Project 2: Structure of white dwarf stars

Computational Physics PHYC30012
Simon Hudson 767507

Aim

White dwarf stars are incredibly dense objects composed of heavy nuclei and their electrons. The most stable nucleus, ^{56}Fe usually dominates. The density and composition of the white dwarf depends on the size of the original star and when it collapsed - the larger the star, the greater the central density of the resulting white dwarf. If the star collapses into a white dwarf prematurely, before the fusion process has had a chance to run its course, there may also be some ^{12}C present.

Finding the mass and radius of a white dwarf involves solving two coupled differential equations simultaneously. Computationally, two of the most important methods used to solve ordinary differential equations are those of Euler (low accuracy but simple algorithm) and Runge-Kutta (high accuracy, more complicated algorithm).

In this experiment, we have the following coupled set of differential equations for the scaled mass and density of a white dwarf star.

$$\frac{d\bar{\rho}}{d\bar{r}} = -\frac{\bar{m}\bar{\rho}}{\gamma(\bar{\rho}^{1/3})\bar{r}^2} \qquad \text{(Equation 1)}$$

$$\frac{d\bar{m}}{d\bar{r}} = \bar{r}^2\bar{\rho} \qquad \text{(Equation 2)}$$

Where $\gamma(\bar{\rho}^{1/3})$ is given by

$$\gamma(x) = x^2 \Big/ \left(3\sqrt{1+x^2}\right) \qquad \text{(Equation 3)}$$

with $x = (\bar{\rho})^{1/3}$.

Euler's Method

Euler's method for coupled systems can be expressed as:

$$x_{n+1} = x_n + h \qquad \text{(Equation 4)}$$

$$y_{n+1} = y_n + hf(x_n, y_n, z_n) \qquad \text{(Equation 5)}$$

$$z_{n+1} = z_n + hg(x_n, y_n, z_n) \qquad \text{(Equation 6)}$$

Runge-Kutta Method

The Runge-Kutta method for coupled systems can be expressed as:

$$x_{n+1} = x_n + h \qquad \text{(Equations 7)}$$

$$y_{n+1} = y_n + \frac{h}{6}(f_1 + 2f_2 + 2f_3 + f_4)$$

$$z_{n+1} = z_n + \frac{h}{6}(g_1 + 2g_2 + 2g_3 + g_4)$$

where

$$f_1 \equiv f(x_n, y_n, z_n) \qquad \text{(Equations 8)}$$

$$g_1 \equiv g(x_n, y_n, z_n)$$

$$f_2 \equiv f(x_n + h/2, y_n + (h/2)f_1, z_n + (h/2)g_1)$$

$$g_2 \equiv g(x_n + h/2, y_n + (h/2)f_1, z_n + (h/2)g_1)$$

$$f_3 \equiv f(x_n + h/2, y_n + (h/2)f_2, z_n + (h/2)g_2)$$

$$g_3 \equiv g(x_n + h/2, y_n + (h/2)f_2, z_n + (h/2)g_2)$$

$$f_4 \equiv f(x_n + h, y_n + hf_3, z_n + hg_3)$$

$$g_4 \equiv g(x_n + h, y_n + hf_3, z_n + hg_3)$$

The initial conditions are determined at $\bar{r} = h$ from a Taylor series analysis which gives:

$$\bar{m}(h) = \frac{h^3}{3}\bar{\rho}_c \qquad \text{(Equation 9)}$$

$$\bar{\rho}(h) = \bar{\rho}_c\left(1 - \frac{h^2\bar{\rho}_c}{6\gamma(\bar{\rho}_c^{\,1/3})}\right) \qquad \text{(Equation 10)}$$

where $\bar{\rho}_c$ is the scaled central density (the scaled density at r = 0).

Here, the functions f and g correspond to our coupled differential equations thus:

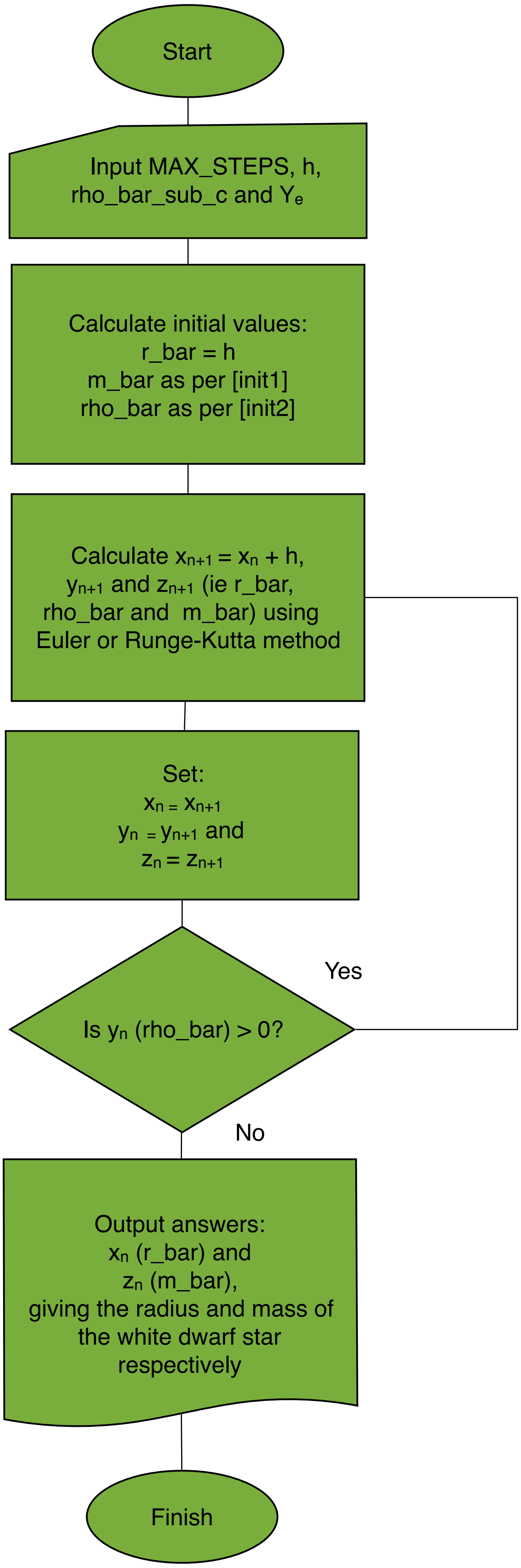
$$f(x_n, y_n, z_n) = \frac{d\bar{\rho}}{d\bar{r}}(\bar{r}, \bar{\rho}, \bar{m}) \qquad \text{(Equation 11)}$$

$$g(x_n, y_n, z_n) = \frac{d\bar{m}}{d\bar{r}}(\bar{r}, \bar{\rho}) \qquad \text{(Equation 12)}$$

Method and Results

1. The flowchart showing an algorithm for solving two coupled differential equations using the Euler and Runge-Kutta methods was implemented in code (see Figure 1).

FIGURE 1 - A FLOWCHART FOR A PROGRAM TO DETERMINE THE MASS AND RADIUS OF A WHITE DWARF STAR, GIVEN AN INITIAL CENTRAL SCALED DENSITY $\bar{\rho}_c$, ATOMIC CONSTITUTION Y_e (NUMBER OF ELECTRONS PER NUCLEON), STEP SIZE H AND MAX_STEPS.

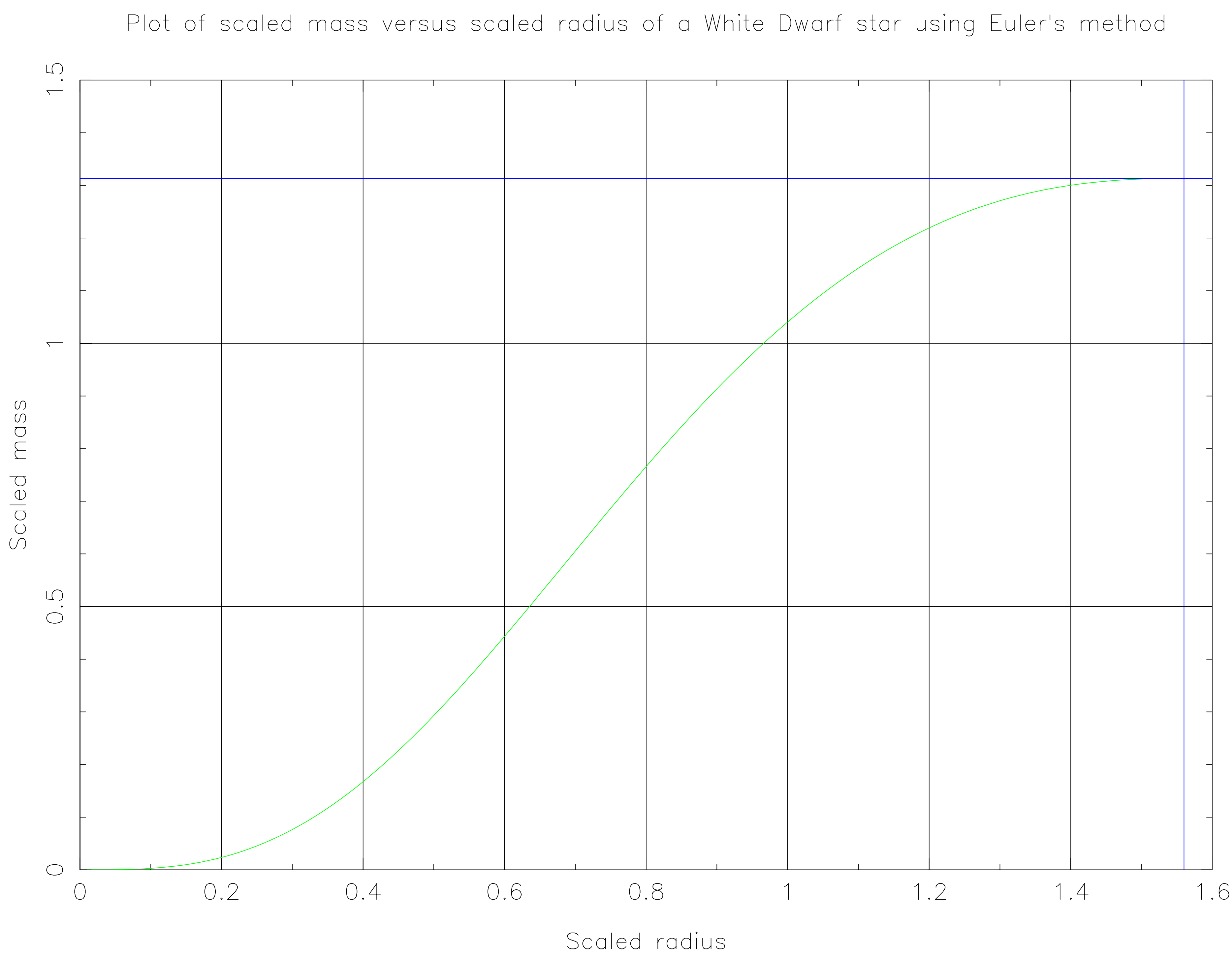


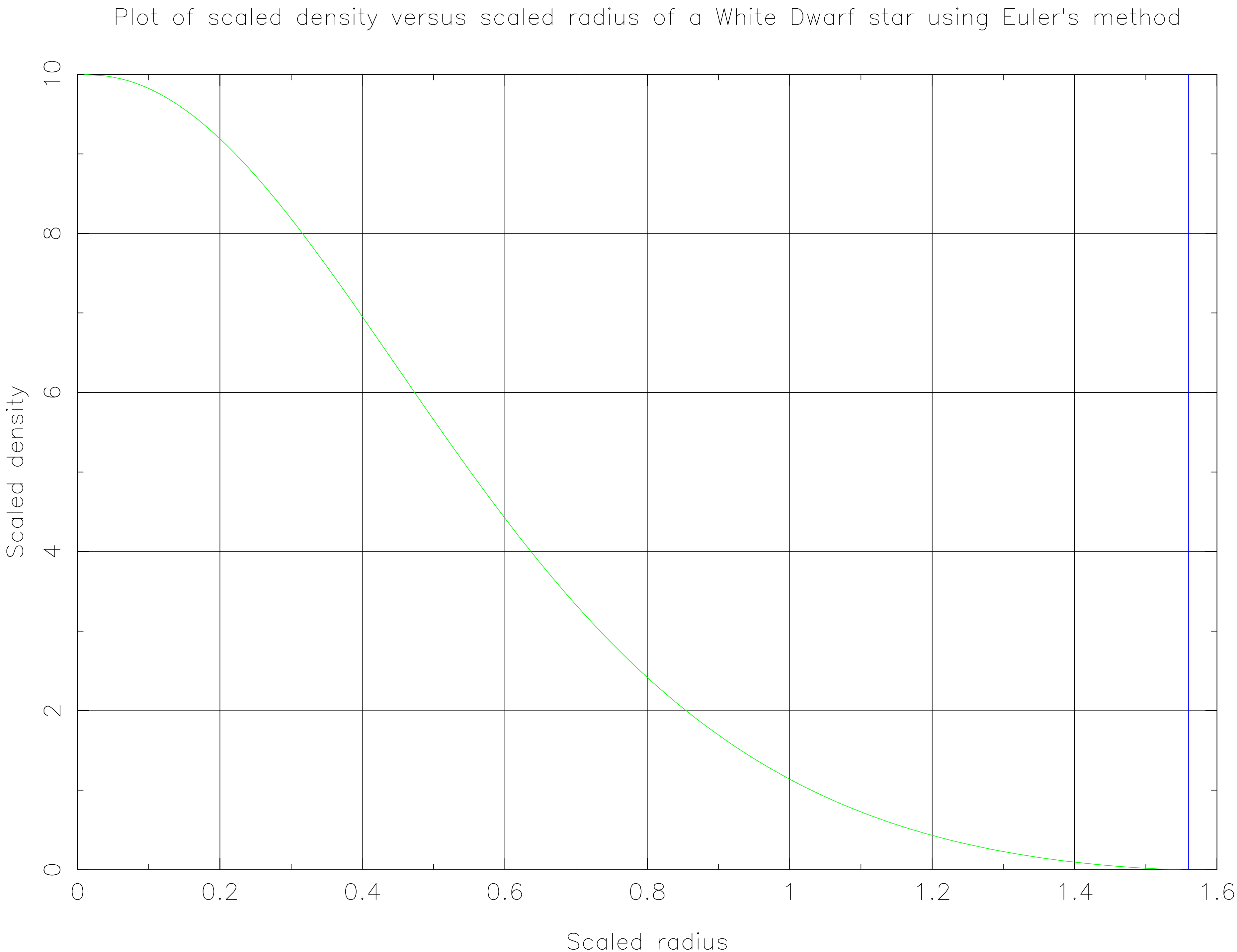
2. The calculation for mass and radius were performed using Euler and Runge-Kutta using $\bar{\rho}_c$ = 10, Y_e = 1 and step-size h = 0.00001. The resulting final values are shown in Table 1, and plots are shown in Figure 2.

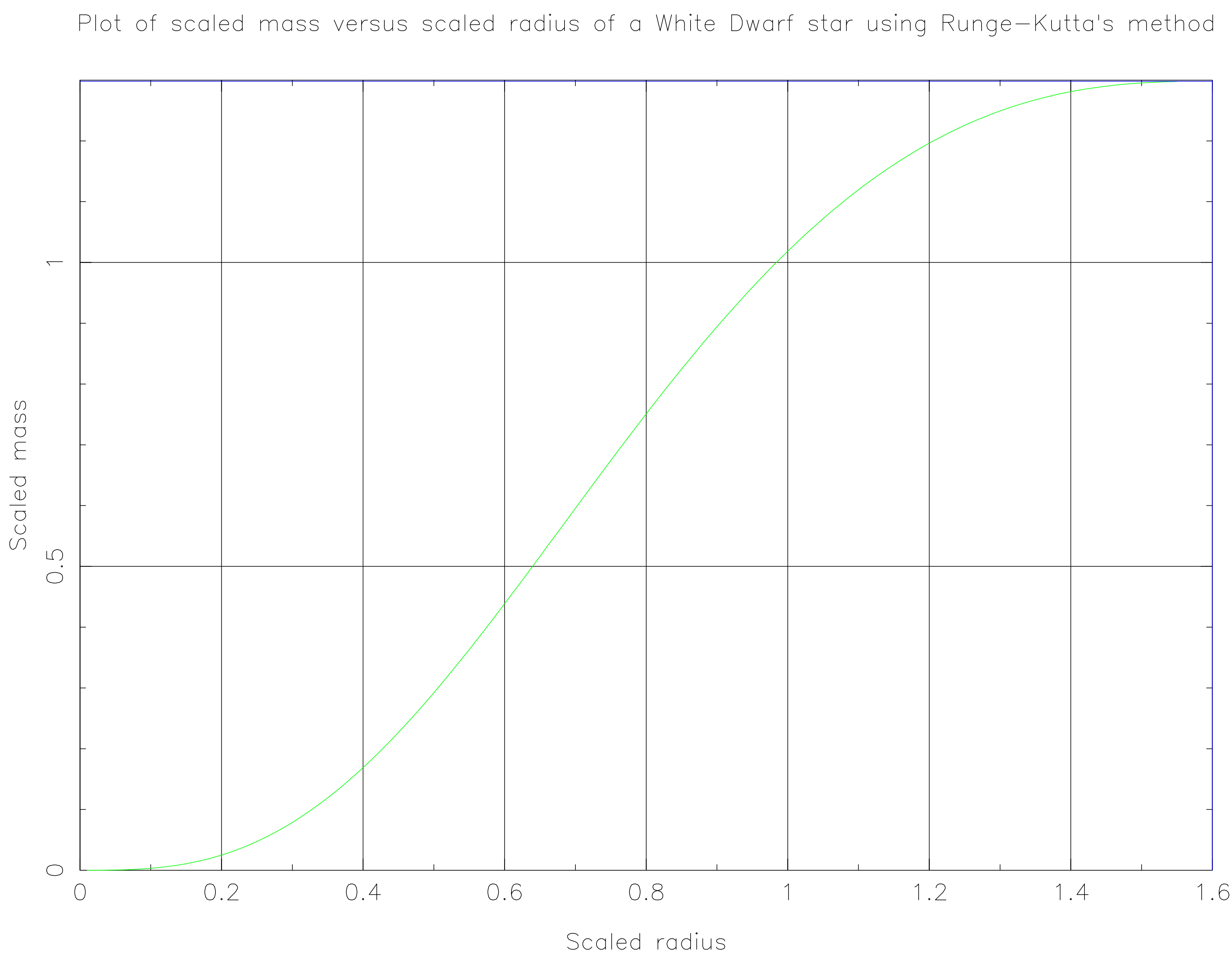
TABLE 1 - TABLE OF STABILITY OF SOLUTIONS FOR SCALED MASS AND RADIUS OF A WHITE DWARF STAR FOR $\bar{\rho}_c$ = 10 AND Y_E = 1 WITH MAX_STEPS = 10^7 AGAINST STEP SIZE H.

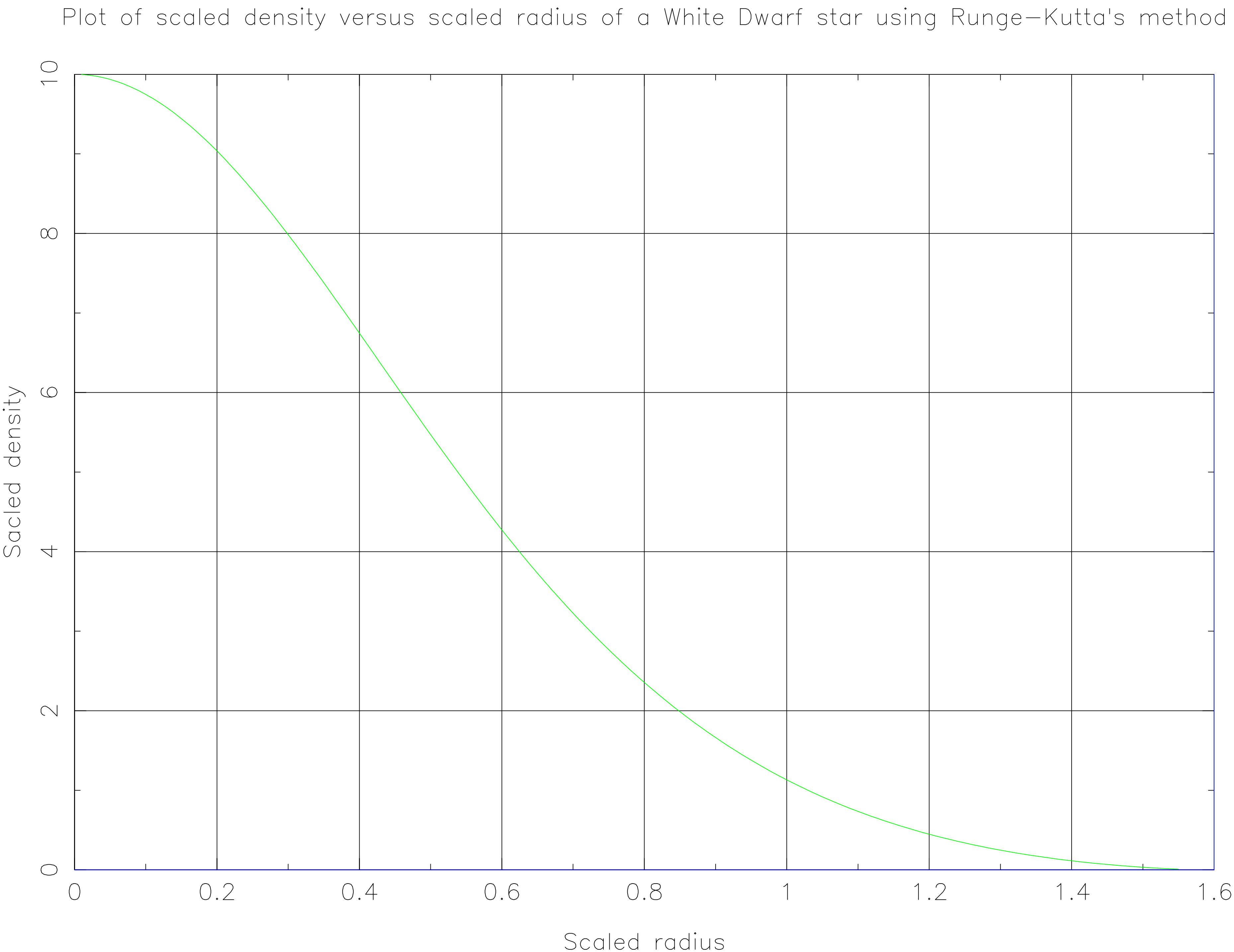
Final values	$\bar{r}$	$\bar{m}$
Euler method	1.591580	1.298028
Runge-Kutta method	1.298013	1.591630

FIGURE 2 - PLOTS FOR SCALED MASS VERSUS SCALED RADIUS, AND SCALED DENSITY VERSUS SCALED RADIUS USING EULER AND RUNGE-KUTTA USING $\bar{\rho}_c$ = 10, Y_E = 1 AND STEP-SIZE H = 0.00001.


Plot of scaled mass versus scaled radius of a White Dwarf star using Euler's method
Scaled mass
Scaled radius
0
0.5
1
1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6


Plot of scaled density versus scaled radius of a White Dwarf star using Euler's method
Scaled density
0
2
4
6
8
10
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
Scaled radius


Plot of scaled mass versus scaled radius of a White Dwarf star using Runge–Kutta's method
Scaled mass
Scaled radius
0
0.5
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6


Plot of scaled density versus scaled radius of a White Dwarf star using Runge-Kutta's method
Sacled density
Scaled radius
0
2
4
6
8
10
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

3. The stability of the solutions was tested against step-size h, still for $\bar{\rho}_c$ = 10 and Y_e = 1 and the results tabulated for total scaled mass and radius in Table 2. It was found that using Runge-Kutta, as step size h was decreased from 1 by orders of magnitude as shown, that the scaled mass and scaled radius converged to the following values:

(Equation 13)
$$\bar{m} = 1.298013$$
(Equation 14)
$$\bar{r} = 1.591630$$

Beyond h = 10^{-5} there was no change in these values, therefore this value of h will be used from here on. Columns are included in Table 2 for the differences between each value and the value that that result converges to. Using Euler there was no convergence up to h = 10^{-6}. Hence we conclude that this is a less accurate method, and Runge-Kutta will be used for all calculations from here on.

TABLE 2 - RESULTS FOR TEST OF STABILITY OF SOLUTIONS FOR SCALED MASS AND RADIUS OF A WHITE DWARF STAR FOR $\bar{\rho}_c$ = 10 AND Y_E = 1 WITH MAX_STEPS = 10^7 AGAINST STEP SIZE H.

	Euler				Runge-Kutta			
Step	$\bar{m}$	$\bar{r}$	% $\bar{m}$ Diff	% $\bar{r}$ Diff	M_bar	$\bar{r}$	% $\bar{m}$ Diff	% $\bar{r}$ Diff
1.000000	3.333333	1.000000	156.802770	-37.171328	3.333333	1.000000	156.802770	-37.171328
0.100000	1.487193	1.400000	14.574586	-12.039859	1.297828	1.600000	-0.014276	0.525876
0.010000	1.313219	1.560000	1.171498	-1.987271	1.298001	1.600000	-0.000900	0.525876
0.001000	1.299510	1.588000	0.115331	-0.228068	1.298013	1.592000	-0.000005	0.023247
0.000100	1.298163	1.591200	0.011522	-0.027016	1.298013	1.591700	-0.000004	0.004398
0.000010	1.298028	1.591580	0.001149	-0.003141	1.298013	1.591630	-0.000004	0.000000
0.000001	1.298014	1.591624	0.000112	-0.000377	1.298013	1.591630	-0.000004	0.000000

6. Solution were obtained for Runge-Kutta for a range of scaled central densities $\bar{\rho}_c$ from 0.1 to 10^8 with Y_e = 1, as shown in Table 3. We can see that $\bar{m}$ and r_bar vary widely with $\bar{\rho}_c$. This can be explained by the fact that this model is valid only for white dwarf stars, and as $\bar{\rho}_c$ is increased, we are getting further and further away from conditions that are possible inside a white dwarf. The largest density possible is ~10^{10} g cm^{-3} [1]. This corresponds to the row with a dark background in the table, where ρ_c = 9,790,000,000 ≃ 10^{10}, corresponding to $\bar{\rho}_c$ = 10,000. This corresponds to a value for ρ_c $_{sol}$ ≃ 6×10^7, or sixty million times the density of our sun. We conclude that values of $\bar{\rho}_c$ larger than 10,000 are not physical, and therefore should not be used in this model of the white dwarf. We will limit $\bar{\rho}_c$ to this value from here on.

[1] Lattimer J 2010, Compact Stars – White Dwarfs, Planets, Neutron Stars, Stony Brook University, viewed 4th Sep 2016, <http://www.astro.sunysb.edu/lattimer/PHY521/degen.pdf>.

TABLE 3 - RESULTS FOR SCALED MASS AND RADIUS, DENSITY ρ_c, AND DENSITY (IN SOLAR UNITS) ρ_c SOL OF A WHITE DWARF STAR FOR VARYING $\bar{\rho}_c$ FROM 0.1 TO 10^8 WITH $Y_E = 1$ AND MAX_STEPS = 10^7.

$\bar{\rho}_c$	ρ_c	ρ_c sol	$\bar{m}$	$\bar{r}$
0.1	97900	652.666667	0.279824	3.759500
1	979000	6526.666667	0.707066	2.498190
10	9790000	65266.666667	1.298013	1.591630
100	97900000	652666.666667	1.735533	0.953950
1000	979000000	6526666.666667	1.932841	0.533560
10000	9790000000	65266666.666667	1.996584	0.279230
100000	97900000000	652666666.666667	2.013253	0.138770
1000000	979000000000	6526666666.666667	2.017138	0.066730
10000000	9790000000000	65266666666.666664	2.017998	0.031520
100000000	97900000000000	652666666666.666626	2.018185	0.014750

7. Various values of $\bar{\rho}_c$ and Y_e were then used to probe the sensitivity of the solutions, and these results compared to the masses and radii of white dwarves Sirius B and 40 Eri B, being (1.053 ± 0.028, 0.0074 ± 0.0006) and (0.48 ± 0.02, 0.0124 ± 0.0005) solar units, respectively. Shown in

TABLE 4 - RESULTS FOR COMPUTED MASS (IN SOLAR UNITS) AND THE PERCENTAGE DIFFERENCE BETWEEN EACH VALUE AND THE OBSERVED MASS OF WHITE DWARF 40 ERI B.

Eri B Mass				
$\bar{\rho}_c$	Y_e	ρ_c sol	m^{sol}	mass % Diff
1.300000	0.466000	18207.439199	0.479843	0.032763
1.130000	0.477000	15461.495458	0.480103	0.021477
1.300000	0.466100	18203.532861	0.480049	0.010146
1.167000	0.474400	16055.269815	0.480000	0.000005
1.219000	0.471000	16891.733900	0.480003	0.000668
1.023100	0.485000	13767.902405	0.480000	0.000007

TABLE 5 - RESULTS FOR COMPUTED RADIUS (IN SOLAR UNITS) AND THE PERCENTAGE DIFFERENCE BETWEEN EACH VALUE AND THE OBSERVED RADIUS OF WHITE DWARF 40 ERI B.

Eri B Radius				
$\overline{\rho}_c$	Y_e	ρ_c sol	r^{sol}	radius % Diff
1.500000	0.482000	20311.203320	0.012402	0.012125
1.420000	0.477000	19429.489867	0.012400	0.000166
1.280000	0.467800	17858.344022	0.012399	0.005096
1.401000	0.475800	19217.864649	0.012400	0.000064
1.435700	0.478000	19603.211994	0.012400	0.000004

TABLE 6 - RESULTS FOR COMPUTED MASS (IN SOLAR UNITS) AND THE PERCENTAGE DIFFERENCE BETWEEN EACH VALUE AND THE OBSERVED MASS OF WHITE DWARF SIRIUS B.

Sirius B Mass				
$\overline{\rho}_c$	Y_e	ρ_c sol	m^{sol}	mass % Diff
22.000000	0.499000	287748.830995	1.053041	0.003923
24.700000	0.495000	325674.074074	1.052985	0.001381
22.630000	0.498000	296583.266399	1.053005	0.000461
23.982000	0.496000	315569.596774	1.053000	0.000005

TABLE 7 - RESULTS FOR COMPUTED RADIUS (IN SOLAR UNITS) AND THE PERCENTAGE DIFFERENCE BETWEEN EACH VALUE AND THE OBSERVED RADIUS OF WHITE DWARF SIRIUS B.

Sirius B Radius				
$\overline{\rho}_c$	Y_e	ρ_c sol	r^{sol}	radius % Diff
22.000000	0.495000	290074.074074	0.007400	0.003109
18.000000	0.474000	247848.101266	0.007400	0.002295
18.900000	0.479000	257524.008351	0.007400	0.000224
21.590000	0.493000	285822.988506	0.007400	0.000122
21.586000	0.493000	285770.033807	0.007400	0.000122

It is assumed that white dwarves consist of large nuclei and their electrons, primarily ${}^{56}Fe$ and ${}^{12}C$, and with Y_e being the number of electrons per nucleon, this will be $\frac{26}{56}$ = 0.464 and $\frac{1}{2}$ = 0.5 respectively. The equations for the composition of a white dwarf are as follows:

$$Y_e = 0.464 P_{Fe} + 0.5 P_c \qquad \text{(Equation 15)}$$

$$P_C = 1 - P_{Fe} \qquad \text{(Equation 16)}$$

where P_{Fe} is the proportion of ${}^{56}Fe$, and P_C is the proportion of ${}^{12}C$. Solving for P_{Fe}, we get the following:

$$P_{Fe} = \frac{Y_e - 0.5}{0.464 - 0.5} \qquad \text{(Equation 17)}$$

As shown, the results varied considerably for mass and radius - value of $\bar{\rho}_c$ and Y_e that resulted in a very low error in mass didn't necessarily result in a low error in radius, and vice versa. Without a more comprehensive method to optimise for both mass and radius simultaneously, the best that can be done is choose values that appear to minimise both the errors to the same degree.

The estimated central densities and compositions of Sirius B and 40 Eri B are thus shown in Table 8.

TABLE 8 - ESTIMATED SCALED CENTRAL DENSITY $\bar{\rho}_c$, CENTRAL DENSITY (SOLAR UNITS) $\bar{\rho}_c$ ^{SOL}, YE, PROPORTION OF ${}^{56}Fe$, PROPORTION OF ${}^{12}C$, MASS (SOLAR UNITS), RADIUS (SOLAR UNITS) AND PERCENTAGE DIFFERENCES BETWEEN M^{SOL}, R^{SOL} AND THEIR EXPERIMENTALLY OBSERVED VALUES (CENTRAL VALUE).

		Y_e		0.464	0.5				
Star	$\bar{\rho}_c$	Y_e	ρ_c ^{sol}	**Prop^n ${}^{56}Fe$**	**Prop^n ${}^{12}C$**	m^{sol}	% m^{sol} diff	r^{sol}	% r^{sol} diff
40 Eri B	1.280000	0.467800	1.785834E+04	89.444444%	10.55555	0.481127	0.234702	0.012399	0.005096
Sirius B	22.630000	0.498000	2.965833E+05	5.555556%	94.44444	1.053005	0.000461	0.007399	0.017519

Conclusion

We have evaluated the validity of our tools in determining the mass and radius of two white dwarf stars.

First, the accuracy of the Euler and Runge-Kutta methods of solving two coupled differential equations was analysed. Upon varying the step size through orders of magnitude, it was found that the Runge-Kutta method converged rapidly on a solution, indicating that it is a more accurate method.

This method and the indicated best final step size were then used to investigate how mass and radius varied with values of scaled central density. We discarded values of scaled central density above a certain threshold because they are unphysical and do not correspond to valid solutions of this model of a white dwarf.

We thus arrived at a final computational setup through investigation and elimination of unnecessary or unphysical conditions. We analysed how computational mass and radius values compared with experimental values for two white dwarves, Sirius B and 40 Eri B, and thus attempted to arrive at values for the central densities and compositions of these stars.

Without a more sophisticated method to optimise the calculation for both mass and radius, we chose values for of $\bar{\rho}_c$ and Y_e that minimised the errors in mass and radius most equally, and thus derived guesses at the central densities and composition of the white dwarves Sirius B and 40 Eri B.