column 2 are > 0, so we are free to choose i. Suppose we pick $i = 3$.

New tableau. The last row is all nonpositive, so the BFS is dually feasible.
Phase 2: Two entries in last column are < 0. Choose either one as the pivot row i.

y_1	1	2	20
y_2	2	2	30
y_3	2	1^*	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

$\rightarrow$

y_1	-3	-2	-30
y_2	-2	-2	-20
s_2	2	1	25
-1	-2	-3	-75
	$= s_1$	$= y_3$	$= g$

Solve by using the dual SA

Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$, subject to

$$\begin{aligned} y_1 + 2y_2 + 2y_3 &\geqslant 4 \\ 2y_1 + 2y_2 + y_3 &\geqslant 3 \\ y_1, y_2, y_3 &\geqslant 0 \end{aligned}$$

Phase 1: Tableau is dual infeasible: Here $c_2 = 3$ is the rightmost positive entry so $j = 2$.
This is the last column, so the pivot column is $j = 2$.
All three entries in column 2 are > 0, so we are free to choose i. Suppose we pick $i = 3$.

New tableau. The last row is all nonpositive, so the BFS is dually feasible.
Phase 2: Two entries in last column are < 0. Choose either one as the pivot row i.
Suppose $i = 2$. Both entries in row i are negative. Column $j = 1$ has the least ratio $\frac{-2}{-2} < \frac{-3}{-2}$.

y_1	1	2	20
y_2	2	2	30
y_3	2	1*	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

$\rightarrow$

y_1	-3	-2	-30
y_2	-2	-2	-20
s_2	2	1	25
-1	-2	-3	-75
	$= s_1$	$= y_3$	$= g$

Solve by using the dual SA

Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$, subject to

$$\begin{aligned} y_1 + 2y_2 + 2y_3 &\geqslant 4 \\ 2y_1 + 2y_2 + y_3 &\geqslant 3 \\ y_1, y_2, y_3 &\geqslant 0 \end{aligned}$$

Phase 1: Tableau is dual infeasible: Here $c_2 = 3$ is the rightmost positive entry so $j = 2$.
This is the last column, so the pivot column is $j = 2$.
All three entries in column 2 are > 0, so we are free to choose i. Suppose we pick $i = 3$.

New tableau. The last row is all nonpositive, so the BFS is dually feasible.
Phase 2: Two entries in last column are < 0. Choose either one as the pivot row i.
Suppose $i = 2$. Both entries in row i are negative. Column $j = 1$ has the least ratio $\frac{-2}{-2} < \frac{-3}{-2}$.
Pivot on cell (i, j). New tableau.

y_1	1	2	20
y_2	2	2	30
y_3	2	1*	25
-1	4	3	0
	$= s_1$	$= s_2$	$= g$

$\rightarrow$

y_1	-3	-2	-30
y_2	-2*	-2	-20
s_2	2	1	25
-1	-2	-3	-75
	$= s_1$	$= y_3$	$= g$

$\rightarrow$

y_1	-3/2	1	0
s_1	-1/2	1	10
s_2	1	-1	5
-1	-1	-1	-55
	$= y_2$	$= y_3$	$= g$

Solve by using the dual SA

Minimize $g(y_1, y_2, y_3) = 20y_1 + 30y_2 + 25y_3$, subject to

$$\begin{aligned} y_1 + 2y_2 + 2y_3 &\geqslant 4 \\ 2y_1 + 2y_2 + y_3 &\geqslant 3 \\ y_1, y_2, y_3 &\geqslant 0 \end{aligned}$$

Phase 1: Tableau is dual infeasible: Here $c_2 = 3$ is the rightmost positive entry so $j = 2$.
This is the last column, so the pivot column is $j = 2$.
All three entries in column 2 are > 0, so we are free to choose i. Suppose we pick $i = 3$.

New tableau. The last row is all nonpositive, so the BFS is dually feasible.
Phase 2: Two entries in last column are < 0. Choose either one as the pivot row i.
Suppose $i = 2$. Both entries in row i are negative. Column $j = 1$ has the least ratio $\frac{-2}{-2} < \frac{-3}{-2}$.
Pivot on cell (i, j). New tableau.

Last column is non-negative. Tableau is both feasible and dual feasible, so it is optimal.

$$\begin{array}{c|cc|c|}
\hline
y_1 & 1 & 2 & 20 \\
y_2 & 2 & 2 & 30 \\
y_3 & 2 & 1^* & 25 \\
\hline
-1 & 4 & 3 & 0 \\
\hline
\multicolumn{1}{c}{} & = s_1 & = s_2 & \multicolumn{1}{c}{= g}
\end{array}
\rightarrow
\begin{array}{c|cc|c|}
\hline
y_1 & -3 & -2 & -30 \\
y_2 & -2^* & -2 & -20 \\
s_2 & 2 & 1 & 25 \\
\hline
-1 & -2 & -3 & -75 \\
\hline
\multicolumn{1}{c}{} & = s_1 & = y_3 & \multicolumn{1}{c}{= g}
\end{array}
\rightarrow
\begin{array}{c|cc|c|}
\hline
y_1 & -3/2 & 1 & 0 \\
s_1 & -1/2 & 1 & 10 \\
s_2 & 1 & -1 & 5 \\
\hline
-1 & -1 & -1 & -55 \\
\hline
\multicolumn{1}{c}{} & = y_2 & = y_3 & \multicolumn{1}{c}{= g}
\end{array}$$

Strong Duality Theorem

Theorem

If either one of the two dual linear programs

$$(P) \quad \text{Maximize } f(\boldsymbol{x}) = \boldsymbol{c}^\top \boldsymbol{x} - d \qquad\qquad (D) \quad \text{Minimize } g(\boldsymbol{y}) = \boldsymbol{b}^\top \boldsymbol{y} - d$$
$$\text{subject to } \boldsymbol{A}\boldsymbol{x} \leqslant \boldsymbol{b} \qquad\qquad \text{subject to } \boldsymbol{A}^\top \boldsymbol{y} \geqslant \boldsymbol{c}$$
$$\boldsymbol{x} \geqslant \boldsymbol{0} \qquad\qquad \boldsymbol{y} \geqslant \boldsymbol{0}$$

has an optimum solution, then its dual linear program also has an optimum solution. Moreover, such optimal solutions $\boldsymbol{x}^*$, $\boldsymbol{y}^*$ *satisfy* $f(\boldsymbol{x}^*) = g(\boldsymbol{y}^*)$

Proof.

If we execute either SA or DSA to their dual canonical tableau, then it will find optimal solutions for both (P) and (D), say $\boldsymbol{x}^*$, $\boldsymbol{y}^*$. The South-East cell of the final tableau shows that

$$f(\boldsymbol{x}^*) = g(\boldsymbol{y}^*).$$

□