MACM 201 - Discrete Mathematics

Graph Theory 6 - Hamiltonian paths/cycles

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Hamiltonian

Definition

Let G be a graph. A path of G is **Hamiltonian** if it contains every vertex of G. Similarly, a cycle of G is **Hamiltonian** if it contains every vertex of G.

Examples:

Note

The definition of Hamiltonian is very similar to Eulerian except that each vertex appears exactly once instead of each edge appearing once.

Necessary and sufficient conditions

Definition

Let P be a property of graphs and let $\mathcal{C}$ be a set of conditions.

(1) $\mathcal{C}$ is **necessary** for P if every graph satisfying P also satisfies $\mathcal{C}$.

(2) $\mathcal{C}$ is **sufficient** for P if every graph that satisfies $\mathcal{C}$ also satisfies P.

(3) If $\mathcal{C}$ is both necessary and sufficient for P, then a graph G satisfies P if and only if G satisfies $\mathcal{C}$. In this case $\mathcal{C}$ **characterizes** when P is satisfied.

Examples

(1) It is necessary for a graph to be connected to have a Hamiltonian path.

(2) Being a complete graph is a sufficient condition to have a Hamiltonian path.

(3) It is necessary and sufficient to be connected and have all vertices of even degree in order to have an Euler circuit.

Hamiltonian vs. Eulerian

Although the properties of having a Hamiltonian cycle or having an Euler circuit look superficially similar, they are actually quite different.

(1) There is a fast algorithm that takes a graph $G = (V, E)$ and determines if it has an Euler circuit, where the running time is a linear function of $|V| + |E|$.

(2) The problem of deciding if a graph has a Hamiltonian path/cycle is NP-complete. So, it is widely believed that there does not exist an algorithm that takes an arbitrary graph $G = (V, E)$ and determines if G has a Hamiltonian path/cycle where the running time is bounded by a polynomial function of $|V| + |E|$.

Note

Assuming there is no polynomial time algorithm to decide if a graph has a Hamiltonian path/cycle, there will not be a "nice" set of conditions $\mathcal{C}$ so that

Every graph has a Hamiltonian cycle if and only if it satisfies $\mathcal{C}$.

Therefore, we will content ourselves with finding some necessary conditions and some sufficient conditions, but will not attempt to find a characterization.

Necessary conditions

Theorem

If $G = (V, E)$ is a graph with a Hamiltonian cycle, then $G - v$ is connected for every $v \in V$.

Proof.

Theorem

Let $G = (V, E)$ be a bipartite graph with bipartition (V_1, V_2). If G has a Hamiltonian cycle, then $|V_1| = |V_2|$.

Proof.

A sufficient condition

Theorem

Let $G = (V, E)$ be a loopless graph with $|V| = n$. If the following condition is satisfied, then G has a Hamiltonian path.

$$\deg(x) + \deg(y) \geq n - 1 \qquad \text{for all } x, y \in V \text{ with } x \text{ not adjacent to } y \text{ and } x \neq y.$$

Proof.

A sufficient condition

Proof (continued)

Corollary

If $G = (V, E)$ is a graph with $|V| = n$ and $\deg(v) \geq \frac{n-1}{2}$ holds for every $v \in V$, then G has a Hamiltonian path.

Proof. In this case G satisfies the condition from the previous theorem since for all $x, y \in V$ we have $\deg(x) + \deg(y) \geq \frac{n-1}{2} + \frac{n-1}{2} = n - 1$.

This corollary is tight in the sense that the statement becomes false when the degree bound is weakened to $\deg(v) \geq \frac{n-2}{2}$. Here is a graph with $n = 2k$ vertices for which all vertices have degree $\frac{n-2}{2} = k - 1$ but there is no Hamiltonian path.