MATH 308 D200, Fall 2019

9. Simplex algorithm for maximum tableau

(based on notes from Dr. J. Hales, Dr. L. Stacho, and Dr. L. Godyyn)

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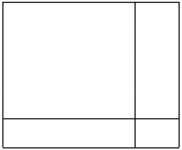
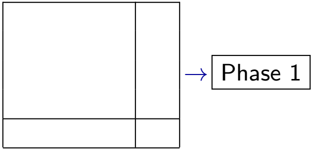
SFU Burnaby

Simplex Algorithm for Max Tableau: Two Phases, Three Outcomes



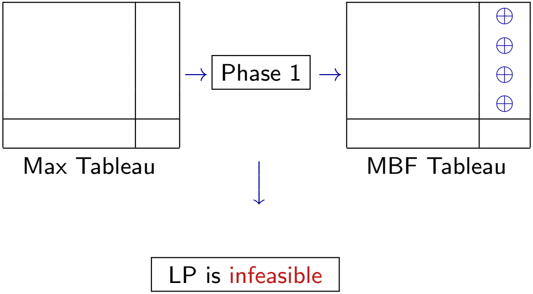
Max Tableau

Simplex Algorithm for Max Tableau: Two Phases, Three Outcomes



Max Tableau

Simplex Algorithm for Max Tableau: Two Phases, Three Outcomes



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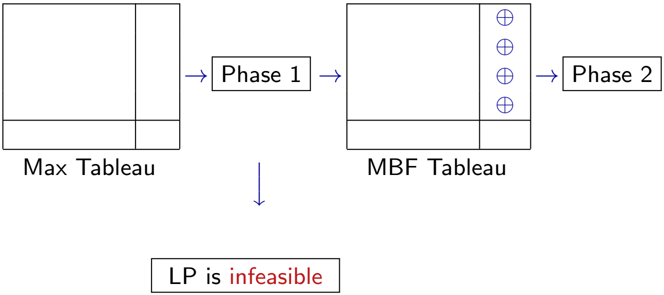

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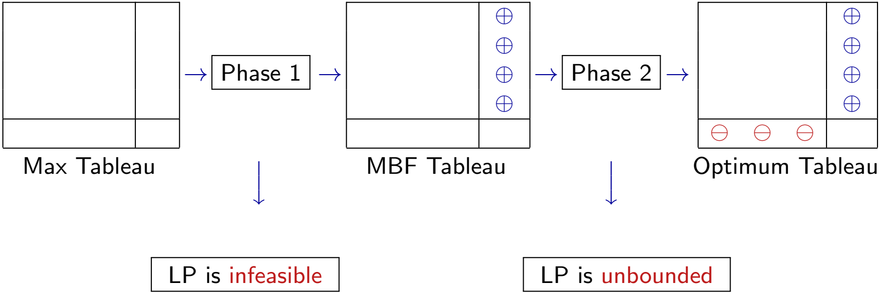
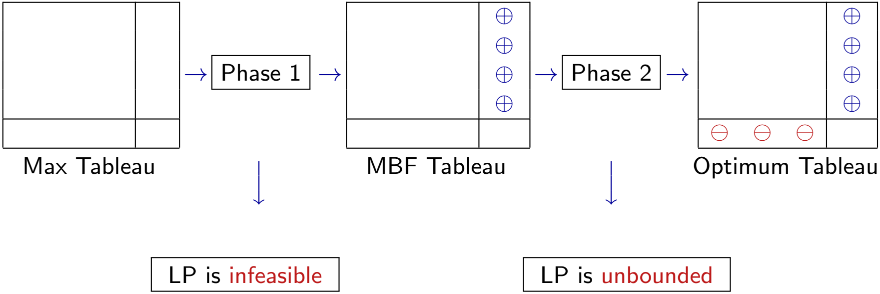


Simplex Algorithm for Max Tableau: Two Phases, Three Outcomes



Phase 2 is "SA for MBFT" (Section 8)

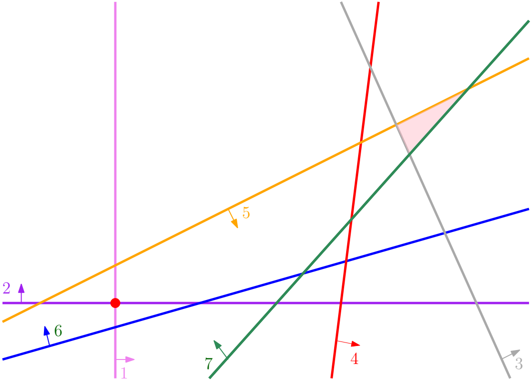
Algorithm (SA for Max Tableau (**Phase 1**))

$$
\begin{array}{cccc|c|l}
 & \textit{(ind var's)} & & & -1 & \\
\hline
a_{11} & a_{12} & \cdots & a_{1n} & b_1 & \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & = -\textit{(dep var's)} \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m & \\
\hline
c_1 & c_2 & \cdots & c_n & d & = f \\
\hline
\end{array}
$$

1. *We have maximum Tucker tableau.*
2. *If* $b_1, b_2, \ldots, b_m \geqslant 0$, *then go to* **Step 6.**
3. *Choose* $b_\ell < 0$ *such that* ℓ *is maximal.*
4. *If* $a_{\ell 1}, a_{\ell 2}, \ldots, a_{\ell n} \geqslant 0 \Longrightarrow$ **STOP**; *the problem is infeasible.*
5. *If* $\ell = m$, *choose* $a_{mj} < 0$, *pivot on* a_{mj}, *and go to* **Step 1.**
   If $\ell < m$, *choose* $a_{\ell j} < 0$, *compute*

   $$\alpha = \min(\{b_\ell / a_{\ell j}\} \cup \{b_k / a_{kj} : k > \ell, a_{kj} > 0\}),$$

   and choose any p *with* $b_p / a_{pj} = \alpha$. *Pivot on* a_{pj} *and go to* **Step 1.**
6. *The tableau is a MBFT. Apply the SA for MBFT (***Phase 2***).*

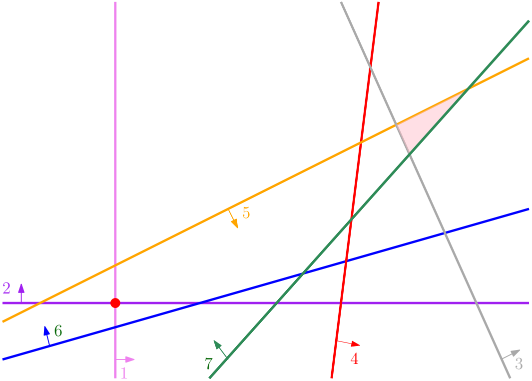


$$
\begin{array}{cc|c|l}
x_1 & x_2 & -1 & \\
\hline
 & & - & = -x_3 \\
 & & - & = -x_4 \\
 & & + & = -x_5 \\
 & & + & = -x_6 \\
 & & + & = -x_7 \\
\hline
 & & & = f
\end{array}
$$

Notes:

▷ Initially:
· The BS is $(x_1, x_2) = (0, 0)$.
· Inequalities $x_3 \geq 0$, $x_4 \geq 0$ are violated

Visual example of Phase 1



$$
\begin{array}{cc|c|l}
x_1 & x_2 & -1 & \\
\hline
 & & - & = -x_3 \\
 & & - & = -x_4 (\ell) \\
 & & + & = -x_5 \\
 & & + & = -x_6 \\
 & & + & = -x_7 \\
\hline
 & & & = f \\
\hline
\end{array}
$$

Notes:

- Initially:
  - The BS is $(x_1, x_2) = (0, 0)$.
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- $\ell = 2$

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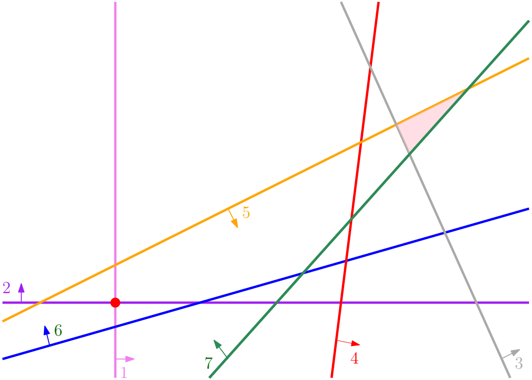

$$\begin{array}{cc|c|l}
x_1 & x_2 & -1 & \\
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Notes:

- ▷ Initially:
  - · The BS is $(x_1, x_2) = (0, 0)$.
  - · Inequalities $x_3 \geq 0$, $x_4 \geq 0$ are violated
- ▷ $\ell = 2$

  Goal:
  - · Move toward the x_4-line
  - · Keep x_5, x_6 and x_7 feasible

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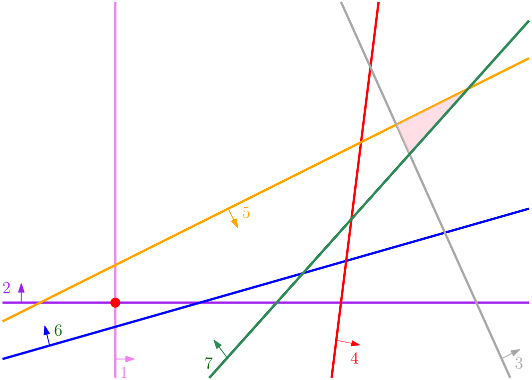
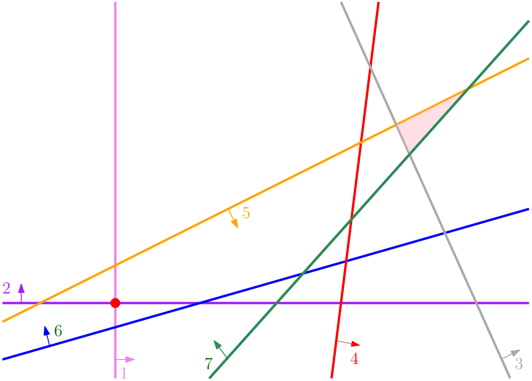


x_1	x_2	-1	
		$-$	$= -x_3$
$-$	$+$	$-$	$= -x_4(\ell)$
		$+$	$= -x_5$
		$+$	$= -x_6$
		$+$	$= -x_7$
			$= f$

Notes:

- Interpretation:
  $-$: increasing x_1 moves toward x_4-feasibility

  $+$: increasing x_2 moves away from x_4-feasibility.



$x_1(j)$	x_2	-1	
		$-$	$= -x_3$
$-$	$+$	$-$	$= -x_4(\ell)$
		$+$	$= -x_5$
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  $+$: increasing x_2 moves away from x_4-feasibility.
- x_1 will leave the basis $(j = 2)$

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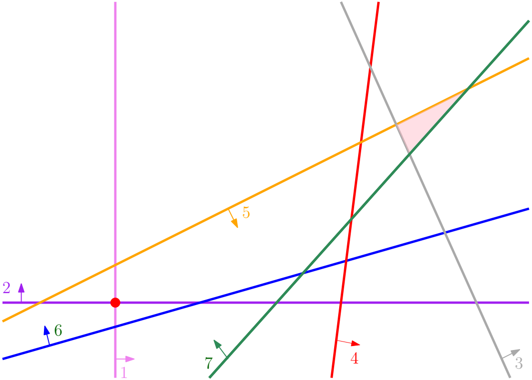

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  $+$: increasing x_2 moves away from x_4-feasibility.

- x_1 will leave the basis $(j = 2)$
- BS moves toward lines x_4, x_6, x_7, and away from line x_5.

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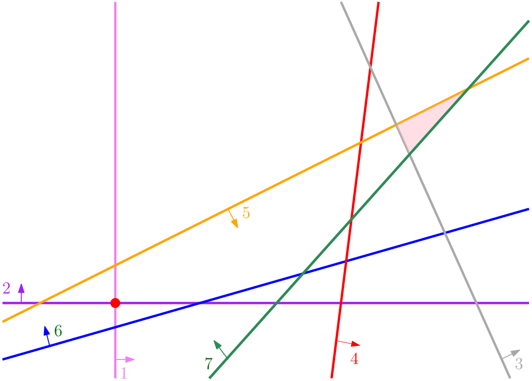

$x_1(j)$	x_2	-1		ratios
		$-$	$= -x_3$	
$-$	$+$	$-$	$= -x_4(\ell)$	$-/-$
$-$		$+$	$= -x_5$	N/A
$+$		$+$	$= -x_6$	$+$/-
$+$		$+$	$= -x_7$	$+$/-
			$= f$	

Notes:

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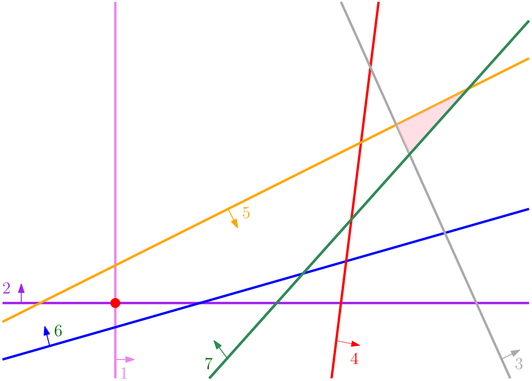
  $+$: increasing x_2 moves away from x_4-feasibility.
- x_1 will leave the basis $(j = 2)$
- BS moves toward lines x_4, x_6, x_7, and away from line x_5.
- Compare ratios



$x_1(j)$	x_2	-1		ratios
		$-$	$= -x_3$	
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$-$		$+$	$= -x_5$	N/A
$+^*$		$+$	$= -x_6(p)$	+/-
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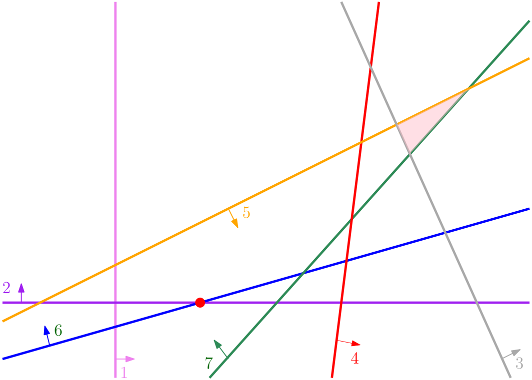

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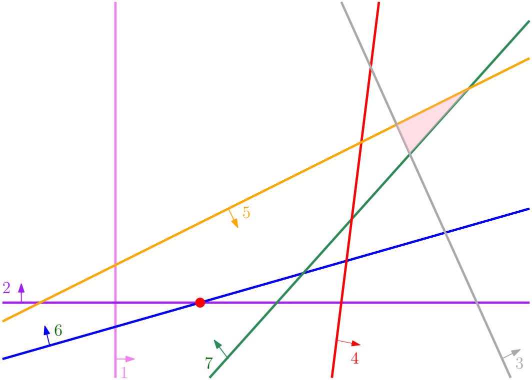
- x_1 will leave the basis $(j = 2)$
- BS moves toward lines x_4, x_6, x_7, and away from line x_5.
- Compare ratios
- Pivot $x_1 \leftrightarrow x_6$

$x_1(j)$	x_2	-1		ratios
		$-$	$= -x_3$	
$-$	$+$	$-$	$= -x_4(\ell)$	$-/-$
$-$		$+$	$= -x_5$	N/A
$+^*$		$+$	$= -x_6(p)$	$+/-$
$+$		$+$	$= -x_7$	$+/-$
			$= f$	

$\mapsto$

x_6	x_2	-1		ratios
			$= -x_3$	
			$= -x_4$	
			$= -x_5$	
			$= -x_1$	
			$= -x_7$	
			$= f$	



Notes:

- ▷ x_4 still infeasible. Again $\ell = 2$.

$x_1(j)$	x_2	-1		ratios
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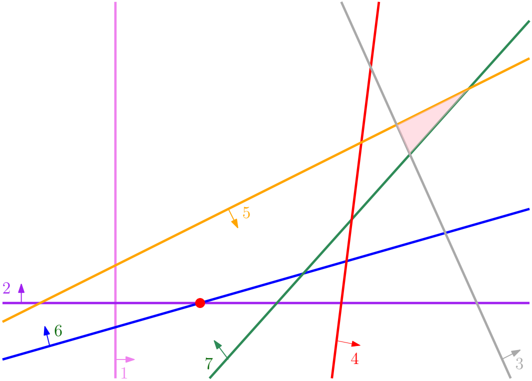

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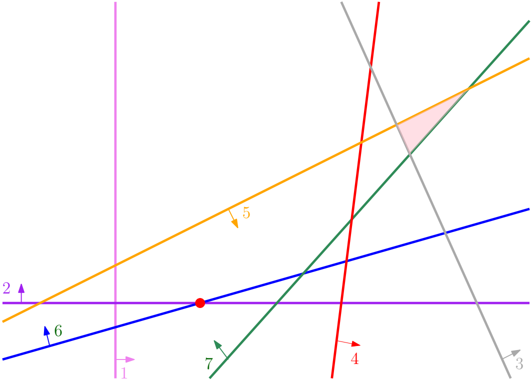

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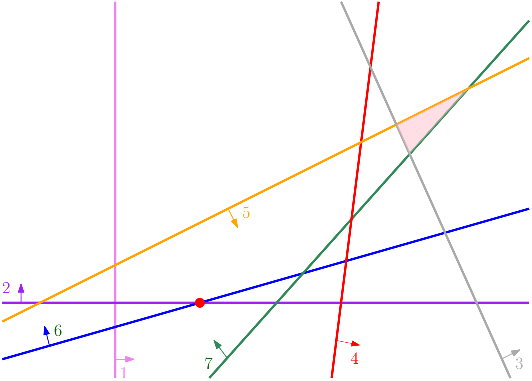

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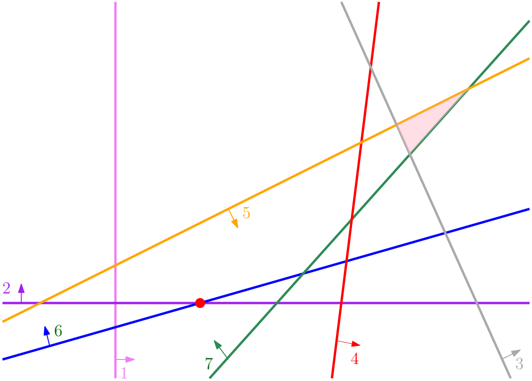

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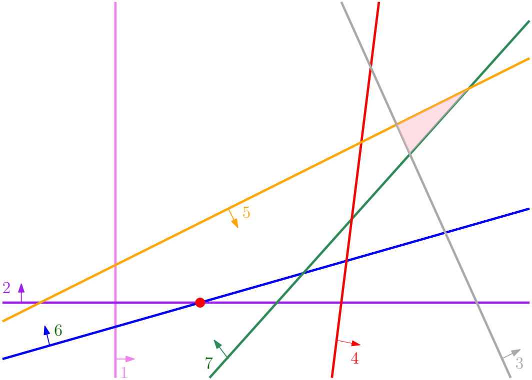

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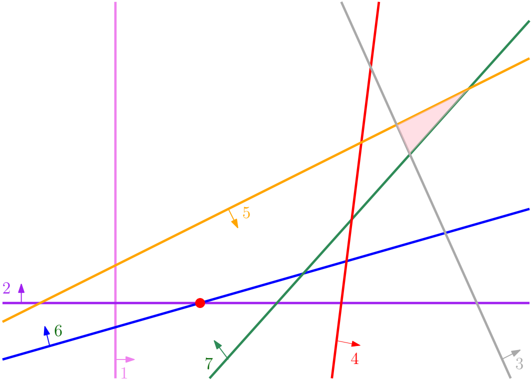

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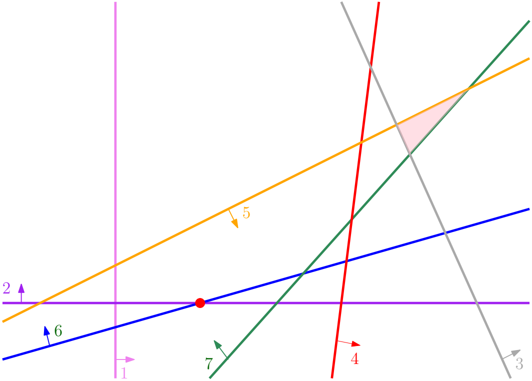

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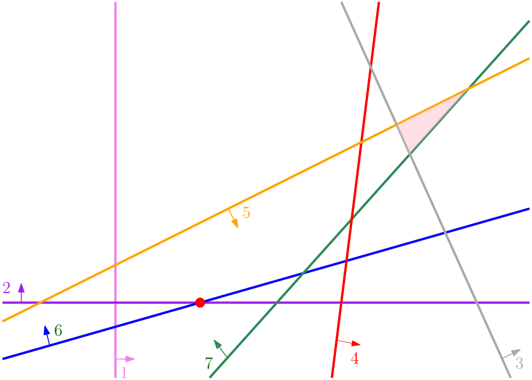

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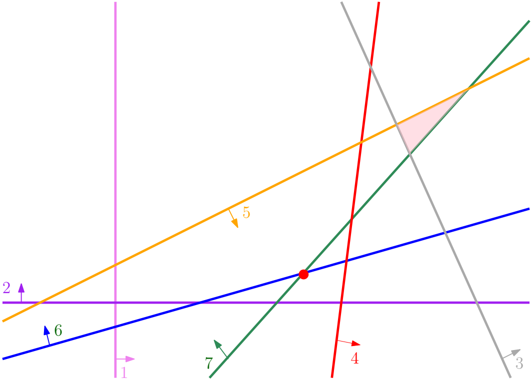

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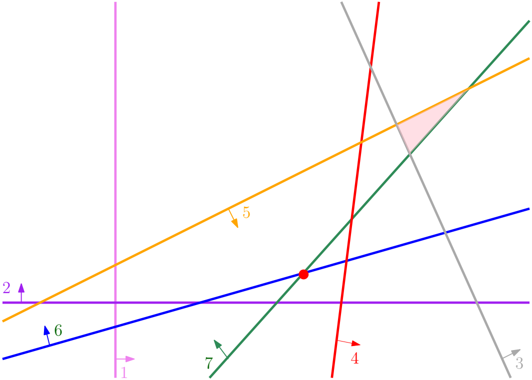

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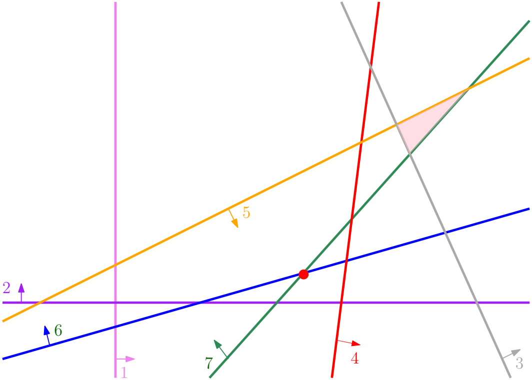

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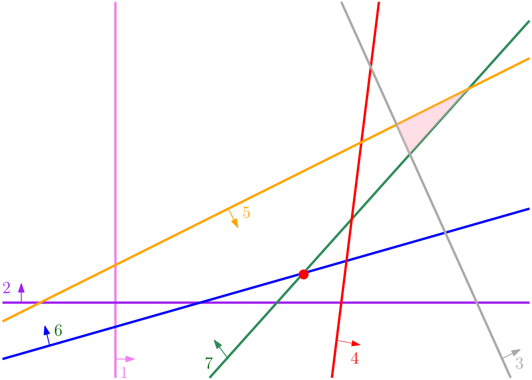
		$-$	$= -x_3$	
$-$	$+$	$-$	$= -x_4(\ell)$	
		$+$	$= -x_5$	
		$+$	$= -x_1$	
		$+$	$= -x_2$	
			$= f$	

Visual example of Phase 1



Notes:

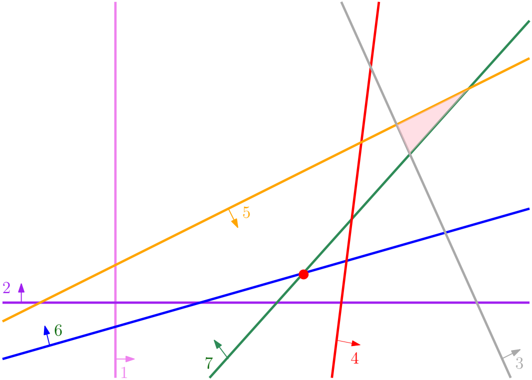
- $\triangleright$ x_4 still infeasible. Again $\ell = 2$.

$\cdots \mapsto$

x_6	$x_2(j)$	-1		ratios
		$-$	$= -x_3$	
$+$	$-$	$-$	$= -x_4(\ell)$	$-/-$
	$-$	$+$	$= -x_5$	N/A
	$-$	$+$	$= -x_1$	N/A
	$+^*$	$+$	$= -x_7(p)$	$+/+$
			$= f$	

$\mapsto$

$x_6(j)$	x_7	-1		ratios
		$-$	$= -x_3$	
$-$	$+$	$-$	$= -x_4(\ell)$	
		$+$	$= -x_5$	
		$+$	$= -x_1$	
		$+$	$= -x_2$	
			$= f$	



Notes:

- ▷ x_4 still infeasible. Again $\ell = 2$.

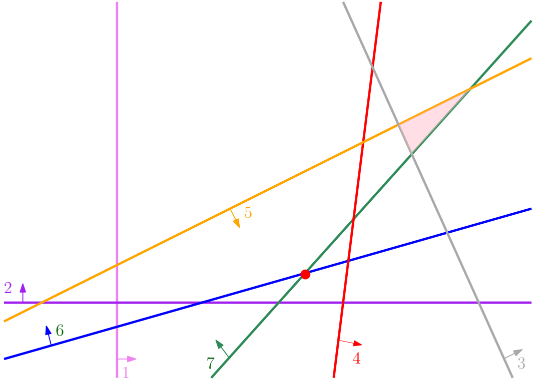
$\cdots \mapsto$

x_6	$x_2(j)$	-1		ratios
		$-$	$= -x_3$	
$+$	$-$	$-$	$= -x_4(\ell)$	$-/-$
	$-$	$+$	$= -x_5$	N/A
	$-$	$+$	$= -x_1$	N/A
	$+^*$	$+$	$= -x_7(p)$	$+/+$
			$= f$	

$\mapsto$

$x_6(j)$	x_7	-1		ratios
		$-$	$= -x_3$	
$-$	$+$	$-$	$= -x_4(\ell)$	
$+$		$+$	$= -x_5$	
$-$		$+$	$= -x_1$	
$-$		$+$	$= -x_2$	
			$= f$	

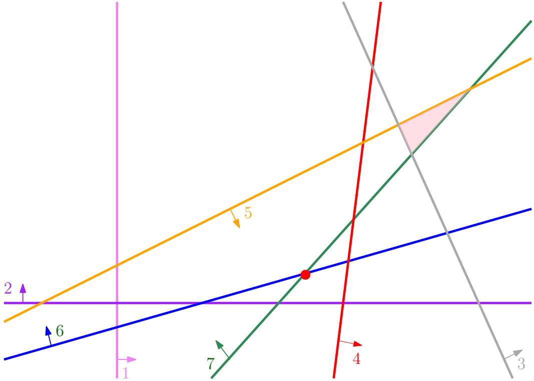
Visual example of Phase 1



Notes:

- ▷ x_4 still infeasible. Again $\ell = 2$.

$$
\cdots \mapsto
\begin{array}{cc|c|ll}
x_6 & x_2(j) & -1 & & \text{ratios} \\
\hline
 & & - & = -x_3 & \\
+ & - & - & = -x_4(\ell) & -/- \\
 & - & + & = -x_5 & \text{N/A} \\
 & - & + & = -x_1 & \text{N/A} \\
 & +^* & + & = -x_7(p) & +/+ \\
\hline
 & & & = f &
\end{array}
\mapsto
\begin{array}{cc|c|ll}
x_6(j) & x_7 & -1 & & \text{ratios} \\
\hline
 & & - & = -x_3 & \\
- & + & - & = -x_4(\ell) & -/- \\
+ & & + & = -x_5 & +/+ \\
- & & + & = -x_1 & \text{N/A} \\
- & & + & = -x_2 & \text{N/A} \\
\hline
 & & & = f &
\end{array}
$$



Notes:

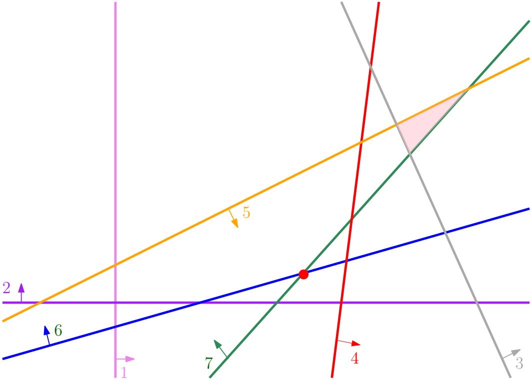
- ▷ x_4 still infeasible. Again $\ell = 2$.

$\cdots \mapsto$

x_6	$x_2(j)$	-1		ratios
		$-$	$= -x_3$	
$+$	$-$	$-$	$= -x_4(\ell)$	$-/-$
	$-$	$+$	$= -x_5$	N/A
	$-$	$+$	$= -x_1$	N/A
	$+^*$	$+$	$= -x_7(p)$	$+/+$
			$= f$	

$\mapsto$

$x_6(j)$	x_7	-1		ratios
		$-$	$= -x_3$	
$-^*$	$+$	$-$	$= -x_4(\ell, p)$	$-/-$
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
			$= f$	



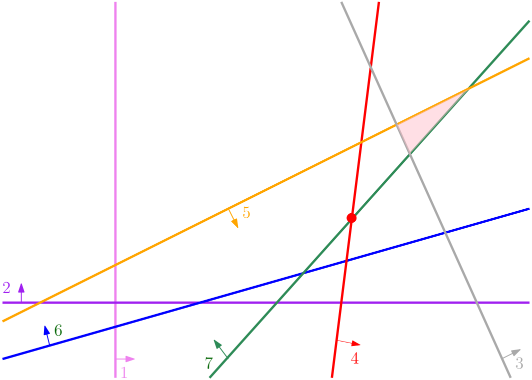
$\cdots \mapsto$

$x_6(j)$	x_7	-1		ratios
		$-$	$= -x_3$	
$-^{*}$	$+$	$-$	$= -x_4(\ell, p)$	$-/-$
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
			$= f$	

Notes:

- x_4 still infeasible. Again $\ell = 2$.

Visual example of Phase 1

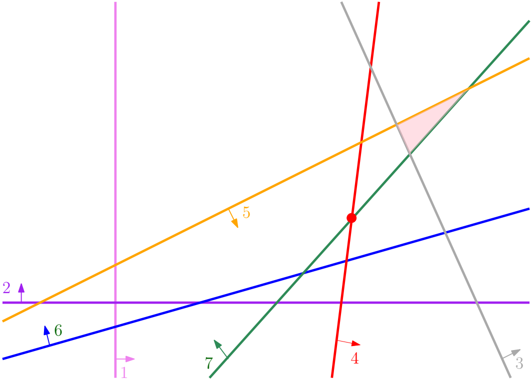


Notes:

- ▷ x_4 still infeasible. Again $\ell = 2$.

	$x_6(j)$	x_7	-1		ratios
			$-$	$= -x_3$	
	$-^*$	$+$	$-$	$= -x_4(\ell, p)$	$-/-$
$\cdots \mapsto$	$+$		$+$	$= -x_5$	$+/+$
	$-$		$+$	$= -x_1$	N/A
	$-$		$+$	$= -x_2$	N/A
				$= f$	

	x_4	x_7	-1		ratios
				$= -x_3$	
				$= -x_6$	
$\mapsto$				$= -x_5$	
				$= -x_1$	
				$= -x_2$	
				$= f$	



Notes:

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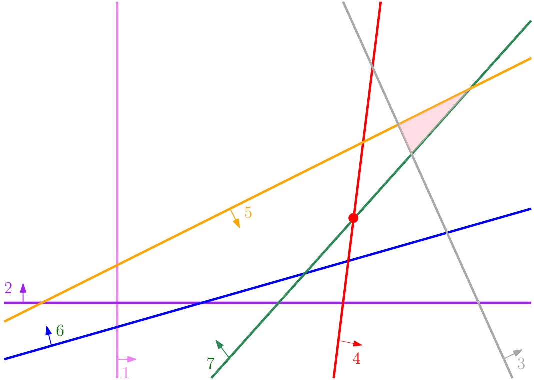
$\cdots \mapsto$

	$x_6(j)$	x_7	-1		ratios
			$-$	$= -x_3$	
	$-^*$	$+$	$-$	$= -x_4(\ell, p)$	$-/-$
	$+$		$+$	$= -x_5$	$+/+$
	$-$		$+$	$= -x_1$	N/A
	$-$		$+$	$= -x_2$	N/A
				$= f$	

$\mapsto$

x_4	x_7	-1		ratios
		$-$	$= -x_3$	
		$+$	$= -x_6$	
		$+$	$= -x_5$	
		$+$	$= -x_1$	
		$+$	$= -x_2$	
			$= f$	



Notes:

- ▷ Finally x_4 is feasible.

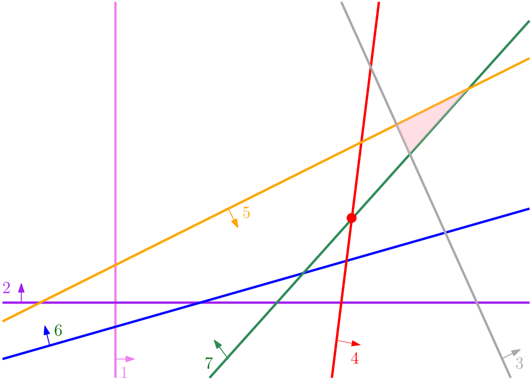
$\cdots \mapsto$

$x_6(j)$	x_7	-1		ratios
		$-$	$= -x_3$	
$-^*$	$+$	$-$	$= -x_4(\ell, p)$	$-/-$
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
			$= f$	

$\mapsto$

x_4	x_7	-1		ratios
		$-$	$= -x_3(\ell)$	
		$+$	$= -x_6$	
		$+$	$= -x_5$	
		$+$	$= -x_1$	
		$+$	$= -x_2$	
			$= f$	

Visual example of Phase 1



Notes:

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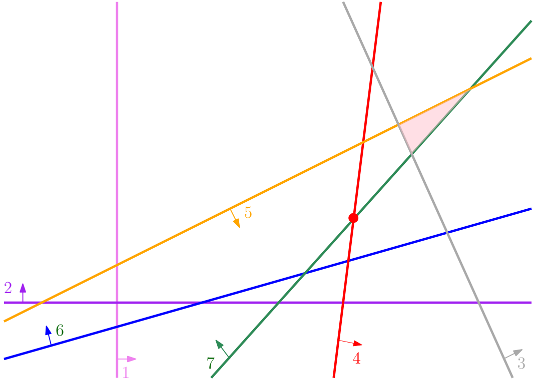
$\cdots \mapsto$

$x_6(j)$	x_7	-1		ratios
		$-$	$= -x_3$	
$-^*$	$+$	$-$	$= -x_4(\ell, p)$	$-/-$
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
			$= f$	

$\mapsto$

x_4	x_7	-1		ratios
$-$	$-$	$-$	$= -x_3(\ell)$	
		$+$	$= -x_6$	
		$+$	$= -x_5$	
		$+$	$= -x_1$	
		$+$	$= -x_2$	
			$= f$	

Visual example of Phase 1

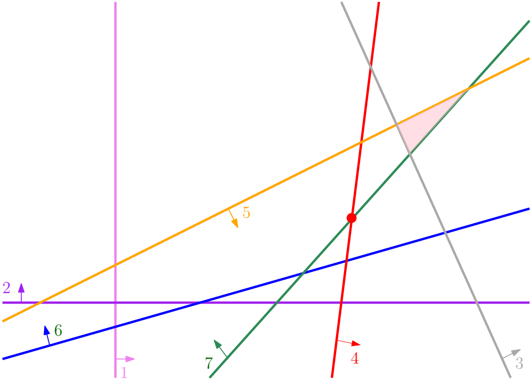


Notes:

- ▷ Finally x_4 is feasible.
- ▷ $\ell = 1$.

$$\cdots \mapsto \begin{array}{c|c|l|c} x_6(j) & x_7 & -1 & & \text{ratios} \\ \hline & & - & = -x_3 & \\ -^{*} & + & - & = -x_4(\ell, p) & -/- \\ + & & + & = -x_5 & +/+ \\ - & & + & = -x_1 & \text{N/A} \\ - & & + & = -x_2 & \text{N/A} \\ \hline & & & = f & \end{array} \mapsto \begin{array}{c|c|l|c} x_4(j) & x_7 & -1 & & \text{ratios} \\ \hline - & - & - & = -x_3(\ell) & \\ & & + & = -x_6 & \\ & & + & = -x_5 & \\ & & + & = -x_1 & \\ & & + & = -x_2 & \\ \hline & & & = f & \end{array}$$

Visual example of Phase 1



Notes:

- ▷ Finally x_4 is feasible.
- ▷ $\ell = 1$.

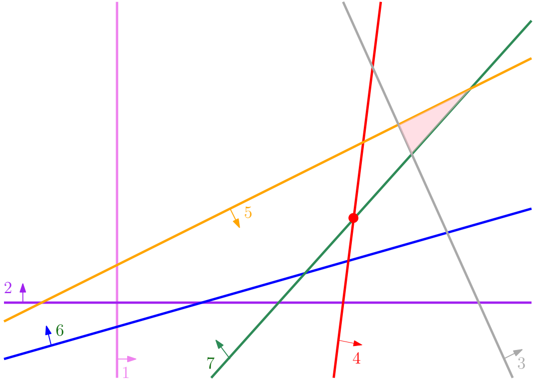
$\cdots \mapsto$

$x_6(j)$	x_7	-1		ratios
		$-$	$= -x_3$	
$-^*$	$+$	$-$	$= -x_4(\ell, p)$	$-/-$
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
			$= f$	

$\mapsto$

$x_4(j)$	x_7	-1		ratios
$-$	$-$	$-$	$= -x_3(\ell)$	
$-$		$+$	$= -x_6$	
$+$		$+$	$= -x_5$	
$-$		$+$	$= -x_1$	
$-$		$+$	$= -x_2$	
			$= f$	

Visual example of Phase 1



Notes:

- ▷ Finally x_4 is feasible.
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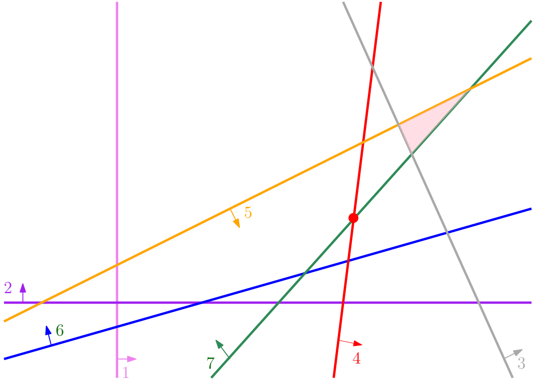
$\cdots \mapsto$

$x_6(j)$	x_7	-1		ratios
		$-$	$= -x_3$	
$-^*$	$+$	$-$	$= -x_4(\ell, p)$	$-/-$
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
			$= f$	

$\mapsto$

$x_4(j)$	x_7	-1		ratios
$-$	$-$	$-$	$= -x_3(\ell)$	$-/-$
$-$		$+$	$= -x_6$	N/A
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
			$= f$	

Visual example of Phase 1



Notes:

- ▷ Finally x_4 is feasible.
- ▷ $\ell = 1$.

$\cdots \mapsto$

$x_6(j)$	x_7	-1		ratios
		$-$	$= -x_3$	
$-^*$	$+$	$-$	$= -x_4(\ell, p)$	$-/-$
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
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$\mapsto$

$x_4(j)$	x_7	-1		ratios
$-^*$	$-$	$-$	$= -x_3(\ell, p)$	$-/-$
$-$		$+$	$= -x_6$	N/A
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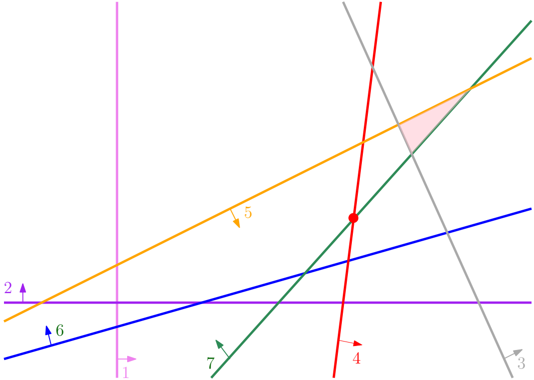
$-$		$+$	$= -x_2$	N/A
			$= f$	

Visual example of Phase 1

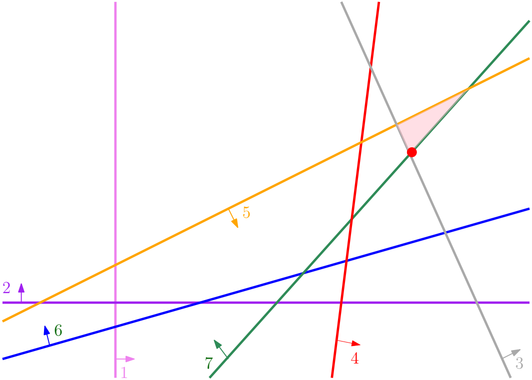


Notes:

- ▷ Finally x_4 is feasible.
- ▷ $\ell = 1$.

$$
\cdots \mapsto
\begin{array}{cc|c|l l}
x_4(j) & x_7 & -1 & & \text{ratios} \\
\hline
-^{*} & - & - & = -x_3(\ell,p) & -/- \\
- & & + & = -x_6 & \text{N/A} \\
+ & & + & = -x_5 & +/+ \\
- & & + & = -x_1 & \text{N/A} \\
- & & + & = -x_2 & \text{N/A} \\
\hline
 & & & = f & \\
\hline
\end{array}
$$

Visual example of Phase 1



Notes:

- ▷ Finally x_4 is feasible.
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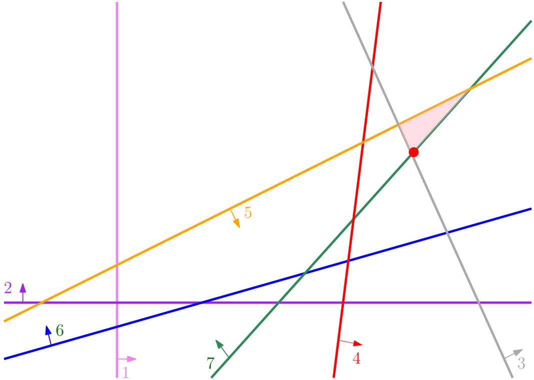
$\cdots \mapsto$

$x_4(j)$	x_7	-1		ratios
$-^*$	$-$	$-$	$= -x_3(\ell,p)$	$-/-$
$-$		$+$	$= -x_6$	N/A
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
			$= f$	

$\mapsto$

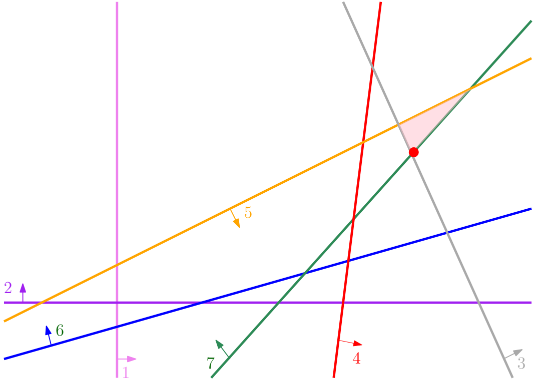
x_3	x_7	-1	
			$= -x_4$
			$= -x_6$
			$= -x_5$
			$= -x_1$
			$= -x_2$
			$= f$



Notes:

- Finally x_4 is feasible.
- $\ell = 1$.

$$
\cdots \mapsto
\begin{array}{cc|c|l l}
x_4(j) & x_7 & -1 & & \text{ratios} \\
\hline
-^{*} & - & - & = -x_3(\ell,p) & -/- \\
- & & + & = -x_6 & \text{N/A} \\
+ & & + & = -x_5 & +/+ \\
- & & + & = -x_1 & \text{N/A} \\
- & & + & = -x_2 & \text{N/A} \\
\hline
 & & & = f &
\end{array}
\mapsto
\begin{array}{cc|c|l}
x_3 & x_7 & -1 & \\
\hline
 & & + & = -x_4 \\
 & & + & = -x_6 \\
 & & + & = -x_5 \\
 & & + & = -x_1 \\
 & & + & = -x_2 \\
\hline
 & & & = f
\end{array}
$$



Notes:

- Finally x_4 is feasible.
- $\ell = 1$.
- Finally x_3 is feasible.

$\cdots \mapsto$

$x_4(j)$	x_7	-1		ratios
$-^*$	$-$	$-$	$= -x_3(\ell,p)$	$-/-$
$-$		$+$	$= -x_6$	N/A
$+$		$+$	$= -x_5$	$+/+$
$-$		$+$	$= -x_1$	N/A
$-$		$+$	$= -x_2$	N/A
			$= f$	

$\mapsto$

x_3	x_7	-1	
		$+$	$= -x_4$
		$+$	$= -x_6$
		$+$	$= -x_5$
		$+$	$= -x_1$
		$+$	$= -x_2$
			$= f$

BFS!

Phase 1 complete

Correctness of SA Phase 1

$$
\begin{array}{cccc|c|l}
 & \text{(ind var's)} & & & -1 & \\
\hline
a_{11} & a_{12} & \dots & a_{1n} & b_1 & \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & = -\text{(dep var's)} \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m & \\
\hline
c_1 & c_2 & \dots & c_n & d & = f \\
\hline
\end{array}
$$

Lemma

If the algorithm stops at Step 4., then LP is infeasible.

$$
\begin{array}{cccc|c}
\multicolumn{4}{c}{\text{(ind var's)}} & -1 \\
\hline
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\
\hline
c_1 & c_2 & \dots & c_n & d \\
\hline
\end{array}
\begin{array}{l}
\\ \\ \\ = -\text{(dep var's)} \\ \\ = f
\end{array}
$$

Lemma

If the algorithm stops at Step 4., then LP is infeasible.

Proof.

Assume $a_{i1}, a_{i2}, \dots, a_{in} \geqslant 0$ and $b_i < 0$. Suppose that $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ is feasible. Then each x_i is nonnegative so we have

$$0 \leqslant a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leqslant b_i,$$

This contradicts $b_1 < 0$ Therefore there is not feasible vector $\boldsymbol{x}$. □

Correctness of SA Phase 1

Lemma

Suppose Step 5 of Phase 1 changes $\boldsymbol{b} = (b_1, \dots b_m)$ *into* $\tilde{\boldsymbol{b}} = (\tilde{b}_1, \dots \tilde{b}_m)$. *Let*

$$\ell = \max\{i \,:\, b_i < 0\} \quad \textit{and} \quad \tilde{\ell} = \max\{i \,:\, \tilde{b}_i < 0\}.$$

Then

1. $b_\ell \leq \tilde{b}_\ell \leq 0$,
2. $\tilde{\ell} \leq \ell$.

Correctness of SA Phase 1

Lemma

Suppose Step 5 of Phase 1 changes $\boldsymbol{b} = (b_1, \ldots b_m)$ *into* $\tilde{\boldsymbol{b}} = (\tilde{b}_1, \ldots \tilde{b}_m)$. *Let*

$$\ell = \max\{i \,:\, b_i < 0\} \quad \textit{and} \quad \tilde{\ell} = \max\{i \,:\, \tilde{b}_i < 0\}.$$

Then

1. $b_\ell \leq \tilde{b}_\ell \leq 0$,
2. $\tilde{\ell} \leq \ell$.

Corollary

If Phase 1 does not terminate, then Step 5 is never changing the value of b_ℓ.

Summary of (Maximization) Simplex Algorithm

Phase 1:

a) If all rows are feasible, then go to Phase 2

b) Find the last infeasible row, ℓ

c) If row ℓ has no other $-$, then LP infeasible.

d) Choose a pivot column j with $-$ in row ℓ

e) Choose a pivot row i, with $\ell \leq i \leq m$, and having the least non-negative ratio*.

f) Pivot at (i, j) and go to step a)

Phase 2:

g) If last row is all ≤ 0, then optimal solution

h) Choose a positive entry as pivot column j

i) If col. j has no other $+$, LP is unbounded.

j) Choose a pivot row i with the least non-negative ratio*.

k) Pivot at (i, j) and go to step g).

*Treat $\frac{0}{a}$ as a "negative ratio" if $a < 0$.

$$\begin{array}{ccc|c|l}
x_1 & \dots & x_n & -1 & \\
\hline
 & & & & = -t_1 \\
 & & & & = \\
 & & & & = \\
 & & & & = \\
 & & & & = \\
 & & & & = -t_m \\
\hline
 & & & & = f \\
\hline
\end{array}$$

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$$\begin{array}{ccc|c|l}
x_1 & \dots & x_n & -1 & \\
\hline
 & & & \oplus & = -t_1 \\
 & & & \oplus & = \\
 & & & \oplus & = \\
 & & & \oplus & = \\
 & & & \oplus & = \\
 & & & \oplus & = -t_m \\
\hline
 & & & & \\
\hline
\end{array}$$

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$$
\begin{array}{ccc|c|l}
x_1 & \dots & x_n & -1 & \\
\hline
 & & & & = -t_1 \\
 & & & & = \\
 & & & - & = -t_\ell \\
 & & & \oplus & = \\
 & & & \oplus & = \\
 & & & \oplus & = -t_m \\
\hline
 & & & & \\
\hline
\end{array}
$$

Phase 1:

a) If all rows are feasible, then go to Phase 2

b) Find the last infeasible row, ℓ

c) If row ℓ has no other $-$, then LP infeasible.

d) Choose a pivot column j with $-$ in row ℓ

e) Choose a pivot row i, with $\ell \leq i \leq m$, and having the least non-negative ratio*.

f) Pivot at (i, j) and go to step a)

Phase 2:

g) If last row is all ≤ 0, then optimal solution

h) Choose a positive entry as pivot column j

i) If col. j has no other $+$, LP is unbounded.

j) Choose a pivot row i with the least non-negative ratio*.

k) Pivot at (i, j) and go to step g).

*Treat $\frac{0}{a}$ as a "negative ratio" if $a < 0$.

$$
\begin{array}{cccc|c|l}
x_1 & \oplus & \dots & x_n & -1 & \\
\hline
 & & & & & = -t_1 \\
 & & & & & = \\
\oplus & \oplus & \dots & \oplus & - & = -t_\ell \\
 & & & & & = \\
 & & & & & = \\
 & & & & & = -t_m \\
\hline
 & & & & & \\
\hline
\end{array}
$$

Summary of (Maximization) Simplex Algorithm

Phase 1:

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$$\begin{array}{ccc|c|l}
x_1 & x_j & x_n & -1 & \\
\hline
 & & & & = \\
 & & & & = \\
 & - & & - & = -t_\ell \\
 & & & \oplus & = \\
 & & & \oplus & = \\
 & & & \oplus & = -t_m \\
\hline
 & & & & \\
\hline
\end{array}$$

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k) Pivot at (i, j) and go to step g).

x_1	x_j	x_n	-1		ratios
				$=$	
				$=$	
	$-$		$-$	$= -t_\ell$	$\frac{-3}{-2}$
	$+$		$\oplus$	$=$	$\frac{1}{3}$
	$\ominus$		$\oplus$	$=$	N/A
	$+$		$\oplus$	$= -t_m$	$\frac{7}{2}$
				$= f$	

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Summary of (Maximization) Simplex Algorithm

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x_1	x_j	x_n	-1		ratios
				$=$	
				$=$	
	$-$		$-$	$= -t_\ell$	$\frac{-3}{-2}$
	$+^*$		$\oplus$	$= -t_i$	$\frac{1}{3}$
	$\ominus$		$\oplus$	$=$	N/A
	$+$		$\oplus$	$= -t_m$	$\frac{7}{2}$
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k) Pivot at (i, j) and go to step g).

$$
\begin{array}{ccc|c|l}
x_1 & x_j & x_n & -1 & \\
\hline
 & & & & = \\
 & & & & = \\
 & & & - & = -t_\ell \\
 & p^* & & \oplus & = -t_i \\
 & & & \oplus & = \\
 & & & \oplus & = -t_m \\
\hline
 & & & & = f \\
\hline
\end{array}
$$

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$$
\begin{array}{cccc|c|l}
x_1 & \cdots & & x_n & -1 & \\
\hline
 & & & & \oplus & = -t_1 \\
 & & & & \oplus & = \\
 & & & & \oplus & = \\
 & & & & \oplus & = \\
 & & & & \oplus & = -t_m \\
\hline
 & & & & & = f \\
\hline
\end{array}
$$

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$$
\begin{array}{cccc|c|l}
x_1 & \dots & & x_n & -1 & \\
\hline
 & & & & \oplus & = -t_1 \\
 & & & & \oplus & = \\
 & & & & \oplus & = \\
 & & & & \oplus & = \\
 & & & & \oplus & = -t_m \\
\hline
\ominus & \ominus & \dots & \ominus & & = f \\
\hline
\end{array}
$$

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$$
\begin{array}{ccc|c|l}
x_1 & \dots & x_n & -1 & \\
\hline
 & & & \oplus & = -t_1 \\
 & & & \oplus & = \\
 & & & \oplus & = \\
 & & & \oplus & = \\
 & & & \oplus & = -t_m \\
\hline
\ominus & + & + & & = f \\
\hline
\end{array}
$$

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$$
\begin{array}{cccc|c|l}
x_1 & x_j & x_n & & -1 & \\
\hline
 & & & & \oplus & = -t_1 \\
 & & & & \oplus & = \\
 & & & & \oplus & = \\
 & & & & \oplus & = \\
 & & & & \oplus & = -t_m \\
\hline
 & + & & & & = f \\
\hline
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$$
\begin{array}{ccc|c|l}
x_1 & x_j & x_n & -1 & \\
\hline
 & \ominus & & \oplus & = -t_1 \\
 & \ominus & & \oplus & = \\
 & \ominus & & \oplus & = \\
 & \ominus & & \oplus & = \\
 & \ominus & & \oplus & = -t_m \\
\hline
 & + & & & = f \\
\hline
\end{array}
$$

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x_1	x_j	x_n	-1		ratios
	$+$		$\oplus$	$= -t_1$	$\frac{8}{3}$
	$+$		$\oplus$	$=$	$\frac{4}{5}$
	$+$		$\oplus$	$=$	$\frac{1}{3}$
	$\ominus$		$\oplus$	$=$	N/A
	$+$		$\oplus$	$= -t_m$	$\frac{6}{5}$
	$+$			$= f$	

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g) If last row is all ≤ 0, then optimal solution

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j) Choose a pivot row i with the least non-negative ratio*.

k) Pivot at (i, j) and go to step g).

$$
\begin{array}{ccc|c|ll}
x_1 & x_j & x_n & -1 & & \text{ratios} \\
\hline
 & + & & \oplus & = -t_1 & \frac{8}{3} \\
 & + & & \oplus & = & \frac{4}{5} \\
 & +^* & & \oplus & = -t_i & \frac{1}{3} \\
 & \ominus & & \oplus & = & \text{N/A} \\
 & + & & \oplus & = -t_m & \frac{6}{5} \\
\hline
 & + & & & = f & \\
\hline
\end{array}
$$

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$$
\begin{array}{ccc|c|l}
x_1 & x_j & x_n & -1 & \\
\hline
 & & & \oplus & = -t_1 \\
 & & & \oplus & = \\
 & p^* & & \oplus & = -t_i \\
 & & & \oplus & = \\
 & & & \oplus & = -t_m \\
\hline
 & + & & & = f \\
\hline
\end{array}
$$

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SA for MT - used to illustrate next example

Algorithm (SA for MT)

$$
\begin{array}{c}
\text{(ind var's)} \qquad\qquad -1 \\
\begin{array}{|cccc|c|}
\hline
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\
\hline
c_1 & c_2 & \dots & c_n & d \\
\hline
\end{array}
\begin{array}{l}
\\ \\ = -\text{(dep var's)} \\ \\ = f
\end{array}
\end{array}
$$

1. *We have maximum Tucker tableau.*
2. *If* $b_1, b_2, \dots, b_m \geqslant 0$, *go to* **Step 6.**
3. *Choose* $b_i < 0$ *such that* i *is maximal.*
4. *If* $a_{i1}, a_{i2}, \dots, a_{in} \geqslant 0 \Longrightarrow$ **STOP**; *the problem is infeasible.*
5. *If* $i = m$, *choose* $a_{mj} < 0$, *pivot on* a_{mj}, *and go to* **Step 1.**
   If $i < m$, *choose* $a_{ij} < 0$, *compute*

   $$\alpha = \min(\{b_i/a_{ij}\} \cup \{b_k/a_{kj} : k > i, a_{kj} > 0\}),$$

   and choose any p *with* $b_p/a_{pj} = \alpha$. *Pivot on* a_{pj} *and go to* **Step 1.**
6. *Apply the SA for MBFT.*

First examples on SA for MT:

a)

1	3	2
-1	2	-1
1	-1	2
-1	5	-2

b)

1	3	2
1	2	-1
1	-1	5
-1	5	-2

c)

1	3	2
1	2	-1
1	-1	-5
-1	5	-2

d)

1	3	2
-1	2	-2
1	-1	1
-1	5	-2

SA for MT - used to illustrate next example

Algorithm (SA for MT)

$$
\begin{array}{c}
\begin{array}{cccc|c}
 & \text{(ind var's)} & & & -1 \\
\end{array} \\
\begin{array}{|cccc|c|}
\hline
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\
\hline
c_1 & c_2 & \dots & c_n & d \\
\hline
\end{array}
\begin{array}{l}
\\ \\ = -\text{(dep var's)} \\ \\ = f
\end{array}
\end{array}
$$

1. *We have maximum Tucker tableau.*
2. *If* $b_1, b_2, \dots, b_m \geqslant 0$, *go to* **Step 6.**
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   If $i < m$, *choose* $a_{ij} < 0$, *compute*

   $$\alpha = \min(\{b_i/a_{ij}\} \cup \{b_k/a_{kj} : k > i, a_{kj} > 0\}),$$

   and choose any p *with* $b_p/a_{pj} = \alpha$. *Pivot on* a_{pj} *and go to* **Step 1.**
6. *Apply the SA for MBFT.*

Apply the simplex algorithm to the maximum tableau:

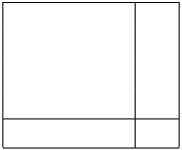
$$\begin{array}{cc|c|l} x_1 & x_2 & -1 & \\ \hline -1 & -2 & -3 & = -t_1 \\ 1 & 1 & 3 & = -t_2 \\ 1 & 1 & 2 & = -t_3 \\ \hline -2 & 4 & 0 & = f \\ \hline \end{array}$$

$$\begin{array}{cc|c|l} & & -1 & \\ \hline & & & = \\ & & & = \\ & & & = \\ \hline & & & = f \\ \hline \end{array}$$

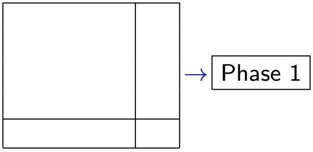
$$\begin{array}{cc|c|l} & & -1 & \\ \hline & & & = \\ & & & = \\ & & & = \\ \hline & & & = f \\ \hline \end{array}$$

$$\begin{array}{cc|c|l} & & -1 & \\ \hline & & & = \\ & & & = \\ & & & = \\ \hline & & & = f \\ \hline \end{array}$$

Simplex Algorithm: Two Phases, Three Outcomes

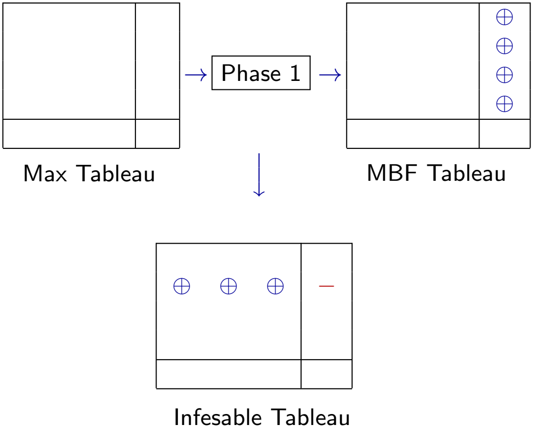


Max Tableau

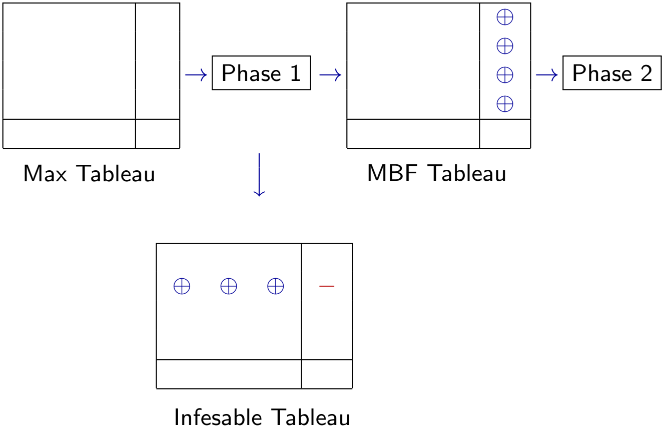


Max Tableau

Simplex Algorithm: Two Phases, Three Outcomes



Simplex Algorithm: Two Phases, Three Outcomes



Simplex Algorithm: Two Phases, Three Outcomes