MACM 201 - Discrete Mathematics

Graph Theory 2 - induction and Euler circuits

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Proofs by induction

Let's begin by considering an easy proof by induction.

Theorem

For every positive integer n, the sum of the first n odd numbers is n^2. So

$$(\star) \qquad \sum_{k=1}^{n}(2k-1) = n^2$$

To prove this by induction, we imagine doing the proof one number at a time. We first prove it for 1, then 2, 3, 4, $\ldots$ The verification splits into two parts:

Base case. Check that the theorem holds for $n = 1$

Inductive step. Prove that the formula holds for n under the assumption that it holds for $n-1$.

As you can see, if we can prove the base case and the inductive step, then the statement must hold true for every positive integer.

Proof of Theorem

We prove this theorem by induction on n.

Base Case. When $n = 1$ equation ($\star$) holds because $\sum_{k=1}^{1}(2k-1) = 1 = 1^2$.

Inductive Step. Let $n \geq 2$ and assume that equation ($\star$) is true for all numbers less than n. (We will prove it must then be true for n.) We have

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1) &= (2n-1) + \sum_{k=1}^{n-1}(2k-1) \\
&= (2n-1) + (n-1)^2 \qquad \text{by the inductive hypothesis} \\
&= (2n-1) + n^2 - 2n + 1 \\
&= n^2
\end{aligned}$$

Induction in more general settings

Suppose that you wish to prove that a certain property P holds for an infinite number of things. Here is an inductive way to achieve this:

Size. Suppose that each thing has a certain size. Assume (for simplicity) that the smallest possible size is 1. Instead of trying to prove property P holds for all things at once, we can break things up according to size. Define

$$P(n): \text{ property } P \text{ holds for all things of size } n$$

To prove that P holds by induction, it suffices to show the following:

Base Case. $P(1)$ is true (i.e. property P holds for all things of size 1)

Inductive Step. If $n \geq 2$ and $P(1), P(2), \ldots, P(n-1)$ are true, then $P(n)$ is true. (i.e. if P is true for all things of size $< n$ then P is also true for all things of size n.)

As is apparent, if the base case and inductive step have been proven, then the result holds in general.

Notes about induction

(1) The smallest possible size might be 0 or 2 or something else.

(2) In general there may be **many** things of size n.

(3) It is possible that the base case needs to handle more than just the smallest size. In general the base case must take care of all of the instances that are not covered by the inductive step.

Written format When writing a proof by induction you don't need to explicitly write out the expression $P(n)$. You state the property you will prove and indicate that you are going by induction on "size" (you get to choose what size to use!). Your proof will have a base case to handle the smallest size (or sizes) and an inductive step.

Alert! When writing the inductive step, you must take an arbitrary object of size n and then show that it satisfies your property assuming that your property holds for all smaller size objects. Do not start with an object of size $n-1$ and build up, start with size n and go down.

Theorem

For every multigraph $G = (V, E)$ *we have*

$$2|E| = \sum_{v \in V} \deg(v)$$

Proof.

continued

Theorem

Every connected graph $G = (V, E)$ with $|V| \geq 2$ has at least two vertices v_1, v_2 so that $G - v_1$ and $G - v_2$ are both connected.

Proof.

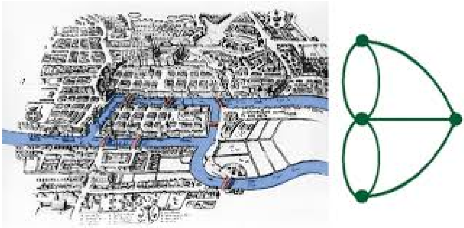
continued

Lemma

If $G = (V, E)$ is a graph and $\deg(v) \geq 2$ holds for every $v \in V$, then G contains a cycle.

Proof.

The bridges of Konigsberg



Question: Is it possible to walk back and forth across the bridges so that each bridge is crossed exactly once?

Definition

An **Euler circuit** of a graph $G = (V, E)$ is a walk

$$W = v_1, e_1, v_2, e_2, \ldots, v_k$$

with $v_1 = v_k$ (so the walk is closed) and the property that every edge appears exactly once in W (i.e. for every $e \in E$ there is exactly one index $1 \leq i \leq k-1$ so that $e = e_i$).

Theorem (Euler)

A connected graph has an Euler circuit if and only if every vertex has even degree.

Proof.

continued