

Composition Lemma for Lean4

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Composition Lemma

1.1. Overview

The *Composition Lemma* was developed and refined over 6 years, beginning in 2018, as a novel approach to settle in the affirmative the *Graceful Tree Conjecture*. The first of such papers was posted in [3] by Gngang. A further developed series of papers resolving the same conjecture again appeared in [4] and [5]. Recently, the same method has been applied to settle other longstanding conjectures in [1] and [2]. We comment that the series of papers shared on the open-source platform arXiv reflect the evolving landscape of Gngang's thought process, and the frequent re-uploads were driven by the natural progression and refinement of ideas. However, we recognize that these numerous edits may have unintentionally caused confusion and raised questions regarding the success of the method. In the current work, we aim to address these concerns by presenting a detailed blueprint of the proof, with the goal of formalizing it in Lean4.

1.2. Functional Directed Graphs

For notational convenience, let \mathbb{Z}_n denote the set whose members are the first n natural numbers, i.e.,

$$(1.2.1) \quad \mathbb{Z}_n := \{0, \dots, n-1\}.$$

For a function $f : \mathbb{Z}_m \rightarrow \mathbb{Z}_n$, we write $f \in \mathbb{Z}_n^{\mathbb{Z}_m}$. For $X \subseteq \mathbb{Z}_m$, $f(X)$ denotes the image of X under f , i.e.,

$$(1.2.2) \quad f(X) = \{f(i) : i \in X\},$$

and $|f(X)|$ denotes its cardinality. For $Y \subseteq \mathbb{Z}_n$, $f^{-1}(Y)$ denotes the pre-image of Y under f i.e.

$$(1.2.3) \quad f^{-1}(Y) = \{j \in \mathbb{Z}_m : f(j) \in Y\}$$

DEFINITION 1.2.4 (Functional digraphs). For an arbitrary $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$, the *functional directed graph* prescribed by f , denoted G_f , is such that the vertex set $V(G_f)$ and the directed edge set $E(G_f)$ are respectively as follows:

$$V(G_f) = \mathbb{Z}_n, \quad E(G_f) = \{(v, f(v)) : v \in \mathbb{Z}_n\}.$$

DEFINITION 1.2.5 (Graceful functional digraphs). The functional directed graph prescribed by $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ is graceful if there exist a bijection $\sigma \in S_n \subset \mathbb{Z}_n^{\mathbb{Z}_n}$ such that

$$(1.2.6) \quad \{|\sigma f \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\} = \mathbb{Z}_n.$$

If $\sigma = \text{id}$ (the identity function), then G_f — the functional directed graph prescribed by f — is gracefully labeled.

DEFINITION 1.2.7 (Automorphism group). For a functional directed graph G_f , its automorphism group, denoted $\text{Aut}(G_f)$, is defined as follows:

$$\text{Aut}(G_f) = \{\sigma \in S_n : \{(i, f(i)) : i \in \mathbb{Z}_n\} = \{(j, \sigma f \sigma^{-1}(j)) : j \in \mathbb{Z}_n\}\}.$$

For a polynomial $P \in \mathbb{C}[x_0, \dots, x_{n-1}]$, its automorphism group, denoted $\text{Aut}(P)$, is defined as follows:

$$\text{Aut}(P) = \{\sigma \in S_n : P(x_0, \dots, x_i, \dots, x_{n-1}) = P(x_{\sigma(0)}, \dots, x_{\sigma(i)}, \dots, x_{\sigma(n-1)})\}.$$

DEFINITION 1.2.8 (Graceful re-labelings). The set of distinct gracefully labeled functional directed graphs isomorphic to G_f is

$$\text{GrL}(G_f) := \left\{ G_{\sigma f \sigma^{-1}} : \begin{array}{l} \sigma \text{ is a representative of a coset in } S_n / \text{Aut}(G_f) \text{ and} \\ \mathbb{Z}_n = \{|\sigma f \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\} \end{array} \right\}$$

DEFINITION 1.2.9 (Complementary labeling involution). If $\varphi = n - 1 - \text{id}$, i.e. $\varphi \in \mathbb{Z}_n^{\mathbb{Z}_n}$ such that

$$\varphi(i) = n - 1 - i, \forall i \in \mathbb{Z}_n,$$

then for an arbitrary $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ the complementary labeling involution is defined as the map

$$f \mapsto \varphi f \varphi^{-1}$$

Observe that for all $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ the complementary labeling involution fixes the induced edge label of each edge as seen from the equality

$$(1.2.10) \quad |f(i) - i| = |\varphi f(i) - \varphi(i)|, \quad \forall i \in \mathbb{Z}_n.$$

In other words, induced edge labels are fixed by the vertex relabeling effected by φ . We call this induced edge label symmetry the *complementary labeling symmetry* of the functional directed graph G_f .

1.3. Quotient-Remainder Theorem and Lagrange Interpolation

PROPOSITION 1.3.1 (Multivariate Quotient-Remainder). Let $d(x) \in \mathbb{C}[x]$ be a degree n monic polynomial with simple roots, i.e.,

$$(1.3.2) \quad d(x) = \prod_{i \in \mathbb{Z}_n} (x - \alpha_i) \quad \text{and} \quad 0 \neq \prod_{0 \leq u < v < n} (\alpha_v - \alpha_u),$$

where $\{\alpha_u : u \in \mathbb{Z}_n\} \subset \mathbb{C}$. For all $P \in \mathbb{C}[x_0, \dots, x_{m-1}]$, there exists a unique remainder $r(x_0, \dots, x_{m-1}) \in \mathbb{C}[x_0, \dots, x_{m-1}]$ of degree at most $n - 1$ in each variable such that for quotients: $\{q_k(x_0, \dots, x_{n-1}) : k \in \mathbb{Z}_n\} \subset \mathbb{C}[x_0, \dots, x_{n-1}]$, we have

$$(1.3.3) \quad P(x_0, \dots, x_{m-1}) = r(x_0, \dots, x_{m-1}) + \sum_{u \in \mathbb{Z}_m} q_u(x_0, \dots, x_{m-1}) d(x_u).$$

PROOF. We prove by induction on the number of variables that the remainder admits the expansion

$$(1.3.4) \quad r(x_0, \dots, x_{m-1}) = \sum_{g \in \mathbb{Z}_n^m} P(\alpha_g) \prod_{i \in \mathbb{Z}_m} \left(\prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} \left(\frac{x_i - \alpha_{j_i}}{\alpha_{g(i)} - \alpha_{j_i}} \right) \right),$$

where for notational convenience $P(\alpha_g) := P(\alpha_{g(0)}, \dots, \alpha_{g(m-1)})$. The base case stems from the univariate quotient-remainder theorem over the field \mathbb{C} . The univariate-quotient remainder theorem over the field \mathbb{C} asserts that there exist a unique quotient-remainder pair $(q(x_0), r(x_0)) \in \mathbb{C}[x_0] \times \mathbb{C}[x_0]$ subject to

$$(1.3.5) \quad H(x_0) = q(x_0) d(x_0) + r(x_0),$$

where $r(x_0) \in \mathbb{C}[x_0]$ is of degree at most $n - 1$. It is completely determined by its evaluation over $\{\alpha_i : i \in \mathbb{Z}_n\}$, and by Lagrange interpolation we have

$$(1.3.6) \quad r(x_0) = \sum_{g \in \mathbb{Z}_n^1} H(\alpha_{g(0)}) \prod_{j_0 \in \mathbb{Z}_n \setminus \{g(0)\}} \left(\frac{x_0 - \alpha_{j_0}}{\alpha_{g(0)} - \alpha_{j_0}} \right),$$

thus establishing the claim in the base case. For the induction step, assume as our induction hypothesis that for all $F \in \mathbb{C}[x_0, \dots, x_{m-1}]$, we have

$$(1.3.7) \quad F = \sum_{k \in \mathbb{Z}_m} q_k(x_0, \dots, x_{m-1}) d(x_k) + \sum_{g \in \mathbb{Z}_n^m} F(\alpha_g) \prod_{i \in \mathbb{Z}_m} \left(\prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} \left(\frac{x_i - \alpha_{j_i}}{\alpha_{g(i)} - \alpha_{j_i}} \right) \right).$$

We proceed to show that the hypothesis implies that every polynomial in $m + 1$ variables also admits a similar expansion, thus establishing the desired claim. Consider a polynomial $H \in \mathbb{C}[x_0, \dots, x_m]$. We view H as a univariate polynomial in the variable x_m whose coefficients lie in the field of fraction $\mathbb{C}(x_0, \dots, x_{m-1})$. The univariate quotient-remainder theorem over the field of fractions $\mathbb{C}(x_0, \dots, x_{m-1})$ asserts that there exist a unique quotient-remainder pair

$$(q(x_m), r(x_m)) \in (\mathbb{C}(x_0, \dots, x_{m-1}))[x_m] \times (\mathbb{C}(x_0, \dots, x_{m-1}))[x_m]$$

subject to

$$(1.3.8) \quad H(x_0, \dots, x_m) = q(x_0, \dots, x_m) d(x_m) + r(x_0, \dots, x_m),$$

where $r(x_0, \dots, x_m) \in (\mathbb{C}(x_0, \dots, x_{m-1}))[x_m]$ is of degree at most $n-1$ in the variable x_m . We write

$$(1.3.9) \quad r(x_0, \dots, x_m) = \sum_{k \in \mathbb{Z}_n} a_k(x_0, \dots, x_{m-1}) (x_m)^k.$$

We now show that coefficients $\{a_k(x_0, \dots, x_{m-1}) : k \in \mathbb{Z}_n\}$ all lie in the polynomial ring $\mathbb{C}[x_0, \dots, x_{m-1}]$ via the equality

$$(1.3.10) \quad \left(\text{Vander} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_u \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \right) \cdot \begin{pmatrix} a_0(x_0, \dots, x_{m-1}) \\ \vdots \\ a_u(x_0, \dots, x_{m-1}) \\ \vdots \\ a_{n-1}(x_0, \dots, x_{m-1}) \end{pmatrix} = \begin{pmatrix} H(x_0, \dots, x_{m-1}, \alpha_0) \\ \vdots \\ H(x_0, \dots, x_{m-1}, \alpha_u) \\ \vdots \\ H(x_0, \dots, x_{m-1}, \alpha_{n-1}) \end{pmatrix},$$

where

$$(1.3.11) \quad \left(\text{Vander} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_u \\ \vdots \\ \alpha_u \end{pmatrix} \right) [i, j] = (\alpha_i)^j, \quad \forall 0 \leq i, j < n.$$

Since the Vandermonde matrix is invertible by the fact

$$(1.3.12) \quad 0 \neq \det \left(\text{Vander} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_u \\ \vdots \\ \alpha_u \end{pmatrix} \right) = \prod_{0 \leq u < v < n} (\alpha_v - \alpha_u),$$

we indeed have

$$(1.3.13) \quad \begin{pmatrix} a_0(x_0, \dots, x_{m-1}) \\ \vdots \\ a_u(x_0, \dots, x_{m-1}) \\ \vdots \\ a_{n-1}(x_0, \dots, x_{m-1}) \end{pmatrix} = \left(\text{Vander} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_u \\ \vdots \\ \alpha_u \end{pmatrix} \right)^{-1} \cdot \begin{pmatrix} H(x_0, \dots, x_{m-1}, \alpha_0) \\ \vdots \\ H(x_0, \dots, x_{m-1}, \alpha_u) \\ \vdots \\ H(x_0, \dots, x_{m-1}, \alpha_{n-1}) \end{pmatrix}.$$

Therefore, we have

$$(1.3.14) \quad H(x_0, \dots, x_m) = q_m(x_0, \dots, x_m) d(x_m) + \sum_{g(m) \in \mathbb{Z}_n} H(x_0, \dots, x_{m-1}, \alpha_{g(m)}) \prod_{j \in \mathbb{Z}_n \setminus \{g(m)\}} \left(\frac{x_m - \alpha_{j_m}}{\alpha_{g(m)} - \alpha_{j_m}} \right).$$

Applying the induction hypothesis to coefficients

$$\{H(x_0, \dots, x_{m-1}, \alpha_{g(m)}) : \alpha_{g(m)} \in \mathbb{C}\} \subset \mathbb{C}[x_0, \dots, x_{m-1}]$$

yields the desired expansion. Finally, quotients $\{q_k(x_0, \dots, x_{m-1}) : k \in \mathbb{Z}_m\}$ lie in the polynomial ring $\mathbb{C}[x_0, \dots, x_{m-1}]$ since the polynomial $H(x_0, \dots, x_{m-1}) - r(x_0, \dots, x_{m-1})$ lies in the ideal generated by members of the set $\{d(x_u) : u \in \mathbb{Z}_m\}$. \square

PROPOSITION 1.3.15 (Ring Homomorphism). *For an arbitrary $H \in \mathbb{C}[x_0, \dots, x_{n-1}]$, let \overline{H} denote the remainder of the congruence class*

$$H \text{ modulo the ideal generated by } \{d(x_i) : i \in \mathbb{Z}_n\},$$

where

$$d(x) = \prod_{i \in \mathbb{Z}_n} (x - \alpha_i) \quad \text{and} \quad 0 \neq \prod_{0 \leq u < v < n} (\alpha_v - \alpha_u),$$

Then the following hold:

- (i) For all $g \in \mathbb{Z}_n^{\mathbb{Z}_n}$, we have $\overline{H}(\alpha_g) = H(\alpha_g)$.
- (ii) If $H = H_0 + H_1$, where $H_0, H_1 \in \mathbb{C}[x_0, \dots, x_{n-1}]$, then $\overline{H_0} + \overline{H_1} = \overline{H}$.
- (iii) If $H = H_0 \cdot H_1$, where $H_0, H_1 \in \mathbb{C}[x_0, \dots, x_{n-1}]$, then $\overline{H} \equiv \overline{H_0} \cdot \overline{H_1}$.

PROOF. The first claim follows from Proposition 1.3.1 for we see that the divisor vanishes over the lattice. To prove the second claim we recall that

$$\begin{aligned} \overline{H} &= \sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_n}} H(\alpha_g) \prod_{i \in \mathbb{Z}_n} \left(\prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} \left(\frac{x_i - \alpha_{j_i}}{\alpha_{g(i)} - \alpha_{j_i}} \right) \right), \\ \Rightarrow \overline{H} &= \sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_n}} (H_0(\alpha_g) + H_1(\alpha_g)) \prod_{i \in \mathbb{Z}_n} \left(\prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} \left(\frac{x_i - \alpha_{j_i}}{\alpha_{g(i)} - \alpha_{j_i}} \right) \right), \\ \Rightarrow \overline{H} &= \sum_{k \in \mathbb{Z}_2} \sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_n}} H_k(\alpha_g) \prod_{i \in \mathbb{Z}_n} \left(\prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} \left(\frac{x_i - \alpha_{j_i}}{\alpha_{g(i)} - \alpha_{j_i}} \right) \right). \end{aligned}$$

Thus $\overline{H_0} + \overline{H_1} = \overline{H}$ as claimed. Finally the fact (iii) is a straightforward consequence of Proposition 1.3.16, which is proved next. \square

PROPOSITION 1.3.16. *Let $f, g \in \mathbb{Z}_n^{\mathbb{Z}_n}$. For congruence classes prescribed modulo the ideal generated by $\{d(x_i) : i \in \mathbb{Z}_n\}$, if*

$$d(x) = \prod_{i \in \mathbb{Z}_n} (x - \alpha_i) \text{ such that } 0 \neq \prod_{0 \leq u < v < n} (\alpha_v - \alpha_u),$$

then

$$L_f(\mathbf{x}) \cdot L_g(\mathbf{x}) \equiv \begin{cases} L_f(\mathbf{x}) & \text{if } f = g \\ 0 & \text{otherwise,} \end{cases}$$

PROOF. Observe that

$$L_f(\mathbf{x}) \cdot L_g(\mathbf{x}) = \prod_{i \in \mathbb{Z}_n} \left((c_{i,f} \frac{d(x_i)}{x_i - \alpha_{f(i)}}) (c_{i,g} \frac{d(x_i)}{x_i - \alpha_{g(i)}}) \right),$$

where

$$c_{i,f} = \prod_{j_i \in \mathbb{Z}_n \setminus \{f(i)\}} (\alpha_{f(i)} - \alpha_{j_i})^{-1} \quad \text{and} \quad c_{i,g} = \prod_{j_i \in \mathbb{Z}_n \setminus \{g(i)\}} (\alpha_{g(i)} - \alpha_{j_i})^{-1}.$$

If $f \neq g$, then there exists $j \in \mathbb{Z}_n$ such that $f(j) \neq g(j)$ and $L_f(\mathbf{x}) \cdot L_g(\mathbf{x})$ is a multiple of $(x_j)^n$, as a result of which we obtain $L_f(\mathbf{x}) \cdot L_g(\mathbf{x}) \equiv 0$. Alternatively if $f = g$, then

$$L_f(\mathbf{x}) \cdot L_g(\mathbf{x}) = (L_f(\mathbf{x}))^2 = L_f(\mathbf{x}) + \left((L_f(\mathbf{x}))^2 - L_f(\mathbf{x}) \right).$$

We now show that $(L_f(\mathbf{x}))^2 - L_f(\mathbf{x}) \equiv 0$ modulo the ideal generated by $\{d(x_i) : i \in \mathbb{Z}_n\}$.

$$\begin{aligned} (L_f(\mathbf{x}))^2 - L_f(\mathbf{x}) &= L_f(\mathbf{x}) (L_f(\mathbf{x}) - 1) \\ &= L_f(\mathbf{x}) \left(L_f(\mathbf{x}) - \sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_n}} L_g(\mathbf{x}) \right) \\ &= -L_f(\mathbf{x}) \left(\sum_{g \in \mathbb{Z}_n^{\mathbb{Z}_n} \setminus \{f\}} L_g(\mathbf{x}) \right) \\ &\equiv 0, \end{aligned}$$

where the latter congruence identity stems from the prior setting where $f \neq g$. \square

DEFINITION 1.3.17 (Polynomial of Grace). We define $P_f \in \mathbb{C}[x_0, \dots, x_{n-1}]$ for all $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ as follows:

$$(1.3.18) \quad P_f(\mathbf{x}) := \underbrace{\prod_{0 \leq u < v < n} (x_v - x_u)}_{V(x_0, \dots, x_{n-1})} \underbrace{\prod_{0 \leq u < v < n} ((x_{f(v)} - x_v)^2 - (x_{f(u)} - x_u)^2)}_{E_f(x_0, \dots, x_{n-1})}.$$

DEFINITION 1.3.19 (Congruence class). For polynomials $P, Q \in \mathbb{C}[x_0, \dots, x_{n-1}]$, if

$$(1.3.20) \quad P(\mathbf{x}) \equiv Q(\mathbf{x}) \pmod{\left\{ \prod_{j \in \mathbb{Z}_n} (x_i - j) : i \in \mathbb{Z}_n \right\}},$$

we simply write $P \equiv Q$.

Unless otherwise stated, all subsequent congruence identities are prescribed modulo the ideal of polynomials generated by members of the set

$$\left\{ \prod_{j \in \mathbb{Z}_n} (x_i - j) : i \in \mathbb{Z}_n \right\}$$

PROPOSITION 1.3.21 (Certificate of Grace). Let $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$. The functional directed graph G_f prescribed by f is graceful if and only if $P_f(\mathbf{x}) \not\equiv 0$.

PROOF. Observe that the vertex Vandermonde factor $V(\mathbf{x})$ is of degree exactly $n-1$ in each variable and therefore equal to its remainder, i.e.,

$$(1.3.22) \quad V(\mathbf{x}) = \sum_{\theta \in S_n} \text{sgn}(\theta) \prod_{i \in \mathbb{Z}_n} (x_i)^{\theta(i)} = \prod_{v \in \mathbb{Z}_n} (v!) \sum_{\theta \in S_n} \text{sgn}(\theta) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\theta(i)\}}} \left(\frac{x_i - j_i}{\theta(i) - j_i} \right),$$

where

$$(1.3.23) \quad \text{sgn}(\theta) := \prod_{0 \leq u < v < n} \left(\frac{\theta(v) - \theta(u)}{v - u} \right), \quad \forall \theta \in S_n.$$

When $n > 2$, for every $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$, the induced edge label Vandermonde factor $E_f(\mathbf{x})$ is of degree $> (n-1)$ in some of its variables. Therefore, by Proposition 1.3.1, we have

$$(1.3.24) \quad E_f(\mathbf{x}) = \sum_{l \in \mathbb{Z}_n} q_l(\mathbf{x}) \prod_{k \in \mathbb{Z}_n} (x_l - k) + \prod_{v \in \mathbb{Z}_n} (v!) \frac{(n-1+v)!}{(2v)!} \sum_{\substack{g \in \mathbb{Z}_n^{\mathbb{Z}_n} \\ |gf - g| \in S_n}} \text{sgn}(|gf - g|) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{g(i)\}}} \left(\frac{x_i - j_i}{g(i) - j_i} \right).$$

Observe that by the expansions in 1.3.22 and 1.3.24,

$$(1.3.25) \quad P_f(\mathbf{x}) = \sum_{l \in \mathbb{Z}_n} q_l(\mathbf{x}) V(\mathbf{x}) \prod_{k \in \mathbb{Z}_n} (x_l - k) + \left(\prod_{v \in \mathbb{Z}_n} v! \sum_{\theta \in S_n} \text{sgn}(\theta) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\theta(i)\}}} \left(\frac{x_i - j_i}{\theta(i) - j_i} \right) \right) \left(\prod_{v \in \mathbb{Z}_n} (v!) \frac{(n-1+v)!}{(2v)!} \sum_{\substack{g \in \mathbb{Z}_n^{\mathbb{Z}_n} \\ |gf - g| \in S_n}} \text{sgn}(|gf - g|) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{g(i)\}}} \left(\frac{x_i - j_i}{g(i) - j_i} \right) \right).$$

is congruent to

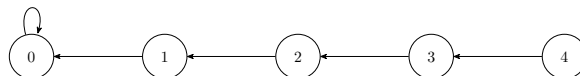
$$(1.3.25) \quad \prod_{v \in \mathbb{Z}_n} (v!)^2 \frac{(n-1+v)!}{(2v)!} \sum_{\substack{\sigma \in S_n \\ \text{s.t.} \\ |\sigma f - \sigma| \in S_n}} \text{sgn}(\sigma |\sigma f - \sigma|) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left(\frac{x_i - j_i}{\sigma(i) - j_i} \right),$$

where the permutation $|\sigma f - \sigma|$ denotes the induced edge label permutation associated with a graceful relabeling $G_{\sigma f \sigma^{-1}}$ of G_f . The congruence above stems from Prop. 1.3.16. A graceful labeling necessitates the integer coefficient

$$\prod_{0 \leq i < j < n} (j-i)(j^2-i^2) = \prod_{0 \leq i < j < n} (j-i)^2(j+i) = \prod_{v \in \mathbb{Z}_n} (v!)^2 \frac{(n-1+v)!}{(2v)!} \neq 0,$$

thus establishing the desired claim. \square

EXAMPLE 1.3.26. We present an example of a path on 5 vertices. This is known to be graceful, so we expect a non-zero remainder.



Run the SageMath script `ex1325.sage` to verify.

PROPOSITION 1.3.27 (Complementary Labeling Symmetry). *Let $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ and the remainder of P_f be*

$$(1.3.28) \quad \bar{P}_f(\mathbf{x}) := \prod_{v \in \mathbb{Z}_n} (v!)^2 \frac{(n-1+v)!}{(2v)!} \sum_{\substack{\sigma \in S_n \\ \text{s.t.} \\ |\sigma f - \sigma| \in S_n}} \text{sgn}(\sigma |\sigma f - \sigma|) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left(\frac{x_i - j_i}{\sigma(i) - j_i} \right).$$

The complementary labeling map $x_i \mapsto x_{n-1-i}$, for all $i \in \mathbb{Z}_n$, fixes \bar{P}_f up to sign.

PROOF. For notational convenience, let $\mathbf{x}_\varphi := (x_{\varphi(0)}, \dots, x_{\varphi(i)}, \dots, x_{\varphi(n-1)})$. Observe that for any permutation $\varphi \in S_n$, the action of φ on P_f yields equalities

$$\begin{aligned} P_f(\mathbf{x}_\varphi) &= \prod_{0 \leq u < v < n} (x_{\varphi(v)} - x_{\varphi(u)}) ((x_{\varphi f(v)} - x_{\varphi(v)})^2 - (x_{\varphi f(u)} - x_{\varphi(u)})^2), \\ &= \prod_{0 \leq \varphi^{-1}(i) < \varphi^{-1}(j) < n} (x_j - x_i) ((x_{\varphi f \varphi^{-1}(j)} - x_j)^2 - (x_{\varphi f \varphi^{-1}(i)} - x_i)^2). \end{aligned}$$

The last equality above features the indexing change of variable $u = \varphi^{-1}(i)$ and $v = \varphi^{-1}(j)$. If $\varphi \in \text{Aut}(G_f)$ then $P_f(x_{\varphi(0)}, \dots, x_{\varphi(n-1)})$ is up to sign equal to $P_{\varphi f \varphi^{-1}}$, in accordance with Definition 1.3.17. Furthermore, by the proof of Proposition 1.3.21, the action of φ on P_f yields the congruence identity

$$P_f(\mathbf{x}_\varphi) \equiv \bar{P}_f(\mathbf{x}_\varphi).$$

Hence,

$$\begin{aligned} \bar{P}_f(\mathbf{x}_\varphi) &= \prod_{v \in \mathbb{Z}_n} ((v!)^2 \frac{(n-1+v)!}{(2v)!}) \sum_{\substack{\sigma \in S_n \\ \text{s.t.} \\ |\sigma f - \sigma| \in S_n}} \text{sgn}(\sigma |\sigma f - \sigma|) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left(\frac{x_{\varphi(i)} - j_i}{\sigma(i) - j_i} \right), \\ &= \text{sgn}(\varphi) \prod_{v \in \mathbb{Z}_n} \left((v!)^2 \frac{(n-1+v)!}{(2v)!} \right) \sum_{\substack{\sigma \in S_n \\ \text{s.t.} \\ |\sigma f - \sigma| \in S_n}} \text{sgn}(\sigma |\sigma f - \sigma| \varphi^{-1}) \prod_{\substack{u \in \mathbb{Z}_n \\ v_u \in \mathbb{Z}_n \setminus \{\sigma \varphi^{-1}(u)\}}} \left(\frac{x_u - v_u}{\sigma \varphi^{-1}(u) - v_u} \right). \end{aligned}$$

If $\varphi = n-1 - \text{id}$, then, by the complementary labeling symmetry, we have

$$G_{\sigma f \sigma^{-1}} \in \text{GrL}(G_f) \iff G_{\sigma \varphi^{-1} f (\sigma \varphi^{-1})^{-1}} \in \text{GrL}(G_f)$$

Let \mathfrak{G} denote the subgroup of S_n whose members are $\{\text{id}, \varphi\}$. We write

$$\begin{aligned} \bar{P}_f(\mathbf{x}_\varphi) &= \\ &\prod_{v \in \mathbb{Z}_n} ((v!)^2 \frac{(n-1+v)!}{(2v)!}) \sum_{\substack{\sigma \in S_n / \mathfrak{G} \\ \gamma = |\sigma f - \sigma| \in S_n}} \text{sgn}(\sigma \gamma) \left(\text{sgn}(\varphi^{-1}) \prod_{\substack{u \in \mathbb{Z}_n \\ v_u \in \mathbb{Z}_n \setminus \{\sigma(u)\}}} \left(\frac{x_u - v_u}{\sigma(u) - v_u} \right) + \prod_{\substack{u \in \mathbb{Z}_n \\ v_u \in \mathbb{Z}_n \setminus \{\sigma \varphi^{-1}(u)\}}} \left(\frac{x_u - v_u}{\sigma \varphi^{-1}(u) - v_u} \right) \right). \end{aligned}$$

Similarly,

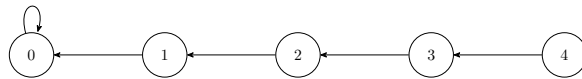
$$\begin{aligned} \bar{P}_f(\mathbf{x}) &= \\ &\prod_{v \in \mathbb{Z}_n} ((v!)^2 \frac{(n-1+v)!}{(2v)!}) \sum_{\substack{\sigma \in S_n / \mathfrak{G} \\ \gamma = |\sigma f - \sigma| \in S_n}} \text{sgn}(\sigma \gamma) \left(\prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}}} \left(\frac{x_i - j_i}{\sigma(i) - j_i} \right) + \text{sgn}(\varphi^{-1}) \prod_{\substack{i \in \mathbb{Z}_n \\ j_i \in \mathbb{Z}_n \setminus \{\sigma \varphi^{-1}(i)\}}} \left(\frac{x_i - j_i}{\sigma \varphi^{-1}(i) - j_i} \right) \right). \end{aligned}$$

We conclude that the complementary labeling symmetry yields the equality

$$\bar{P}_f(\mathbf{x}) = \text{sgn}(\varphi) \bar{P}_f(\mathbf{x}_\varphi) = \bar{P}_{\varphi f \varphi^{-1}}(\mathbf{x}),$$

thus establishing the desired claim. \square

EXAMPLE 1.3.29. We present an example of a path on 5 vertices.



Run the SageMath script `ex1328.sage` to verify.

1.4. The Composition Lemma

LEMMA 1.4.1 (Transposition Invariance). *Let $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ be such that its functional directed graph G_f has at least two sibling leaf nodes, i.e., G_f has vertices $u, v \in \mathbb{Z}_n$ such that $f^{-1}(\{u, v\}) = \emptyset$ and $f(u) = f(v)$. If the transposition $\tau \in S_n$ exchanges u and v , i.e.,*

$$\tau(i) = \begin{cases} v & \text{if } i = u \\ u & \text{if } i = v \\ i & \text{otherwise} \end{cases} \quad \forall i \in \mathbb{Z}_n.$$

Then

$$(1.4.2) \quad \tau \in \text{Aut}(P_f(\mathbf{x})),$$

where P_f is the polynomial certificate of grace as defined in 1.3.17.

PROOF. Stated otherwise, the claim asserts that the polynomial P_f is fixed by a transposition of any pair of variables associated with sibling leaf vertices. By construction of $P_f(\mathbf{x})$, the changes in its Vandermonde factors induced by the action of τ are as follows:

$$(1.4.3) \quad P_f(x_{\tau(0)}, \dots, x_{\tau(i)}, \dots, x_{\tau(n-1)}) = \prod_{0 \leq i < j < n} (x_{\tau(j)} - x_{\tau(i)}) \prod_{0 \leq i < j < n} ((x_{\tau f(j)} - x_{\tau(j)})^2 - (x_{\tau f(i)} - x_{\tau(i)})^2).$$

Note that there is a bijection

$$(1.4.4) \quad x_i \mapsto (x_{f(i)} - x_i)^2, \quad \forall i \in \mathbb{Z}_n.$$

Hence, the transposition $\tau \in \text{Aut}(G_f)$ of the leaf nodes induces a transposition τ of the corresponding leaf edges outgoing from the said leaf nodes. More precisely, the maps

$$\begin{pmatrix} x_0 & , \dots , & x_i & , \dots , & x_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ x_{\tau(0)} & , \dots , & x_{\tau(i)} & , \dots , & x_{\tau(n-1)} \end{pmatrix}$$

and

$$\begin{pmatrix} (x_{f(0)} - x_0)^2 & , \dots , & (x_{f(i)} - x_i)^2 & , \dots , & (x_{f(n-1)} - x_{n-1})^2 \\ \downarrow & & \downarrow & & \downarrow \\ (x_{\tau f(0)} - x_{\tau(0)})^2 & , \dots , & (x_{\tau f(i)} - x_{\tau(i)})^2 & , \dots , & (x_{\tau f(n-1)} - x_{\tau(n-1)})^2 \end{pmatrix}$$

prescribe the same permutation τ of the vertex variables and induced edges label binomials respectively. Observe that

$$\begin{aligned} & P_f(x_{\tau(0)}, \dots, x_{\tau(i)}, \dots, x_{\tau(n-1)}) = \\ & \left(\prod_{0 \leq i < j < n} \frac{x_{\tau(j)} - x_{\tau(i)}}{x_j - x_i} \prod_{0 \leq i < j < n} (x_j - x_i) \right) \left(\prod_{0 \leq i < j < n} \frac{(x_{\tau f(j)} - x_{\tau(j)})^2 - (x_{\tau f(i)} - x_{\tau(i)})^2}{(x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2} \prod_{0 \leq i < j < n} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \right) \\ & = \left(\text{sgn}(\tau) \prod_{0 \leq i < j < n} (x_j - x_i) \right) \left(\text{sgn}(\tau) \prod_{0 \leq i < j < n} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \right) \\ & = \left((-1) \prod_{0 \leq i < j < n} (x_j - x_i) \right) \left((-1) \prod_{0 \leq i < j < n} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \right) \\ (1.4.5) \quad & \implies P_f(x_{\tau(0)}, \dots, x_{\tau(n-1)}) = P_f(x_0, \dots, x_{n-1}), \end{aligned}$$

thus establishing the desired claim. \square

PROPOSITION 1.4.6 (Composition Inequality). *Consider an arbitrary $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ subject to the fixed point condition $|f^{(n-1)}(\mathbb{Z}_n)| = 1$. The following statements are equivalent:*

(i)

$$\max_{\sigma \in S_n} |\{\sigma f^{(2)} \sigma^{-1}(i) - i : i \in \mathbb{Z}_n\}| \leq \max_{\sigma \in S_n} |\{\sigma f \sigma^{-1}(i) - i : i \in \mathbb{Z}_n\}|.$$

(ii)

$$P_{f^{(2)}}(\mathbf{x}) \not\equiv 0 \implies P_f(\mathbf{x}) \not\equiv 0.$$

(iii)

$$GrL(G_f) \neq \emptyset$$

PROOF. If $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ is identically constant, then G_f is graceful. We see this from the fact that the functional digraph of the identically zero function is gracefully labeled and the fact that functional digraphs of identically constant functions are all isomorphic. It follows that all functional directed graphs having diameter less than 3 are graceful. Consequently, all claims hold for all functional digraphs of diameter less than 3. We now turn our attention to functional trees of diameter greater or equal to 3. It follows by definition

$$(1.4.7) \quad n = \max_{\sigma \in S_n} |\{|\sigma f \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}| \iff P_f(\mathbf{x}) \not\equiv 0 \iff GrL(G_f) \neq \emptyset.$$

We now proceed to show (i) \iff (iii). The backward claim is the simplest of the two claims. We see that if f is contractive, so too is $f^{(2)}$. Then the assertions

$$(1.4.8) \quad n = \max_{\sigma \in S_n} |\{|\sigma f^{(2)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}| \text{ and } n = \max_{\sigma \in S_n} |\{|\sigma f \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}|$$

indeed implies the inequality

$$(1.4.9) \quad \max_{\sigma \in S_n} |\{|\sigma f^{(2)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}| \leq \max_{\sigma \in S_n} |\{|\sigma f \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}|.$$

We now establish the forward claim by contradiction. Assume for the sake of establishing a contradiction that for some contractive map $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$ we have

$$(1.4.10) \quad n > \max_{\sigma \in S_n} |\{|\sigma f^{(2)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}|,$$

for we know by the number of edges being equal to n that it is impossible that

$$(1.4.11) \quad n < \max_{\sigma \in S_n} |\{|\sigma f^{(2)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}|.$$

Note that the range of f is a proper subset of \mathbb{Z}_n . By the premise that f is contractive, it follows that $f^{(\lceil 2^{\lg(n-1)} \rceil)}$ is identically constant and thus

$$(1.4.12) \quad n = \max_{\sigma \in S_n} |\{|\sigma f^{(\lceil 2^{\lg(n-1)} \rceil)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}|,$$

where \lg denotes the logarithm base 2. Consequently there must be some integer $0 \leq \kappa < \lg(n-1)$ such that

$$(1.4.13) \quad \max_{\sigma \in S_n} |\{|\sigma f^{(\lceil 2^\kappa \rceil)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}| > \max_{\sigma \in S_n} |\{|\sigma f^{(\lceil 2^{\kappa-1} \rceil)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}|.$$

This contradicts the assertion of statement (i), thereby establishing the backward claim. The exact same reasoning as above establishes (ii) \iff (iii), for we have

$$(1.4.14) \quad P_{f^{(\lceil 2^{\lg(n-1)} \rceil)}}(\mathbf{x}) \not\equiv 0.$$

□

Having assembled together the pieces required to prove our main result, we proceed to fit the pieces together to state and prove the *Composition Lemma*.

LEMMA 1.4.15 (Composition Lemma). *For all contractive $f \in \mathbb{Z}_n^{\mathbb{Z}_n}$, i.e., subject to $|f^{(n-1)}(\mathbb{Z}_n)| = 1$, we have*

$$(1.4.16) \quad \max_{\sigma \in S_n} |\{|\sigma f^{(2)} \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}| \leq \max_{\sigma \in S_n} |\{|\sigma f \sigma^{-1}(i) - i| : i \in \mathbb{Z}_n\}|.$$

PROOF. Owing to Proposition 1.4.6, we prove the statement by establishing

$$P_{f^{(2)}}(\mathbf{x}) \not\equiv 0 \implies P_f(\mathbf{x}) \not\equiv 0.$$

For simplicity, we prove a generalization of the desired claim. Assume without loss of generality that the vertex labeled $(n-1)$ is at furthest edge distance from the root in G_f (i.e. the fixed point). Given that the diameter of G_f is greater than 2, we may also assume without loss of generality that $f^{-1}(\{n-1\}) = \emptyset$ and $f^{(2)}(n-1) \neq f(n-1)$. Let the contractive map $g \in \mathbb{Z}_n^{\mathbb{Z}_n}$ be devised from f and an arbitrary nonempty subset $S \subseteq f^{-1}(\{f(n-1)\})$ such that

$$(1.4.17) \quad g(i) = \begin{cases} f^{(2)}(i) & \text{if } i \in S \\ f(i) & \text{otherwise} \end{cases}, \forall i \in \mathbb{Z}_n.$$

267 We show that

$$268 \quad (1.4.18) \quad P_g(\mathbf{x}) \not\equiv 0 \implies P_f(\mathbf{x}) \not\equiv 0.$$

269 Note that the assertion immediately above generalizes the composition lemma since, f is only partially iterated. More
 270 precisely, we iterate f on a subset S subject to $\emptyset \neq S \subseteq f^{-1}(\{f(n-1)\}) \subset \mathbb{Z}_n$. In turn, iterating (at most $\binom{n}{2}$ times)
 271 this generalization of the composition lemma yields that all functional trees are graceful, which in turn implies that the
 272 *Composition Lemma* as stated in Lemma 1.4.16 holds. For notational convenience, assume without loss of generality that

$$273 \quad (1.4.19) \quad f(n-1) = n - |f^{-1}(\{f(n-1)\})| - 1, \quad f^{-1}(\{f(n-1)\}) = \mathbb{Z}_n \setminus \mathbb{Z}_{1+f(n-1)} \text{ and } S = \{n-1, \dots, n-|S|\}.$$

274 If the conditions stated above are not met, we relabel the vertices of G_f to ensure that such is indeed the case. Note that
 275 such a relabeling does not affect the property we seek to prove. It suffices to show that the claim holds when $S = \{n-1\}$.
 276 We prove by contradiction the contrapositive claim

$$277 \quad (1.4.20) \quad P_f(\mathbf{x}) \equiv 0 \implies P_g(\mathbf{x}) \equiv 0.$$

278 By construction, the polynomial

$$279 \quad (1.4.21) \quad P_f(\mathbf{x}) = \prod_{0 \leq i < j < n-1} (x_j - x_i)((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \times \\ \prod_{u \in \mathbb{Z}_{n-1}} (x_{n-1} - x_u)((x_{f(n-1)} - x_{n-1})^2 - (x_{f(u)} - x_u)^2).$$

280 differs only slightly from

$$281 \quad (1.4.22) \quad P_g(\mathbf{x}) = \prod_{0 \leq i < j < n-1} (x_j - x_i)((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \times \\ \prod_{u \in \mathbb{Z}_{n-1}} (x_{n-1} - x_u)((x_{f^{(2)}(n-1)} - x_{n-1})^2 - (x_{f(u)} - x_u)^2).$$

282 Assume for the sake of establishing a contradiction that for some choice of function f subject to the premise above $P_g \not\equiv 0$
 283 and $P_f \equiv 0$. We setup a variable *telescoping* within each induced edge label binomial $(x_{f^{(2)}(v)} - x_v)$ where $v \in S$ (i.e. induced
 284 edge binomials of edges outgoing from the subset of vertices where f is iterated) as follows:

$$285 \quad \underbrace{(x_{f^{(2)}(n-1)} - x_{n-1})}_{x_{n-1} \longrightarrow x_{f^{(2)}(n-1)}} = \underbrace{(x_{f(n-1)} - x_{n-1})}_{x_{n-1} \longrightarrow x_{f(n-1)}} + \underbrace{(x_{f^{(2)}(v)} - x_{f(n-1)})}_{x_{f(n-1)} \longrightarrow x_{f^{(2)}(n-1)}}.$$

286 We start by breaking the symmetry among chromatic variables to distinguish more clearly distinct canceling partners and
 287 formally define the transpositions which generate the Θ -involution. We rewrite the telescoping setup as follows

$$288 \quad P_g(\mathbf{x}) = V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \times \\ \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} ((x_{f^{(2)}(n-1)} - x_{u,t,f(n-1)}) + (x_{u,t,f(n-1)} - x_{n-1}) + (-1)^t(x_{f(u)} - x_u)).$$

289 The telescoping setup introduces the $2(n-1)$ pair of canceling blue red variables:

$$290 \quad \left\{ (-x_{u,t,f(n-1)}, x_{u,t,f(n-1)}) : \begin{array}{l} u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\} \end{array} \right\}$$

291 into the expression of P_g . Invoking the multibinomial identity in the telescoping setup above yields a multibinomial expansion
 292 of P_g of the form

$$293 \quad P_g(\mathbf{x}) = V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \left[\prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} ((x_{u,t,f(n-1)} - x_{n-1}) + (-1)^t(x_{f(u)} - x_u)) + \right.$$

$$\sum_{\substack{s_{ut} \in \{0,1\} \\ 0 = \prod_{u,t} s_{ut}}} \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((\textcolor{red}{x}_{u,t,f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \right)^{s_{ut}} (x_{f^{(2)}(n-1)} \textcolor{blue}{-} x_{u,t,f(n-1)})^{1-s_{ut}},$$

which features $4^{(n-1)}$ chromatic multibinomial summands. Each chromatic multibinomial summand is associated with a choice of $s_{ut} \in \{0,1\}$ for each $(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}$ and is of the form

$$\prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((\textcolor{red}{x}_{u,t,f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \right)^{s_{ut}} (x_{f^{(2)}(n-1)} \textcolor{blue}{-} x_{u,t,f(n-1)})^{1-s_{ut}}$$

In both expressions of P_g above (which feature blue and red variables), chromatic variables cancel each other out by construction. Thus evaluations of chromatic variables play no role in evaluations of P_g overall. Consider the Abelian permutation group \mathcal{G} of order $4^{(n-1)}$ generated by the $2(n-1)$ disjoint transpositions:

$$(x_{f^{(2)}(n-1)} \textcolor{blue}{-} x_{u,t,f(n-1)}) \begin{matrix} \rightarrow \\ \leftarrow \end{matrix} (\textcolor{red}{x}_{u,t,f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \quad \forall (u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}.$$

Each transposition exchanges a distinct occurrence of a “blue” binomial (so named because the binomial features a distinct blue variable $\textcolor{blue}{-}x_{u,t,f(n-1)}$) with its corresponding complementary “red” quadrinomial partner (so named because the quadrinomial features the corresponding distinct canceling partner variable $\textcolor{red}{x}_{u,t,f(n-1)}$) and vice versa. This group acts upon expressions in the algebra \mathcal{A} generated by arbitrary sums and products of blue binomials, red quadrinomials in the set

$$\left\{ (x_{f^{(2)}(n-1)} \textcolor{blue}{-} x_{u,t,f(n-1)}), (\textcolor{red}{x}_{u,t,f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) : (u,t) \in \mathbb{Z}_{n-1} \times \{0,1\} \right\}.$$

Note that both expressions of P_g above which feature blue and red variables lie in \mathcal{A} . Let Θ — *involution* denote the special member of \mathcal{G} induced by enacting all $2(n-1)$ transpositions

$$(x_{f^{(2)}(n-1)} \textcolor{blue}{-} x_{u,t,f(n-1)}) \begin{matrix} \rightarrow \\ \leftarrow \end{matrix} (\textcolor{red}{x}_{u,t,f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \quad \forall (u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}.$$

Observe that \mathcal{G} acts transitively on the $4^{(n-1)}$ chromatic multibinomial summands featured in the multibinomial expansion of P_g .

For the purposes of the argument we restrict chromatic variables via assignments prescribed for all $(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}$ by

$$\textcolor{red}{x}_{u,t,f(n-1)} = \textcolor{red}{x}_{f(n-1)} \quad \text{and} \quad \textcolor{blue}{x}_{u,t,f(n-1)} = \textcolor{blue}{x}_{f(n-1)}.$$

In the present setting the $2(n-1)$ transpositions which induce the Θ -involution become

$$(x_{f^{(2)}(n-1)} \textcolor{blue}{-} \textcolor{blue}{x}_{f(n-1)}) \begin{matrix} \rightarrow \\ \leftarrow \end{matrix} (\textcolor{red}{x}_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \quad \forall (u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}.$$

The ensuing telescoping setup and multibinomial expansion of P_g are respectively

$$P_g(\mathbf{x}) = V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \times \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} ((x_{f^{(2)}(n-1)} \textcolor{blue}{-} \textcolor{blue}{x}_{f(n-1)}) + (\textcolor{red}{x}_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u))$$

and

$$P_g(\mathbf{x}) = V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \left[\prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} ((\textcolor{red}{x}_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u)) + \right.$$

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$$\sum_{\substack{s_{ut} \in \{0,1\} \\ 0 = \prod_{u,t} s_{ut}}} \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((x_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \right)^{s_{ut}} (x_{f(2)(n-1)} - x_{f(n-1)})^{1-s_{ut}} \Bigg].$$

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We now proceed to describe the *chromatic argument*. Let $R_{f,g}$ denote the polynomial expression which results from the removal of the monochromatic red summand from the multibinomial expansion of P_g . Namely

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$$R_{f,g}(\mathbf{x}) = V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \times \sum_{\substack{s_{ut} \in \{0,1\} \\ 0 = \prod_{u,t} s_{ut}}} \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} \left((x_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \right)^{s_{ut}} (x_{f(2)(n-1)} - x_{f(n-1)})^{1-s_{ut}}.$$

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By construction the expression of $R_{f,g}$ immediately above lies in \mathcal{A} and is not fixed by the Θ -involution. Indeed the Θ -involution maps $R_{f,g}$ to the polynomial which results from the removal of the monochromatic blue summand from the multibinomial expansion of P_g . More precisely, the Θ -involution maps $R_{f,g}$ to the polynomial expression

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$$V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \left[\prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((x_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \right) + \right.$$

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$$\left. \sum_{\substack{s_{ut} \in \{0,1\} \\ (\prod_{u,t} s_{ut}) = 0 \neq (\sum_{u,t} s_{ut})}} \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((x_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \right)^{s_{ut}} (x_{f(2)(n-1)} - x_{f(n-1)})^{1-s_{ut}} \right].$$

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Consequently chromatic variables no longer cancel each other out in the expression of $R_{f,g}$ but possibly cancel out in $\overline{R}_{f,g}$. We show that this is not the case. By contrast, on the multibinomial expansion of P_g , the Θ -involution transposes the unique monochromatic red summand

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$$P_f(\mathbf{x}) = V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} \left((x_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \right),$$

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with the unique monochromatic blue summand

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$$V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} (x_{f(2)(n-1)} - x_{f(n-1)}).$$

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Furthermore the Θ -involution transposes each multibinomial bichromatic summand of the form

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$$V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((x_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \right)^{s_{ut}} (x_{f(2)(n-1)} - x_{f(n-1)})^{1-s_{ut}},$$

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with the corresponding complementary multibinomial bichromatic summand

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$$V(\mathbf{x}) \prod_{0 \leq i < j < n-1} ((x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2) \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((x_{f(n-1)} - x_{n-1}) + (-1)^t (x_{f(u)} - x_u) \right)^{1-s_{u,t}} (x_{f(2)(n-1)} - x_{f(n-1)})^{s_{u,t}},$$

343

where

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$$s_{ut} \in \{0,1\} \text{ where } (u,t) \in \mathbb{Z}_{n-1} \times \{0,1\} \text{ and } \left(\prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} s_{ut} \right) = 0 \neq \left(\sum_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} s_{ut} \right).$$

We now describe the Θ -induced action multibinomial expansions of \overline{P}_g and $\overline{R}_{f,g}$ expressed with respect to the Lagrange basis. We start from congruence identities

$$\begin{aligned} \overline{P}_g(\mathbf{x}) \equiv & \sum_{\sigma \in S_n} \left(V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) L_\sigma \right) \left[\prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((\sigma f(n-1) - \sigma(n-1) + (-1)^t(\sigma f(u) - \sigma(u))) L_\sigma \right) + \right. \\ & \left. \sum_{\substack{s_{ut} \in \{0,1\} \\ 0 = \prod_{u,t} s_{ut}}} \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((\sigma f(n-1) - \sigma(n-1) + (-1)^t(\sigma f(u) - \sigma(u))) L_\sigma \right)^{s_{ut}} \left((\sigma f^{(2)}(n-1) - \sigma f(n-1)) L_\sigma \right)^{1-s_{ut}} \right]. \end{aligned}$$

and

$$\begin{aligned} \overline{R}_{f,g}(\mathbf{x}) \equiv & \sum_{\sigma \in S_n} \left(V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) L_\sigma \right) \times \\ & \left[\sum_{\substack{s_{ut} \in \{0,1\} \\ 0 = \prod_{u,t} s_{ut}}} \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left((\sigma f(n-1) - \sigma(n-1) + (-1)^t(\sigma f(u) - \sigma(u))) L_\sigma \right)^{s_{ut}} \left((\sigma f^{(2)}(n-1) - \sigma f(n-1)) L_\sigma \right)^{1-s_{ut}} \right]. \end{aligned}$$

Observe that transpositions

$$(x_{f^{(2)}(n-1)} - x_{u,t,f(n-1)}) \xrightarrow[\leftarrow]{\rightarrow} (x_{u,t,f(n-1)} - x_{n-1}) + (-1)^t(x_{f(u)} - x_u) \quad \forall (u,t) \in \mathbb{Z}_{n-1} \times \{0,1\},$$

induce an action on scaled Lagrange basis summand prescribed for each permutation $\sigma \in S_n$ and $(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}$

$$(\sigma f^{(2)}(n-1) - \sigma f(n-1)) L_\sigma(x_0, \dots, x_{f(n-1)}, x_{n-1}) \xrightarrow[\leftarrow]{\rightarrow} (\sigma f(n-1) - \sigma(n-1)) + (-1)^t(\sigma f(u) - \sigma(u)) L_\sigma(x_0, \dots, x_{f(n-1)}, x_{n-1}).$$

since

$$(x_{f^{(2)}(n-1)} - x_{f(n-1)}) = \sum_{h \in \mathbb{Z}_n^{Z_n}} (h f^{(2)}(n-1) - h f(n-1)) L_h(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}),$$

as well as the fact that for all $(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}$ we have

$$(x_{f(n-1)} - x_{n-1}) + (-1)^t(x_{f(u)} - x_u) = \sum_{h \in \mathbb{Z}_n^{Z_n}} (h f(n-1) - h(n-1)) + (-1)^t(h f(u) - h(u)) L_h(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}).$$

364 Invoking the orthonormality property of the Lagrange basis as well rules

$$\left\{ \begin{array}{l}
 L_{\sigma}(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}) \cdot \textcolor{red}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{red}{x}_{f(n-1)}, x_{n-1}) \text{ is replaced by } \textcolor{red}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{red}{x}_{f(n-1)}, x_{n-1}), \\
 L_{\sigma}(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}) \cdot \textcolor{blue}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{blue}{x}_{f(n-1)}, x_{n-1}) \text{ is replaced by } \textcolor{blue}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{blue}{x}_{f(n-1)}, x_{n-1}), \\
 L_{\sigma}(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}) \cdot \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}) \text{ is replaced by } \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}), \\
 \textcolor{red}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{red}{x}_{f(n-1)}, x_{n-1}) \cdot \textcolor{blue}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{blue}{x}_{f(n-1)}, x_{n-1}) \text{ is replaced by } \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}), \\
 \textcolor{blue}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{blue}{x}_{f(n-1)}, x_{n-1}) \cdot \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}) \text{ is replaced by } \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}), \\
 \textcolor{red}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{red}{x}_{f(n-1)}, x_{n-1}) \cdot \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}) \text{ is replaced by } \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}), \\
 \textcolor{red}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{red}{x}_{f(n-1)}, x_{n-1}) \cdot \textcolor{red}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{red}{x}_{f(n-1)}, x_{n-1}) \text{ is replaced by } \textcolor{red}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{red}{x}_{f(n-1)}, x_{n-1}), \\
 \textcolor{blue}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{blue}{x}_{f(n-1)}, x_{n-1}) \cdot \textcolor{blue}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{blue}{x}_{f(n-1)}, x_{n-1}) \text{ is replaced by } \textcolor{blue}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{blue}{x}_{f(n-1)}, x_{n-1}), \\
 \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}) \cdot \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}) \text{ is replaced by } \textcolor{violet}{L}_{\sigma}(x_0, x_1, \dots, \textcolor{violet}{x}_{f(n-1)}, x_{n-1}),
 \end{array} \right.$$

366 devised to record occurrences and colors of chromatic variables. By repeated use of these rules, we devise the following
 367 multibinomial expansions for \overline{P}_g and $\overline{R}_{f,g}$ respectively given by

$$\begin{aligned}
 \overline{P}_g = & \sum_{\sigma \in S_n} V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \left[\prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} (\sigma f^{(2)}(n-1) - \textcolor{blue}{\sigma f(n-1)}) \textcolor{blue}{L}_{\sigma} + \right. \\
 & \sum_{\substack{s_{ut} \in \{0,1\} \\ (\prod_{u,t} s_{ut}) = 0 \neq (\sum_{u,t} s_{ut})}} \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left(\textcolor{red}{\sigma f(n-1)} - \sigma(n-1) + (-1)^t (\sigma f(u) - \sigma(u)) \right)^{s_{ut}} (\sigma f^{(2)}(n-1) - \textcolor{blue}{\sigma f(n-1)})^{1-s_{ut}} \textcolor{violet}{L}_{\sigma} + \\
 & \left. \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left(\textcolor{red}{\sigma f(n-1)} - \sigma(n-1) + (-1)^t (\sigma f(u) - \sigma(u)) \right) \textcolor{red}{L}_{\sigma} \right]
 \end{aligned}$$

373 and

$$\begin{aligned}
 \overline{R}_{f,g} = & \sum_{\sigma \in S_n} V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \left[\prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} (\sigma f^{(2)}(n-1) - \textcolor{blue}{\sigma f(n-1)}) \textcolor{blue}{L}_{\sigma} + \right. \\
 & \sum_{\substack{s_{ut} \in \{0,1\} \\ (\prod_{u,t} s_{ut}) = 0 \neq (\sum_{u,t} s_{ut})}} \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left(\textcolor{red}{\sigma f(n-1)} - \sigma(n-1) + (-1)^t (\sigma f(u) - \sigma(u)) \right)^{s_{ut}} (\sigma f^{(2)}(n-1) - \textcolor{blue}{\sigma f(n-1)})^{1-s_{ut}} \textcolor{violet}{L}_{\sigma} \left. \right].
 \end{aligned}$$

In both expansion immediately above, for each permutation $\sigma \in S_n$, we partition scaled Lagrange basis according to their colors. For each permutation $\sigma \in S_n$, the expressions features three distinct types of chromatic scaled Lagrange basis summands. The first type is a scaling of a red Lagrange basis element defined as

$$L_\sigma(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}) := \prod_{j_{f(n-1)} \in \mathbb{Z}_n \setminus \{\sigma f(n-1)\}} \left(\frac{x_{f(n-1)} - j_{f(n-1)}}{\sigma f(n-1) - j_{f(n-1)}} \right) \prod_{i \in \mathbb{Z}_n \setminus \{f(n-1)\}} \left(\prod_{j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}} \left(\frac{x_i - j_i}{\sigma(i) - j_i} \right) \right),$$

which is scaled by a corresponding “red” coefficient (so named because it features the evaluation of red variable) of the precise form

$$P_f(\sigma) = V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} \left(\sigma f(n-1) - \sigma(n-1) + (-1)^t (\sigma f(u) - \sigma(u)) \right).$$

The second type of scaled chromatic Lagrange basis summand is a scaling of a blue Lagrange basis element defined as

$$L_\sigma(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}) := \prod_{j_{f(n-1)} \in \mathbb{Z}_n \setminus \{\sigma f(n-1)\}} \left(\frac{x_{f(n-1)} - j_{f(n-1)}}{\sigma f(n-1) - j_{f(n-1)}} \right) \prod_{i \in \mathbb{Z}_n \setminus \{f(n-1)\}} \left(\prod_{j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}} \left(\frac{x_i - j_i}{\sigma(i) - j_i} \right) \right),$$

which is scaled by the corresponding “blue” coefficient of the precise form

$$V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} (\sigma f^{(2)}(n-1) - \sigma f(n-1)).$$

The third and last type of scaled chromatic Lagrange basis summand is a scaling of a purple Lagrange basis element defined as

$$L_\sigma(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}) := \prod_{j_{f(n-1)} \in \mathbb{Z}_n \setminus \{\sigma f(n-1)\}} \left(\frac{x_{f(n-1)} - j_{f(n-1)}}{\sigma f(n-1) - j_{f(n-1)}} \right) \prod_{i \in \mathbb{Z}_n \setminus \{f(n-1)\}} \left(\prod_{j_i \in \mathbb{Z}_n \setminus \{\sigma(i)\}} \left(\frac{x_i - j_i}{\sigma(i) - j_i} \right) \right),$$

which is scaled by the corresponding (red and blue) bichromatic coefficient of the precise form

$$\prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} \left(\sigma f(n-1) - \sigma(n-1) + (-1)^t (\sigma f(u) - \sigma(u)) \right)^{s_{ut}} (\sigma f^{(2)}(n-1) - \sigma f(n-1))^{1-s_{ut}},$$

such that

$$s_{ut} \in \{0,1\} \text{ where } (u,t) \in \mathbb{Z}_{n-1} \times \{0,1\} \text{ and } \left(\prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} s_{ut} \right) = 0 \neq \left(\sum_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} s_{ut} \right).$$

For every $\sigma \in S_n$ the involutive action induced by the Θ -involution maps the scaled red Lagrange basis summand

$$P_f(\sigma) L_\sigma(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1})$$

to the blue scaled Lagrange basis summand

$$V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} (\sigma f^{(2)}(n-1) - \sigma f(n-1)) L_\sigma(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1})$$

and vice versa. Furthermore, the involutive action induced by the Θ -involution maps each scaled purple scaled Lagrange basis summand

$$V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \times \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} \left(\sigma f(n-1) - \sigma(n-1) + (-1)^t (\sigma f(u) - \sigma(u)) \right)^{s_{ut}} (\sigma f^{(2)}(n-1) - \sigma f(n-1))^{1-s_{ut}} L_\sigma(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}),$$

to the corresponding purple scaled Lagrange basis summand

$$V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \times \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} \left(\sigma f(n-1) - \sigma(n-1) + (-1)^t (\sigma f(u) - \sigma(u)) \right)^{1-s_{ut}} (\sigma f^{(2)}(n-1) - \sigma f(n-1))^{s_{ut}} L_\sigma(x_0, x_1, \dots, x_{f(n-1)}, x_{n-1}),$$

and vice versa. the action of Θ -involution pairs up (by the involution map) purple scaled Lagrange basis summands among themselves while simultaneously pairs up each red scaled Lagrange basis summand with a corresponding blue scaled Lagrange basis summand and vice versa. By our premise, the Θ -involution symmetry holds for $\bar{P}_g = \bar{R}_{f,g}$. Given that coefficients of red Lagrange basis summands all vanish i.e. $P_f(\sigma) = 0$ for all $\sigma \in S_n$, a necessary condition for the Θ -involution symmetry is that

$$V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} (\sigma f^{(2)}(n-1) - \sigma f(n-1)) = 0,$$

for all $\sigma \in S_n$. However the premise $P_g \neq 0$ implies

$$V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \prod_{(u,t) \in \mathbb{Z}_{n-1} \times \{0,1\}} (\sigma f^{(2)}(n-1) - \sigma f(n-1)) \neq 0,$$

for all $\sigma \in S_n$ subject to $G_{\sigma g \sigma^{-1}} \in \text{GrL}(G_g)$. This fact constitutes an obstruction to the Θ -involution symmetry. Indeed owing to the fact that chromatic variables should play no role in evaluations of \bar{P}_g we know that for each permutation $\sigma \in S_n$ the linear combination of chromatic evaluations featured in the coefficients of the corresponding Lagrange basis must vanish. However in the multibinomial expansion expressing $\bar{R}_{f,g}$, for each permutation $\sigma \in S_n$ the linear combination of chromatic evaluations featured in coefficients of Lagrange basis elements indexed by σ is equal to

$$-V(\sigma) \prod_{0 \leq i < j < n-1} ((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2) \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} (\sigma f(n-1)).$$

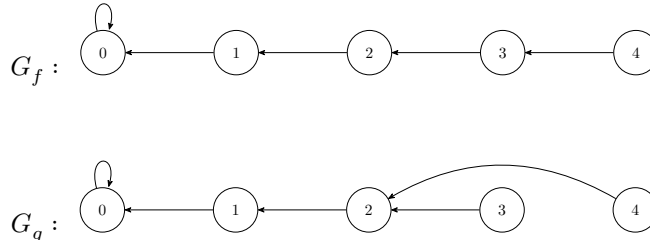
We call the quantity immediately above the chromatic defect. The chromatic defect is the left over after cancellation of evaluation of telescoping variables featured in the coefficient of the Lagrange basis indexed by σ . For we see that for each permutation σ the corresponding linear combinations of evaluations of chromatic variables which features as coefficient of the Lagrange basis indexed by $\sigma \in S_n$ is given by

$$0 = V(\sigma) \prod_{0 \leq i < j < n-1} \left((\sigma f(j) - \sigma(j))^2 - (\sigma f(i) - \sigma(i))^2 \right) \times \left[\prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} (\sigma f(n-1)) + \sum_{\substack{s_{ut} \in \{0,1\} \\ 0 = \prod_{u,t} s_{ut}}} \prod_{\substack{u \in \mathbb{Z}_{n-1} \\ t \in \{0,1\}}} (\sigma f(n-1))^{s_{ut}} (-\sigma f(n-1))^{1-s_{ut}} \right].$$

Roughly speaking the chromatic defect quantifies for a given permutation $\sigma \in S_n$ the extent to which the Θ -involution symmetry is broken. For the Θ -involution symmetry to hold the chromatic defect must vanish. However by the complementary involution symmetry there must be at least one permutation $\sigma \in \{\gamma, n-1-\gamma\}$ for which $\sigma f(n-1) \neq 0$ whenever $G_{\gamma g \gamma^{-1}} \in \text{GrL}(G_g)$. Thus we established that the chromatic defect is non-vanishing which in turn breaks the Θ -involution symmetry and affords a non-trivial role to evaluations of chromatic variables when evaluating $R_{f,g}$ over $\mathbb{Z}_n^{\mathbb{Z}_n}$. Thus contradicting the premise that $P_f \equiv 0$. By which we conclude that $P_g \neq R_{f,g} \implies P_f \neq 0$ as claimed. Given that \mathcal{G} acts transitively on chromatic multibinomial summands, a similar argument establishes that none of the chromatic summands is congruent to zero

□

EXAMPLE 1.4.23. We present a verification of Lemma 1.4.15 with an example of a path on 5 vertices.



Run the SageMath script `ex1434.sage` to verify.

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