Group Coursework Submission Form

Specialist Masters Programme

Please list all names of group members: (Surname, first name) 1. Gupta Mayankita 2. Laryea Grace 3. Lin Yuanbin Gloria	4. Yao Muyan **GROUP NUMBER:** 2
MSc in: Quantitative Finance, Financial Mathematics	
Module Code: SMM272	
Module Title: RISK ANALYSIS	
Lecturer: GIANLUCA FUSAI	**Submission Date:** 18th MARCH,2024

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Risk Analysis Report

Group 2: Gupta Mayankita, Laryea Grace, Lin Yuanbin Gloria, Yao Muyan

March 18, 2024

1 Question 1

This section elaborates on the statistical analysis of the daily log returns of the following stocks: JP Morgan Chase (JPM), Intel Corp (INTC), Alcoa Corp (AA), Procter & Gamble Co (PG), and Microsoft (MSFT), as well as the equally weighted portfolio labelled with a 50:50 allocation from January 1st, 2014, to December 31st, 2023.

1.1 Statistical Analysis of Stock Prices and Returns

From Figure 1, we observe an upward-sloping trend over the 9 years, indicating gains from investing in this stock. MSFT showed the largest price dispersion among the four stocks, while JPM had the smallest average price.

Figure 1: Plot of the closing prices.

Table 1 summarises the basic statistics of each of the stocks, with the mean log return ranging from 0.01% to 0.10% and variance from 0.01% to 0.11%. The minimum return ranges from-23.66% to -9.14%, with the largest spread observed in PG compared to the second smallest value of -15.95% for MSFT. Based on Table 1, we also observe that all our stock returns are leptokurtic, indicated by their Kurtosis > 0. Moreover, in terms of skewness, all values except AA showed left-skewed distributions.

Ticker	Minimum	Maximum	Q1	Q2	Q3	Skewness	Kurtosis	Mean	Variance
INTC	-19.90%	17.83%	-0.90%	0.08%	1.03%	-0.523	14.76	0.04%	0.04%
JPM	-16.21%	16.56%	-0.73%	0.05%	0.84%	-0.21	8.35	0.05%	0.03%
AA	-23.66%	24.86%	-1.71%	0.00%	1.73%	0.0023	15.21	0.01%	0.11%
PG	-9.14%	11.34%	0.48%	0.06%	0.60%	-0.069	16.67	0.04%	0.01%
MSFT	-15.95%	13.29%	-0.68%	0.09%	0.98%	-0.15	11.14	0.10%	0.03%
50:50	-14.60%	13.07%	-0.65%	0.08%	0.77%	-0.53	16.48	0.05%	0.02%

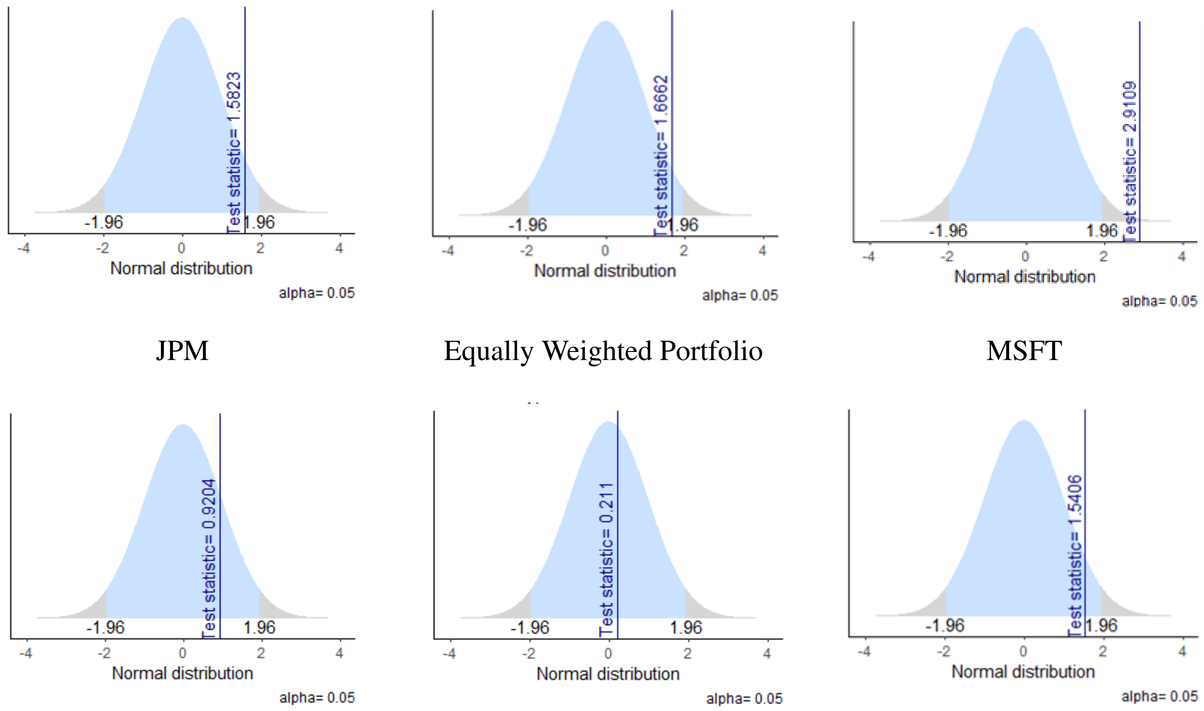
Table 1: Descriptive statistics of log returns of stocks.

Further analysis included conducting a Ljung-Box test to check for evidence of serial correlation in squared log returns and log returns. All stocks exhibited serial correlation in their squared log returns, and all except stock AA showed serial correlation in their log returns. Serial correlation, sometimes referred to as serial dependence in finance, explains the linear relationship of how past observations influence future observations.

In order to determine whether the mean of our log returns significantly differs from 0 at a 95% confidence interval, we use a two-tailed t-test at a 5% significance level. In MATLAB, we use the `ttest()` function, while in R, we use the `ggttest(t.test())` function to plot the t-statistic.

$$H_0 : \text{True mean is equal to } 0$$
$$H_1 : \text{True mean is not equal to } 0$$

After conducting the one-sample t-test, we found that INTC, JPM, AA, PG, and the equally weighted portfolio are statistically insignificant to 0, indicating that their true mean is equal to 0. Conversely, MSFT's true mean is statistically significant compared to 0 (MarinStatsLectures [9]).



JPM Equally Weighted Portfolio MSFT

INTC AA PG

Figure 2: One-sample T-test results for the stocks and portfolio.

After testing the significance of the stock log returns, we proceeded to identify the model that best fits our log returns. Understanding how returns behave is crucial in risk management. The procedure was conducted in Python using the automated and machine learning package, `distfit()`. The `distfit()` function operates based on a goodness-of-fit test, wherein it assesses the RSS value across multiple distributions and selects the one with the smallest RSS as the best fit.

Based on the test results, all stocks and the equally weighted portfolio were best fitted by a T distribution, except for MSFT, which was best fitted by a Weibull distribution. This is understandable, as one dress size does not fit all individuals. Plots of the fitted distributions can be found in the appendix (Figure A1 to A6). Furthermore, given that all stocks and the equally weighted portfolio were best fitted by a T distribution, it may be advisable to also fit a skewed generalized t-distribution to these variables. This can be accomplished in R using the 'sgt' package.

1.2 VaR at 90% and 99% Confidence Levels

This section uses three different models to forecast the daily VaR (Value at Risk) of the equally weighted portfolio over a 180-day rolling window. Each stock in the portfolio carries a weight of 20%. We consider 30 days equivalent to one month, thus 180 days represents 6 months. VaR indicates the maximum loss a portfolio can incur at a certain significance level. Using the VaR models, we define a violation as a loss exceeding the current VaR. We then conduct a backtest of the equally weighted portfolio using various models, as detailed in the following sections. The methods we used are based on MathWorks [12], MathWorks [10], Nieppola [13], and Danielsson [3].

1.2.1 EWMA - Daily VaR at 90% and 99% Confidence Level

We forecasted our first VaR using the exponentially weighted moving average model as per JP Morgan's Riskmetrics system. It was simulated with an optimal decay factor of φ = 0.94, which is used for daily frequency data by Riskmetrics. The optimal decay factor ensures that correlations remain within the range of -1 to 1.

We used this model due to the (Christoffersen [2]),

1. Greater importance to recent observations.
2. Easy to implement, as it relies on one parameter.
3. More robust to estimation error.

$$\sigma_n^2 = \varphi\sigma_{n-1}^2 + (1-\varphi)\mu_{n-1}^2$$
$$\sigma_n^2 = 0.94\sigma_{n-1}^2 + (1-0.94)\mu_{n-1}^2$$

Where σ is the variance and μ is the mean, φ = 0.94 as per risk metrics.

In the EWMA model, covariances and variances are estimated using an exponential weighting scheme. Figure 3 illustrates the estimation of VaR for the EWMA model, showing that the two tend to follow the same pattern. However, the 99% VaR exhibits more dispersion compared to the 90% VaR.

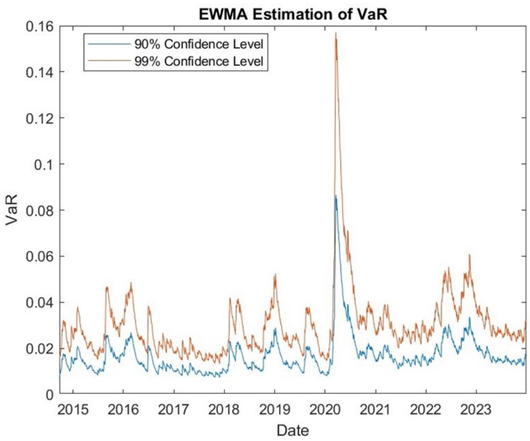


Figure 3: Plot of EWMA estimation of VaR.

1.2.2 Gaussian - Daily VaR at 90% and 99% Confidence Level

The Gaussian model assumes that all our daily VaR values follow a normal distribution, $N \sim (\mu, \sigma^2)$. Based on this assumption, we simulated the probabilities using the `norminv()` function and used them to calculate our VaR values for both the 90% and 99% confidence intervals (Nieppola [13]).

From Figure 4, we observe that the 90% confidence interval VaR generally falls within the 99% VaR.

Additionally, they tend to follow the same trend, but the 99% VaR exhibits wider dispersion compared to the 90% VaR. Based on the 99% confidence interval, we find the maximum VaR to be approximately 0.075, while for the 90% confidence interval, it is around 0.035 (MathWorks [12]).

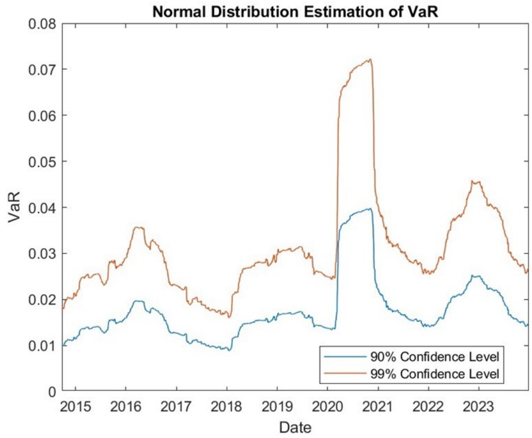


Figure 4: Plot of normal distribution estimation of VaR.

1.2.3 Historical - Daily VaR at 90% and 99% Confidence Level

Historical VaR is calculated based on the percentile formulation since the VaR Backtesting model does not have a predefined distribution, using both 90% and 99% confidence levels. The percentile formulation entails arranging data in ascending order, plotting a histogram, selecting the observation, and interpolating to determine the corresponding VaR value. Further details can be found in the MATLAB file and the reference link (Danielsson [3]).

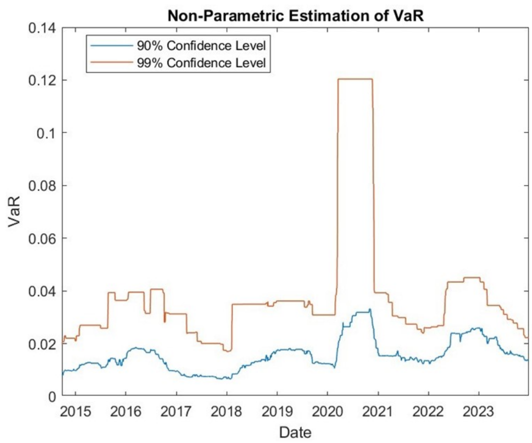


Figure 5: Plot of non-parametric estimation of VaR.

1.3 VaR Violations and Forecasts

After running the various VaR models, we compared the number of violations obtained. As seen in Table 2 using the command `exceptions()` in MATLAB on our data produced from `varbacktest()`.

VAR Forecast	Confidence Level	
	$90\%, \max = 233.3$	$99\%, \max = 23.3$
	Violations	Violations
EWMA	221	48
Gaussian	206	53
Historical	249	33

Table 2: Number of violations in VaR models.

Based on the results in Table 2, the 90% CI VaR models exhibit more violations compared to the 99% CI models, with the 90% Historical VaR identifying the most violations. This aligns with our expectations, as the Historical VaR, with its non-parametric nature and less strictness, is expected to identify the most violations when losses exceed the VaR level. The methods we used are based on MathWorks [12], MathWorks [10], and Nieppola [13].

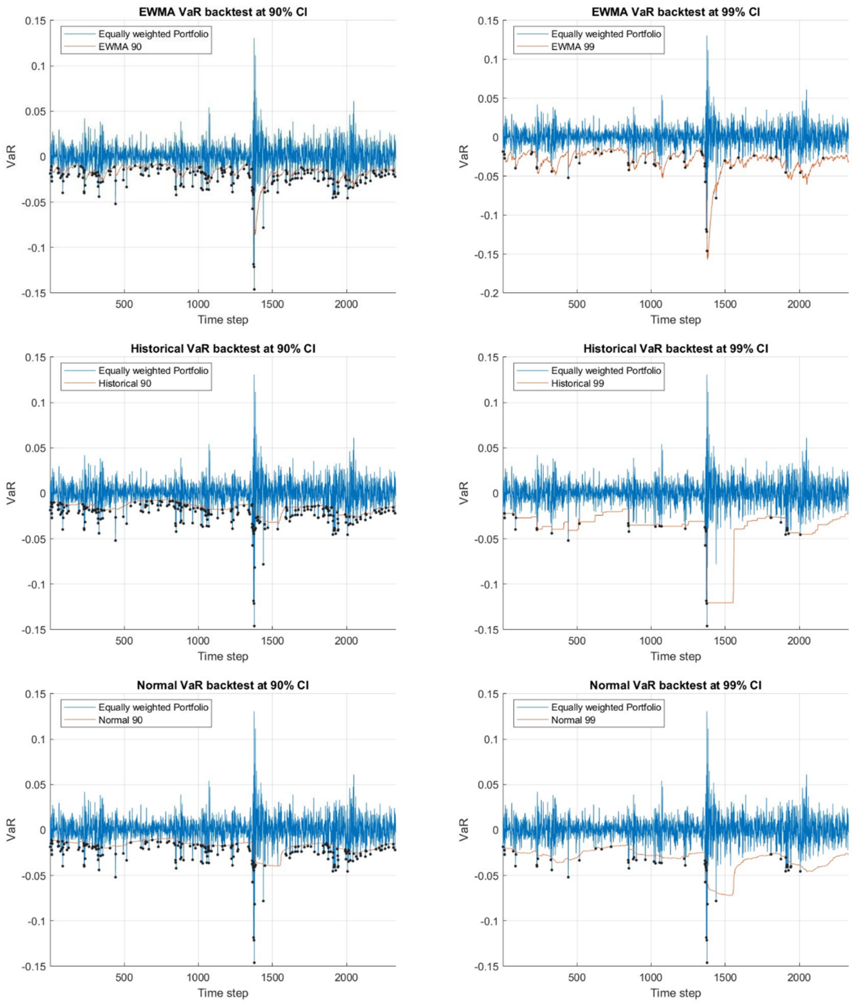


Figure 6: EWMA, Historical, and Normal VaR backtesting; (L): 90% confidence interval; (R): 99% confidence interval.

1.4 Unconditional Coverage Test (Kupiec Test)

This test examines the number of violations without considering whether violations are independent of each other. Violations in this model are modelled as a Bernoulli binomial random variable with a theoretical probability of $1 - \alpha$ and an estimated probability of $1 - \hat{\alpha}$ (Jorion [7]).

$$H_0 : \hat{\alpha} = \alpha, \text{ risk model is correct on average}$$
$$H_1 : \hat{\alpha} \neq \alpha, \text{ risk model is inaccurate on average}$$
$$LR_{\text{POF}} = -2 \ln \left(\frac{(1-\alpha)^j (\alpha)^{n-j}}{(1-\hat{\alpha})^j (\hat{\alpha})^{n-j}} \right) \sim \chi^2_{1,0.95}$$

Where j = number of exceptions, n = number of observations.

We used MATLAB's built-in functions, `varbacktest()` and `pof()`, for this model. The Kupiec test has two variants: Time Until First Failure (TUFF) and Proportion of Failures (POF). We opted to conduct the POF test, which assesses the proportion of failures, as opposed to TUFF, which focuses solely on the timing of the first failure, potentially omitting relevant information. The equation for the likelihood ratio is provided below.

$$X_1^2 = 3.841 \text{ at } 95\% \text{ probability level}$$

To identify whether our VaR models are well calibrated, we compare our test statistics or log-likelihood ratios with the critical value, as shown below. Based on this we can accept our null hypothesis.

$$LR(j, n, 1 - \frac{j}{n}, \alpha) \leq \chi^2_{1,0.95}$$

As seen in Table 3, all the test statistics for models at the 90% confidence interval are less than the critical value. Therefore, we accept the models. However, for models at the 99% confidence interval, we reject both the EWMA and Gaussian models and fail to reject the Historical model.

VAR Forecast ($n = 2333$)	Confidence Level			
	90%, max $= 233.3$		99%, max $= 23.3$	
	LR	Max violation	LR	Max violation
EWMA	0.7321	Accept	20.185	Reject
Gaussian	3.6804	Accept	28.021	Reject
Historical	1.1513	Accept	3.5872	Accept

Table 3: Test statistics and results of Kupiec test.

According to Chapter **Backtesting and Stress Testing** in Christofferson's book, a higher value of LR_{POF} indicates a lower likelihood of the null hypothesis being true. This trend is evident in Table 3, where both the EWMA and Gaussian models are rejected at the 99% confidence level. The Kupiec test exhibits a high level of Type I or Type II errors, depending on the chosen significance level. Increasing the significance level results in larger Type I errors and smaller Type II errors. Therefore, to be safe we kept the significance level at 95% compared to 90% and 99%.

1.5 Serial Dependence

If our models have a serial dependency, the conditional coverage independent test devised by Christoffersen (Christoffersen [2]) can be used. Our task is to establish a test that will be able to reject a VaR with clustered violations. For this purpose, we assume that the hit sequence exhibits temporal dependence and can be characterized as a first-order Markov sequence with a transition probability matrix. Based on our test results from Christoffersen, we reject all models, indicating the absence of serial dependence in our VaR models. Further details of the tests are provided in the next section.

Based on Ballotta and Fusai, "Indeed, in the EWMA model only 1-period returns are Gaussian. Moreover, they are uncorrelated, but not independent, because they are characterized by a serial dependence through the variance dynamics." (Ballotta and Fusai [1]).

1.6 Conditional Coverage Independence Test (Christoffersen Book) & Distributional Test

The CCI test is conducted in MATLAB using the `cci(VaR_Backtest)` function, and a summary can be generated using the `varbacktest()` function.

Christoffersen's Interval Forecast Tests
Christoffersen [2] proposed a test to measure whether the probability of observing an exception on a particular day depends on whether an exception occurred, i.e., test that exceptions/violations are serially independent (Ballotta and Fusai [1]). Unlike the unconditional probability of observing an exception, Christoffersen's test measures the dependency between consecutive days only.

$$H_0 : \text{VaR violations do not follow a Markov property}$$
$$H_1 : \text{VaR violations follow a Markov property}$$
$$\text{LR}_{\text{cci}} = -2\ln\left(\frac{(1-\pi)^{n_{00}+n_{10}}\pi^{n_{01}+n_{11}}}{(1-\pi_0)^{n_{00}}\pi_0^{n_{10}}(1-\pi_1)^{n_{10}}\pi_1^{n_{11}}}\right) \sim \chi_1^2$$

Where $\pi_0 = \frac{n_{01}}{n_{00}+n_{01}}, \pi_1 = \frac{n_{11}}{n_{10}+n_{01}}, \pi = \frac{n_{01}+n_{11}}{n_{00}+n_{01}+n_{10}+n_{11}}$,
$n_{00} =$ number of periods with no violation followed by a period with no violation,
$n_{01} =$ number of periods with no violation followed by a period with violation,
$n_{11} =$ number of periods with a violation followed by a period with a violation,
$n_{10} =$ number of periods with violation followed by a period with no violation,
$\pi_0 =$ probability of having a violation on period t, given no failure occurred at period $t-1$,
$\pi_1 =$ probability of having a violation on period t, given failure occurred at period $t-1$,
$\pi =$ probability of having a violation on period t.

All the models have been rejected based on the CCI (Conditional Coverage Independence) test. Based on this, we can conclude that all the VaR models adhere to the Markov property. Below is the table showing the test statistics and the critical values (Christoffersen [2] and MathWorks [11]).

VAR Forecast (n = 2333)	Confidence Level			
	$90\%, \max = 233.3$		$99\%, \max = 23.3$	
	LR	Violation	LR	Violation
EWMA	9.9592	221	8.09035	48
Gaussian	12.19	206	18.59	53
Historical	9.8974	249	6.5024	33

Table 4: Test statistics and results of CCI test.

Moreover, since all models have been rejected by Christoffersen, we can apply the Distributional tests. These tests are based on the entire distribution, unlike Christoffersen and Kupiec tests, which focus on violations or exceptions. This broader approach provides a more comprehensive view of the forecasted outcomes of losses in our trade/portfolio (Fusai [5]).

The hypothesis for the distributional tests is,

$$H_0 : \widetilde{\mathrm{p}_{t+\Delta}} \sim \text{ i.i.d. and } \mathrm{U}(0,1)$$
$$H_1 : \widetilde{\mathrm{p}_{t+\Delta}} \text{ not i.i.d. and } \mathrm{U}(0,1)$$

Based on this, we can run a distributional test for our Gaussian VaR model, since it follows a $N(0, \sigma)$.

1. Find the mean and volatility of the sample.
2. Create a CDF forecast $F_t(x) = \Phi_{\mu,\sigma}(x)$, where x is a return value you choose.
3. Solve the probability transform.

Based on the Historical VaR, as slightly different approach is used, as model is built on percentiles.

1. Re-arrange data of returns in ascending order.
2. Create a CDF based on the percentiles, as this will be used to implement the Berkowitz statistic.
3. Set the return, you want to run the distributional test on, it can be any value in or outside the sample.
4. Find where your chosen return falls, it should be between 2 values.
5. Use linear interpolation to calculate the probability based on the CDF in step 2 .

For the EWMA VaR model, we utilize a method similar to the Historical VaR distributional test. However, it's important to note that our returns follow an exponential distribution.

If the probability of the chosen return is independently and identically distributed (i.i.d.) over as a Uniform (0,1), we can conclude our VaR model is accurate based on the distributional test. One way to test for iid behaviour is by plotting the quantile-quantile (Q-Q) plots which should be a $45°$ line.

2 Question 2

In this section, our dataset was divided into two equal parts: the first part covers January 2nd, 2014, to January 2nd, 2018, and the second part from January 3rd, 2019, to December 29th, 2023. The first part of the data was used for constructing the risk parity portfolio, while the second part was used for comparing the risk-parity portfolio with the equally weighted portfolio. Additionally, we used the DSG1 dataset, which provides daily percentage quoted prices of the yearly yield. To obtain the daily yields, we divided by 25,000, assuming 250 days and 100 percent.

2.1 Building the Risk Parity Portfolio (1st Half)

By creating a risk parity portfolio, we aim to enhance portfolio optimization by allocating optimal weights across all stocks (INTC, JPM, AA, PG, MSFT), ensuring that each stock contributes equally to the VaR of the portfolio. To achieve this, we begin by computing the daily log returns of the stocks (Table 5) and constructing a covariance matrix including all stocks. Subsequently, we start with an equally weighted portfolio and proceed to determine the covariance weight and Component VaR for each asset. These CVaRs combine to form 100%, representing the aggregate VaR of the portfolio. The distribution of CVaRs for each stock is illustrated in Figure 7, with JPM accounting for 17%, AA for 34%, INTC for 21%, PG for 8%, and MSFT for 19%.

	JPM	AA	INTC	PG	MSFT
JPM	0.0173%	—	—	—	—
AA	0.0130%	0.0638%	—	—	—
INTC	0.0091%	0.0130%	0.0241%	—	—
PG	0.0037%	0.0043%	0.0044%	0.0086%	—
MSFT	0.0093%	0.0111%	0.0131%	0.0049%	0.0213%

Table 5: Variance-covariance matrix.

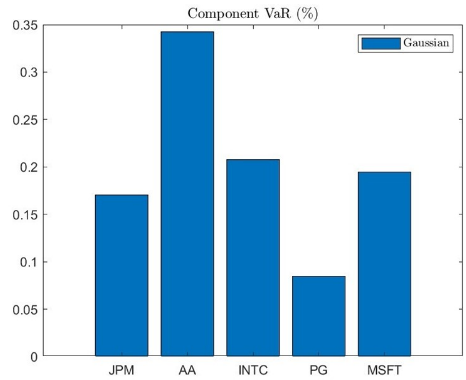


Figure 7: Distribution of CVaRs.

Next, we assess deviations from the ideal case where each stock contributes proportionately to the portfolio's VaR. Leveraging these deviations, we refine the portfolio for optimization. The weight distribution of each stock within the risk parity portfolio is presented in Figure 8: AA accounts for 12%, INTC for 17%, JPM for 20%, MSFT for 18%, and PG for 30%. Figure 8 also illustrates the weight distributions in the equally weighted portfolio and the minimum variance portfolio.

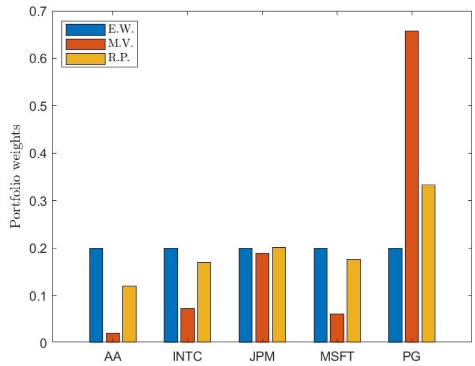


Figure 8: Portfolio weights of equally weighted (EW), minimum variance (MV) and risk parity (RP) portfolios.

2.2 Building the Equally Weighted Portfolio (2nd Half)

By creating an equally weighted portfolio, our aim is to enhance portfolio optimization by allocating optimal weights across all stocks (INTC, JPM, AA, PG, MSFT), ensuring that each stock contributes equally to the VaR of the portfolio. To achieve this, we begin by computing the daily log returns of the stocks and constructing a covariance matrix including all stocks. Subsequently, we start with an equally weighted portfolio, assigning each stock a weight of 20%, and the average daily log returns as shown in Table 6 and 7.

Summary Stats	JPM	AA	INTC	PG	MSFT	DGS1	Portfolio
Mean	0.06%	0.01%	0.01%	0.05%	0.10%	0.17%	0.05%

Table 6: Average daily log returns.

Matrix	JPM	AA	INTC	PG	MSFT
JPM	0.040%	0.043%	0.022%	0.010%	0.018%
AA	0.043%	0.152%	0.034%	0.010%	0.023%
INTC	0.022%	0.034%	0.061%	0.012%	0.028%
PG	0.010%	0.010%	0.012%	0.018%	0.012%
MSFT	0.018%	0.023%	0.028%	0.012%	0.037%

Table 7: Variance-covariance matrix.

Thus, we proceeded to determine the covariance weight and Component VaR for each asset. These CVaRs combine to form 100%, representing the aggregate VaR of the portfolio. The distribution of CVaRs for each stock is as follows: JPM accounts for 18%, AA for 36%, INTC for 21%, PG for 9%, and MSFT for 16%.

2.3 Comparison Between the Two Portfolios

To compare the risk parity (RP) portfolio and the equally weighted (EW) portfolio daily log returns, we subsequently determine the expected return for each portfolio. For the risk parity portfolio, the expected return is 0.051487%, whereas for the equally weighted portfolio, it is 0.04637%. This comparison indicates that the risk parity portfolio potentially yields greater returns compared to the equally weighted portfolio.

2.3.1 Performance Measures

For a more comprehensive comparison between the two portfolios, we analyze from three key aspects: the Sharpe ratio, maximum drawdown, and VaR, as well as checking the number of VaR violations at a 95% confidence level.

2.3.2 Sharpe Ratio

We use the Sharpe ratio formula:

$$\text{Sharpe Ratio} = \frac{R_x - R_f}{\sigma_x}$$

To calculate the ratios for both portfolios, we first need to determine R_f, which represents the risk-free rate of the period. We obtained this rate from a 1-year treasury bill sourced from the FRED Louis database, using the ticker DGS1 for calculation. The average daily R_f yield in this case is 0.01% for the 2nd half and 0.08% for the 1st half.

R_x represents the expected annual portfolio return. From the previous section, we already have the daily log return, making it easy to convert to the expected annual portfolio return. For the RP portfolio, it is 14.643%, and for the EW portfolio, it is 14.297%. We also need to compute the standard deviation of the portfolios' returns. This can be accomplished by calculating the standard deviation of each stock and then determining the portfolio's standard deviation by considering the relative weight of each stock.

The Sharpe ratio for the EW portfolio is 2.22, and for the RP portfolio, it is 4.58. Both portfolios have high Sharpe ratios, greater than 2, but the RP portfolio outperforms the EW portfolio by a factor of two. This demonstrates that the RP portfolio has delivered better risk-adjusted returns compared to the equally weighted portfolio.

2.3.3 Maximum Drawdown

The next performance measure is the maximum drawdown. Maximum drawdown refers to the largest peak-to-trough decline in the value of a portfolio or investment over a specific period, typically expressed as a percentage. It measures the worst possible loss an investor would have experienced if they had invested at the peak and sold at the lowest point before a new peak is achieved. For the EW portfolio, it is 27.67%, occurring between March 13 to March 16, 2020, and for the RP portfolio, it is 8.91%, occurring between August 24, 2015, to June 20, 2016. This shows that the RP portfolio is significantly better than the EW portfolio in terms of maximum drawdown, demonstrating its stability.

2.3.4 VaR Violations at 95% Confidence Level

The last performance measure is the VaR violations at 95% confidence level. The VaR for the equally weighted portfolio is 0.0282, whereas for the risk parity portfolio, it is 0.0076, which is significantly lower compared to the equally weighted portfolio. This is further demonstrated in Table 8, indicating that the risk parity portfolio remains a better investment compared to the equally weighted portfolio due to its substantially lower VaR.

Backtest Model	Number of Violations	
	Equally weighted portfolio (n=1383)	Risk Parity portfolio (n=1248)
EWMA	60	56
Gaussian	62	58
Historical	74	69

Table 8: Violations of VaR.

Based on the VaR backtesting, we observe a slight difference in the number of violations, with the Equally Weighted portfolio having more violations compared to the Risk Parity portfolio. This indicates a higher likelihood of losses. Additionally, it's important to note that the number of observations is not equal due to data cleaning.

In general, the RP portfolio outperforms the equally weighted portfolio in all three performance benchmarks, as the risk-parity approach aims to equalize each asset's risk contribution where the risks are not perfectly correlated. Therefore, the weights refer to risk rather than the dollar amount invested in each asset, with greater weights allocated to assets with lower risk and vice versa. By applying the risk-parity approach, greater diversification can be achieved not only through the types of assets included but also through the risks encompassed in the entire portfolio. Consequently, with greater portfolio diversification, higher performance measures can be achieved compared to the equally weighted portfolio. However, other performance measures can be used to assert the superiority of the risk-parity approach over these two portfolios, such as the Sortino ratio or the CAGR. It should also be noted that some criticisms of the risk parity approach have emerged. It has been pointed out that the risk-parity approach focuses only on risk without considering the assets' return and uses volatility as the sole measure of risk, effectively excluding any credit-related risks.

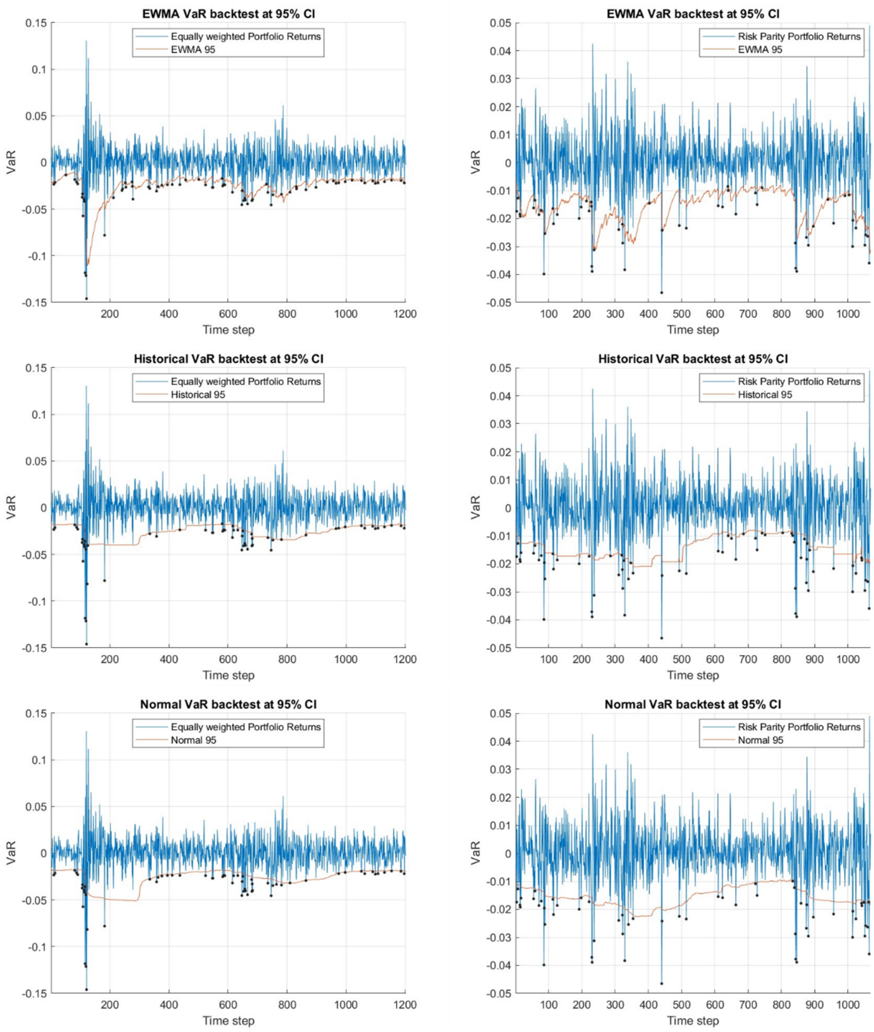


Figure 9: EWMA, Historical, and Normal VaR backtesting; (L): equally weighted; (R): risk parity.

3 Question 4

3.1 Var of a Portfolio with Options

In this section, we explain how a portfolio of 4 stock options affects each other, as well as detailing their Expected Shortfall, Value at Risk, and Black-Scholes Price of their options. The analysis considered data from 4 stocks: INTC, JPM, AA, PG, over a period of 2 years, from February 22, 2022, to February 21, 2024. The details are shown in Table 9.

The portfolio consists of long positions in 6 JPM stock put options, and 6 AA call stock options as well as a short positions in 3 INTC stock call options and 2 PG stock put options. As we learned from lectures, a short position in a call option has unlimited downside and limited upside, which is received as premium. This also applies to the long put option in JPM and INTC. Conversely, by longing the call option, the buyer has limited downside and unlimited upside profit potential, based on the intrinsic value at expiry. This principle also applies to the short-put options in PG and AA.

Over the 2 years, there was no particular trend in the stock prices for PG, while AA showed an upward trend. PG and INTC exhibited constant price movements due to their low volatilities, as observed in their variance-covariance matrix. Moreover, this could be attributed to recent market events in the global economy, such as the recovery from the COVID-19 pandemic, UK Treasury troubles, the collapse of Credit Suisse, and the Ukrainian war.

	JPM	AA	INTC	PG
Stock Price (S_0)	\$180.90	\$27.86	\$43.47	\$160.40
K Rate	$S0$	1.05(S0)	0.9(S0)	1.1(S0)
K	\$180.90	\$29.25	\$39.12	\$176.44
Maturity (T)	6 months	12 months	9 months	9 months
Option	Puts	Calls	Calls	Puts
Position	Long	Long	Short	Short

Table 9: Details of underlying stocks and options 21st February 2024.

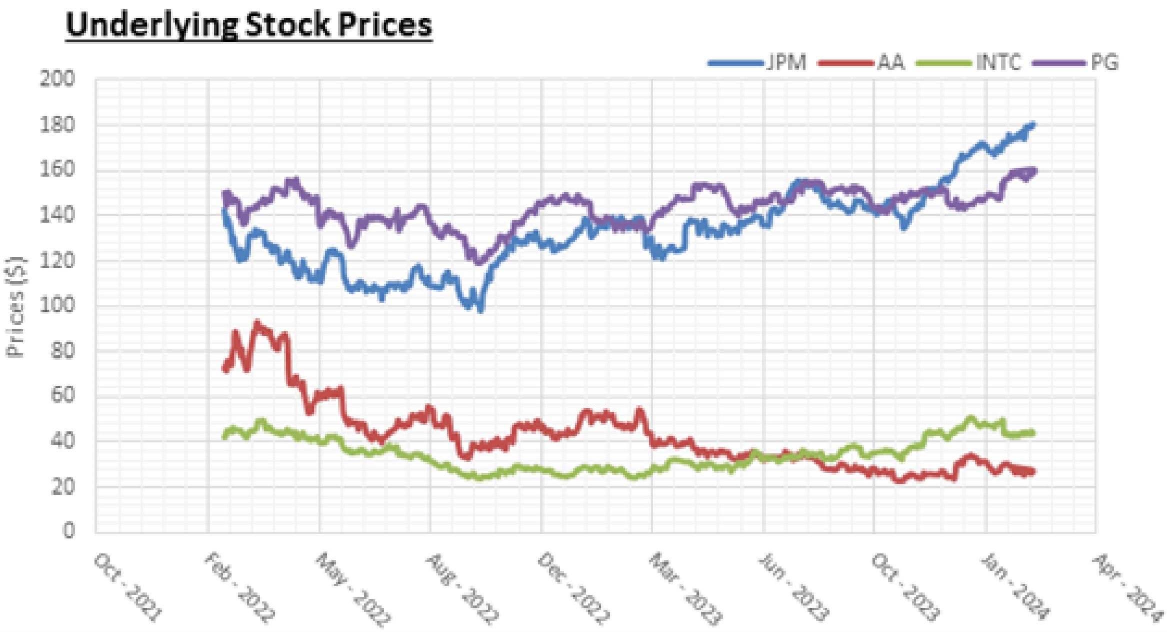


Figure 10: Underlying stock prices.

3.2 Pricing of Stock Options Using Black-Scholes Model

As we need to calculate the Black-Scholes price of the stock options, the variance-covariance matrix was computed to estimate the annualized historical volatility, σ, of the stocks. Moreover, we observe that there are no negative covariances, indicating that all our stocks have no negative relationship with each other. Their daily historical variances seem quite low as they are all below 1%.

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,\mathbf{1}} & \cdots & \sigma_n^n \end{bmatrix}$$

Variance-Covariance	JPM	AA	INTC	PG
JPM	0.024%	-	-	-
AA	0.025%	0.141%	-	-
INTC	0.014%	0.028%	0.059%	-
PG	0.006%	0.008%	0.008%	0.014%

Table 10: Variance-covariance matrix of stocks.

Based on the variance-covariance matrix in Table 10, we now estimate the annualised volatility $\left(\sigma_{\Delta}^{2}\right)$ by, assuming a total trading day of 250 days and $\sqrt{(\sigma^{2}xn)}$ as σ_{Δ}^{2}. The details are shown in Table 11.

Ticker	JPM	AA	INTC	PG
σ_{Δ}^{2}	0.5949	0.1852	0.2436	0.3856

Table 11: Annualized volatility of stocks.

To price our stock options, we use the Black-Scholes formula as detailed below,

$$
\begin{aligned}
C(S,t) &= S_{t}N\left(d_{1}\right) - N\left(d_{2}\right)Ke^{-r(T-t)} \\
P(S,t) &= N\left(d_{2}\right)Ke^{-rT} - N\left(d_{1}\right)S_{t} \\
d_{1} &= \frac{\ln\left(\frac{S_{t}}{K}\right) + \left(r + \frac{\sigma_{\Delta}^{2}}{2}\right)(T-t)}{\sigma_{\Delta}^{2}\sqrt{T-t}} \\
d_{2} &= d_{1} - \sigma_{\Delta}\sqrt{T-t}
\end{aligned}
$$

With assumptions that (Macroption [8]),

1. No dividends are paid in the life span of option.
2. Implied volatility is equal to the annualised historical volatility.
3. No transaction costs in buying the option.
4. Risk free rate of 4% and volatility is constant.
5. Stock prices are all stochastic and follow a random walk, etc.

Based on these formulas and assumptions, we obtain the following exact Black-Scholes prices as shown in Table 12.

Stocks	AA	INTC	JPM	PG	Portfolio
BS Price	\$6.42	\$8.58	\$10.58	\$16.88	\$42.52

Table 12: Black-Scholes prices of stocks.

From this, we calculate the value of our portfolio based on the number of positions held in our option portfolio: long positions in 6 JPM stock put options, and 6 AA call stock options as well as short positions in 3 INTC stock call options and 2 PG stock put options. This calculation yields a value of \$42.52 (numerical solution).

3.3 P&L Gaussian Distribution of Individual Stocks and Portfolio

However, since we are estimating our Value at Risk and Expected Shortfall, we need an accurate measure to compare against our numerically solved value. Therefore, we use Monte Carlo simulation to obtain a more robust and consistent value. This approach provides an accurate measure to compare against our numerically solved value. Based on this approach, we assume that our returns are normally distributed with $\mu = 0$ and σ as our annualized historical volatility. We then use this assumption to run 10,000 simulations on the returns for each underlying stock using the `randn()` function in MATLAB. The simulations were controlled by setting a seed of 123 using the `rng()` function. Based on our simulated returns, we reprice our stock options, which will be used to recalculate our portfolio value.

The Profit and Loss values are obtained as the difference between the Monte Carlo simulated prices and the actual prices from the Black-Scholes formula. This depends on whether the derivative position is monotonically increasing or decreasing. As seen from Figure 11, we find that AA and INTC are monotonically increasing, while PG and JPM are decreasing monotonically.

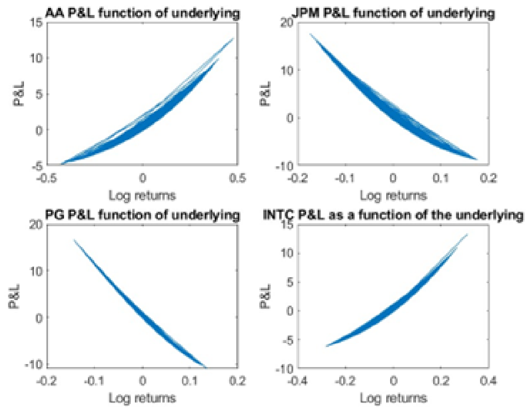


Figure 11: P&L vs Log returns of AA, JPM, PG and INTC.

From the plots in Figure 12, we observe that the histograms of the simulated P&L plots for the various stock options vary greatly.

1. AA call option: This option is an out-of-the-money option, with strike of $180.90, showing a right-skewed distribution, where its mode > median > mean. Moreover, simulated returns ranged from -28 to 62, with all probabilities below 0.035. Additionally, the distribution seems to be leptokurtic. As previously stated, taking a long position in AA has an unlimited upside, which is evident from the histogram plots.

2. PG put option: This option is an in-the-money short put with a strike price of $176.44. Additionally, the distribution seems to be leptokurtic and left-skewed, leading to more downside losses as seen in the plots.

3. JPM put option: This option is a long at-the-money put option with a strike price of $180.90. Additionally, the distribution seems to be leptokurtic and right-tailed. JPM also has unlimited upside, even though it appears to be right-skewed. Upon closer inspection, it is noticeable that more negative returns are being made.

4. INTC call option: This option is a short in-the-money call option with a strike price of $39.12. Additionally, the distribution seems to be leptokurtic, with a left-tailed distribution, leading to capped upside and unlimited downside potential for profits.

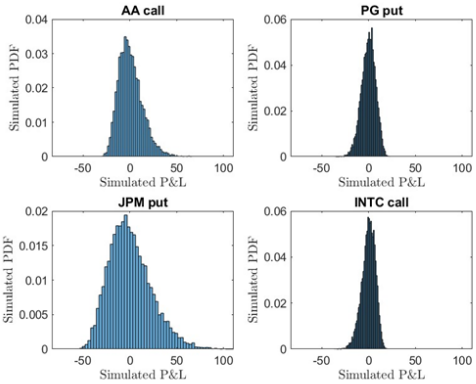


Figure 12: Histograms of simulated P&L plots of stocks.

Portfolio of options

In comparison, the simulated profit and loss distribution of the portfolio seems to be much more centered around the mean of 0, but slightly right-skewed due to its long tail on the right side of the histogram plot. Based on the portfolio, the majority of our profits and losses fall in the -20 to 50 regions, with a few gains surpassing 100.

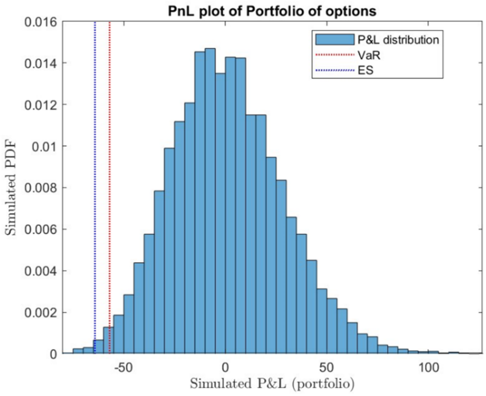


Figure 13: Histograms of simulated P&L plot of the portfolio.

Based on our simulation, we are able to obtain our Gaussian VaR and ES (Expected Shortfall) as follows in Table 13.

Stocks	AA call	INTC call	JPM put	PG put	Portfolio
VaR	3.65	4.54	6.79	7.75	56.84
ES	3.92	4.99	7.37	8.64	63.64

Table 13: Gaussian VaR and ES of individual stocks and portfolio.

According to the Basel committee specifications (Daníelsson [4]),

1. Time horizon of 10 trading days.
2. 99% confidence interval.
3. Method of probability distribution construction is aligned to the intuition of calculation.

Thus, based on the Basel assumptions, we obtained the individual stock and portfolio VaR and ES illustrated by the P&L distribution, the VaR of the portfolio is observed on the red line and ES the blue line

(Hull [6]).

On the other hand, our returns were log returns that followed the Gaussian distribution, $r(t) = r(t, t + \Delta) \sim \mathrm{N}\left(, \mu_\Delta, \sigma_\Delta^2\right)$, and were serially uncorrelated. They were estimated using the equation, $\mathrm{VaR}_\alpha(t, t+n\Delta) = -\left(\mu_\Delta n + z_{1-\alpha}\sigma_\Delta\sqrt{n}\right)$, where n represents 10 days, thus we estimated the 10-day VaR. Therefore, since we are using the profit and loss distribution, the actual VaR formula is represented by,

$$\mathrm{VaR}_\alpha{}^{pl}(t, t+n\Delta) = P(t)x\left(1 - e^{-VaR_a(t,t+n\Delta)}\right)$$

And the Expected Shortfall using the Gaussian approach as,

$$ES_\alpha^r(t, t+\mathrm{n}\Delta) = -\left(\mu_\Delta n - \sigma_\Delta\sqrt{n}\frac{\phi\left(z_{1-\alpha}\right)}{1-\alpha}\right)$$
$$ES_\alpha^{rpl}(t, t+\mathrm{n}\Delta) = P(t)\left(1 - e^{-ES_\alpha(t,t+n\Delta)}\right)$$

Since VaR does not always provide a complete picture of the portfolio's riskiness, the expected shortfall is calculated by averaging all the returns that are lower than the VaR in the stock and also for our portfolio. In this scenario, we calculated the expected shortfall over a 99% confidence level, implying the average worst return in 1% of our return's distribution. Note that the VaR calculated is the P&L VaR and not the return VaR; the same assumptions apply to ES calculations.

3.4 Non-Gaussian Marginal and Component VaR/ES

MVaR for a given asset is the asset conditional expected return given that the portfolio loss is at VaR level (Gianluca Fusai). MVaR can be defined as the change in portfolio VaR from taking additional exposure to each given asset. In mathematical terms, it is the first partial derivative of VaR with respect to individual weighting

$$MVaR_i = \frac{\partial VaR_\alpha}{\partial w_i} = E\left(r_i \mid w' = -\mathrm{VaR}_\alpha(w)\right), i = 1, \ldots, N$$

Since we are calculating the non-Gaussian approach, we use Monte Carlo simulation.

Moreover, Marginal Expected Shortfall (MES) is akin to marginal VaR, but it involves considering the partial derivative of expected shortfall with respect to the weighting of each asset in the overall portfolio (RiskNet [14]).

$$MES_i = \frac{\partial ES_\propto}{\partial w_i} = E\left(r_i \mid w' = -ES_\propto(w)\right), i = 1, \ldots, N$$
$$\text{Portfolio value } = S0g\left(P, S, T, \widehat{\sigma_\Delta}\sqrt{250}, r\right)$$

The CVaR and ES represent the impact on portfolio risk if a specific asset were to be removed from the portfolio.

In general, MVaR can help us manage portfolio VaR more effectively. Essentially, if our goal is to maximize portfolio returns, we should take a long position on the asset with the lowest MVaR value and a short position on the asset with the highest MVaR value. This strategy aims to bring our portfolio as close as possible to the efficient frontier, thereby maximizing the portfolio's Sharpe ratio.

$$\text{CVaR} = \text{weighting of each asset} \times \text{MVaR}$$
$$\text{CES} = \text{weighting of each asset} \times \text{MES}$$

Based on the table for the MVaR and CVaR, we observe both negative and positive values. In the MVaR for stocks, we notice that adding JPM and INTC to our portfolio diversifies and reduces the total portfolio risk compared to AA and PG, which increases the risk by significant amounts. Ideally, it would be beneficial to hold some long positions in INTC and JPM stock options in addition to our original positions. This would certainly halve the risk in our portfolio.

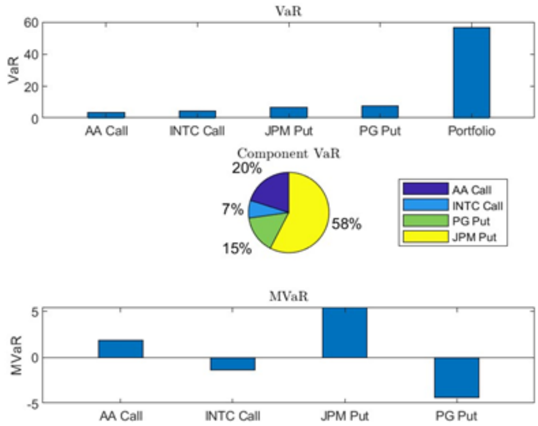


Figure 14: (Top): VaR of individual options and overall portfolio; (Middle): Component VaR of individual options; (Bottom): Marginal VaR of individual options.

Stocks	AA call	INTC call	JPM put	PG put
MVaR	1.90	-1.35	-4.31	5.45
Component Var	11.42	4.04	32.72	8.63

Table 14: MVaR and CVaR of individual stocks.

Moreover, to check if our Component VaR is accurate, we find the total CVaR of the portfolio as 56.84 and sum the individual CVaR to obtain 56.80, which is similar to the total CVaR. Thus, our model is accurate. As our model is accurate, we notice that PG contributes most to our portfolio risk. The order of risk contribution is JPM $>$ AA $>$ PG $>$ INTC, corresponding to our MVaR values, with JPM reducing portfolio risk the most. Counterintuitively, the CVaR of JPM is large since it reduces the risk most and seems to be a higher volatile stock option.

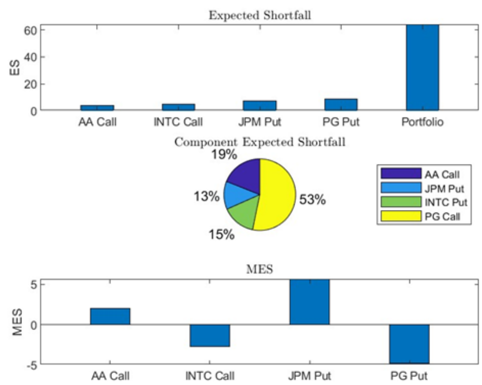


Figure 15: (Top): ES of individual options and overall portfolio; (Middle): Component VaR ES of individual options; (Bottom): Marginal ES of individual options.

Stocks	AA call	INTC call	JPM put	PG put
Marginal ES	2.02	-2.68	5.65	-4.82
Component ES	12.09	8.05	33.87	9.64

Table 15: Marginal ES and Component ES of individual stocks.

Furthermore, to check if our Component ES is very accurate, we find the total CES of the portfolio as 63.64 and sum the individual ES to obtain 63.64, which is similar to the total CES. The Expected Shortfall, MES, and CES over the portfolio showed a similar pattern to the VaR, MVaR, and CVaR, as JPM seemed to have a high ES, MES, and CES, showing that JPM put contributes significantly to the Portfolio risk and INTC call contributes the least risk. Overall, even though we had some high Component VaRs, the portfolio seems quite diversified.

Appendix

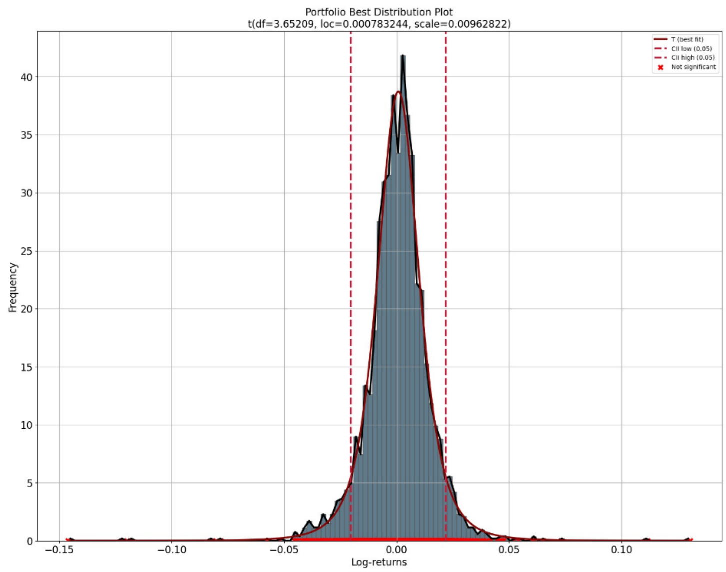


Figure A1: Fitted distribution of the equally weighted portfolio.

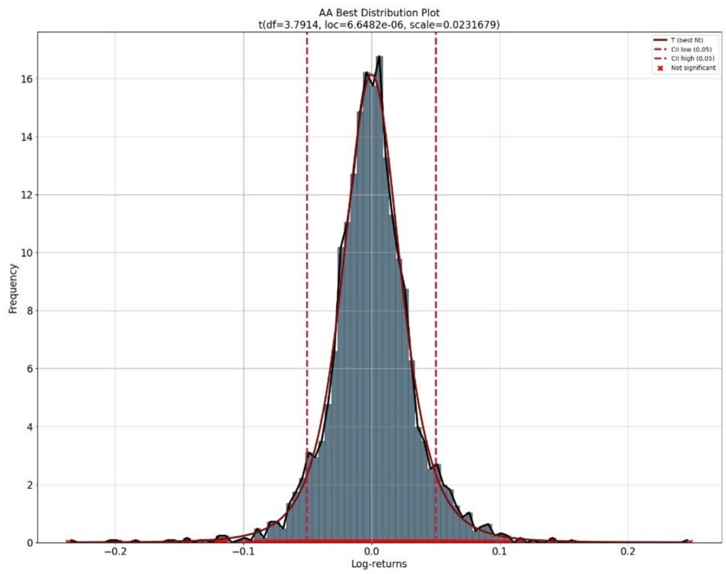


Figure A2: Fitted distribution of AA.

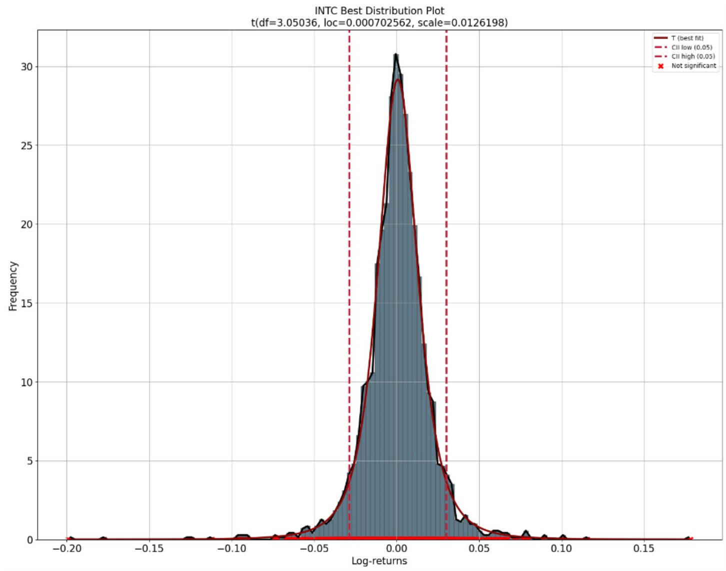


Figure A3: Fitted distribution of INTC.

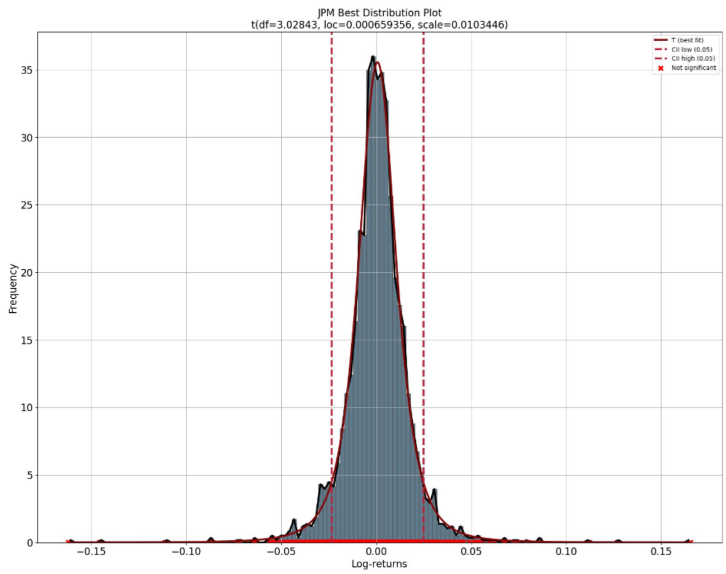


Figure A4: Fitted distribution of JPM.

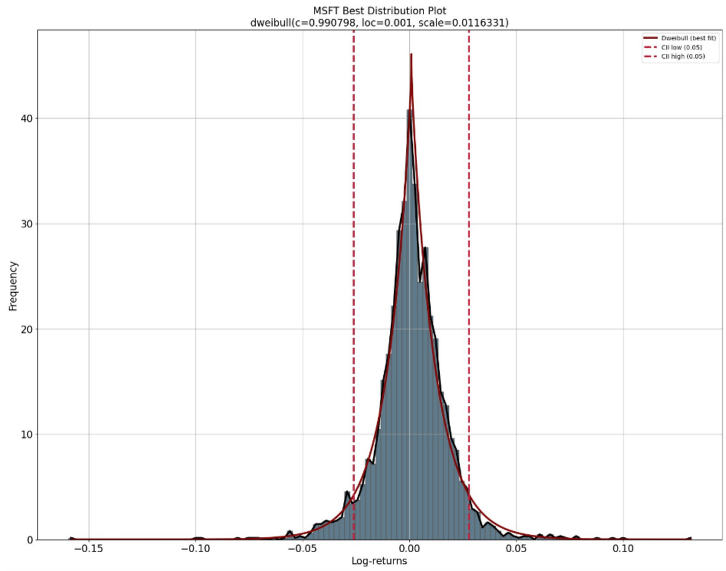


Figure A5: Fitted distribution of MSFT.

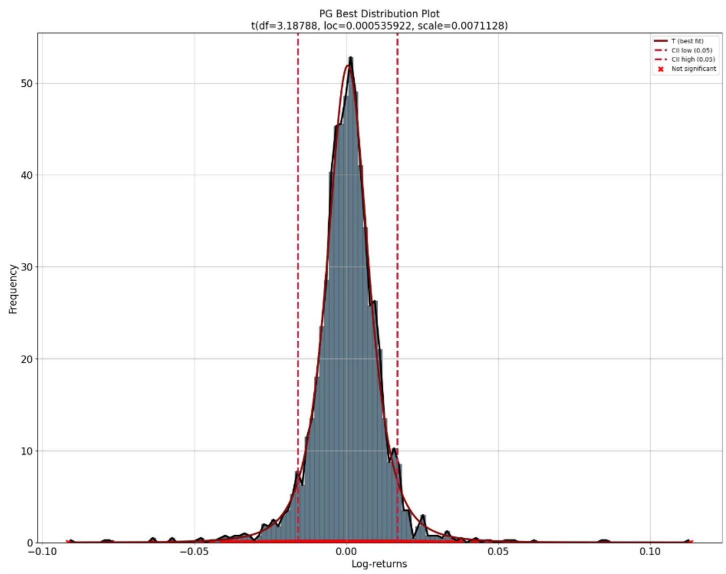


Figure A6: Fitted distribution of PG.

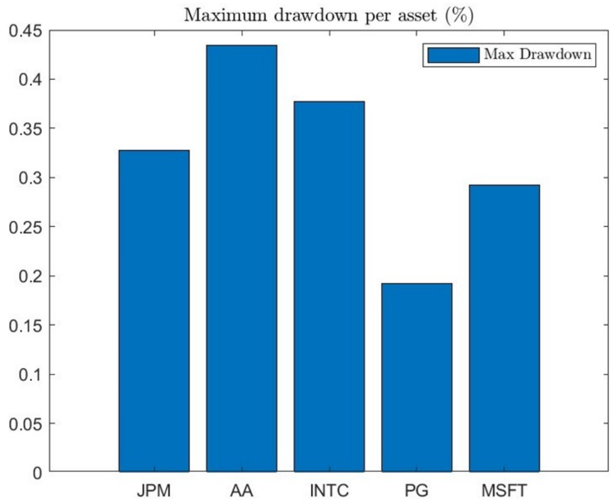


Figure A7: Maximum drawdown of the equally weighted portfolio (Whole sample of data).

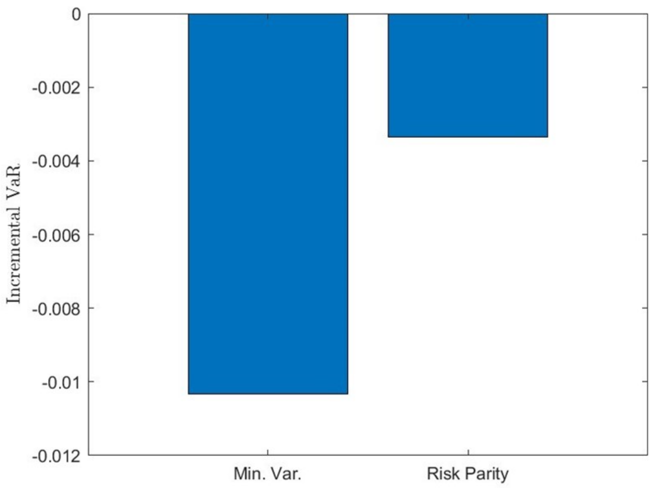


Figure A8: Incremental VaR of the minimum variance portfolio and the risk parity portfolio.

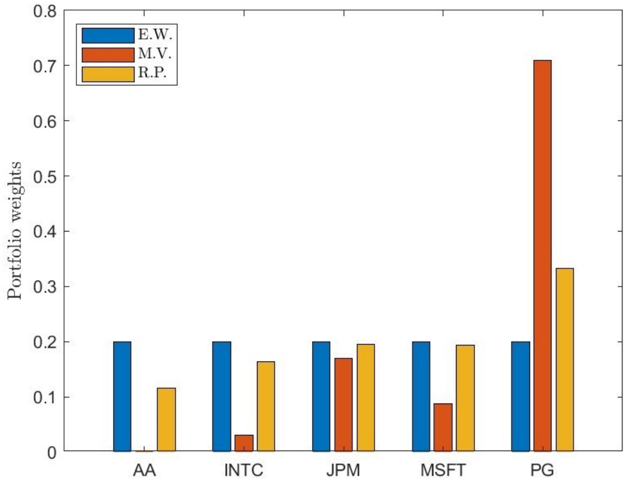


Figure A9: Portfolio weights of equally weighted (EW), minimum variance (MV) and risk parity (RP) portfolio.

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