# Cmposition Lemma for Lean

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CHAPTER 1

# Composition Lemma

## 1.1. Overview

The *Composition Lemma* was developed and refined over 6 years, beginning in 2018, as a novel approach to settle in the affirmative the *Graceful Tree Conjecture*. The first of such papers was posted in [3] by Gnang. A further developed series of papers resolving the same conjecture again appeared in [4] and [5]. Recently, the same method has been applied to settle other longstanding conjectures in [1] and [2]. We comment that the series of papers shared on the open-source platform arXiv reflect the evolving landscape of Gnang's thought process, and the frequent re-uploads were driven by the natural progression and refinement of ideas. However, we recognize that these numerous edits may have unintentionally caused confusion and raised questions regarding the success of the method. In the current work, we aim to address these concerns by presenting a detailed blueprint of the proof, with the goal of formalizing it in Lean4.

## 1.2. Functional Directed Graphs

For notational convenience, let $\mathbb{Z}_n$ denote the set whose members are the first $n$ natural numbers, i.e.,

$$\mathbb{Z}_n := \{0, 1, \dots, n-1\}. \tag{1.2.1}$$

For a function $f : \mathbb{Z}_m \to \mathbb{Z}_n$, we write $f \in \mathbb{Z}_n^{\mathbb{Z}_m}$. For $X \subseteq \mathbb{Z}_m$, $f(X)$ denotes the image of $X$ under $f$, i.e.,

$$f(X) = \{f(i) : i \in X\}, \tag{1.2.2}$$

and $|f(X)|$ denotes its cardinality. For $Y \subseteq \mathbb{Z}_n$, $f^{-1}(Y)$ denotes the pre-image of $Y$ under $f$ i.e.

$$f^{-1}(Y) = \{j \in \mathbb{Z}_m : f(j) \in Y\} \tag{1.2.3}$$

Definition 1.2.4 (Functional digraphs). For an arbitrary $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$, the *functional directed graph* prescribed by $f$, denoted $G_f$, is such that the vertex set $V(G_f)$ and the directed edge set $E(G_f)$ are respectively as follows:

$$V(G_f) = \mathbb{Z}_n,\ E(G_f) = \{(v, f(v)) : v \in \mathbb{Z}_n\}.$$

Definition 1.2.5 (Graceful functional digraphs). The functional directed graph prescribed by $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ is graceful if there exist a bijection $\sigma \in \mathrm{S}_n \subset \mathbb{Z}_n^{\mathbb{Z}_n}$ such that

$$\left\{ \left|\sigma f \sigma^{-1}(i) - i\right| : i \in \mathbb{Z}_n \right\} = \mathbb{Z}_n. \tag{1.2.6}$$

If $\sigma = \mathrm{id}$ (the identity function), then $G_f$ — the functional directed graph prescribed by $f$ — is gracefully labeled.

Definition 1.2.7 (Automorphism group). For a functional directed graph $G_f$, its automorphism group, denoted $\mathrm{Aut}\left(G_f\right)$, is defined as follows:

$$\mathrm{Aut}\left(G_f\right) = \{\sigma \in \mathrm{S}_n : \{(i, f(i)) : i \in \mathbb{Z}_n\} = \{(j, \sigma f \sigma^{-1}(j)) : j \in \mathbb{Z}_n\}\}.$$

For a polynomial $P \in \mathbb{C}[x_0, \dots, x_{n-1}]$, its automorphism group, denoted $\mathrm{Aut}(P)$, is defined as follows:

$$\mathrm{Aut}\left(P\right) = \{\sigma \in \mathrm{S}_n : P\left(x_0, \dots, x_i, \dots, x_{n-1}\right) = P\left(x_{\sigma(0)}, \dots, x_{\sigma(i)}, \dots, x_{\sigma(n-1)}\right)\}.$$

Definition 1.2.8 (Graceful re-labelings). The set of distinct gracefully labeled functional directed graphs isomorphic to $G_f$ is

$$\mathrm{GrL}\left(G_f\right) := \left\{ G_{\sigma f \sigma^{-1}} : \begin{array}{c} \sigma \text{ is a representative of a coset in } {}^{\mathrm{S}_n}/_{\mathrm{Aut}(G_f)} \text{ and} \\ \mathbb{Z}_n = \left\{\left|\sigma f \sigma^{-1}\left(i\right) - i\right| \,:\, i \in \mathbb{Z}_n\right\} \end{array} \right\}$$

Definition 1.2.9 (Complementary labeling symmetry). If $\varphi = n - 1 - \mathrm{id}$, then for all $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ we have

$$|f(i) - i| = |\varphi f(i) - \varphi(i)|\,, \quad \forall\, i \in \mathbb{Z}_n. \tag{1.2.10}$$

In other words, induced edge labels are fixed by the vertex relabeling effected by $\varphi$. We call this induced edge label symmetry the *complementary labeling symmetry* of the functional directed graph $G_f$.

## 1.3. Quotient-Remainder Theorem and Lagrange Interpolation

Proposition 1.3.1 (Multivariate Quotient-Remainder). *Let $d(x) \in \mathbb{C}[x]$ be a degree $n$ monic polynomial with simple roots, i.e.,*

$$d(x) = \prod_{u \in \mathbb{Z}_n} (x - \alpha_u) \;\; \text{and} \;\; 1 = GCD(d(x), \frac{d}{dx} d(x)), \tag{1.3.2}$$

*where $\{\alpha_u : u \in \mathbb{Z}_n\} \subset \mathbb{C}$. For all $P \in \mathbb{C}[x_0, \dots, x_{m-1}]$, there exists a unique remainder $r(x_0, \dots, x_{m-1}) \in \mathbb{C}[x_0, \dots, x_{m-1}]$ of degree at most $n-1$ in each variable such that*

$$P(x_0, \dots, x_{m-1}) = \sum_{u \in \mathbb{Z}_m} q_u(x_0, \dots, x_{m-1})\, d(x_u) \; + \; r(x_0, \dots, x_{m-1}). \tag{1.3.3}$$

Proof. We prove by induction on the number of variables that

$$r(x_0, \dots, x_{m-1}) = \sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_m}} P(\alpha_g) \prod_{i \in \mathbb{Z}_m} \left( \prod_{j_i \in \mathbb{Z}_n \backslash \{g(i)\}} \left( \frac{x_i - \alpha_{j_i}}{\alpha_{g(i)} - \alpha_{j_i}} \right) \right), \tag{1.3.4}$$

where for notational convenience $P(\alpha_g) := P(\alpha_{g(0)}, \dots, \alpha_{g(m-1)})$. The base case stems from the univariate quotient-remainder theorem. The univariate-quotient remainder theorem asserts that there exit a unique quotient-remainder pair $(q(x_0),\, r(x_0)) \in \mathbb{C}[x_0] \times \mathbb{C}[x_0]$ subject to

$$H(x_0) = q(x_0)\, d(x_0) + r(x_0), \tag{1.3.5}$$

where $r(x_0) \in \mathbb{C}[x_0]$ is of degree at most $n-1$. It is completely determined by its evaluation over $\{\alpha_i : i \in \mathbb{Z}_n\}$, and by Lagrange interpolation we have

$$r(x_0) = \sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_1}} H(\alpha_{g(0)}) \prod_{j_0 \in \mathbb{Z}_n \backslash \{g(0)\}} \left( \frac{x_0 - \alpha_{j_0}}{\alpha_{g(0)} - \alpha_{j_0}} \right), \tag{1.3.6}$$

thus establishing the claim in the base case. For the induction step, assume as our induction hypothesis that for all $F \in \mathbb{Q}\left[x_0, \dots, x_{m-1}\right]$, we have

$$F = \sum_{k \in \mathbb{Z}_m} q_k(x_0, \dots, x_{m-1})\, d(x_k) + \sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_m}} F(\alpha_g) \prod_{i \in \mathbb{Z}_m} \left( \prod_{j_i \in \mathbb{Z}_n \backslash \{g(i)\}} \left( \frac{x_i - \alpha_{j_i}}{\alpha_{g(i)} - \alpha_{j_i}} \right) \right). \tag{1.3.7}$$

We proceed to show the hypothesis implies that every polynomial in $m+1$ variables also admits a similar expansion, thus establishing the desired claim. Consider a polynomial $H \in \mathbb{C}[x_0, \dots, x_m]$, we view $H$ as a univariate polynomial in the variable $x_m$ whose coefficients lie in the polynomial ring $\mathbb{C}[x_0, \dots, x_{m-1}]$. The univariate quotient-remainder theorem over this ring asserts that there exit a unique quotient-remainder pair

$$(q(x_m),\, r(x_m)) \in (\mathbb{C}[x_0, \dots, x_{m-1}])[x_m] \times (\mathbb{C}[x_0, \dots, x_{m-1}])[x_m]$$

subject to

$$H(x_0, \dots, x_m) = q(x_0, \dots, x_m)\, d(x_m) + r(x_0, \dots, x_m), \tag{1.3.8}$$

where $r\left(x_0, \dots, x_m\right) \in (\mathbb{C}[x_0, \dots, x_{m-1}])[x_m]$ is of degree at most $n-1$ in the variable $x_m$. We write

$$r\left(x_0, \dots, x_m\right) = \sum_{k \in \mathbb{Z}_n} a_k\left(x_0, \dots, x_{m-1}\right) \left(x_m\right)^k. \tag{1.3.9}$$

We see that $a_k(x_0, \dots, x_{m-1}) \in \mathbb{C}[x_0, \dots, x_{m-1}]$ from the equality

$$\left( \text{Vander} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_u \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \right) \cdot \begin{pmatrix} a_0\left(x_0, \dots, x_{m-1}\right) \\ \vdots \\ a_u\left(x_0, \dots, x_{m-1}\right) \\ \vdots \\ a_{n-1}\left(x_0, \dots, x_{m-1}\right) \end{pmatrix} = \begin{pmatrix} H(x_0, \dots, x_{m-1}, \alpha_0) \\ \vdots \\ H(x_0, \dots, x_{m-1}, \alpha_u) \\ \vdots \\ H(x_0, \dots, x_{m-1}, \alpha_{n-1}) \end{pmatrix}, \tag{1.3.10}$$

where

$$\left(\text{Vander}\begin{pmatrix}\alpha_0\\ \vdots\\ \alpha_u\\ \vdots\\ \alpha_u\end{pmatrix}\right)[i,j]=(\alpha_i)^j,\ \forall\, 0\le i,j<n. \tag{1.3.11}$$

Since the Vandermonde matrix is invertible by the fact

$$0\ne\det\left(\text{Vander}\begin{pmatrix}\alpha_0\\ \vdots\\ \alpha_u\\ \vdots\\ \alpha_u\end{pmatrix}\right)=\prod_{0\le u<v<n}(\alpha_v-\alpha_u), \tag{1.3.12}$$

we indeed have

$$\begin{pmatrix}a_0\left(x_0,\dots,x_{m-1}\right)\\ \vdots\\ a_u\left(x_0,\dots,x_{m-1}\right)\\ \vdots\\ a_{n-1}\left(x_0,\dots,x_{m-1}\right)\end{pmatrix}=\left(\text{Vander}\begin{pmatrix}\alpha_0\\ \vdots\\ \alpha_u\\ \vdots\\ \alpha_u\end{pmatrix}\right)^{-1}\cdot\begin{pmatrix}H(x_0,\dots,x_{m-1},\alpha_0)\\ \vdots\\ H(x_0,\dots,x_{m-1},\alpha_u)\\ \vdots\\ H(x_0,\dots,x_{m-1},\alpha_{n-1})\end{pmatrix}. \tag{1.3.13}$$

Therefore, we have

$$H(x_0,\dots,x_m)=q_m(x_0,\dots,x_m)\,d(x_m)+\sum_{g(m)\in\mathbb{Z}_n}H(x_0,\dots,x_{m-1},\alpha_{g(m)})\prod_{j\in\mathbb{Z}_n\backslash\{g(m)\}}\left(\frac{x_m-\alpha_{j_m}}{\alpha_{g(m)}-\alpha_{j_m}}\right). \tag{1.3.14}$$

Applying the induction hypothesis to coefficients

$$\left\{H\left(x_0,\dots,x_{m-1},g\left(m\right)\right):g\left(m\right)\in\mathbb{Z}_n\right\}\subset\mathbb{C}[x_0,\dots,x_{m-1}]$$

yields the desired claim.

□

Definition 1.3.15 (Polynomial of Grace). We define $P_f\in\mathbb{C}[x_0,\dots,x_{n-1}]$ for all $f\in\mathbb{Z}_n^{\mathbb{Z}_n}$ as follows:

$$P_f:=\underbrace{\prod_{0\le u<v<n}(x_v-x_u)}_{V_f(x_0,\dots,x_{n-1})}\ \underbrace{\prod_{0\le u<v<n}\left((x_{f(v)}-x_v)^2-(x_{f(u)}-x_u)^2\right)}_{E_f(x_0,\dots,x_{n-1})}. \tag{1.3.16}$$

Definition 1.3.17 (Congruence class). For polynomials $P,Q\in\mathbb{C}[x_0,\dots,x_{n-1}]$, if

$$P(\mathbf{x})\equiv\ Q(\mathbf{x})\quad\text{mod}\left\{\prod_{j\in\mathbb{Z}_n}(x_i-j):i\in\mathbb{Z}_n\right\}, \tag{1.3.18}$$

we simply say $P\equiv Q$. All congruence relations in this paper are modulo the set immediately above.

Proposition 1.3.19 (Certificate of Grace). *Let $f\in\mathbb{Z}_n^{\mathbb{Z}_n}$. The functional directed graph $G_f$ prescribed by $f$ is graceful if and only if $0\not\equiv\ P_f(\mathbf{x})$.*

Proof. Observe that the vertex Vandermonde factor $V_f(\mathbf{x})$ is of degree exactly $n-1$ in each variable and therefore equal to its remainder, i.e.,

$$V_f(\mathbf{x})=\sum_{\theta\in\mathrm{S}_n}\text{sgn}(\theta)\prod_{i\in\mathbb{Z}_n}(x_i)^{\theta(i)}=\prod_{v\in\mathbb{Z}_n}v!\sum_{\theta\in\mathrm{S}_n}\text{sgn}(\theta)\prod_{\substack{i\in\mathbb{Z}_n\\ j_i\in\mathbb{Z}_n\backslash\{\theta(i)\}}}\left(\frac{x_i-j_i}{\theta(i)-j_i}\right), \tag{1.3.20}$$

where

$$\text{sgn}(\theta):=\prod_{0\le u<v<n}\left(\frac{\theta(v)-\theta(u)}{v-u}\right),\quad\forall\,\theta\in\mathrm{S}_n. \tag{1.3.21}$$

The induced edge label Vandermonde factor $E_f(\mathbf{x})$ is of degree $> n-1$ in some of its variables. Therefore, by Proposition 1.3.1, we have

$$E_f(\mathbf{x}) = \sum_{l \in \mathbb{Z}_m} q_l(\mathbf{x}) \prod_{k \in \mathbb{Z}_n} (x_l - k) + \sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_n}} \prod_{0 \le u < v < n} \left( \left(gf(v) - g(v)\right)^2 - \left(gf(u) - g(u)\right)^2 \right) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{g(i)\}}} \left( \frac{x_i - j_i}{g(i) - j_i} \right). \tag{1.3.22}$$

Observe that by the expansions in 1.3.20 and 1.3.22,

$$P_f(\mathbf{x}) = \sum_{l \in \mathbb{Z}_m} q_l(\mathbf{x}) V_f(\mathbf{x}) \prod_{k \in \mathbb{Z}_n} (x_l - k) +$$

$$\left( \prod_{v \in \mathbb{Z}_n} v! \sum_{\theta \in \mathrm{S}_n} \mathrm{sgn}(\theta) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\theta(i)\}}} \left( \frac{x_i - j_i}{\theta(i) - j_i} \right) \right) \left( \sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_n}} \prod_{0 \le u < v < n} \left( \left(gf(v) - g(v)\right)^2 - \left(gf(u) - g(u)\right)^2 \right) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{g(i)\}}} \left( \frac{x_i - j_i}{g(i) - j_i} \right) \right).$$

is congruent to

$$\prod_{v \in \mathbb{Z}_n} (v!)^2 \frac{(n-1+v)!}{(2v)!} \sum_{\substack{\sigma \in \mathrm{S}_n \\ \gamma = |\sigma f \sigma^{-1} - \mathrm{id}| \in \mathrm{S}_n}} \mathrm{sgn}(\gamma\sigma) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left( \frac{x_i - j_i}{\sigma(i) - j_i} \right), \tag{1.3.23}$$

where the permutation $\gamma$ denotes the induced edge label permutation associated with a graceful relabeling $G_{\sigma f \sigma^{-1}}$ of $G_f$. The congruence above stems from the congruence identity

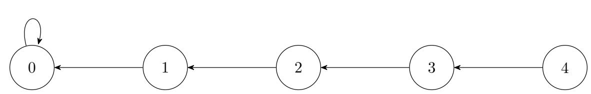
$$\prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\theta(i)\}}} \left( \frac{x_i - j_i}{\theta(i) - j_i} \right) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{g(i)\}}} \left( \frac{x_i - j_i}{g(i) - j_i} \right) \equiv \begin{cases} \displaystyle\prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\theta(i)\}}} \left( \tfrac{x_i - j_i}{\theta(i) - j_i} \right) & \text{if } \theta = g \\ 0 & \text{otherwise} \end{cases} \quad \forall\, (\theta, g) \in \mathrm{S}_n \times \mathbb{Z}_n^{\mathbb{Z}_n}$$

and a graceful labeling necessitates the integer coefficient

$$\prod_{0 \le i < j < n} (j-i)(j^2 - i^2) = \prod_{0 \le i < j < n} (j-i)^2 (j+i) = \prod_{v \in \mathbb{Z}_n} (v!)^2 \frac{(n-1+v)!}{(2v)!} \neq 0,$$

thus establishing the desired claim. □

Example 1.3.24. We present an example of a path on 5 vertices. This is known to be graceful, so we expect a non-zero remainder.




Run the SageMath script `ex1324.sage` to verify.

Proposition 1.3.25 (Complementary Labeling Symmetry). *Let $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ and*

$$\overline{P}_f(\mathbf{x}) := \prod_{v \in \mathbb{Z}_n} (v!)^2 \frac{(n-1+v)!}{(2v)!} \sum_{\substack{\sigma \in S_n \\ \gamma = |\sigma f \sigma^{-1} - id| \in S_n}} sgn(\gamma\sigma) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left( \frac{x_i - j_i}{\sigma(i) - j_i} \right). \tag{1.3.26}$$

*The complementary labeling map $x_i \mapsto x_{n-1-i}$, for all $i \in \mathbb{Z}_n$, fixes $\overline{P}_f$ up to sign.*

Proof. For notational convenience let $\mathbf{x}_\varphi := x_{\varphi(0)}, \dots, x_{\varphi(i)}, \dots, x_{\varphi(n-1)}$. Observe that for any permutation $\varphi \in \mathrm{S}_n$, the action of $\varphi$ on $P_f$ yields equalities

$$\begin{aligned} P_f(\mathbf{x}_\varphi) &= \prod_{0 \le u < v < n} (x_{\varphi(v)} - x_{\varphi(u)})\big((x_{\varphi f(v)} - x_{\varphi(v)})^2 - (x_{\varphi f(u)} - x_{\varphi(u)})^2\big), \\ &= \prod_{0 \le \varphi^{-1}(i) < \varphi^{-1}(j) < n} (x_j - x_i)\big((x_{\varphi f \varphi^{-1}(j)} - x_j)^2 - (x_{\varphi f \varphi^{-1}(i)} - x_i)^2\big). \end{aligned}$$

The last equality above features the indexing change of variable $u = \varphi^{-1}(i)$ and $v = \varphi^{-1}(j)$. Thus, $P_f(x_{\varphi(0)}, \dots, x_{\varphi(n-1)})$ is up to sign equal to $P_{\varphi f \varphi^{-1}}$, in accordance with Definition 1.3.15. Furthermore, by the proof of Proposition 1.3.19, the action of $\varphi$ on $P_f$ yields the congruence identity

$$P_f(\mathbf{x}_\varphi) \equiv \overline{P}_f(\mathbf{x}_\varphi).$$

Hence,

$$\begin{aligned}\overline{P}_f(\mathbf{x}_\varphi) &= \prod_{v\in\mathbb{Z}_n}\left((v!)^2\,\frac{(n-1+v)!}{(2v)!}\right)\sum_{\substack{\sigma\in\mathrm{S}_n\\ \gamma=|\sigma f\sigma^{-1}-\mathrm{id}|\in\mathrm{S}_n}}\mathrm{sgn}(\gamma\sigma)\prod_{\substack{i\in\mathbb{Z}_n\\ j_i\in\mathbb{Z}_n\setminus\{\sigma(i)\}}}\left(\frac{x_{\varphi(i)}-j_i}{\sigma(i)-j_i}\right).\\ &= \mathrm{sgn}(\varphi)\prod_{v\in\mathbb{Z}_n}\left((v!)^2\,\frac{(n-1+v)!}{(2v)!}\right)\sum_{\substack{\sigma\in\mathrm{S}_n\\ \gamma=|\sigma f\sigma^{-1}-\mathrm{id}|\in\mathrm{S}_n}}\mathrm{sgn}(\gamma\sigma\varphi^{-1})\prod_{\substack{u\in\mathbb{Z}_n\\ v_u\in\mathbb{Z}_n\setminus\{\sigma\varphi^{-1}(u)\}}}\left(\frac{x_u-v_u}{\sigma\varphi^{-1}(u)-v_u}\right).\end{aligned}$$

If $\varphi = n-1-\mathrm{id}$, then, by the complementary labeling symmetry, we have

$$G_{\sigma f\sigma^{-1}} \in \mathrm{GrL}\left(G_f\right) \iff G_{\sigma\varphi^{-1}f(\sigma\varphi^{-1})^{-1}} \in \mathrm{GrL}\left(G_f\right)$$

Let $\mathfrak{G}$ denote the subrgoup of $\mathrm{S}_n$ whose members are $\{\mathrm{id},\ \varphi\}$. We write

$$\overline{P}_f(\mathbf{x}_\varphi) =$$

$$\prod_{v\in\mathbb{Z}_n}\left((v!)^2\,\frac{(n-1+v)!}{(2v)!}\right)\sum_{\substack{\sigma\in\mathrm{S}_n/\mathfrak{G}\\ \gamma=|\sigma f\sigma^{-1}-\mathrm{id}|\in\mathrm{S}_n}}\mathrm{sgn}(\gamma\sigma)\left(\mathrm{sgn}(\varphi^{-1})\prod_{\substack{u\in\mathbb{Z}_n\\ v_u\in\mathbb{Z}_n\setminus\{\sigma(u)\}}}\left(\frac{x_u-v_u}{\sigma(u)-v_u}\right)+\prod_{\substack{u\in\mathbb{Z}_n\\ v_u\in\mathbb{Z}_n\setminus\{\sigma\varphi^{-1}(u)\}}}\left(\frac{x_u-v_u}{\sigma\varphi^{-1}(u)-v_u}\right)\right).$$

Similarly,

$$\overline{P}_f(\mathbf{x}) =$$

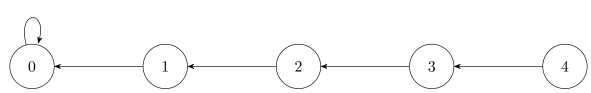
$$\prod_{v\in\mathbb{Z}_n}\left((v!)^2\,\frac{(n-1+v)!}{(2v)!}\right)\sum_{\substack{\sigma\in\mathrm{S}_n/\mathfrak{G}\\ \gamma=|\sigma f\sigma^{-1}-\mathrm{id}|\in\mathrm{S}_n}}\mathrm{sgn}(\gamma\sigma)\left(\prod_{\substack{i\in\mathbb{Z}_n\\ j_i\in\mathbb{Z}_n\setminus\{\sigma(i)\}}}\left(\frac{x_i-j_i}{\sigma(i)-j_i}\right)+\mathrm{sgn}(\varphi^{-1})\prod_{\substack{i\in\mathbb{Z}_n\\ j_i\in\mathbb{Z}_n\setminus\{\sigma\varphi^{-1}(i)\}}}\left(\frac{x_i-j_i}{\sigma\varphi^{-1}(i)-j_i}\right)\right).$$

We conclude that the complementary labeling symmetry yields the equality

$$\overline{P}_f(\mathbf{x}) = \mathrm{sgn}(\varphi)\,\overline{P}_f(\mathbf{x}_\varphi) = \overline{P}_{\varphi f\varphi^{-1}}(\mathbf{x}),$$

thus establishing the desired claim. □

Example 1.3.27. We present an example of a path on 5 vertices.




Run the SageMath script `ex1328.sage` to verify.

Lemma 1.3.28 (Variable dependency). *Let $P \in \mathbb{Q}[x_0, \dots, x_{n-1}]$ and $S \subsetneq \mathbb{Z}_n$. If*

$$P(\mathbf{x}) = \sum_{g\in\mathbb{Z}_n^S} c_g \prod_{i\in S}(x_i)^{g(i)}, \tag{1.3.29}$$

*where $c_g \in \mathbb{C}$ for all $g \in \mathbb{Z}_n^S$, then for any positive integer $m$, the polynomial $\left(P(\mathbf{x})\right)^m$ admits a quotient-remainder expansion of the form*

$$\left(P(x_0,\dots,x_{n-1})\right)^m = \sum_{j\in\mathbb{Z}_m} q_j(x_0,\dots,x_{n-1})\prod_{k\in\mathbb{Z}_n}(x_j-\alpha_k) + \sum_{g\in\mathbb{Z}_n^S} a_g \prod_{i\in S}(x_i)^{g(i)} \tag{1.3.30}$$

*where $\alpha_k, a_g \in \mathbb{C}$ for all $k \in \mathbb{Z}_n$ such that $n = \left|\{\alpha_k : k \in \mathbb{Z}_n\}\right|$and $g \in \mathbb{Z}_n^S$.*

PROOF. By the premise, the polynomial $P(\mathbf{x})$ is of degree at most $n-1$ in its variables. Thus by Proposition 1.3.1, the polynomial $P(\mathbf{x})$ is equal to its remainder, i.e.,

$$P(\mathbf{x}) \;=\; \sum_{g\in\mathbb{Z}_n^S} c_g \prod_{i\in S} (x_i)^{g(i)} \;=\; \sum_{g\in\mathbb{Z}_n^S} P(g) \prod_{\substack{i\in S\\ j_i\in\mathbb{Z}_n\setminus\{g(i)\}}} \left(\frac{x_i-\alpha_{j_i}}{\alpha_{g(i)}-\alpha_{j_i}}\right). \tag{1.3.31}$$

The remainder of $(P(\mathbf{x}))^m$ is obtained by repeatedly replacing each occurrence of $(x_i)^n$ with $(x_i)^n - \prod_{k\in\mathbb{Z}_n}(x_i-\alpha_k)$, followed by expanding the resulting polynomials, starting from the expanded form of

$$\left(\sum_{g\in\mathbb{Z}_n^S} c_g \prod_{i\in S}(x_i)^{g(i)}\right)^m, \tag{1.3.32}$$

until we obtain a polynomial of degree at most $n-1$ in each variable. The transformation never introduces a variable indexed by a member of the complement of $S$. We obtain that

$$\left(\sum_{g\in\mathbb{Z}_n^S} c_g \prod_{i\in S}(x_i)^{g(i)}\right)^m = \sum_{j\in\mathbb{Z}_m} q_j(\mathbf{x}) \prod_{k\in\mathbb{Z}_n}(x_j-\alpha_k) + \sum_{g\in\mathbb{Z}_n^S} (P(g))^m \prod_{\substack{i\in S\\ j_i\in\mathbb{Z}_n\setminus\{g(i)\}}} \left(\frac{x_i-\alpha_{j_i}}{\alpha_{g(i)}-\alpha_{j_i}}\right)$$

by which it follows that

$$\left(\sum_{g\in\mathbb{Z}_n^S} c_g \prod_{i\in S}(x_i)^{g(i)}\right)^m = \sum_{j\in\mathbb{Z}_m} q_j(\mathbf{x}) \prod_{k\in\mathbb{Z}_n}(x_j-\alpha_k) + \sum_{g\in\mathbb{Z}_n^S} a_g \prod_{i\in S}(x_i)^{g(i)}, \tag{1.3.33}$$

where $\alpha_k, a_g \in \mathbb{C}$ for all $k\in\mathbb{Z}_n$ and $n = \left|\{\alpha_k : k\in\mathbb{Z}_n\}\right|$ as claimed. □

LEMMA 1.3.34 (Monomial support). *Let $P\in\mathbb{Q}[x_0,\dots,x_{n-1}]$ be such that it is not identically constant. If*

$$P(\mathbf{x}) \;=\; \sum_{\sigma\in S_n} a_\sigma \prod_{\substack{i\in\mathbb{Z}_n\\ j_i\in\mathbb{Z}_n\setminus\{\sigma(i)\}}} \left(\frac{x_i-j_i}{\sigma(i)-j_i}\right), \tag{1.3.35}$$

*then there exist a minimal non-empty set $\mathcal{M}_P \subset \mathbb{Z}_n^{\mathbb{Z}_n}$ subject to $\left|f^{-1}(\{0\})\right| \le 1$ for all $f\in\mathcal{M}_P$ such that*

$$P(\mathbf{x}) = \sum_{f\in\mathcal{M}_P} c_f \prod_{i\in\mathbb{Z}_n} x_i^{f(i)}, \tag{1.3.36}$$

*where $c_f\in\mathbb{Q}\setminus\{0\}$.*

PROOF. Stated otherwise, every term in the expanded form of $P$ is a multiple of at least $n-1$ distinct variables. Consider a Lagrange basis polynomial associated with an arbitrary $\sigma\in\mathrm{S}_n$:

$$L_\sigma(\mathbf{x}) = \prod_{\substack{i\in\mathbb{Z}_n\\ j_i\in\mathbb{Z}_n\setminus\{\sigma(i)\}}} \left(\frac{x_i-j_i}{\sigma(i)-j_i}\right) \;=\; \prod_{\substack{i\in\mathbb{Z}_n\setminus\{\sigma^{-1}(0)\}\\ j_i\in\mathbb{Z}_n\setminus\{\sigma(i)\}}} \left(\frac{x_i-j_i}{\sigma(i)-j_i}\right) \prod_{j_{\sigma^{-1}(0)}\in\mathbb{Z}_n\setminus\{0\}} \left(\frac{x_{\sigma^{-1}(0)}-j_{\sigma^{-1}(0)}}{0-j_{\sigma^{-1}(0)}}\right).$$

On the right-hand side of the second equal sign immediately above, the univariate polynomial in $x_{\sigma^{-1}(0)}$ encompassed within the scope of the second $\Pi$ indexed by $j_{\sigma^{-1}(0)}\in\mathbb{Z}_n\setminus\{0\}$ has (in its expanded form) a non-vanishing constant term equal to one. However, the constant term vanishes within the expanded form of each univariate factors

$$\prod_{j_i\in\mathbb{Z}_n\setminus\{\sigma(i)\}} \left(\frac{x_i-j_i}{\sigma(i)-j_i}\right)$$

encompassed within the scope of the first $\Pi$ indexed by $i\in\mathbb{Z}_n\setminus\{\sigma^{-1}(0)\}$. Indeed, we have

$$L_\sigma(\mathbf{x}) = \underbrace{\prod_{\substack{i\in\mathbb{Z}_n\setminus\{\sigma^{-1}(0)\}\\ j_i\in\mathbb{Z}_n\setminus\{\sigma(i)\}}} \left(\frac{x_i-j_i}{\sigma(i)-j_i}\right)}_{\text{does not feature the variable } x_{\sigma^{-1}(0)}} \left(\frac{(x_{\sigma^{-1}(0)})^{n-1}+\dots+(-1)^{n-1}(n-1)!}{(-1)^{n-1}(n-1)!}\right).$$

Observe that each summand term in the expanded form of the Lagrange basis polynomial $L_\sigma(\mathbf{x})$ above which is a non-vanishing monomial multiple of $x_{\sigma^{-1}(0)}$ is a multiple of every variable in $\{x_0,\dots,x_{n-1}\}$. By contrast, every non-vanishing monomial summand term which is not a multiple of $x_{\sigma^{-1}(0)}$ is a multiple of every other variables, i.e., variables in the set $\{x_0, \dots, x_{n-1}\} \setminus \left\{x_{\sigma^{-1}(0)}\right\}$. Applying the same argument to each $\sigma \in \mathrm{S}_n$ yields the desired claim. □

## 1.4. The Composition Lemma

LEMMA 1.4.1 (Transposition Invariance). *Let $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ be such that its functional directed graph $G_f$ has at least two sibling leaf nodes, i.e., $G_f$ has vertices $u, v \in \mathbb{Z}_n$ such that $f^{-1}(\{u, v\}) = \emptyset$ and $f(u) = f(v)$. If the transposition $\tau \in S_n$ exchanges $u$ and $v$, i.e.,*

$$\tau(i) = \begin{cases} v & \text{if } i = u \\ u & \text{if } i = v \\ i & \text{otherwise} \end{cases} \quad \forall\, i \in \mathbb{Z}_n.$$

*Then we have*

$$\tau \in Aut\left(P_f(\mathbf{x})\right), \tag{1.4.2}$$

*where $P_f$ is the polynomial of grace as defined in 1.3.15.*

PROOF. Stated otherwise, the claim asserts that the polynomial $P_f$ is fixed by a transposition of any pair of variables associated with sibling leaf vertices. By construction of $P_f(\mathbf{x})$, the changes in its Vandermonde factors induced by the action of $\tau$ are as follows

$$P_f(x_{\tau(0)}, \dots, x_{\tau(i)}, \dots, x_{\tau(n-1)}) = \prod_{0 \le i < j < n} (x_{\tau(j)} - x_{\tau(i)}) \prod_{0 \le i < j < n} \left((x_{\tau f(j)} - x_{\tau(j)})^2 - (x_{\tau f(i)} - x_{\tau(i)})^2\right)$$

Note that there is a bijection

$$x_i \mapsto (x_{f(i)} - x_i)^2, \quad \forall\, i \in \mathbb{Z}_n. \tag{1.4.3}$$

Hence, the transposition $\tau$ of the leaf nodes induces a transposition $\tau$ of the corresponding leaf edges outgoing from the said leaf nodes. More precisely, the maps

$$\begin{pmatrix} x_0 & , \dots , & x_i & , \dots , & x_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ x_{\tau(0)} & , \dots , & x_{\tau(i)} & , \dots , & x_{\tau(n-1)} \end{pmatrix}$$

and

$$\begin{pmatrix} (x_{f(0)} - x_0)^2 & , \dots , & (x_{f(i)} - x_i)^2 & , \dots , & (x_{f(n-1)} - x_{n-1})^2 \\ \downarrow & & \downarrow & & \downarrow \\ (x_{\tau f(0)} - x_{\tau(0)})^2 & , \dots , & (x_{\tau f(i)} - x_{\tau(i)})^2 & , \dots , & (x_{\tau f(n-1)} - x_{\tau(n-1)})^2 \end{pmatrix}$$

prescribe the same permutation $\tau$ of the vertex variables and induced edges label binomials respectively. Observe that

$$\begin{aligned} & P_f(x_{\tau(0)}, \dots, x_{\tau(i)}, \dots, x_{\tau(n-1)}) = \\ & \left(\prod_{0 \le i < j < n} \frac{x_{\tau(j)} - x_{\tau(i)}}{x_j - x_i} \prod_{0 \le i < j < n} (x_j - x_i)\right) \left(\prod_{0 \le i < j < n} \frac{(x_{\tau f(j)} - x_{\tau(j)})^2 - (x_{\tau f(i)} - x_{\tau(i)})^2}{(x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2} \prod_{0 \le i < j < n} \left((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2\right)\right) \\ & = \left(\operatorname{sgn}(\tau) \prod_{0 \le i < j < n} (x_j - x_i)\right) \left(\operatorname{sgn}(\tau) \prod_{0 \le i < j < n} \left((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2\right)\right) \\ & = \left((-1) \prod_{0 \le i < j < n} (x_j - x_i)\right) \left((-1) \prod_{0 \le i < j < n} \left((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2\right)\right) \end{aligned}$$

$$\implies P_f(x_{\tau(0)}, \dots, x_{\tau(n-1)}) = P_f(x_0, \dots, x_{n-1}), \tag{1.4.4}$$

thus establishing the desired claim.

□

PROPOSITION 1.4.5 (Composition Inequality). *Consider an arbitrary $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ subject to the fixed point condition $\left|f^{(n-1)}\left(\mathbb{Z}_n\right)\right| = 1$. The following statements are equivalent:*

(i)

$$\max_{\sigma \in S_n} \left|\left\{\left|\sigma f^{(2)} \sigma^{-1}(i) - i\right| : i \in \mathbb{Z}_n\right\}\right| \le \max_{\sigma \in S_n} \left|\left\{\left|\sigma f \sigma^{-1}(i) - i\right| : i \in \mathbb{Z}_n\right\}\right|.$$

(ii)

$$P_{f^{(2)}}(\mathbf{x}) \not\equiv 0 \implies P_f(\mathbf{x}) \not\equiv 0.$$

(iii)

$$GrL(G_f) \neq \emptyset$$

PROOF. If $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ is identically constant, then $G_f$ is graceful. We see this from the fact that the functional digraphs of the identically zero function is gracefully labeled and the fact that functional digraph of identically constant functions are all isomorphic. It follows that all functional directed graphs having diameter less than 3 are graceful. Consequently, all claima hold for all functional digraph of diameter less than 3. We now turn our attention to functional trees of diameter greater or equal to 3. It follows by definition

$$n = \max_{\sigma \in \mathrm{S}_n} \left| \left\{ \left| \sigma f \sigma^{-1}(i) - i \right| : i \in \mathbb{Z}_n \right\} \right| \iff P_f(\mathbf{x}) \not\equiv 0 \iff \mathrm{GrL}(G_f) \neq \emptyset. \tag{1.4.6}$$

We now proceed to show (i) $\iff$ (iii). The backward claim is the simplest of the two claims. We see that if $f$ is contractive, so too is $f^{(2)}$. Then the assertions

$$n = \max_{\sigma \in \mathrm{S}_n} \left| \left\{ |\sigma f^{(2)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n \right\} \right| \text{ and } n = \max_{\sigma \in \mathrm{S}_n} \left| \left\{ |\sigma f \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n \right\} \right| \tag{1.4.7}$$

indeed implies the inequality

$$\max_{\sigma \in \mathrm{S}_n} \left| \left\{ |\sigma f^{(2)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n \right\} \right| \leq \max_{\sigma \in \mathrm{S}_n} \left| \left\{ |\sigma f \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n \right\} \right|. \tag{1.4.8}$$

We now establish the forward claim by contradiction. Assume for the sake of establishing a contradiction that for some contractive map $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ we have

$$n > \max_{\sigma \in \mathrm{S}_n} \left| \left\{ |\sigma f^{(2)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n \right\} \right|, \tag{1.4.9}$$

for we know by the number of edges being equal to $n$ that it is impossible that

$$n < \max_{\sigma \in \mathrm{S}_n} \left| \left\{ |\sigma f^{(2)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n \right\} \right|. \tag{1.4.10}$$

Note that the range of $f$ is a proper subset of $\mathbb{Z}_n$. By the premise that $f$ is contractive, it follows that $f^{(\lceil 2^{\lg(n-1)} \rceil)}$ is identically constant and thus

$$n = \max_{\sigma \in \mathrm{S}_n} \left| \left\{ |\sigma f^{(\lceil 2^{\lg(n-1)} \rceil)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n \right\} \right|, \tag{1.4.11}$$

where lg denotes the logarithm base 2. Consequently there must be some integer $0 \leq \kappa < \lg(n-1)$ such that

$$\max_{\sigma \in \mathrm{S}_n} \left| \left\{ |\sigma f^{(\lceil 2^{\kappa} \rceil)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n \right\} \right| > \max_{\sigma \in \mathrm{S}_n} \left| \left\{ |\sigma f^{(\lceil 2^{\kappa-1} \rceil)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n \right\} \right|. \tag{1.4.12}$$

This contradicts the assertion of the composition lemma, thereby establishing the backward claim. The exact same reasoning establishes (ii) $\iff$ (iii), for we have

$$P_{f^{(\lceil 2^{\lg(n-1)} \rceil)}}(\mathbf{x}) \not\equiv 0. \tag{1.4.13}$$

$\square$

Having assembled together the pieces required to prove our main result, we proceed to fit the pieces together to state and prove the *Composition Lemma*.

LEMMA 1.4.14 (Composition Lemma). *For all $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ subject to the contraction $\left| f^{(n-1)}(\mathbb{Z}_n) \right| = 1$, we have*

$$\max_{\sigma \in S_n} \left| \left\{ \left| \sigma f^{(2)} \sigma^{-1}(i) - i \right| : i \in \mathbb{Z}_n \right\} \right| \leq \max_{\sigma \in S_n} \left| \left\{ \left| \sigma f \sigma^{-1}(i) - i \right| : i \in \mathbb{Z}_n \right\} \right|. \tag{1.4.15}$$

PROOF. Owing to Proposition 1.4.5, we prove the statement by establishing

$$P_{f^{(2)}}(\mathbf{x}) \not\equiv 0 \implies P_f(\mathbf{x}) \not\equiv 0.$$

For simplicity, we prove a generalization of the desired claim. Given that the diameter of $G_f$ is greater than 2, we may assume without loss of generality that $f^{-1}(\{n-1\}) = \emptyset$ and $f^{(2)}(n-1) \neq f(n-1)$. Let the contractive map $g \in \mathbb{Z}_n^{\mathbb{Z}_n}$ be devised from $f$ such that

$$g(i) = \begin{cases} f^{(2)}(i) & \text{if } i \in f^{-1}(\{f(n-1)\}) \\ f(i) & \text{otherwise} \end{cases}, \ \forall\, i \in \mathbb{Z}_n. \tag{1.4.16}$$

We show that

$$P_g(\mathbf{x}) \not\equiv 0 \implies P_f(\mathbf{x}) \not\equiv 0. \tag{1.4.17}$$

Note that the assertion immediately above generalizes the composition lemma since, the function $f$ is only partially iterated. More precisely, $f$ is iterated only on the restriction $f^{-1}(\{f(n-1)\}) \subset \mathbb{Z}_n$. Iterating this slight generalization of the composition lemma yields that all functional trees are graceful, which in turn implies that the *Composition Lemma* as stated in Lemma 1.4.14 holds. Just as in the previous case, the assertion **??**. For notational convenience, we assume without loss of generality that

$$f^{-1}(\{f(n-1)\}) = \left\{n-1,\, n-2,\, \dots,\, n - \left|f^{-1}(\{f(n-1)\})\right|\right\} \quad \text{and}$$

$$f\left(n - \left|f^{-1}(\{f(n-1)\})\right|\right) = n - \left|f^{-1}(\{f(n-1)\})\right| - 1. \tag{1.4.18}$$

For if the condition above was not met, we relabel the vertices of $G_f$ to ensure that this was the case. Note that such a relabeling does not affect the property we seek to prove. We prove by contradiction the contrapositive claim

$$P_f(\mathbf{x}) \equiv 0 \implies P_g(\mathbf{x}) \equiv 0.$$

By construction, the polynomial

$$\begin{aligned} P_f(\mathbf{x}) = \prod_{0 \le i < j < n} \left(x_j - x_i\right) \times \\ \prod_{\substack{0 \le u < v \le f(n-1) \\ t \in \{0,1\}}} \left(x_{f(v)} - x_v + (-1)^t \left(x_{f(u)} - x_u\right)\right) \times \\ \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ 0 \le u \le f(n-1) \\ t \in \{0,1\}}} \left(x_{f(v)} - x_v + (-1)^t \left(x_{f(u)} - x_u\right)\right) \times \\ \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v \\ t \in \{0,1\}}} \left(x_{f(v)} - x_v + (-1)^t \left(x_{f(u)} - x_u\right)\right), \end{aligned} \tag{1.4.19}$$

differs only slightly from

$$\begin{aligned} P_g(\mathbf{x}) = \prod_{0 \le i < j < n} \left(x_j - x_i\right) \times \\ \prod_{\substack{0 \le u < v \le f(n-1) \\ t \in \{0,1\}}} \left(x_{f(v)} - x_v + (-1)^t \left(x_{f(u)} - x_u\right)\right) \times \\ \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ 0 \le u \le f(n-1) \\ t \in \{0,1\}}} \left(x_{f^{(2)}(v)} - x_v + (-1)^t \left(x_{f(u)} - x_u\right)\right) \times \\ \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v \\ t \in \{0,1\}}} \left(x_{f^{(2)}(v)} - x_v + (-1)^t \left(x_{f^{(2)}(u)} - x_u\right)\right). \end{aligned} \tag{1.4.20}$$

We setup a variable telescoping within each binomial $x_{f^{(2)}(v)} - x_v$ for all $v \in f^{-1}(\{f(n-1)\})$ (i.e., vertices associated with an iterated edge) as follows:

$$\underbrace{\left(x_{f^{(2)}(v)} - x_v\right)}_{x_v \longrightarrow x_{f^{(2)}(v)}} = \underbrace{\left(x_{f(v)} - x_v\right)}_{x_v \longrightarrow x_{f(v)}} + \underbrace{\left(x_{f^{(2)}(v)} - x_{f(v)}\right)}_{x_{f(v)} \longrightarrow x_{f^{(2)}(v)}},$$

$$\left(x_{f^{(2)}(v)} - x_v\right) = \left(x_{f^{(2)}(v)} - x_{f(v)}\right) + \left(x_{f(v)} - x_v\right) = \left(x_{f^{(2)}(n-1)} - x_{f(n-1)}\right) + \left(x_{f(v)} - x_v\right),$$

where the last equality immediately above results from the fact that $f(v) = f(n-1)$ for all $v \in f^{-1}(\{f(n-1)\})$. Thus

$$\begin{aligned} P_g = & \prod_{0 \le i < j < n} \left(x_j - x_i\right) \times \\ & \prod_{\substack{0 \le u < v \le f(n-1) \\ t \in \{0,1\}}} \left(x_{f(v)} - x_v + (-1)^t \left(x_{f(u)} - x_u\right)\right) \times \\ & \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ 0 \le u \le f(n-1) \\ t \in \{0,1\}}} \left(\left(x_{f^{(2)}(n-1)} - x_{f(n-1)}\right) + \left(x_{f(v)} - x_v\right) + (-1)^t \left(x_{f(u)} - x_u\right)\right) \times \\ & \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v \\ t \in \{0,1\}}} \left(\left(x_{f^{(2)}(n-1)} - x_{f(n-1)}\right) + \left(x_{f(v)} - x_v\right) + (-1)^t \left(\left(x_{f^{(2)}(n-1)} - x_{f(n-1)}\right) + \left(x_{f(u)} - x_u\right)\right)\right). \end{aligned}$$

For notational convenience, we write

$$\begin{aligned} P_g = & \prod_{0 \le i < j < n} \left(x_j - x_i\right) \times \\ & \prod_{0 \le u < v \le f(n-1)} \left(\left(x_{f(v)} - x_v\right)^2 - \left(x_{f(u)} - x_u\right)^2\right) \times \\ & \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ 0 \le u \le f(n-1) \\ t \in \{0,1\}}} \left(b_{u,v,t} + a_{f(n-1)}\right) \times \\ & \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} \left(b_v + b_u + 2\, a_{f(n-1)}\right) \times \\ & \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} \left(b_v - b_u + 0\, a_{f(n-1)}\right), \end{aligned} \tag{1.4.21}$$

where

$$a_{f(n-1)} = \left(x_{f^{(2)}(n-1)} - x_{f(n-1)}\right),$$

$$b_i = \left(x_{f(i)} - x_i\right), \quad \forall\, i \in f^{-1}\left(\{f(n-1)\}\right),$$

and

$$b_{u,v,t} = \left(x_{f(v)} - x_v\right) + (-1)^t \left(x_{f(u)} - x_u\right), \quad \forall \begin{smallmatrix} v \in f^{-1}(\{f(n-1)\}) \\ 0 \le u \le f(n-1) \\ t \in \{0,1\}. \end{smallmatrix}$$

Note that adopting the same notation, we may also re-write $P_f$ in equation (1.4.19) as follows:

$$P_f = \prod_{0\le i<j<n} \left(x_j - x_i\right) \times \prod_{0\le u<v\le f(n-1)} \left(\left(x_{f(v)} - x_v\right)^2 - \left(x_{f(u)} - x_u\right)^2\right) \times \prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ 0\le u\le f(n-1)\\ t\in\{0,1\}}} b_{u,v,t} \times \prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ f(n-1)<u<v}} \left(b_v - b_u\right)\left(b_v + b_u\right). \tag{1.4.22}$$

Invoking the multi-binomial identity on the two bichromatic factors of $P_g$ in equation (1.4.21) yields equalities

$$\prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ f(n-1)<u<v}} \left(b_v + b_u + 2a_{f(n-1)}\right) =$$

$$\prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ f(n-1)<u<v}} \left(b_v + b_u\right) + \sum_{\substack{r_{u,v}\in\{0,1\}\\ 0=\prod r_{u,v}}} \prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ f(n-1)<u<v}} \left(b_v + b_u\right)^{r_{u,v}} \left(2a_{f(n-1)}\right)^{1-r_{u,v}}.$$

and

$$\prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ 0\le u\le f(n-1)\\ t\in\{0,1\}}} \left(b_{u,v,t} + a_{f(n-1)}\right) =$$

$$\prod_{\substack{t\in\{0,1\}\\ v\in f^{-1}(\{f(n-1)\})\\ 0\le u\le f(n-1)}} b_{u,v,t} + \sum_{\substack{s_{u,v,t}\in\{0,1\}\\ 0=\prod s_{u,v,t}}} \prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ 0\le u\le f(n-1)\\ t\in\{0,1\}}} \left(b_{u,v,t}\right)^{s_{u,v,t}} \left(a_{f(n-1)}\right)^{1-s_{u,v,t}}.$$

Substituting equalities immediately above into equation (1.4.21) yields an expression of $P_g$ of the form

$$P_g = P_f + R_{f,g}. \tag{1.4.23}$$

The monochromatic red expressions in the multi-binomial expansion collect to result in $P_f$ as written in equation (1.4.22). The second part denoted $R_{f,g}$ simply collects the remaining bichromatic summands and is given by

$$\begin{aligned}
R_{f,g} = &\prod_{0\le i<j<n} \left(x_j - x_i\right) \prod_{0\le u<v\le f(n-1)} \left((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2\right) \prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ f(n-1)<u<v}} \left(b_v - b_u\right) \times \\
&\left[\left(\prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ f(n-1)<u<v}} \left(b_v + b_u\right)\right)\left(\sum_{\substack{s_{u,v,t}\in\{0,1\}\\ 0=\prod s_{u,v,t}}} \prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ 0\le u\le f(n-1)\\ t\in\{0,1\}}} \left(b_{u,v,t}\right)^{s_{u,v,t}} \left(a_{f(n-1)}\right)^{1-s_{u,v,t}}\right) + \right. \\
&\left(\prod_{\substack{t\in\{0,1\}\\ v\in f^{-1}(\{f(n-1)\})\\ 0\le u\le f(n-1)}} b_{u,v,t}\right)\left(\sum_{\substack{r_{u,v}\in\{0,1\}\\ 0=\prod r_{u,v}}} \prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ f(n-1)<u<v}} \left(b_v + b_u\right)^{r_{u,v}} \left(2a_{f(n-1)}\right)^{1-r_{u,v}}\right) + \\
&\left.\left(\sum_{\substack{s_{u,v,t}\in\{0,1\}\\ 0=\prod s_{u,v,t}}} \prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ 0\le u\le f(n-1)\\ t\in\{0,1\}}} \left(b_{u,v,t}\right)^{s_{u,v,t}} \left(a_{f(n-1)}\right)^{1-s_{u,v,t}}\right)\left(\sum_{\substack{r_{u,v}\in\{0,1\}\\ 0=\prod r_{u,v}}} \prod_{\substack{v\in f^{-1}(\{f(n-1)\})\\ f(n-1)<u<v}} \left(b_v + b_u\right)^{r_{u,v}} \left(2a_{f(n-1)}\right)^{1-r_{u,v}}\right)\right]
\end{aligned}$$

The color scheme introduced here is meant to help track the location of telescoping variables. We now proceed with the main contradiction argument. Assume for the sake of establishing a contradiction that the contrapositive of claim (1.4.17) is false, i.e., for some $f$ subject to conditions described in our premise, we have

$$0 \equiv P_f \text{ and } 0 \not\equiv P_g. \tag{1.4.24}$$

Owing to equation (1.4.23), we obtain

$$P_g \equiv R_{f,g} \not\equiv 0. \tag{1.4.25}$$

Observe that every summand in $R_{f,g}$ is a multiple of a positive power of $a_{f(n-1)} = (x_{f^{(2)}(n-1)} - x_{f(n-1)})$. We focus in particular on the summand within $R_{f,g}$ which is a multiple of the largest possible power of the blue binomial $(x_{f^{(2)}(n-1)} - x_{f(n-1)})$, namely the summand associated with binary exponent assignments

$$s_{u,v,t} = 0, \text{ for all } \begin{array}{c} v \in f^{-1}\left(\{f\left(n-1\right)\}\right) \\ 0 \le u \le f\left(n-1\right) \\ t \in \{0,1\} \end{array} \text{ as well as } r_{u,v} = 0, \text{ for all } \begin{array}{c} v \in f^{-1}\left(\{f\left(n-1\right)\}\right) \\ 0 \le u \le f\left(n-1\right) \end{array}. \tag{1.4.26}$$

The said summand is

$$c \prod_{0 \le i < j < n} (x_j - x_i) \prod_{0 \le u < v \le f(n-1)} \left((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2\right) \left( \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} (b_v - b_u) \right) \left(a_{f(n-1)}\right)^m,$$

where

$$m = \left| \left\{ \begin{array}{c} v \in f^{-1}\left(\{f\left(n-1\right)\}\right) \\ f\left(n-1\right) < u < v \end{array} \right\} \right| + \left| \left\{ \begin{array}{c} v \in f^{-1}\left(\{f\left(n-1\right)\}\right) \\ 0 \le u \le f\left(n-1\right) \\ t \in \{0,1\} \end{array} \right\} \right| \text{ and } c = 2^{\left| \left\{ \begin{array}{c} v \in f^{-1}\left(\{f\left(n-1\right)\}\right) \\ f\left(n-1\right) < u < v \end{array} \right\} \right|}. \tag{1.4.27}$$

The said summand is thus given by

$$c \prod_{0 \le i < j < n} (x_j - x_i) \prod_{0 \le u < v \le f(n-1)} \left((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2\right) \left( \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} (x_u - x_v) \right) (x_{f^{(2)}(n-1)} - x_{f(n-1)})^m. \tag{1.4.28}$$

It follows from the premise $0 \not\equiv P_g$ that the remainder of the chosen summand is non–vanishing. Observe that the factor

$$\prod_{0 \le i < j < n} (x_j - x_i) \prod_{0 \le u < v \le f(n-1)} \left((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2\right) \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} (b_v - b_u) \tag{1.4.29}$$

is common to every summand in $R_{f,g}$. Factoring out the common factor, we write

$$R_{f,g}\left(\mathbf{x}\right) = \left( \prod_{0 \le i < j < n} (x_j - x_i) \prod_{0 \le u < v \le f(n-1)} \left((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2\right) \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} (b_v - b_u) \right) R1_{f,g}(\mathbf{x}),$$

where

$$\begin{aligned} R1_{f,g} = & \left( \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} (b_v + b_u) \right) \left( \sum_{\substack{s_{u,v,t} \in \{0,1\} \\ 0 = \prod s_{u,v,t}}} \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ 0 \le u \le f(n-1) \\ t \in \{0,1\}}} \left(b_{u,v,t}\right)^{s_{u,v,t}} \left(a_{f(n-1)}\right)^{1 - s_{u,v,t}} \right) + \\ & \left( \prod_{\substack{t \in \{0,1\} \\ v \in f^{-1}(\{f(n-1)\}) \\ 0 \le u \le f(n-1)}} b_{u,v,t} \right) \left( \sum_{\substack{r_{u,v} \in \{0,1\} \\ 0 = \prod r_{u,v}}} \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} (b_v + b_u)^{r_{u,v}} \left(2 a_{f(n-1)}\right)^{1 - r_{u,v}} \right) + \\ & \left( \sum_{\substack{s_{u,v,t} \in \{0,1\} \\ 0 = \prod s_{u,v,t}}} \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ 0 \le u \le f(n-1) \\ t \in \{0,1\}}} \left(b_{u,v,t}\right)^{s_{u,v,t}} \left(a_{f(n-1)}\right)^{1 - s_{u,v,t}} \right) \left( \sum_{\substack{r_{u,v} \in \{0,1\} \\ 0 = \prod r_{u,v}}} \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} (b_v + b_u)^{r_{u,v}} \left(2 a_{f(n-1)}\right)^{1 - r_{u,v}} \right). \end{aligned}$$

Observe that for each $\sigma \in \left\{\theta \in \mathrm{S}_n : G_{\theta g \theta^{-1}} \in \mathrm{GrL}(G_g)\right\}$, there is a non-vanishing integer evaluation

$$\begin{aligned} v_\sigma = & \prod_{0 \le i < j < n} (\sigma(j) - \sigma(i)) \times \\ & \prod_{0 \le u < v \le f(n-1)} \left((\sigma f(v) - \sigma(v))^2 - (\sigma f(u) - \sigma(u))^2\right) \times \\ & \prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} (\sigma(u) - \sigma(v)) \,. \end{aligned} \tag{1.4.30}$$

More generally, by Proposition 1.3.1, the premise $P_g \equiv R_{f,g}$ implies

$$P_g(h) = R_{f,g}(h) = v_h \cdot R1_{f,g}(h) \,, \; \forall \, h \in \mathbb{Z}_n^{\mathbb{Z}_n}. \tag{1.4.31}$$

By Lemma 1.4.1, the same premise implies that any transposition $\tau \in \mathrm{S}_n$ which exchanges $x_{f(n-1)}$ with $x_v$ where $v \in f^{-1}(\{f(n-1)\})$ fixes the remainder of $R_{f,g}$ is congruent to given by

$$\sum_{\substack{\sigma \in \mathrm{S}_n \\ G_{\sigma g \sigma^{-1}} \in \mathrm{GrL}(G_g)}} v_\sigma \cdot R1_{f,g}(\sigma) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left(\frac{x_i - j_i}{\sigma(i) - j_i}\right), \tag{1.4.32}$$

which is congruent to

$$\left(\sum_{\substack{\sigma \in \mathrm{S}_n \\ G_{\sigma g \sigma^{-1}} \in \mathrm{GrL}(G_g)}} v_\sigma \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left(\frac{x_i - j_i}{\sigma(i) - j_i}\right)\right) \left(\sum_{\substack{\sigma \in \mathrm{S}_n \\ G_{\sigma g \sigma^{-1}} \in \mathrm{GrL}(G_g)}} R1_{f,g}(\sigma) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left(\frac{x_i - j_i}{\sigma(i) - j_i}\right)\right)$$

and in turn by averaging implies that such a transposition $\tau$ also fixes

$$\left(\sum_{\substack{\sigma \in \mathrm{S}_n \\ G_{\sigma g \sigma^{-1}} \in \mathrm{GrL}(G_g)}} v_\sigma\right) \sum_{\substack{\sigma \in \mathrm{S}_n \\ G_{\sigma g \sigma^{-1}} \in \mathrm{GrL}(G_g)}} R1_{f,g}(\sigma) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left(\frac{x_i - j_i}{\sigma(i) - j_i}\right). \tag{1.4.33}$$

As a consequence of the multi-binomial expansion, the sum expressing $R1_{f,g}$ features as one of its summand a unique monochromatic blue binomial summand given by

$$\left(a_{f(n-1)}\right)^m = c\,(x_{f^{(2)}(n-1)} - x_{f(n-1)})^m.$$

This monochromatic summand makes up the blue part of the chosen bichromatic summand of $R_{f,g}$ given by

$$c \prod_{0 \le i < j < n} (x_j - x_i) \prod_{0 \le u < v \le f(n-1)} \left((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2\right) \left(\prod_{\substack{v \in f^{-1}(\{f(n-1)\}) \\ f(n-1) < u < v}} (b_v - b_u)\right) \left(a_{f(n-1)}\right)^m,$$

which is also described by equation (1.4.28). In fact, by Lemma 1.3.28, the summand of $R_{f,g}$ described in equation (1.4.28) is congruent to

$$\begin{aligned} & c \left(\sum_{\substack{\sigma \in \mathrm{S}_n \\ G_{\sigma g \sigma^{-1}} \in \mathrm{GrL}(G_g)}} v_\sigma \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left(\frac{x_i - j_i}{\sigma(i) - j_i}\right)\right) \sum_{\substack{\sigma \in \mathrm{S}_n \\ G_{\sigma g \sigma^{-1}} \in \mathrm{GrL}(G_g)}} (\sigma f^{(2)}(n-1) - \sigma f(n-1))^m \times \\ & \prod_{j_{f^{(2)}(n-1)} \in \mathbb{Z}_n \setminus \{\sigma f^{(2)}(n-1)\}} \left(\frac{x_{f^{(2)}(n-1)} - j_{f^{(2)}(n-1)}}{\sigma f^{(2)}(n-1) - j_{f^{(2)}(n-1)}}\right) \prod_{j_{f(n-1)} \in \mathbb{Z}_n \setminus \{\sigma f(n-1)\}} \left(\frac{x_{f(n-1)} - j_{f(n-1)}}{\sigma f(n-1) - j_{f(n-1)}}\right) \end{aligned}$$

As prescribed by equation (1.4.33), we apply the averaging argument to each bichromatic summand of $R_{f,g}$. In particular, applying the averaging argument to the chosen summand yields the polynomial

$$c \left(\sum_{\substack{\sigma \in \mathrm{S}_n \\ G_{\sigma g \sigma^{-1}} \in \mathrm{GrL}(G_g)}} v_\sigma\right) \sum_{\substack{\sigma \in \mathrm{S}_n \\ G_{\sigma g \sigma^{-1}} \in \mathrm{GrL}(G_g)}} (\sigma f^{(2)}(n-1) - \sigma f(n-1))^m \times$$

$$\prod_{j_{f^{(2)}(n-1)}\in\mathbb{Z}_n\setminus\{\sigma f^{(2)}(n-1)\}}\left(\frac{x_{f^{(2)}(n-1)}-j_{f^{(2)}(n-1)}}{\sigma f^{(2)}\left(n-1\right)-j_{f^{(2)}(n-1)}}\right)\prod_{j_{f(n-1)}\in\mathbb{Z}_n\setminus\{\sigma f(n-1)\}}\left(\frac{x_{f(n-1)}-j_{f(n-1)}}{\sigma f\left(n-1\right)-j_{f(n-1)}}\right).$$

Let us now focus on the action of a transposition on individual summands of some polynomial resulting from an arbitrary but fixed partition of its non-vanishing monomial terms. There are exactly three distinct ways that a candidate transposition of variables can lie in the automorphism group of a given polynomial. Assume that we reason about a particular summand denoted as $P$.

(i) Option 1: The candidate transposition of a pair of variables fixes the chosen summand $P$. This occurs when $P$ is symmetric in the chosen pair of variables being transposed.

(ii) Option 2: The candidate transposition of the chosen pair of variables does not fix $P$ (i.e., Option 1 does not apply) but induces in turn a transposition which exchanges the chosen summand $P$ with some other summand from the partition say $Q$. This occurs, for instance, if we consider the sum $P+Q$ where $P=(x_0)^2\,x_1$ and $Q=x_0\,(x_1)^2$. In this example, we see that transposition which exchanges variables $x_0$ with $x_1$ does not fix $P$, but it induces a transposition which exchanges the summand $P$ with the summand $Q$.

(iii) Option 3: The candidate transposition of a pair of variables neither fixes $P$ nor does it induce a transposition which exchanges $P$ with some other summand (i.e., neither Option 1 nor Option 2 applies). Instead, $P$ is such that a symmetry broadening cancellation occurs. Such a cancellation must involve interaction between the non-vanishing monomials within the monomial support of $P$ with the non-vanishing monomials within the support of other summands. Option 3 occurs, for instance, if we take $P=-x_1$ and $Q=x_0+2x_1$. We see that in this example neither Option 1 nor Option 2 applies when the candidate transposition is the transposition which exchanges variables $x_0$ with $x_1$. However $P+Q=x_0+x_1$ is symmetric and thus admits the said transposition in its automorphism group. This fact is due to the symmetry broadening cancellation of like terms : $-x_1+2x_1$.

By Lemma 1.3.34, the monomial support of the polynomial immediately above is not fixed by any transposition $\tau\in\mathrm{S}_n$ which exchanges $x_{f(n-1)}$ with $x_v$ where $v\in f^{-1}\left(\{f\left(n-1\right)\}\right)$. This first observation accounts for Option 1. Also note that the remainder of the chosen summand does not exchange with the remainder of any other summands when we exchange $x_{f(n-1)}$ with $x_v$ where $v\in f^{-1}\left(\{f\left(n-1\right)\}\right)$ since the non-vanishing remainders of other bi-chromatic summand in $R1_{f,g}$ depends on 3 or more variables. This second observation accounts for Option 2. We now account for Option 3 and show that there are no symmetry-broadening cancellations which adjoin $\tau$ to the automorphism group. Again by Lemma 1.3.34, such a symmetry broadening cancellation can occur only for Lagrange bases

$$\prod_{j_{f^{(2)}(n-1)}\in\mathbb{Z}_n\setminus\{\sigma f^{(2)}(n-1)\}}\left(\frac{x_{f^{(2)}(n-1)}-j_{f^{(2)}(n-1)}}{\sigma f^{(2)}\left(n-1\right)-j_{f^{(2)}(n-1)}}\right)\prod_{j_{f(n-1)}\in\mathbb{Z}_n\setminus\{\sigma f(n-1)\}}\left(\frac{x_{f(n-1)}-j_{f(n-1)}}{\sigma f\left(n-1\right)-j_{f(n-1)}}\right).$$

where $\sigma\in\left\{\theta\in\mathrm{S}_n:G_{\theta g\theta^{-1}}\in\mathrm{GrL}\left(G_g\right)\right\}$ is subject to $\sigma\left(n-1\right)=0$ and $G_f$ is such that $1=\left|f^{-1}\left(\{f\left(n-1\right)\}\right)\right.$. In that setting, non-vanishing monomials occurring in the expanded form of said Lagrange bases summands possibly cancel out non-vanishing monomials occurring in the expanded form of Lagrange bases expressing canonical representative of bi-chromatic summands in $R1_{f,g}$ of the form

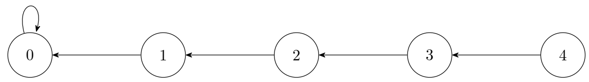
$$\left(b_{f(n-1),n-1,t}\right)^r\left(a_{f(n-1)}\right)^s=\left(\left(x_{f(n-1)}-x_{n-1}\right)+\left(-1\right)^t\left(x_{f^{(2)}(n-1)}-x_{f(n-1)}\right)\right)^r\left(x_{f^{(2)}(n-1)}-x_{f(n-1)}\right)^s.$$

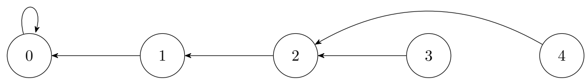
However, the restriction imposed by $\sigma\in\left\{\theta\in\mathrm{S}_n:G_{\theta g\theta^{-1}}\in\mathrm{GrL}\left(G_g\right)\right\}$ where $\sigma\left(n-1\right)=0$ breaks the complementary-labeling symmetry. Indeed by Proposition 1.3.25, the remainder is up to sign invariant to the involution prescribed by the map: $x_i\mapsto x_{n-1-i}$ for all $i\in\mathbb{Z}_n$. But the complementary labeling involution maps any Lagrange basis associated with $\sigma\in\left\{\theta\in\mathrm{S}_n:G_{\theta g\theta^{-1}}\in\mathrm{GrL}\left(G_g\right)\right\}$ such that $\sigma\left(n-1\right)=0$ to different Lagrange basis associated $\sigma'\in\left\{\theta\in\mathrm{S}_n:G_{\theta g\theta^{-1}}\in\mathrm{GrL}\left(G_g\right)\right\}$ such that $\sigma'\left(n-1\right)=n-1$ and thus negates the symmetry broadening cancellations. We see that a symmetry broadening cancellation which adjoins $\tau$ to the automorphism group of the canonical representative of $R_{f,g}$ would break the complementary labeling symmetry, thereby resulting in the contradiction

$$\tau\notin\mathrm{Aut}\left(\overline{P}_g\right).$$

We conclude that the desired claim $\emptyset\neq\mathrm{GrL}\left(G_g\right)\implies\mathrm{GrL}\left(G_f\right)\neq\emptyset$ holds. □

Example 1.4.34. We present a verification of Lemma 1.4.14 with an example of a path on 5 vertices.

$G_f$ : 


$G_g$ : 


Run the contents of `ex1434.sage` to verify.

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