Project 3: Monte-Carlo simulation of a neutrino beam

Computational Physics PHYC30012
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Aim

Monte Carlo methods are computational methods relying on repeated random sampling to obtain numerical results[1]. They can be used to computationally simulate systems which contain an element of chance - an event may or may not occur, or the time or place at which it occurs may follow a probabilistic distribution.

Here we look at a system which produces a neutrino beam through meson decay. A beam consisting of π-mesons and K-mesons (pions and kaons) is allowed to enter a decay tunnel. Here as the beam propagates, it undergoes meson decay into a muon and a neutrino.

Monte Carlo methods are useful here because many of the events leading to the production of the neutrino beam are probabilistic. These include:

1. The initial momenta of the mesons
2. The time at which the mesons decay
3. The direction in which the resulting neutrino propagates.

Meson Decay

In this experiment it is assumed that the initial meson beam is focussed (point-like) and directed along the x-axis. The meson decay occurs with the following continuous random time probability distribution (time spectrum):

$$f(t) = \frac{1}{\tau_0} e^{-1/\tau_0} \qquad \text{Equation 1}$$

where for pions, $\tau_0 = 2.608 \times 10^{-8}$ s and for kaons $\tau_0 = 1.237 \times 10^{-8}$ s.

The distance s in meters travelled by a meson in time t with momentum p and mass m before it decays (in the laboratory frame) is given by:

$$s = \frac{p}{m} ct \qquad \text{Equation 2}$$

where $c = 2.9979 \times 10^8$ (to account for the difference in units between p and m). For a pion, m = 0.1396 GeV/c^2 and for a kaon, m = 0.4937 GeV/c^2. Here p is in units of GeV/c.

Neutrino Momentum

Of the possible decay modes that produce neutrinos, only the following decay modes are considered:

$$\pi^+ \rightarrow \mu^+ + \upsilon_\mu \qquad \text{Equation 3}$$

$$K^+ \rightarrow \mu^+ + \upsilon_\mu \qquad \text{Equation 4}$$

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[1] Wikipedia, 2016, *Monte Carlo method*, viewed 17th September 2016, <https://en.wikipedia.org/wiki/Monte_Carlo_method>

We calculate the momentum magnitude of the resulting neutrinos thus. By conservation laws (for the pion, for example):

$$\vec{p}_{\mu} = -\vec{p}_{\upsilon} \qquad \text{Equation 5}$$

$$m_{\pi} = \sqrt{p_{\mu}^2 + m_{\mu}^2} + \sqrt{p_{\upsilon}^2 + m_{\upsilon}^2} \qquad \text{Equation 6}$$

Since we assume neutrinos are massless, their momentum magnitude is given by:

$$|p_{\upsilon}| = \frac{m_{\pi}^2 - m_{\mu}^2}{2m_{\pi}} \qquad \text{Equation 7}$$

where m_{μ} = 0.1057 GeV/c^2.

The momentum magnitude is now know, but the direction is yet to be determined. The meson decay is isotropic i.e. all directions have equal probability. If we examine a neutrino momentum $\vec{p}$ making angle θ with the x-axis, we see the problem has spherical symmetry with respect to ϕ (the angle away from the z-axis swept out by the transverse momentum $\vec{p}_t$).

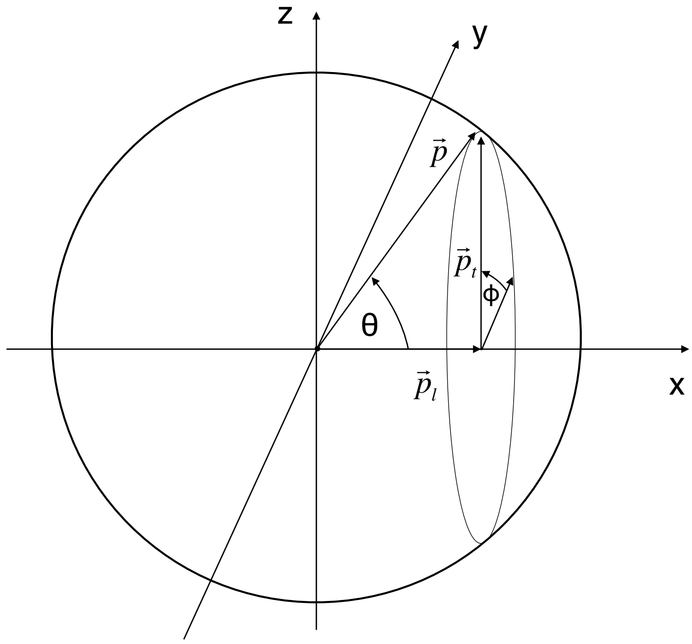

FIGURE 1 - ANALYSIS OF NEUTRINO DIRECTION AFTER MESON DECAY

To simulate this, we generate cos θ from a uniform distribution between -1 and 1, where θ is the angle the neutrino's momentum makes with the x axis.

The reason we generate cos θ from a uniform distribution is as follows. For a given value of θ, $\vec{p}_t$ could be in any direction ϕ. The circumference C of the circle that $\vec{p}_t$ traces out is C = $2\pi|\vec{p}_t|$ i.e. $C \propto p_t$ Now for larger values of C, the possible ϕ values of the neutrino direction are being spread over a larger (infinitesimal) area i.e. $P(\theta) \propto C$. Therefore

$P(\theta) \propto p_t$, and since $p_t \propto \sin\theta \Rightarrow P(\theta) \propto \sin\theta$. But $p_l^2 + p_t^2 = p \;\Rightarrow\; p_l = \sqrt{p^2 - p_t^2}$. Normalising to a value of p of 1, we have $p_l = \sqrt{1 - p_t^2}$ and therefore $P(\theta) \propto \cos\theta$.

We calculate the transverse and longitudinal components of the momentum thus:

$$p_l = p\cos\theta \qquad \text{Equation 8}$$

$$p_t = p\sin\theta \qquad \text{Equation 9}$$

Also, the energy of the neutrino is now given by $E_{rest} = |p|$. However these are all in the rest frame of the neutrino - they need to be transformed into the laboratory frame. This is done using the following Lorentz transformation:

$$\begin{pmatrix} p_l \\ p_t \\ E \end{pmatrix}_{lab} = \begin{pmatrix} \gamma & 0 & \beta\gamma \\ 0 & 1 & 0 \\ \beta\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_l \\ p_t \\ E \end{pmatrix}_{rest} \qquad \text{Equation 10}$$

where β is the velocity of the meson in the laboratory frame, expressed in units of the speed of light.

Thus:

$$(p_l)_{lab} = \gamma (p_l)_{rest} + \beta\gamma (E)_{rest} \qquad \text{Equation 11}$$

$$(p_t)_{lab} = (p_t)_{rest} \qquad \text{Equation 12}$$

$$(E)_{lab} = \beta\gamma (p_l)_{rest} + \gamma (E)_{rest} \qquad \text{Equation 13}$$

β and γ are calculated as follows (for the pion, for example):

$$\beta = \frac{|p_\pi|}{\sqrt{p_\pi^2 + m_\pi^2}} \qquad \text{Equation 14}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \qquad \text{Equation 15}$$

The simulated experimental apparatus is:

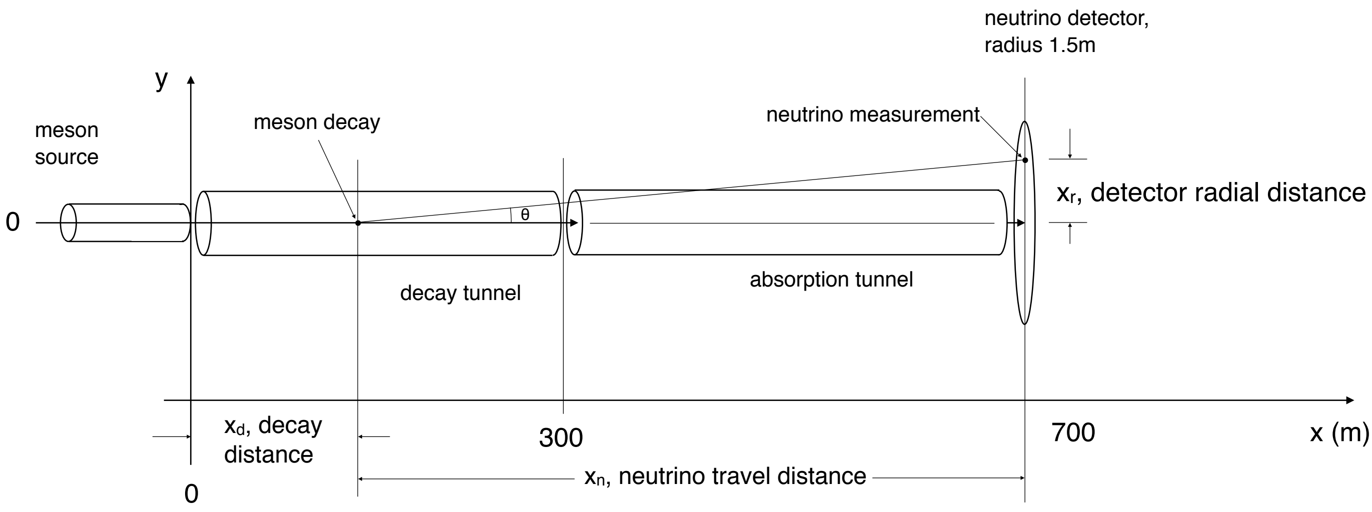

FIGURE 2 - EXPERIMENTAL APPARATUS FOR COMPUTATIONALLY SIMULATED GENERATION OF NEUTRINOS FROM π- AND K-MESONS

Generating Uniform Random Numbers

Generating truly random numbers on a computer, or indeed anywhere, is impossible. However generating distributions of numbers that are very close to random, or pseudo-random, can be done using a variety of methods. One of these is the linear congruential sequence given by:

$$x_{n+1} = \left|(a \times x_n + c), m\right| \qquad \text{Equation 16}$$

where $x_0 > 0$ is an arbitrary starting value (the seed), $a > 0$ is the multiplication factor, $c \geq 0$ is the increment, (m-1) gives the maximum value of a number from the sequence, and x_{n+1} is the next pseudo-random number generated. This is a algorithm used by the drand48() function in c, using a linear congruential algorithm and 48-bit integer arithmetic[2], where the seed is set by the function srand48(), passing in a number with which to begin the sequence. In our program we pass in the current time (in Unix epoch time).

Generating Discrete Non-uniform Random Numbers

A discrete binomial distribution can be generated as follows. Say we want a distribution where P(0) = a and P(1) = b = 1 - a. We can use the following algorithm - take a number u from the uniform generator. If u < a, generate 0. Otherwise (ie if u ≥ a), generate 1. We will use this several times in our procedure.

Generating Continuous Non-uniform Random Numbers

One way of doing this is using the inverse transform method as follows. If we have the continuous distribution f(x) on the interval (x_{min}, x_{max}), we determine the cumulative distribution:

$$F(x) = \int_{x_{min}}^{x_{max}} f(x')\,dx' \qquad \text{Equation 17}$$

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[2] The Open Group 1997, *The Single UNIX ® Specification, Version 2*, viewed 18th Sep 2016, <http://pubs.opengroup.org/onlinepubs/7908799/xsh/drand48.html>

We then take u from the uniform generator and generate x by solving u = F(x), i.e. x = $F^{-1}(u)$.

For some distributions, the cumulative distribution has no analytical form. An example of this is the normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \text{Equation 18}$$

In this case, the Box Muller method may be used. The exact Box Muller method is as follows. Take x_1 and x_2 from the uniform distribution between 0 and 1. Then calculate:

$$y_1 = \sqrt{-2\ln x_1}.\cos(2\pi x_2) \qquad \text{Equation 19}$$

$$y_2 = \sqrt{-2\ln x_1}.\sin(2\pi x_2) \qquad \text{Equation 20}$$

This method is exact but not very fast. The fast direct method is as follows. Take x_1 and x_2 from the uniform distribution between 0 and 1. Then calculate:

$$u_1 = 2x_1 - 1 \qquad \text{Equation 21}$$

$$u_2 = 2x_2 - 1 \qquad \text{Equation 22}$$

$$d = u_1^2 + u_2^2 \qquad \text{Equation 23}$$

If $d > 1$, reject u_1 and u_2. If $d \le 1$, calculate:

$$y_1 = u_1\sqrt{-2\ln(d)/d} \qquad \text{Equation 24}$$

$$y_2 = u_2\sqrt{-2\ln(d)/d} \qquad \text{Equation 25}$$

Now it can be shown that y_1 and y_2 are independent and follow the standard normal distribution.

Arbitrary Normal Distribution

To transform y calculated above to an arbitrary normal distribution with mean value a and standard deviation σ, calculate:

$$z = a + \sigma y \qquad \text{Equation 26}$$

Method and Results

1. A flowchart for generating a normal distribution of pions and kaons, each with a mean of 200 and standard deviation 10 is shown in Figure 1.

   These were generated using the Box-Muller method. We want a meson beam that contains 86% pions and 14% kaons. We use the method described in the section entitled Generating Discrete Non-uniform Random Numbers above, to generate a discrete binomial distribution to implement this.

2. Plots of the histograms of momentum of the decaying π and K mesons are shown in Figures 4 and 5 respectively.

   We see that we have generated more pions than kaons (there are 86% pions compared to 14% kaons), therefore the peak of the pion histogram in Figure 4 is higher at around 2×10^5 counts, compared to the peak of the kaon histogram in Figure 5 of around 3.5×10^4.

FIGURE 1 - FLOWCHART OF AN ALGORITHM FOR COMPUTATIONALLY GENERATING A NORMAL DISTRIBUTION WITH MEAN 200 AND STANDARD DEVIATION 10

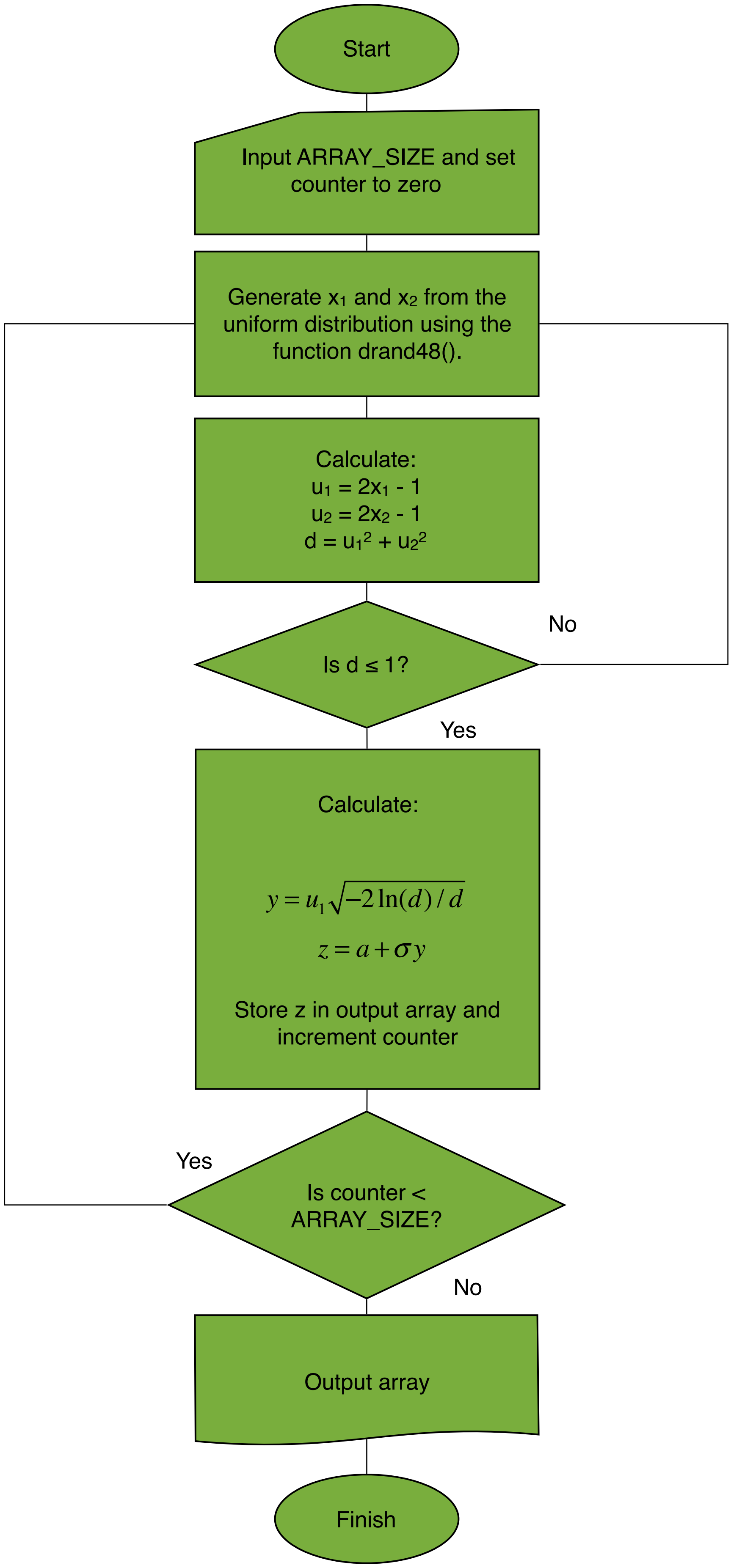


Figure 4 – Histogram of pion momentum distribution

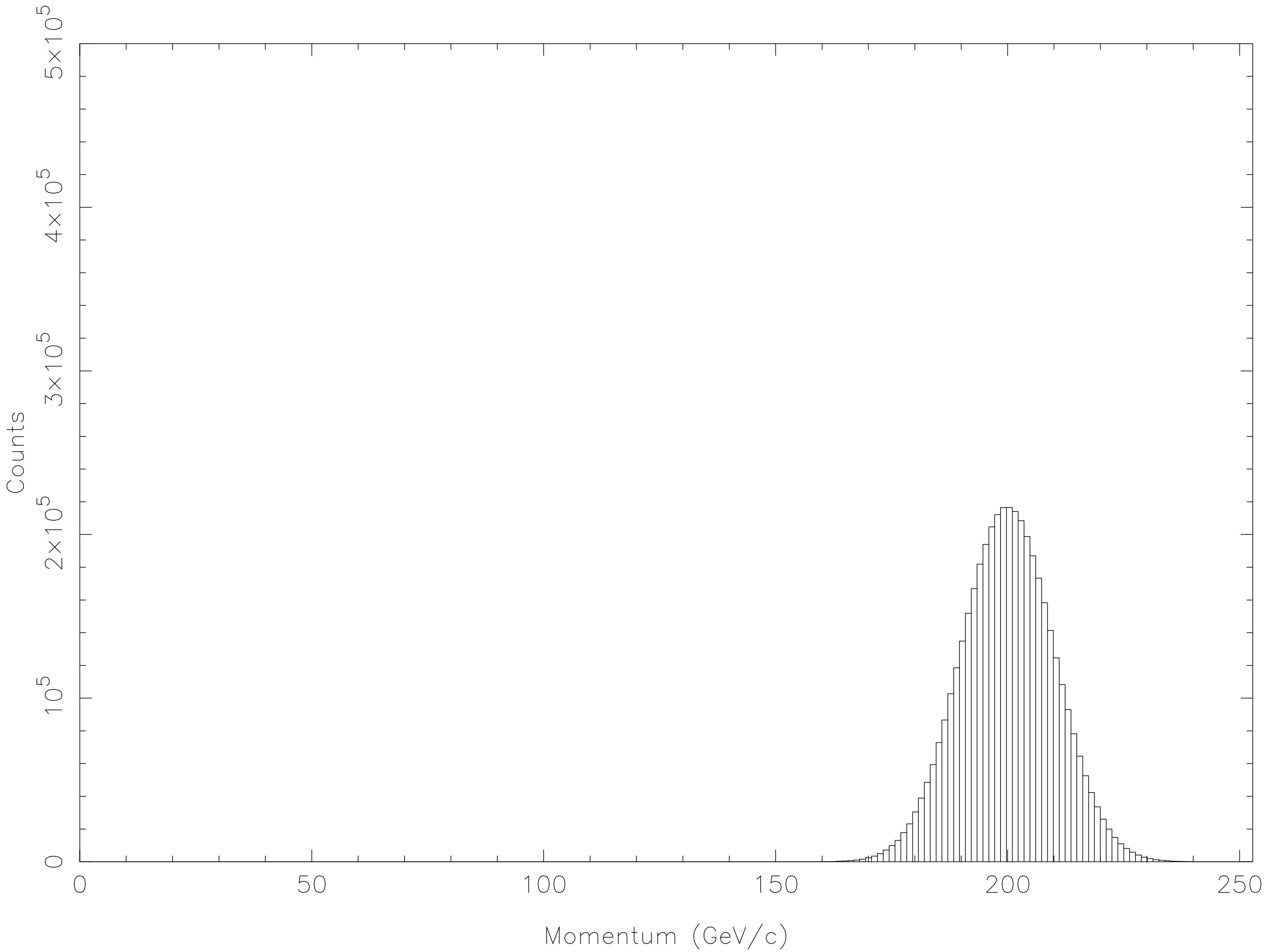
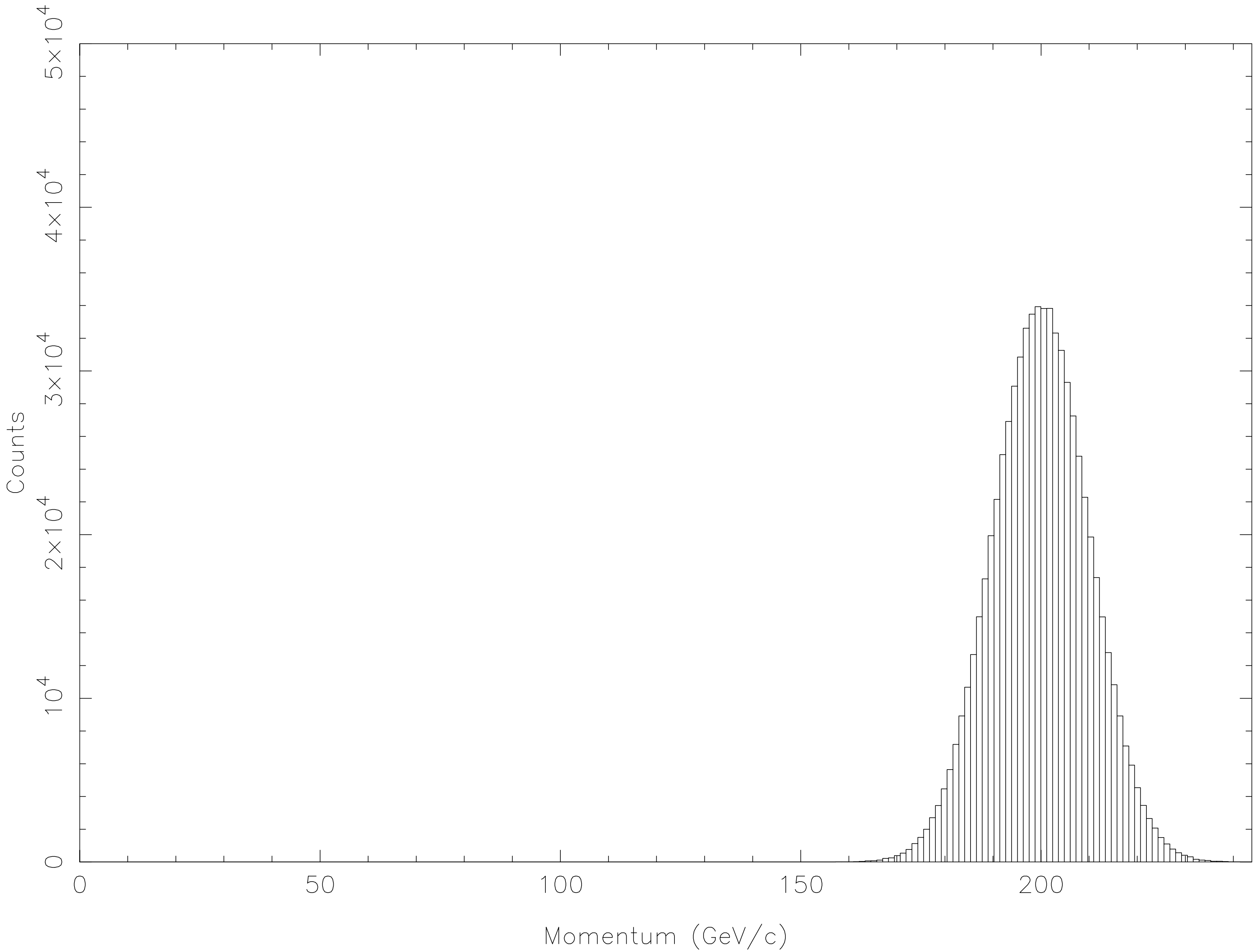


Figure 5 – Histogram of Kaon momentum distribution



3. Plots of the histograms of decay distance of the pions and kaons are shown in Figures 6 and 7 respectively. They are plotted to an arbitrary limit of 10^4 m, ignoring for now the finite length of the decay tunnel.

   The distributions are different in that the pions are seen to decay over much longer distances than the kaons. This is explained by the fact that the pion decay lifetime τ_0 = 2.608 x 10^{-8} s is larger than the kaon value, τ_0 = 1.237 x 10^{-8} s (roughly twice as large). Additionally, we have more pions than kaons. Therefore the histogram of pion decay distance compared to kaon decay distance will be higher in general, and decay more slowly.

Figure 6 – Histogram of pion decay distance (up to arbitrary limit of 10,000m)

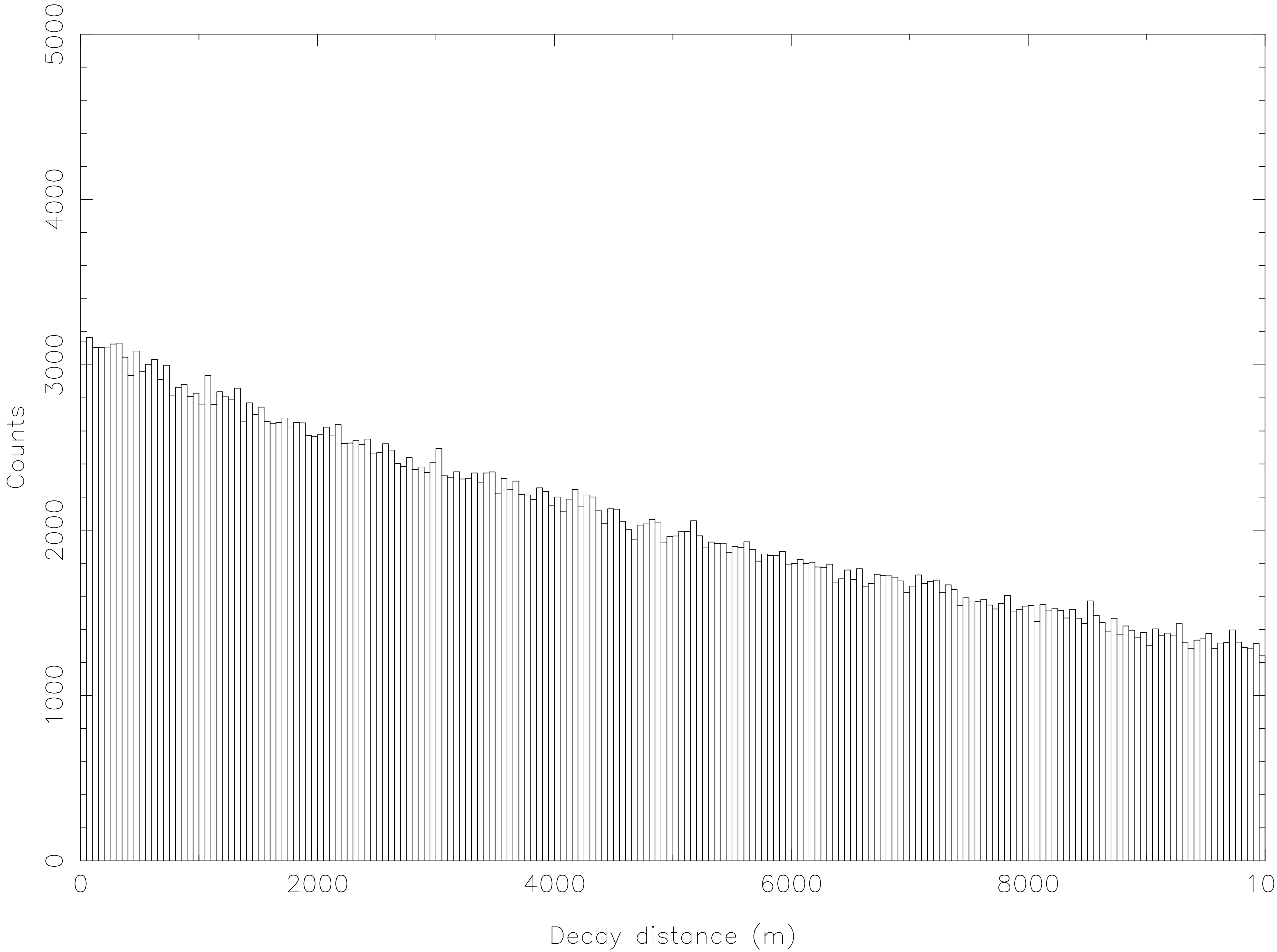
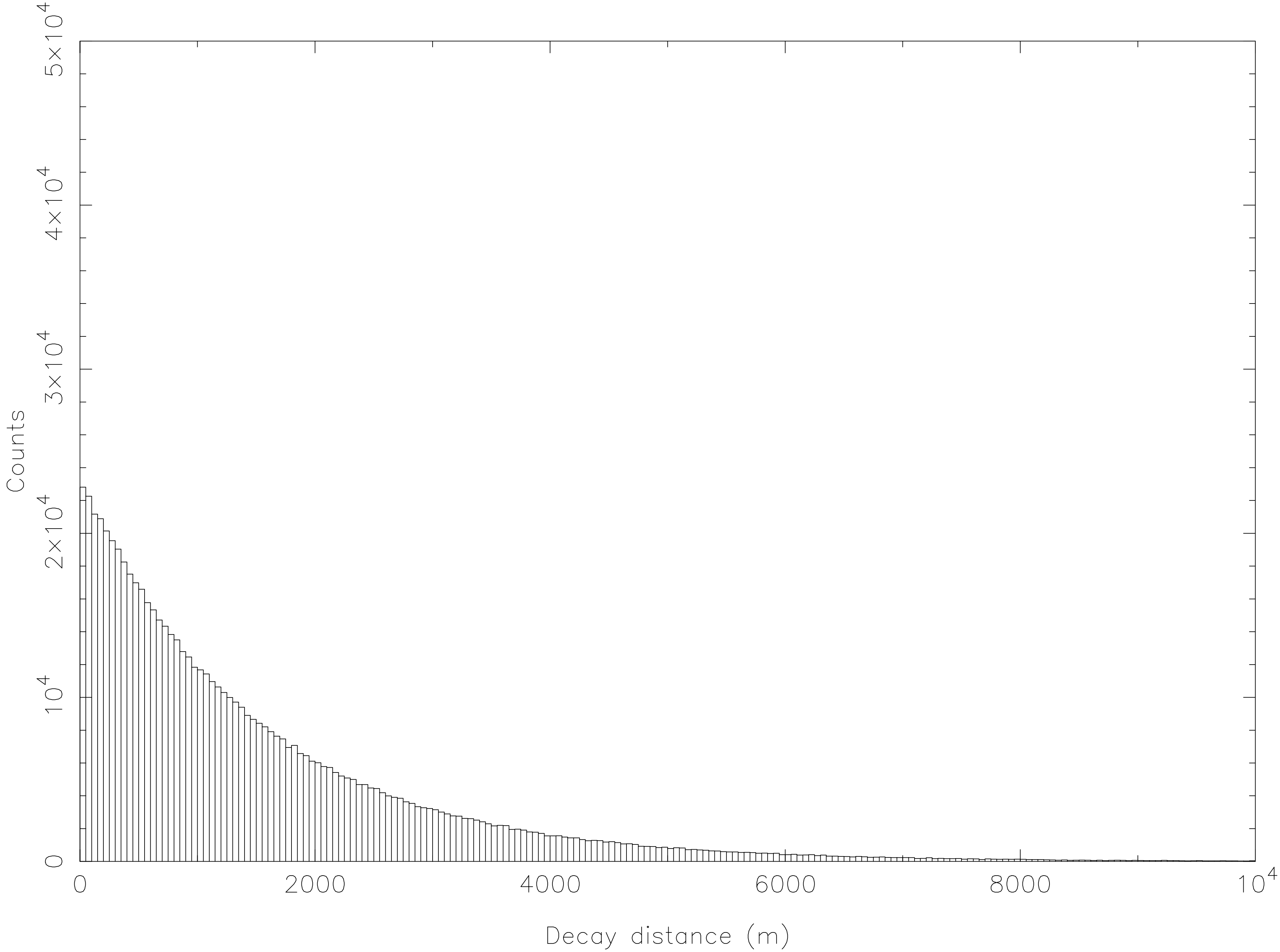


Figure 7 – Histogram of kaon decay distance (up to arbitrary limit of 10,000m)



4. Plots of the longitudinal and transverse energy distributions of neutrinos from decaying pions and kaons in the laboratory frame are shown in Figures 8-11 (ignoring the finite size of the detector).

   Additionally, because only 64% of kaon decay modes produce neutrinos we are interested in, we use the method described in the section entitled Generating Discrete Non-uniform Random Numbers above, to generate a discrete binomial distribution to implement this.

Longitudinal

The shape of the histograms of pion and kaon neutrino longitudinal momentum p_l indicate that p_l is roughly even across the momentum spectrum up to a point i.e. there are roughly similar numbers of neutrinos with all p_l values across the momentum spectrum, up until a certain value. This is as expected, because $|\vec{p}_l|$ is proportional to cos θ, as per Equation 8, therefore $|\vec{p}_l|$ is proportional to a uniform distribution, and should result in a flat histogram, since all values are equally likely. Of course, the histogram will cut off at a certain value, which corresponds to the momentum resulting from the neutrinos being directed along the x axis i.e. $\theta \simeq 0$. This is the highest neutrino momentum possible in this system.

Also, it is expected that the histogram plateau for pion neutrinos will be higher than kaon neutrinos. This is because there are more pion neutrinos than kaon neutrinos in general, firstly because we generated more pions than kaons, and secondly because only 64% of kaon decays result in the kind of neutrinos we want.

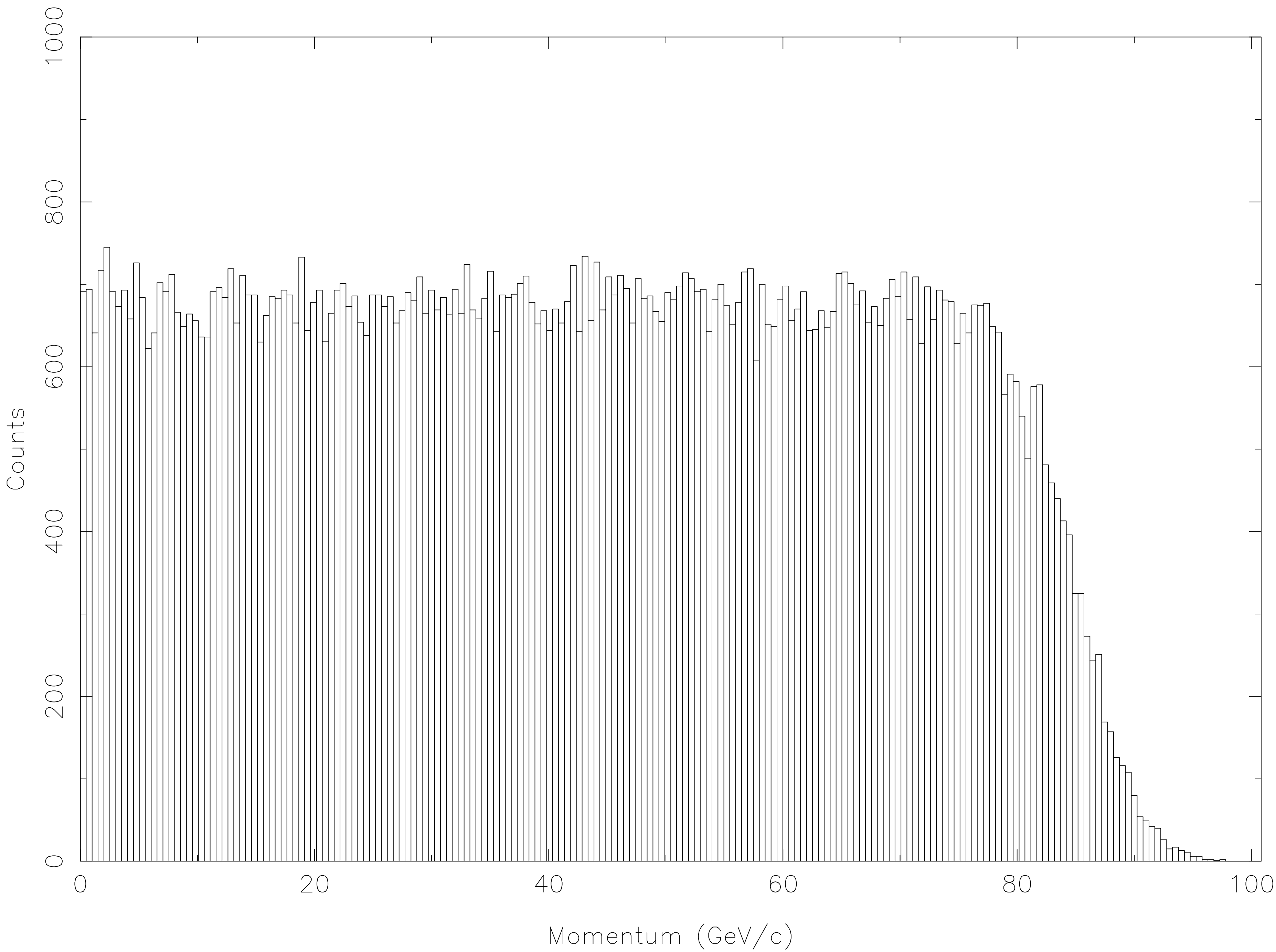
Transverse

The shape of the histograms for pion and kaon transverse momentum p_t indicate that p_t is slewed towards the higher end i.e. there are more neutrinos with high p_t than with low.

The final transverse neutrino momentum comes solely from neutrino momentum (as opposed to the contribution from meson momentum), because there was no transverse component of initial meson momentum. These transverse neutrino momenta are unchanged by the Lorentz transformation, since they are perpendicular to the direction of the meson velocity β. Therefore they should strictly follow the initial probability distribution established for neutrino momentum as per Equation 8, i.e. they will follow $\sin(\cos^{-1}(\cos\theta))$, where cos θ belongs to a uniform distribution. This will yield more values of θ around 0 and π, and less around $\pi/2$. Therefore there will be less values where p_t is a maximum, since $p_t \propto \sin\theta$. Therefore this result is not what we would expect.

However these values may be slewed by the Lorentz transformation into the lab frame, because for those neutrinos with momentum directed close to the x-axis, their longitudinal momentum will be Lorentz transformed and hence smaller in the lab frame than in the meson rest frame. Therefore the transverse momentum of these neutrinos will be under-represented in the transfers histogram.

Figure 8 — Pion neutrino longitudinal momentum distribution (ignoring finite size of detector)



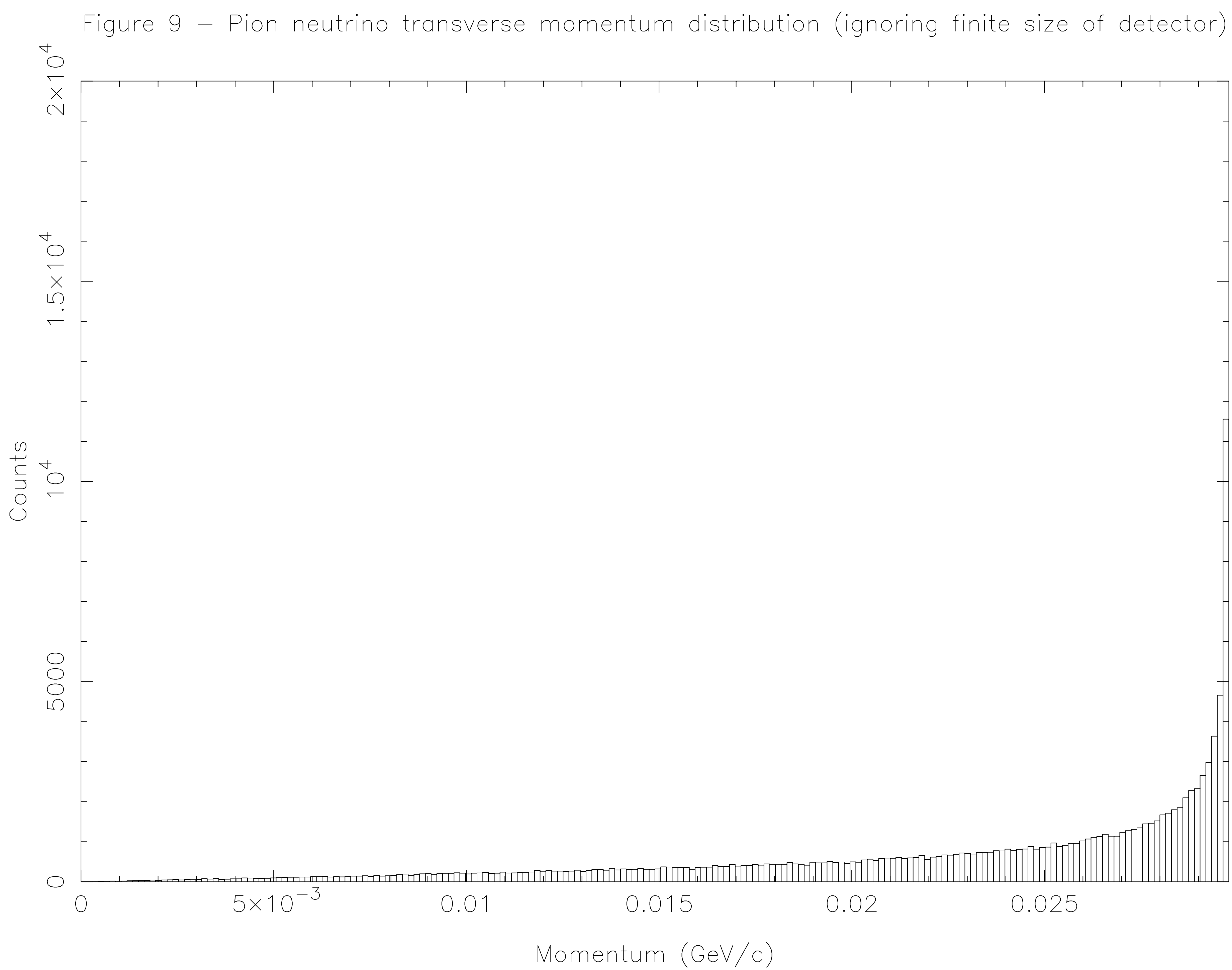


Figure 9 – Pion neutrino transverse momentum distribution (ignoring finite size of detector)
Counts
0
5000
10^4
1.5×10^4
2×10^4
0
5×10^-3
0.01
0.015
0.02
0.025
Momentum (GeV/c)

Figure 10 – Kaon neutrino longitudinal momentum distribution (ignoring finite size of detector)

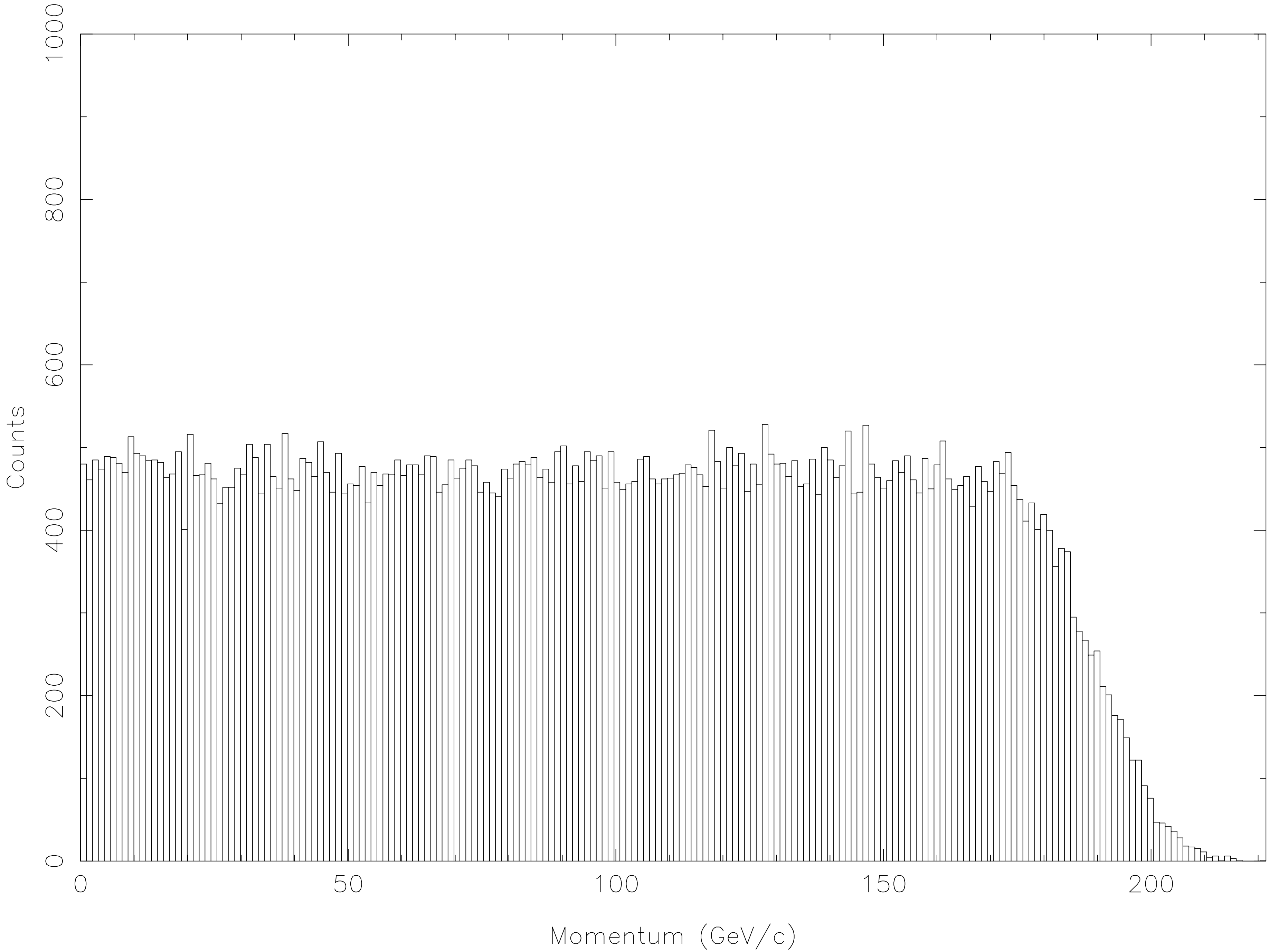
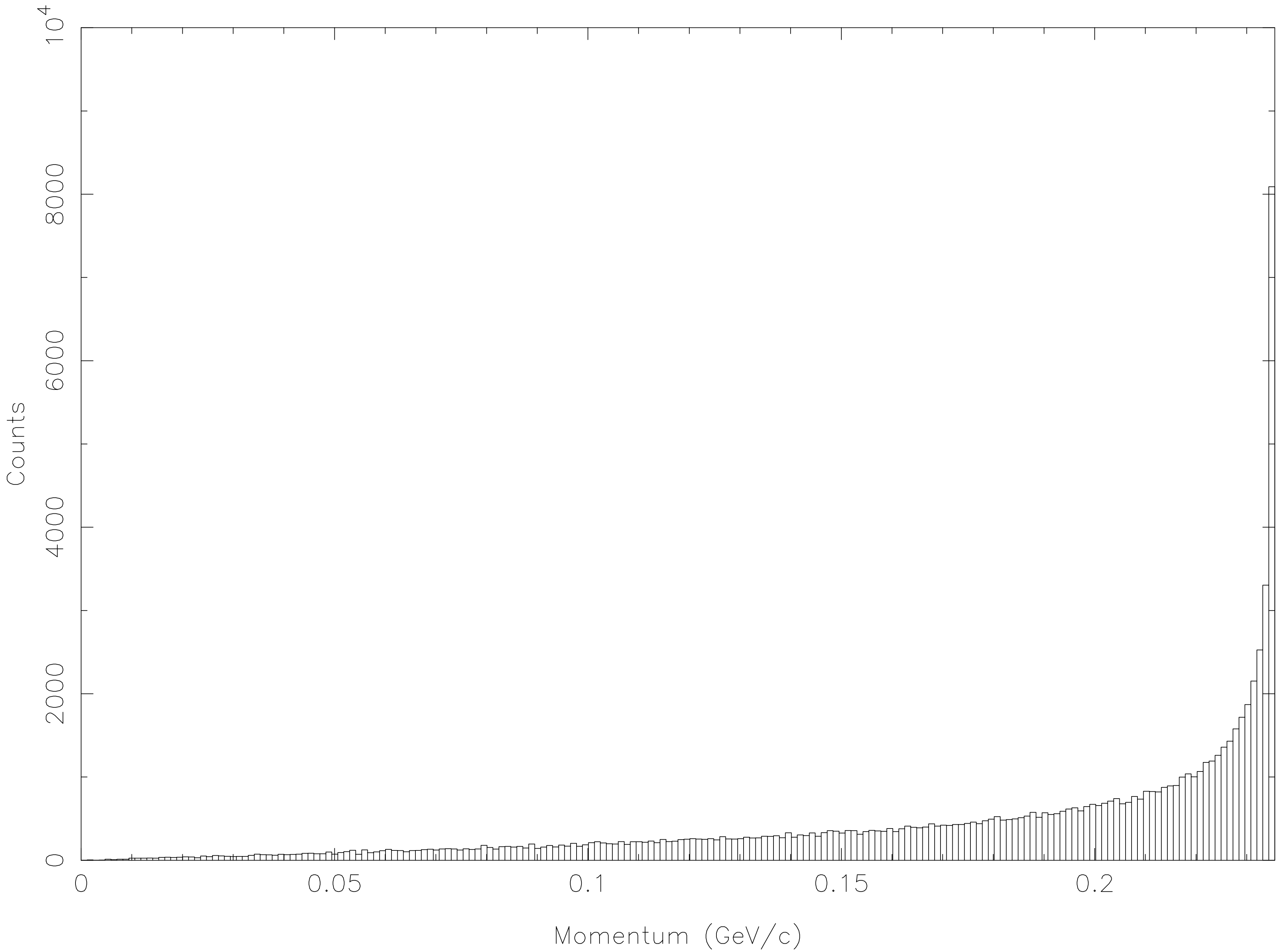


Figure 11 — Kaon neutrino transverse momentum distribution (ignoring finite size of detector)



5. A plot of the energy versus radial position in the detector for neutrinos from pion and kaon decay is shown in Figure 13.

   The radial position on the detector, x_r, was calculated with reference to Figure 2 and Figure 12 as follows.

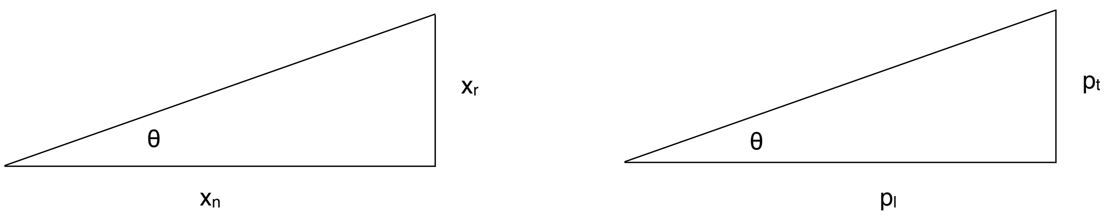


FIGURE 12 - CALCULATION OF NEUTRINO RADIAL POSITION ON DETECTOR

By similar triangles, the following ratio holds:

$$\frac{x_r}{x_n} = \frac{p_t}{p_l} \qquad \text{Equation 27}$$

Therefore:

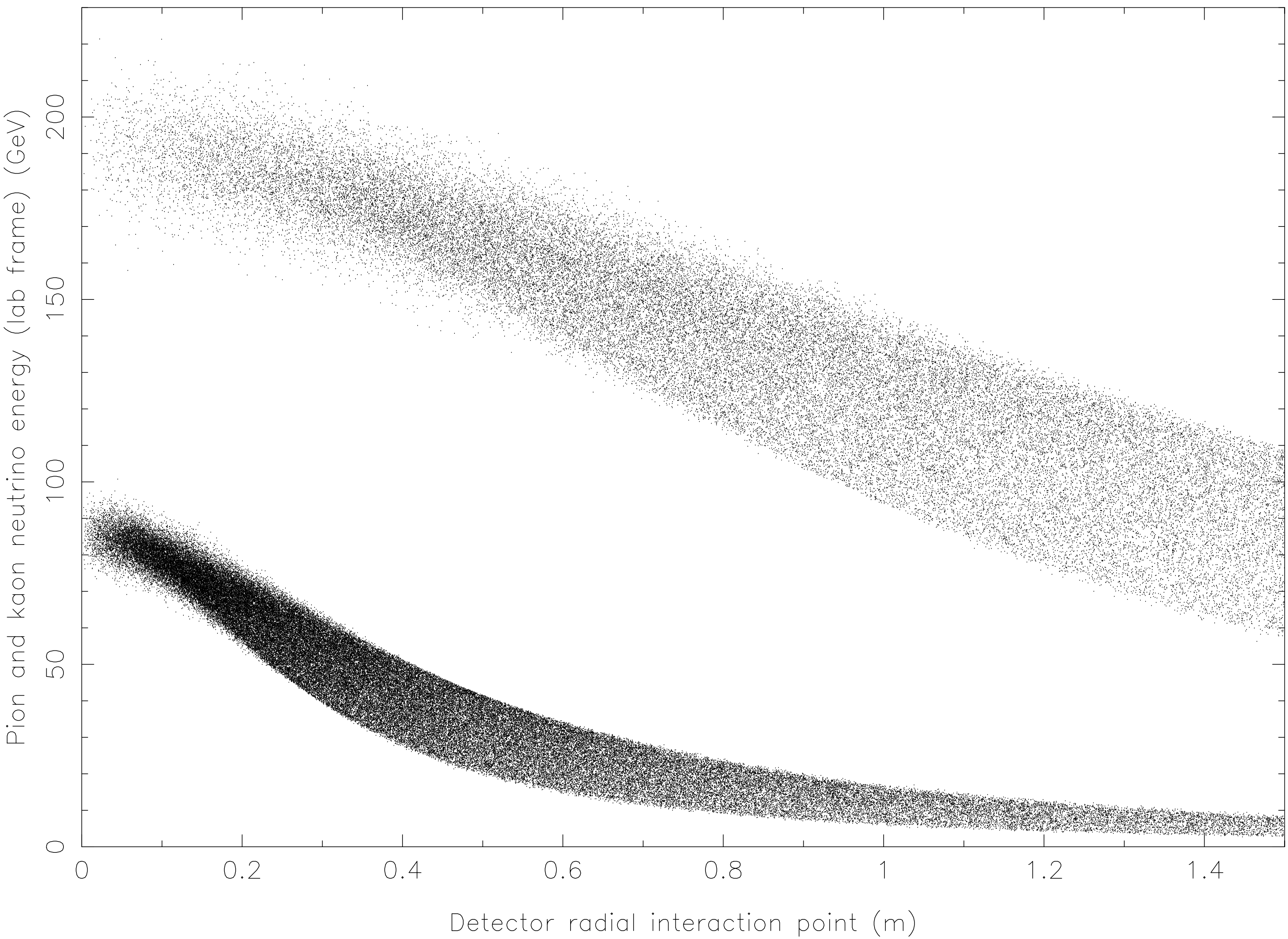
$$x_r = \frac{p_t}{p_l} x_n = \frac{p_t}{p_l}(700 - x_d) \qquad \text{Equation 28}$$

In producing this plot, firstly negative longitudinal neutrino momenta were discarded, because they can never reach the detector, and in Equation 13 may result in negative x_r values. Then, the x_r values were restricted to only those smaller than or equal to the detector size, 1.5m.

There is a significant correlation between the energy of the neutrinos and their radial position x_r. This is indicated by the fact that both bands of neutrino plot points form definite shapes with obvious clustering into regions. If there were no correlation, the plot would be covered evenly with points with no discernible shape or clustering.

The correlation for both types of neutrinos is that the higher the neutrino energy, the smaller the radial position i.e. the closer to the centre of the detector. This can be explained by the fact that the variation in neutrino energy comes from three sources: 1) the variation in neutrino momentum magnitude from the decay of different meson types, which is fixed depending on the type of meson, resulting in the two bands, 2) the variation in initial meson momentum, and 3) the radial variation in neutrino momentum direction. Therefore in general the higher energy neutrinos came from mesons that had higher initial momenta, and/or decay directions that were more closely aligned with the x-axis. Both of these causes result in smaller radial position.

Figure 13 – pion & kaon neutrino energy (lab frame) vs detector radial interaction point



In the plot of Figure 13, the pion neutrinos form the lower band, and the kaon neutrinos the upper band. This is because in the mesons' rest frame, the kaon neutrinos have higher momenta (at 0.235535 GeV/c) than the pion neutrinos (at 0.029784 GeV/c). When Lorentz-transformed into the lab frame, this would still be true.

6. The percentage $p_{p\&k}$ of initial pions and kaons that decay into neutrinos that reach the detector was 3.034139%, calculated thus:

$$p_{p\&k} = \frac{\text{number of on-target neutrinos}}{\text{total number of initial mesons}} \times 100\% \qquad \text{Equation 29}$$

The percentage p_p of initial pions that decay into neutrinos that reach the detector was 2.484119%, calculated thus:

$$p_p = \frac{\text{number of on-target neutrinos from pions}}{\text{number of initial pions}} \times 100\% \qquad \text{Equation 30}$$

The percentage p_k of initial kaons that decay into neutrinos that reach the detector was 6.380430%, calculated thus:

$$p_k = \frac{\text{number of on-target neutrinos from kaons}}{\text{number of initial kaons}} \times 100\% \qquad \text{Equation 31}$$

7. The percentage of neutrinos reaching the detector that came from the decay of pions was 70.509601%, calculated thus:

$$p_{p\&k} = \frac{\text{number of neutrinos from pion decay}}{\text{total number of neutrinos}} \times 100\% \qquad \text{Equation 32}$$

Conclusion

We have simulated a neutrino beam produced from the decay of π- and K-mesons using Monte Carlo methods. This is a simplified version of a neutrino beam used at CERN.

First we simulated a beam of mesons consisting of 86% π-mesons (pions) and 14% K-mesons (kaons). These were assumed to be perfectly focussed and moving parallel to the x-axis. The momenta of these mesons was plotted. The decay of these mesons was then simulated, considering 100% of pions and 64% of kaons with the kind of decay that resulted in neutrinos. These decay distances were then plotted for each meson type, ignoring the finite size of the simulated experimental apparatus i.e. up to an (arbitrary) distance of 10^4m. The resulting neutrinos were then simulated, only accounting for those mesons decaying within the allowed 300m of the decay tunnel of the simulated apparatus, and the longitudinal and transverse momenta of neutrinos from each meson type plotted. The radial positions of the neutrinos on the detector were then simulated, further accounting for factors within the experimental apparatus by removing those neutrinos moving in the negative x-direction, and only accounting for those actually impacting the 1.5m radius detector. The radial positions of these surviving neutrinos were then plotted against neutrino energy.

We have discovered that there is a correlation between neutrino energy and radial position on the detector, for neutrinos from both pion and kaons - the highest energy neutrinos resulted in the lowest radial position values i.e. they impacted the detector closest to the axis of the original meson beam.

The analysis of neutrinos reaching the detector and their origins can be summarised thus:

Neutrinos reaching the detector	
	%
Percentage of initial pions and kaons that decay into neutrinos that reach the detector	3.034139
Percentage of initial pions that decay into neutrinos that reach the detector	2.484119
Percentage of initial kaons that decay into neutrinos that reach the detector	6.380430
Percentage of neutrinos reaching the detector that came from the decay of pions	70.509601

FIGURE 14 - ANALYSIS OF SIMULATED NEUTRINOS REACHING THE DETECTOR AND THEIR ORIGINS