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# CHAPTER 4

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## Flexure Elements

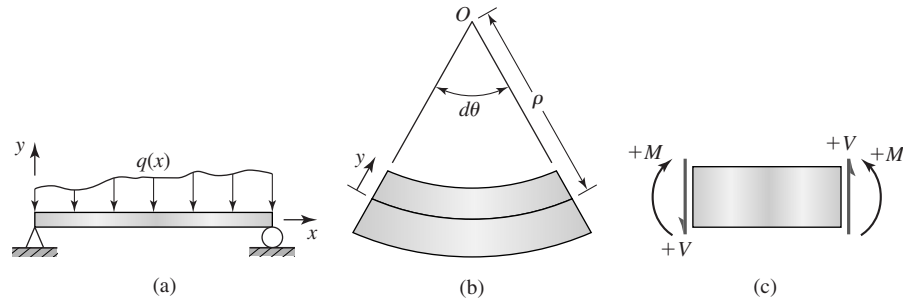
### 4.1 INTRODUCTION

The one-dimensional, axial load-only elements discussed in Chapters 2 and 3 are quite useful in analyzing the response to load of many simple structures. However, the restriction that these elements are not capable of transmitting bending effects precludes their use in modeling more commonly encountered structures that have welded or riveted joints. In this chapter, elementary beam theory is applied to develop a *flexure* (beam) element capable of properly exhibiting transverse bending effects. The element is first presented as a line (one-dimensional) element capable of bending in a plane. In the context of developing the discretized equations for this element, we present a general procedure for determining the interpolation functions using an assumed polynomial form for the field variable. The development is then extended to two-plane bending and the effects of axial loading and torsion are added.

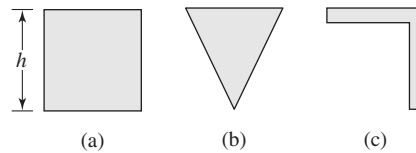
### 4.2 ELEMENTARY BEAM THEORY

Figure 4.1a depicts a simply supported beam subjected to a general, distributed, transverse load  $q(x)$  assumed to be expressed in terms of force per unit length. The coordinate system is as shown with  $x$  representing the axial coordinate and  $y$  the transverse coordinate. The usual assumptions of elementary beam theory are applicable here:

1. The beam is loaded only in the  $y$  direction.
2. Deflections of the beam are small in comparison to the characteristic dimensions of the beam.
3. The material of the beam is linearly elastic, isotropic, and homogeneous.
4. The beam is prismatic and the cross section has an axis of symmetry in the plane of bending.

**Figure 4.1**

(a) Simply supported beam subjected to arbitrary (negative) distributed load. (b) Deflected beam element. (c) Sign convention for shear force and bending moment.



**Figure 4.2** Beam cross sections: (a) and (b) satisfy symmetry conditions for the simple bending theory, (c) does not satisfy the symmetry requirement.

The ramifications of assumption 4 are illustrated in Figure 4.2, which depicts two cross sections that satisfy the assumption and one cross section that does not. Both the rectangular and triangular cross sections are symmetric about the  $xy$  plane and bend only in that plane. On the other hand, the L-shaped section possesses no such symmetry and bends out of the  $xy$  plane, even under loading only in that plane. With regard to the figure, assumption 2 can be roughly quantified to mean that the maximum deflection of the beam is much less than dimension  $h$ . A generally applicable rule is that the maximum deflection is less than  $0.1h$ .

Considering a differential length  $dx$  of a beam after bending as in Figure 4.1b (with the curvature greatly exaggerated), it is intuitive that the top surface has decreased in length while the bottom surface has increased in length. Hence, there is a “layer” that must be undeformed during bending. Assuming that this layer is located distance  $\rho$  from the center of curvature  $O$  and choosing this layer (which, recall, is known as the *neutral surface*) to correspond to  $y = 0$ , the length after bending at any position  $y$  is expressed as

$$ds = (\rho - y) d\theta \quad (4.1)$$

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and the bending strain is then

$$\epsilon_x = \frac{ds - dx}{dx} = \frac{(\rho - y) d\theta - \rho d\theta}{\rho d\theta} = -\frac{y}{\rho} \quad (4.2)$$

From basic calculus, the radius of curvature of a planar curve is given by

$$\rho = \frac{\left[1 + \left(\frac{dv}{dx}\right)^2\right]^{3/2}}{\frac{d^2v}{dx^2}} \quad (4.3)$$

where  $v = v(x)$  represents the deflection curve of the neutral surface.

In keeping with small deflection theory, slopes are also small, so Equation 4.3 is approximated by

$$\rho = \frac{1}{\frac{d^2v}{dx^2}} \quad (4.4)$$

such that the normal strain in the direction of the longitudinal axis as a result of bending is

$$\epsilon_x = -y \frac{d^2v}{dx^2} \quad (4.5)$$

and the corresponding normal stress is

$$\sigma_x = E\epsilon_x = -Ey \frac{d^2v}{dx^2} \quad (4.6)$$

where  $E$  is the modulus of elasticity of the beam material. Equation 4.6 shows that, at a given cross section, the normal stress varies linearly with distance from the neutral surface.

As no net axial force is acting on the beam cross section, the resultant force of the stress distribution given by Equation 4.6 must be zero. Therefore, at any axial position  $x$  along the length, we have

$$F_x = \int_A \sigma_x dA = - \int_A Ey \frac{d^2v}{dx^2} dA = 0 \quad (4.7)$$

Noting that at an arbitrary cross section the curvature is constant, Equation 4.7 implies

$$\int_A y dA = 0 \quad (4.8)$$

which is satisfied if the  $xz$  plane ( $y = 0$ ) passes through the centroid of the area. Thus, we obtain the well-known result that the neutral surface is perpendicular to the plane of bending and passes through the centroid of the cross-sectional area.

Similarly, the internal bending moment at a cross section must be equivalent to the resultant moment of the normal stress distribution, so

$$M(x) = - \int_A y \sigma_x \, dA = E \frac{d^2 v}{dx^2} \int_A y^2 \, dA \quad (4.9)$$

The integral term in Equation 4.9 represents the moment of inertia of the cross-sectional area about the  $z$  axis, so the bending moment expression becomes

$$M(x) = EI_z \frac{d^2 v}{dx^2} \quad (4.10)$$

Combining Equations 4.6 and 4.10, we obtain the normal stress equation for beam bending:

$$\sigma_x = - \frac{M(x)y}{I_z} = -yE \frac{d^2 v}{dx^2} \quad (4.11)$$

Note that the negative sign in Equation 4.11 ensures that, when the beam is subjected to positive bending moment per the convention depicted in Figure 4.1c, compressive (negative) and tensile (positive) stress values are obtained correctly depending on the sign of the  $y$  location value.

### 4.3 FLEXURE ELEMENT

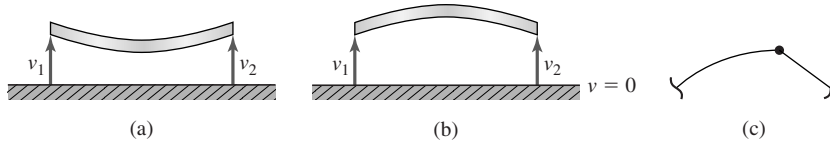
Using the elementary beam theory, the 2-D beam or *flexure* element is now developed with the aid of the first theorem of Castigliano. The assumptions and restrictions underlying the development are the same as those of elementary beam theory with the addition of

1. The element is of length  $L$  and has two nodes, one at each end.
2. The element is connected to other elements only at the nodes.
3. Element loading occurs only at the nodes.

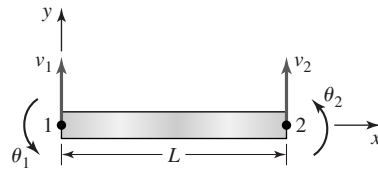
Recalling that the basic premise of finite element formulation is to express the continuously varying field variable in terms of a finite number of values evaluated at element nodes, we note that, for the flexure element, the field variable of interest is the transverse displacement  $v(x)$  of the neutral surface away from its straight, undeflected position. As depicted in Figure 4.3a and 4.3b, transverse deflection of a beam is such that the variation of deflection along the length is not adequately described by displacement of the end points only. The end deflections can be identical, as illustrated, while the deflected shape of the two cases is quite different. Therefore, the flexure element formulation must take into account the slope (rotation) of the beam as well as end-point displacement. In addition to avoiding the potential ambiguity of displacements, inclusion of beam element nodal rotations ensures compatibility of rotations at nodal connections between

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**Figure 4.3**

(a) and (b) Beam elements with identical end deflections but quite different deflection characteristics. (c) Physically unacceptable discontinuity at the connecting node.

**Figure 4.4** Beam element nodal displacements shown in a positive sense.

elements, thus precluding the physically unacceptable discontinuity depicted in Figure 4.3c.

In light of these observations regarding rotations, the nodal variables to be associated with a flexure element are as depicted in Figure 4.4. Element nodes 1 and 2 are located at the ends of the element, and the nodal variables are the transverse displacements  $v_1$  and  $v_2$  at the nodes and the slopes (rotations)  $\theta_1$  and  $\theta_2$ . The nodal variables as shown are in the positive direction, and it is to be noted that the slopes are to be specified in radians. For convenience, the superscript ( $e$ ) indicating element properties is not used at this point, as it is understood in context that the current discussion applies to a single element. When multiple elements are involved in examples to follow, the superscript notation is restored.

The displacement function  $v(x)$  is to be discretized such that

$$v(x) = f(v_1, v_2, \theta_1, \theta_2, x) \quad (4.12)$$

subject to the boundary conditions

$$v(x = x_1) = v_1 \quad (4.13)$$

$$v(x = x_2) = v_2 \quad (4.14)$$

$$\left. \frac{dv}{dx} \right|_{x=x_1} = \theta_1 \quad (4.15)$$

$$\left. \frac{dv}{dx} \right|_{x=x_2} = \theta_2 \quad (4.16)$$

Before proceeding, we assume that the element coordinate system is chosen such that  $x_1 = 0$  and  $x_2 = L$  to simplify the presentation algebraically. (This is not at all restrictive, since  $L = x_2 - x_1$  in any case.)

Considering the four boundary conditions and the one-dimensional nature of the problem in terms of the independent variable, we assume the displacement function in the form

$$v(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad (4.17)$$

The choice of a cubic function to describe the displacement is not arbitrary. While the general requirements of interpolation functions is discussed in Chapter 6, we make a few pertinent observations here. Clearly, with the specification of four boundary conditions, we can determine no more than four constants in the assumed displacement function. Second, in view of Equations 4.10 and 4.17, the second derivative of the assumed displacement function  $v(x)$  is linear; hence, the bending moment varies linearly, at most, along the length of the element. This is in accord with the assumption that loads are applied only at the element nodes, as indicated by the bending moment diagram of a loaded beam element shown in Figure 4.5. If a distributed load were applied to the element across its length, the bending moment would vary at least quadratically.

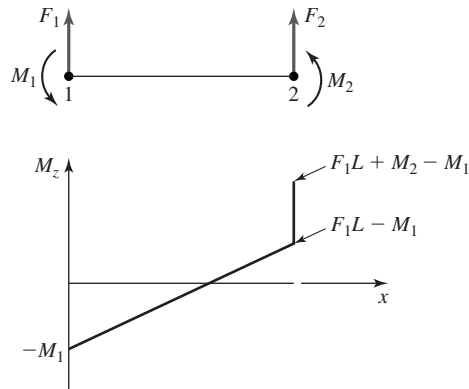
Application of the boundary conditions 4.13–4.16 in succession yields

$$v(x = 0) = v_1 = a_0 \quad (4.18)$$

$$v(x = L) = v_2 = a_0 + a_1L + a_2L^2 + a_3L^3 \quad (4.19)$$

$$\left. \frac{dv}{dx} \right|_{x=0} = \theta_1 = a_1 \quad (4.20)$$

$$\left. \frac{dv}{dx} \right|_{x=L} = \theta_2 = a_1 + 2a_2L + 3a_3L^2 \quad (4.21)$$



**Figure 4.5** Bending moment diagram for a flexure element. Sign convention per the strength of materials theory.

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Equations 4.18–4.21 are solved simultaneously to obtain the coefficients in terms of the nodal variables as

$$a_0 = v_1 \quad (4.22)$$

$$a_1 = \theta_1 \quad (4.23)$$

$$a_2 = \frac{3}{L^2}(v_2 - v_1) - \frac{1}{L}(2\theta_1 + \theta_2) \quad (4.24)$$

$$a_3 = \frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\theta_1 + \theta_2) \quad (4.25)$$

Substituting Equations 4.22–4.25 into Equation 4.17 and collecting the coefficients of the nodal variables results in the expression

$$\begin{aligned} v(x) = & \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right)v_1 + \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right)\theta_1 \\ & + \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)v_2 + \left(\frac{x^3}{L^2} - \frac{x^2}{L}\right)\theta_2 \end{aligned} \quad (4.26)$$

which is of the form

$$v(x) = N_1(x)v_1 + N_2(x)\theta_1 + N_3(x)v_2 + N_4(x)\theta_2 \quad (4.27a)$$

or, in matrix notation,

$$v(x) = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = [N] \{\delta\} \quad (4.27b)$$

where  $N_1$ ,  $N_2$ ,  $N_3$ , and  $N_4$  are the interpolation functions that describe the distribution of displacement in terms of nodal values in the nodal displacement vector  $\{\delta\}$ .

For the flexure element, it is convenient to introduce the dimensionless length coordinate

$$\xi = \frac{x}{L} \quad (4.28)$$

so that Equation 4.26 becomes

$$\begin{aligned} v(x) = & (1 - 3\xi^2 + 2\xi^3)v_1 + L(\xi - 2\xi^2 + \xi^3)\theta_1 + (3\xi^2 - 2\xi^3)v_2 \\ & + L\xi^2(\xi - 1)\theta_2 \end{aligned} \quad (4.29)$$

where  $0 \leq \xi \leq 1$ . This form proves more amenable to the integrations required to complete development of the element equations in the next section.

As discussed in Chapter 3, displacements are important, but the engineer is most often interested in examining the stresses associated with given loading conditions. Using Equation 4.11 in conjunction with Equation 4.27b, the normal

stress distribution on a cross section located at axial position  $x$  is given by

$$\sigma_x(x, y) = -yE \frac{d^2[N]}{dx^2} \{\delta\} \quad (4.30)$$

Since the normal stress varies linearly on a cross section, the maximum and minimum values on any cross section occur at the outer surfaces of the element, where distance  $y$  from the neutral surface is largest. As is customary, we take the maximum stress to be the largest tensile (positive) value and the minimum to be the largest compressive (negative) value. Hence, we rewrite Equation 4.30 as

$$\sigma_x(x) = y_{\max} E \frac{d^2[N]}{dx^2} \{\delta\} \quad (4.31)$$

and it is to be understood that Equation 4.31 represents the maximum and minimum normal stress values at any cross section defined by axial coordinate  $x$ . Also  $y_{\max}$  represents the largest distances (one positive, one negative) from the neutral surface to the outside surfaces of the element. Substituting for the interpolation functions and carrying out the differentiations indicated, we obtain

$$\begin{aligned} \sigma_x(x) = y_{\max} E \left[ \left( \frac{12x}{L^3} - \frac{6}{L^2} \right) v_1 + \left( \frac{6x}{L^2} - \frac{4}{L} \right) \theta_1 + \left( \frac{6}{L^2} - \frac{12x}{L^3} \right) v_2 \right. \\ \left. + \left( \frac{6x}{L^2} - \frac{2}{L} \right) \theta_2 \right] \end{aligned} \quad (4.32)$$

Observing that Equation 4.32 indicates a linear variation of normal stress along the length of the element and since, once the displacement solution is obtained, the nodal values are known constants, we need calculate only the stress values at the cross sections corresponding to the nodes; that is, at  $x = 0$  and  $x = L$ . The stress values at the nodal sections are given by

$$\sigma_x(x = 0) = y_{\max} E \left[ \frac{6}{L^2} (v_2 - v_1) - \frac{2}{L} (2\theta_1 + \theta_2) \right] \quad (4.33)$$

$$\sigma_x(x = L) = y_{\max} E \left[ \frac{6}{L^2} (v_1 - v_2) + \frac{2}{L} (2\theta_2 + \theta_1) \right] \quad (4.34)$$

The stress computations are illustrated in following examples.

## 4.4 FLEXURE ELEMENT STIFFNESS MATRIX

We may now utilize the discretized approximation of the flexure element displacement to examine stress, strain, and strain energy exhibited by the element under load. The total strain energy is expressed as

$$U_e = \frac{1}{2} \int_V \sigma_x \epsilon_x dV \quad (4.35)$$



## 4.4 Flexure Element Stiffness Matrix

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where  $V$  is total volume of the element. Substituting for the stress and strain per Equations 4.5 and 4.6,

$$U_e = \frac{E}{2} \int_V y^2 \left( \frac{d^2 v}{dx^2} \right)^2 dV \quad (4.36)$$

which can be written as

$$U_e = \frac{E}{2} \int_0^L \left( \frac{d^2 v}{dx^2} \right)^2 \left( \int_A y^2 dA \right) dx \quad (4.37)$$

Again recognizing the area integral as the moment of inertia  $I_z$  about the centroidal axis perpendicular to the plane of bending, we have

$$U_e = \frac{EI_z}{2} \int_0^L \left( \frac{d^2 v}{dx^2} \right)^2 dx \quad (4.38)$$

Equation 4.38 represents the strain energy of bending for any constant cross-section beam that obeys the assumptions of elementary beam theory. For the strain energy of the finite element being developed, we substitute the discretized displacement relation of Equation 4.27 to obtain

$$U_e = \frac{EI_z}{2} \int_0^L \left( \frac{d^2 N_1}{dx^2} v_1 + \frac{d^2 N_2}{dx^2} \theta_1 + \frac{d^2 N_3}{dx^2} v_2 + \frac{d^2 N_4}{dx^2} \theta_2 \right)^2 dx \quad (4.39)$$

as the approximation to the strain energy. We emphasize that Equation 4.39 is an approximation because the discretized displacement function is not in general an exact solution for the beam flexure problem.

Applying the first theorem of Castigliano to the strain energy function with respect to nodal displacement  $v_1$  gives the transverse force at node 1 as

$$\frac{\partial U_e}{\partial v_1} = F_1 = EI_z \int_0^L \left( \frac{d^2 N_1}{dx^2} v_1 + \frac{d^2 N_2}{dx^2} \theta_1 + \frac{d^2 N_3}{dx^2} v_2 + \frac{d^2 N_4}{dx^2} \theta_2 \right) \frac{d^2 N_1}{dx^2} dx \quad (4.40)$$

while application of the theorem with respect to the rotational displacement gives the moment as

$$\frac{\partial U_e}{\partial \theta_1} = M_1 = EI_z \int_0^L \left( \frac{d^2 N_1}{dx^2} v_1 + \frac{d^2 N_2}{dx^2} \theta_1 + \frac{d^2 N_3}{dx^2} v_2 + \frac{d^2 N_4}{dx^2} \theta_2 \right) \frac{d^2 N_2}{dx^2} dx \quad (4.41)$$

For node 2, the results are

$$\frac{\partial U_e}{\partial v_2} = F_2 = EI_z \int_0^L \left( \frac{d^2 N_1}{dx^2} v_1 + \frac{d^2 N_2}{dx^2} \theta_1 + \frac{d^2 N_3}{dx^2} v_2 + \frac{d^2 N_4}{dx^2} \theta_2 \right) \frac{d^2 N_3}{dx^2} dx \quad (4.42)$$

$$\frac{\partial U_e}{\partial \theta_2} = M_2 = EI_z \int_0^L \left( \frac{d^2 N_1}{dx^2} v_1 + \frac{d^2 N_2}{dx^2} \theta_1 + \frac{d^2 N_3}{dx^2} v_2 + \frac{d^2 N_4}{dx^2} \theta_2 \right) \frac{d^2 N_4}{dx^2} dx \quad (4.43)$$

Equations 4.40–4.43 algebraically relate the four nodal displacement values to the four applied nodal forces (here we use *force* in the general sense to include applied moments) and are of the form

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} \quad (4.44)$$

where  $k_{mn}$ ,  $m, n = 1, 4$  are the coefficients of the element stiffness matrix. By comparison of Equations 4.40–4.43 with the algebraic equations represented by matrix Equation 4.44, it is seen that

$$k_{mn} = k_{nm} = EI_z \int_0^L \frac{d^2 N_m}{dx^2} \frac{d^2 N_n}{dx^2} dx \quad m, n = 1, 4 \quad (4.45)$$

and the element stiffness matrix is symmetric, as expected for a linearly elastic element.

Prior to computing the stiffness coefficients, it is convenient to convert the integration to the dimensionless length variable  $\xi = x/L$  by noting

$$\int_0^L f(x) dx = \int_0^1 f(\xi) L d\xi \quad (4.46)$$

$$\frac{d}{dx} = \frac{1}{L} \frac{d}{d\xi} \quad (4.47)$$

so the integrations of Equation 4.45 become

$$k_{mn} = k_{nm} = EI_z \int_0^L \frac{d^2 N_m}{dx^2} \frac{d^2 N_n}{dx^2} dx = \frac{EI_z}{L^3} \int_0^1 \frac{d^2 N_m}{d\xi^2} \frac{d^2 N_n}{d\xi^2} d\xi \quad m, n = 1, 4 \quad (4.48)$$

The stiffness coefficients are then evaluated as follows:

$$\begin{aligned}
 k_{11} &= \frac{EI_z}{L^3} \int_0^1 (12\xi - 6)^2 d\xi = \frac{36EI_z}{L^3} \int_0^1 (4\xi^2 - 4\xi + 1) d\xi \\
 &= \frac{36EI_z}{L^3} \left( \frac{4}{3} - 2 + 1 \right) = \frac{12EI_z}{L^3} \\
 k_{12} &= k_{21} = \frac{EI_z}{L^3} \int_0^1 (12\xi - 6)(6\xi - 4)L d\xi = \frac{6EI_z}{L^2} \\
 k_{13} &= k_{31} = \frac{EI_z}{L^3} \int_0^1 (12\xi - 6)(6 - 12\xi) d\xi = -\frac{12EI_z}{L^3} \\
 k_{14} &= k_{41} = \frac{EI_z}{L^3} \int_0^1 (12\xi - 6)(6\xi - 2)L d\xi = \frac{6EI_z}{L^2}
 \end{aligned}$$

Continuing the direct integration gives the remaining stiffness coefficients as

$$\begin{aligned}
 k_{22} &= \frac{4EI_z}{L} \\
 k_{23} &= k_{32} = -\frac{6EI_z}{L^2} \\
 k_{24} &= k_{42} = \frac{2EI_z}{L} \\
 k_{33} &= \frac{12EI_z}{L^3} \\
 k_{34} &= k_{43} = -\frac{6EI_z}{L^3} \\
 k_{44} &= \frac{4EI_z}{L}
 \end{aligned}$$

The complete stiffness matrix for the flexure element is then written as

$$[k_e] = \frac{EI_z}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (4.49)$$

Symmetry of the element stiffness matrix is apparent, as previously observed. Again, the element stiffness matrix can be shown to be singular since rigid body motion is possible unless the element is constrained in some manner. The element stiffness matrix as given by Equation 4.49 is valid in any consistent system of units *provided* the rotational degrees of freedom (slopes) are expressed in *radians*.

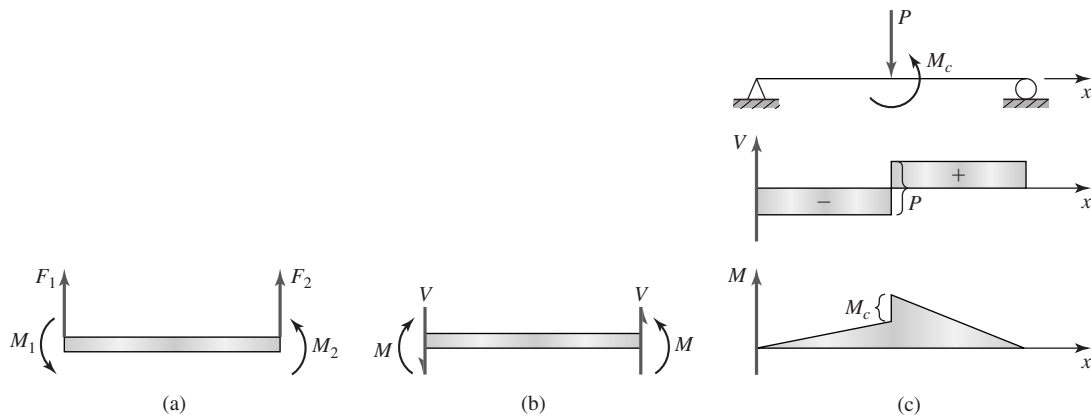
## 4.5 ELEMENT LOAD VECTOR

In Equations 4.40–4.43, the element forces and moments were treated as required by the first theorem of Castigliano as being in the direction of the associated displacements. These directions are in keeping with the assumed positive directions of the nodal displacements. However, as depicted in Figures 4.6a and 4.6b, the usual convention for shear force and bending moment in a beam are such that

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} -V_1 \\ -M_1 \\ V_2 \\ M_2 \end{Bmatrix} \quad (4.50)$$

In Equation 4.50, the column matrix (vector) on the left represents positive nodal forces and moments per the finite element formulations. The right-hand side contains the corresponding signed shear forces and bending moments per the beam theory sign convention.

If two flexure elements are joined at a common node, the internal shear forces are equal and opposite *unless* an external force is applied at that node, in which case the sum of the internal shear forces must equal the applied load. Therefore, when we assemble the finite element model using flexure elements, the force at a node is simply equal to any external force at that node. A similar argument holds for bending moments. At the juncture between two elements (i.e., a node), the internal bending moments are equal and opposite, thus self-equilibrating, *unless* a concentrated bending moment is applied at that node. In this event, the internal moments sum to the applied moment. These observations are illustrated in Figure 4.6c, which shows a simply supported beam subjected to a concentrated force and concentrated moment acting at the midpoint of the



**Figure 4.6**

(a) Nodal load positive convention. (b) Positive convention from the strength of materials theory. (c) Shear and bending moment diagrams depicting nodal load effects.

beam length. As shown by the shear force diagram, a jump discontinuity exists at the point of application of the concentrated force, and the magnitude of the discontinuity is the magnitude of the applied force. Similarly, the bending moment diagram shows a jump discontinuity in the bending moment equal to the magnitude of the applied bending moment. Therefore, if the beam were to be divided into two finite elements with a connecting node at the midpoint, the net force at the node is the applied external force and the net moment at the node is the applied external moment.

**EXAMPLE 4.1**

Figure 4.7a depicts a statically indeterminate beam subjected to a transverse load applied at the midspan. Using two flexure elements, obtain a solution for the midspan deflection.

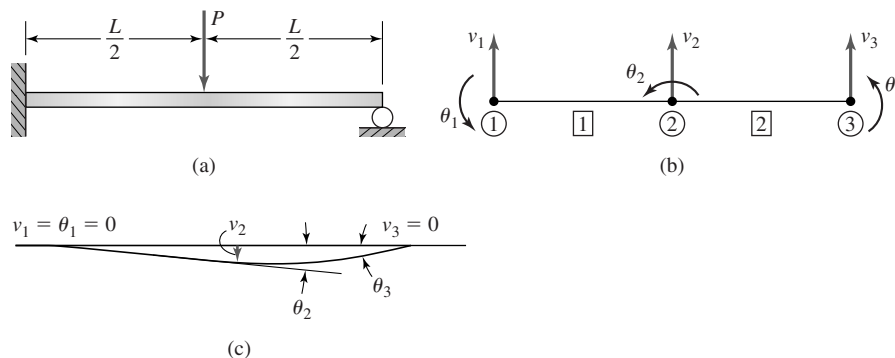
**■ Solution**

Since the flexure element requires loading only at nodes, the elements are taken to be of length  $L/2$ , as shown in Figure 4.7b. The individual element stiffness matrices are then

$$[k^{(1)}] = [k^{(2)}] = \frac{EI_z}{(L/2)^3} \begin{bmatrix} 12 & 6L/2 & -12 & 6L/2 \\ 6L/2 & 4L^2/4 & -6L/2 & 2L^2/4 \\ -12 & -6L/2 & 12 & -6L/2 \\ 6L/2 & 2L^2/4 & -6L/2 & 4L^2/4 \end{bmatrix}$$

$$= \frac{8EI_z}{L^3} \begin{bmatrix} 12 & 3L & -12 & 3L \\ 3L & L^2 & -3L & L^2/2 \\ -12 & -3L & 12 & -3L \\ 3L & L^2/2 & -3L & L^2 \end{bmatrix}$$

Note particularly that the length of each element is  $L/2$ . The appropriate boundary conditions are  $v_1 = \theta_1 = v_3 = 0$  and the element-to-system displacement correspondence table is Table 4.1.

**Figure 4.7**

(a) Loaded beam of Example 4.1. (b) Element and displacement designations. (c) Displacement solution.

**Table 4.1** Element-to-System Displacement Correspondence

Global Displacement	Element 1	Element 2
1	1	0
2	2	0
3	3	1
4	4	2
5	0	3
6	0	4

Assembling the global stiffness matrix per the displacement correspondence table we obtain in order (and using the symmetry property)

$$K_{11} = k_{11}^{(1)} = \frac{96EI_z}{L^3}$$

$$K_{12} = k_{12}^{(1)} = \frac{24EI_z}{L^2}$$

$$K_{13} = k_{13}^{(1)} = \frac{-96EI_z}{L^3}$$

$$K_{14} = k_{14}^{(1)} = \frac{24EI_z}{L^2}$$

$$K_{22} = k_{22}^{(1)} = \frac{8EI_z}{L}$$

$$K_{23} = k_{23}^{(1)} = \frac{-24EI_z}{L^2}$$

$$K_{24} = k_{24}^{(1)} = \frac{4EI_z}{L}$$

$$K_{25} = K_{26} = 0$$

$$K_{33} = k_{33}^{(1)} + k_{11}^{(2)} = \frac{192EI_z}{L^3}$$

$$K_{34} = k_{34}^{(1)} + k_{12}^{(2)} = 0$$

$$K_{35} = k_{13}^{(2)} = \frac{-96EI_z}{L^3}$$

$$K_{36} = k_{14}^{(2)} = \frac{24EI_z}{L^2}$$

$$K_{44} = k_{44}^{(1)} + k_{22}^{(1)} = \frac{16EI_z}{L}$$

$$K_{45} = k_{23}^{(2)} = \frac{-24EI_z}{L^2}$$

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$$K_{46} = k_{24}^{(2)} = \frac{4EI_z}{L}$$

$$K_{55} = k_{33}^{(2)} = \frac{96EI_z}{L^3}$$

$$K_{56} = k_{34}^{(2)} = \frac{-24EI_z}{L^2}$$

$$K_{66} = k_{44}^{(2)} = \frac{8EI_z}{L}$$

Using the general form

$$[K]\{U\} = \{F\}$$

we obtain the system equations as

$$\frac{EI_z}{L^3} \begin{bmatrix} 96 & 24L & -96 & 24L & 0 & 0 \\ 24L & 8L^2 & -24L & 4L^2 & 0 & 0 \\ -96 & -24L & 192 & 0 & -96 & 24L \\ 24L & 4L^2 & 0 & 16L^2 & -24L & 4L^2 \\ 0 & 0 & -96 & -24L & 96 & 24L \\ 0 & 0 & 24L & 4L^2 & 24L & 8L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix}$$

Invoking the boundary conditions  $v_1 = \theta_1 = v_3 = 0$ , the reduced equations become

$$\frac{EI_z}{L^3} \begin{bmatrix} 192 & 0 & 24L \\ 0 & 16L^2 & 4L^2 \\ 24L & 4L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix}$$

Yielding the nodal displacements as

$$v_2 = \frac{-7PL^3}{768EI_z} \quad \theta_2 = \frac{-PL^2}{128EI_z} \quad \theta_3 = \frac{PL^2}{32EI_z}$$

The deformed beam shape is shown in superposition with a plot of the undeformed shape with the displacements noted in Figure 4.7c. Substitution of the nodal displacement values into the constraint equations gives the reactions as

$$F_1 = \frac{EI_z}{L^3}(-96v_2 + 24L\theta_2) = \frac{11P}{16}$$

$$F_3 = \frac{EI_z}{L^3}(-96v_2 - 24L\theta_2 - 24L\theta_3) = \frac{5P}{16}$$

$$M_1 = \frac{EI_z}{L^3}(-24Lv_2 + 4L^2\theta_2) = \frac{3PL}{16}$$

Checking the overall equilibrium conditions for the beam, we find

$$\sum F_y = \frac{11P}{16} - P + \frac{5P}{16} = 0$$

and summing moments about node 1,

$$\sum M = \frac{3PL}{16} - P \frac{L}{2} + \frac{5P}{16}L = 0$$

Thus, the finite element solution satisfies global equilibrium conditions.

The astute reader may wish to compare the results of Example 4.1 with those given in many standard beam deflection tables, in which case it will be found that the results are in exact agreement with elementary beam theory. In general, the finite element method is an approximate method, but in the case of the flexure element, the results are exact in certain cases. In this example, the deflection equation of the neutral surface is a cubic equation and, since the interpolation functions are cubic, the results are exact. When distributed loads exist, however, the results are not necessarily exact, as will be discussed next.

## 4.6 WORK EQUIVALENCE FOR DISTRIBUTED LOADS

The restriction that loads be applied only at element nodes for the flexure element must be dealt with if a distributed load is present. The usual approach is to replace the distributed load with nodal forces and moments such that the mechanical work done by the nodal load system is equivalent to that done by the distributed load. Referring to Figure 4.1, the mechanical work performed by the distributed load can be expressed as

$$W = \int_0^L q(x)v(x) dx \quad (4.51)$$

The objective here is to determine the equivalent nodal loads so that the work expressed in Equation 4.51 is the same as

$$W = \int_0^L q(x)v(x) dx = F_{1q}v_1 + M_{1q}\theta_1 + F_{2q}v_2 + M_{2q}\theta_2 \quad (4.52)$$

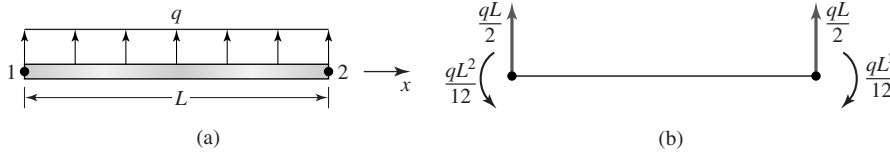
where  $F_{1q}$ ,  $F_{2q}$  are the equivalent forces at nodes 1 and 2, respectively, and  $M_{1q}$  and  $M_{2q}$  are the equivalent nodal moments. Substituting the discretized displacement function given by Equation 4.27, the work integral becomes

$$W = \int_0^L q(x)[N_1(x)v_1 + N_2(x)\theta_1 + N_3(x)v_2 + N_4(x)\theta_2] dx \quad (4.53)$$



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**Figure 4.8** Work-equivalent nodal forces and moments for a uniform distributed load.

Comparison of Equations 4.52 and 4.53 shows that

$$F_{1q} = \int_0^L q(x) N_1(x) dx \quad (4.54)$$

$$M_{1q} = \int_0^L q(x) N_2(x) dx \quad (4.55)$$

$$F_{2q} = \int_0^L q(x) N_3(x) dx \quad (4.56)$$

$$M_{2q} = \int_0^L q(x) N_4(x) dx \quad (4.57)$$

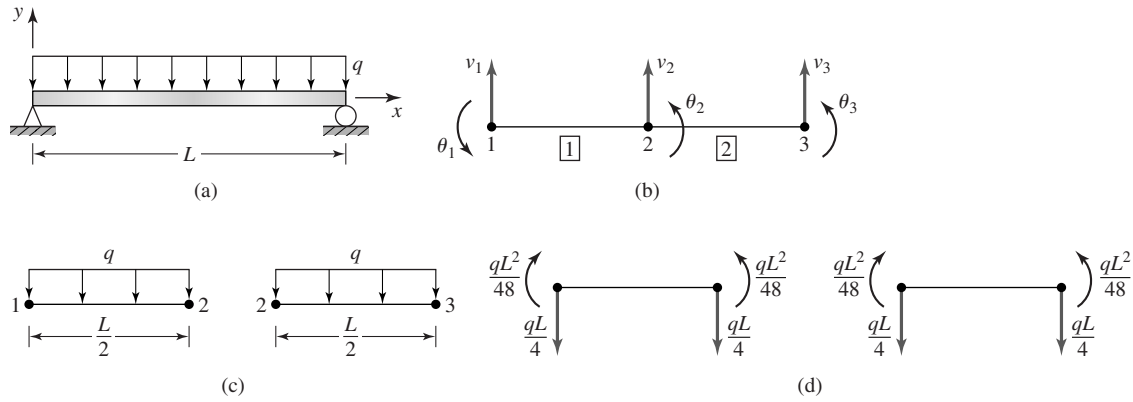
Hence, the nodal force vector representing a distributed load on the basis of work equivalence is given by Equations 4.54–4.57. For example, for a uniform load  $q(x) = q = \text{constant}$ , integration of these equations yields

$$\begin{Bmatrix} F_{1q} \\ M_{1q} \\ F_{2q} \\ M_{2q} \end{Bmatrix} = \begin{Bmatrix} \frac{qL}{2} \\ \frac{qL^2}{12} \\ \frac{qL}{2} \\ -\frac{qL^2}{12} \end{Bmatrix} \quad (4.58)$$

The equivalence of a uniformly distributed load to the corresponding nodal loads on an element is shown in Figure 4.8.

**EXAMPLE 4.2**

The simply supported beam shown in Figure 4.9a is subjected to a uniform transverse load, as shown. Using two equal-length elements and work-equivalent nodal loads, obtain a finite element solution for the deflection at midspan and compare it to the solution given by elementary beam theory.

**Figure 4.9**

(a) Uniformly loaded beam of Example 4.2. (b) Node, element, and displacement notation. (c) Element loading. (d) Work-equivalent nodal loads.

### ■ Solution

Per Figure 4.9b, we number the nodes and elements as shown and note the boundary conditions  $v_1 = v_3 = 0$ . We could also note the symmetry condition that  $\theta_2 = 0$ . However, in this instance, we let that fact occur as a result of the solution process. The element stiffness matrices are identical, given by

$$\begin{aligned}
 [k^{(1)}] &= [k^{(2)}] = \frac{EI_z}{(L/2)^3} \begin{bmatrix} 12 & 6L/2 & -12 & 6L/2 \\ 6L/2 & 4L^2/4 & -6L/2 & 2L^2/4 \\ -12 & -6L/2 & 12 & -6L/2 \\ 6L/2 & 2L^2/4 & -6L/2 & 4L^2/4 \end{bmatrix} \\
 &= \frac{8EI_z}{L^3} \begin{bmatrix} 12 & 3L & -12 & 3L \\ 3L & L^2 & -3L & L^2/2 \\ -12 & -3L & 12 & -3L \\ 3L & L^2/2 & -3L & L^2 \end{bmatrix}
 \end{aligned}$$

(again note that the individual element length  $L/2$  is used to compute the stiffness terms), and Table 4.2 is the element connectivity table, so the assembled global stiffness matrix is

$$[K] = \frac{8EI_z}{L^3} \begin{bmatrix} 12 & 3L & -12 & 3L & 0 & 0 \\ 3L & L^2 & -3L & L^2/2 & 0 & 0 \\ -12 & -3L & 24 & 0 & -12 & 3L \\ 3L & L^2/2 & 0 & 2L^2 & -3L & L^2/2 \\ 0 & 0 & -12 & -3L & 12 & -3L \\ 0 & 0 & 3L & L^2/2 & -3L & L^2 \end{bmatrix}$$

The work-equivalent loads for each element are computed with reference to Figure 4.9c and the resulting loads shown in Figure 4.9d. Observing that there are reaction forces at both nodes 1 and 3 in addition to the equivalent forces from the distributed load, the

## 4.6 Work Equivalence for Distributed Loads

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**Table 4.2** Element Connectivity

Global Displacement	Element 1	Element 2
1	1	0
2	2	0
3	3	1
4	4	2
5	0	3
6	0	4

global equilibrium equations become

$$[K] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} \frac{-qL}{4} + F_1 \\ \frac{-qL^2}{48} \\ \frac{-qL}{2} \\ 0 \\ \frac{-qL}{4} + F_3 \\ \frac{qL^2}{48} \end{Bmatrix}$$

where the work-equivalent nodal loads have been utilized per Equation 4.58, with each element length =  $L/2$  and  $q(x) = -q$ , as shown in Figure 4.9c. Applying the constraint and symmetry conditions, we obtain the system

$$\frac{8EI_z}{L^3} \begin{bmatrix} L^2 & -3L & L^2/2 & 0 \\ -3L & 24 & 0 & 3L \\ L^2/2 & 0 & 2L^2 & L^2/2 \\ 0 & 3L & L^2/2 & L^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} \frac{-qL^2}{48} \\ \frac{-qL}{2} \\ 0 \\ \frac{qL^2}{48} \end{Bmatrix}$$

which, on simultaneous solution, gives the displacements as

$$\begin{aligned} \theta_1 &= -\frac{qL^3}{24EI_z} \\ \theta_2 &= 0 \\ v_2 &= -\frac{5qL^4}{384EI_z} \\ \theta_3 &= \frac{qL^3}{24EI_z} \end{aligned}$$

As expected, the slope of the beam at midspan is zero, and since the loading and support conditions are symmetric, the deflection solution is also symmetric, as indicated by

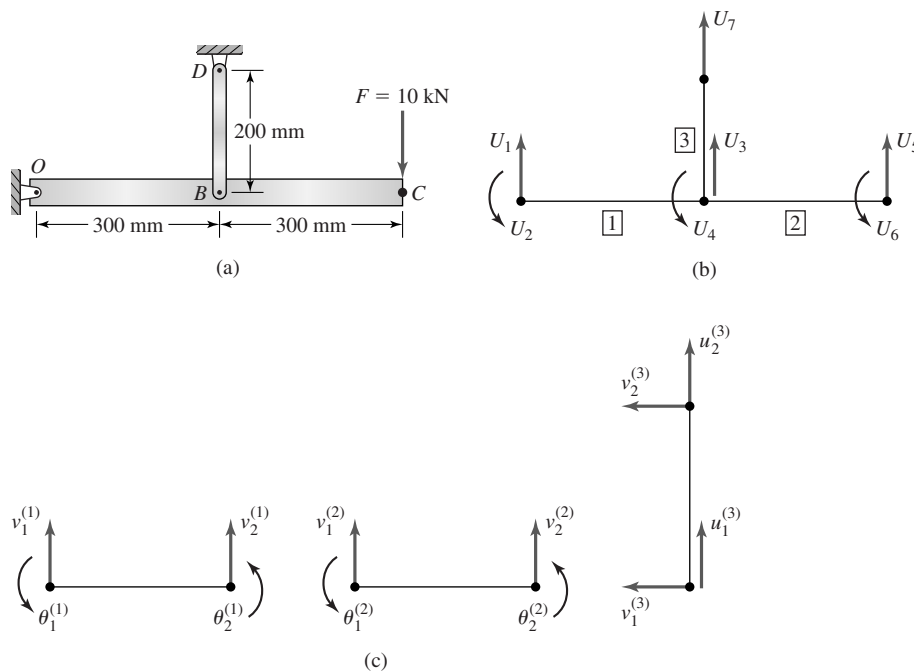
the end slopes. The nodal displacement results from the finite element analysis of this example are exactly the results obtained by a strength of materials approach. This is due to applying the work-equivalent nodal loads. However, the general deflected shape as given by the finite element solution is *not* the same as the strength of materials result. The equation describing the deflection of the neutral surface is a quartic function of  $x$  and, since the interpolation functions used in the finite element model are cubic, the deflection curve varies somewhat from the exact solution.

**EXAMPLE 4.3**

In Figure 4.10a, beam  $OC$  is supported by a smooth pin connection at  $O$  and supported at  $B$  by an elastic rod  $BD$ , also through pin connections. A concentrated load  $F = 10$  kN is applied at  $C$ . Determine the deflection of point  $C$  and the axial stress in member  $BD$ . The modulus of elasticity of the beam is 207 GPa (steel) and the dimensions of the cross section are 40 mm  $\times$  40 mm. For elastic rod  $BD$ , the modulus of elasticity is 69 GPa (aluminum) and the cross-sectional area is 78.54 mm<sup>2</sup>.

**■ Solution**

This is the first example in which we use multiple element types, as the beam is modeled with flexure elements and the elastic rod as a bar element. Clearly, the horizontal member

**Figure 4.10**

(a) Supported beam. (b) Global coordinate system and variables. (c) Individual element displacements.

**Table 4.3** Displacement Scheme

Global	Figure 4.10b	Element 1	Element 2	Element 3
1	$U_1$	$v_1^{(1)}$	0	0
2	$U_2$	$\theta_1^{(1)}$	0	0
3	$U_3$	$v_2^{(1)}$	$v_1^{(2)}$	$u_1^{(3)}$
4	$U_4$	$\theta_2^{(1)}$	$\theta_1^{(2)}$	0
5	$U_5$	0	$v_2^{(2)}$	0
6	$U_6$	0	$\theta_2^{(2)}$	0
7	$U_7$	0	0	$u_2^{(3)}$

**Table 4.4** Element-Displacement Correspondence

Global Displacement	Element 1	Element 2	Element 3
1	1	0	0
2	2	0	0
3	3	1	1
4	4	2	0
5	0	3	0
6	0	4	0
7	0	0	3

is subjected to bending loads, so the assumptions of the bar element do not apply to this member. On the other hand, the vertical support member is subjected to only axial loading, since the pin connections cannot transmit moment. Therefore, we use two different element types to simplify the solution and modeling. The global coordinate system and global variables are shown in Figure 4.10b, where the system is divided into two flexure elements (1 and 2) and one spar element (3). For purposes of numbering in the global stiffness matrix, the displacement scheme in Table 4.3 is used.

While the notation shown in Figure 4.10b may appear to be inconsistent with previous notation, it is simpler in terms of the global equations to number displacements successively. By proper assignment of element displacements to global displacements, the distinction between linear and rotational displacements are clear. The individual element displacements are shown in Figure 4.10c, where we show the bar element in its general 2-D configuration, even though, in this case, we know that  $v_1^{(3)} = v_2^{(3)} = 0$  and those displacements are ignored in the solution.

The element displacement correspondence is shown in Table 4.4. For the beam elements, the moment of inertia about the  $z$  axis is

$$I_z = \frac{bh^3}{12} = \frac{40(40^3)}{12} = 213333 \text{ mm}^4$$

For elements 1 and 2,

$$\frac{EI_z}{L^3} = \frac{207(10^3)(213333)}{300^3} = 1635.6 \text{ N/mm}$$

**Table 4.5** Global Stiffness Matrix

	1	2	3	4	5	6	7
1	19,627.2	$2.944 \times 10^6$	-19,627.2	$2.944 \times 10^6$	0	0	0
2	$2.944 \times 10^6$	$5.888 \times 10^8$	$-2.944 \times 10^6$	$2.944 \times 10^8$	0	0	0
3	-19,627.2	$-2.944 \times 10^6$	66,350.4	0	-19,627.2	$2.944 \times 10^6$	-27,096
4	$2.944 \times 10^6$	$2.944 \times 10^8$	0	$11.78 \times 10^8$	$-2.944 \times 10^6$	$2.944 \times 10^8$	0
5	0	0	-19,627.2	$-2.944 \times 10^6$	19,627.2	$-2.944 \times 10^6$	0
6	0	0	$2.944 \times 10^6$	$2.944 \times 10^8$	$-2.944 \times 10^6$	$5.889 \times 10^8$	0
7	0	0	-27,096	0	0	0	27,096

so the element stiffness matrices are (per Equation 4.48)

$$[k^{(1)}] = [k^{(2)}] = 1,635.6 \begin{bmatrix} 12 & 1,800 & -12 & 1,800 \\ 1,800 & 360,000 & -1,800 & 180,000 \\ -12 & -1,800 & 12 & -1,800 \\ 1,800 & 180,000 & -1,800 & 360,000 \end{bmatrix}$$

while for element 3,

$$\frac{AE}{L} = \frac{78.54(69)(10^3)}{200} = 27096 \text{ N/mm}$$

so the stiffness matrix for element 3 is

$$[k^{(3)}] = 27,096 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assembling the global stiffness matrix per the displacement correspondence table (noting that we use a “short-cut” for element 3, since the stiffness of the element in the global  $X$  direction is meaningless), we obtain the results in Table 4.5. The constraint conditions are  $U_1 = U_7 = 0$  and the applied force vector is

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ 0 \\ -10,000 \\ 0 \\ R_4 \end{Bmatrix}$$

where we use  $R$  to indicate a reaction force. If we apply the constraint conditions and solve the resulting  $5 \times 5$  system of equations, we obtain the results

$$\theta_1 = 9.3638(10^{-4}) \text{ rad}$$

$$v_2 = -0.73811 \text{ mm}$$

$$\theta_2 = -0.0092538 \text{ rad}$$

$$v_3 = -5.5523 \text{ mm}$$

$$\theta_3 = -0.019444 \text{ rad}$$

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(Note that we intentionally carry more significant decimal digits than necessary to avoid “round-off” inaccuracies in secondary calculations.) To obtain the axial stress in member  $BD$ , we utilize Equation 3.52 with  $\theta^{(3)} = \pi/2$ :

$$\sigma_{BD} = 69(10^3) \begin{bmatrix} -\frac{1}{200} & \frac{1}{200} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.7381 \\ 0 \\ 0 \end{Bmatrix} = 254.6 \text{ MPa}$$

The positive result indicates tensile stress.

The reaction forces are obtained by substitution of the computed displacements into the first and seventh equations (the constraint equations):

$$\begin{aligned} R_1 &= 2.944(10^6)[9.3638(10^{-4})] - 19,627.2(-0.73811) \\ &\quad + 2.944(10^6)(-0.0092538) \approx -10,000 \text{ N} \\ R_4 &= -27,096(-0.73811) + 27,096(0) = 20,000 \text{ N} \end{aligned}$$

and within the numerical accuracy used in this example, the system is in equilibrium. The reader is urged to check moment equilibrium about the left-hand node and note that, by statics alone, the force in element 3 should be 20,000 N and the axial stress computed by  $F/A$  is 254.6 MPa.

The bending stresses at nodes 1 and 2 in the flexure elements are computed via Equations 4.33 and 4.34, respectively, noting that for the square cross section  $y_{\max/\min} = 20 \text{ mm}$ . For element 1,

$$\begin{aligned} \sigma_x^{(1)}(x=0) &= \pm 20(207)(10^3) \left[ \frac{6}{300^2}(-0.738 - 0) - \frac{2}{300}(-(2)0.00093 - 0.0092) \right] \\ &\approx 0 \end{aligned}$$

at node 1. Note that the computed stress at node 1 should be identically zero, since this node is a pin joint and cannot support bending moment.

For node 2 of element 1, we find

$$\begin{aligned} \sigma_x^{(1)}(x=L) &= \pm 20(207)(10^3) \left[ \frac{6}{300^2}(0 + 0.738) + \frac{2}{300}(-(2)0.0092 - 0.00093) \right] \\ &\approx \pm 281.3 \text{ MPa} \end{aligned}$$

For element 2, we similarly compute the stresses at each node as

$$\begin{aligned} \sigma_x^{(2)}(x=0) &= \pm 20(207)(10^3) \\ &\quad \times \left[ \frac{6}{300^2}(-5.548 + 0.738) - \frac{2}{300}(-(2)0.0092 - 0.0194) \right] \approx \pm 281.3 \text{ MPa} \\ \sigma_x^{(2)}(x=L) &= \pm 20(207)(10^3) \\ &\quad \times \left[ \frac{6}{300^2}(-0.73811 + 5.5523) + \frac{2}{300}(-(2)0.019444 - 0.009538) \right] \approx 0 \text{ MPa} \end{aligned}$$

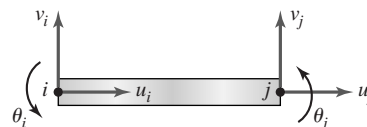
and the latter result is also to be expected, as the right end of the beam is free of bending moment. We need to carefully observe here that the bending stress is the same at the

juncture of the two flexure elements; that is, at node 2. This is *not* the usual situation in finite element analysis. The formulation requires displacement and slope continuity but, in general, no continuity of higher-order derivatives. Since the flexure element developed here is based on a cubic displacement function, the element does not often exhibit moment (hence, stress) continuity. The convergence of derivative functions is paramount to examining the accuracy of a finite element solution to a given problem. We must examine the numerical behavior of the derived variables as the finite element “mesh” is refined.

## 4.7 FLEXURE ELEMENT WITH AXIAL LOADING

The major shortcoming of the flexure element developed so far is that force loading must be transverse to the axis of the element. Effectively, this means that the element can be used only in end-to-end modeling of linear beam structures. If the element is formulated to also support axial loading, the applicability is greatly extended. Such an element is depicted in Figure 4.11, which shows, in addition to the nodal transverse deflections and rotations, axial displacements at the nodes. Thus, the element allows axial as well as transverse loading. It must be pointed out that there are many ramifications to this seemingly simple extension. If the axial load is compressive, the element could buckle. If the axial load is tensile and significantly large, a phenomenon known as *stress stiffening* can occur. The phenomenon of stress stiffening can be likened to tightening of a guitar string. As the tension is increased, the string becomes more resistant to motion perpendicular to the axis of the string.

The same effect occurs in structural members in tension. As shown in Figure 4.12, in a beam subjected to both transverse and axial loading, the effect of the axial load on bending is directly related to deflection, since the deflection at a specific point becomes the moment arm for the axial load. In cases of small elastic deflection, the additional bending moment attributable to the axial loading is negligible. However, in most finite element software packages, buckling and stress stiffening analyses are available as *options* when such an element is used in an analysis. (The reader should be aware that buckling and stress stiffening effects are checked *only if the software user so specifies*.) For the present purpose, we assume the axial loads are such that these secondary effects are not of concern and the axial loading is independent of bending effects.

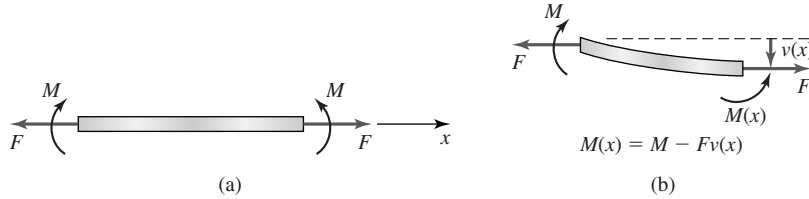


**Figure 4.11** Nodal displacements of a beam element with axial stiffness.



## 4.7 Flexure Element with Axial Loading

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**Figure 4.12**

(a) Beam with bending moment and axial load. (b) Section of beam, illustrating how tensile load reduces bending moment, hence, “stiffening” the beam.

This being the case, we can simply add the spar element stiffness matrix to the flexure element stiffness matrix to obtain the  $6 \times 6$  element stiffness matrix for a flexure element with axial loading as

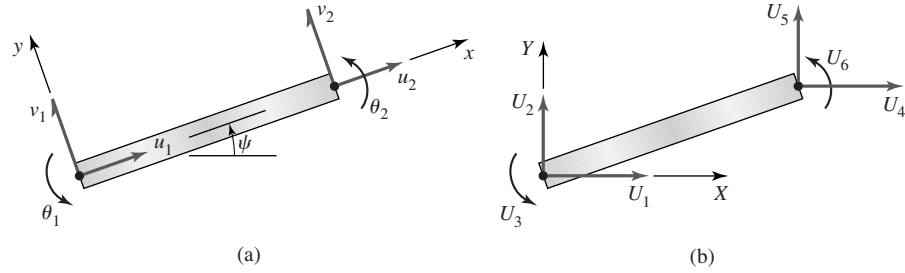
$$[k_e] = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} & 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & \frac{AE}{L} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ 0 & 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \quad (4.59)$$

which is seen to be simply

$$[k_e] = \begin{bmatrix} [k_{\text{axial}}] & [0] \\ [0] & [k_{\text{flexure}}] \end{bmatrix} \quad (4.60)$$

and is a noncoupled superposition of axial and bending stiffnesses.

Adding axial capability to the beam element eliminates the restriction that such elements be aligned linearly and enables use of the element in the analysis of planar frame structures in which the joints have bending resistance. For such applications, orientation of the element in the global coordinate system must be considered, as was the case with the spar element in trusses. Figure 4.13a depicts an element oriented at an arbitrary angle  $\psi$  from the X axis of a global reference frame and shows the element nodal displacements. Here, we use  $\psi$  to indicate the orientation angle to avoid confusion with the nodal slope, denoted  $\theta$ . Figure 4.13b shows the assigned global displacements for the element, where again we have adopted a single symbol for displacement with a numerically increasing subscript from node to node. Before proceeding, note that it is convenient here to reorder the element stiffness matrix given by Equation 4.59 so that the element

**Figure 4.13**

(a) Nodal displacements in the element coordinate system. (b) Nodal displacements in the global coordinate system.

displacement vector in the element reference frame is given as

$$\{\delta\} = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \quad (4.61)$$

and the element stiffness matrix becomes

$$[k_e] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \quad (4.62)$$

Using Figure 4.13, the element displacements are written in terms of the global displacements as

$$\begin{aligned} u_1 &= U_1 \cos \psi + U_2 \sin \psi \\ v_1 &= -U_1 \sin \psi + U_2 \cos \psi \\ \theta_1 &= U_3 \\ u_2 &= U_4 \cos \psi + U_5 \sin \psi \\ v_2 &= -U_4 \sin \psi + U_5 \cos \psi \\ \theta_2 &= U_6 \end{aligned} \quad (4.63)$$

## 4.7 Flexure Element with Axial Loading

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Equations 4.63 can be written in matrix form as

$$\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi & 0 & 0 & 0 & 0 \\ -\sin \psi & \cos \psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \psi & \sin \psi & 0 \\ 0 & 0 & 0 & -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = [R]\{U\} \quad (4.64)$$

where  $[R]$  is the transformation matrix that relates element displacements to global displacements. In a manner exactly analogous to that of Section 3.3, it is readily shown that the  $6 \times 6$  element stiffness matrix in the global system is given by

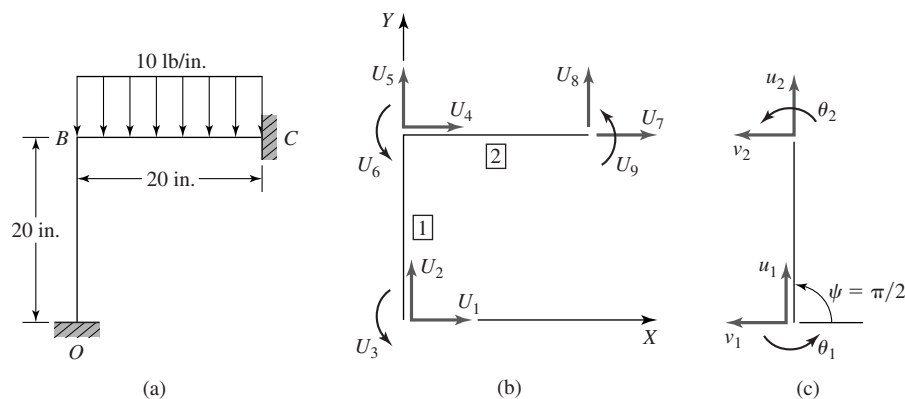
$$[K_e] = [R]^T [k_e] [R] \quad (4.65)$$

Owing to its algebraic complexity, Equation 4.65 is not expanded here to obtain a general result. Rather, the indicated computations are best suited for specific element characteristics and performed by computer program.

Assembly of the system equations for a finite element model using the beam-axial element is accomplished in an identical fashion to the procedures followed for trusses as discussed in Chapter 3. The following simple example illustrates the procedure.

**EXAMPLE 4.4**

The frame of Figure 4.14a is composed of identical beams having a 1-in. square cross section and a modulus of elasticity of  $10 \times 10^6$  psi. The supports at  $O$  and  $C$  are to be considered completely fixed. The horizontal beam is subjected to a uniform load of intensity 10 lb/in., as shown. Use two beam-axial elements to compute the displacements and rotation at  $B$ .

**Figure 4.14**

(a) Frame of Example 4.4. (b) Global coordinate system and displacement numbering. (c) Transformation of element 1.

## ■ Solution

Using the specified data, The cross-sectional area is

$$A = 1(1) = 1 \text{ in.}^2$$

And the area moment of inertia about the  $z$  axis is

$$I_z = bh^3/12 = 1/12 = 0.083 \text{ in.}^4$$

The characteristic axial stiffness is

$$AE/L = 1(10 \times 10^6)/20 = (5 \times 10^5) \text{ lb/in.}$$

and the characteristic bending stiffness is

$$EI_z/L^3 = 10 \times 10^6(0.083)/20^3 = 104.2 \text{ lb/in.}$$

Denoting member  $OB$  as element 1 and member  $BC$  as element 2, the stiffness matrices in the element coordinate systems are identical and given by

$$[k^{(1)}] = [k^{(2)}] = \begin{bmatrix} 5(10^5) & 0 & 0 & -5(10^5) & 0 & 0 \\ 0 & 1,250.4 & 12,504 & 0 & -1,250.4 & 12,504 \\ 0 & 12,504 & 166,720 & 0 & -12,504 & 83,360 \\ -5(10^5) & 0 & 0 & 5(10^5) & 0 & 0 \\ 0 & -1,250.4 & -12,504 & 0 & 1,250.4 & -12,504 \\ 0 & 12,504 & 83,360 & 0 & -12,504 & 166,720 \end{bmatrix}$$

Choosing the global coordinate system and displacement numbering as in Figure 4.14b, we observe that element 2 requires no transformation, as its element coordinate system is aligned with the global system. However, as shown in Figure 4.14c, element 1 requires transformation. Using  $\psi = \pi/2$ , Equations 4.64 and 4.65 are applied to obtain

$$[K^{(1)}] = \begin{bmatrix} 1,250.4 & 0 & -12,504 & 1,250.4 & 0 & -12,504 \\ 0 & 5(10^5) & 0 & 0 & -5(10^5) & 0 \\ -12,504 & 0 & 166,720 & 12,504 & 0 & 83,360 \\ 1,250.4 & 0 & 12,504 & 1,250.4 & 0 & 12,504 \\ 0 & -5(10^5) & 0 & 0 & 5(10^5) & 0 \\ -12,504 & 0 & 83,360 & 12,504 & 0 & 166,720 \end{bmatrix}$$

Note particularly how the stiffness matrix of element 1 changes as a result of the  $90^\circ$  rotation. The values of individual components in the stiffness matrix are unchanged. The positions of the terms in the matrix are changed to reflect, quite simply, the directions of bending and axial displacements of the element when described in the global (system) coordinate system.

The displacement correspondence table is shown in Table 4.6 and the assembled system stiffness matrix, by the direct assembly procedure, is in Table 4.7. Note, as usual, the “overlap” of the element stiffness matrices at the displacements associated with the common node. At these positions in the global stiffness matrix, the stiffness terms from the individual element stiffness matrices are additive.

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**Table 4.6** Displacement Correspondence

Global	Element 1	Element 2
1	1	0
2	2	0
3	3	0
4	4	1
5	5	2
6	6	3
7	0	4
8	0	5
9	0	6

**Table 4.7** System Stiffness Matrix

$[K] =$	$\begin{bmatrix} 1,250.4 & 0 & -12,504 & 1,250.4 & 0 & -12,504 & 0 & 0 & 0 \\ 0 & 500,000 & 0 & 0 & -500,000 & 0 & 0 & 0 & 0 \\ -12,504 & 0 & 166,720 & 12,504 & 0 & 833,360 & 0 & 0 & 0 \\ 1,250.4 & 0 & 12,504 & 501,250.4 & 0 & 12,504 & -500,000 & 0 & 0 \\ 0 & -500,000 & 0 & 0 & 501,250.4 & 12,504 & 0 & -1,250.4 & 12,504 \\ -12,504 & 0 & 83,360 & 12,504 & 12,504 & 333,440 & 0 & -12,504 & 83,360 \\ 0 & 0 & 0 & -500,000 & 0 & 0 & 500,000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1,250.4 & -12,504 & 0 & 1,250.4 & -12,504 \\ 0 & 0 & 0 & 0 & 12,504 & 83,360 & 0 & -12,504 & 166,720 \end{bmatrix}$
---------	--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

Using the system stiffness matrix, the assembled system equations are

$$[K] \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \end{Bmatrix} = \begin{Bmatrix} R_{X1} \\ R_{Y1} \\ M_{R1} \\ 0 \\ -100 \\ -333.3 \\ R_{X3} \\ R_{Y3} - 100 \\ M_{R3} + 333.3 \end{Bmatrix}$$

where we denote the forces at nodes 1 and 3 as reaction components, owing to the displacement constraints  $U_1 = U_2 = U_3 = U_7 = U_8 = U_9 = 0$ . Taking the constraints into account, the equations to be solved for the active displacements are then

$$\begin{bmatrix} 501,250.4 & 0 & 12,504 \\ 0 & 501,250.4 & 12,504 \\ 12,504 & 12,504 & 333,440 \end{bmatrix} \begin{Bmatrix} U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -100 \\ -16.7 \end{Bmatrix}$$

Simultaneous solution gives the displacement values as

$$U_4 = 2.47974(10^{-5}) \text{ in.}$$

$$U_5 = -1.74704(10^{-4}) \text{ in.}$$

$$U_6 = -9.94058(10^{-4}) \text{ rad}$$

As usual, the reaction components can be obtained by substituting the computed displacements into the six constraint equations.

For the beam element with axial capability, the stress computation must take into account the superposition of bending stress and direct axial stress. For element 1, for example, we use Equation 4.63 with  $\psi = \pi/2$  to compute the element displacement as

$$\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.74704(10^{-4}) \\ -2.47974(10^{-5}) \\ -9.94058(10^{-4}) \end{Bmatrix}$$

The bending stress is computed at nodes 1 and 2 via Equations 4.33 and 4.34 as

$$\begin{aligned} \sigma_x(x=0) &= \pm 0.5(10)(10^6) \left[ \frac{6}{20^2}(-2.47974)(10^{-5}) - \frac{2}{20}(-9.94058)(10^{-4}) \right] \\ &= \pm 495.2 \text{ psi} \end{aligned}$$

$$\begin{aligned} \sigma_x(x=L) &= \pm 0.5(10)(10^6) \left[ \frac{6}{20^2}(2.47974)(10^{-5}) + \frac{2}{20}(2)(-9.94058)(10^{-4}) \right] \\ &= \pm 992.2 \text{ psi} \end{aligned}$$

and the axial stress is

$$\sigma_{\text{axial}} = 10(10^6) \frac{-1.74704(10^{-4})}{20} = -87.35 \text{ psi}$$

Therefore, the largest stress magnitude occurs at node 2, at which the compressive axial stress adds to the compressive portion of the bending stress distribution to give

$$\sigma = 1079.6 \text{ psi} \quad (\text{compressive})$$

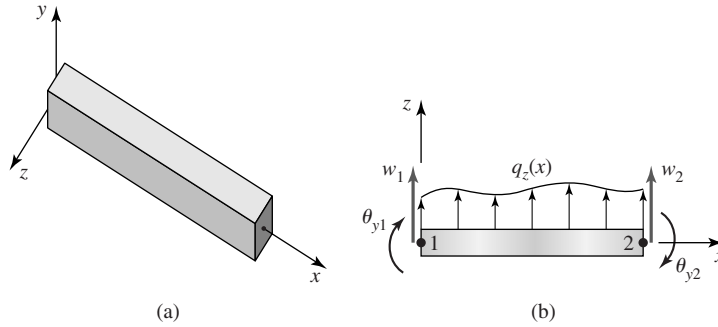
## 4.8 A GENERAL THREE-DIMENSIONAL BEAM ELEMENT

A general three-dimensional beam element is capable of both axial and torsional deflections as well as two-plane bending. To examine the stiffness characteristics of such an element and obtain the element stiffness matrix, we first extend the beam-axial element of the previous section to include two-plane bending, then add torsional capability.

Figure 4.15a shows a beam element with an attached three-dimensional element coordinate system in which the  $x$  axis corresponds to the longitudinal axis

## 4.8 A General Three-Dimensional Beam Element

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**Figure 4.15**

(a) Three-dimensional beam element. (b) Nodal displacements in element  $xz$  plane.

of the beam and is assumed to pass through the centroid of the beam cross section. The  $y$  and  $z$  axes are assumed to correspond to the *principal axes for area moments of inertia of the cross section* [1]. If this is not the case, treatment of simultaneous bending in two planes and superposition of the results as in the following element development will *not* produce correct results [2].

For bending about the  $z$  axis (i.e., the plane of bending is the  $xy$  plane), the element stiffness matrix is given by Equation 4.48. For bending about the  $y$  axis, the plane of bending is the  $xz$  plane, as in Figure 4.15b, which depicts a beam element defined by nodes 1 and 2 and subjected to a distributed load  $q_z(x)$  shown acting in the positive  $z$  direction. Nodal displacements in the  $z$  direction are denoted  $w_1$  and  $w_2$ , while nodal rotations are  $\theta_{y1}$  and  $\theta_{y2}$ . For this case, it is necessary to add the axis subscript to the nodal rotations to specifically identify the axis about which the rotations are measured. In this context, the rotations corresponding to  $xy$  plane bending henceforth are denoted  $\theta_{z1}$  and  $\theta_{z2}$ . It is also important to note that, in Figure 4.15b, the  $y$  axis is perpendicular to the plane of the page with the positive sense *into* the page. Therefore, the rotations shown are positive about the  $y$  axis per the right-hand rule. Noting the difference in the positive sense of rotation relative to the linear displacements, a development analogous to that used for the flexure element in Sections 4.3 and 4.4 results in the element stiffness matrix for  $xz$  plane bending as

$$[k_e]_{xz} = \frac{EI_y}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \quad (4.66)$$

The only differences between the  $xz$  plane bending stiffness matrix and that for  $xy$  plane bending are seen to be sign changes in the off-diagonal terms and the fact that the characteristic stiffness depends on the area moment of inertia  $I_y$ .

Combining the spar element stiffness matrix, the  $xy$  plane flexure stiffness matrix, and the  $xz$  plane stiffness matrix given by Equation 4.60, the element

equilibrium equations for a two-plane bending element with axial stiffness are written in matrix form as

$$\begin{bmatrix} [k_{\text{axial}}] & [0] & [0] \\ [0] & [k_{\text{bending}}]_{xy} & [0] \\ [0] & [0] & [k_{\text{bending}}]_{xz} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \\ w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{x2} \\ f_{y1} \\ M_{z1} \\ f_{y2} \\ M_{z2} \\ f_{z1} \\ M_{y1} \\ f_{z2} \\ M_{y2} \end{Bmatrix} \quad (4.67)$$

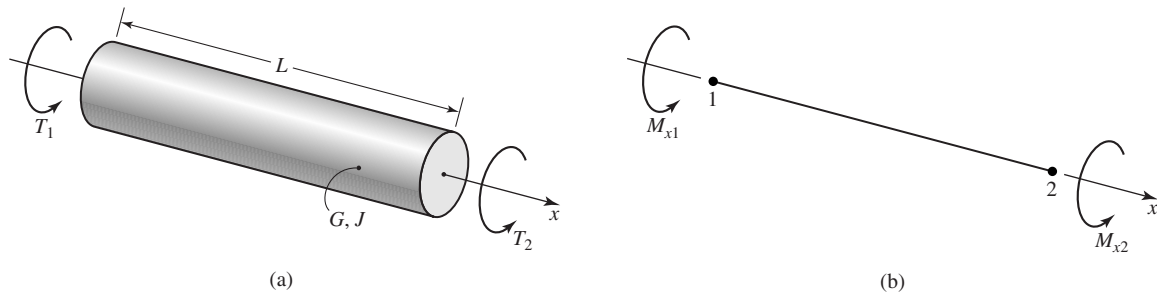
where the  $10 \times 10$  element stiffness matrix has been written in the shorthand form

$$[k_e] = \begin{bmatrix} [k_{\text{axial}}] & [0] & [0] \\ [0] & [k_{\text{bending}}]_{xy} & [0] \\ [0] & [0] & [k_{\text{bending}}]_{xz} \end{bmatrix} \quad (4.68)$$

The equivalent nodal loads corresponding to a distributed load are computed on the basis of work equivalence, as in Section 4.6. For a uniform distributed load  $q_z(x) = q_z$ , the equivalent nodal load vector is found to be

$$\begin{Bmatrix} f_{qz1} \\ M_{qz1} \\ f_{qz2} \\ M_{qz2} \end{Bmatrix} = \begin{Bmatrix} \frac{q_z L}{2} \\ -\frac{q_z L^2}{12} \\ \frac{q_z L}{2} \\ \frac{q_z L^2}{12} \end{Bmatrix} \quad (4.69)$$

The addition of torsion to the general beam element is accomplished with reference to Figure 4.16a, which depicts a circular cylinder subjected to torsion via twisting moments applied at its ends. A corresponding torsional finite element



**Figure 4.16**

(a) Circular cylinder subjected to torsion. (b) Torsional finite element notation.



## 4.8 A General Three-Dimensional Beam Element

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is shown in Figure 4.16b, where the nodes are 1 and 2, the axis of the cylinder is the  $x$  axis, and twisting moments are positive according to the right-hand rule. From elementary strength of materials, it is well known that the angle of twist per unit length of a uniform, elastic circular cylinder subjected to torque  $T$  is given by

$$\phi = \frac{T}{JG} \quad (4.70)$$

where  $J$  is polar moment of inertia of the cross-sectional area and  $G$  is the shear modulus of the material. As the angle of twist per unit length is constant, the total angle of twist of the element can be expressed in terms of the nodal rotations and twisting moments as

$$\theta_{x2} - \theta_{x1} = \frac{TL}{JG} \quad (4.71)$$

or

$$T = \frac{JG}{L}(\theta_{x2} - \theta_{x1}) = k_T(\theta_{x2} - \theta_{x1}) \quad (4.72)$$

Comparison of Equation 4.72 with Equation 2.2 for a linearly elastic spring and consideration of the equilibrium condition  $M_{x1} + M_{x2} = 0$  lead directly to the element equilibrium equations:

$$\frac{JG}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} = \begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} \quad (4.73)$$

so the torsional stiffness matrix is

$$[k_{\text{torsion}}] = \frac{JG}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (4.74)$$

While this development is, strictly speaking, applicable only to a circular cross section, an equivalent torsional stiffness  $J_{eq}G/L$  is known for many common structural cross sections and can be obtained from standard structural tables or strength of materials texts.

Adding the torsional characteristics to the general beam element, the element equations become

$$\begin{bmatrix} [k_{\text{axial}}] & [0] & [0] & [0] \\ [0] & [k_{\text{bending}}]_{xy} & [0] & [0] \\ [0] & [0] & [k_{\text{bending}}]_{xz} & [0] \\ [0] & [0] & [0] & [k_{\text{torsion}}] \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \\ w_1 \\ \theta_{y1} \\ w_2 \\ \theta_{y2} \\ \theta_{x1} \\ \theta_{x2} \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{x2} \\ f_{y1} \\ M_{z2} \\ f_{y2} \\ M_{z2} \\ f_{z1} \\ M_{y1} \\ f_{z2} \\ M_{y2} \\ M_{x1} \\ M_{x2} \end{Bmatrix} \quad (4.75)$$

and the final stiffness matrix for a general 3-D beam element is observed to be a  $12 \times 12$  symmetric matrix composed of the individual stiffness matrices representing axial loading, two-plane bending, and torsion.

The general beam element can be utilized in finite element analyses of three-dimensional frame structures. As with most finite elements, it is often necessary to transform the element matrices from the element coordinate system to the global coordinates. The transformation procedure is quite similar to that discussed for the bar and two-dimensional beam elements, except, of course, for the added algebraic complexity arising from the size of the stiffness matrix and certain orientation details required.

## 4.9 CLOSING REMARKS

In this chapter, finite elements for beam bending are formulated using elastic flexure theory from elementary strength of materials. The resulting elements are very useful in modeling frame structures in two or three dimensions. A general three-dimensional beam element including axial, bending, and torsional effects is developed by, in effect, superposition of a spar element, two flexure elements, and a torsional element.

In development of the beam elements, stiffening of the elements owing to tensile loading, the possibility of buckling under compressive axial loading, and transverse shear effects have not been included. In most commercial finite element software packages, each of these concerns is an option that can be taken into account at the user's discretion.

## REFERENCES

1. Beer, F. P., E. R. Johnston, and J. T. DeWolf. *Mechanics of Materials*, 3rd ed. New York: McGraw-Hill, 2002.
2. Budynas, R. *Advanced Strength and Applied Stress Analysis*, 2nd ed. New York: McGraw-Hill, 1999.

## PROBLEMS

- 4.1 Two identical beam elements are connected at a common node as shown in Figure P4.1. Assuming that the nodal displacements  $v_i$ ,  $\theta_i$  are known, use Equation 4.32 to show that the normal stress  $\sigma_x$  is, in general, discontinuous at the common element boundary (i.e., at node 2). Under what condition(s) would the stress be continuous?

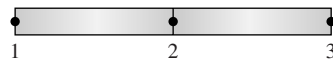


Figure P4.1

- 4.2 For the beam element loaded as shown in Figure P4.2, construct the shear force and bending moment diagrams. What is the significance of these diagrams with respect to Equations 4.10, 4.17, and the relation  $V = \frac{dM}{dx}$  from strength of materials theory?

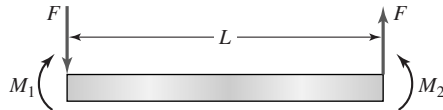


Figure P4.2

- 4.3 For a uniformly loaded beam as shown in Figure P4.3, the strength of materials theory gives the maximum deflection as

$$v_{\max} = -\frac{5qL^4}{384EI_z}$$

at  $x = L/2$ . Treat this beam as a single finite element and compute the maximum deflection. How do the values compare?

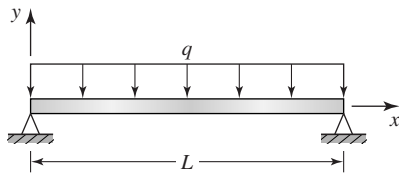


Figure P4.3

- 4.4 The beam element shown in Figure P4.4 is subjected to a linearly varying load of maximum intensity  $q_o$ . Using the work-equivalence approach, determine the nodal forces and moments.

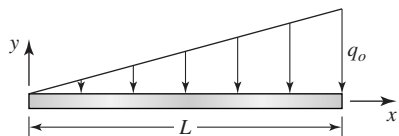


Figure P4.4

- 4.5 Use the results of Problem 4.4 to calculate the deflection at node 2 of the beam shown in Figure P4.5 if the beam is treated as a single finite element.

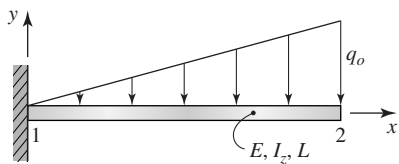


Figure P4.5

- 4.6 For the beam element of Figure P4.5, compute the reaction force and moment at node 1. Compute the maximum bending stress assuming beam height is  $2h$ . How does the stress value compare to the maximum stress obtained by the strength of materials approach?
- 4.7 Repeat Problem 4.5 using two equal length elements. For this problem, let  $E = 30 \times 10^6$  psi,  $I_z = 0.1 \text{ in.}^4$ ,  $L = 10 \text{ in.}$ ,  $q_o = 10 \text{ lb/in.}$
- 4.8 Consider the beam shown in Figure P4.8. What is the minimum number of elements that can be used to model this problem? Construct the global nodal load vector corresponding to your answer.

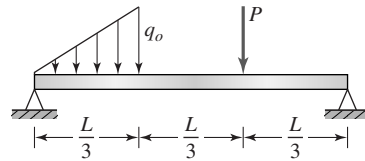


Figure P4.8

- 4.9 What is the justification for writing Equation 4.36 in the form of Equation 4.37?
- 4.10–4.15 For each beam shown in the associated figure, compute the deflection at the element nodes. The modulus of elasticity is  $E = 10 \times 10^6$  psi and the cross section is as shown in each figure. Also compute the maximum bending stress. Use the finite element method with the minimum number of elements for each case.

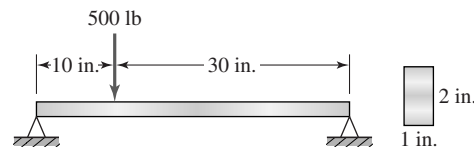


Figure P4.10

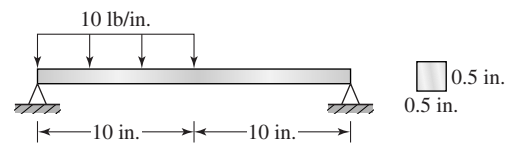


Figure P4.11

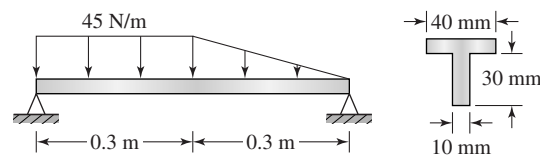
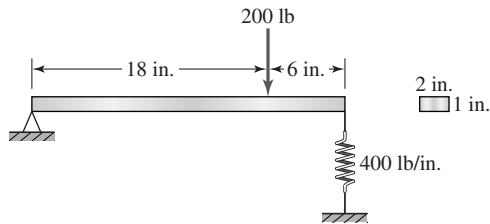
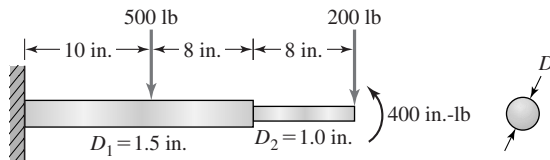
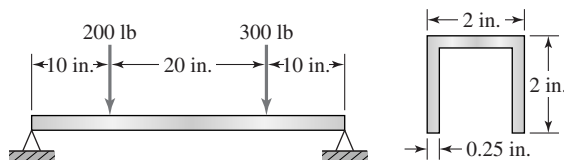
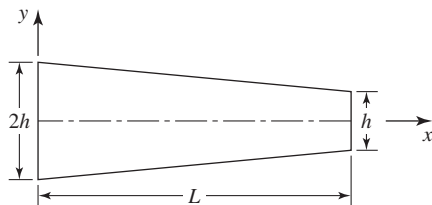


Figure P4.12

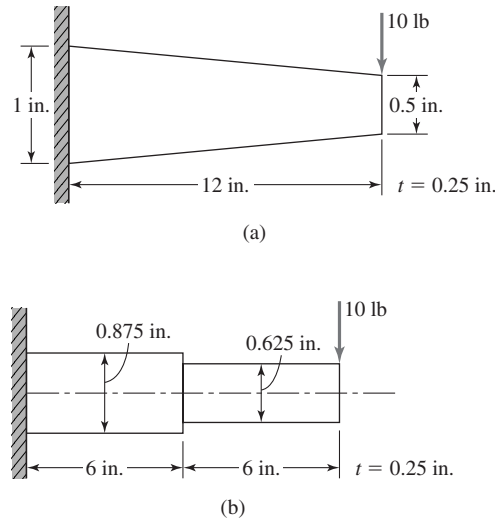

**Figure P4.13**

**Figure P4.14**

**Figure P4.15**

- 4.16** The tapered beam element shown in Figure P4.16 has uniform thickness  $t$  and varies linearly in height from  $2h$  to  $h$ . Beginning with Equation 4.37, derive the strain energy expression for the element in a form similar to Equation 4.39.


**Figure P4.16**

- 4.17** Use the result of Problem 4.16 to derive the value of component  $k_{11}$  of the element stiffness matrix.
- 4.18** The complete stiffness matrix for the tapered element of Figure P4.16 is given by

$$[k] = \frac{Eth^3}{60L^3} \begin{bmatrix} 243 & 156L & -243 & 87L \\ 156L & 56L^2 & -156L & 42L^2 \\ -243 & -156L & 243 & -87L \\ 87L & 42L^2 & -87L & 45L^2 \end{bmatrix}$$

**Figure P4.18**

- Using the given stiffness matrix with  $E = 10(10^6)$ , compute the deflection of node 2 for the tapered element loaded as shown in Figure P4.18a.
  - Approximate the tapered beam using two straight elements, as in Figure P4.18b, and compute the deflection.
  - How do the deflection results compare?
  - How do the stress computations compare?
- 4.19** The six equilibrium equations for a beam-axial element in the element coordinate system are expressed in matrix form as

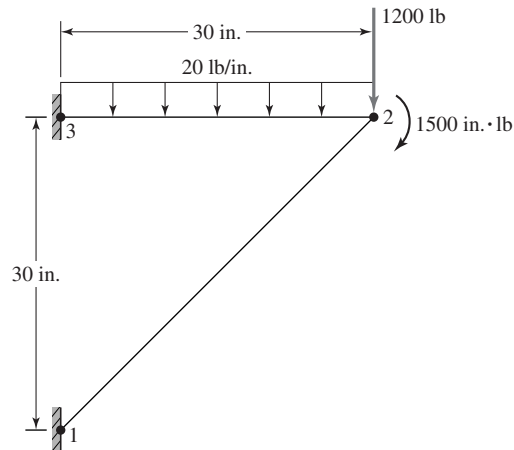
$$[k_e] \{\delta\} = \{f_e\}$$

with  $\{\delta\}$  as given by Equation 4.61,  $[k_e]$  by Equation 4.62, and  $\{f_e\}$  as the nodal force vector

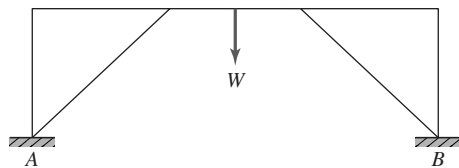
$$\{f_e\} = [f_{1x} \quad f_{1y} \quad M_1 \quad f_{2x} \quad f_{2y} \quad M_2]^T$$

For an element oriented at an arbitrary angle  $\psi$  relative to the global  $X$  axis, convert the equilibrium equations to the global coordinate system and verify Equation 4.65.

- 4.20** Use Equation 4.63 to express the strain energy of a beam-axial element in terms of global displacements. Apply the principle of minimum potential energy to derive the expression for the element equilibrium equations in the global coordinate system. (Warning: This is algebraically tedious.)
- 4.21** The two-dimensional frame structure shown in Figure P4.21 is composed of two  $2 \times 4$  in. steel members ( $E = 10 \times 10^6$  psi), and the 2-in. dimension is perpendicular to the plane of loading. All connections are treated as welded joints. Using two beam-axial elements and the node numbers as shown, determine

**Figure P4.21**

- The global stiffness matrix.
  - The global load vector.
  - The displacement components of node 2.
  - The reaction forces and moments at nodes 1 and 3.
  - Maximum stress in each element.
- 4.22** Repeat Problem 4.21 for the case in which the connection at node 2 is a pin joint.
- 4.23** The frame structure shown in Figure P4.23 is the support structure for a hoist located at the point of application of load  $W$ . The supports at  $A$  and  $B$  are completely fixed. Other connections are welded. Assuming the structure to be modeled using the minimum number of beam-axial elements:

**Figure P4.23**

- How many elements are needed?
- What is the size of the assembled global stiffness matrix?
- What are the constraint (boundary) conditions?
- What is the size of the reduced global stiffness matrix after application of the constraint conditions?
- Assuming a finite element solution is obtained for this problem, what steps could be taken to judge the accuracy of the solution?

4.24 Repeat Problem 4.23 for the frame structure shown in Figure P4.24.

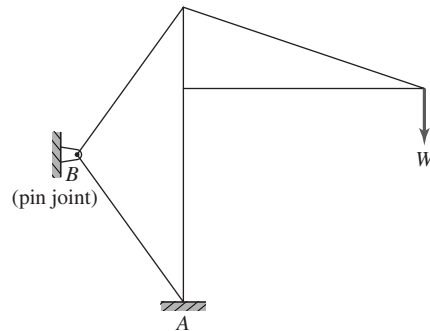


Figure P4.24

4.25 Verify Equation 4.69 by direct calculation.

4.26 The cantilevered beam depicted in Figure P4.26 is subjected to two-plane bending. The loads are applied such that the planes of bending correspond to the principal moments of inertia. Noting that no axial or torsional loadings are present, model the beam as a single element (that is, construct the  $8 \times 8$  stiffness matrix containing bending terms only) and compute the deflections of the free end, node 2. Determine the exact location and magnitude of the maximum bending stress. (Use  $E = 207 \text{ GPa}$ .)

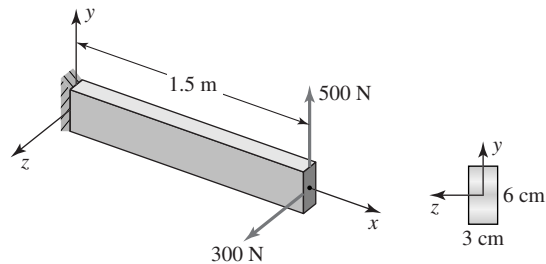


Figure P4.26

4.27 Repeat Problem 4.26 for the case in which the concentrated loads are replaced by uniform distributed loads  $q_y = 6 \text{ N/cm}$  and  $q_z = 4 \text{ N/cm}$  acting in the positive coordinate directions, respectively.



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# CHAPTER 5

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## Method of Weighted Residuals

### 5.1 INTRODUCTION

Chapters 2, 3, and 4 introduced some of the basic concepts of the finite element method in terms of the so-called line elements. The linear elastic spring, the bar element and the flexure element are line elements because structural properties can be described in terms of a single spatial variable that identifies position along the longitudinal axis of the element. The displacement-force relations for the line elements are straightforward, as these relations are readily described using only the concepts of elementary strength of materials. To extend the method of finite element analysis to more general situations, particularly nonstructural applications, additional mathematical techniques are required. In this chapter, the method of weighted residuals is described in general and Galerkin's method of weighted residuals [1] is emphasized as a tool for finite element formulation for essentially any field problem governed by a differential equation.

### 5.2 METHOD OF WEIGHTED RESIDUALS

It is a basic fact that most practical problems in engineering are governed by differential equations. Owing to complexities of geometry and loading, rarely are exact solutions to the governing equations possible. Therefore, approximate techniques for solving differential equations are indispensable in engineering analysis. Indeed, the finite element method is such a technique. However, the finite element method is based on several other, more-fundamental, approximate techniques, one of which is discussed in detail in this section and subsequently applied to finite element formulation.

The *method of weighted residuals* (MWR) is an approximate technique for solving boundary value problems that utilizes *trial functions* satisfying the